Title: The logic of quantum mechanics - take II

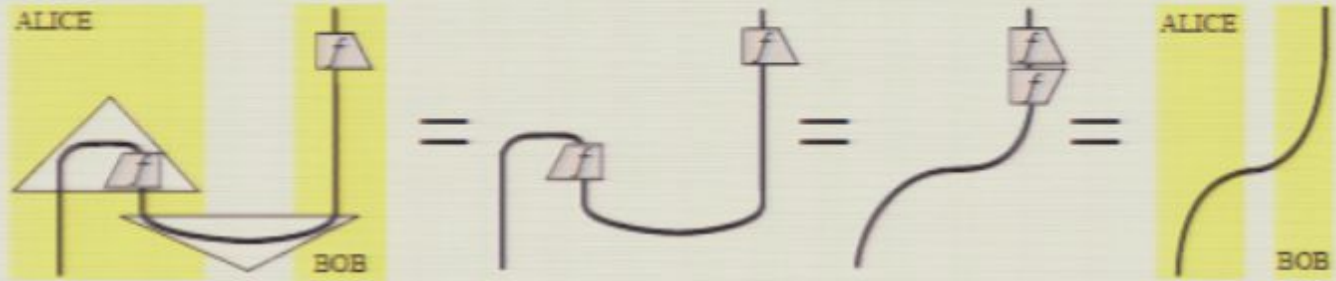
Date: May 10, 2011 10:50 AM

URL: http://pirsa.org/11050032

Abstract: It is now exactly 75 years ago that John von Neumann denounced his own Hilbert space formalism: ``I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic) [1] His reason was that Hilbert space does not elucidate in any direct manner the key quantum behaviors. One year later, together with Birkhoff, they published "The logic of quantum mechanics". However, it is fair to say that this program was never successful nor does it have anything to do with logic. So what is logic? We will conceive logic in two manners: (1) Something which captures the mathematical content of language (cf `and', `or', `no', `if ... then' are captured by Boolean algebra); (2) something that can be encoded in a `machine' and enables it to reason. Recently we have proposed a new kind of `logic of quantum mechanics' [4]. It follows Schrodinger in that the behavior of compound quantum systems, described by the tensor product [2, again 75 years ago], that captures the quantum behaviors. Over the past couple of years we have played the following game: how much quantum phenomena can be derived from `composition + epsilon'. It turned out that epsilon can be taken to be `very little', surely not involving anything like continuum, fields, vector spaces, but merely a `two-dimensional space' of temporal composition (cf `and then') and compoundness (cf `while'), together with some very natural purely operational assertion. In a very short time, this radically different approach has produced a universal graphical language for quantum theory which helped to resolve some open problems. Most importantly, it paved the way to automate quantum reasoning [5,6], and also enables to model meaning for natural languages [7,8]. That is, we are truly taking `quantum logic' now! If time permits, we also discuss how this logical view has helped to solve concrete problems in quantum information. [1] M Redei (1997) Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead). Stud Hist Phil Mod Phys 27, 493-510. [2] G Birkhoff and J von Neumann (1936) The logic of quantum mechanics. Annals of Mathematics 37, 823843. [3] E Schroedinger, (1935) Discussion of probability relations between separated systems. Proc Camb Phil Soc 31, 555-563; (1936) 32, 446-451. [4] B Coecke (2010) Quantum picturalism. Contemporary Physics 51, 59-83. arXiv:0908.1787 [5] L Dixon, R Duncan, A Kissinger and A Merry. http://dream.inf.ed.ac.uk/projects/quantomatic/ [6] L Dixon and R Duncan (2009) Graphical reasoning in compact closed categories for quantum computation. Annals of Mathematics and Articial Intelligence 56, 2342. [7] B Coecke, M Sadrzadeh & S Clark (2010) Linguistic Analysis 36. Mathematical foundations for a compositional distributional model of meaning. arXiv:1003.4394 [8] New scientist (11 Dec 2011) Quantum links let computers read.

The Logic of Quantum Mechanics - take II

Bob Coecke – Oxford Univ. Computing Lab. – Quantum Group



BC (2010) *Quantum picturalism*. Contemporary physics **51**, 59–83. arXiv:0908.1787 BC & **Ross Duncan** (2011) *Interacting quantum observables*. New Journal of Physics **13**, 043016. arXiv:0906.4725 BC, **Mehrnoosh Sadrzadeh** & **Stephen Clark** (2011) *Mathematical foundations for a compositional distributional model of meaning*. Linguistic Analysis - Lambek Festschrift. arXiv:1003.4394 BC & **Aleks Kissinger** (2010) *The compositional structure of multipartite quantum entanglement*. ICALP'10. arXiv:1002.2540 **Lucas Dixon**, R. Duncan, A. Kissinger & **Alex Merry**. `quantomatic`. http://sites.google.com/site/quantomatic/

INTRODUCTION

— *genesis* —

[**von Neumann 1932**] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

[Birkhoff and von Neumann 1936] **The Logic of Quantum Mechanics** in *Annals of Mathematics*.

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

[Birkhoff and von Neumann 1936] **The Logic of Quantum Mechanics** in *Annals of Mathematics*.

[1936 – 2000] many followed them, ...

— *genesis* —

[von Neumann 1932] Formalized quantum mechanics in "Mathematische Grundlagen der Quantenmechanik"

[von Neumann to Birkhoff 1935] "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space no more." (sic)

[Birkhoff and von Neumann 1936] **The Logic of Quantum Mechanics** in *Annals of Mathematics*.

[1936 – 2000] many followed them, ... and **FAILED**.

— *the mathematics of it* —

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

WHY?

von Neumann: only used *it* since *it* was 'available'.

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

WHY?

von Neumann: only used *it* since *it* was 'available'.

Model theory: one can do almost anything with *it*.

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

WHY?

von Neumann: only used *it* since *it* was 'available'.

Model theory: one can do almost anything with *it*.

$\Rightarrow$ ***it* comes with little conceptual content** $\Leftarrow$

— *the mathematics of it* —

Hilber space stuff: continuum, field structure of complex numbers, vector space over it, inner-product, etc.

WHY?

von Neumann: only used *it* since *it* was 'available'.

Model theory: one can do almost anything with *it*.

$\Rightarrow$ ***it* comes with little conceptual content** $\Leftarrow$

SO WHAT ELSE SHOULD ONE USE THEN?

— *the physics of it* —

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

Schrödinger (1935): the stuff which is the true soul of quantum theory is **how quantum systems compose**.

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

Schrödinger (1935): the stuff which is the true soul of quantum theory is **how quantum systems compose**.

Quantum Computer Scientists: Schrödinger is right!

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

Schrödinger (1935): the stuff which is the true soul of quantum theory is **how quantum systems compose**.

Quantum Computer Scientists: Schrödinger is right!

75 years later Birkhoff-von Neumann quantum 'logicians' still fail to elegantly capture composition of quantum systems, ... nor much else neiter, ...

— *the game plan* —

— *the physics of it* —

von Neumann crafted Birkhoff-von Neumann Quantum 'Logic' to capture the concept of **superposition**.

Schrödinger (1935): the stuff which is the true soul of quantum theory is **how quantum systems compose**.

Quantum Computer Scientists: Schrödinger is right!

75 years later Birkhoff-von Neumann quantum 'logicians' still fail to elegantly capture composition of quantum systems, ... nor much else neiter, ...

— *the game plan* —

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize "$\otimes$" without reference to spaces.**

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize** "$\otimes$" **without reference to spaces**.

Task 1. Investigate which assumptions (i.e. which structure) on $\otimes$ is needed to deduce **physical phenomena**?

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize** "$\otimes$" **without reference to spaces**.

Task 1. Investigate which assumptions (i.e. which structure) on $\otimes$ is needed to deduce **physical phenomena**?

Task 2. Investigate wether such an "interaction structure" appear elsewhere in **"our classical reality"**

Outcome 1a:

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize** "$\otimes$" **without reference to spaces**.

Task 1. Investigate which assumptions (i.e. which structure) on $\otimes$ is needed to deduce **physical phenomena**?

Task 2. Investigate wether such an "interaction structure" appear elsewhere in **"our classical reality"**

Outcome 1a:

— *the game plan* —

Task 0. Solve:

$$\frac{\texttt{tensor product structure}}{\texttt{the other stuff}} = ???$$

i.e. **axiomatize** "$\otimes$" **without reference to spaces**.

Task 1. Investigate which assumptions (i.e. which structure) on $\otimes$ is needed to deduce **physical phenomena**?

Task 2. Investigate wether such an "interaction structure" appear elsewhere in **"our classical reality"**

Outcome 1a: "Sheer ratio of results to assumptions"

Hans Halvorson (2010) Editorial to: *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, Cambridge University Press.

Outcome 1a: "Sheer ratio of results to assumptions" confirms that we are probing something very essential.

Hans Halvorson (2010) Editorial to: *Deep Beauty: Understanding the Quantum World through Mathematical Innovation*, Cambridge University Press.

Outcome 1a: "Sheer ratio of results to assumptions" confirms that we are probing something very essential.

Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. 2× ICALP'10)

E.g.: Ross Duncan & Simon Perdrix (2010) *Rewriting measurement-based quantum computations with generalised flow.* ICALP'10.

Outcome 1a: "Sheer ratio of results to assumptions" confirms that we are probing something very essential.

Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. 2× ICALP'10)

Outcome 1c: Framework is a simple intuitive (but rigorous) diagrammatic language.

[1] BC (2010) *Quantum picturalism*. Contemporary Physics **51**, 59–83.

Outcome 1a: "Sheer ratio of results to assumptions" confirms that we are probing something very essential.

Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. 2× ICALP'10)

Outcome 1c: Framework is a simple intuitive (but rigorous) diagrammatic language, meanwhile adopted by others e.g. Lucien Hardy in arXiv:1005.5164:

"... we join the ***quantum picturalism*** revolution [1]"

---

[1] BC (2010) ***Quantum picturalism***. Contemporary Physics **51**, 59–83.

Outcome 2a:

Outcome 1a: "Sheer ratio of results to assumptions" confirms that we are probing something very essential.

Outcome 1b: Exposing this structure has already helped to **solve open problems elsewhere**. (e.g. 2× ICALP'10)

Outcome 1c: Framework is a simple intuitive (but rigorous) diagrammatic language, meanwhile adopted by others e.g. Lucien Hardy in arXiv:1005.5164:

"... we join the ***quantum picturalism*** revolution [1]"

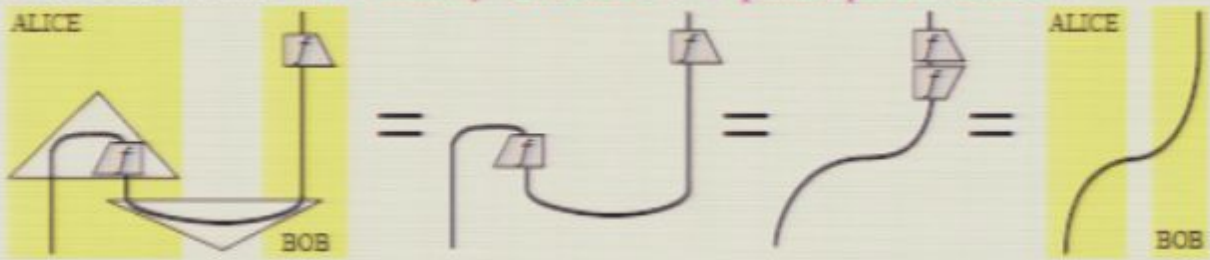
---

[1] BC (2010) ***Quantum picturalism***. Contemporary Physics **51**, 59–83.

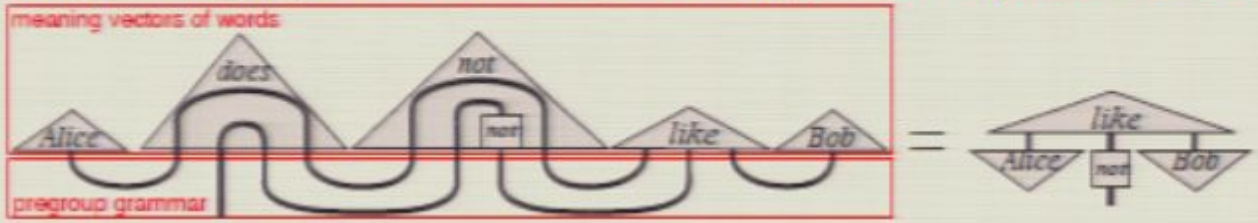
Outcome 2a:

Outcome 2a:

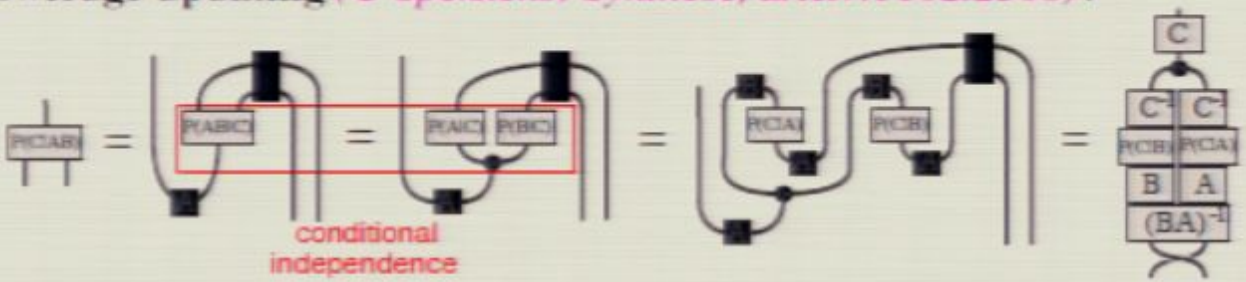
Behaviors of matter *(Abramsky-C; LiCS'04, quant-ph/0402130)*:



Meaning in language *(Clark-C-Sadrzadeh; Linguistic Analysis, arXiv:1003.4394)*:



Knowledge updating *(C-Spekkens; Synthese, arXiv:1102.2368)*:



— *the logic of it* —

— *the logic of it* —

WHAT IS "LOGIC"?

— *the logic of it* —

WHAT IS "LOGIC"?

Definitely not Birkhoff-von Neumann quantum logic, which, by definition, is 'logic without deduction',

— *the logic of it* —

WHAT IS "LOGIC"?

Definitely not Birkhoff-von Neumann quantum logic, which, by definition, is 'logic without deduction', nor is it some metaphysical jargon (cf. Putnam, Omnés, ...) aiming to solve the measurement problem.

— *the logic of it* —

WHAT IS "LOGIC"?

Pragmatic option 1: Logic is structure in language.

— *the logic of it* —

WHAT IS "LOGIC"?

Pragmatic option 1: Logic is structure in language.

"Alice and Bob ate everything or nothing, then got sick."

connectives $(\wedge, \vee)$: *and, or*
negation $(\neg)$: *not (cf. nothing = not something)*
entailment $(\Rightarrow)$: *then*
quantifiers $(\forall, \exists)$: *every(thing), some(thing)*
constants (a, b) : *thing*
variable (x) : *Alice, Bob*
predicates $(P(x), R(x, y))$: *eating, getting sick*
truth valuation $(0, 1)$: ***true, false***

$$(\forall z : Eat(a, z) \wedge Eat(b, z)) \wedge \neg(\exists z : Eat(a, z) \wedge Eat(b, z)) \Rightarrow Sick(a), Sick(b)$$

— *the logic of it* —

WHAT IS "LOGIC"?

Pragmatic option 1: **Logic is structure in language.**

Pragmatic option 2: **Logic lets machines reason.**

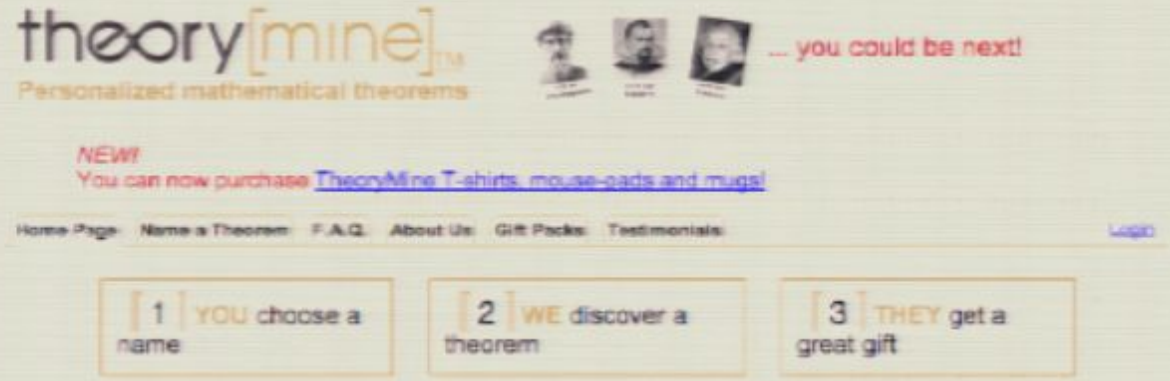
— *the logic of it* —

WHAT IS "LOGIC"?

Pragmatic option 1: Logic is structure in language.

Pragmatic option 2: Logic lets machines reason.

Cf. the *soft* incarnation of AI in robotics, automated theorem proving, automated theory exploration, ...



— *the logic of it* —

WHAT IS "LOGIC"?

Pragmatic option 1: Logic is structure in language.

Pragmatic option 2: Logic lets machines reason.

Our framework appeals to both senses of logic,

— *the logic of it* —

WHAT IS "LOGIC"?

Pragmatic option 1: Logic is structure in language.

Pragmatic option 2: Logic lets machines reason.

Our framework appeals to both senses of logic, and moreover induces important new applications:

From truth to meaning in natural language processing:

— NewScientist (December 2010)

— *the logic of it* —

WHAT IS "LOGIC"?

Pragmatic option 1: Logic is structure in language.

Pragmatic option 2: Logic lets machines reason.

Our framework appeals to both senses of logic, and moreover induces important new applications:

From truth to meaning in natural language processing:

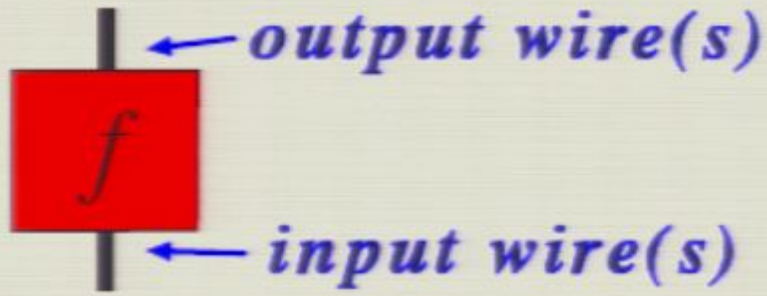
— NewScientist (December 2010)

Automated theorem generation for graphical theories:

— http://sites.google.com/site/quantomatic/

— the data of tensor logic —

$Box :=$

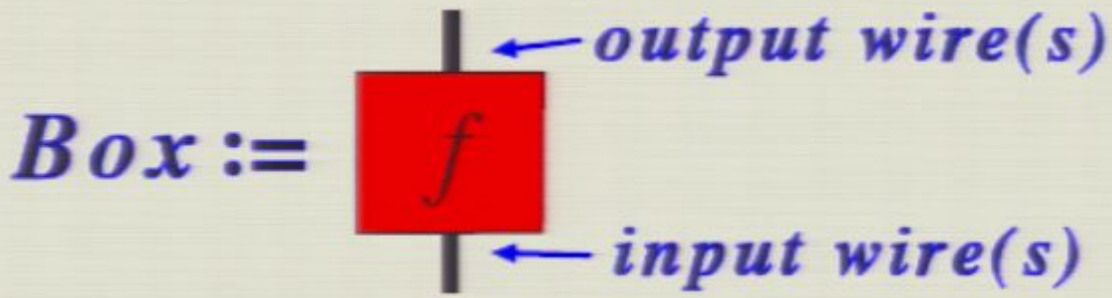


BASIC TENSOR LOGIC

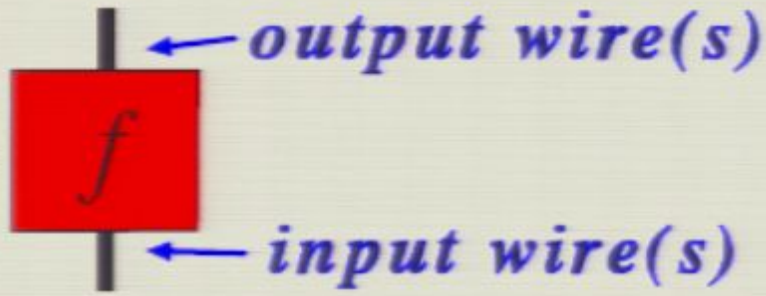
Roger Penrose (1971) *Applications of negative dimensional tensors*. In: Combinatorial Mathematics and its Applications, 221–244. Academic Press.

André Joyal and Ross Street (1991) *The Geometry of tensor calculus* I. Advances in Mathematics **88**, 55–112.

