Title: Is the universe exponentially complicated? A no-go theorem for hidden variable interpretations of quantum theory.

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Abstract: The quantum mechanical state vector is a complicated object. In particular, the amount of data that must be given in order to specify the state vector (even approximately) increases exponentially with the number of quantum systems. Does this mean that the universe is, in some sense, exponentially complicated? I argue that the answer is yes, if the state vector is a one-to-one description of some part of physical reality. This is the case according to both the Everett and Bohm interpretations. But another possibility is that the state vector merely represents information about an underlying reality. In this case, the exponential complexity of the state vector is no more disturbing that that of a classical probability distribution: specifying a probability distribution over N variables also requires an amount of data that is exponential in N. This leaves the following question: does there exist an interpretation of quantum theory such that (i) the state vector merely represents information and (ii) the underlying reality is simple to describe (i.e., not exponential)? Adapting recent results in communication complexity, I will show that the answer is no. Just as any realist interpretation of quantum theory must be non-locally-causal (by Bell's theorem), any realist interpretation must describe an exponentially complicated reality.

Is the quantum state a real thing?

Two no-go theorems for hidden variable interpretations of quantum theory

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Introduction

Consider N classical systems, each of which has only two distinct states – abstractly these are N classical bits.

Two states: $\uparrow \qquad \downarrow$

An example of a joint state: $\underbrace{\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\ \cdots\ \downarrow}_{\text{N systems}}$

A probability distribution over the joint states involves 2^N probabilities

$$P(\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\uparrow\uparrow)$$
$$P(\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots\uparrow\downarrow)$$
$$\vdots$$
$$P(\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\cdots\downarrow\downarrow)$$

Introduction

• The classical probability distribution is a *very complicated* object, being specified by $2^N - 1$ real parameters. But it is not a real thing. It corresponds (depending on one's stance) to an infinite ensemble of sets of N systems, or to the beliefs of an ideally rational agent.

• The real thing – the *ontology* – is simple. A specification of the underlying state of the system does not even require one real parameter. It needs a *linear* number of classical bits.

The quantum case

- Consider N two-level quantum systems. Abstractly, these are N qubits.

- The number of real parameters required to specify a (pure or mixed) quantum state is exponential in N.

Pure state – $2\ 2^N - 2$
Mixed state – $2^{2N} - 1$

- Does this mean that the universe is exponentially complicated?

- The answer is not obvious because it might depend on how we *interpret* the quantum state. If the quantum state is a real thing (alternatively: a mathematical object in 1-1 correspondence to the physical state of a real thing), then the universe is exponentially complicated. This is the case in the Everett, and de Broglie – Bohm theories. But perhaps the quantum state is more like a probability distribution - with an underlying simple reality.

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The idea that the quantum state is best understood as an analogue of a classical probability distribution – as encoding information about reality, rather than describing reality directly – is the *epistemic interpretation* of the quantum state.

One motivation has been described above – if an exponentially complicated universe is implausible, an interpretation of quantum theory with an underlying simple ontology is attractive.

Other motivations:

- Collapse of the quantum state – much more like updating a probability distribution than a real dynamical process.

- Spekkens toy theory.

Ontic models for quantum theory

Suppose that

(i) There is some underlying reality (corresponding to the **hidden**, or **ontic** state)

(ii) A pure quantum state corresponds to a distribution over ontic states.

For a quantum system *S*, an **ontic model** defines:

- A space Λ of possible ontic states.
- For each quantum state $|\psi\rangle$, a probability distribution $\mu_\psi(\lambda)$ on Λ.
- For each hidden state λ, measurement M, and outcome k, a response function $\xi_{k,M}(\lambda)$ on Λ such that $\xi_{k,M}(\lambda)$ = Prob(k | M, λ).

The model reproduces the quantum predictions if

$$\langle \psi \mid E_{k,M} \mid \psi \rangle = \int_\Lambda \xi_{k,M}(\lambda)\, \mu_\psi(\lambda)\, d\lambda$$

This is the positive operator corresponding to outcome k of measurement M.

Part I

On the exponential character of the ontic state.

