Title: Quantum Conditional States, Bayes Rule and Quantum Pooling

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Abstract: Quantum theory can be thought of a noncommutative generalization of classical probability and, from this perspective, it is puzzling that no quantum generalization of conditional probability is in widespread use. In this talk, I discuss one such generalization and show how it can unify the description of ensemble preparations of quantum states, POVM measurements and the description of correlations between quantum systems. The conditional states formalism allows for a description of prepare-and-measure experiments that is neutral with respect to the direction of inference, such that both the retrodictive formalism and the more usual predictive formalism are consequences of a more fundamental description in terms of a conditionally independent tripartite state, and the two formalisms are related by a quantum generalization of Bayes' rule. As an application, I give a generalized argument for the pooling rule proposed by Spekkens and Wiseman that is a direct analog of a result in classical supra-Bayesian pooling.

Quantum conditional states, Bayes' rule, and quantum pooling

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Joint work with R. W. Spekkens (Perimeter)

Perimeter Institute
17th May 2011

Outline

1. Introduction
2. Quantum conditional states
3. Hybrid quantum-classical systems
4. Quantum Bayes' rule
5. Conditional Independence
6. Bayesian Conditioning
7. Quantum state pooling
8. Further results and open questions

Topic

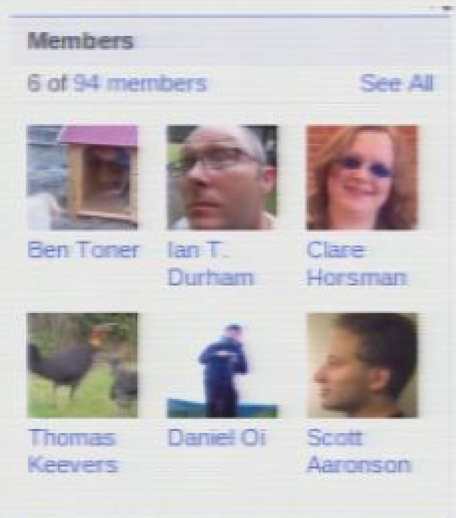
Quantum Theology

- The Church of the Larger Hilbert Space (Smolin)
  - Quantum theory is "about" a pure state vector of the universe that evolves unitarily.
  - Schrödinger, Everett, Zurek, Bennett, . . .
- The Church of the Smaller Hilbert Space
  - Quantum theory is a noncommutative generalization of classical probability theory.
  - Heisenberg, von Neumann, . . .

Quantum Theology on Facebook

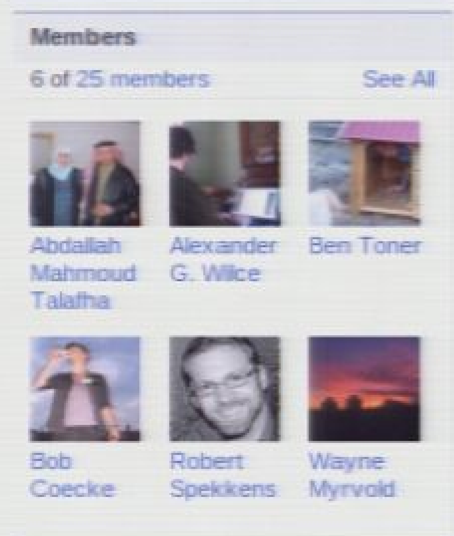
The Church of the Larger Hilbert Space Join

http://www.facebook.com/group.php?gid=5658946617



The Church of The Smaller Hilbert Space

http://www.facebook.com/group.php?gid=5965533115



A Problem for the Smaller Church

- Classical probability theory does not care about causality
  - $P(X, Y, Z, \ldots)$
- Conventional quantum formalism does. . .