— *the data of tensor logic* —

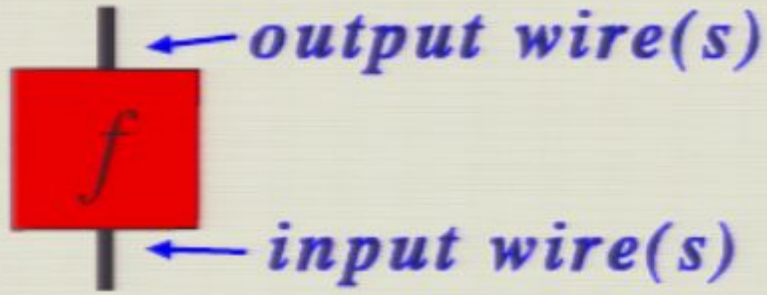


— *the data of tensor logic* —

Box := 

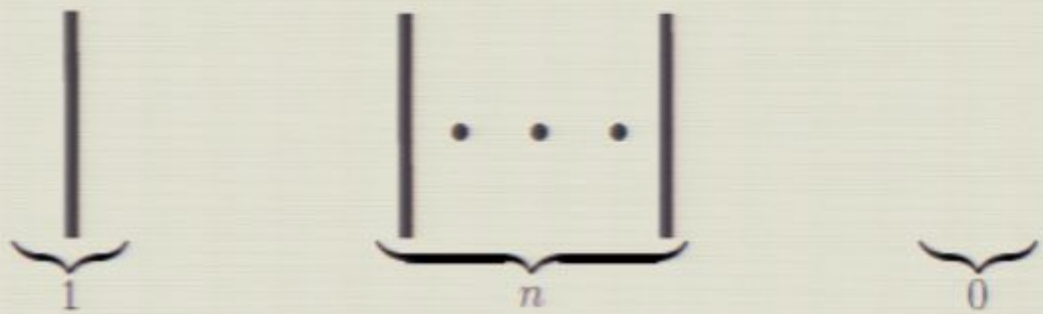
Interpretation: wire := system ; box := **process**

— *the data of tensor logic* —

Box := 

Interpretation: wire := system ; box := **process**

one system: *n subsystems:* no system:



— *the connectives of tensor logic* —

— *the connectives of tensor logic* —

sequential or *causal* or *connected* composition:

$g \circ f \equiv$

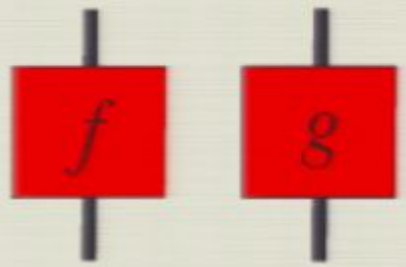
— *the connectives of tensor logic* —

sequential or *causal* or *connected* composition:

$g \circ f \equiv$

parallel or *acausal* or *disconnected* composition:

$f \otimes g \equiv$



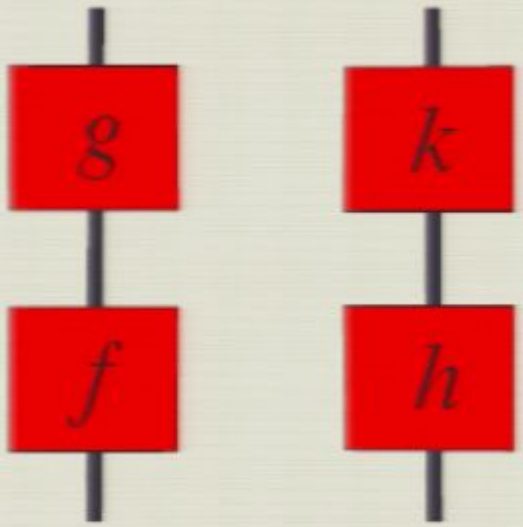
— *merely a new notation?* —

— *merely a new notation?* —

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$

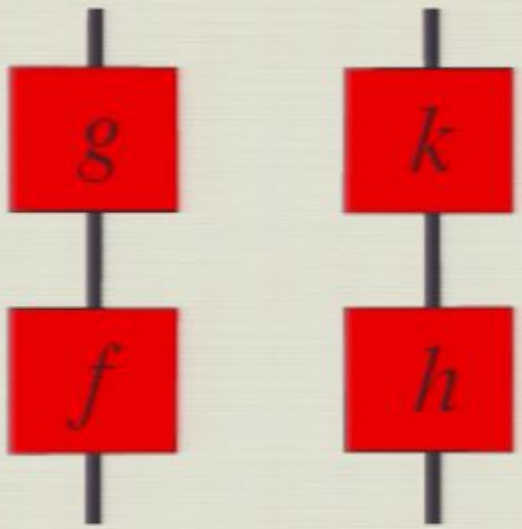
— *merely a new notation?* —

$$(g \circ f) \otimes (k \circ h)$$



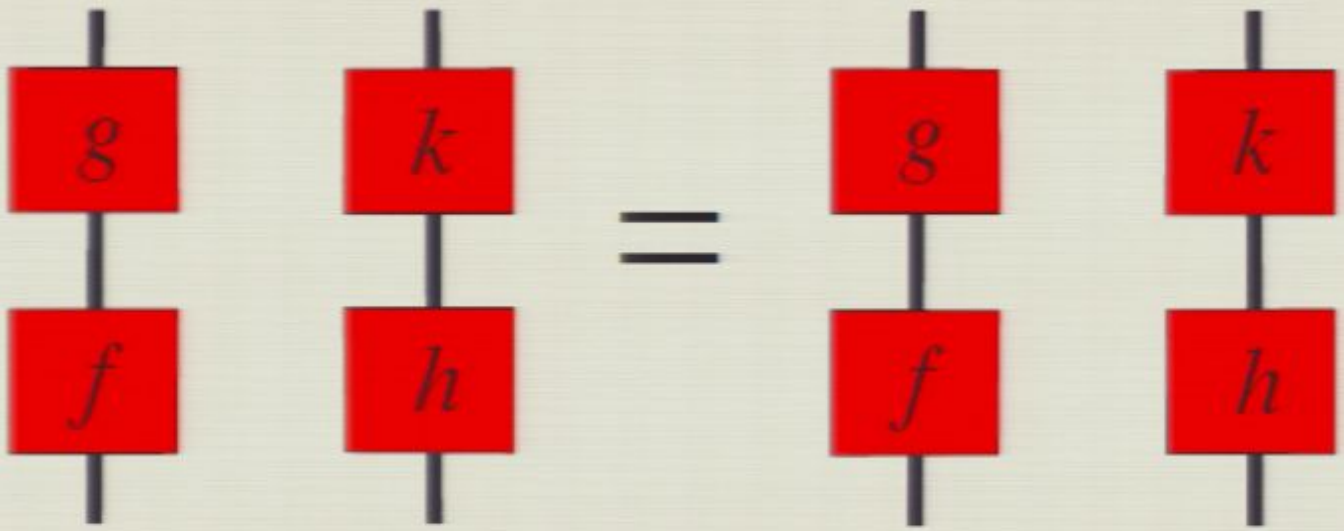
— *merely a new notation?* —

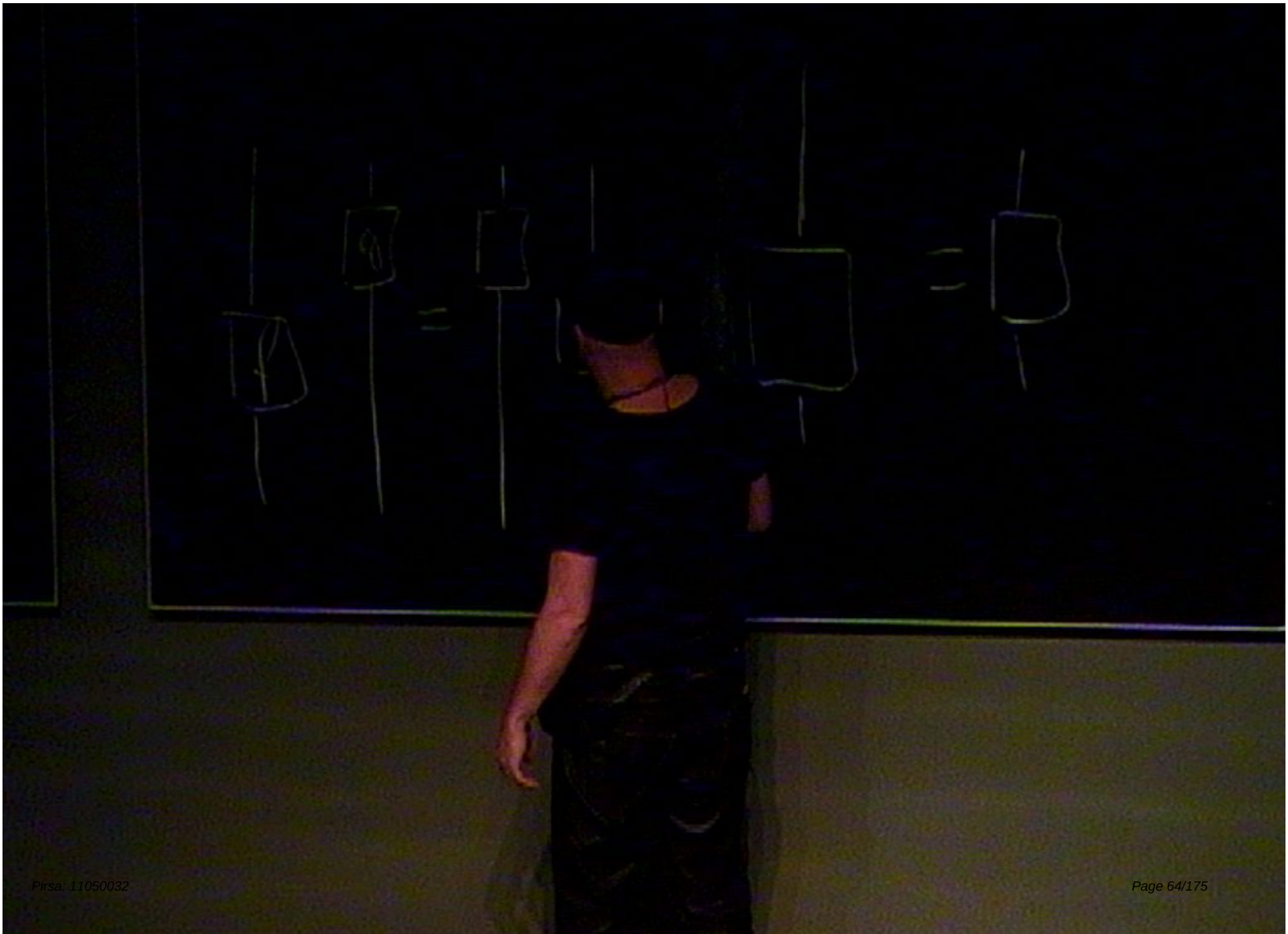
$$(g \otimes k) \circ (f \otimes h)$$

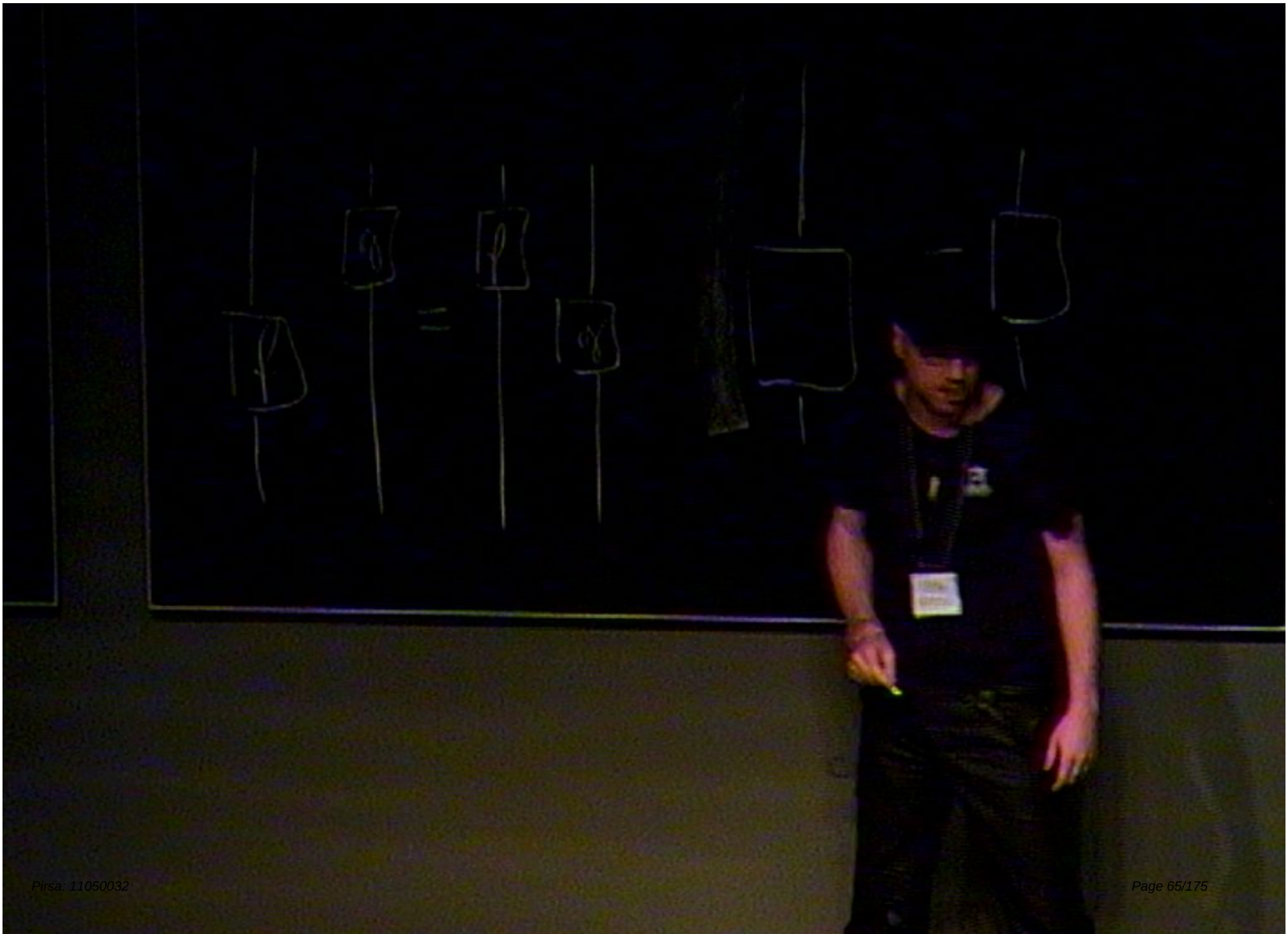


— *merely a new notation?* —

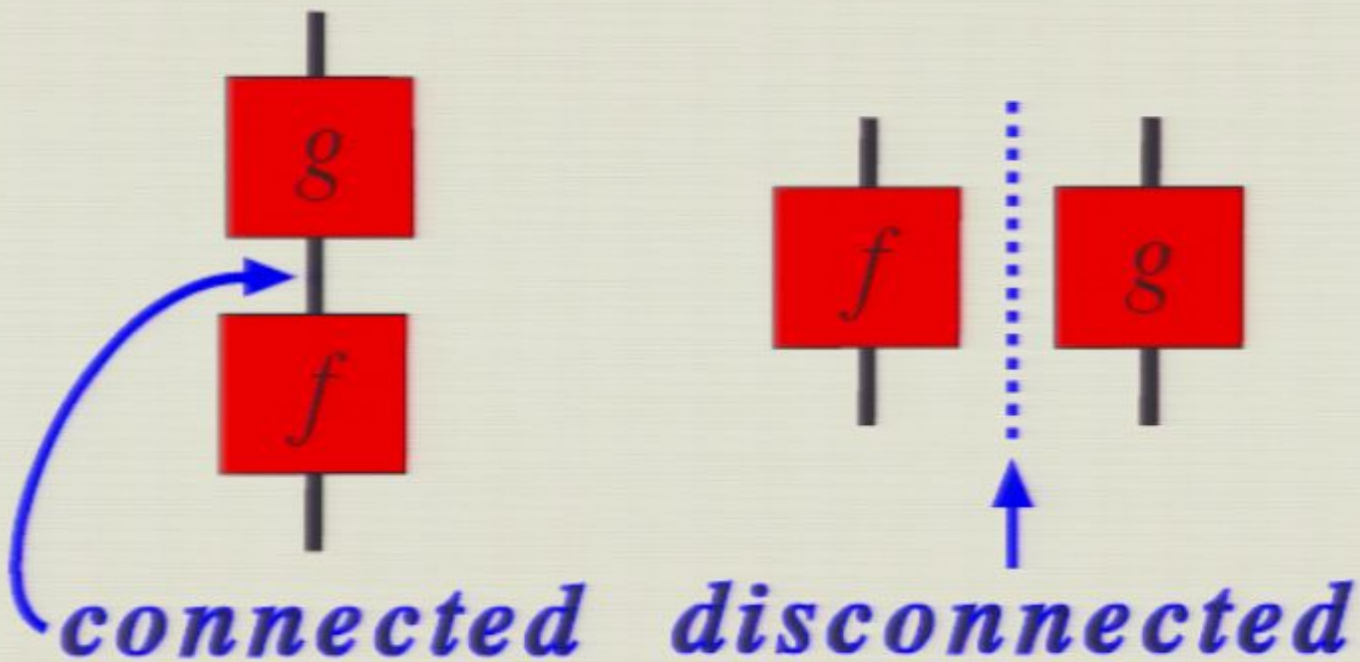
$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



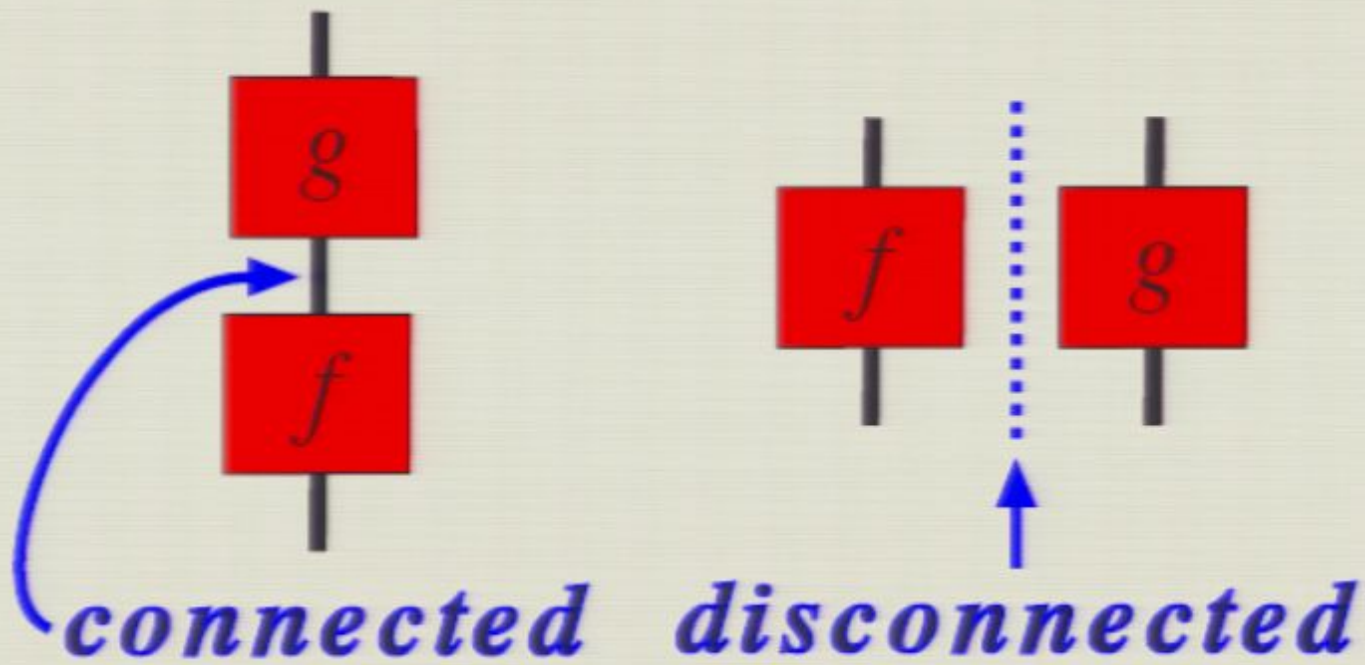




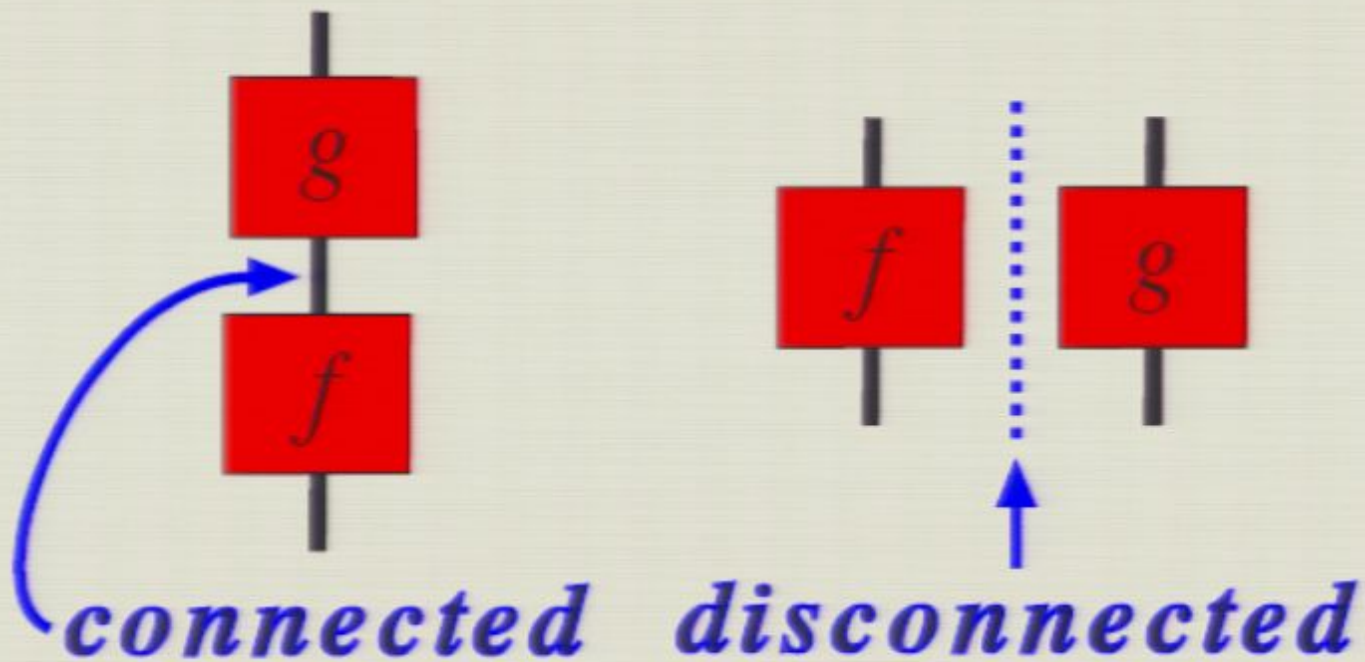
— topological connectedness as a paradigm —



— topological connectedness as a paradigm —

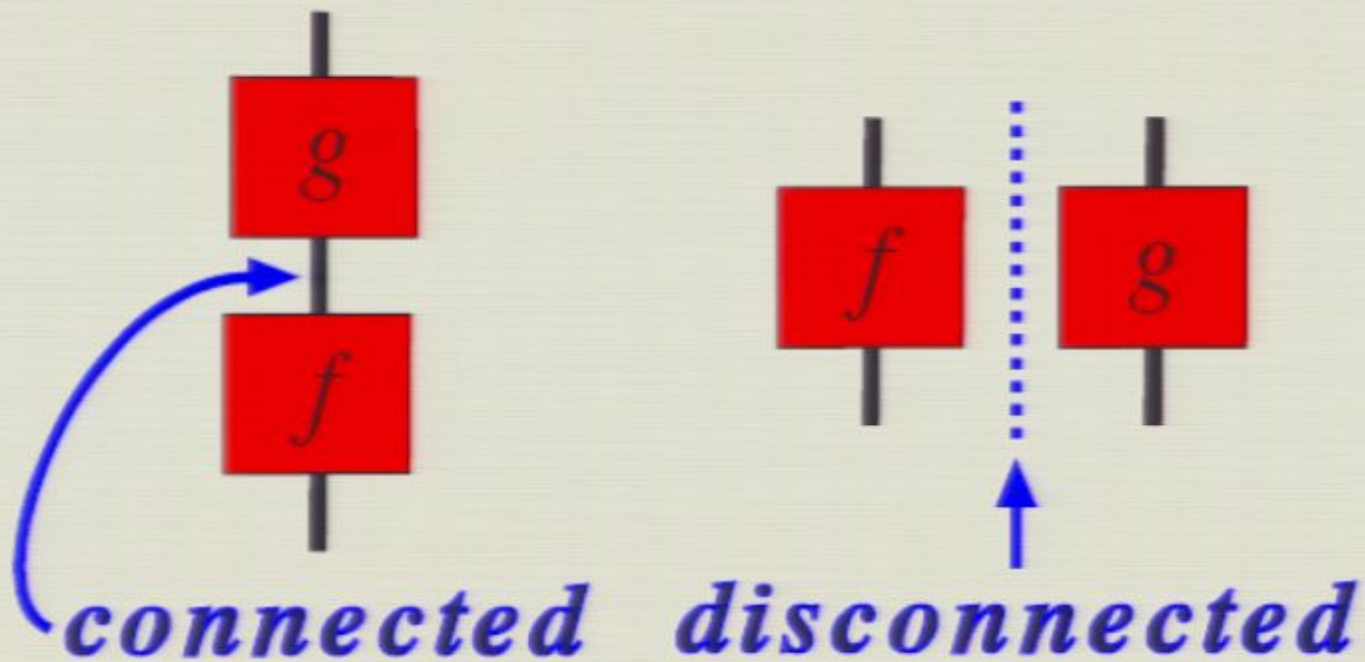


— *topological connectedness as a paradigm* —



This is a digital paradigm:

— *topological connectedness as a paradigm* —



This is a digital paradigm:

$0 \sim$ **no information-flow** $\quad 1 \sim$ **information-flow**

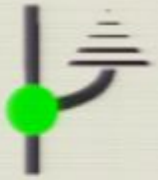
— topological connectedness as a paradigm —

dot =

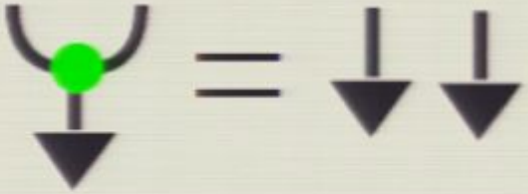
ground =

mixedness

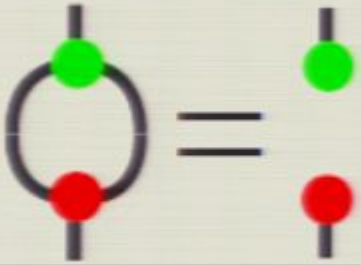
decoherence



eigenstate



complementarity



GHZ-state vs W-state

A MINIMAL LANGUAGE FOR QUANTUM REASONING

Samson Abramsky & BC (2004) *A categorical semantics for quantum protocols.* In: IEEE-LiCS'04. quant-ph/0402130