Joint work with N. Brunner, R. Galego, and C. Gogolin

A question

- Given N quantum systems, is there an ontic model such that an ontic state can be identified with only a linear number of classical bits?

If so, the ontic model would be very similar to the classical analogue discussed above. The quantum state would be complicated in the way that the classical probability distribution is, but the ontology would be very simple. Given the ontological extravagance of the Everett and Bohm theories, the existence of such a simple model would potentially be a powerful argument *against* Everett and Bohm.

Answer: No

The Ontological Excess Baggage Theorem (Hardy)

For an ontic model to reproduce the predictions of a single qubit, the cardinality of Λ must be infinite.

So the quantum case of N qubits is not at all like the classical case of N bits and probability distributions over N bits. Indeed, forget about scaling as the number of systems increases -- even for a single qubit, the number of hidden states must be infinite.

Another question ?

- Instead of counting the number of hidden states (or the number of classical bits required to identify a state), assume a continuum of hidden states and count the **number of real parameters** required to identify one.

- More precisely, let the space Λ have some suitable structure (e.g., a differential manifold) and let suitable continuity conditions hold (e.g, nearby hidden states on the manifold make similar predictions). Does there then exist an ontic model for N quantum systems such that the number of real parameters required to identify a hidden state is linear (or polynomial, or at least sub-exponential) in N?

This question was raised by A. Montina in arXiv:0711.4770

Answer: Unknown

An alternative direction…

The approximate case

Suppose we only require of an epistemic model that quantum probabilities are reproduced up to a tolerance of $\pm\epsilon$?

Hardy's result does not hold.

For a finite dimensional quantum system, can approximately reproduce the quantum predictions with a *finite* number of ontic states.

Question: for fixed ϵ, how quickly must the number of ontic states grow as the number of systems increases?

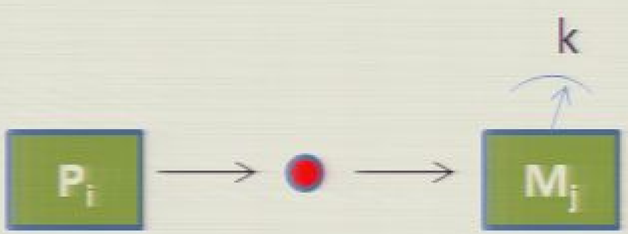
Data tables

- Consider an experimental setup consisting of r different ways of preparing a system, and s different measurements that can be performed on the system, each with t outcomes.

- For each preparation and each measurement, there is a probability of obtaining each outcome.

- Represent the data in the form of a table of probabilities…

		P_1	P_2	…	P_r
M_1	1	$Pr(1\|M_1,P_1)$	$Pr(1\|M_1,P_2)$	…	$Pr(1\|M_1,P_r)$
	2	$Pr(2\|M_1,P_1)$	$Pr(2\|M_1,P_2)$	…	$Pr(2\|M_1,P_r)$
	…	…	…	…	…
	t	$Pr(t\|M_1,P_1)$	$Pr(t\|M_1,P_2)$	…	$Pr(t\|M_1,P_r)$
M_2	1	$Pr(1\|M_2,P_1)$	$Pr(1\|M_2,P_2)$	…	$Pr(1\|M_2,P_r)$
	2	$Pr(2\|M_2,P_1)$	$Pr(2\|M_2,P_2)$	…	$Pr(2\|M_2,P_r)$
	.	…	…	…	…
	t	$Pr(t\|M_2,P_1)$	$Pr(t\|M_2,P_2)$	…	$Pr(t\|M_2,P_r)$
	…	…	…	…	…
M_s	1	$Pr(1\|M_s,P_1)$	$Pr(1\|M_s,P_2)$	…	$Pr(1\|M_s,P_r)$
	2	$Pr(2\|M_s,P_1)$	$Pr(2\|M_s,P_2)$	…	$Pr(2\|M_s,P_r)$
	.	…	…	…	…
	t	$Pr(t\|M_s,P_1)$	$Pr(t\|M_s,P_2)$	…	$Pr(t\|M_s,P_r)$



Bounds on the classical dimension

- Given a data table, suppose we want to construct an ontic model with the specific aim of reproducing the probabilities in that table.

