Title: Vectors and affine forms, straight and twisted

Date: May 24, 2011 04:00 PM

URL: http://pirsa.org/11050024

Abstract: This is a geometric tutorial about straight and twisted vectors and forms (ie, de Rham currents) leading to some wild thoughts about the EM field as a *thing*, ie with properties similar to a piece of matter; and to some even wilder thoughts about a metric-free GR.

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Vectors and affine forms, straight and twisted

with examples of their physically intuitive meaning

P.G.L. Porta Mana

PI

24 May 2011

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PI

24 May 2011

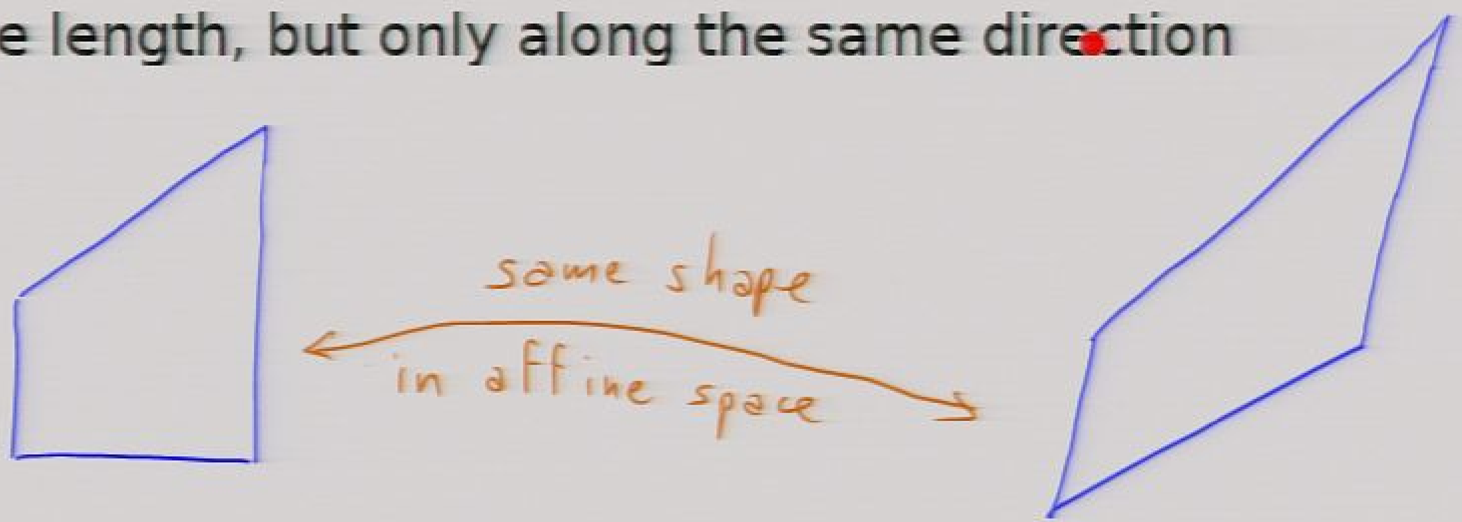
Affine spaces

- Parallelism

- No absolute length

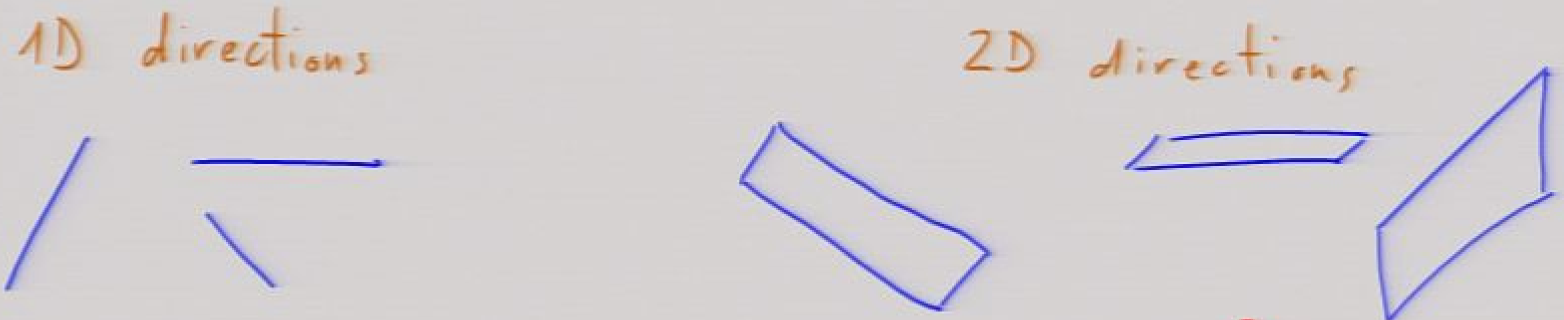
- No angles

- Relative length, but only along the same direction

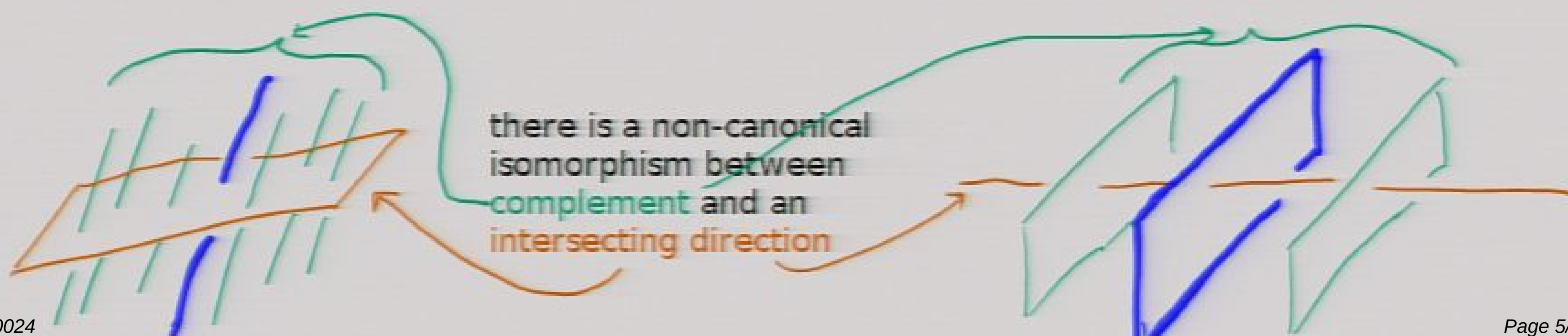


Directions and complement spaces

mD direction = mD subspace (without a definite position)

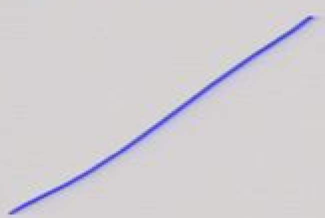


(n-m)D complement space = space of parallel mD directions



Ordinary vectors

1D direction



magnitude

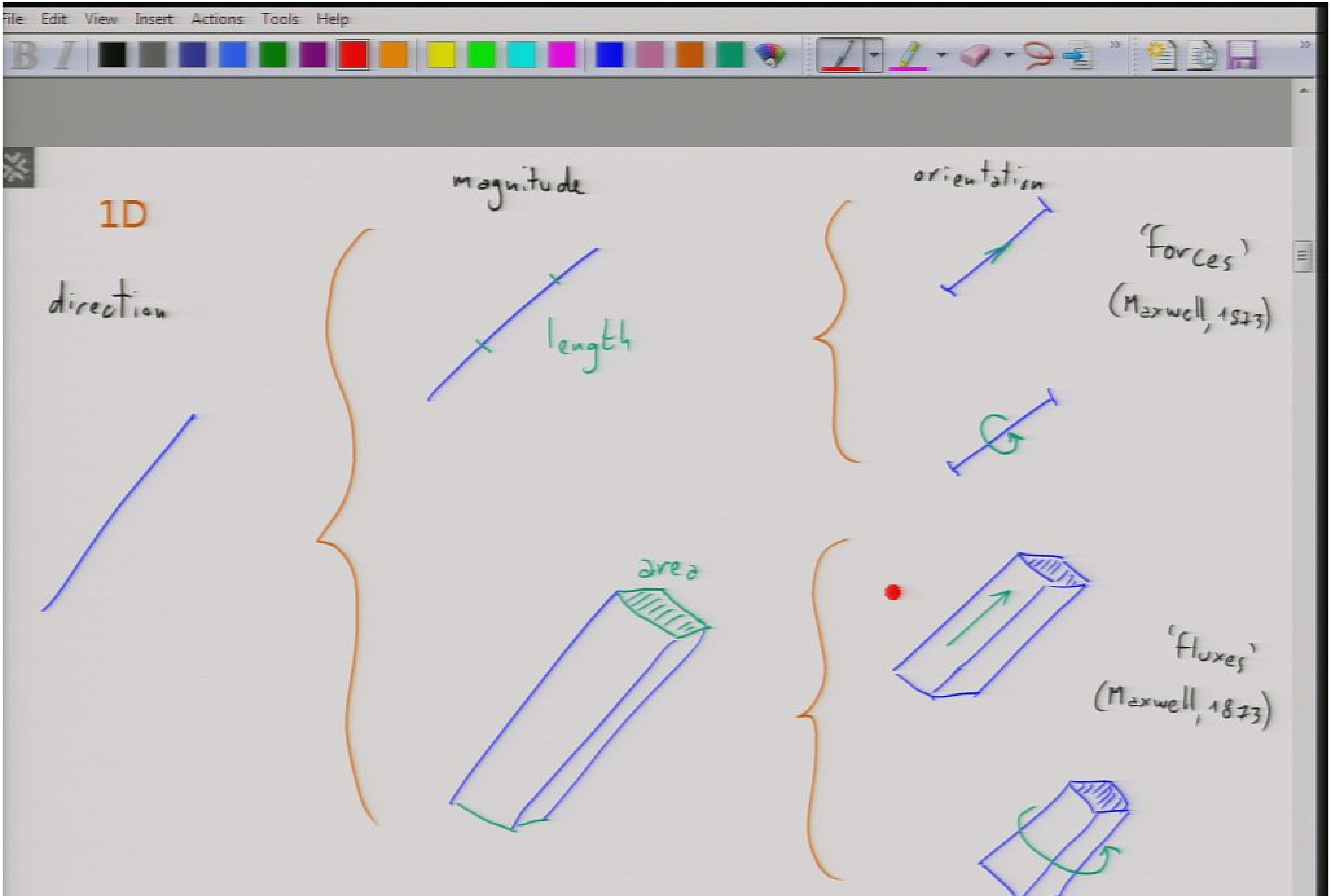
orientation

Generalize:

*n*D direction

magnitude:
- on direction
- or on complement

orientation:
- on direction
- or on complement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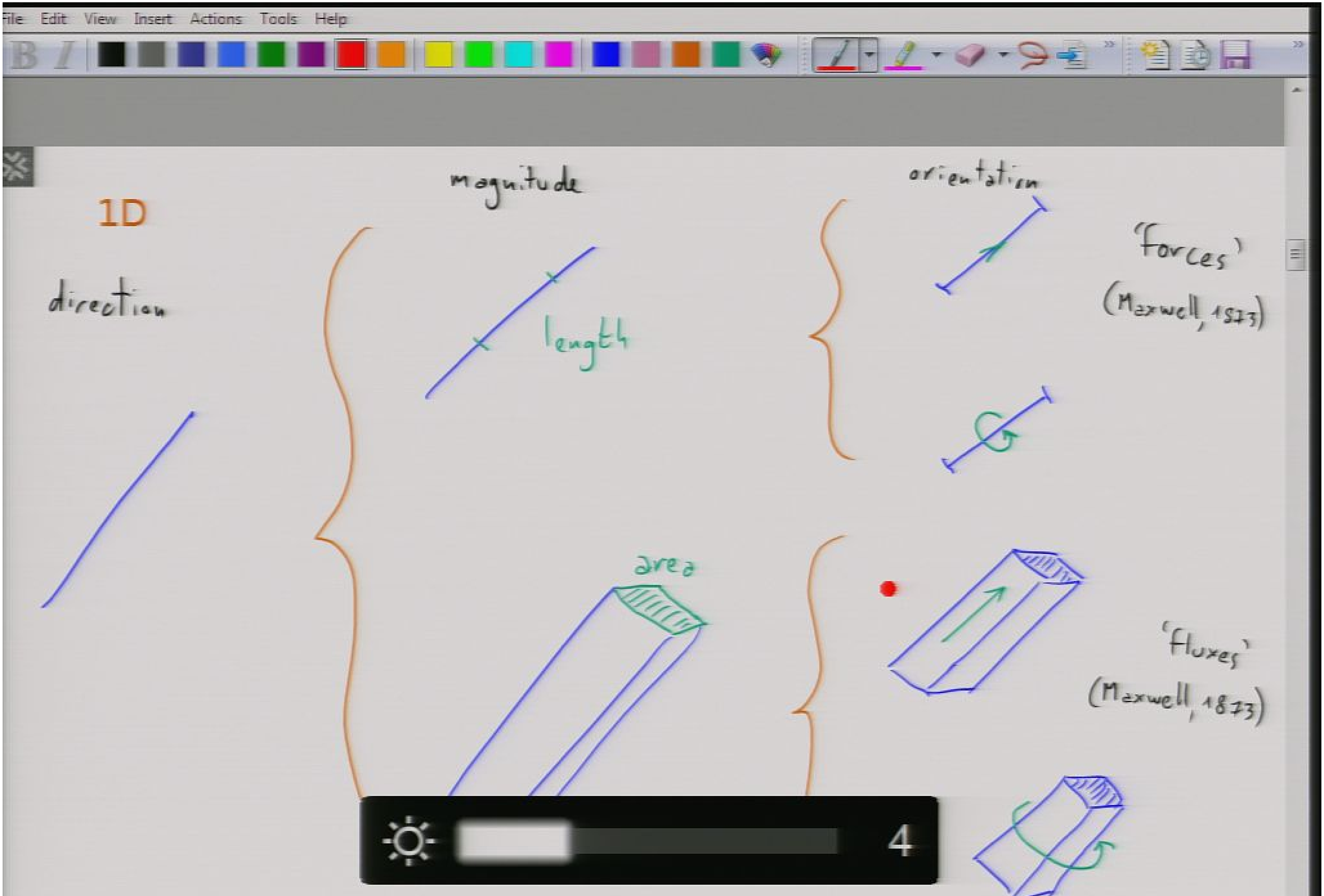
1D
direction
magnitude
length
area
orientation
'forces'
(Maxwell, 1873)
'fluxes'
(Maxwell, 1873)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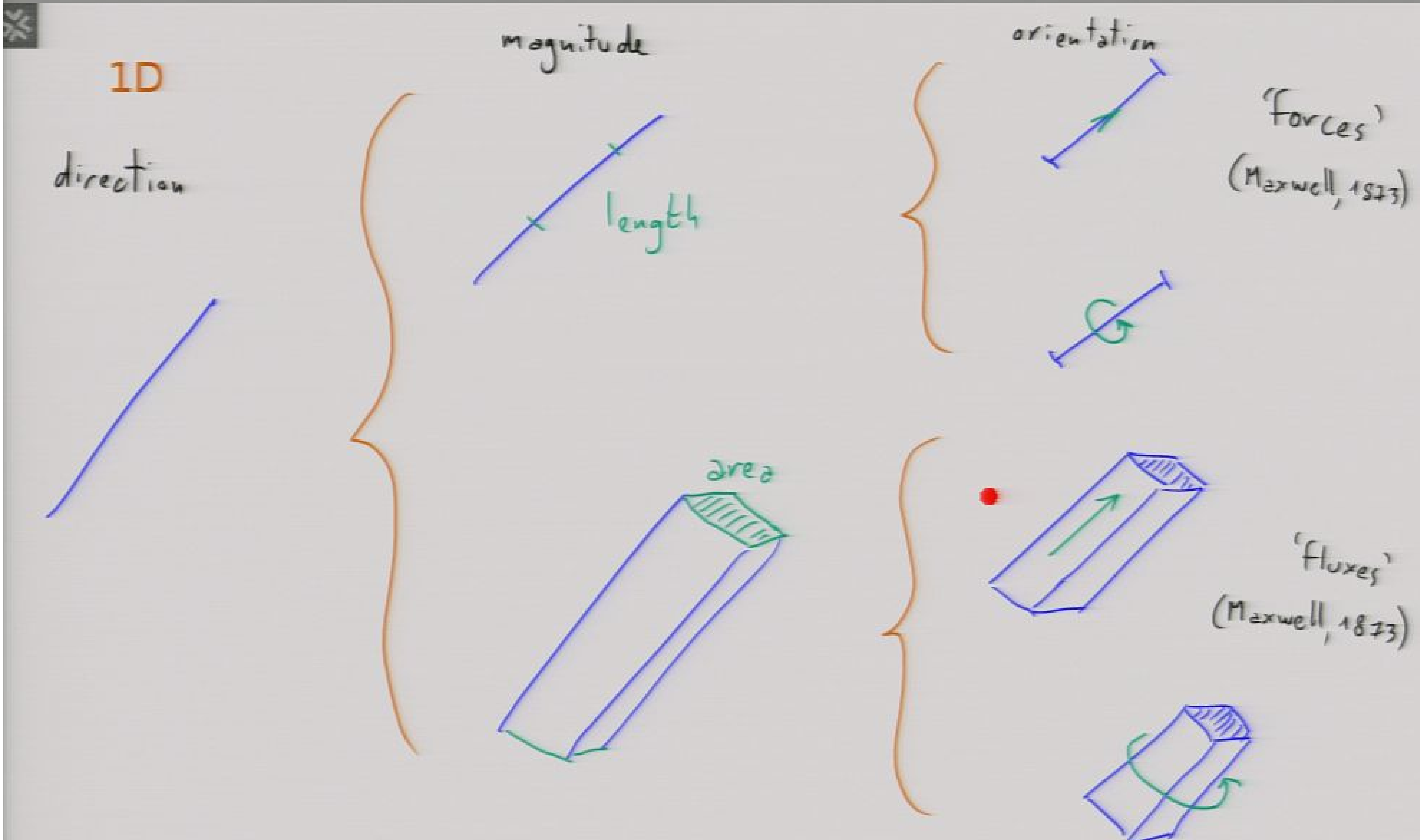
4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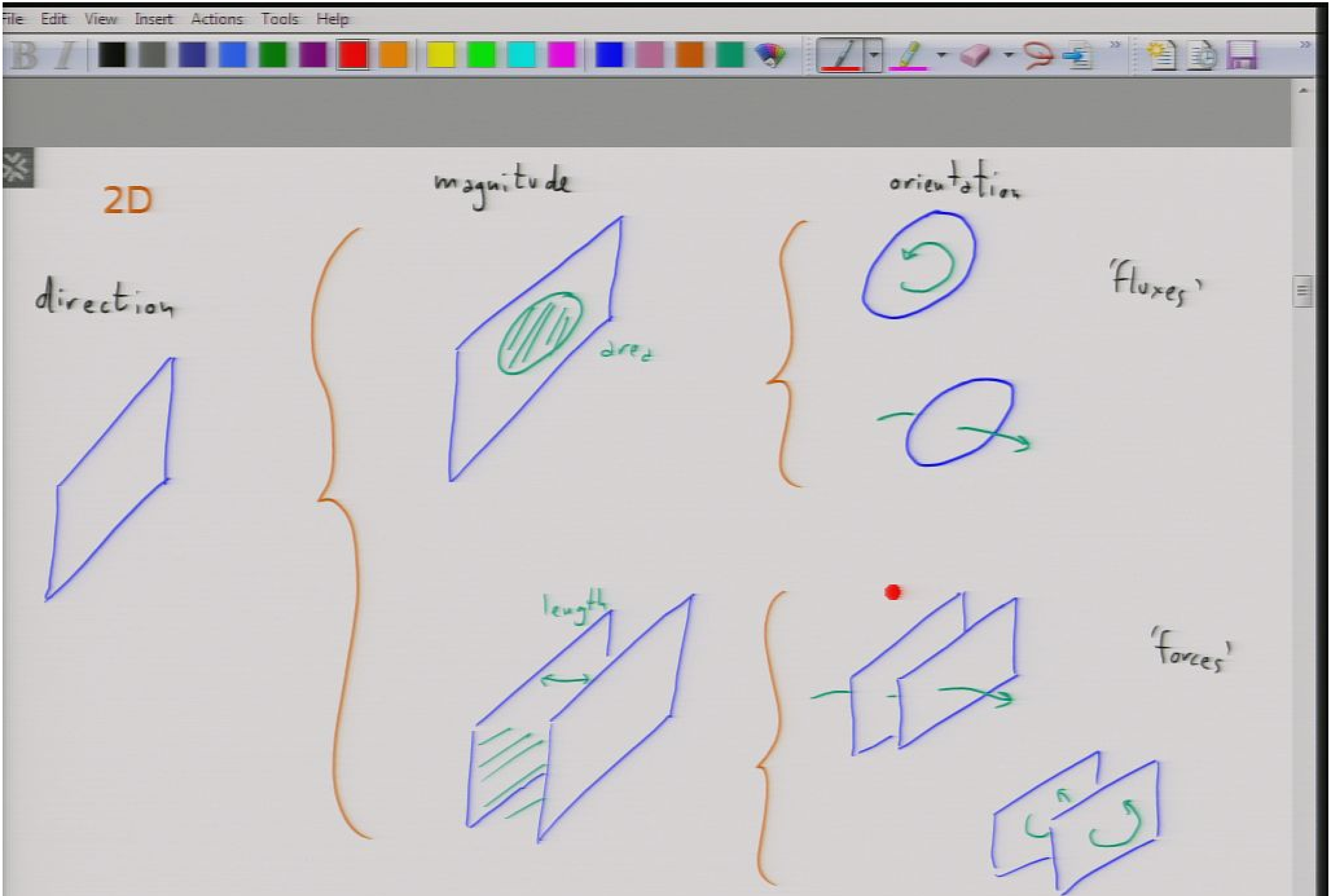
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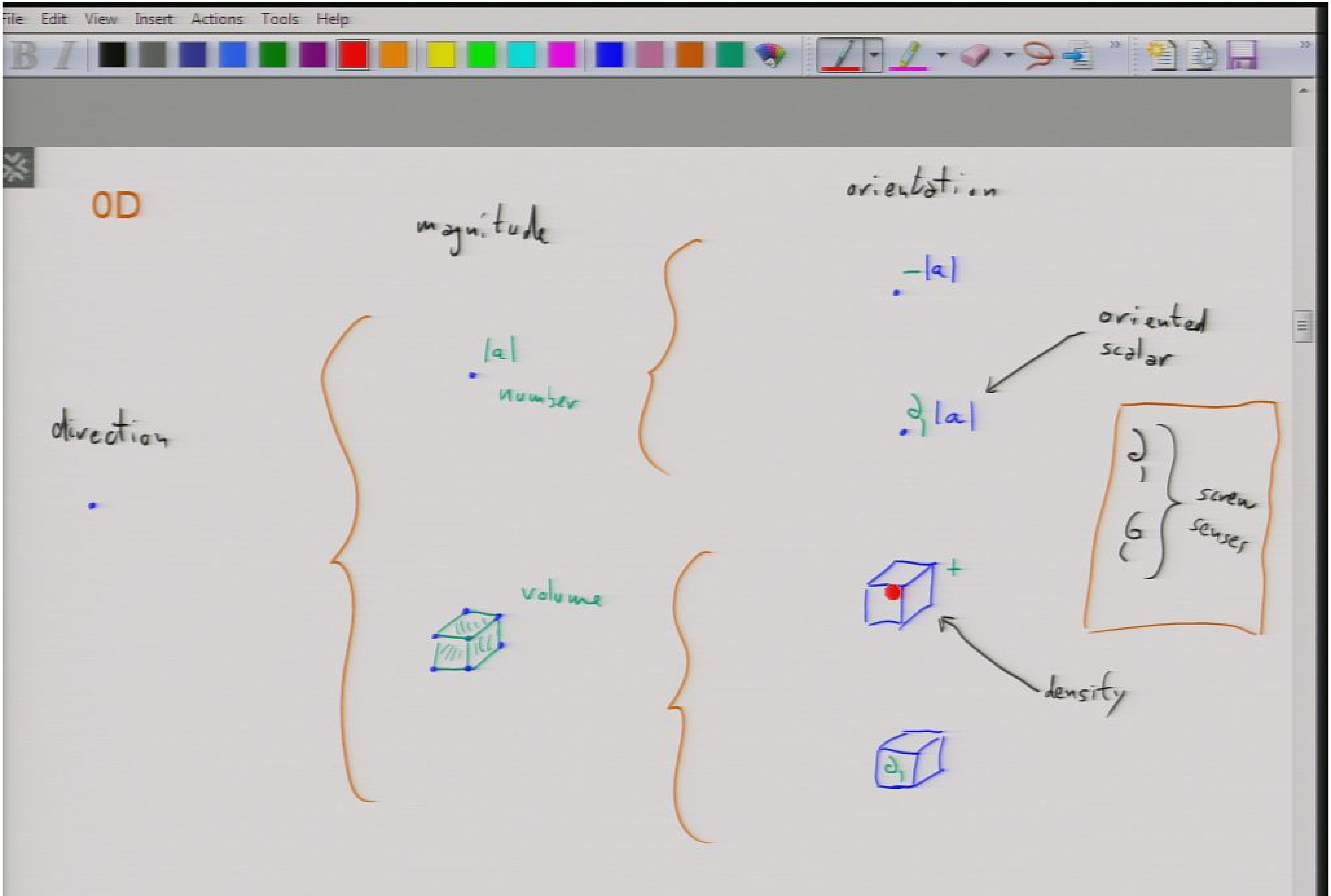
2D
direction
magnitude
orientation
area
'fluxes'
length
'forces'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0D
magnitude
orientation
direction
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oriented scalar
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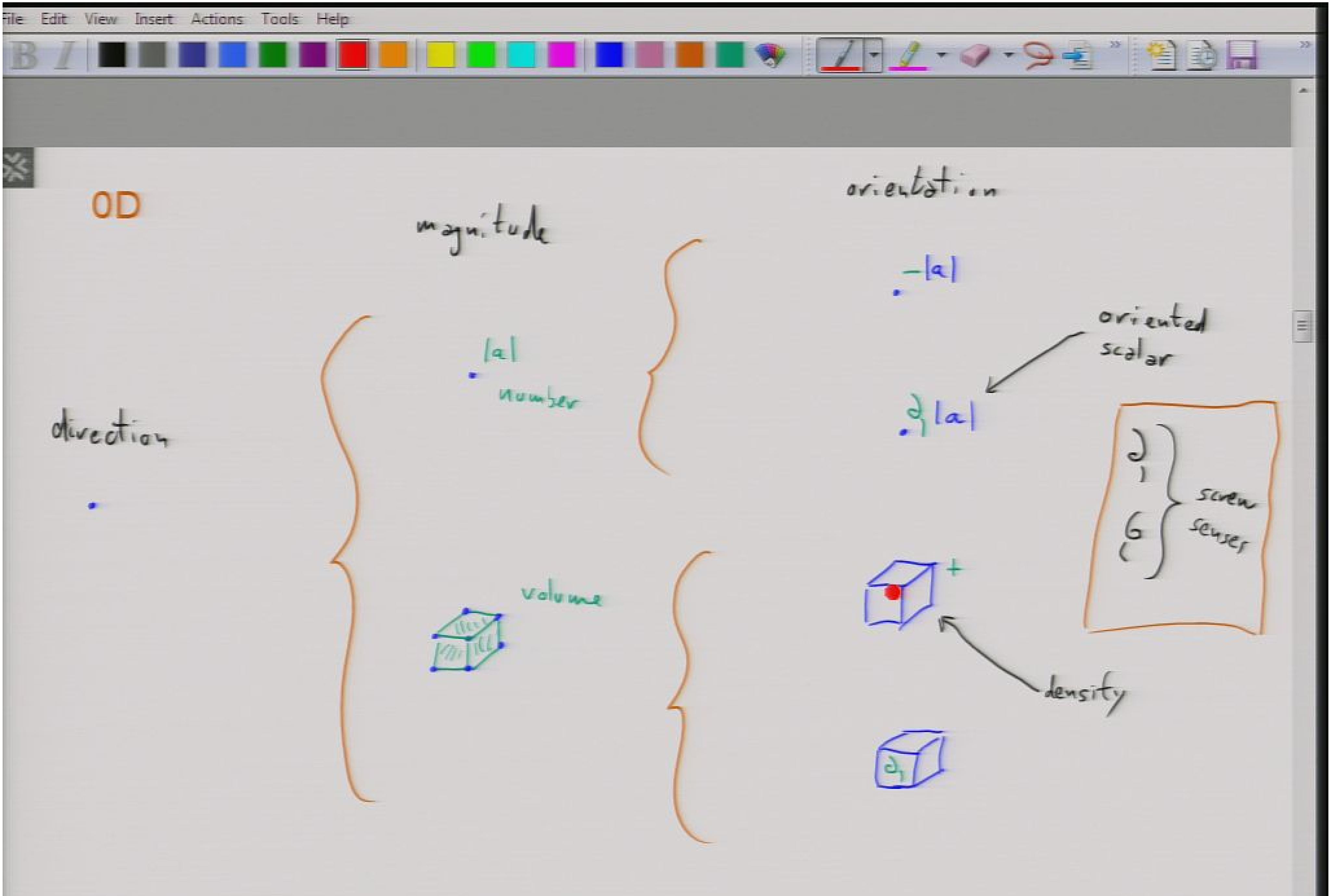
∂₁
6
screw senses
volume
+
density
∂₁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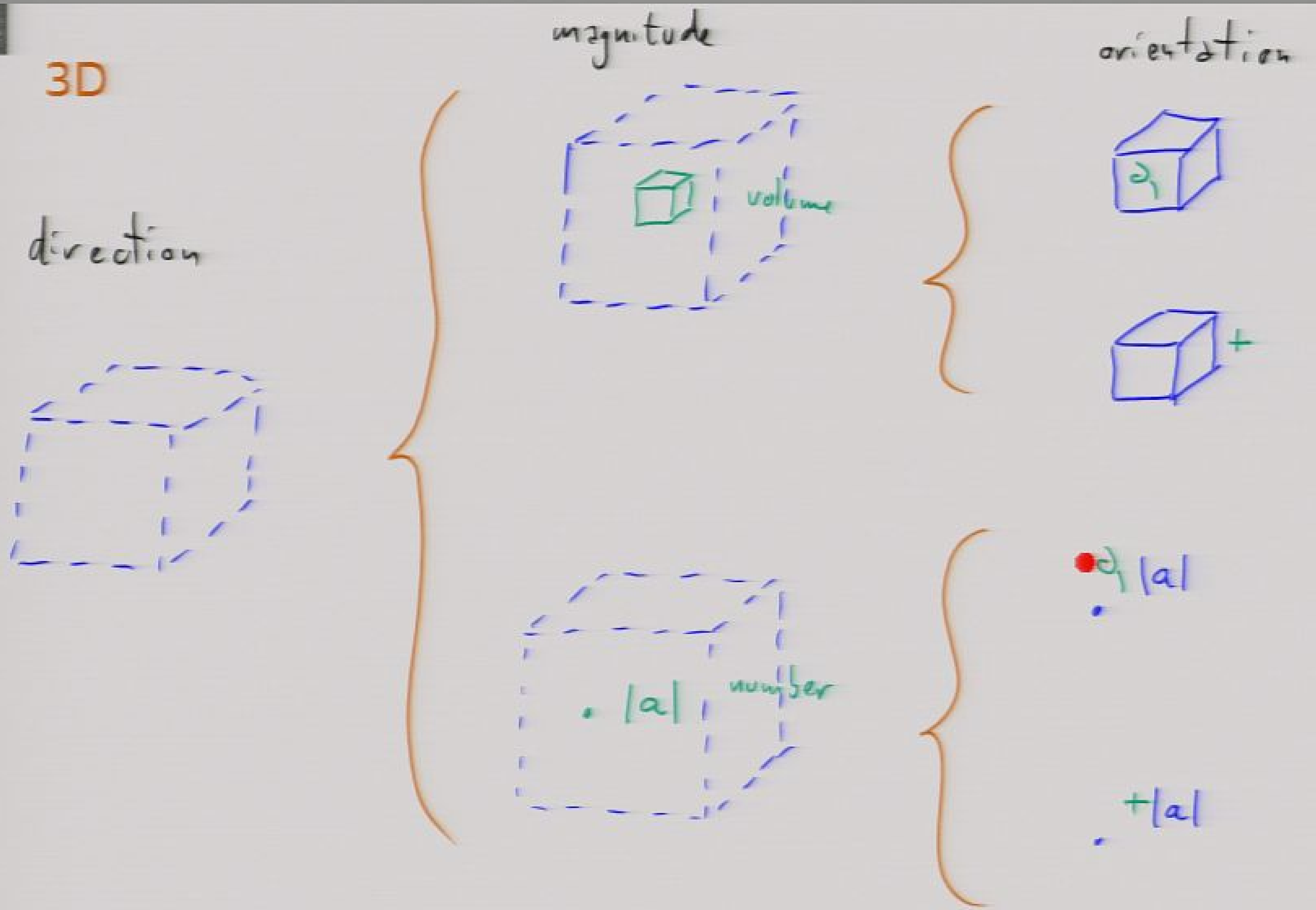