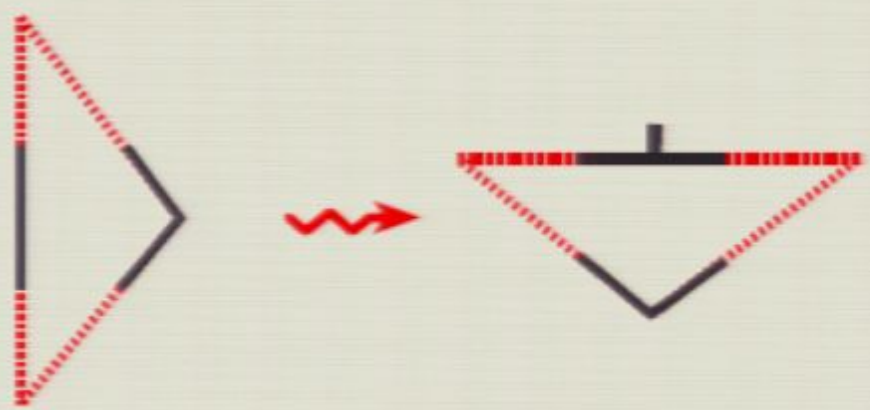
BC (2005) *Kindergarten quantum mechanics.* quant-ph/0510032

— graphical notation for states and effects —

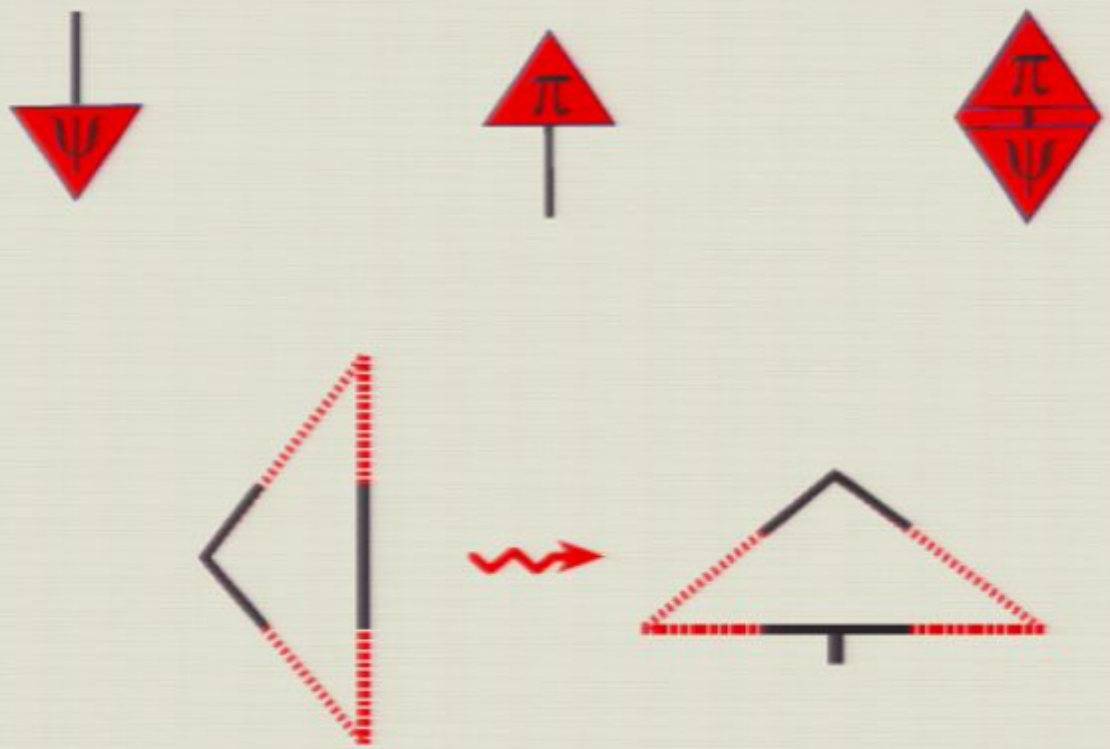
— graphical notation for states and effects —

$|\ \rangle$

— graphical notation for states and effects —

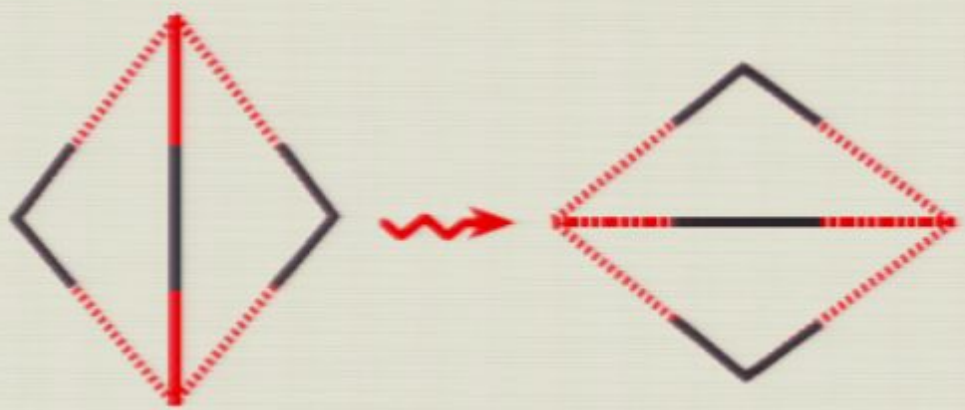


— graphical notation for states and effects —



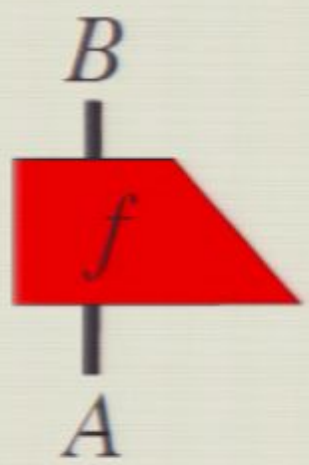
— graphical notation for states and effects —