		P_1	P_2	…	P_r
M_1	1	$Pr(1\|M_1,P_1)$	$Pr(1\|M_1,P_2)$	…	$Pr(1\|M_1,P_r)$
	2	$Pr(2\|M_1,P_1)$	$Pr(2\|M_1,P_2)$	…	$Pr(2\|M_1,P_r)$
	…	…	…	…	…
	t	$Pr(t\|M_1,P_1)$	$Pr(t\|M_1,P_2)$	…	$Pr(t\|M_1,P_r)$
M_2	1	$Pr(1\|M_2,P_1)$	$Pr(1\|M_2,P_2)$	…	$Pr(1\|M_2,P_r)$
	2	$Pr(2\|M_2,P_1)$	$Pr(2\|M_2,P_2)$	…	$Pr(2\|M_2,P_r)$
	.	…	…	…	…
	t	$Pr(t\|M_2,P_1)$	$Pr(t\|M_2,P_2)$	…	$Pr(t\|M_2,P_r)$
	…	…	…	…	…
M_s	1	$Pr(1\|M_s,P_1)$	$Pr(1\|M_s,P_2)$	…	$Pr(1\|M_s,P_r)$
	2	$Pr(2\|M_s,P_1)$	$Pr(2\|M_s,P_2)$	…	$Pr(2\|M_s,P_r)$
	.	…	…	…	…
	t	$Pr(t\|M_s,P_1)$	$Pr(t\|M_s,P_2)$	…	$Pr(t\|M_s,P_r)$

- Finite set Λ of ontic states, $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{d_c}\}$
- Each preparation P_i corresponds to a probability distribution over Λ.
- For each hidden state λ_l, and for each measurement M_j, outcome k, there is a probability $Pr(k \mid M_j, \lambda_l)$.

- Must have $Pr(k|M_j, P_i) = \sum_l Pr(\lambda_l \mid P_i)\, Pr(k|M_j,\lambda_l)$ ($\pm\, \epsilon$ for an approximate model).

For a particular data table, what is the value of $d_c = |\Lambda|$ in the most efficient model?

This question was raised by Harrigan and Rudolph in arXiv:0709.1149.

For a completely arbitrary data table, it may be a difficult question, but they give some lower and upper bounds.

Bounds on the quantum dimension

- Given a data table, suppose we want to construct a quantum model with the specific aim of reproducing the probabilities of that table.

		P_1	P_2	...	P_r
M_1	1	$Pr(1\|M_1,P_1)$	$Pr(1\|M_1,P_2)$	...	$Pr(1\|M_1,P_r)$
	2	$Pr(2\|M_1,P_1)$	$Pr(2\|M_1,P_2)$	...	$Pr(2\|M_1,P_r)$
	...	...	...	...	...
	t	$Pr(t\|M_1,P_1)$	$Pr(t\|M_1,P_2)$	...	$Pr(t\|M_1,P_r)$
M_2	1	$Pr(1\|M_2,P_1)$	$Pr(1\|M_2,P_2)$	...	$Pr(1\|M_2,P_r)$
	2	$Pr(2\|M_2,P_1)$	$Pr(2\|M_2,P_2)$	...	$Pr(2\|M_2,P_r)$
	.	...	...	...	...
	t	$Pr(t\|M_2,P_1)$	$Pr(t\|M_2,P_2)$	...	$Pr(t\|M_2,P_r)$
	...	...	...	...	...
M_s	1	$Pr(1\|M_s,P_1)$	$Pr(1\|M_s,P_2)$	...	$Pr(1\|M_s,P_r)$
	2	$Pr(2\|M_s,P_1)$	$Pr(2\|M_s,P_2)$	...	$Pr(2\|M_s,P_r)$
	.	...	...	...	...
	t	$Pr(t\|M_s,P_1)$	$Pr(t\|M_s,P_2)$	...	$Pr(t\|M_s,P_r)$

- Hilbert space H, dimension d_Q.
- Each preparation P_i corresponds to a density matrix ρ_i on H.
- Each measurement M_j corresponds to a POVM $\{ E^j_1, \ldots, E^j_t \}$ on H.
- Must have $Pr(k|M_j, P_i) = Tr(\rho_i E^j_k)$.