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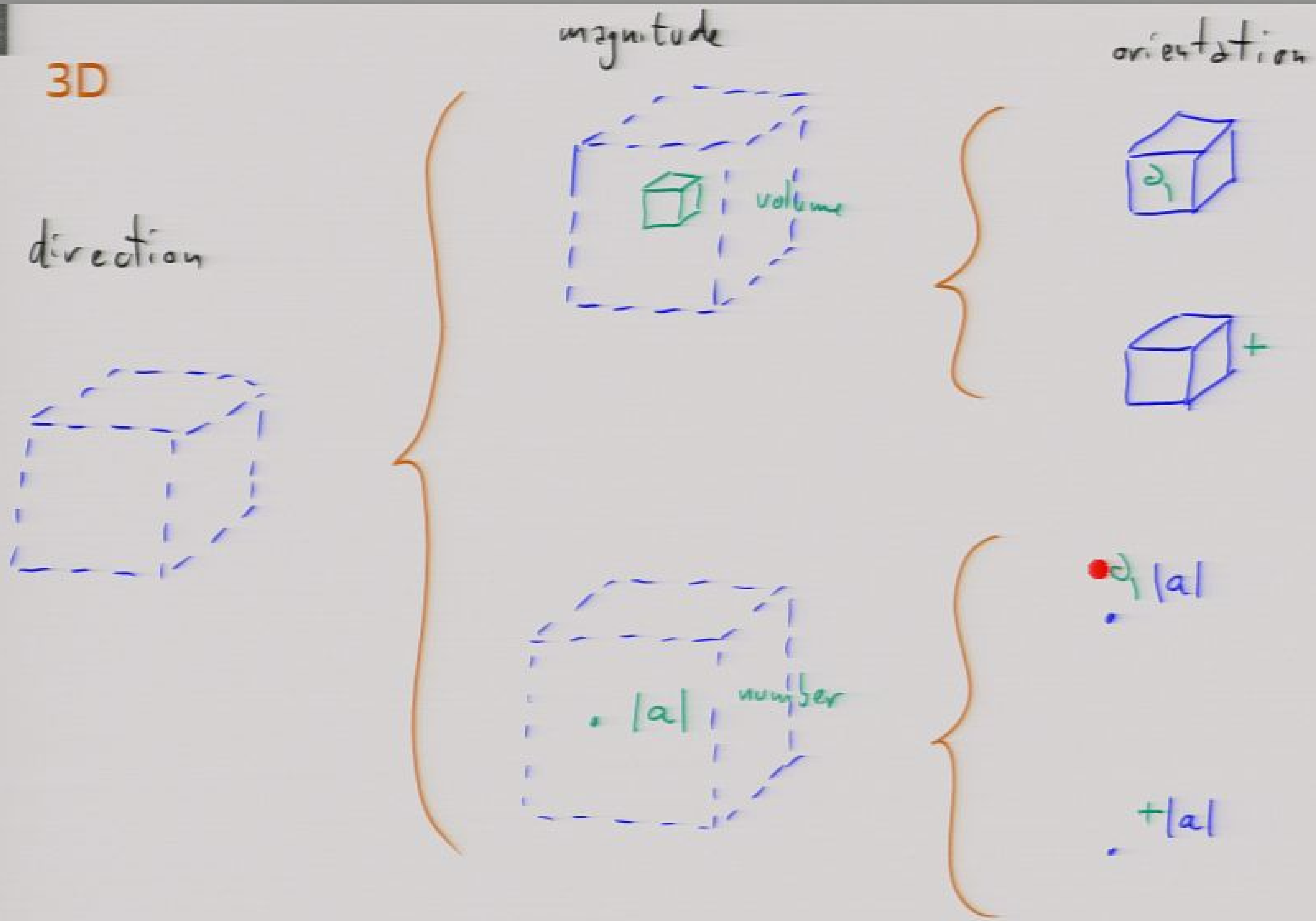
3D



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if magnitude is on:

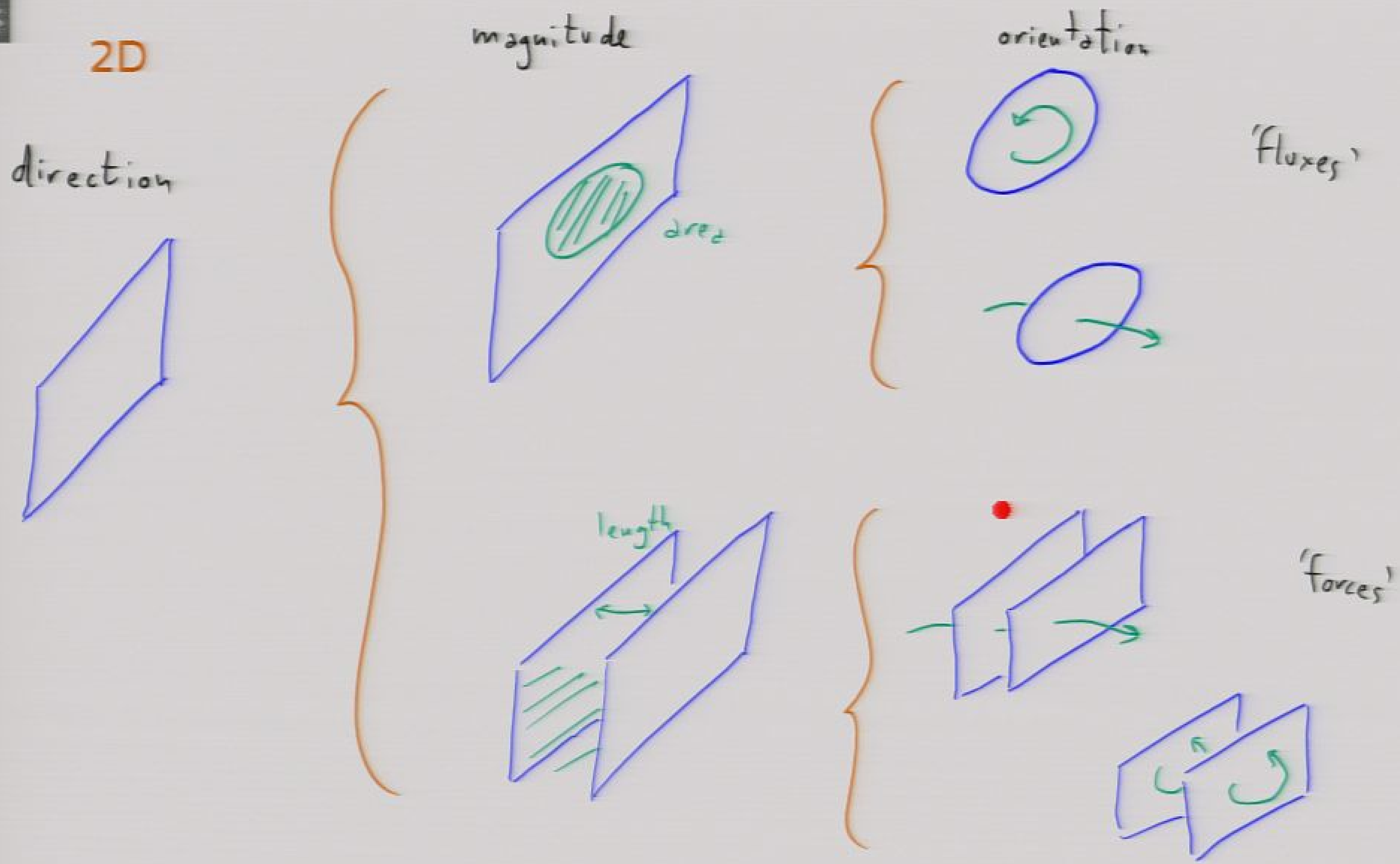
- direction = *vector*
- complement = *form (covector)*

dimension of magnitude:

1-vector, 2-vector (bivector), etc. 1-form, 2-form, etc.

if orientation is on:

- direction = *straight (even)*
- complement = *twisted (odd)*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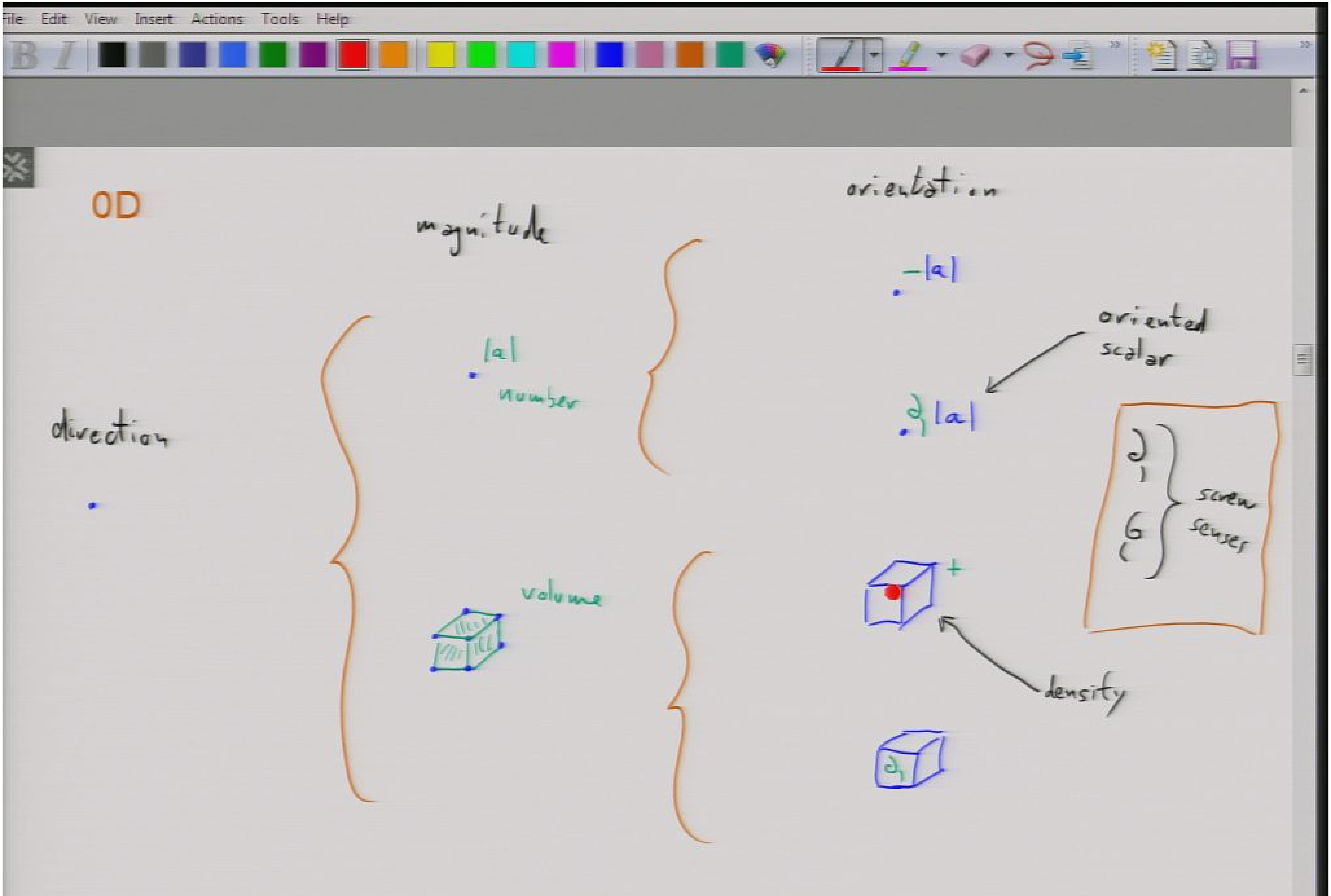
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|a|
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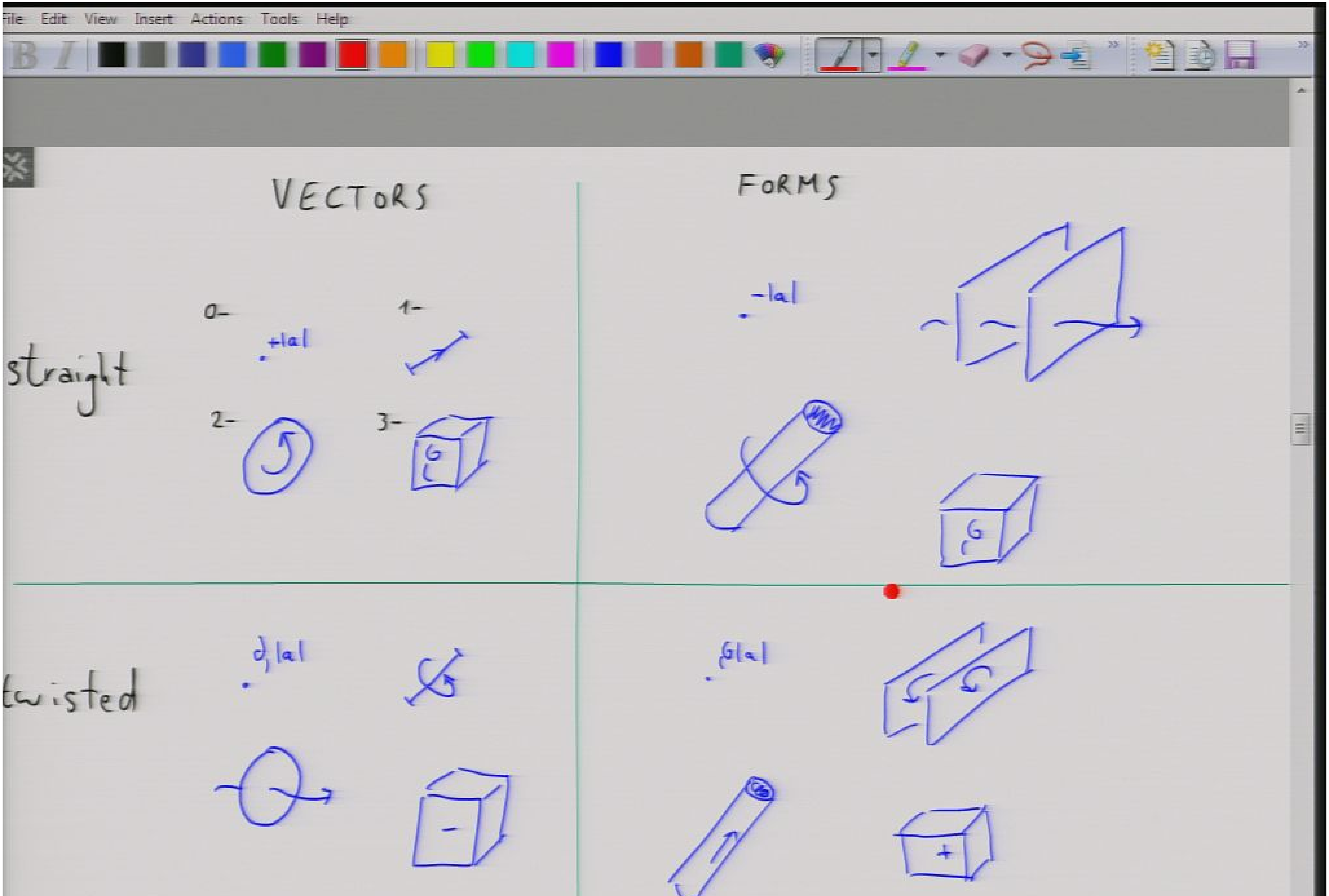
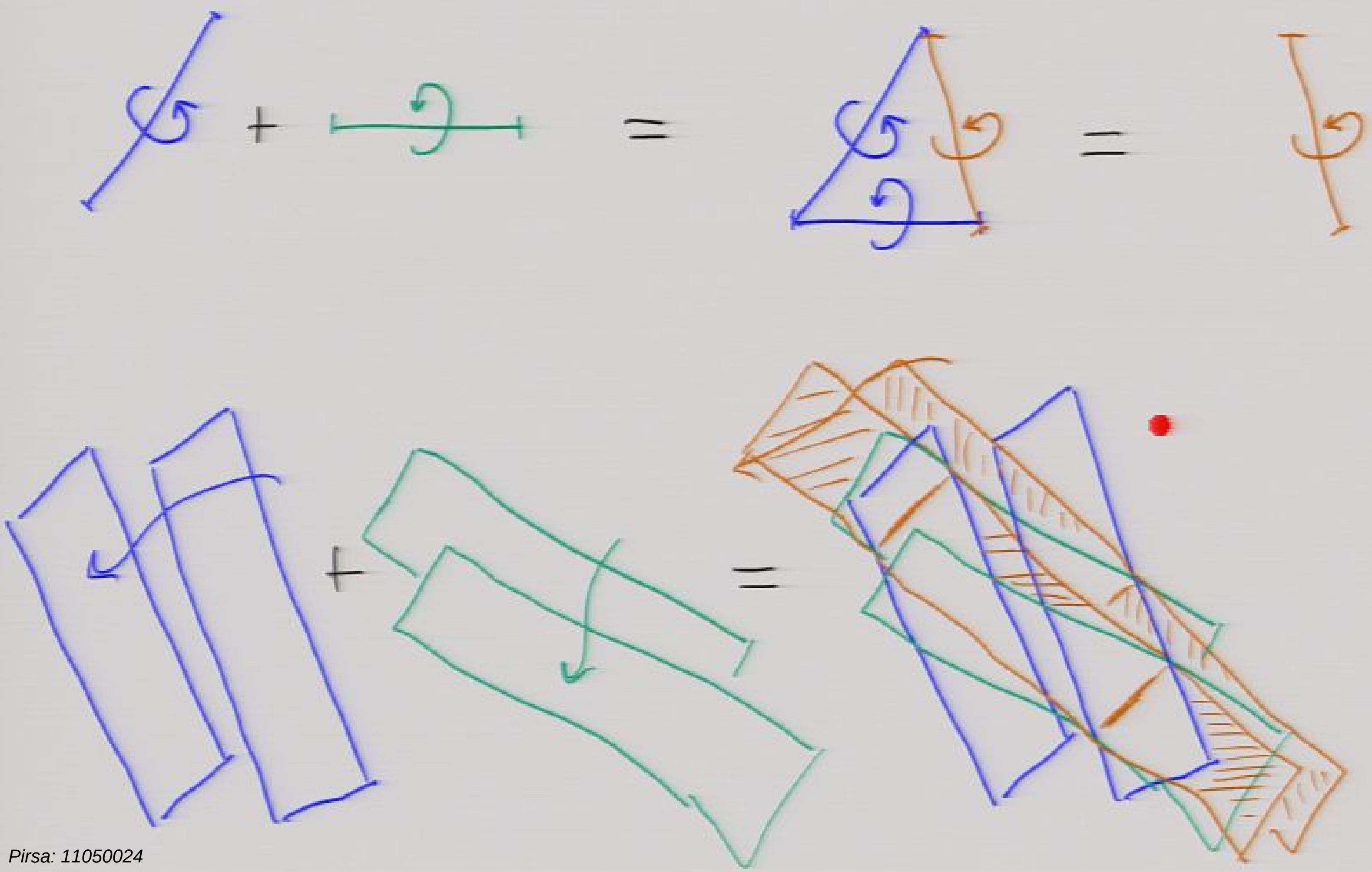
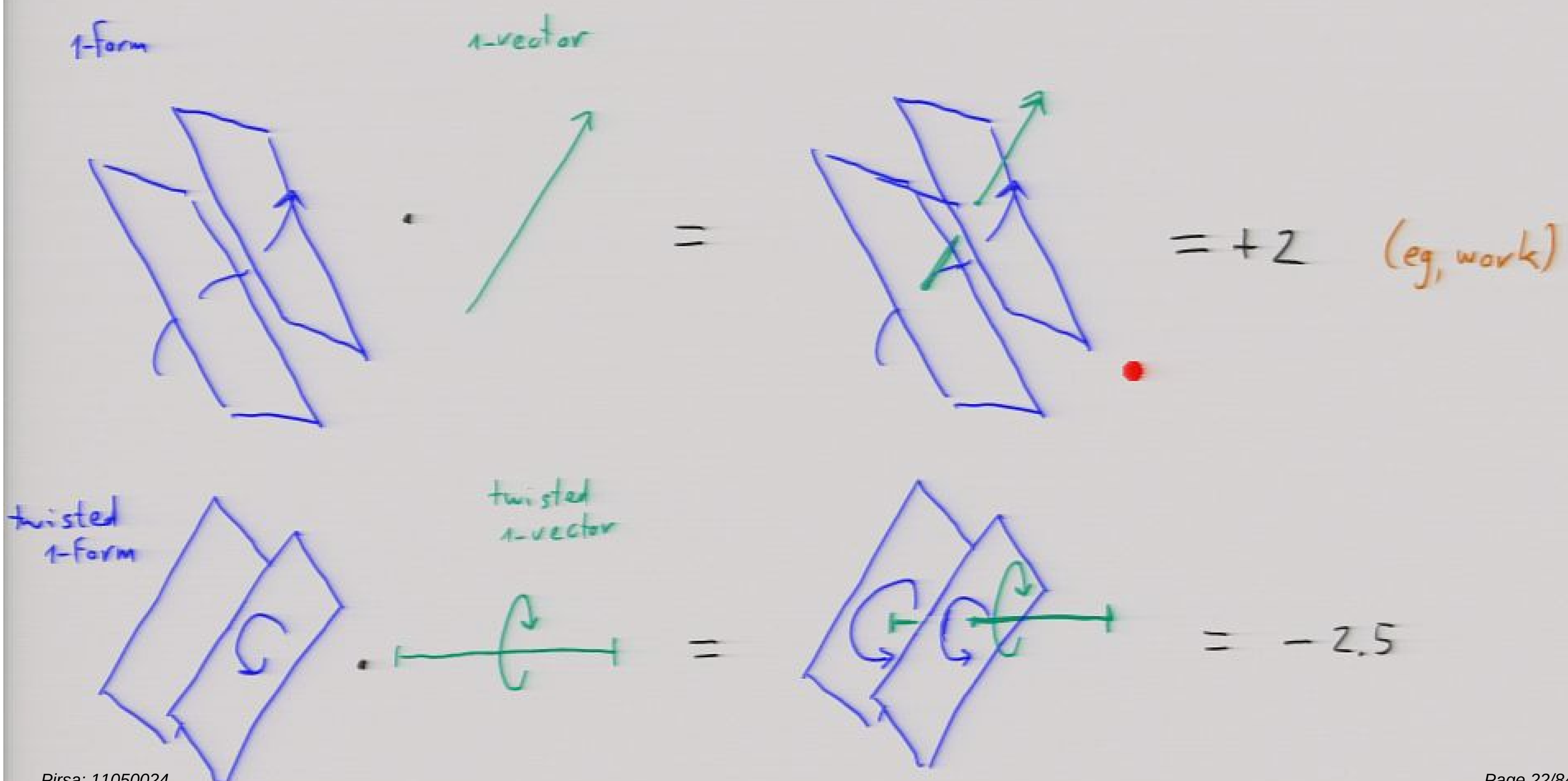