— graphical notation for states and effects —

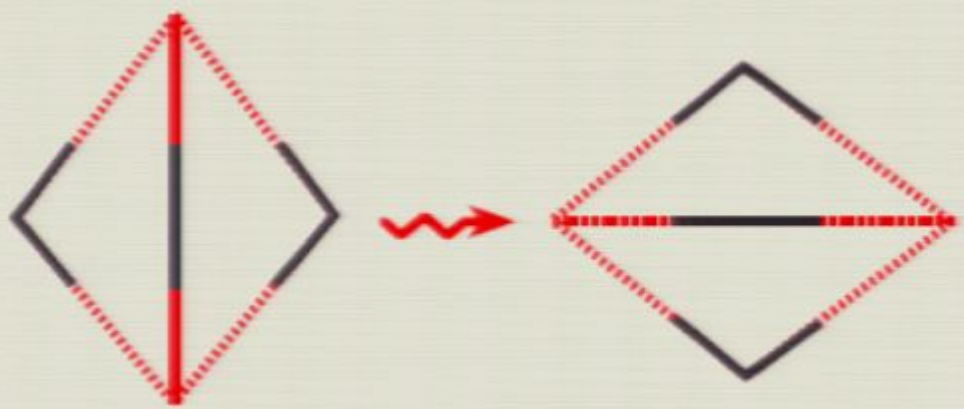


— *adjoint* —

$$f : A \to B$$

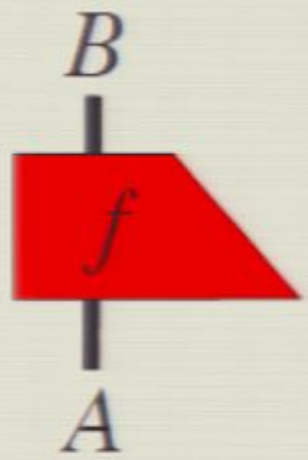


— graphical notation for states and effects —



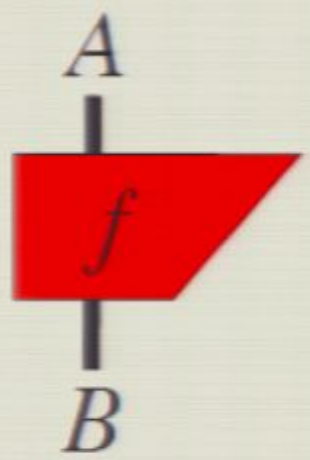
— *adjoint* —

$$f : A \to B$$



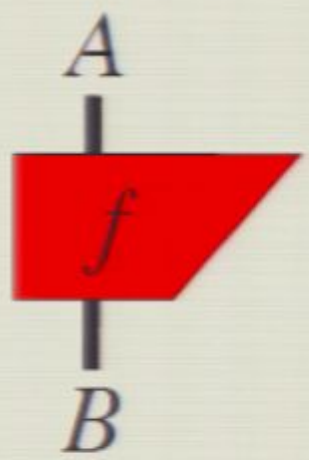
— *adjoint* —

$$f^\dagger : B \to A$$



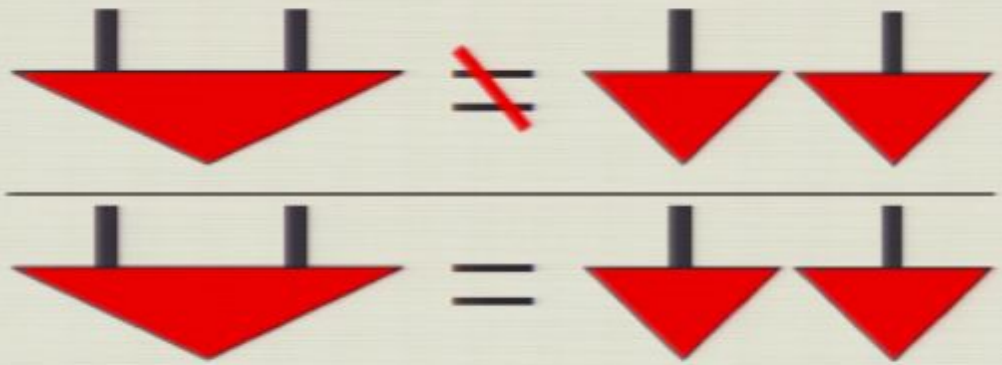
— *adjoint* —

$$f^\dagger : B \to A$$



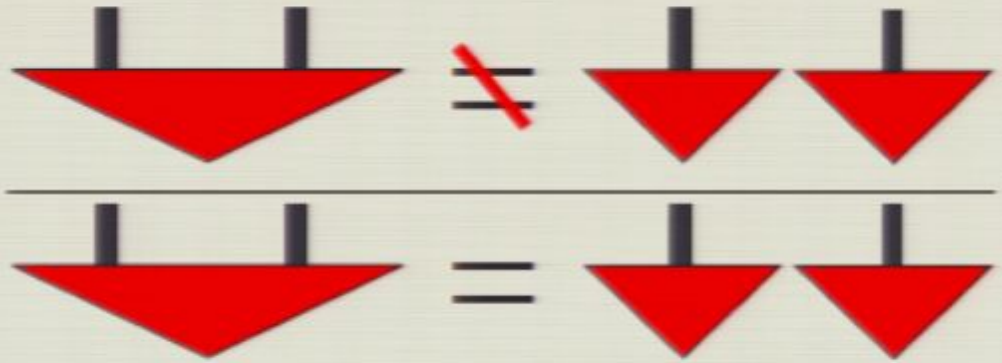
— *asserting (pure) entanglement* —

$$\frac{quantum}{classical} =$$



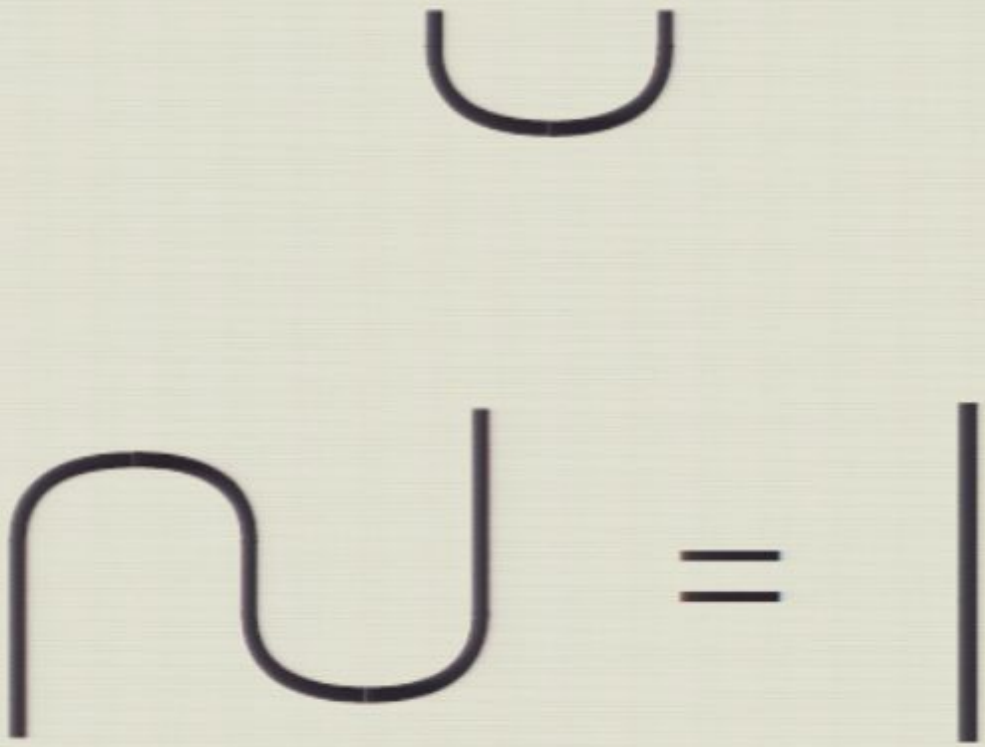
— *asserting (pure) entanglement* —

$$\frac{quantum}{classical} =$$

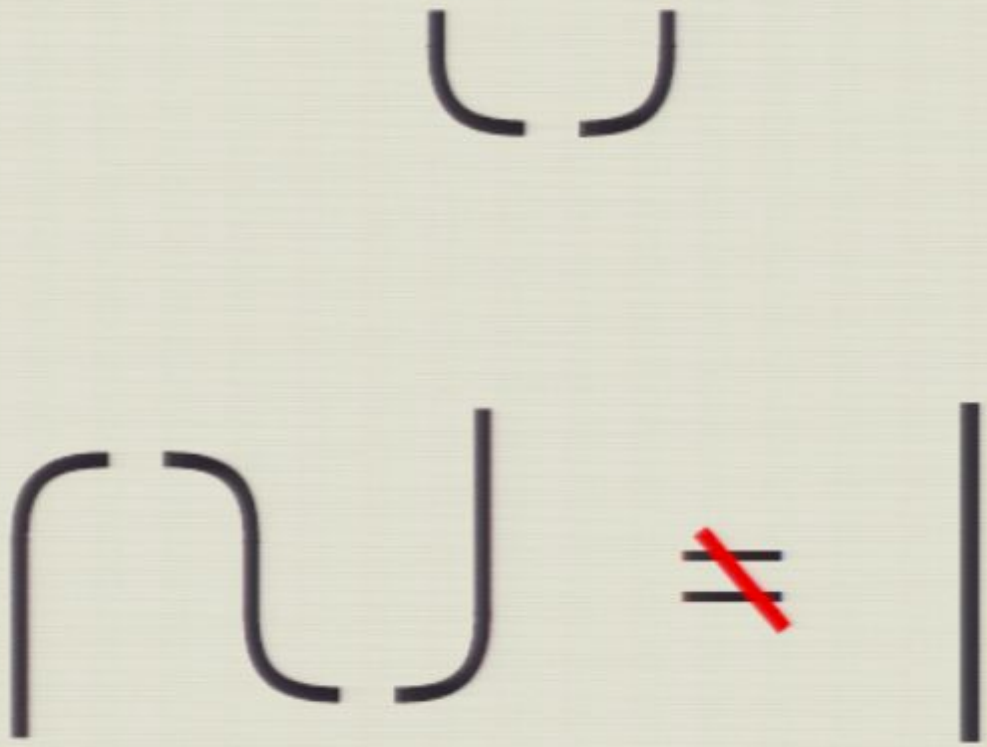


⇒ **introduce 'parallel connection between systems':**

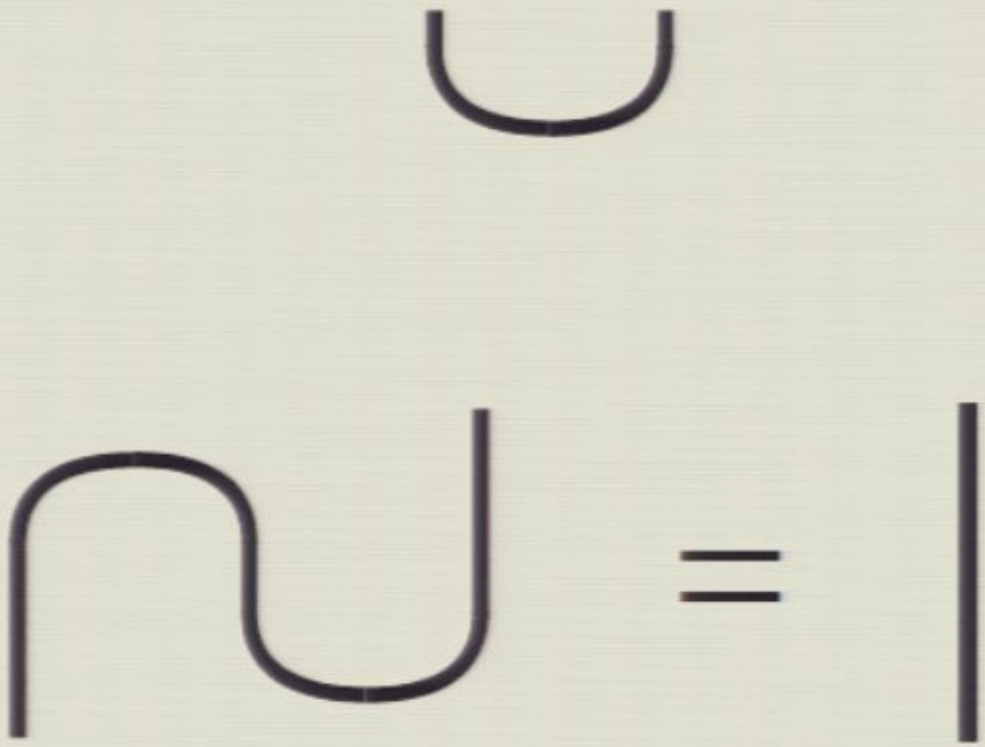
— *quantum-like* —



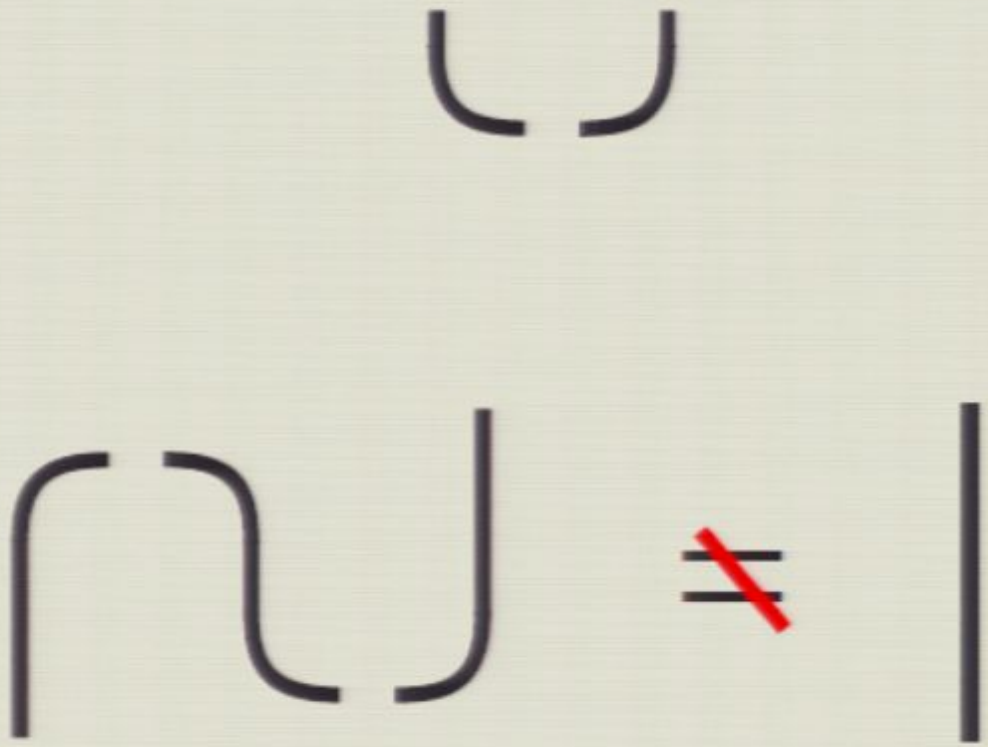
— *quantum-like* —



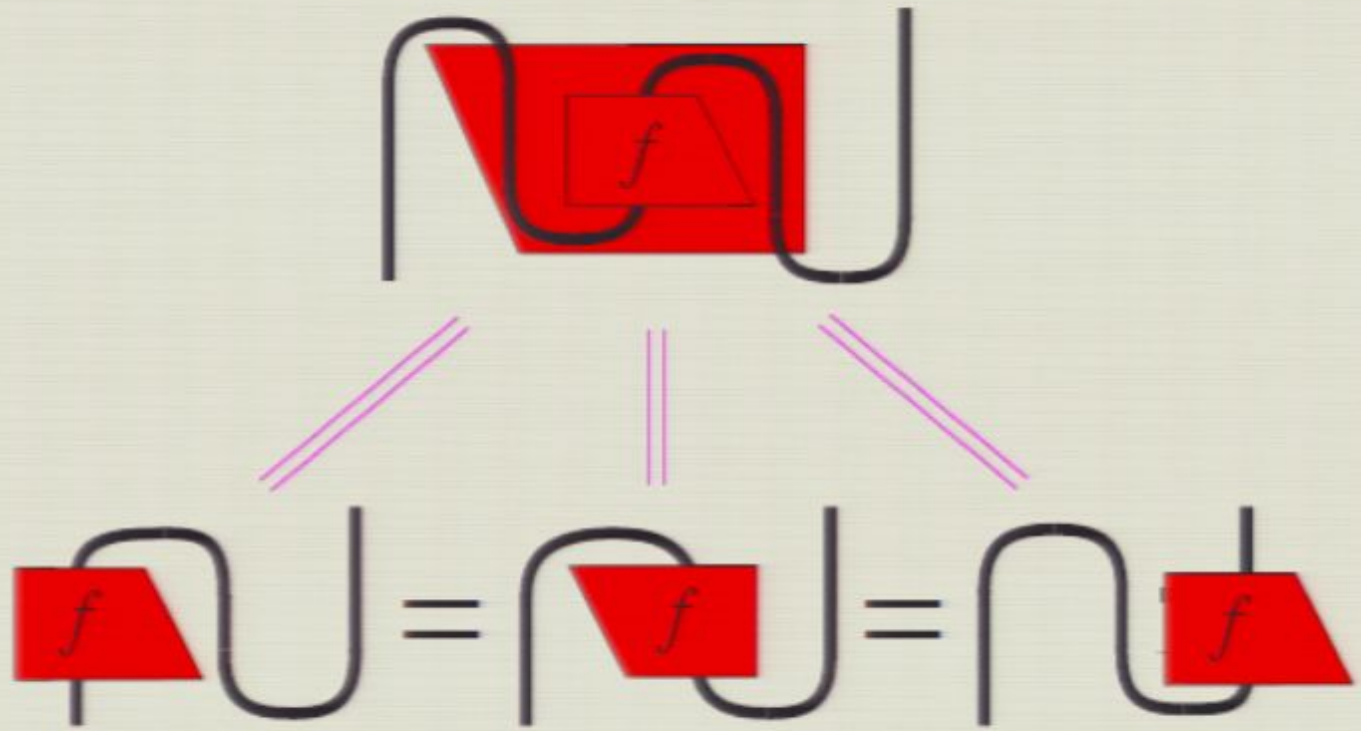
— *quantum-like* —



— *quantum-like* —

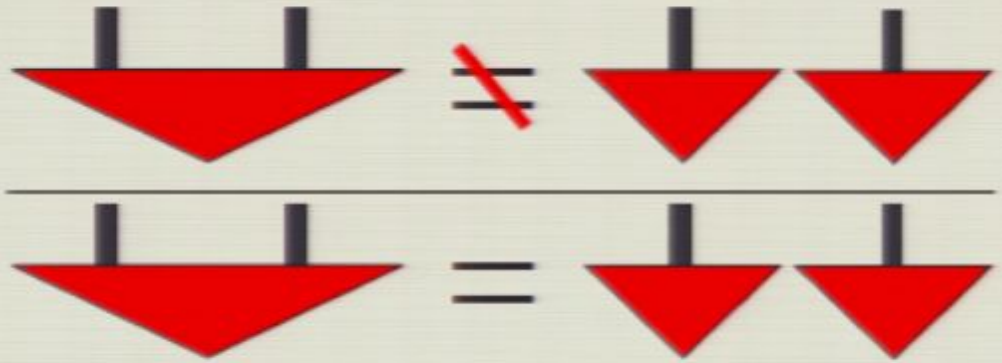


— *sliding* —



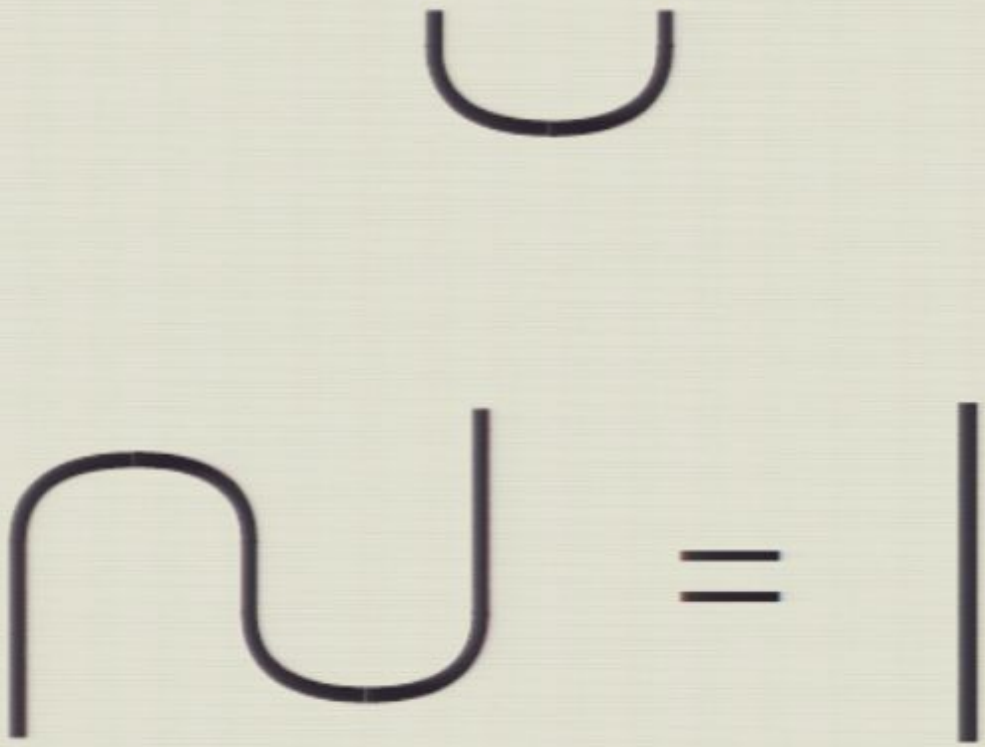
— *asserting (pure) entanglement* —

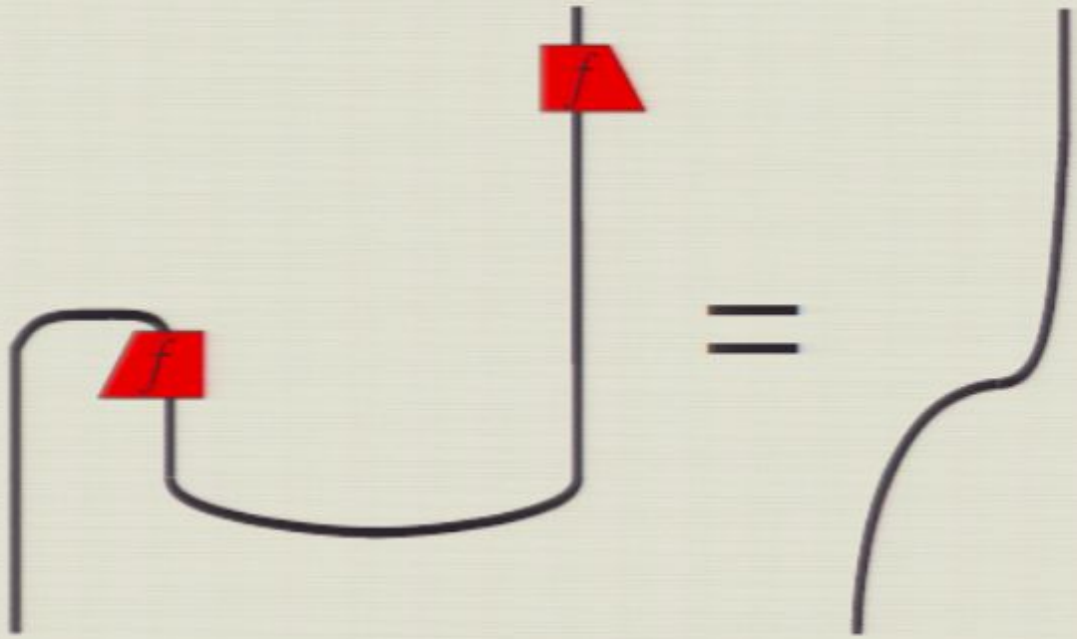
$$\frac{quantum}{classical} =$$



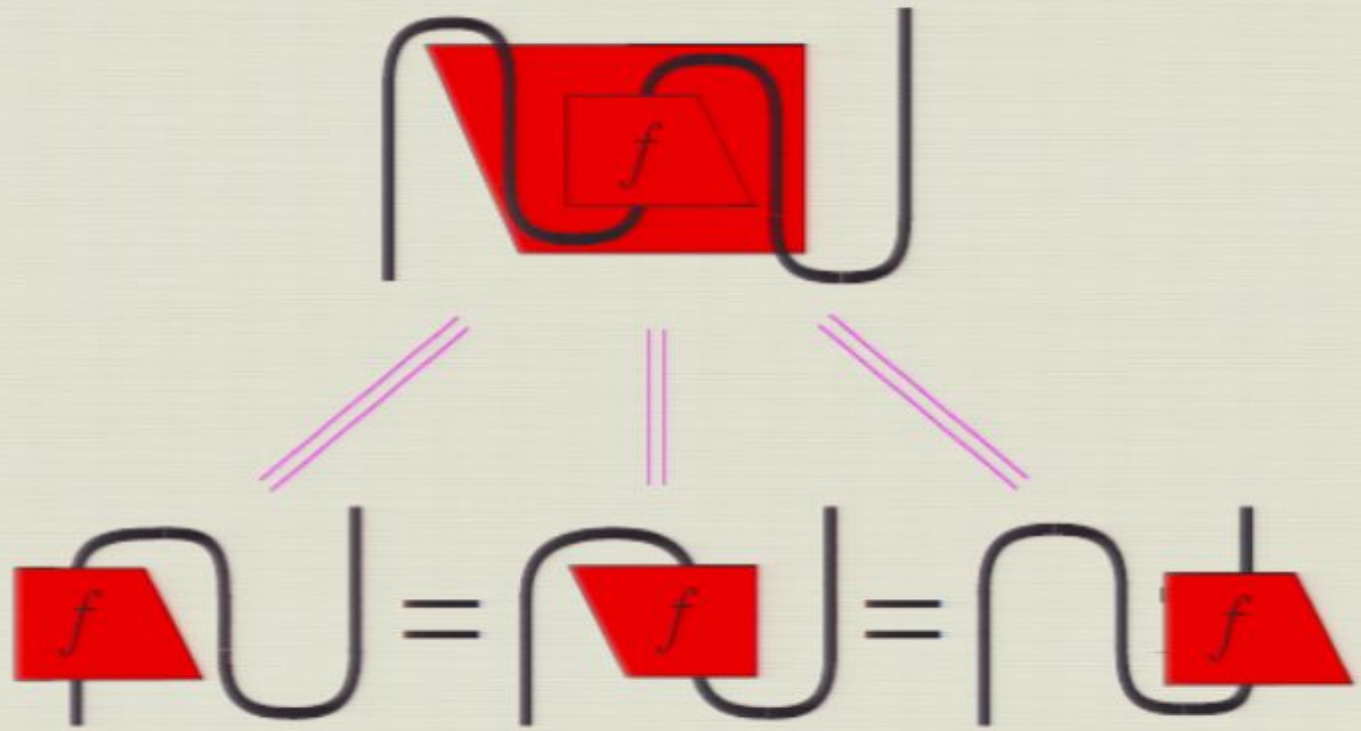
⇒ **introduce 'parallel connection between systems':**

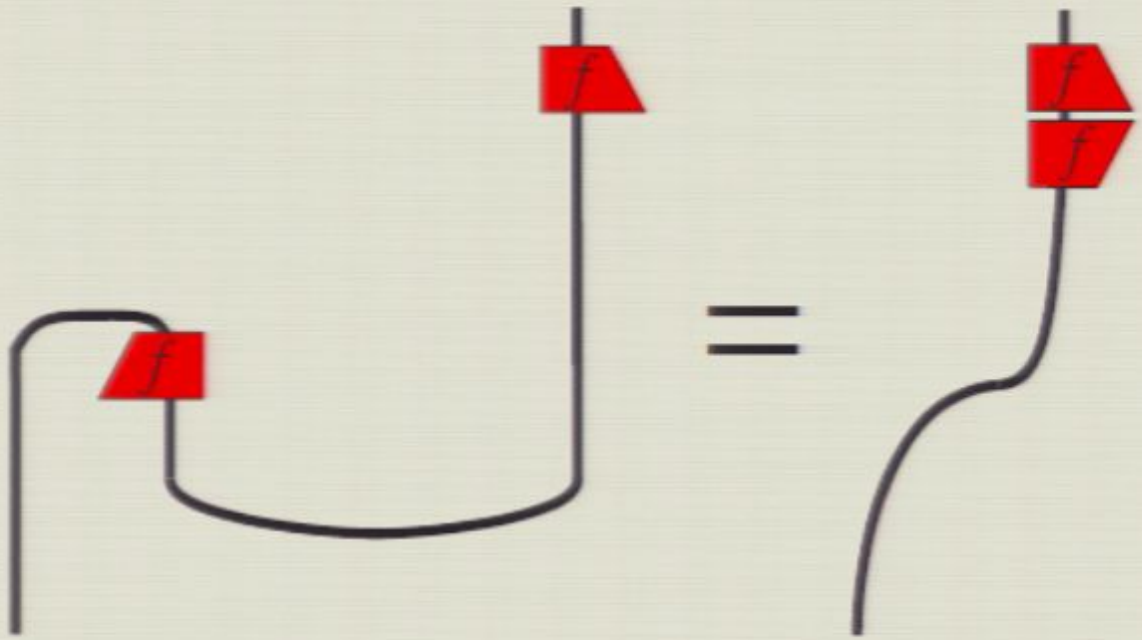
— *quantum-like* —

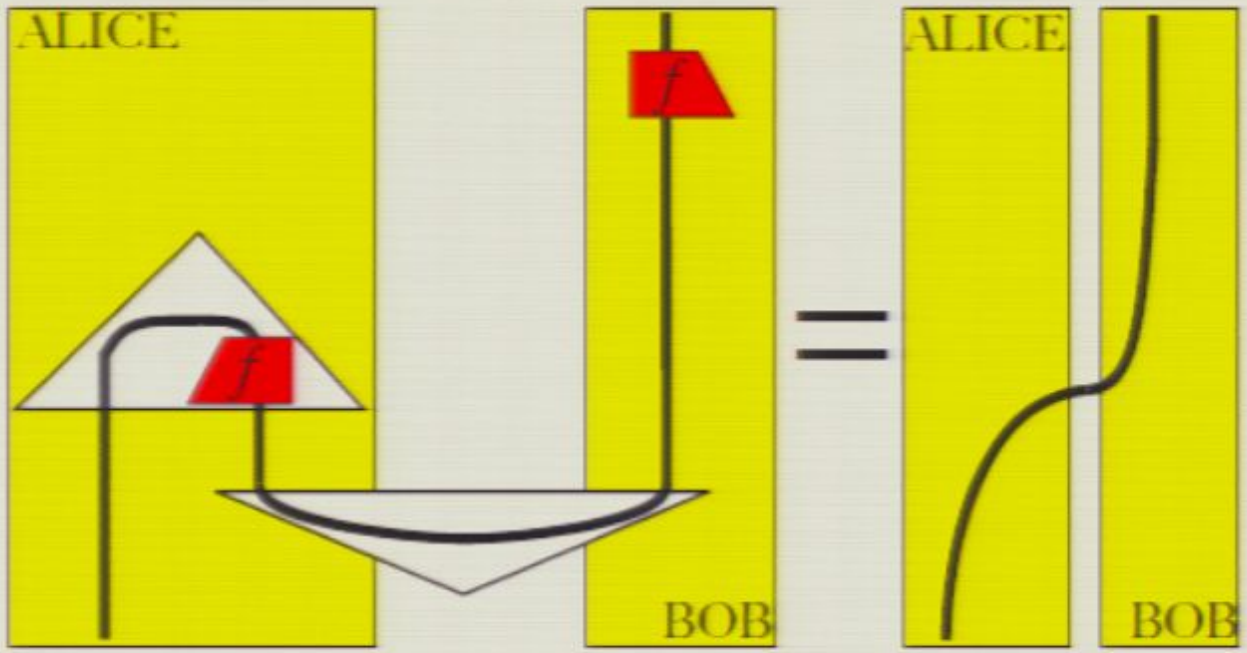



f
f
=

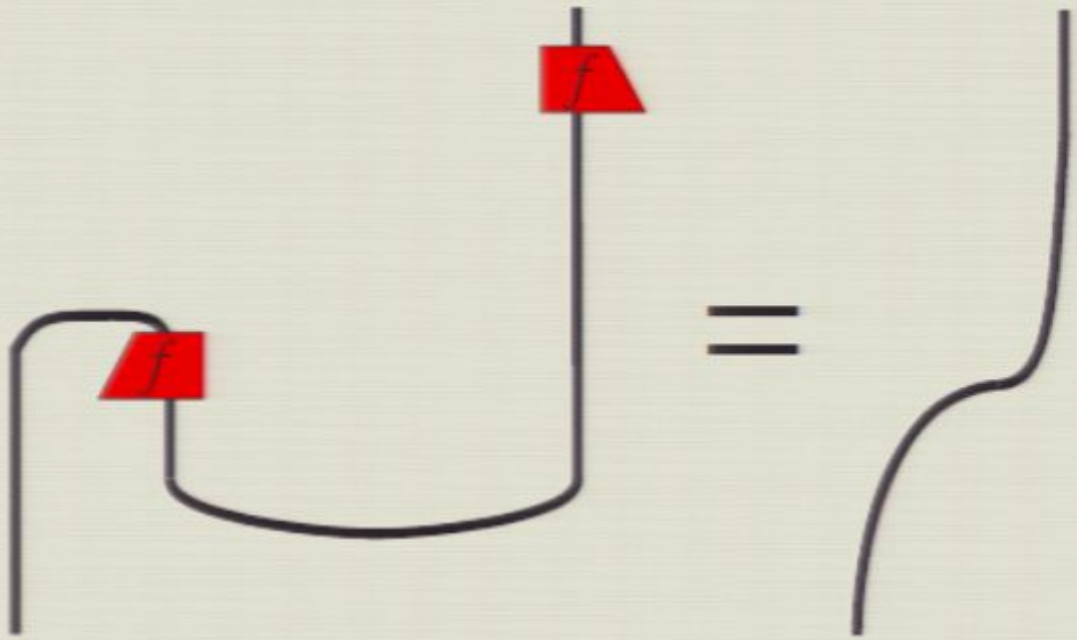
— *sliding* —

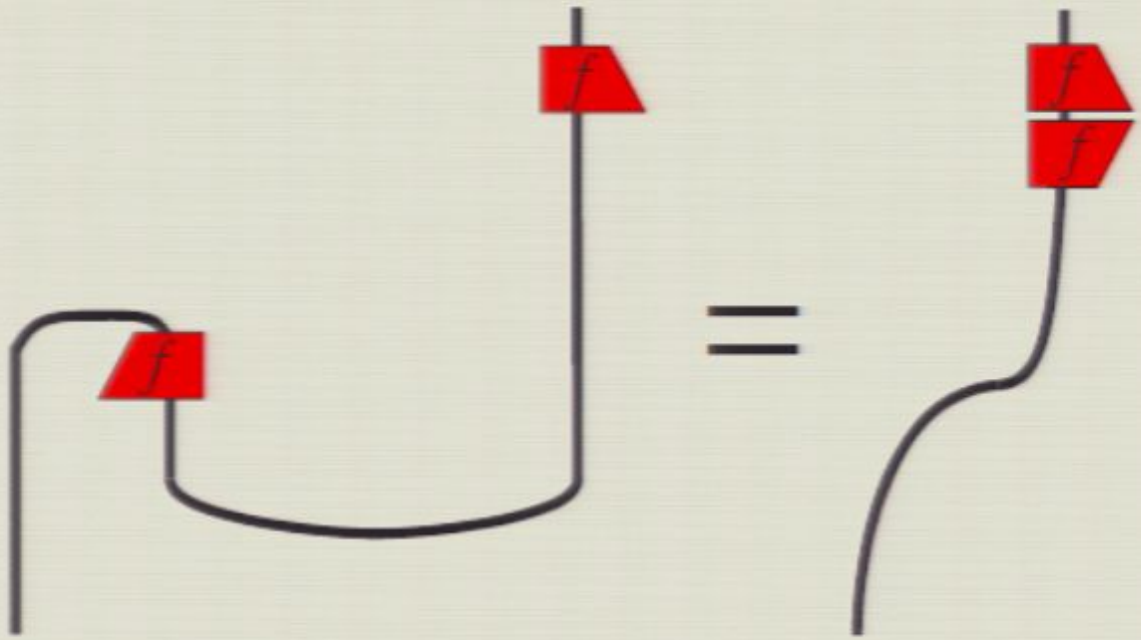


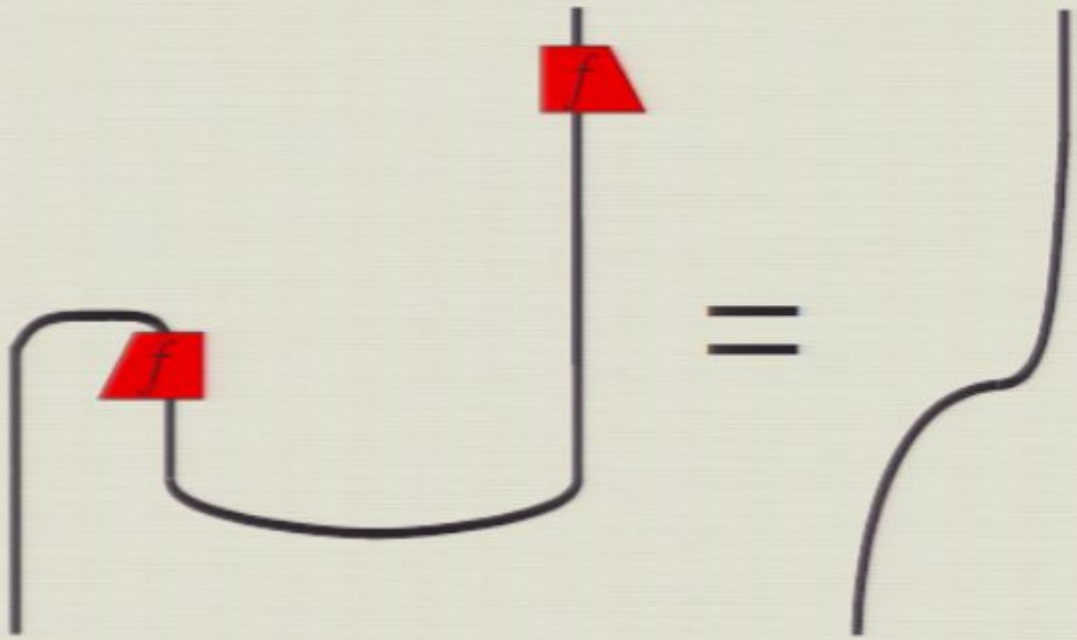

f
f
=
f
f

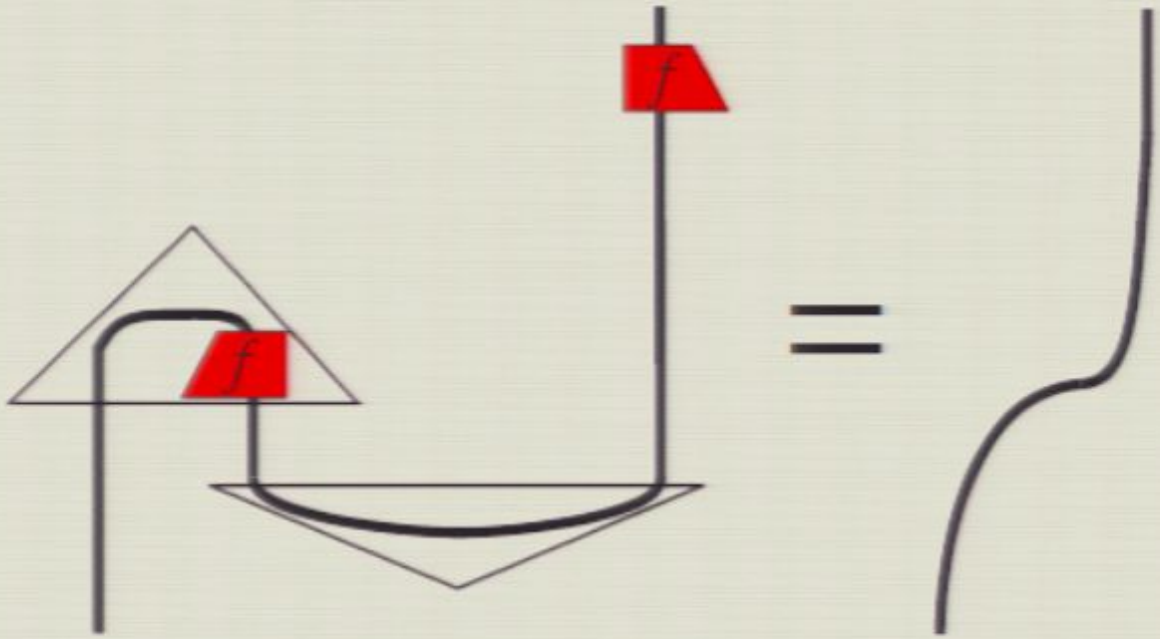


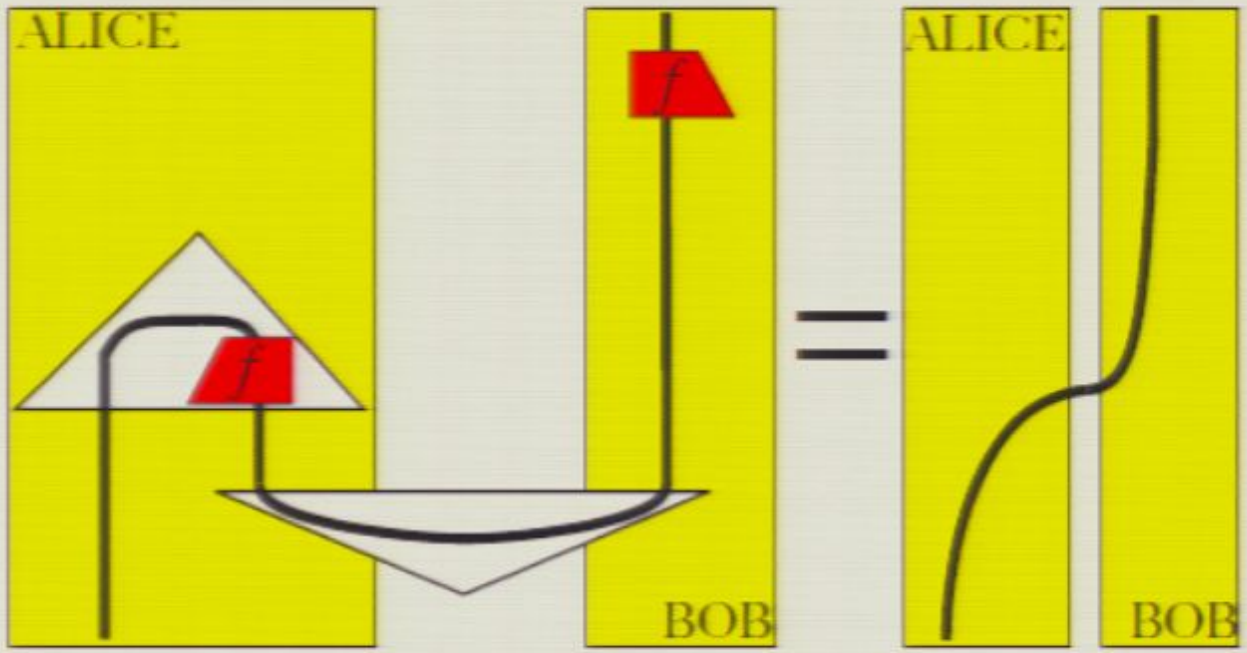
$\Rightarrow$ **quantum teleportation**


f
f
=


f
f
=
f
f


f
f
=


f
f
=



$\Rightarrow$ **quantum teleportation**

— *symbolically: dagger compact categories* —

— *symbolically: dagger compact categories* —

Thm. [Kelly-Laplaza '80; Selinger '05] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the graphical notation via homotopy*.