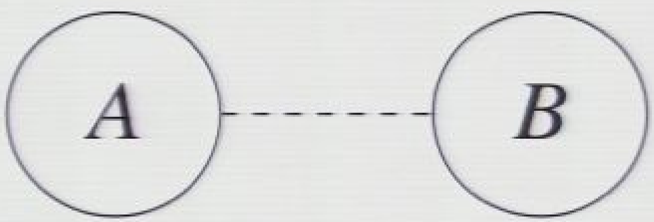


Figure: “Spacelike” correlations

$$\rho_{AB}$$

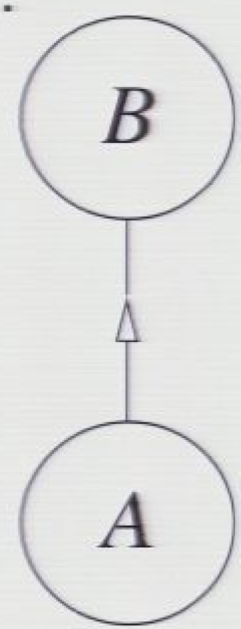


Figure: “Timelike” correlations

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

A note on notation

- Conventional Formalism: Hilbert spaces are attached to systems that persist in time.
  - States are a catalogue of probabilities for potential future measurement outcomes.
- Conditional States Formalism: Hilbert spaces are attached to systems at a specific time, or more generally to spacetime regions.
  - States are a catalogue of probabilities for any classical variables correlated with the region.
  - Always use a distinct label to distinguish input and output systems of a channel.
  - Always combine regions via the tensor product.

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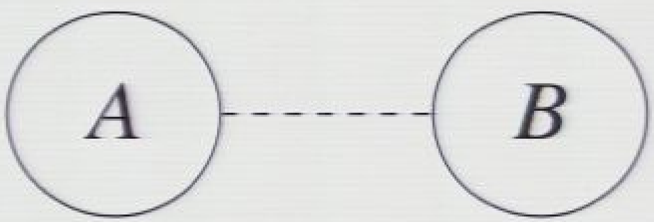

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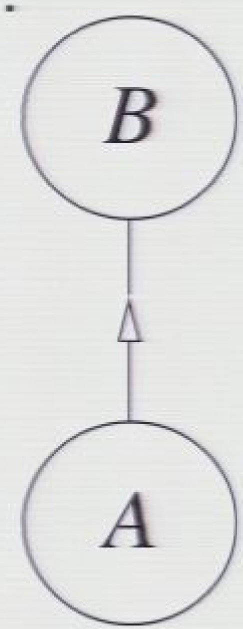
ρ_{AB}



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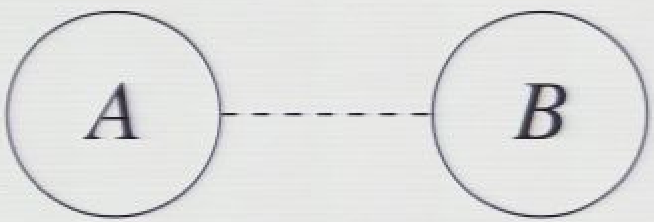

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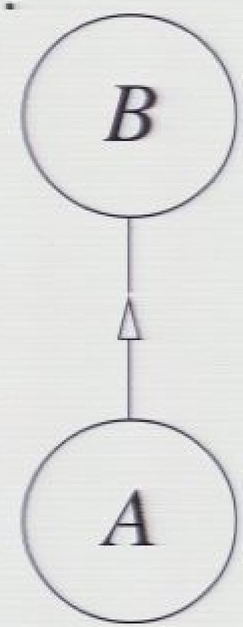

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Classical vs. quantum Probability

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Table: Basic definitions

Classical Probability	Quantum Theory
Sample space $\Omega_X = \{1, 2, \ldots, d_X\}$	Hilbert space $\mathcal{H}_A = \mathbb{C}^{d_A}$ $= \text{span}\left(\lvert 1\rangle, \lvert 2\rangle, \ldots, \lvert d_A\rangle\right)$
Probability distribution $P(X = x) \geq 0$ $\sum_{x \in \Omega_X} P(X = x) = 1$	Quantum state $\rho_A \in \mathfrak{L}^+(\mathcal{H}_A)$ $\text{Tr}_A(\rho_A) = 1$