File Edit View Insert Actions Tools Help
VECTORS
FORMS
straight
twisted
0-
1-
2-
3-
+|a|
-|a|

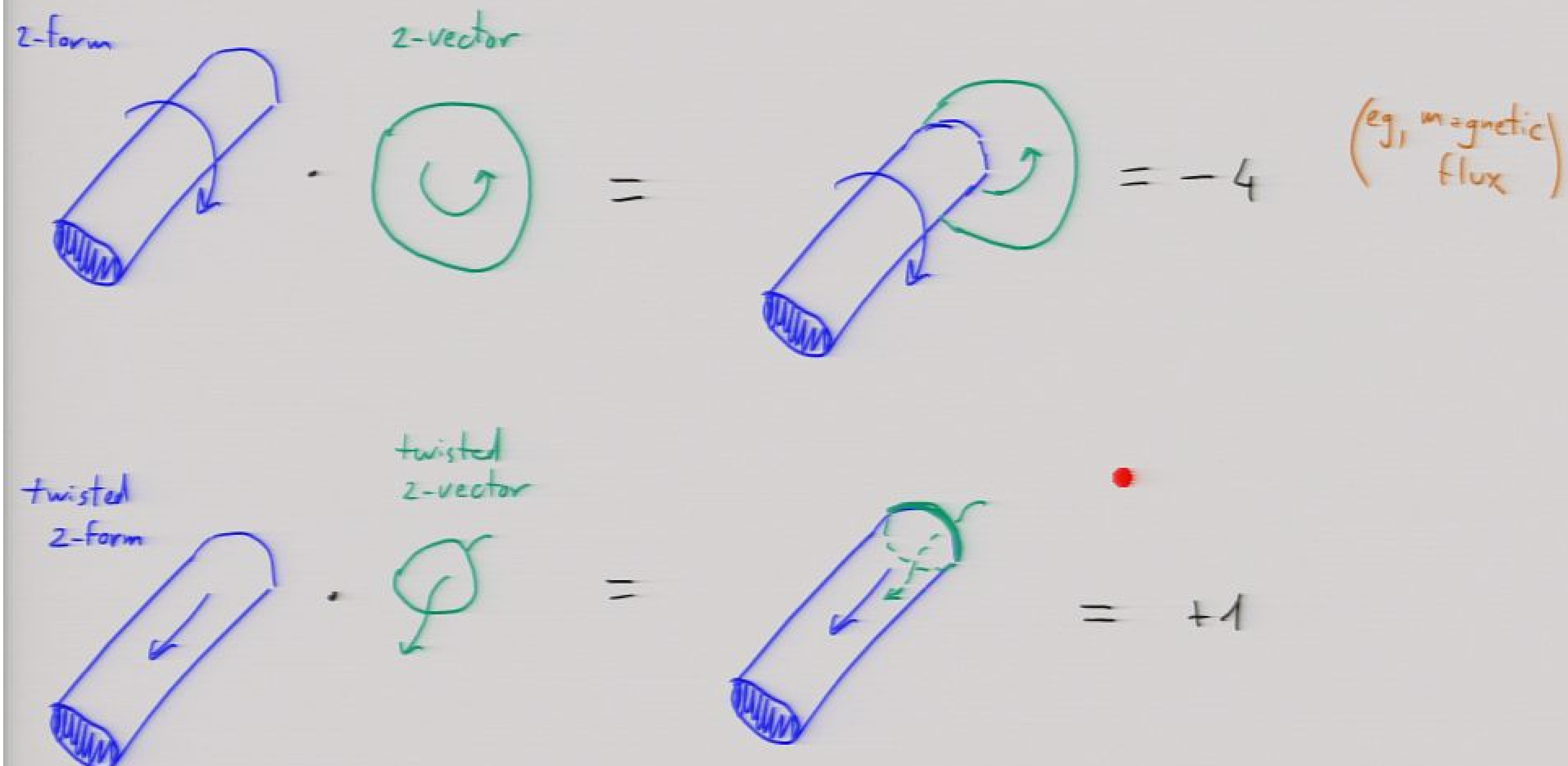
File Edit View Insert Actions Tools Help

These objects can be scaled and added



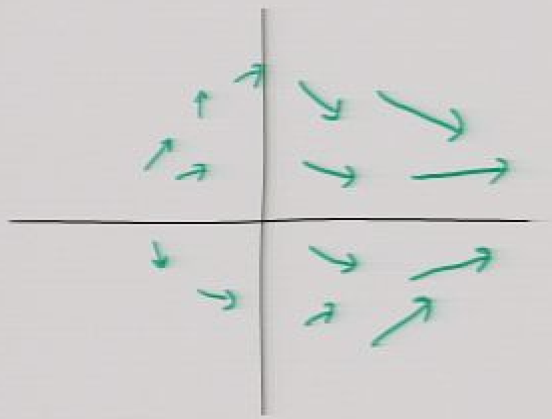
forms and vectors of same dimension and orientation act on each other to give scalars

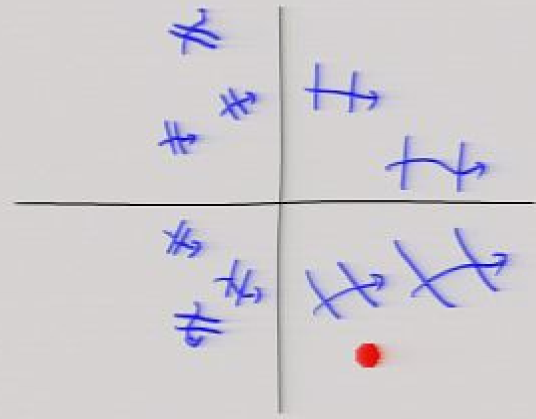



2-form
2-vector
= = −4
(eg, magnetic flux)
twisted 2-form
twisted 2-vector
= = +1

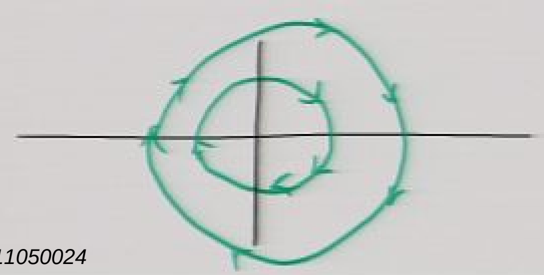
Vector fields and differential forms

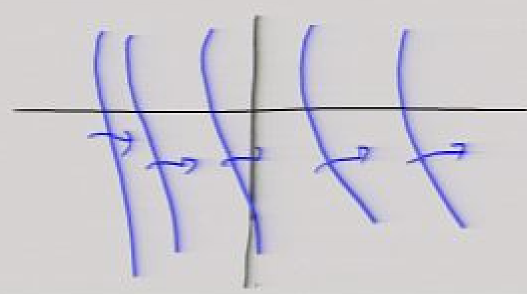
On a differential manifold we can associate a vector or form to each tangent space: vector field or differential-form field

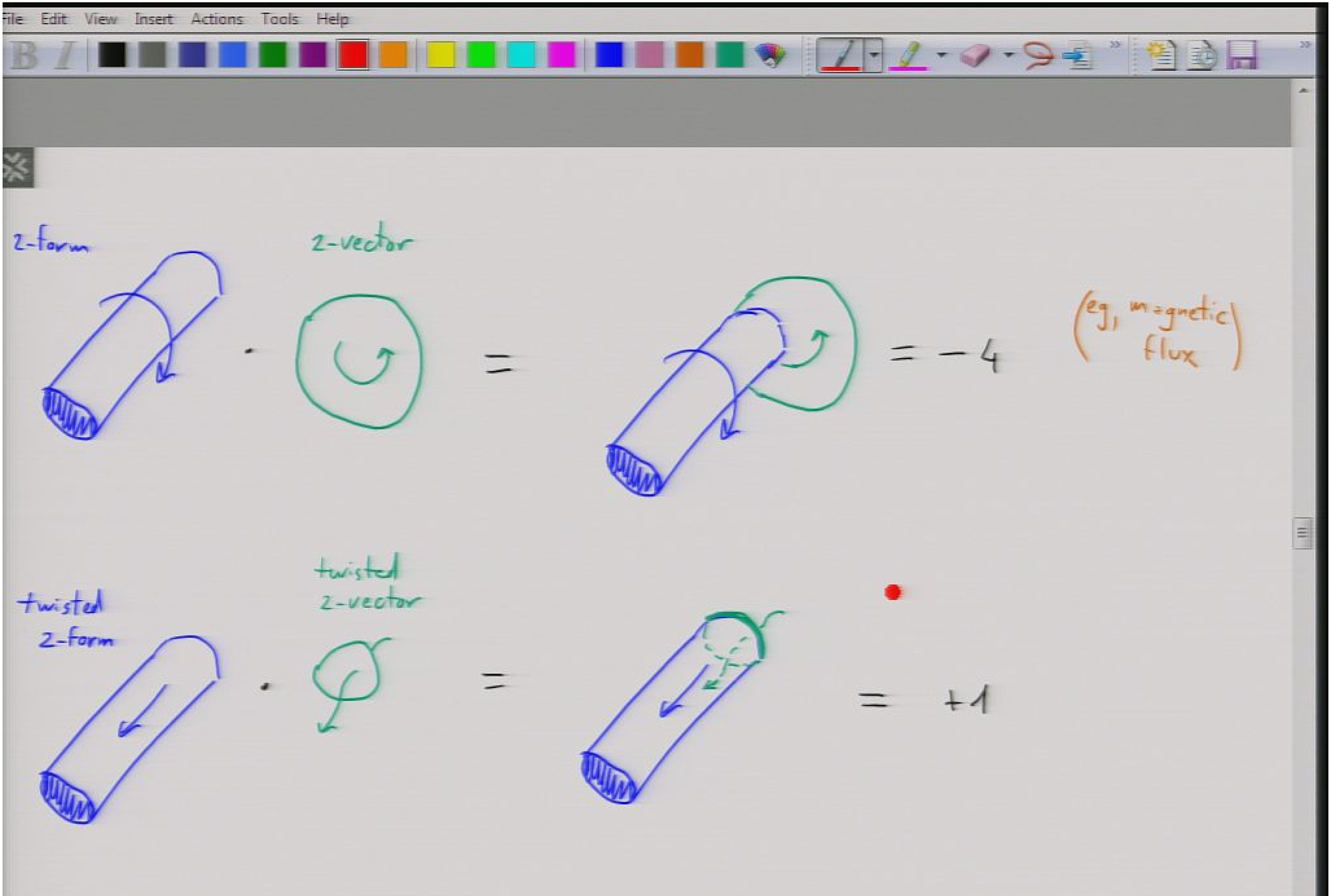




these fields may be tangent to a submanifold (integral manifold)

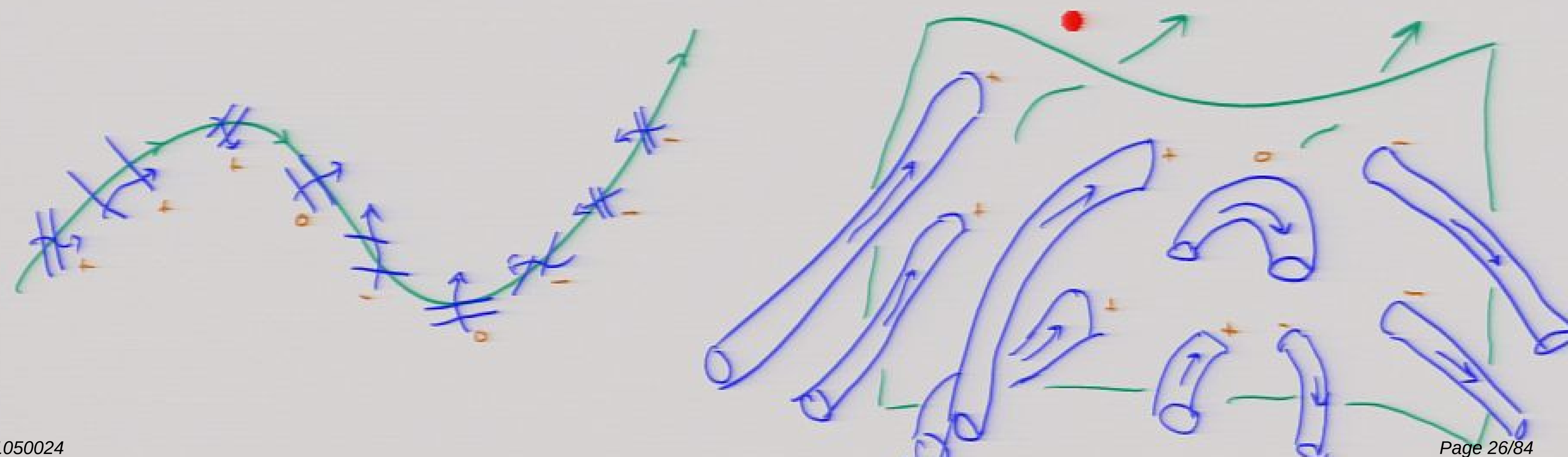





File Edit View Insert Actions Tools Help
2-form
2-vector
= −4
(eg, magnetic flux)
twisted 2-form
twisted 2-vector
= +1