— *symbolically: dagger compact categories* —

Thm. [Kelly-Laplaza '80; Selinger '05] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the graphical notation via homotopy*.

there are many models, i.e. concrete dagger compact categories, e.g. linear maps, relations, ...

— *symbolically: dagger compact categories* —

Thm. [Kelly-Laplaza '80; Selinger '05] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the graphical notation via homotopy*.

Thm. [Selinger '08] *An equational statement between expressions in dagger compact categorical language holds if and only if it is derivable in the dagger compact category of finite dimensional Hilbert spaces, linear maps, tensor product and adjoints*.

— *symbolically: dagger compact categories* —

In words: *Any equation involving:*

- *states, operations, effects*
- *unitarity, adjoints (e.g. self-adjoint), projections*
- *Bell-states/effects, transpose, conjugation*
- *inner-product, trace, Hilbert-Schmidt norm*
- *positivity, completely positive maps, ...*

holds in quantum theory **if and only if** *it can be derived in the graphical language via homotopy.*

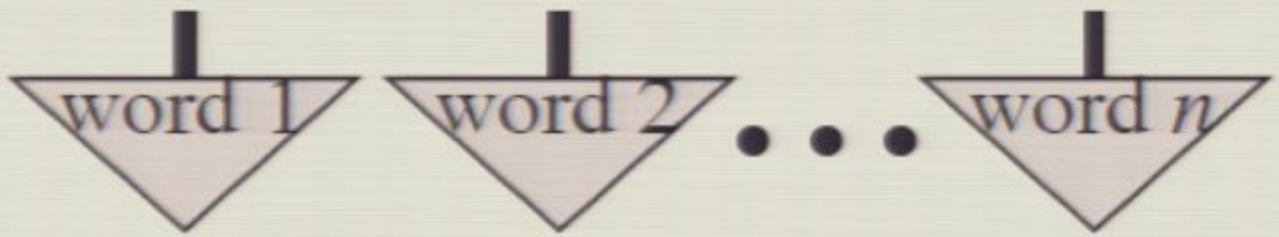
MEANING IN NATURAL LANGUAGE

BC, Mehrnoosh Sadrzadeh & Stephen Clark (2010) *Mathematical foundations for a compositional distributional model of meaning*. arXiv:1003.4394

— *symbolically: dagger compact categories* —

— *the from-words-to-a-sentence process* —

Consider meanings of **words** (for Google these are vectors):

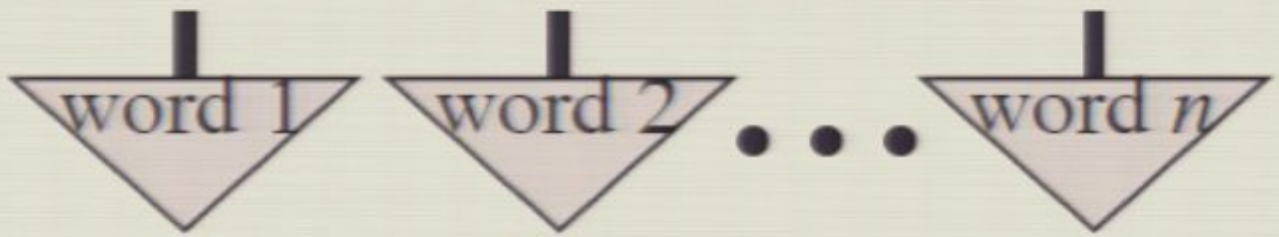


MEANING IN NATURAL LANGUAGE

BC, Mehrnoosh Sadrzadeh & Stephen Clark (2010) *Mathematical foundations for a compositional distributional model of meaning*. arXiv:1003.4394

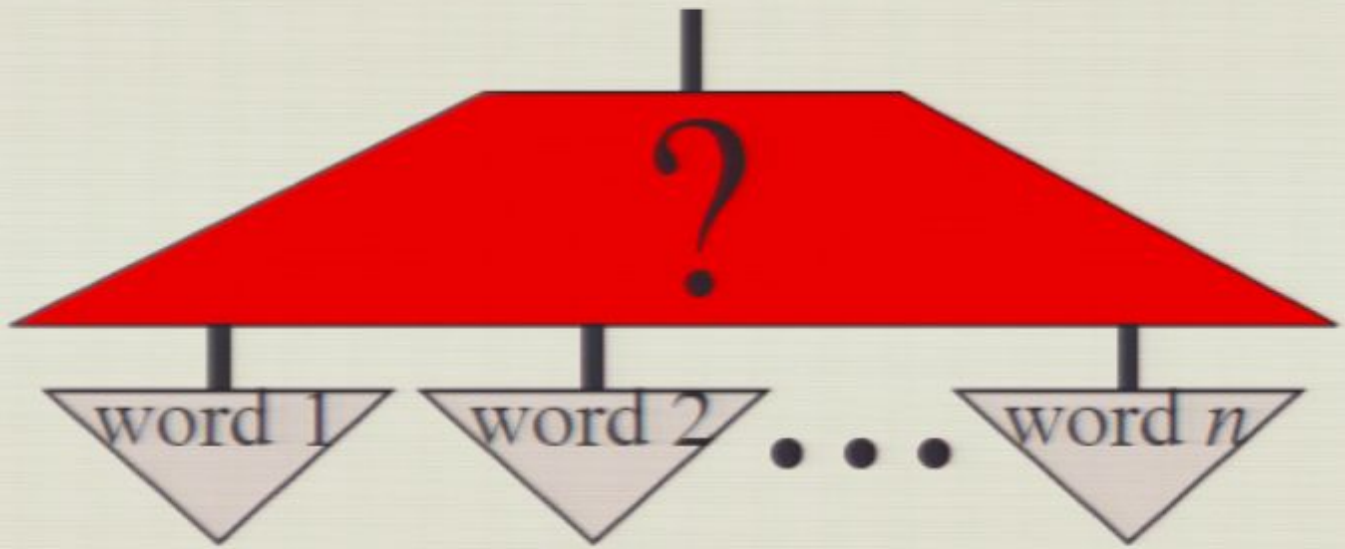
— *the from-words-to-a-sentence process* —

Consider meanings of **words** (for Google these are vectors):



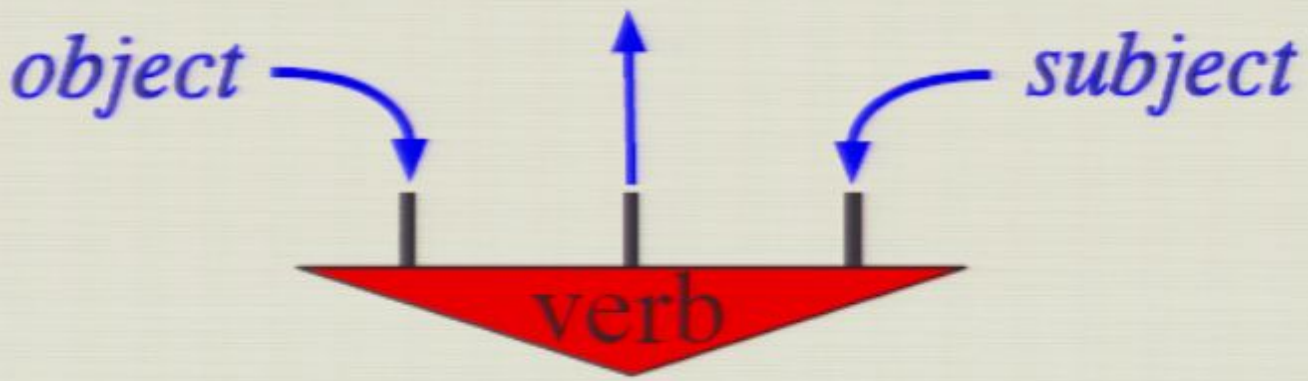
— *the from-words-to-a-sentence process* —

How do we/machines compute meaning of **sentences**?



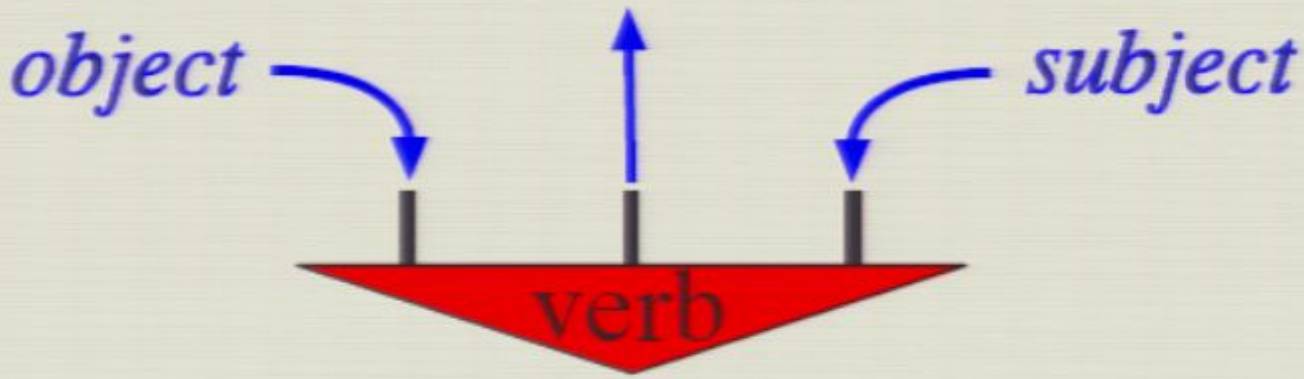
— *the from-words-to-a-sentence process* —

Information flow within a verb:



— *the from-words-to-a-sentence process* —

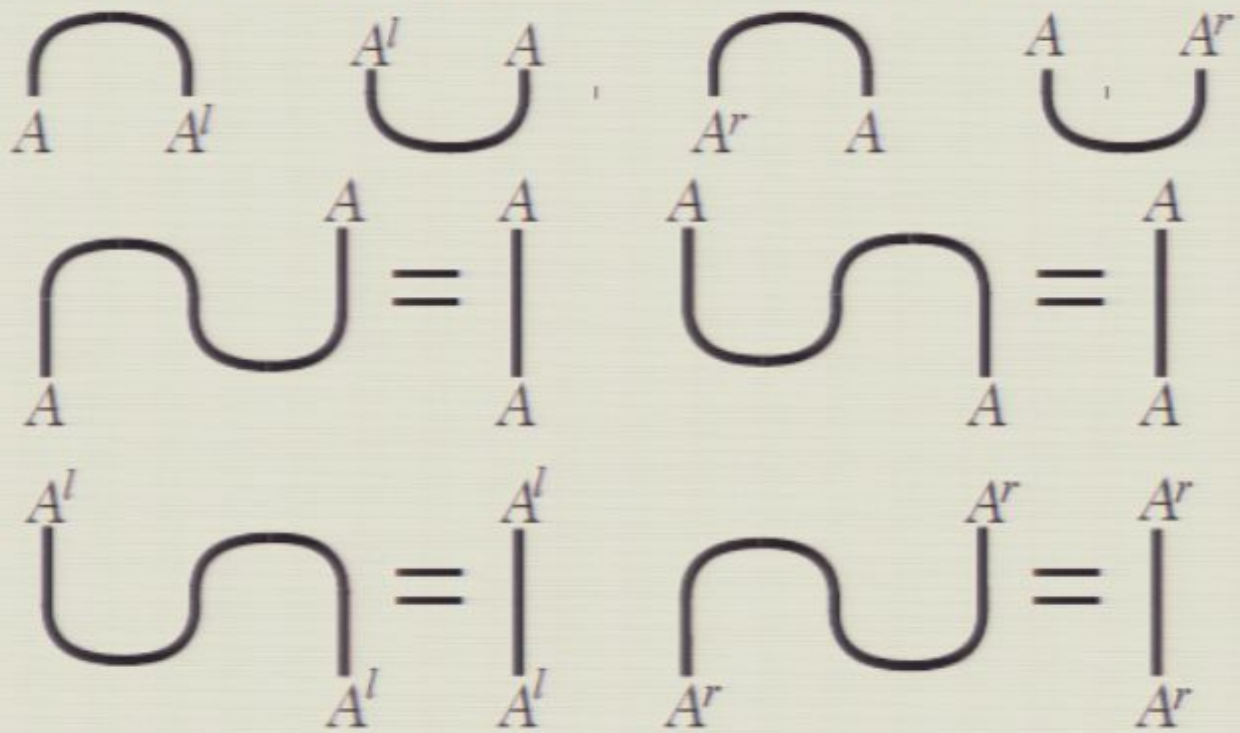
Information flow within a verb:



Again we have:

— grammar as pregroups – Lambek '99 —

$$\mathrm{I} \xrightarrow{\eta^l} A \otimes A^l \qquad A^l \otimes A \xrightarrow{\epsilon^l} \mathrm{I} \qquad \mathrm{I} \xrightarrow{\eta^r} A^r \otimes A \qquad A \otimes A^r \xrightarrow{\epsilon^r} \mathrm{I}$$



— *grammar as pregroups – Lambek '99* —

For noun type "n", verb type is "${}^{-1}n \cdot s \cdot n^{-1}$", so:

$$n \cdot {}^{-1}n \cdot s \cdot n^{-1} \cdot n = s$$

— *grammar as pregroups – Lambek '99* —

For noun type "n", verb type is "${}^{-1}n \cdot s \cdot n^{-1}$", so:

$$n \cdot {}^{-1}n \cdot s \cdot n^{-1} \cdot n = s$$

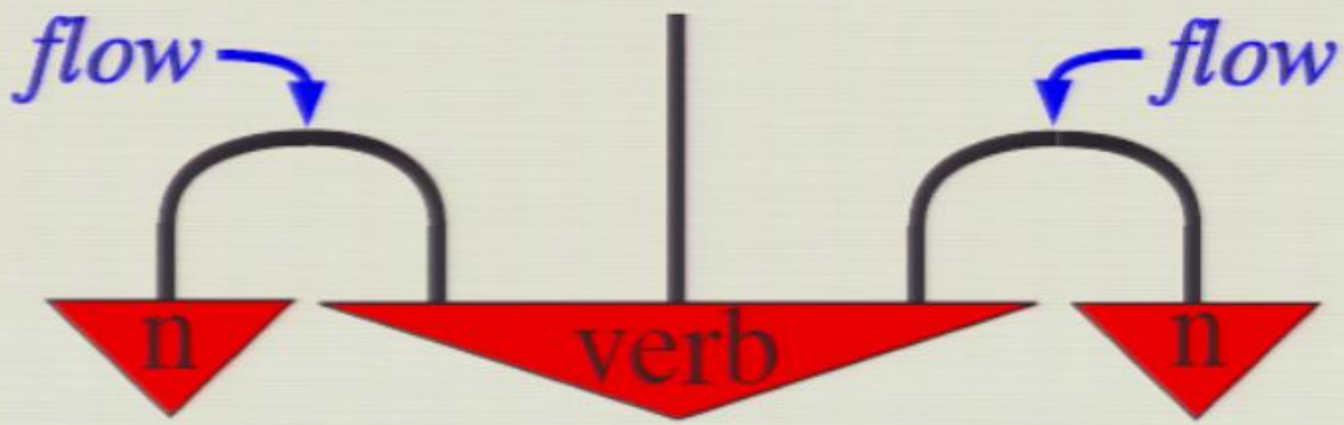
Diagrammatic typing:

— *from gammer to meaning* —

For noun type "n" and verb type "${}^{-1}n \cdot s \cdot n^{-1}$":

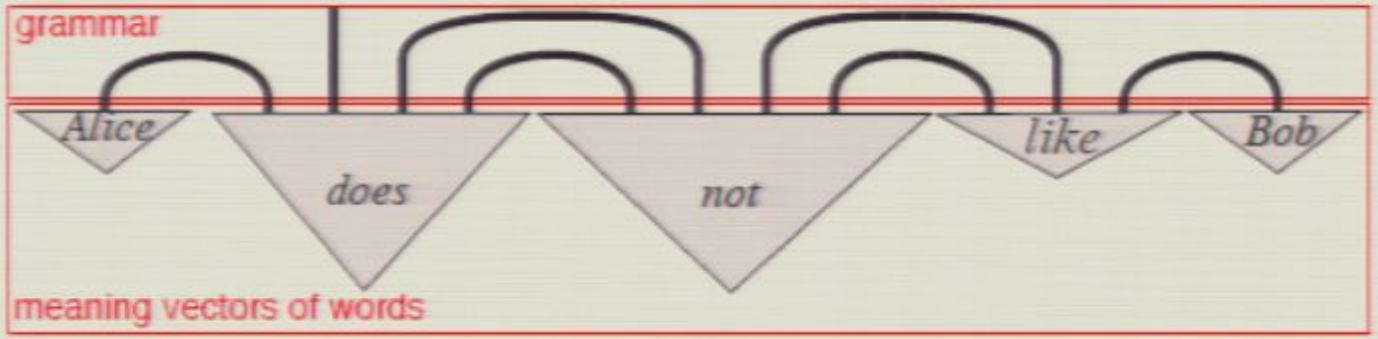
$$n \cdot {}^{-1}n \cdot s \cdot n^{-1} \cdot n = s$$

Diagrammatic meaning:

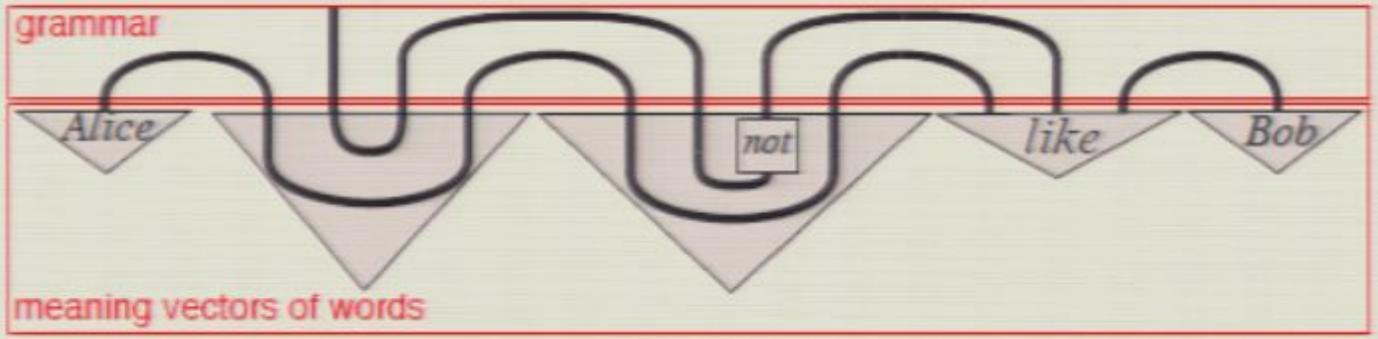


$$\overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob}$$

$$\overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob}$$



$$\overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob}$$



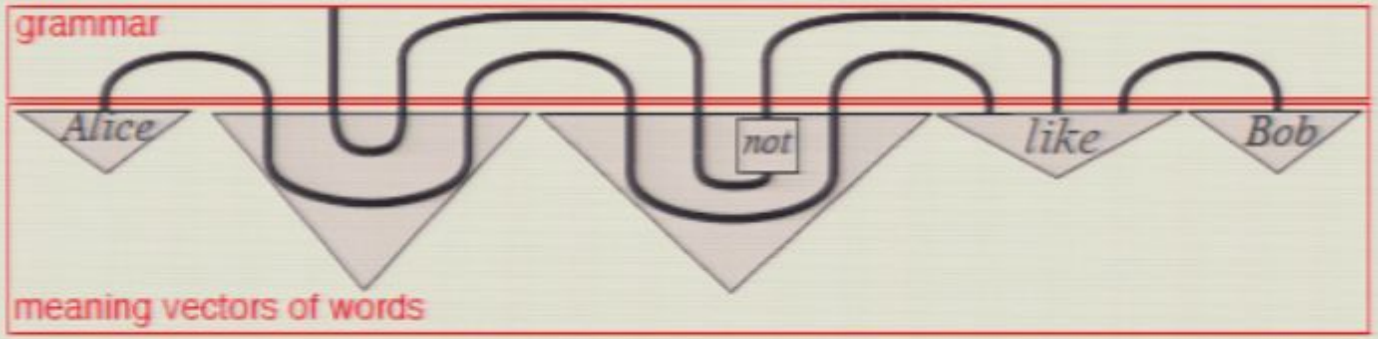
— *experiment: word disambiguation* —

E.g. what is "saw"' in: "Alice saw Bob with a saw".