For a particular data table, what is the value of $d_Q = \dim(H)$ in the most efficient model?

This question was raised by Wehner, Christandl and Doherty in arXiv:0808.3960.

For a completely arbitrary data table, it may be a difficult question, but they give a lower bound.

Final question

?

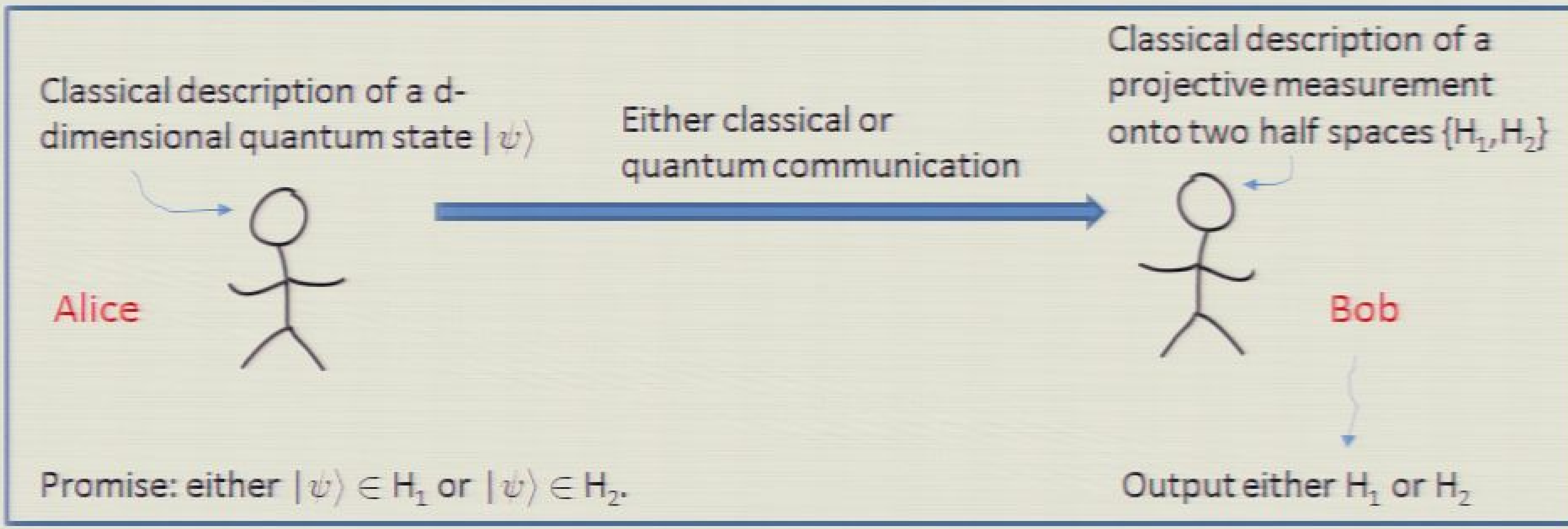
Can we find a sequence of data tables D_1, D_2 , ... of increasing size such that

i) The nth table has a quantum model with dim $(H_n) = n$.
ii) The most efficient hidden variable model for the nth table has $|\Lambda| = O(2^n)$, even if a fixed error ϵ is tolerated.

Answer:
Yes!

Communication complexity

Klartag and Regev (arXiv:1009.3640) consider the following communication complexity problem.



- There is a trivial quantum solution involving only log d qubits – Alice simply sends the state $|\psi\rangle$ to Bob.
- Klartag and Regev show (this is the hard bit) that if Alice and Bob can only send classical bits to one another, then they must send O(d) bits, even if approximation is allowed (they can sometimes output the wrong answer) and even if a two-way conversation is allowed.

Applying the Klartag and Regev result

- It is tailor-made for our problem!

- Construct data tables such that preparations are preparations of d-dimensional quantum states, and measurements are 2-outcome projective measurements onto half-spaces.

- Any ontic model for the data table corresponds to a strategy for the communication complexity problem: Alice simply sends to Bob a classical description of the ontic state.