Integration

- Choose an *m*-form field, choose an *m*-submanifold
- at each point, apply the *m*-form to the tangent *m*-vector
- sum up the scalars
- that's the integral of the form on the submanifold



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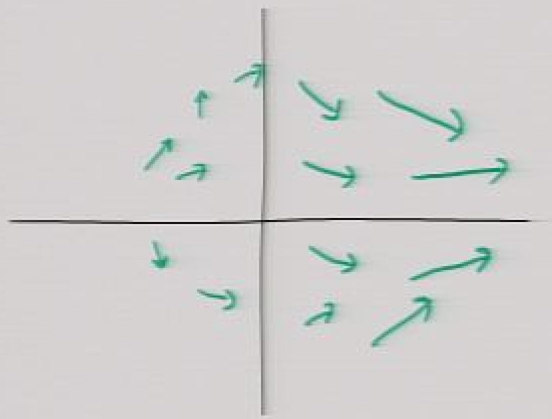

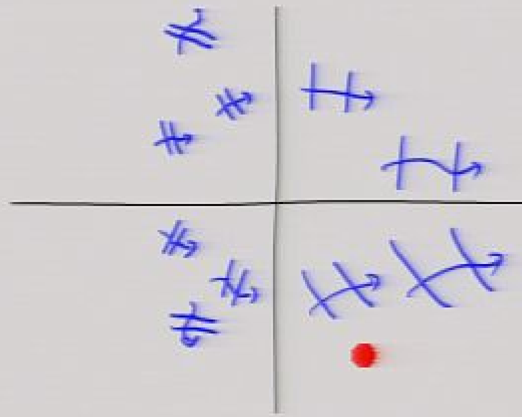

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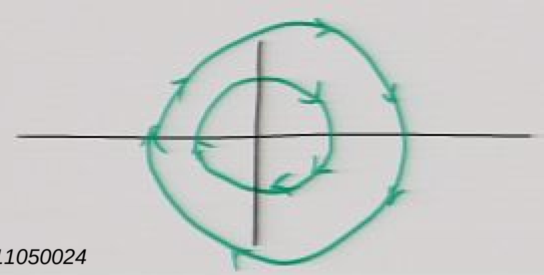

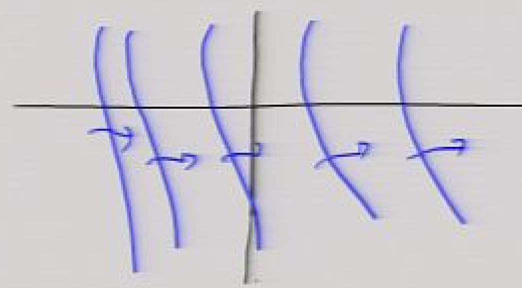

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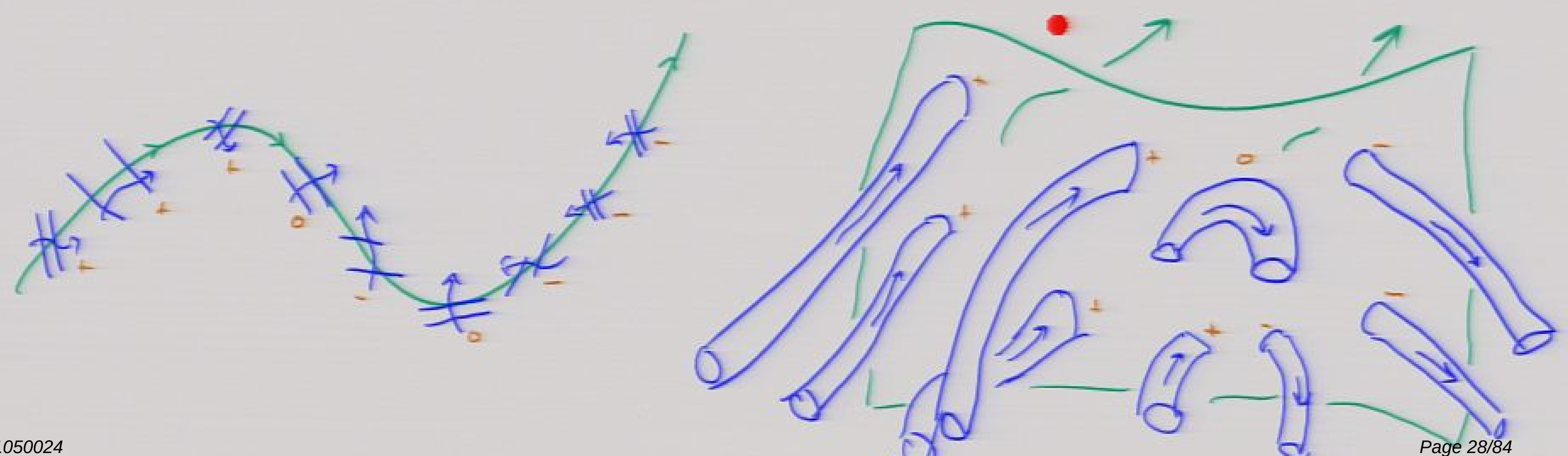

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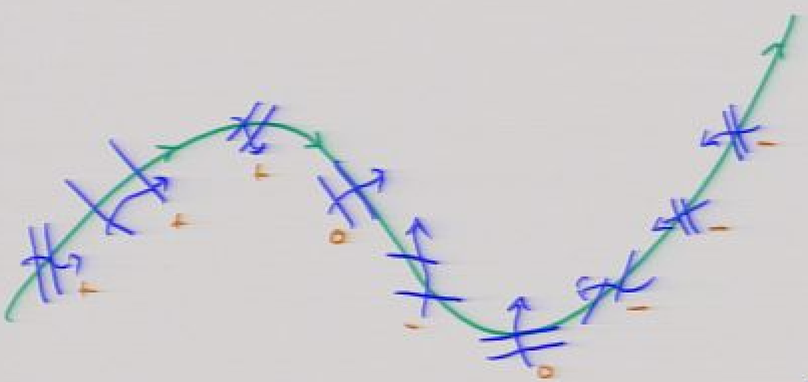

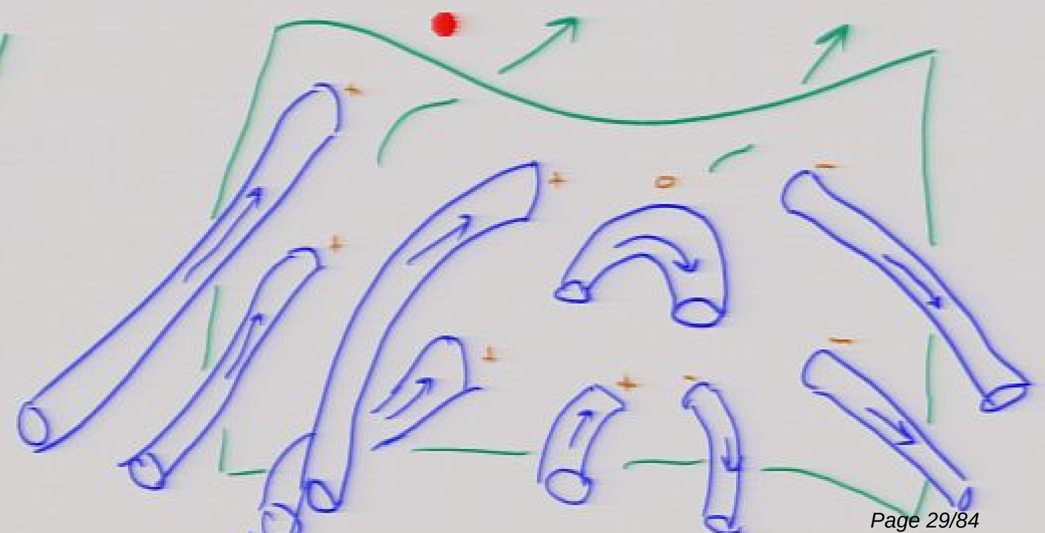

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Note: none of these concepts and operations involves a metric!

Only topology

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Algebraically:

vectors: $\sum a^i \partial_{x^i}$, $\sum a^{ij} \partial_{x^i} \wedge \partial_{x^j}$, ...

forms: $\sum b_i \, dx^i$, $\sum b_{ij} \, dx^i \wedge dx^j$, ...

twisted forms: $\sum c_i \, d\tilde{x}^i$

$\sim$ = 'twisting operator' introduced by Burke (J. Math. Phys, 1983)

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Differential forms and physical intuition

Our representation of matter: something that

- is associated to a (small) bounded 3D region of space, eg a ball

- can be marked and followed around

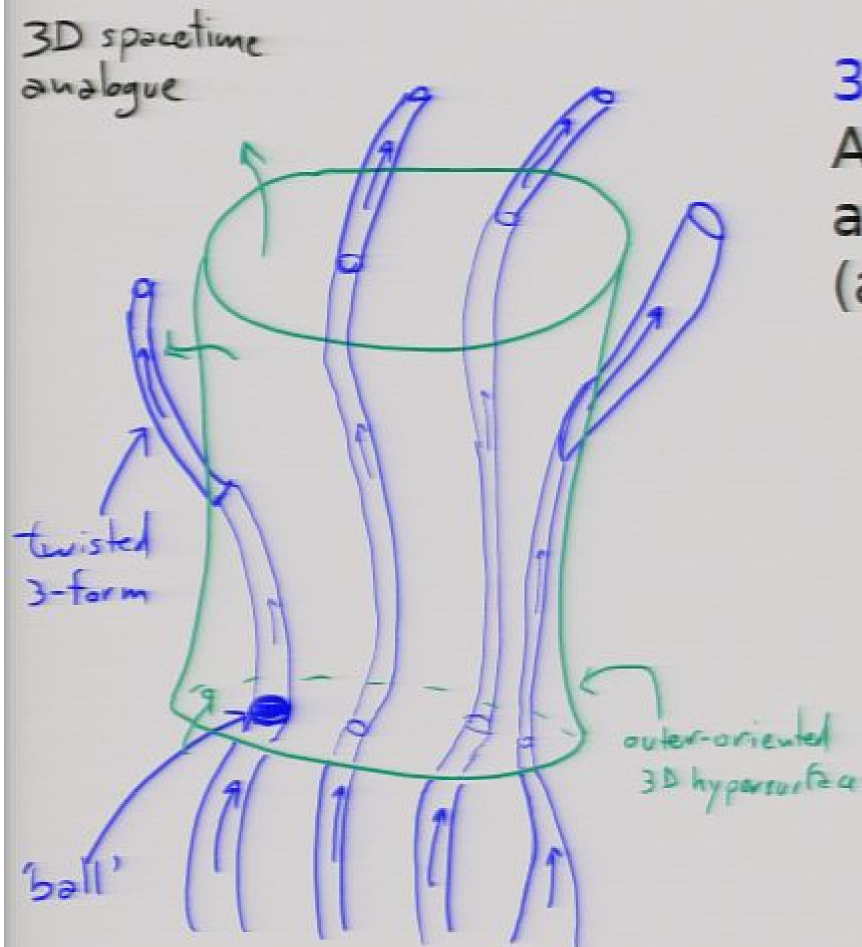
- doesn't disappear

- We can count how many 'balls' are there inside a 3D region

- we can count how many 'balls' cross some 2D surface (in a given time)

'The ball that was then here, is now there...'

Matter/mass (and charge) is a twisted 3-form in spacetime



3-form in spacetime = 'hypertubes'
A 3D observer sees them
as 'balls' moving around in 3D space
(and changing shape)

	balls initially in 3D region
+	balls entering 2D surface
-	balls exiting 2D surface
-	balls finally in 3D region
=	0

This property =
the twisted 3-form is closed
(its tubes never end)

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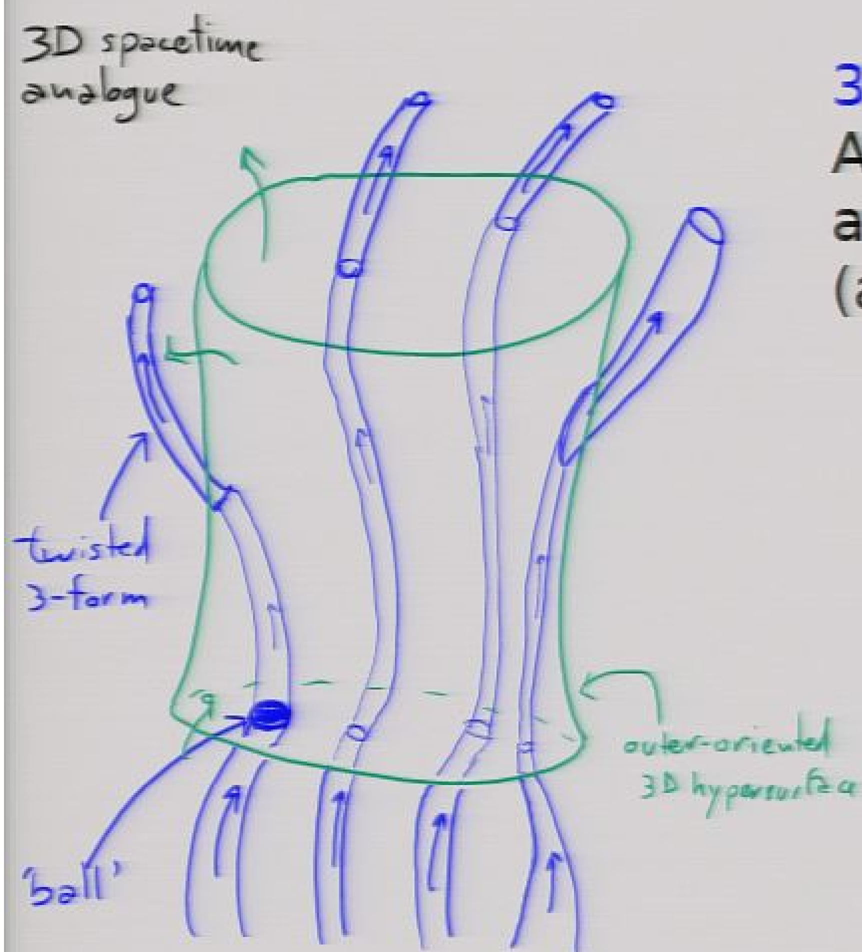

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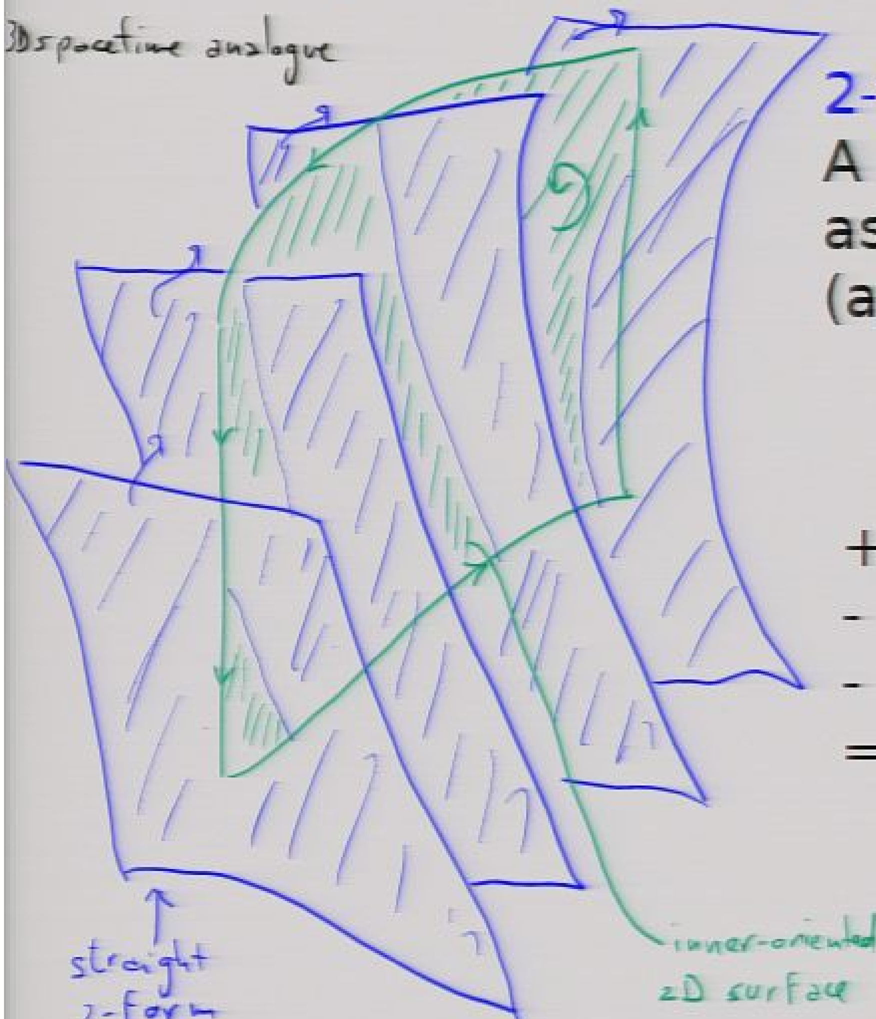