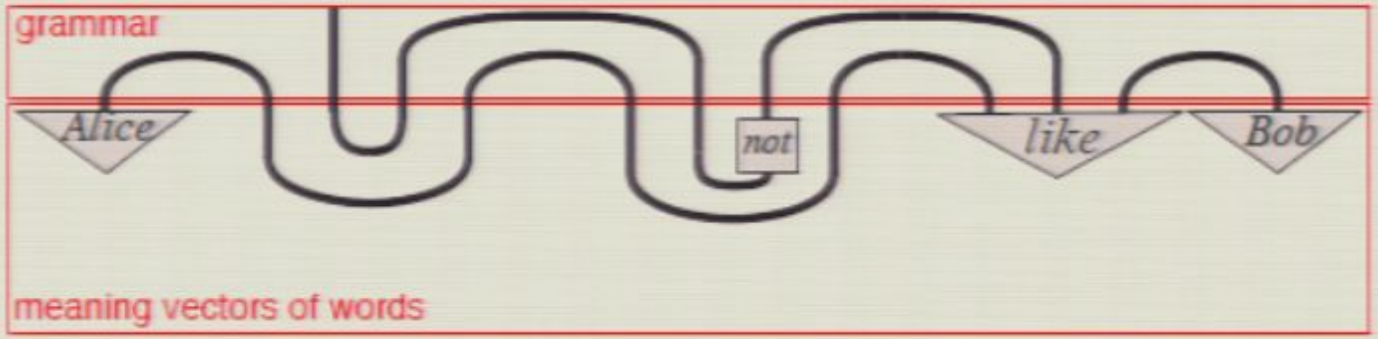
Model	High	Low	ρ
Baseline	0.47	0.44	0.16
Add	0.90	0.90	0.05
Multiply	0.67	0.59	0.17
Categorical	**0.73**	**0.72**	**0.21**

Edward Grefenstette & Mehrnoosh Sadrzadeh (2011) *Experimental support for a categorical compositional distributional model of meaning.* Accepted for: Empirical Methods in Natural Language Processing (EMNLP'11).

$$\overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob}$$



$$\overrightarrow{Alice} \otimes \overrightarrow{does} \otimes \overrightarrow{not} \otimes \overrightarrow{like} \otimes \overrightarrow{Bob}$$

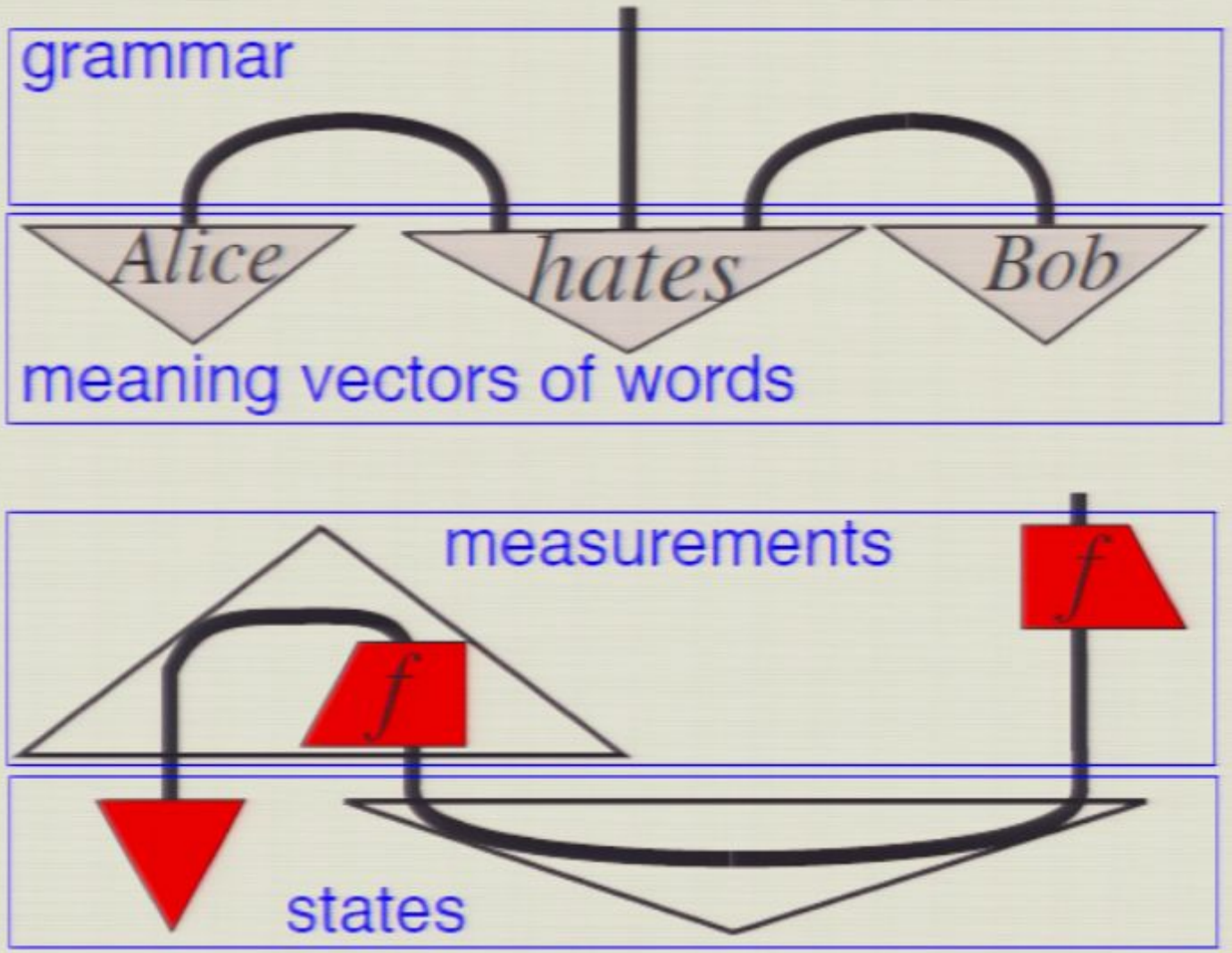


— *experiment: word disambiguation* —

E.g. what is "saw"' in: "Alice saw Bob with a saw".

Model	High	Low	ρ
Baseline	0.47	0.44	0.16
Add	0.90	0.90	0.05
Multiply	0.67	0.59	0.17
Categorical	**0.73**	**0.72**	**0.21**

Edward Grefenstette & Mehrnoosh Sadrzadeh (2011) *Experimental support for a categorical compositional distributional model of meaning.* Accepted for: Empirical Methods in Natural Language Processing (EMNLP'11).


grammar
Alice
hates
Bob
meaning vectors of words
measurements
f
f
states

— *analogy: "non-local" info-flows* —

English (& French):

He	bought	a book	from	the bookshop
π	$(\pi^r s w^l o^l)$	o	$(w\pi^l)$	n

Hindi:

vaha	idhara	se	kitaba	kharidana
π	π	$(\pi^r w)$	o	$(o^r w^r \pi^r s)$

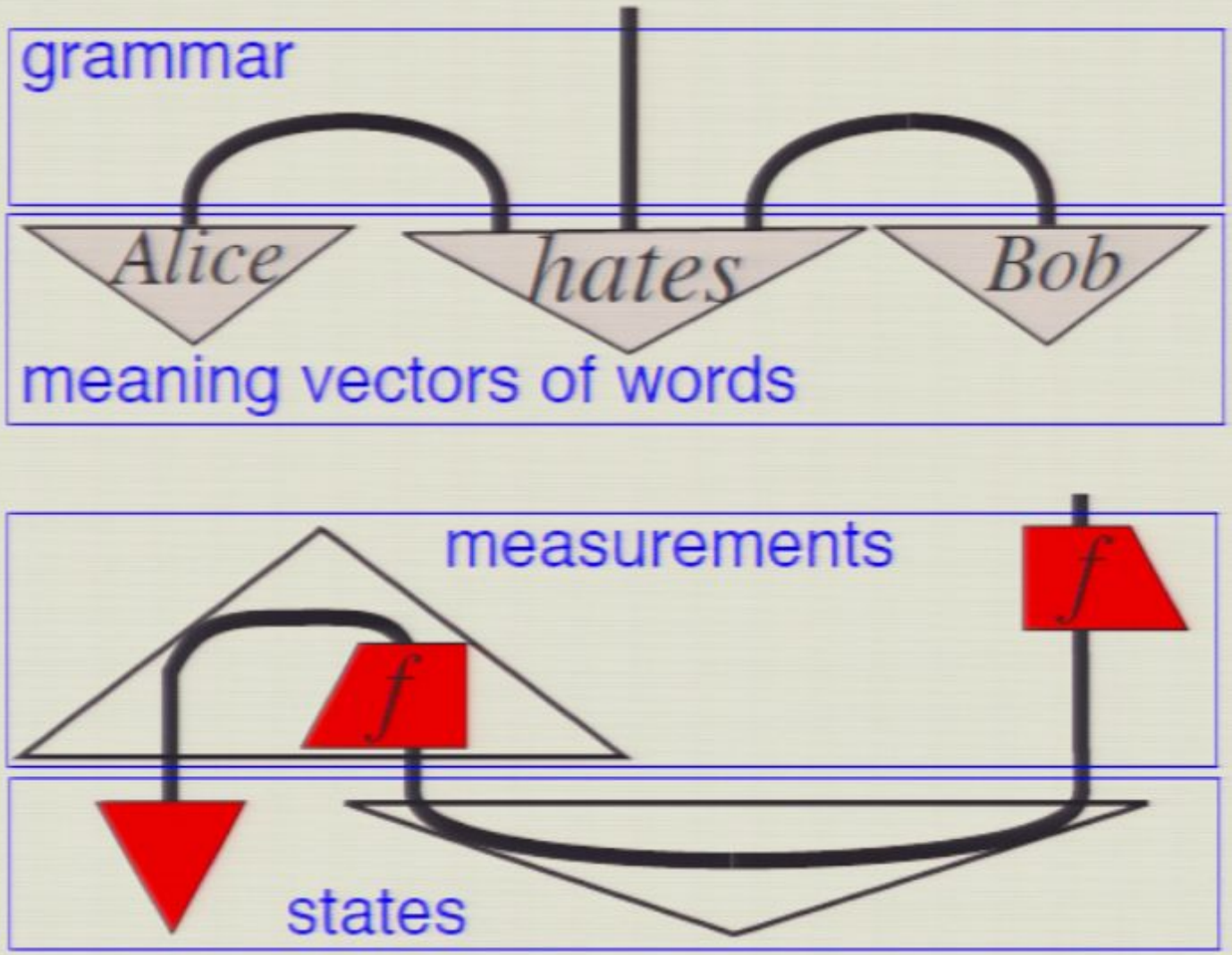
Persian:

u	ketab	ra	az	ketabkhaneh	kharid
π	n	$(\pi^r o)$	$(w\pi^l)$	n	$(w^r o^r \pi^r s)$

Arabic (and Hebrew):

Yashtari	ketaban	men	alsogh.
$(s\, w^l\, o^l)$	o	$w\pi^l$	n

Mehrnoosh Sadrzadeh (2008) *Pregroup analysis of Persian sentences.*


grammar
Alice
hates
Bob
meaning vectors of words
measurements
f
f
states

— *analogy: "non-local" info-flows* —

English (& French):

He	bought	a book	from	the bookshop
π	$(\pi^r s w^l o^l)$	o	$(w\pi^l)$	n

Hindi:

vaha	idhara	se	kitaba	kharidana
π	π	$(\pi^r w)$	o	$(o^r w^r \pi^r s)$

Persian:

u	ketab	ra	az	ketabkhaneh	kharid
π	n	$(\pi^r o)$	$(w\pi^l)$	n	$(w^r o^r \pi^r s)$

Arabic (and Hebrew):

Yashtari	ketaban	men	alsogh.
$(s\, w^l\, o^l)$	o	$w\pi^l$	n

Mehrnoosh Sadrzadeh (2008) *Pregroup analysis of Persian sentences.*

THE EXTENDED LANGUAGE: BASES AND COMPLEMENTARITY

Coecke, Pavlovic & Vicary (2008) *A new description of orthogonal bases.* Mathematical Structures in Computer Science, TA. arXiv:0810.0812

BC & Ross Duncan (IS HERE) (2008) *Interacting quantum observables.* ICALP'08 & New Journal of Physics **13**, 043016. arXiv:0906.4725

— *commutative Frobenius algebras* —

THE EXTENDED LANGUAGE: BASES AND COMPLEMENTARITY

Coecke, Pavlovic & Vicary (2008) *A new description of orthogonal bases.* Mathematical Structures in Computer Science, TA. arXiv:0810.0812

BC & Ross Duncan (IS HERE) (2008) *Interacting quantum observables.* ICALP'08 & New Journal of Physics **13**, 043016. arXiv:0906.4725

— *commutative Frobenius algebras* —

— *commutative Frobenius algebras* —

A commutative monoid is a set X with a binary map

$$- \bullet - : X \times X \to X$$

which is commutative, associative and unital i.e

$$(a \bullet b) \bullet c = a \bullet (b \bullet c) \quad a \bullet b = b \bullet a \quad a \bullet 1 = a$$

commutative monoid	cocommutative comonoid
$+_{\mathbb{Z}_2} :: \begin{cases} (0,0),(1,1) \mapsto 0 \\ (0,1),(1,0) \mapsto 1 \end{cases}$	
$\times_{\mathbb{B}} :: \begin{cases} (0,0),(0,1),(1,0) \mapsto 0 \\ (1,1) \mapsto 1 \end{cases}$	
$\vdots$	
	$\Delta :: \begin{cases} 0 \mapsto (0,0) \\ 1 \mapsto (1,1) \end{cases}$

— *commutative Frobenius algebras* —

A commutative monoid is vect. space V & lin. map

$$\mu : V \otimes V \to V$$

which is commutative, associative and unital i.e

$$\mu \circ (\mu \otimes 1_V) = \mu \circ (1_V \otimes \mu) \quad \mu = \mu \circ \sigma \quad \mu \circ (1_V \otimes e) = 1_V$$

A cocomutative comonoid is vect. space V & lin. map

$$\delta : V \to V \otimes V$$

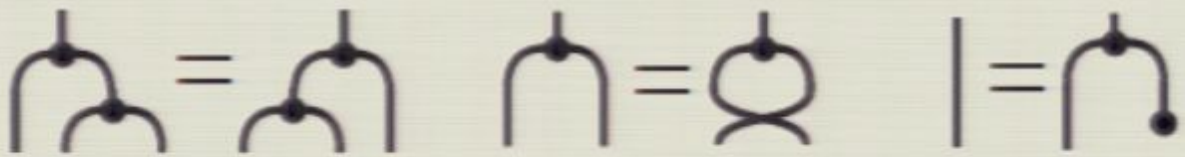
which is cocommutative, coassociative and counital i.e

$$(\delta \otimes 1_V) \circ \delta = (1_V \otimes \delta) \circ \delta \quad \delta = \sigma \circ \delta \quad (1_V \otimes e') \circ \delta = 1_V$$

— *commutative Frobenius algebras* —

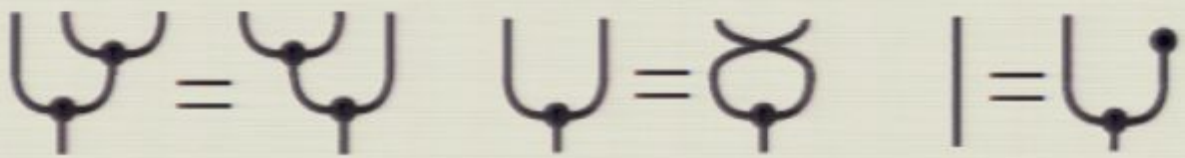
A commutative monoid is:

such that



A cocomutative comonoid is:

such that

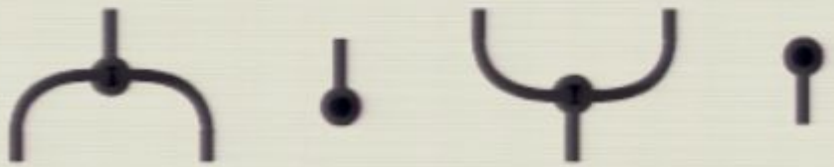


commutative monoid	cocommutative comonoid
$\hat{+}_{\mathbb{Z}_2} :: \begin{cases} \lvert 00\rangle, \lvert 11\rangle \mapsto \lvert 0\rangle \\ \lvert 01\rangle, \lvert 10\rangle \mapsto \lvert 1\rangle \end{cases}$	$\hat{+}^{\dagger}_{\mathbb{Z}_2} :: \begin{cases} \lvert 0\rangle \mapsto \lvert 00\rangle, \lvert 11\rangle \\ \lvert 1\rangle \mapsto \lvert 01\rangle, \lvert 10\rangle \end{cases}$
$\hat{\times}_{\mathbb{B}} :: \cdots$	$\hat{\times}^{\dagger}_{\mathbb{B}} :: \cdots$
$\delta^{\dagger}_{Z} :: \begin{cases} \lvert 00\rangle \mapsto \lvert 0\rangle \\ \lvert 11\rangle \mapsto \lvert 1\rangle \end{cases}$	$\delta_{Z} :: \begin{cases} \lvert 0\rangle \mapsto \lvert 00\rangle \\ \lvert 1\rangle \mapsto \lvert 11\rangle \end{cases}$
$\delta^{\dagger}_{X} :: \begin{cases} \lvert ++\rangle \mapsto \lvert +\rangle \\ \lvert --\rangle \mapsto \lvert -\rangle \end{cases}$	$\delta_{X} :: \begin{cases} \lvert +\rangle \mapsto \lvert ++\rangle \\ \lvert -\rangle \mapsto \lvert --\rangle \end{cases}$
$\delta^{\dagger}_{Y} :: \cdots$	$\delta_{Y} :: \cdots$

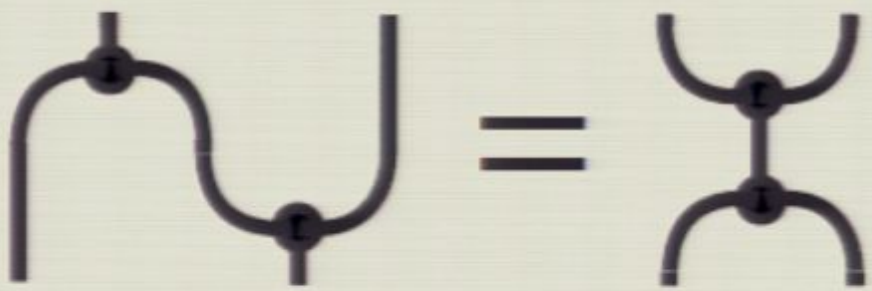
commutative monoid	cocommutative comonoid
$+_{\mathbb{Z}_2} :: \begin{cases} (0,0),(1,1) \mapsto 0 \\ (0,1),(1,0) \mapsto 1 \end{cases}$	
$\times_{\mathbb{B}} :: \begin{cases} (0,0),(0,1),(1,0) \mapsto 0 \\ (1,1) \mapsto 1 \end{cases}$	
$\vdots$	
	$\Delta :: \begin{cases} 0 \mapsto (0,0) \\ 1 \mapsto (1,1) \end{cases}$

— *commutative Frobenius algebras* —

A commutative Frobenius algebra is a commutative monoid and a commutative comonoid:



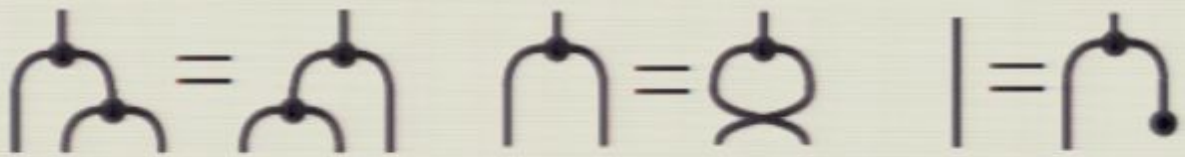
which moreover satisfy:



— *commutative Frobenius algebras* —

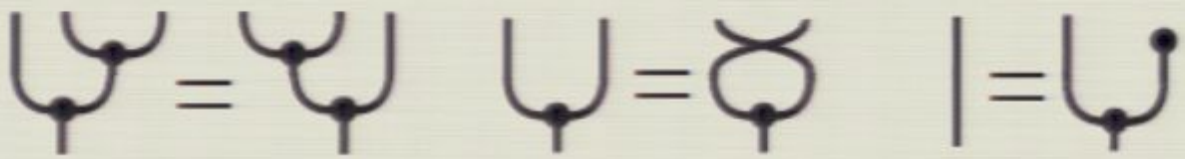
A commutative monoid is:

such that



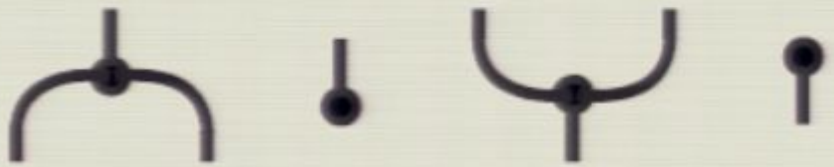
A cocomutative comonoid is:

such that

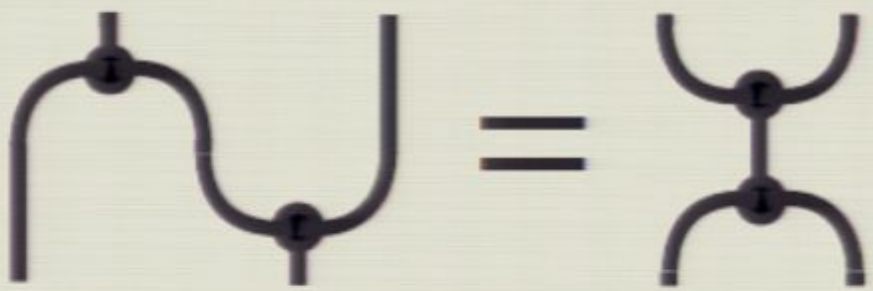


— *commutative Frobenius algebras* —

A commutative Frobenius algebra is a commutative monoid and a commutative comonoid:



which moreover satisfy:



— *bases in tensor logic* —

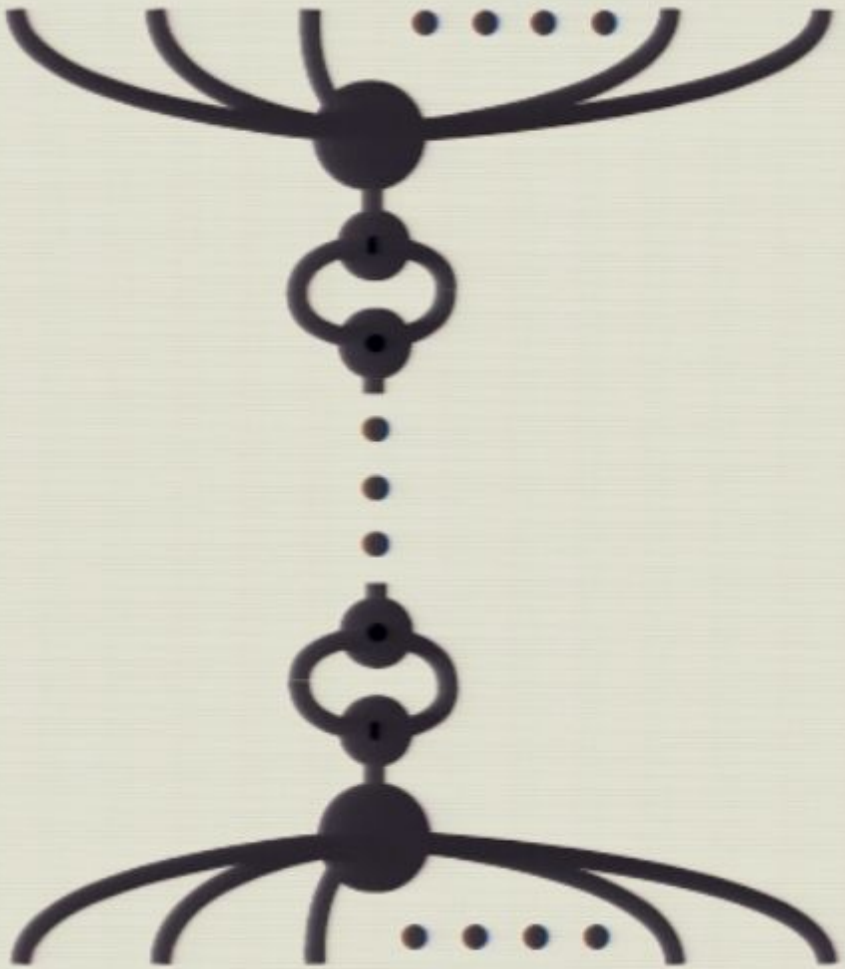
The notion of a(n) (orthonormal) basis can be captured purely in the language of tensor logic:

Thm. *Dagger special* commutative Frobenius algebras:

are the same thing as orthonormal bases; the basis vectors are the vectors copied by the comultiplication.

BC, Dusko Pavlovic & Jamie Vicary (2008) *A new description of orthogonal bases*. Mathematical Structures in Computer Science. arXiv:0810.0812

CFA normal form depends on ♯inputs, ♯outputs, ♯loops:



— *bases in tensor logic* —

The notion of a(n) (orthonormal) basis can be captured purely in the language of tensor logic:

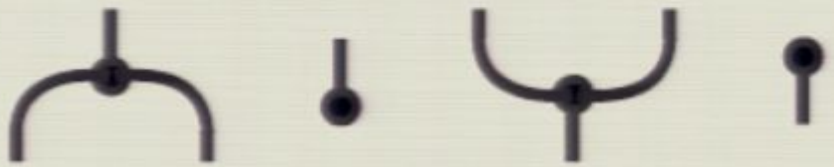
Thm. *Dagger special* commutative Frobenius algebras:

are the same thing as orthonormal bases; the basis vectors are the vectors copied by the comultiplication.