- The Klartag-Regev result shows that any ontic model for such a data table must have a number of ontic states $O(2^d)$. Otherwise the corresponding strategy would solve the communication complexity problem with only a small amount of classical communication, which is impossible.

Can the no-go theorem be tested experimentally?

Part II

On the epistemic character of the quantum state

Joint work with M. Pusey

Reminder - ontic models for quantum theory

Suppose that

(i) There is some underlying reality (corresponding to the **hidden**, or **ontic** state)

(ii) A particular method of preparing a system corresponds to a distribution over ontic states.

For a quantum system *S*, an **ontic model** defines:

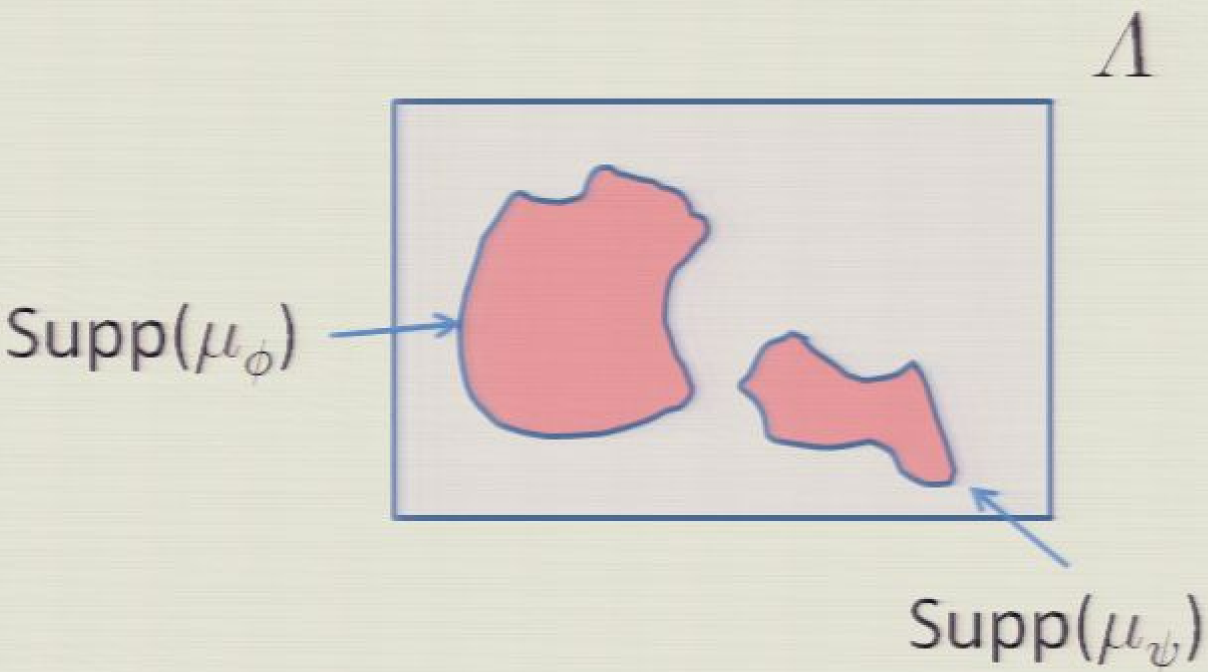
- A space Λ of possible ontic states.
- For each quantum state $|\psi\rangle$, a probability distribution $\mu_\psi(\lambda)$ on Λ.
- For each hidden state λ, measurement M, and outcome k, a response function $\xi_{k,M}(\lambda)$ on Λ such that $\xi_{k,M}(\lambda)$ = Prob(k|M, λ).