2-form in spacetime = 'hypersurfaces'
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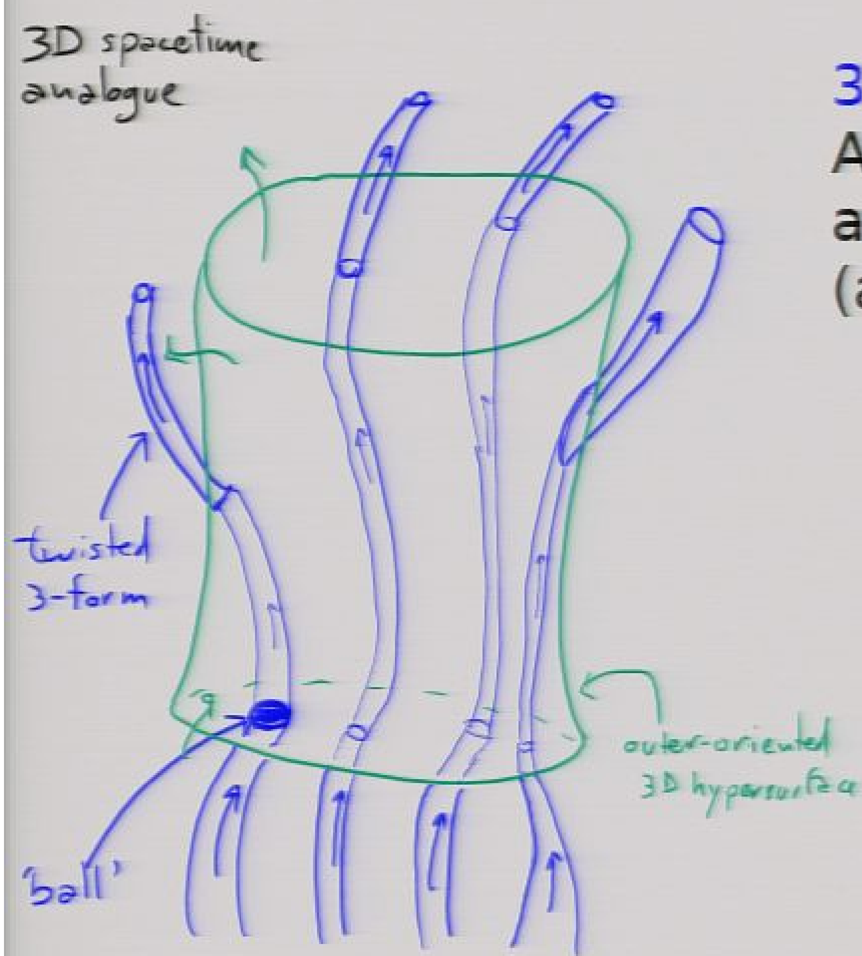

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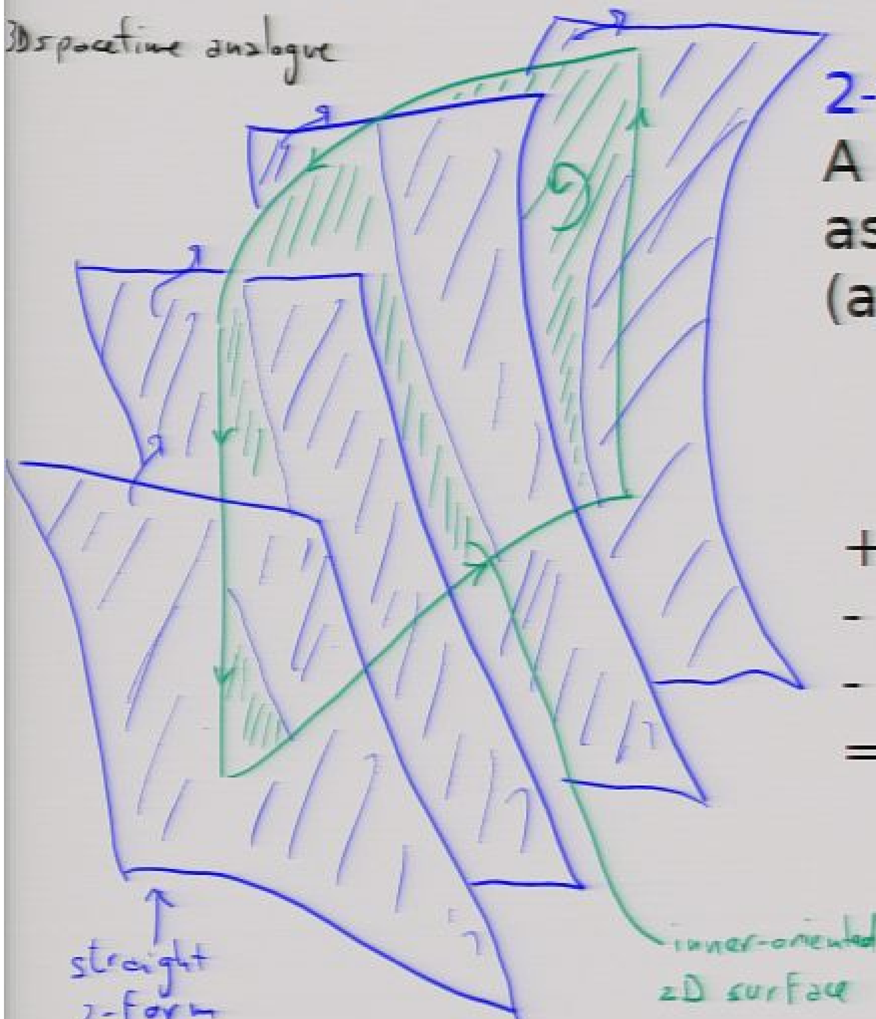

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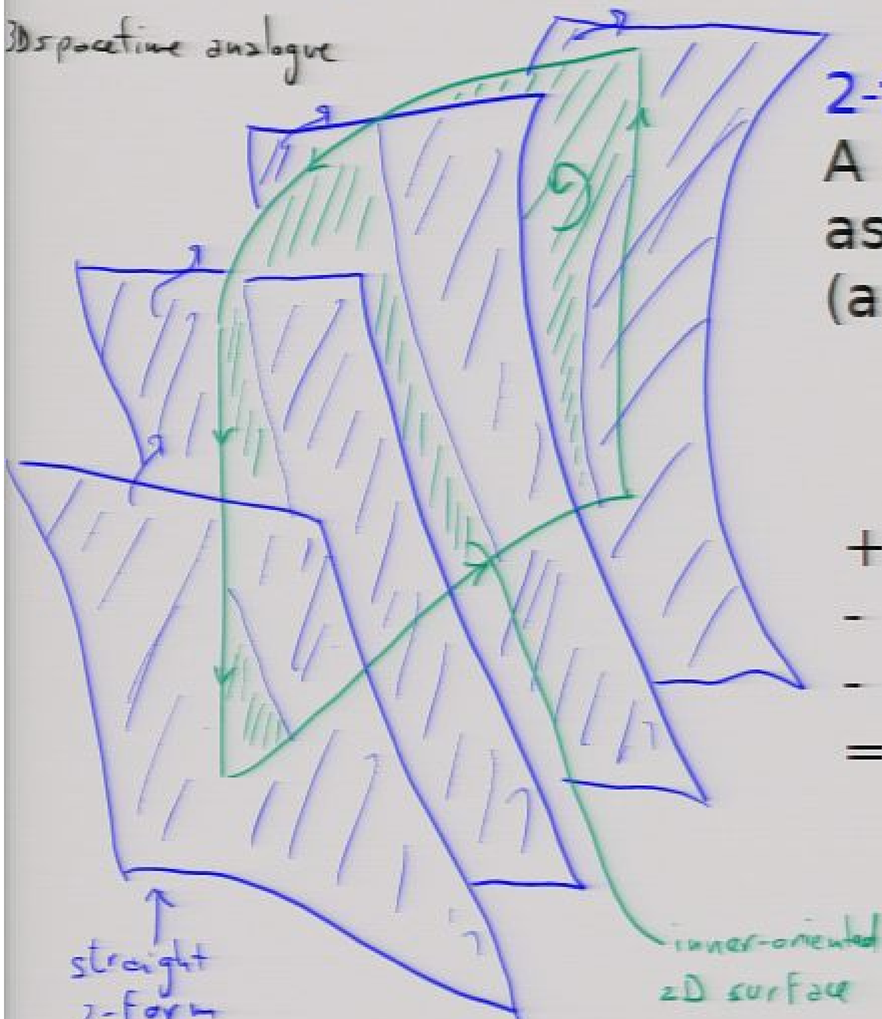
...araday tensor) is a straight 2-form in spacetime



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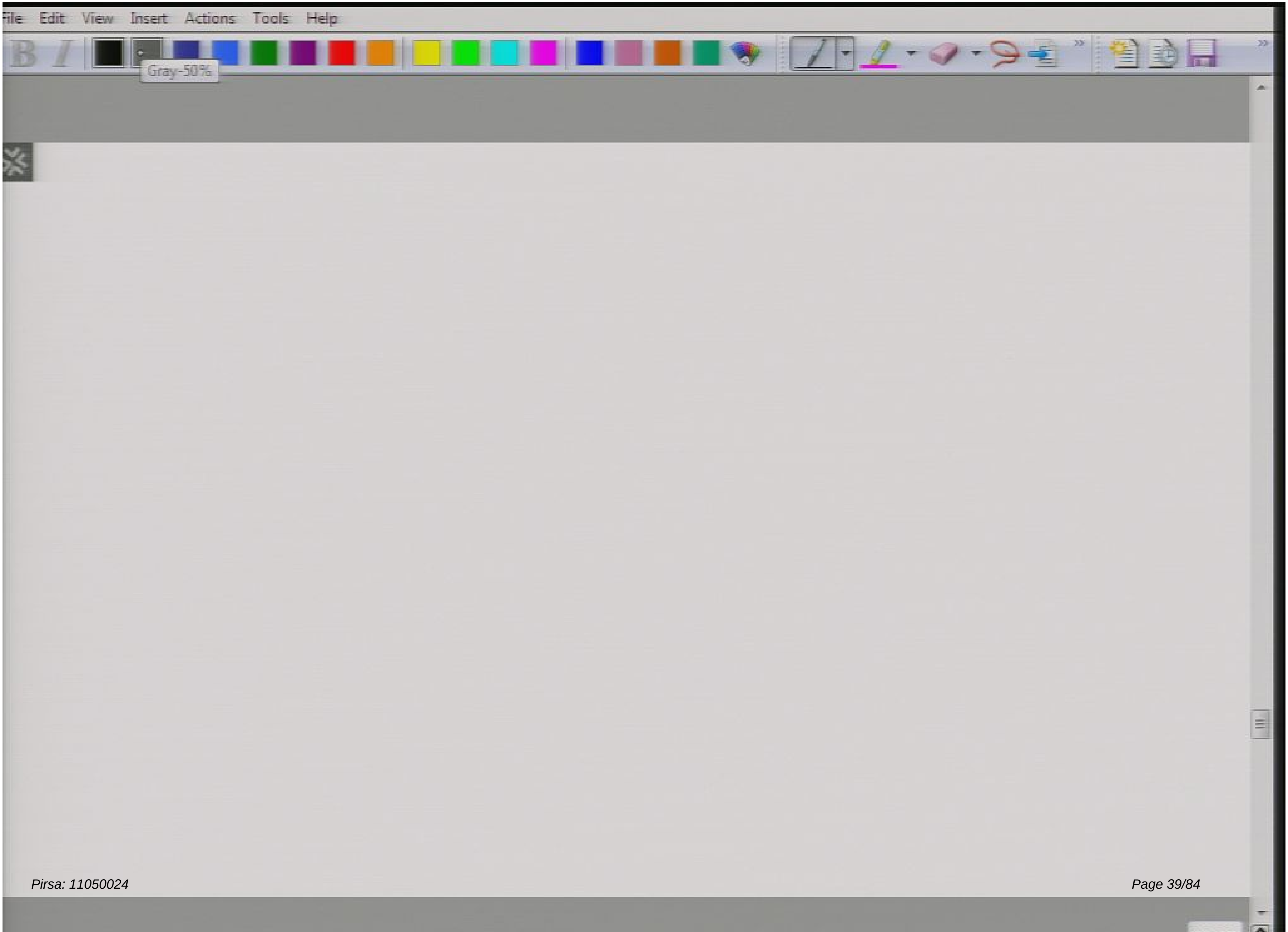
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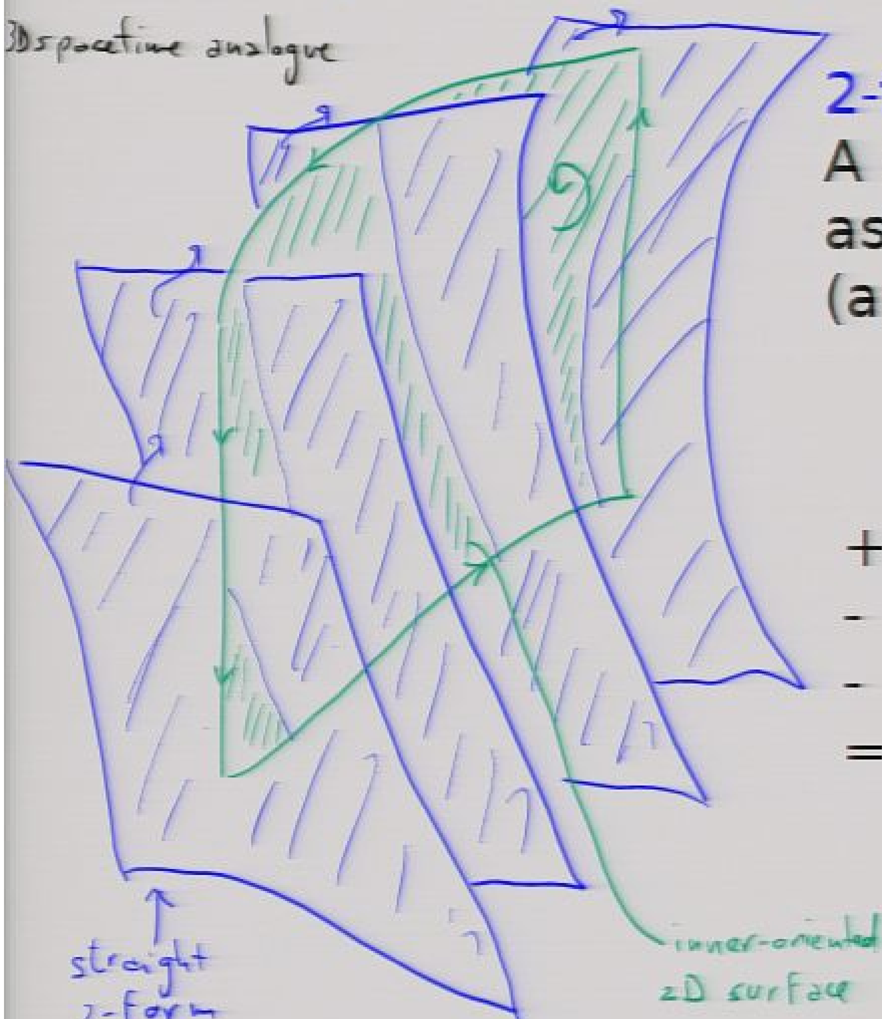

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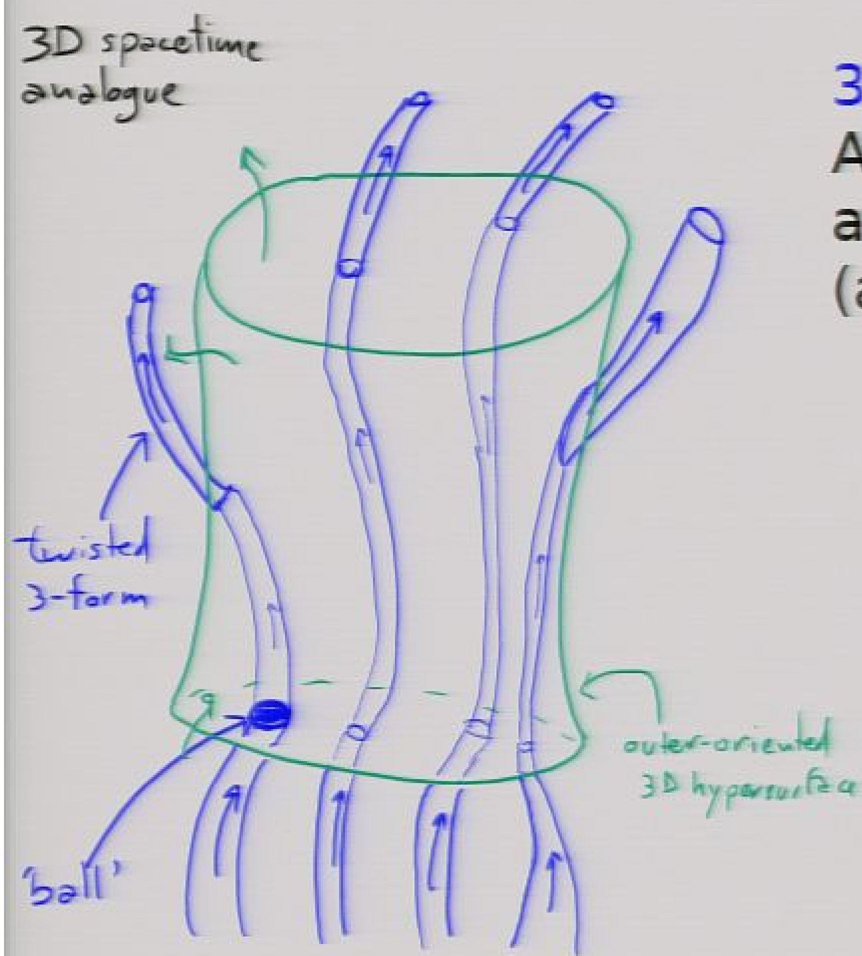