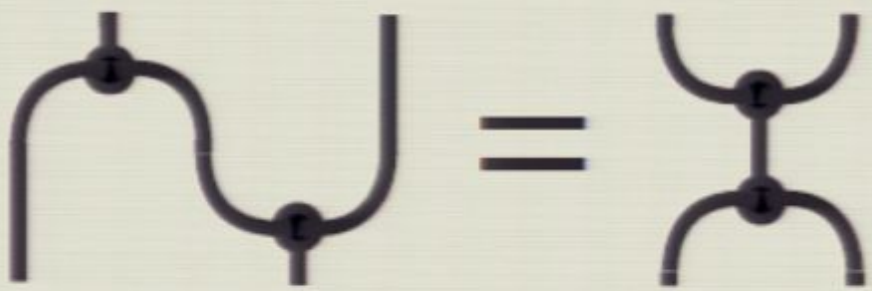
BC, Dusko Pavlovic & Jamie Vicary (2008) *A new description of orthogonal bases*. Mathematical Structures in Computer Science. `arXiv:0810.0812`

— *commutative Frobenius algebras* —

A commutative Frobenius algebra is a commutative monoid and a commutative comonoid:



which moreover satisfy:



— *bases in tensor logic* —

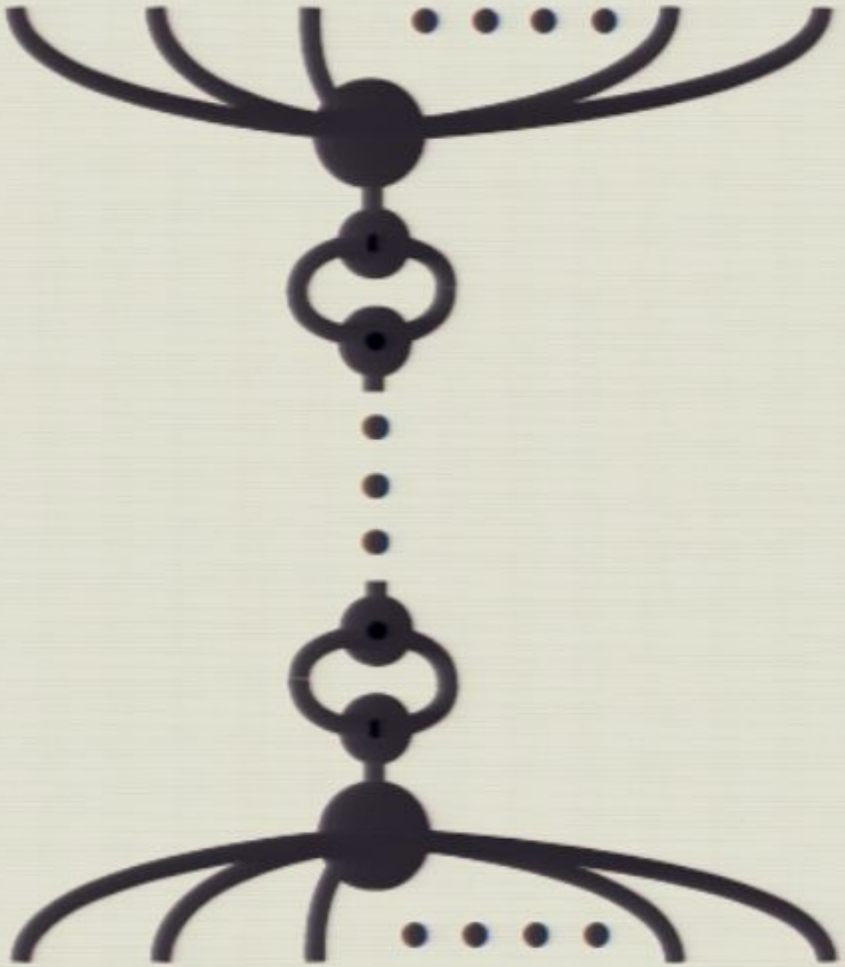
The notion of a(n) (orthonormal) basis can be captured purely in the language of tensor logic:

Thm. *Dagger special* commutative Frobenius algebras:

are the same thing as orthonormal bases; the basis vectors are the vectors copied by the comultiplication.

BC, Dusko Pavlovic & Jamie Vicary (2008) *A new description of orthogonal bases*. Mathematical Structures in Computer Science. arXiv:0810.0812

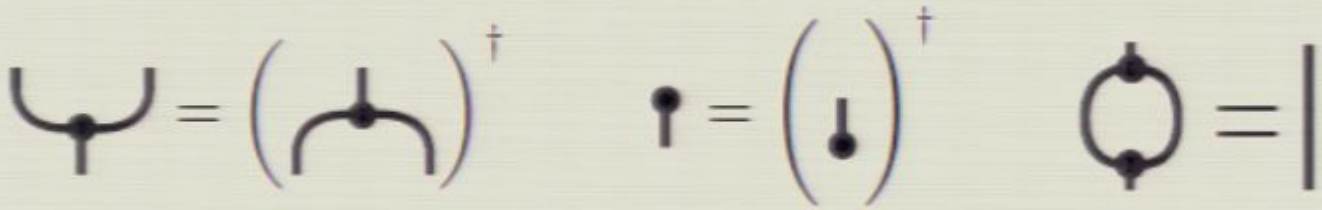
CFA normal form depends on ♯inputs, ♯outputs, ♯loops:



— *bases in tensor logic* —

The notion of a(n) (orthonormal) basis can be captured purely in the language of tensor logic:

Thm. *Dagger special* commutative Frobenius algebras:

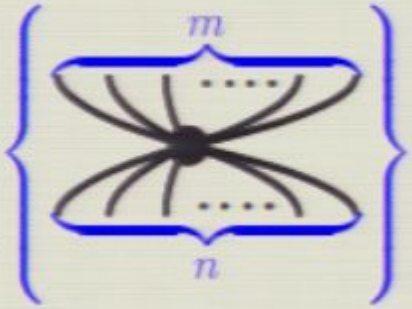


are the same thing as orthonormal bases; the basis vectors are the vectors copied by the comultiplication.

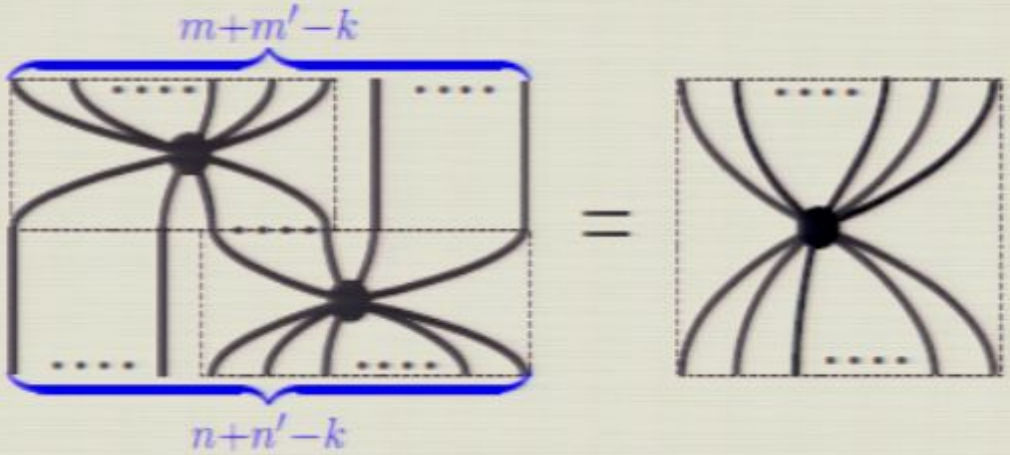
BC, Dusko Pavlovic & Jamie Vicary (2008) *A new description of orthogonal bases*. Mathematical Structures in Computer Science. arXiv:0810.0812

— *spiders* —

'spiders' =

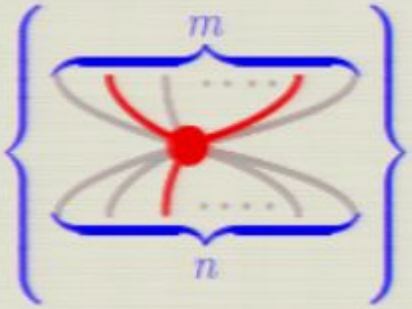


such that, for $k > 0$:

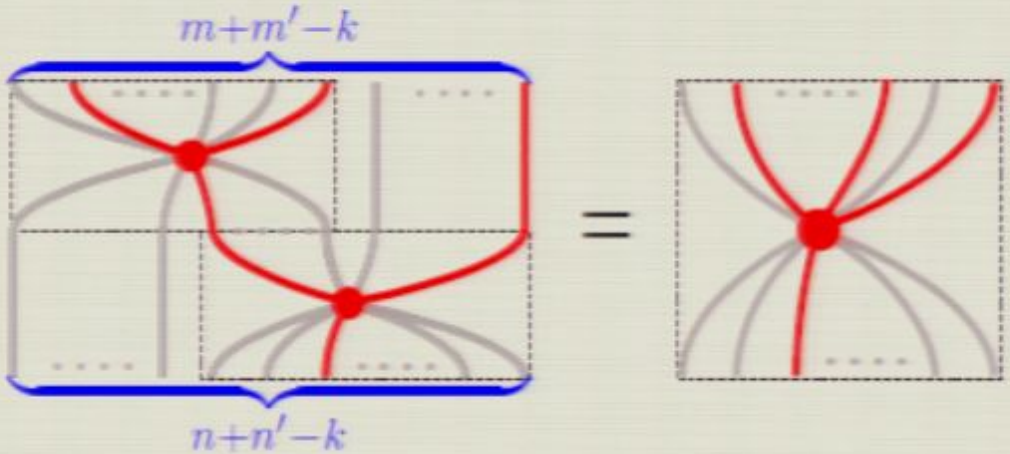


— *spiders* —

'(co-)mult.' =

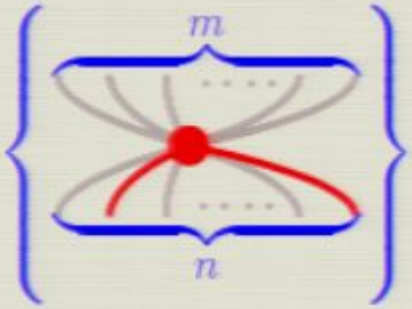


such that, for $k > 0$:

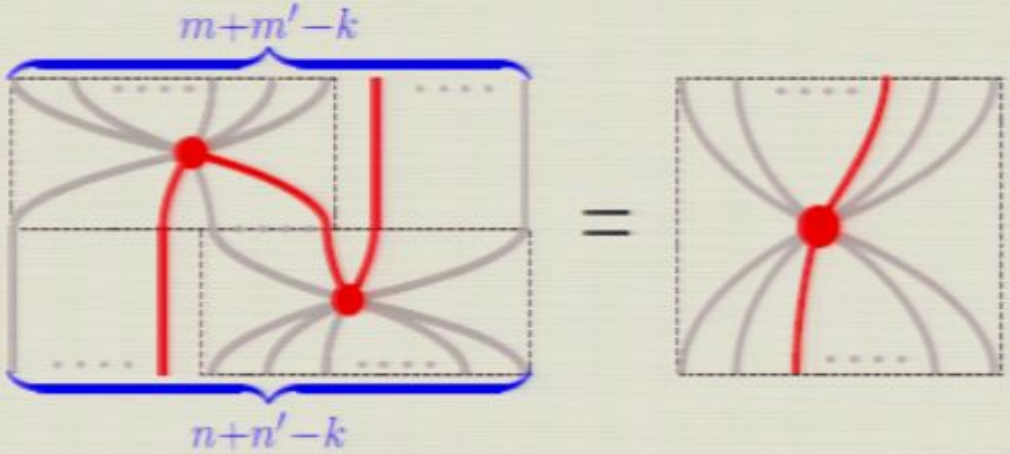


— *spiders* —

'cups/caps' =



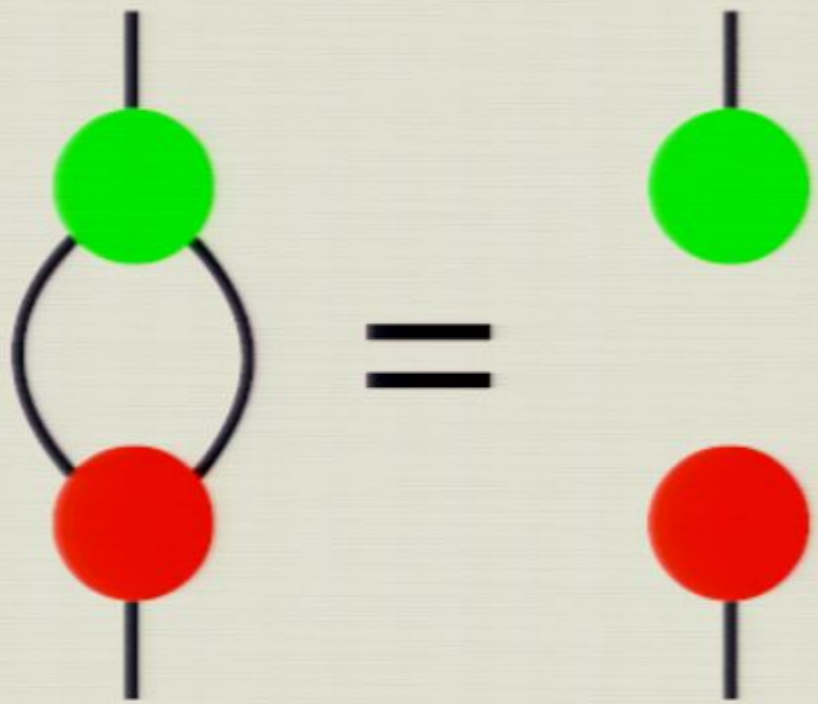
such that, for $k > 0$:



— *complementary bases* —

— *complementary bases* —

Thm.



BC & Ross Duncan (IS HERE) (2008) *Interacting quantum observables*. ICALP'08 & New Journal of Physics **13**, 043016. arXiv:0906.4725

— *quantum gates* —

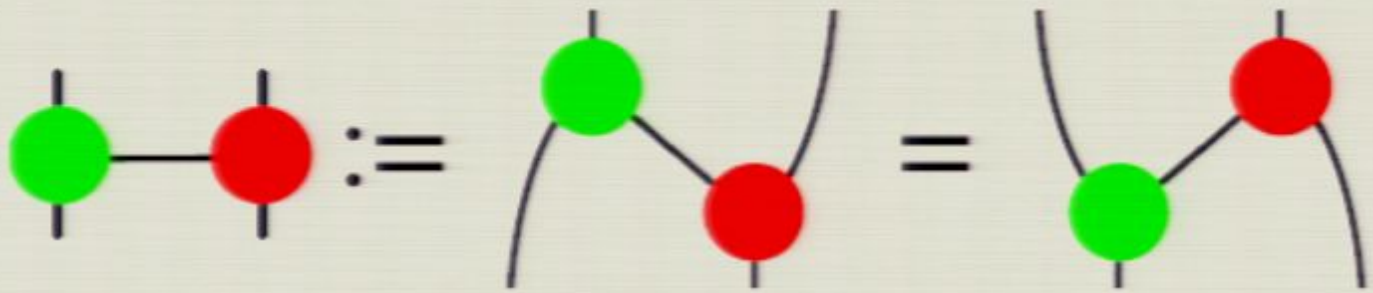
Z-spin:

$$\delta_Z : |i\rangle \mapsto |ii\rangle$$

X-spin:

$$\delta_X : |\pm\rangle \mapsto |\pm\pm\rangle$$

— *quantum gates* —



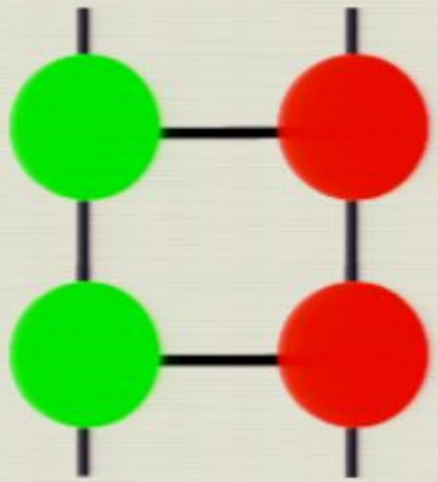
i.e.

$$(\delta_Z^\dagger \otimes 1) \circ (1 \otimes \delta_X) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = CNOT$$