The model reproduces the quantum predictions if

$$\langle \psi | E_{k,M} | \psi \rangle = \int_\Lambda \xi_{k,M}(\lambda)\, \mu_\psi(\lambda)\, d\lambda$$

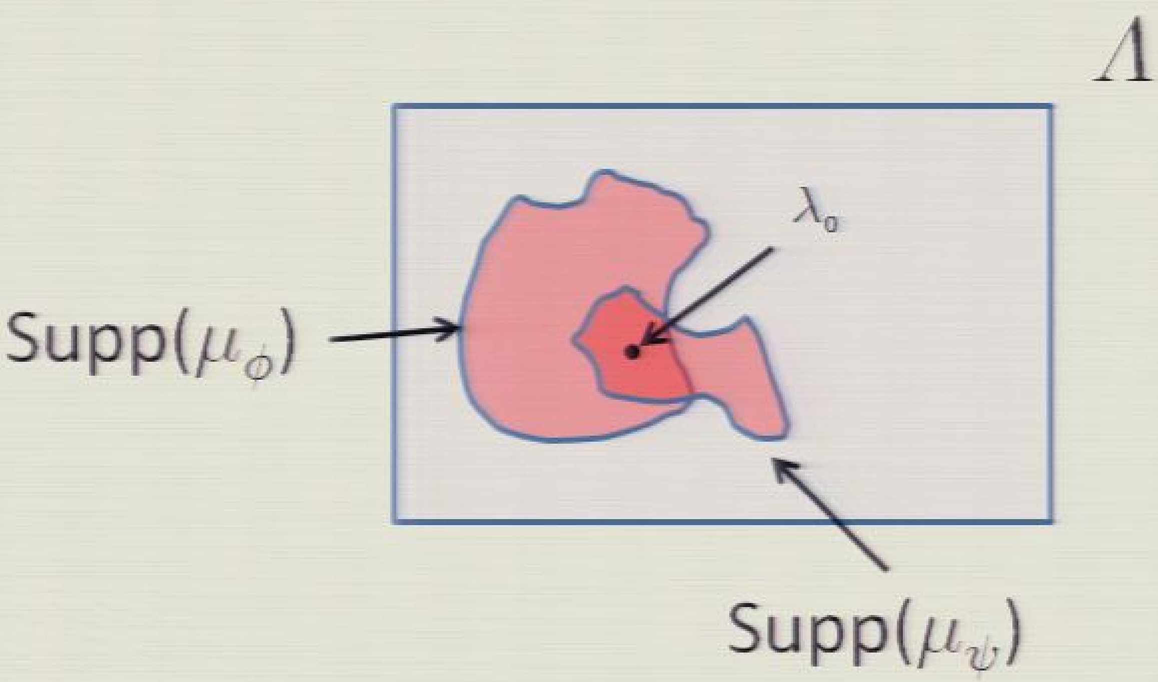
This is the positive operator corresponding to outcome k of measurement M.

ψ-ontic models



- Suppose that for every pair of distinct quantum states $|\phi\rangle$ and $|\psi\rangle$, the distributions μ_ϕ and μ_ψ do not overlap.
- In this case, the quantum state can be inferred from the ontic state. All the information about the quantum state is contained in the ontic state.
- Call such a model ψ-ontic.

ψ-epistemic models

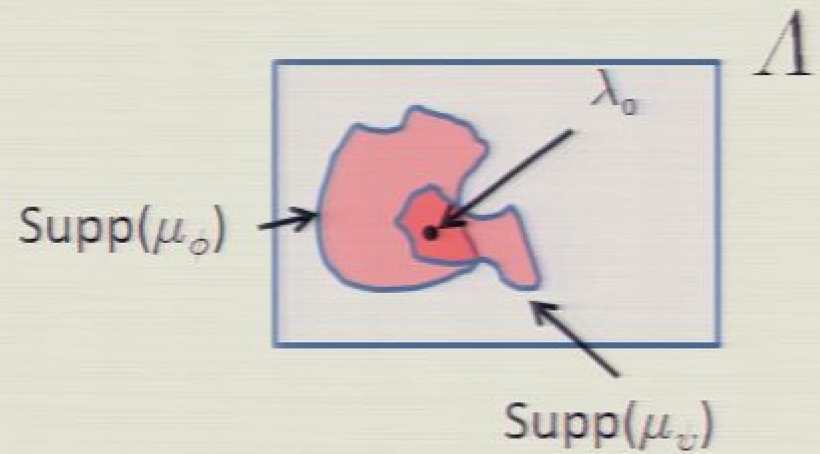


- If there exists some pair of quantum states $|\phi\rangle$ and $|\psi\rangle$, with overlapping distributions μ_ϕ and μ_ψ , then the model is ψ-epistemic.
- If the ontic state lies in the overlap region, e.g., λ_0 above, then it is not possible to infer whether the quantum state $|\phi\rangle$ or $|\psi\rangle$ was prepared.