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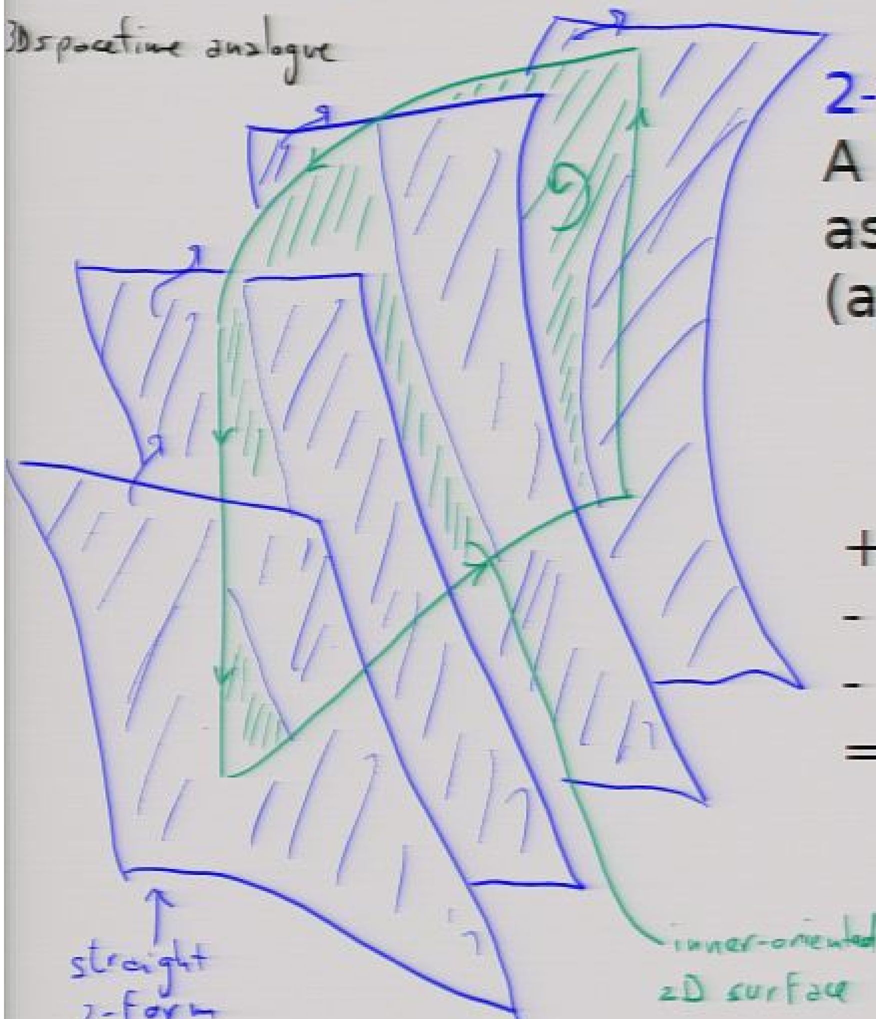

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$dF = 0$

File Edit View Insert Actions Tools Help

- We can count how many 'tubes' intersect a 2D surface

- we can count how many 'tubes' cross a 1D egde

The EM field can be thought of as something that

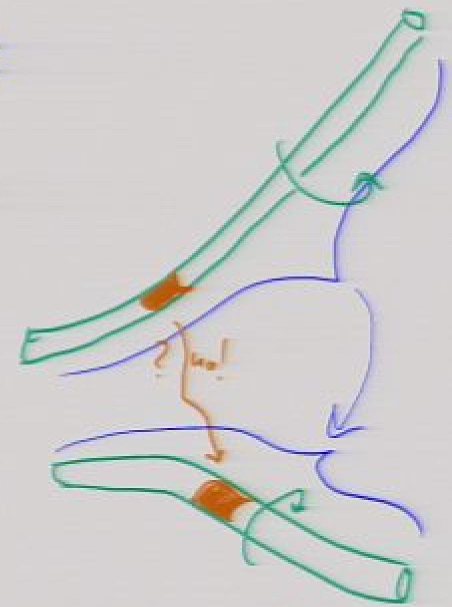
- is associated to a (thin) tubular region of space

- can be marked and followed around

- doesn't disappear

'The tube that was then here, is now there...'

- The whole tube can be identified, but not parts of it!



OK, fine, this may have some pedagogical use and may help visualization...

...but then?

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- Can we generalize the concept of force,
  from something applied to points
  to something applied to lines?

This idea is not so absurd:

- Ericksen (*J. Elasticity*, 2007):
'...the idea that electromagnetic fields can sustain forces'.

- Page (*Introduction to Theoretical Physics*, 1928, Ch. XII)

Force not as a form-valued 2-form,
but as a form-valued 3-form...

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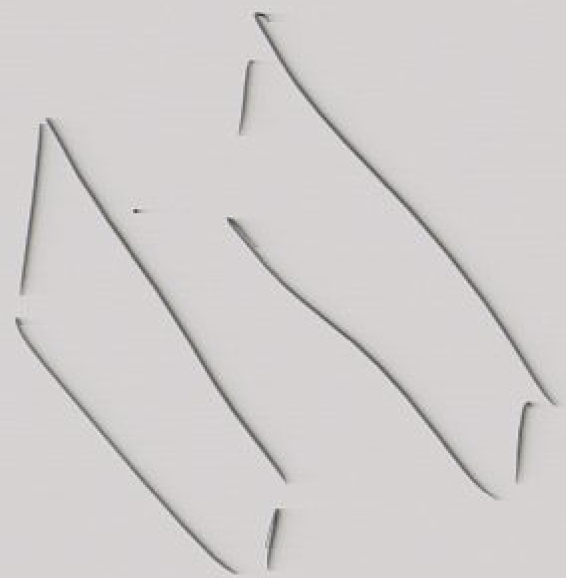

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References

- de Rham
- Schouten
- Truesdell & Toupin
- Burke
- Bossavit
- Misner, Thorne, Wheeler
- Page
- Ericksen

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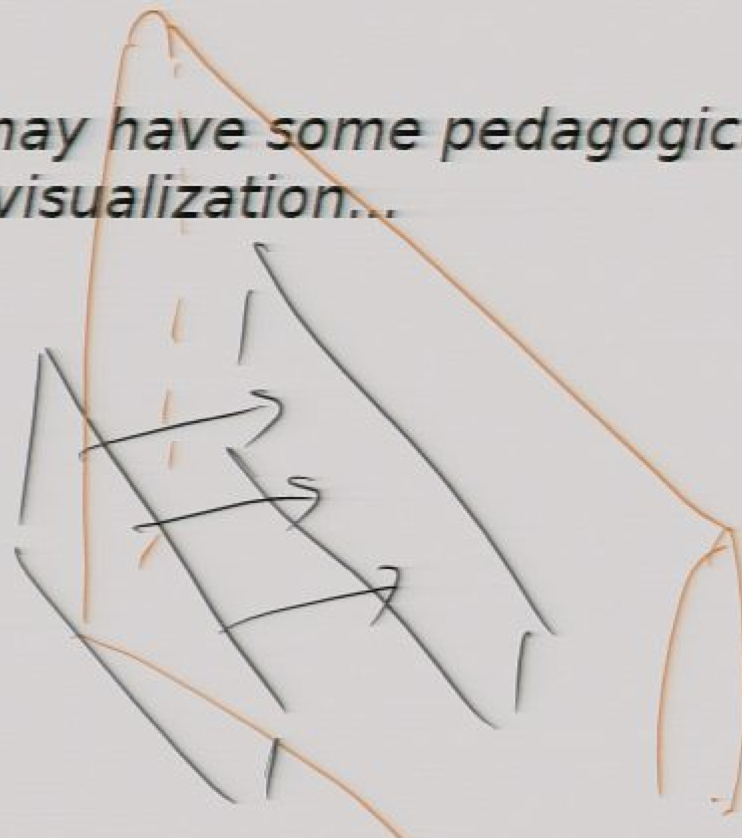

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'...the idea that electromagnetic fields can sustain forces'.

- Page (*Introduction to Theoretical Physics*, 1928, Ch. XII)

Force not as a form-valued 2-form,
but as a form-valued 3-form...

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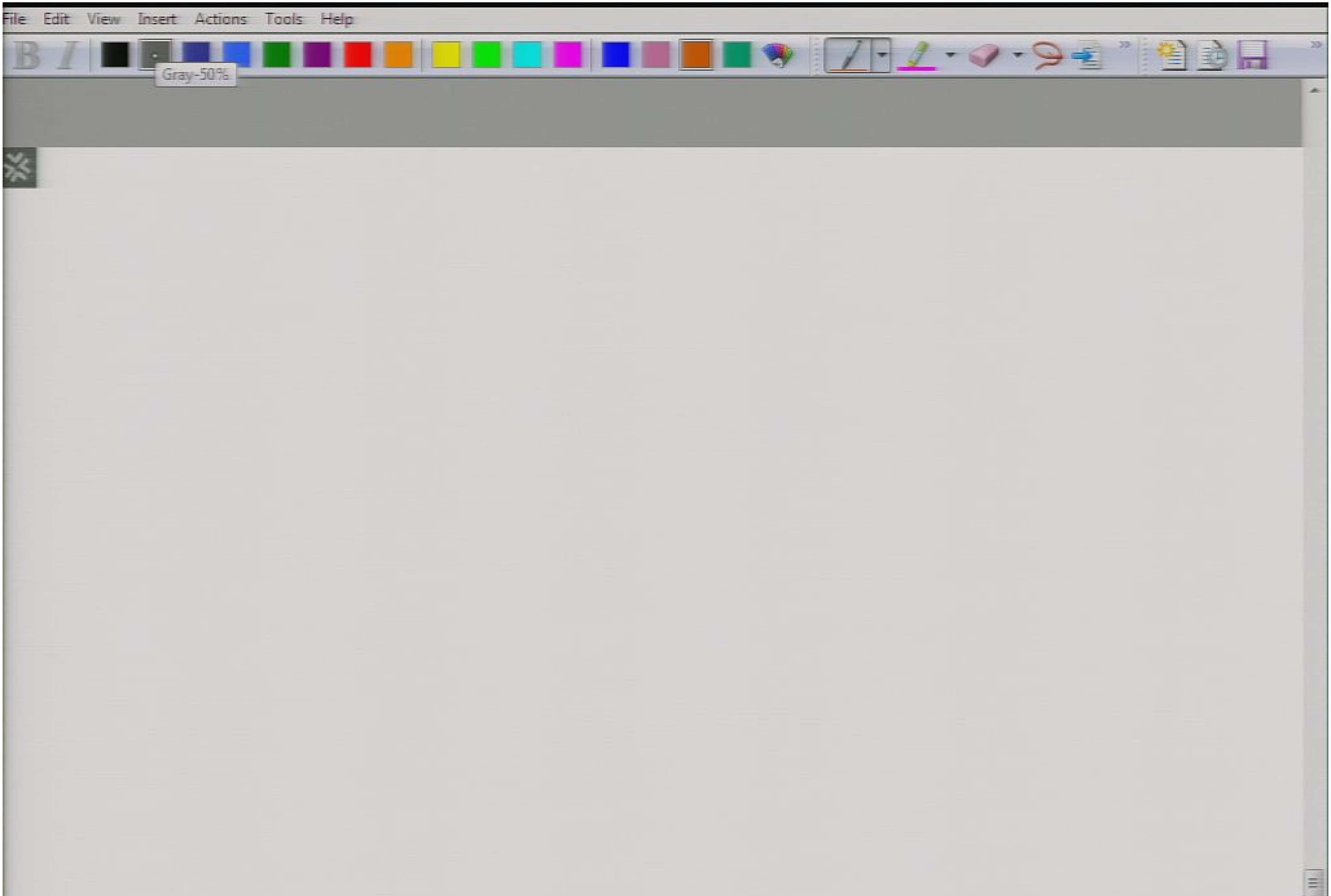
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Kotler 1920

v_2

File Edit View Insert Actions Tools Help

Kottler 1920

van Dantzig

Truesdell

File Edit View Insert Actions Tools Help

Kottler 1920

van Dantzig

Truesdell & Toupin

Hehl

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File Edit View Insert Actions Tools Help

Kottler 1920

van Dantzig

Truesdell & Toupin

Hehl

Bateman

Q .

File Edit View Insert Actions Tools Help

Kottler 1920

van Dantzig

Truesdell & Toupin

Hehl

Bateman

$d\Theta$

Q tw. 3-Form

charge-current

File Edit View Insert Actions Tools Help

Kottler 1920

van Dantzig

Truesdell & Toupin

Hehl

Bateman

$dQ = 0$ ← cons. of charge

Q tw. 3-Form

charge-current

$dQ = 0$.

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Kottler 1920

van Dantzig

Truesdell & Toupin

Hehl

Bateman

$dQ = 0 \leftarrow$ cons. of charge

Q tw. 3-Form

charge-current

$dQ = 0 \Rightarrow \exists M$ 2-form

$Q = dM$

$(\rho, \cdot$

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Kottler 1920

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F 2-form

$dQ = 0 \leftarrow$ cons. of charge

Q tw. 3-Form

charge-current

$dQ = 0 \Rightarrow \exists M$ 2-form

$$\underset{(\rho, J)}{Q} = d\underset{(D, H)}{M}$$

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Kottler 1920

van Dantzig

Truesdell & Toupin

Hehl

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$dF = 0$?⟷ $dQ = 0$ ← cons. of charge

F 2-form ← (E, B)

Q tw. 3-Form

charge-current

$F = dA$

$dQ = 0 \Rightarrow \exists M$ 2-form

$Q = dM$

(ρ, J) → Q

(D, H) → M

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Kottler 1920

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$dF = 0$ $\xleftrightarrow{?}$ $dQ = 0$ ← cons. of charge

F 2-form (E, B)

Q tw. 3-Form

charge-current

$dQ = 0 \Rightarrow \exists M$ 2-form

$F = dA$

vector potential

$Q = dM$

(ρ, J)

(D, H)

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Kottler 1920

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Hehl

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$dF = 0$

strength

F 2-form

(E, B)

?