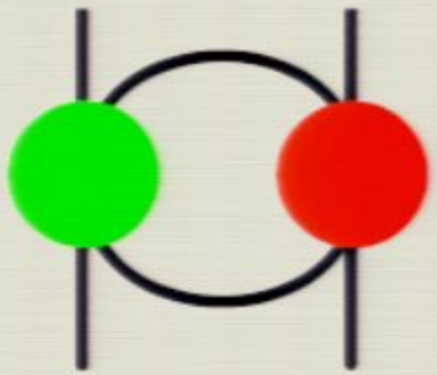
— *quantum gates* —

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = ?$$

— *quantum gates* —

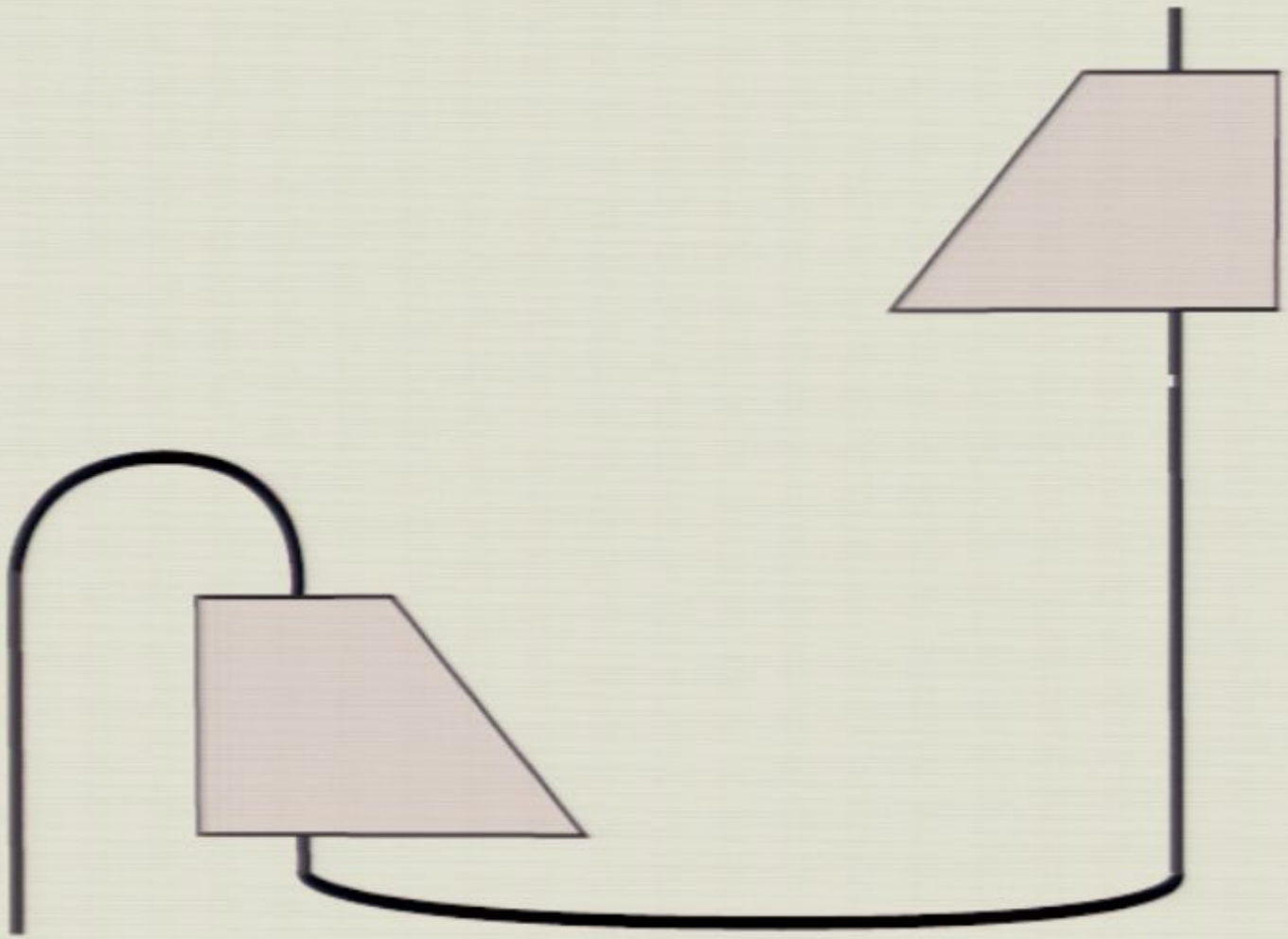


— *quantum gates* —

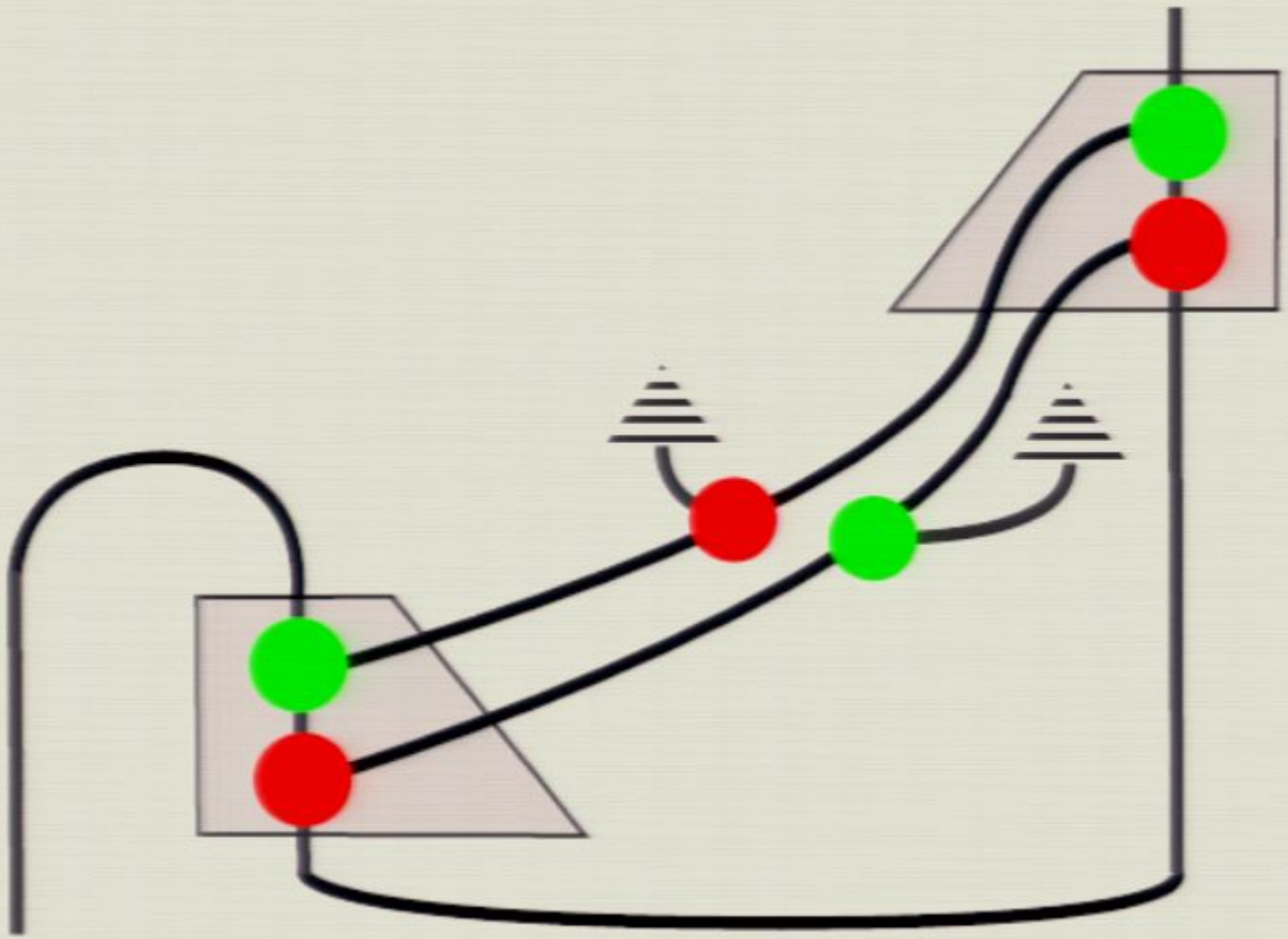


— *quantum gates* —

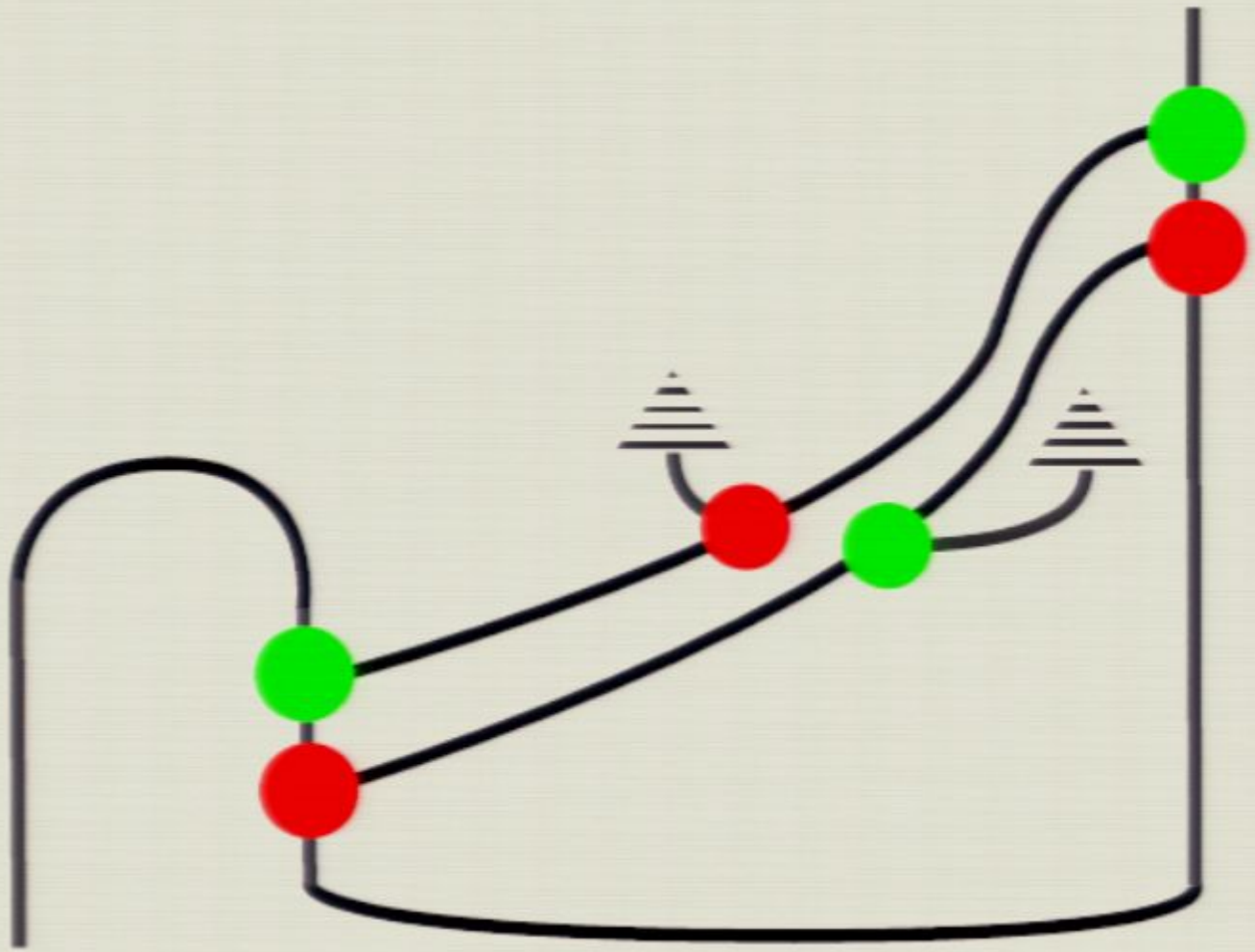
— classical communication and control —



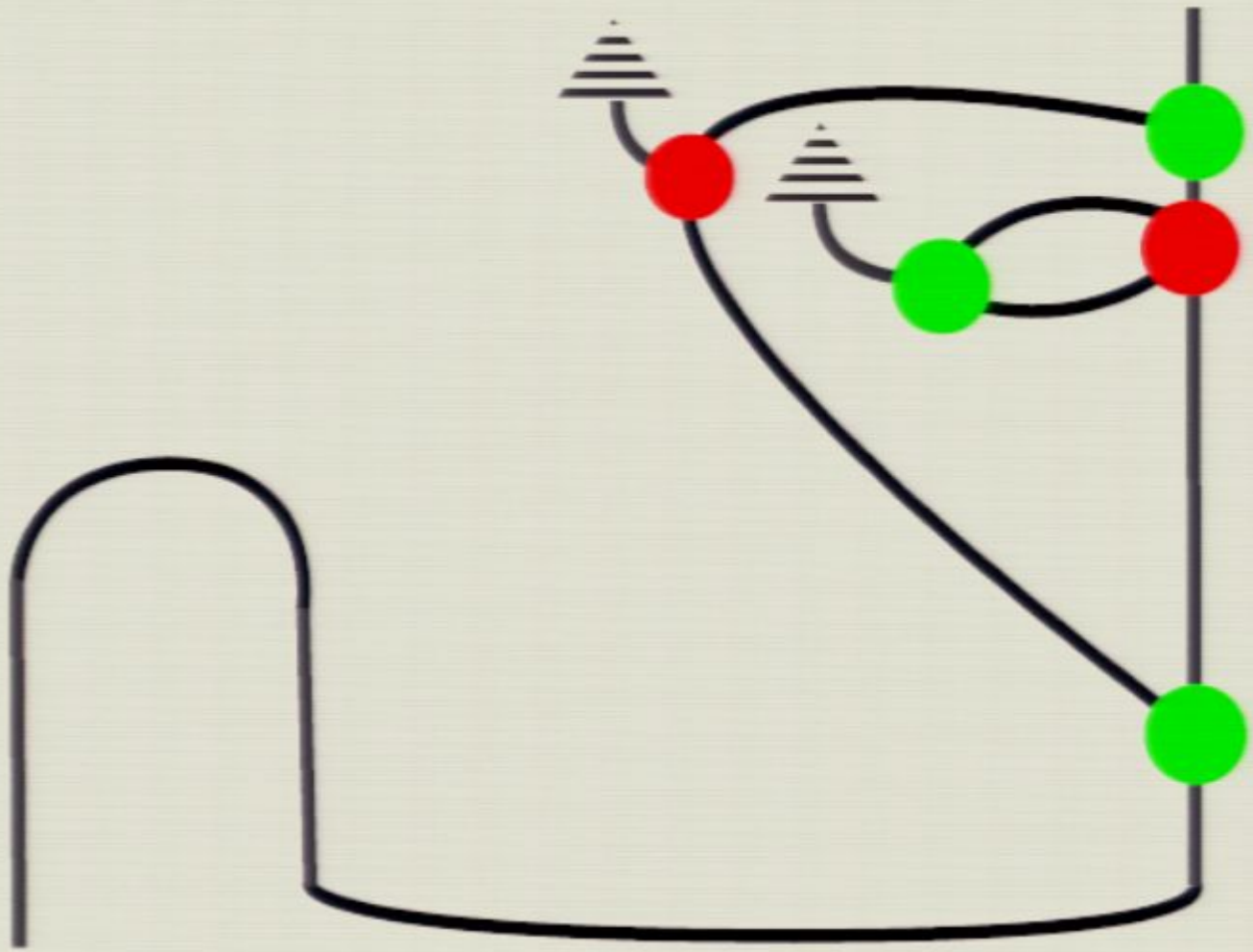
— classical communication and control —



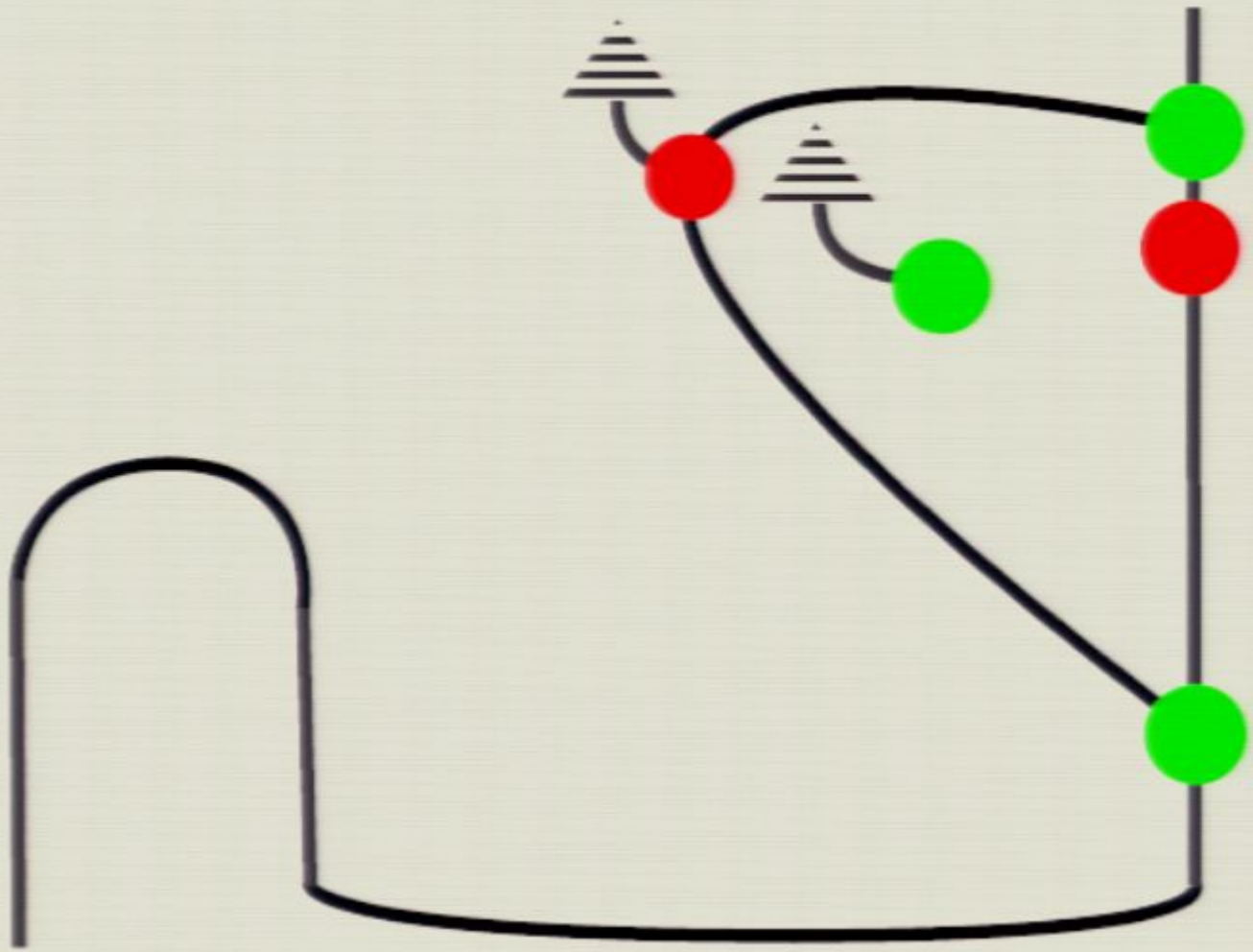
— classical communication and control —



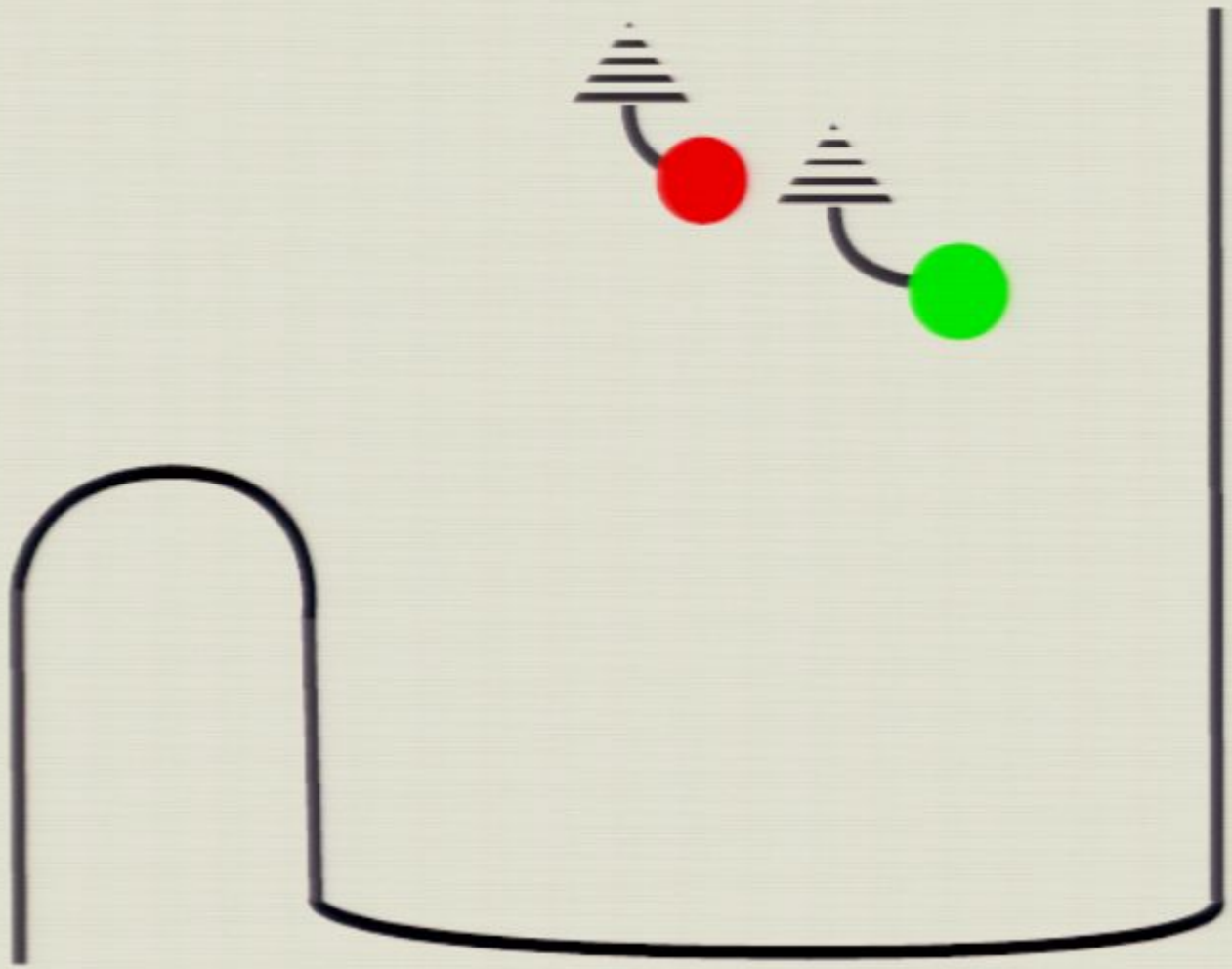
— classical communication and control —



— classical communication and control —

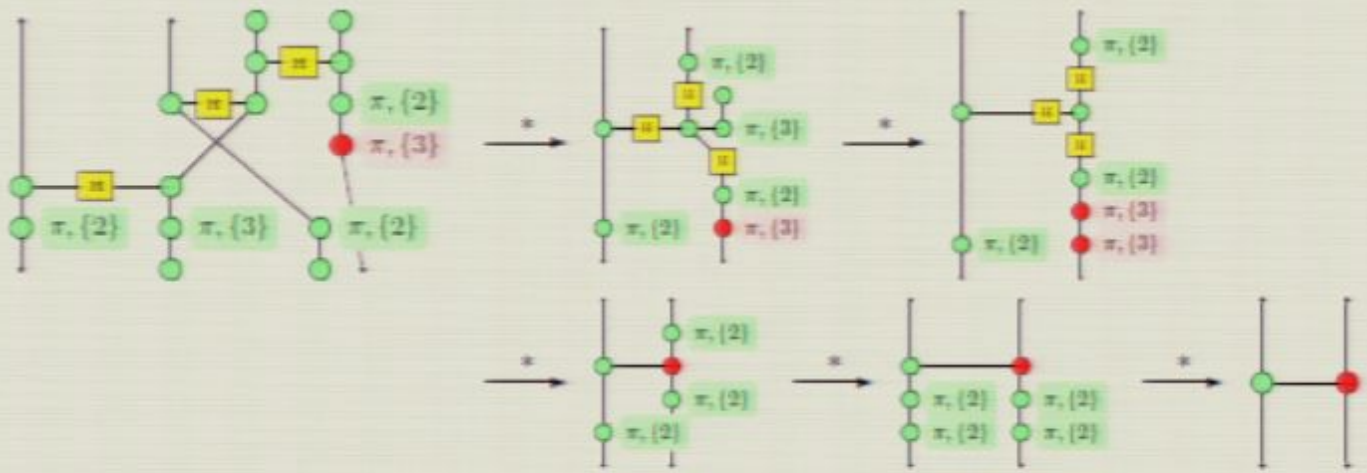


— classical communication and control —



— *applications to QC models* —

Translating measurement based quantum computations to circuits without additional resources:



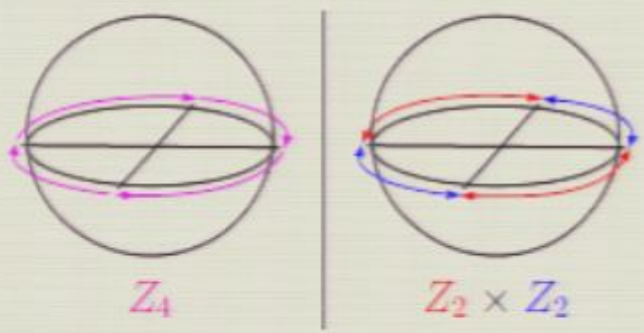
Ross Duncan (IS HERE) & Simon Perdrix (2010) Rewriting measurement-based quantum computations with generalised flow. ICALP'10.

— *applications to quantum foundations* —

Spekkens' toy qubit qubit theory and stabilizer quantum theory within one framework.

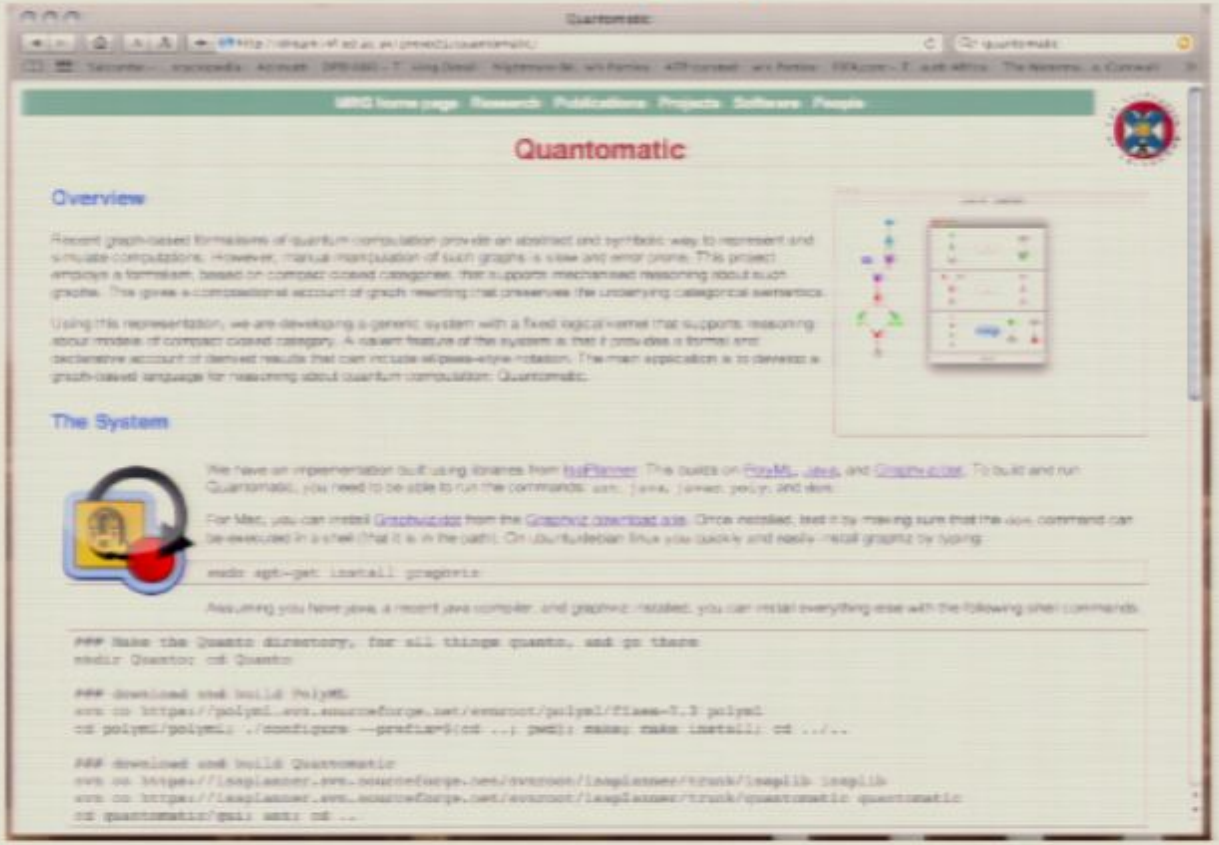
(Non-)locality can be reduced to the 'phase group':

$$\frac{\text{Spekkens' qubit QM}}{\text{stabilizer qubit QM}} = \frac{Z_2 \times Z_2}{Z_4} = \frac{\text{local}}{\text{non-local}}$$



BC, Bill Edwards (COMES HERE) & Robert W. Spekkens (2010) *Phase groups and the origin of non-locality for qubits*. arXiv:1003.5005

— *automated theory exploration* —



Lucas Dixon (Google), Ross Duncan (Brussels, IS HERE), Aleks Kissinger (Oxford, IS HERE AND HAS A POSTER ON THIS) & Alex Merry (Oxford); website: `http://sites.google.com/site/quantomatic/`

THE EXTENDED LANGUAGE II: GHZ vs W ENTANGLEMENT

BC & Aleks Kissinger (IS HERE) (2010) *The compositional structure of multipartite quantum entanglement*. ICALP'10. arXiv:1002.2540

BC, A. Kissinger, Alex Merry & Shibdas Roy (2011) *The GHZ/W-calculus contains rational arithmetic*. arXiv:1103.2812

— *GHZ algebra* —

$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 1 & 1 \end{pmatrix}$

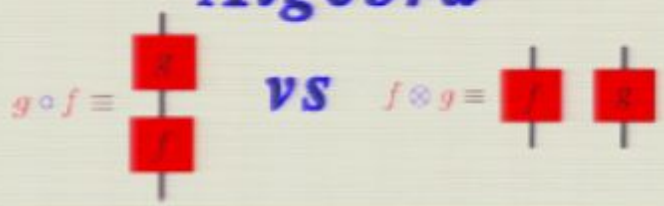
$= \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = \langle 00| + \langle 11|$

— GHZ-W interaction —

Informatics
information flow
vs
no information flow

Topology
connected
vs
disconnected

Algebra



Entanglement
GHZ ***vs*** W

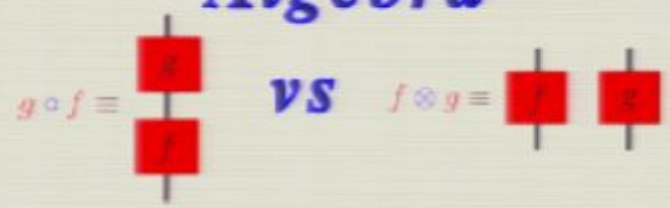
— GHZ-W interaction —

Informatics
information flow
vs
no information flow

Topology
connected
vs
disconnected

Calculus
× *vs* +

Algebra



Entanglement
GHZ *vs* W

Posters on this line of work:

- Aleks Kissinger: *Synthesising physical theories.*
- Ray Lal: *Causal structure in categorical quantum mechanics.*
- Johan Paulsson: *Towards a diagrammatic representation of classical data.*

Also from the Oxford gang:

- Shane Mansfield: *Hardy's non-locality paradox and other possibilistic non-locality conditions.*

Want to learn this kind of category theory? Try:

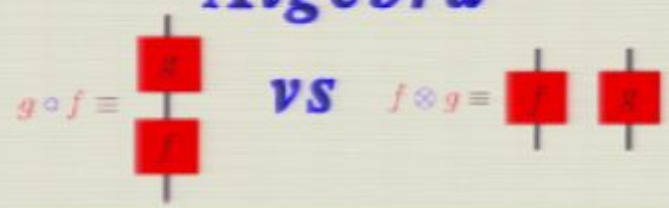
- BC & E. O. Paquette (2010) *Categories for the practicing physicist.* In: New Structures for Physics. BC (Ed.), Springer. `arXiv:0905.3010`

— GHZ-W interaction —

Informatics
information flow
vs
no information flow

Topology
connected
vs
disconnected

Algebra



Entanglement
GHZ ***vs*** W

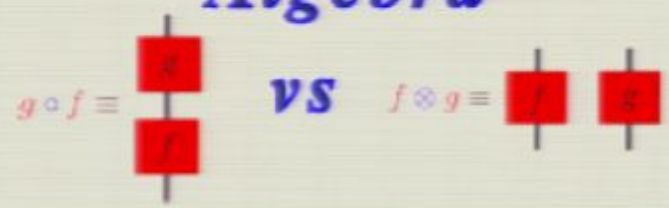
— GHZ-W interaction —

Informatics
information flow
vs
no information flow

Topology
connected
vs
disconnected

Calculus
× *vs* +

Algebra



Entanglement
GHZ *vs* W