A no-go theorem for ψ-epistemic models

Suppose there are distinct quantum states $|\phi\rangle$ and $|\psi\rangle$, and an ontic state λ_o such that:

$\Pr(\lambda_o \mid \phi) \geq q > 0,$
$\Pr(\lambda_o \mid \psi) \geq q > 0.$

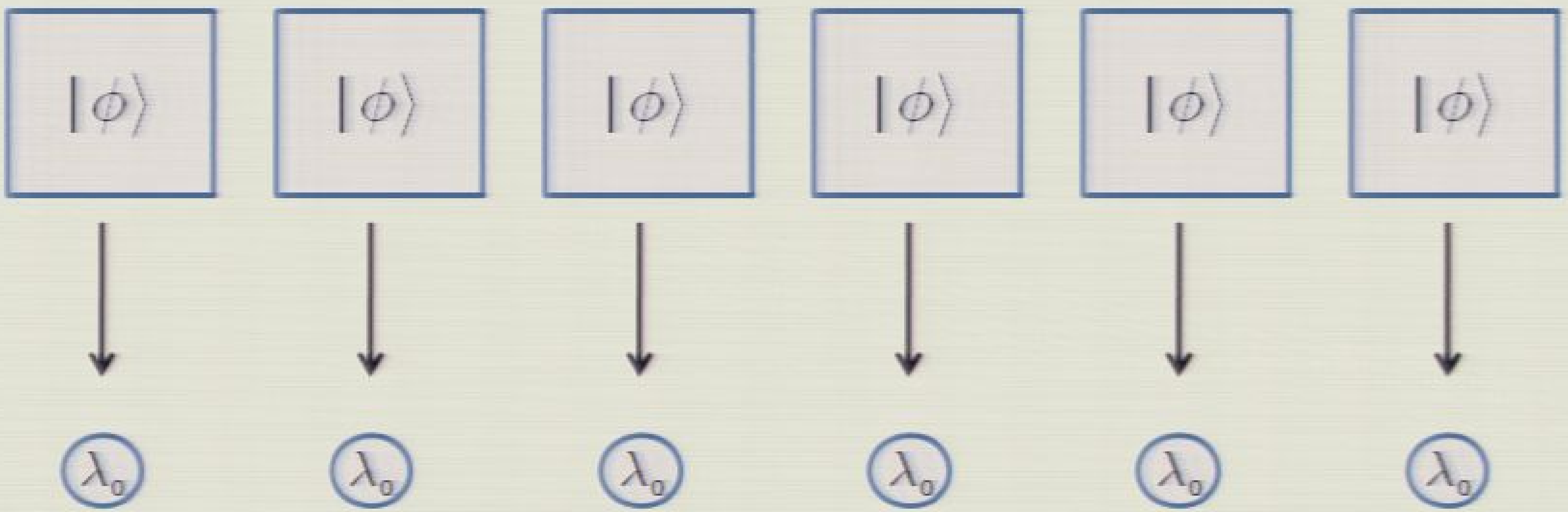


Let $x = |\langle\phi|\psi\rangle|^2$ and choose positive n such that $n \geq \log_x 1/3$.

Consider the following three joint states of 2n systems. Each is a product state and can be generated with 2n completely independent preparations.

$$|\chi_1\rangle = |\phi\rangle^{\otimes 2n}$$
$$|\chi_2\rangle = |\psi\rangle^{\otimes 2n}$$
$$|\chi_3\rangle = |\phi\rangle^{\otimes n} \otimes |\psi\rangle^{\otimes n}$$

When, e.g., $|\chi_1\rangle = |\phi\rangle^{\otimes 2n}$ is prepared, assume that the ontic state is a product of ontic states for each system, and that the distribution over ontic states is a product distribution. This is very natural given that the 2n preparations may have nothing to do with one another. There is then some chance that the ontic state λ_0 is prepared every time:



$$\Pr(\lambda_0 \times \lambda_0 \times \cdots \times \lambda_0 \mid \chi_1) \geq (q)^{2n}$$