$dQ = 0 \leftarrow$ cons. of charge

Q tw. 3-Form

charge-current

$dQ = 0 \Rightarrow \exists M$ twisted 2-form

$F = dA$

vector potential

$Q = dM$

(ρ, J)

(D, H)

$\lambda(F) -$

Kottler 1920

van Dantzig

Truesdell & Toupin

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$dF = 0$

F straight 2-form

(E, B)

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Q tw. 3-Form

charge-current

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(ρ, J)

(D, H)

$\lambda(F) = M$

$\ast \rightarrow$ conformal structure

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(ρ, J)

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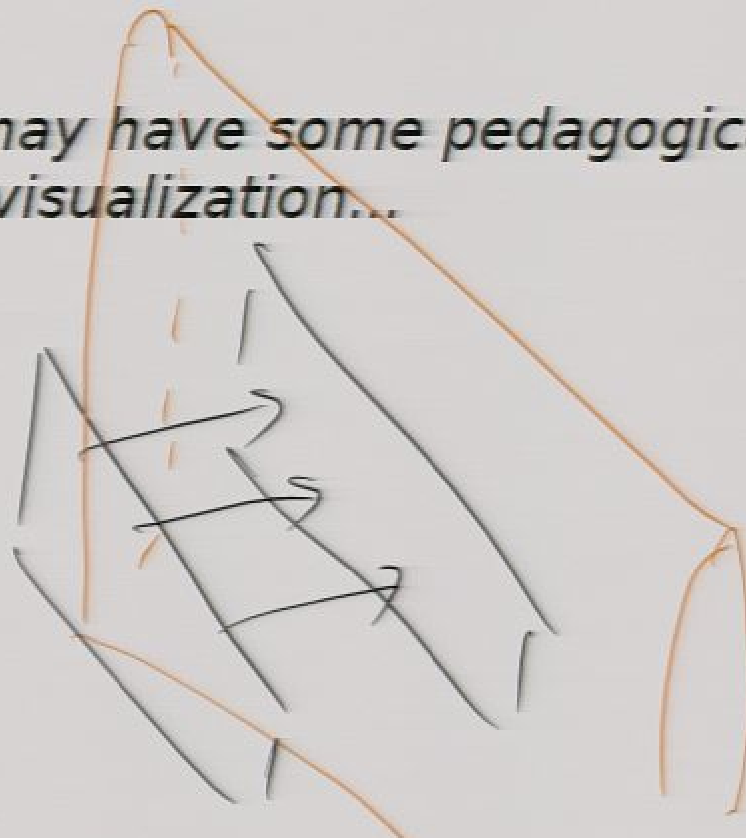

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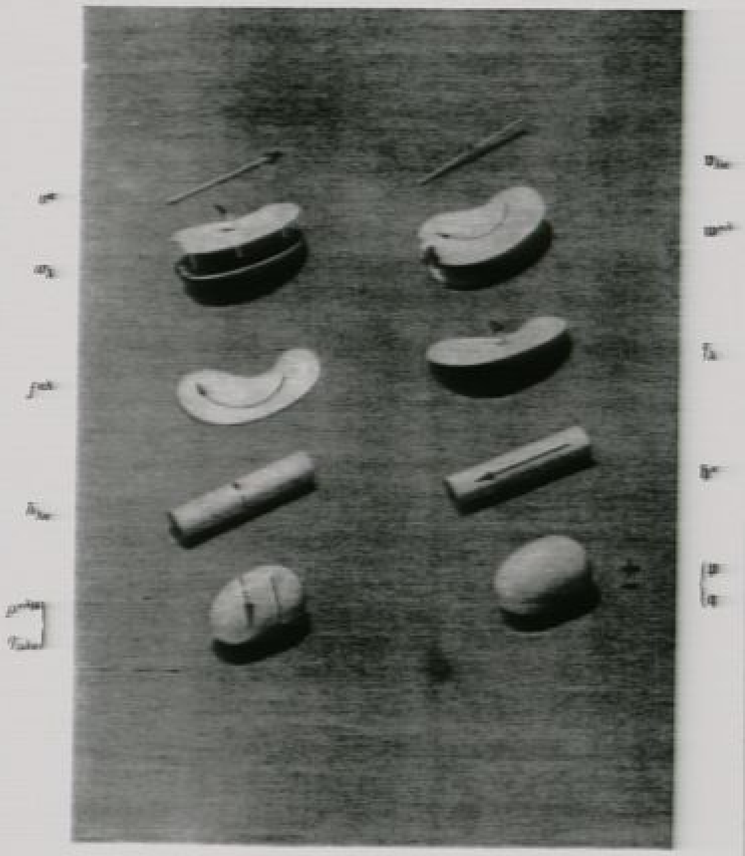


Fig. 6

can be made to agree with the identifications made possible after introduction of a sub-group of G_a. This is to be discussed in Chapter III. If no sub-group of G_a is introduced we may take $\gamma_p = \delta_p = 0$ for all values of p.

(8.14) shows that the components of a contra-(co-)variant W-p-vector may also be considered as components of a co-(contra-)variant $(n-p)$-vector density of weight $-1(+1)$. Hence the geometrical meanings of corresponding quantities do not differ. There is only a difference in notation.

The following table gives a summary of these quantities:

Figure	Second notation	Ordinary notation	Weight	Relations (cycl.)	Number of components	Orientation
	$\tilde{p}$	$\mathfrak{p}_{\lambda\kappa\mu}$	-1	$\mathfrak{p}_{123} = (-1)^{\gamma_0}\tilde{p}$	1	
		$\mathfrak{p}^{\kappa\lambda\mu}$	$+1$	$\mathfrak{p}^{123} = (-1)^{\delta_0}\tilde{p}$		
	$\tilde{v}^{\kappa}$	$\mathfrak{v}_{\lambda\mu}$	-1	$\mathfrak{v}_{23} = (-1)^{\gamma_1}\tilde{v}^{1}$	3 (proj.)	outer
	$\tilde{w}_{\lambda}$	$\mathfrak{w}^{\kappa\lambda}$	$+1$	$\mathfrak{w}^{23} = (-1)^{\delta_1}\tilde{w}_{1}$	$3\left(\frac{1}{\text{sect.}}\right)$	inner
	$\tilde{f}^{\kappa\lambda}$	$\mathfrak{f}_{\lambda}$	-1	$\mathfrak{f}_{1} = (-1)^{\gamma_2}\tilde{f}^{23}$	3 (proj.)	outer
	$\tilde{h}_{\lambda\mu}$	$\mathfrak{h}^{\kappa}$	$+1$	$\mathfrak{h}^{1} = (-1)^{\delta_2}\tilde{h}_{23}$	$3\left(\frac{1}{\text{sect.}}\right)$	inner
	$\tilde{j}^{\kappa\lambda\mu}$	$\mathfrak{j}$	-1	$\mathfrak{j} = (-1)^{\gamma_3}\tilde{j}^{123}$	1 (vol.)	outer
	$\tilde{q}_{\lambda\kappa\mu}$	$\mathfrak{q}$	$+1$	$\mathfrak{q} = (-1)^{\delta_3}\tilde{q}_{123}$	$1\left(\frac{1}{\text{vol.}}\right)$	outer

(8.15)

An ordinary scalar density has no screw-sense. This is the kind of density occurring in physics. For instance a mass density is an ordinary scalar density of weight $+1$. Fig. 6 shows models of the quantities occurring in (8.10) and (8.15).

9. Quantities of valence 2 and matrices (cf. R.C. I § 8)

It is easily proved that

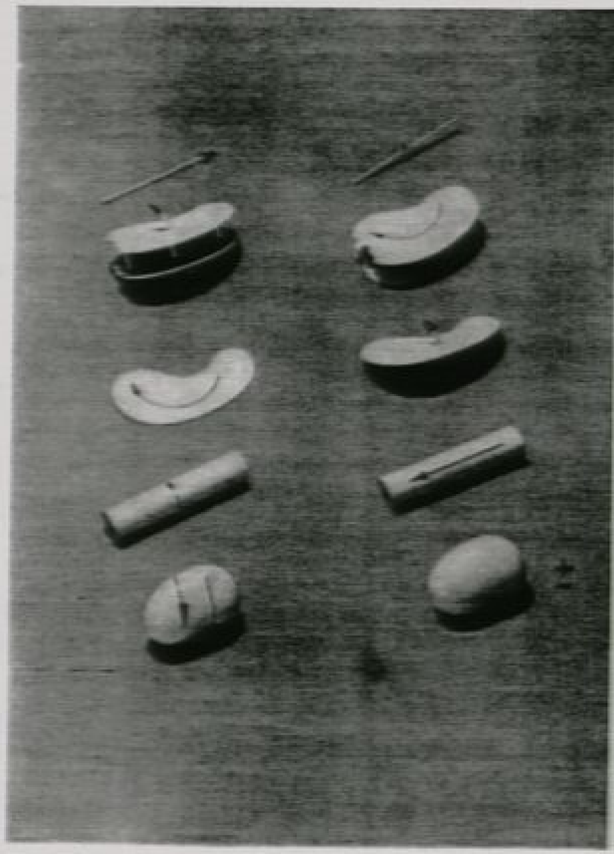
$$\mathrm{Det}(P^{\kappa}_{\cdot\lambda}) = n!\,P^{[1}_{\cdot 1}\cdots P^{n]}_{\cdot n} = n!\,P^{1}_{\cdot[1}\cdots P^{n}_{\cdot n]} = n!\,P^{[1}_{\cdot[1}\cdots P^{n]}_{\cdot n]} \tag{9.1}$$

and that the components of the quantity

$$s!\,P^{[\kappa_1}_{\cdot\cdot[\lambda_1}\cdots P^{\kappa_s]}_{\cdot\cdot\lambda_s]} = s!\,P^{[\kappa_1}_{\cdot\cdot\lambda_1}\cdots P^{\kappa_s]}_{\cdot\cdot\lambda_s} = s!\,P^{\kappa_1}_{\cdot\cdot[\lambda_1}\cdots P^{\kappa_s}_{\cdot\cdot\lambda_s]} \tag{9.2}$$

are the s-rowed sub-determinants in the matrix of $P^{\kappa}_{\cdot\lambda}$. From this it follows that the rank of $P^{\kappa}_{\cdot\lambda}$ is r if and only if

$$P^{[\kappa_1}_{\cdot\cdot[\lambda_1}\cdots P^{\kappa_s]}_{\cdot\cdot\lambda_s]} \begin{cases} \neq 0 & \text{for } s \leq r, \\ = 0 & \text{for } s > r. \end{cases} \tag{9.3}$$



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	$\tilde v^{\kappa}$	$\mathfrak{v}_{\lambda\kappa}$	-1	$\mathfrak{v}_{23} = (-1)^{p+\delta}\tilde v^{1}$	1 (proj.)	outer
	$\tilde w_{\lambda}$	$\mathfrak{w}^{\kappa\lambda}$	$+1$	$\mathfrak{w}^{23} = (-1)^{\delta}\tilde w_{1}$	$1\left(\frac{1}{\text{sect.}}\right)$	inner
	$\tilde f^{\kappa\lambda}$	$\mathfrak{f}_{\lambda}$	-1	$\mathfrak{f}_{1} = (-1)^{p}\tilde f^{23}$	1 (proj.)	outer
	$\tilde h_{\lambda\kappa}$	$\mathfrak{h}^{\kappa}$	$+1$	$\mathfrak{h}^{1} = (-1)^{\delta}\tilde h_{23}$	$1\left(\frac{1}{\text{sect.}}\right)$	inner
	$\tilde p^{\kappa\lambda\mu}$	$\mathfrak{p}$	-1	$\mathfrak{p} = (-1)^{p}\tilde p^{123}$	1 (vol.)	outer
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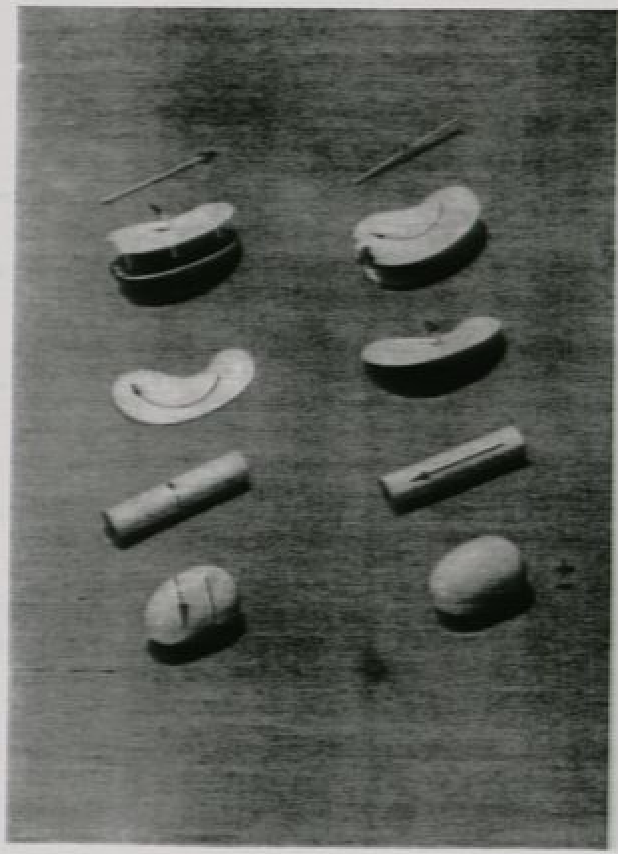

Fig. 6

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	v^{κ}	$\mathfrak{v}_{\lambda}$	$-$
	$\tilde{w}_{\lambda}$	$\mathfrak{w}^{\kappa\lambda}$	$+$
	$f^{\kappa\lambda}$	$\mathfrak{f}_{\lambda}$	$-$
	$\tilde{h}_{\lambda\kappa}$	$\mathfrak{h}^{\nu}$	$+$
	$p^{\kappa\lambda\nu}$	$\mathfrak{p}$	$-$
	$\tilde{q}_{\lambda\kappa\nu}$	$\mathfrak{q}$	$+$

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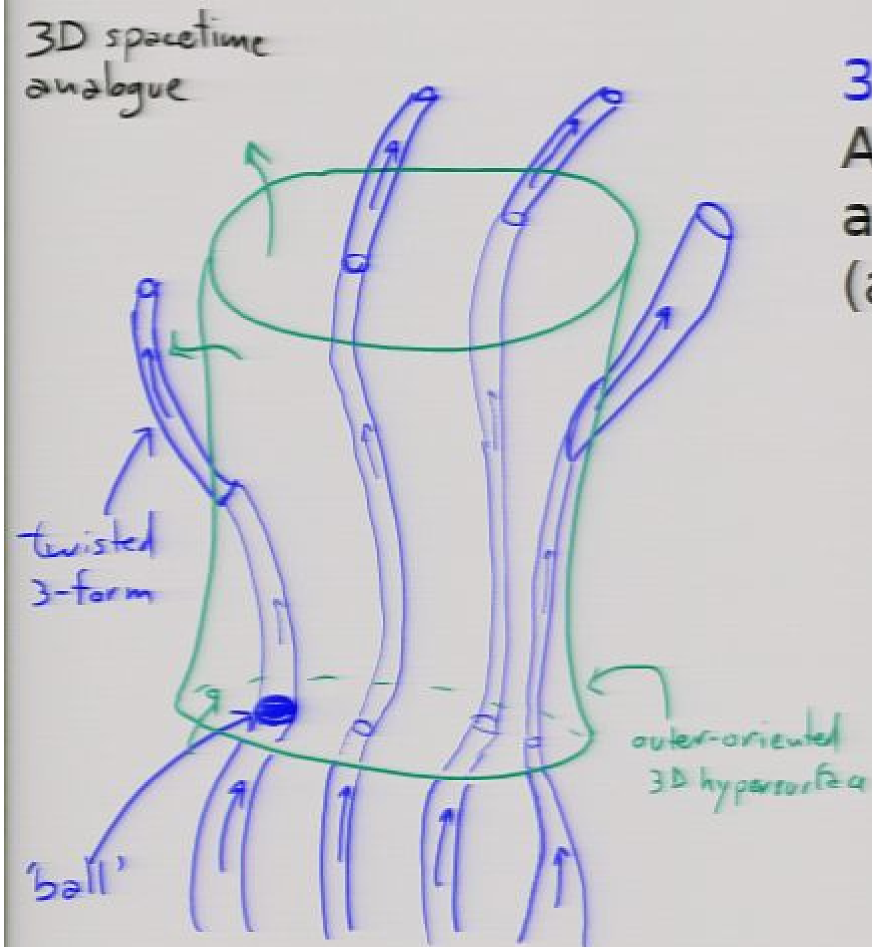
Matter/mass (and charge) is a twisted 3-form in spacetime



3-form in spacetime = 'hypertubes'
A 3D observer sees them
as 'balls' moving around in 3D space
(and changing shape)

balls initially in 3D region
\+ balls entering 2D surface
\- balls exiting 2D surface
\- balls finally in 3D region
= 0

This property =
the twisted 3-form is closed
(its tubes never end) $d\omega = 0$

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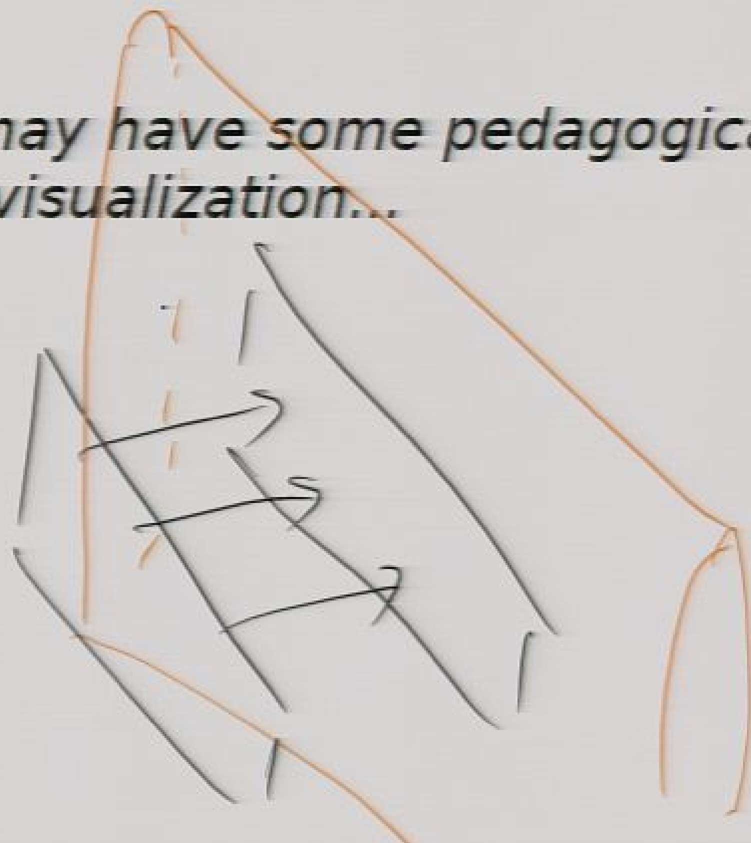
...but then?



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Kottler 1920

van Dantzig

Truesdell & Toupin

Hehl

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$dF = 0$

straight

F 2-form

(E, B)

?

$dQ = 0 \leftarrow$ cons. of charge

Q tw. 3-Form

charge-current

$dQ = 0 \Rightarrow \exists M$ twisted 2-form

$F = dA$

vector potential

$Q = dM$

(ρ, J)

(D, H)

$\lambda(F) = M$

$* \rightarrow$ conformal structure

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F strength 2-form $\longleftrightarrow^{?}$

(E, B)

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