Similarly,

$$\Pr(\lambda_0 \times \lambda_0 \times \cdots \times \lambda_0 \mid \chi_2) \geq (q)^{2n}$$

$$\Pr(\lambda_0 \times \lambda_0 \times \cdots \times \lambda_0 \mid \chi_3) \geq (q)^{2n}$$

- Now here's the problem…

- As long as we choose n large enough ($n \geq \log_x 1/3$), then $|\chi_1\rangle$, $|\chi_2\rangle$ and $|\chi_3\rangle$ are what Caves, Fuchs and Schack call *incompatible state assigments.*
  C. M. Caves, C. A. Fuchs and R. Schack, PRA **66**, 062111 (2002)

- More precisely, there exists a projective measurement $M = \{ Q_1, Q_2, Q_3, Q_0 \}$ across the joint system such that

$$\langle \chi_1 | Q_1 | \chi_1 \rangle = 0$$
$$\langle \chi_2 | Q_2 | \chi_2 \rangle = 0$$
$$\langle \chi_3 | Q_3 | \chi_3 \rangle = 0$$
$$\langle \chi_i | Q_0 | \chi_i \rangle = 0, \quad i = 1,2,3$$

- If the quantum state prepared is $|\chi_1\rangle$, then there is a non-zero probability that the ontic state is $\lambda_o \times \cdots \times \lambda_o$. So if the ontic model is to recover quantum predictions, we must have $\Pr(Q_1 | \lambda_o \times \cdots \times \lambda_o) = 0$.

- Similarly $\Pr(Q_0 | \lambda_o \times \cdots \times \lambda_o) = \Pr(Q_2 | \lambda_o \times \cdots \times \lambda_o) = \Pr(Q_3 | \lambda_o \times \cdots \times \lambda_o) = 0$.

- So given $\lambda_o \times \cdots \times \lambda_o$ there is no probability distribution over the outcomes of M that is consistent with the predictions of quantum theory.

Conclusion

If an ontic model is to reproduce the predictions of quantum theory, distinct quantum states must correspond to non-overlapping distributions over ontic states.

- I have presented a simplified version of the argument.
- A better proof deals with continuum ontic states (where we can't assume $\Pr(\lambda) > 0$ for any individual λ).
- The better proof also deals with the approximate case, where an ontic model only need reproduce the predictions of quantum theory up to ϵ.

Theorem (M. Pusey and J. Barrett, forthcoming):

Suppose that an ontic model approximately reproduces quantum predictions, with probabilities for measurement outcomes within ϵ of the quantum probabilities. Consider quantum states $|\phi\rangle$ and $|\psi\rangle$ with $x = |\langle\phi|\psi\rangle|^2$. Then

$$4\,\epsilon \geq (\,1 - D\,(\mu_\phi, \mu_\psi))^{2n}$$

with D the trace distance, and $n \geq 1$, $n \geq \log_x 1/3$.

Experimental challenge

- For quantum states $|\phi\rangle$ and $|\psi\rangle$ with $|\langle\phi|\psi\rangle|^2$ as close to 1 as possible, establish a lower bound for D(μ_ϕ, μ_ψ) as close to 1 as possible.

Conclusions

- In a hidden variable model that reproduces *exactly* the predictions of quantum theory, there must be an infinity of ontic states. But how many real parameters are needed to characterize an ontic state? Is it exponential in the number of systems? Don't know.

- If a hidden variable model is only required to reproduce *approximately* the predictions of quantum theory, it can be done with a finite set of ontic states. But then, then number of classical bits needed to identify the hidden state underlying N quantum systems is exponential in N.

- If, given n independent preparations, the joint ontic state can be represented as a Cartesian product, with a product distribution, then a genuinely epistemic interpretation of the quantum state is impossible. All information about the quantum state must be contained in the ontic state.

These no-go theorems are not easy to evade.

Their assumptions are not strong.

Either: something pretty radical has to give...

Or: accept an exponential ontology/real quantum state – might as well be Everettian.