Classical vs. quantum Probability

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Table: Composite systems

Classical Probability	Quantum Theory
Cartesian product $\Omega_{XY} = \Omega_X \times \Omega_Y$	Tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$
Joint distribution $P(X, Y)$	Bipartite state ρ_{AB}
Marginal distribution $P(Y) = \sum_{x \in \Omega_X} P(X = x, Y)$	Reduced state $\rho_B = \text{Tr}_A(\rho_{AB})$
Conditional distribution $P(Y\|X) = \frac{P(X,Y)}{P(X)}$	Conditional state $\rho_{B\|A} = ?$

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Definition of QCS

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Definition

A quantum conditional state of B given A is a positive operator $\rho_{B|A}$ on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ that satisfies

$$\text{Tr}_B\left(\rho_{B|A}\right) = I_A.$$

c.f. $P(Y|X)$ is a positive function on $\Omega_{XY} = \Omega_X \times \Omega_Y$ that satisfies

$$\sum_{y \in \Omega_Y} P(Y = y|X) = 1.$$

Relation to reduced and joint States

$$(\rho_A, \rho_{B|A}) \quad \rightarrow \quad \rho_{AB} = (\sqrt{\rho_A} \otimes I_B)\, \rho_{B|A} (\sqrt{\rho_A} \otimes I_B)$$

$$\rho_{AB} \quad \rightarrow \quad \rho_A = \text{Tr}_B (\rho_{AB})$$

$$\rho_{B|A} = \left(\sqrt{\rho_A^{-1}} \otimes I_B\right) \rho_{AB} \left(\sqrt{\rho_A^{-1}} \otimes I_B\right)$$

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$$\rho_{B|A} = \left(\sqrt{\rho_A^{-1}} \otimes I_B\right) \rho_{AB} \left(\sqrt{\rho_A^{-1}} \otimes I_B\right)$$

Note: $\rho_{B|A}$ defined from ρ_{AB} is a QCS on $\mathrm{supp}(\rho_A) \otimes \mathcal{H}_B$.

Relation to reduced and joint States

Table: Comparison of relations between joints, conditionals and marginals

Classical Probability	Quantum Theory
$P(X,Y) = P(Y\|X)P(X)$	$\rho_{AB} = \left(\sqrt{\rho_A} \otimes I_B\right) \rho_{B\|A} \left(\sqrt{\rho_A} \otimes I_B\right)$
$P(Y\|X) = \frac{P(X,Y)}{P(X)}$	$\rho_{B\|A} = \left(\sqrt{\rho_A^{-1}} \otimes I_B\right) \rho_{AB} \left(\sqrt{\rho_A^{-1}} \otimes I_B\right)$

Notation

- Drop implied identity operators, e.g.
  - $I_A \otimes M_{BC} N_{AB} \otimes I_C \quad \rightarrow \quad M_{BC} N_{AB}$
  - $M_A \otimes I_B = N_{AB} \quad \rightarrow \quad M_A = N_{AB}$
- Define non-associative "product"
  - $M \star N = \sqrt{N} M \sqrt{N}$

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Relation to reduced and joint states

Table: Comparison of relations between joints, conditionals and marginals

Classical Probability	Quantum Theory
$P(X, Y) = P(Y\|X)P(X)$	$\rho_{AB} = \rho_{B\|A} \star \rho_A$
$P(Y\|X) = \frac{P(X,Y)}{P(X)}$	$\rho_{B\|A} = \rho_{AB} \star \rho_A^{-1}$

Classical conditional probabilities

Example (classical conditional probabilities)

Given a classical variable X, define a Hilbert space $\mathcal{H}_X$ with a preferred basis $\{|1\rangle_X, |2\rangle_X, \ldots, |d_X\rangle_X\}$ labeled by elements of Ω_X. Then,

$$\rho_X = \sum_{x \in \Omega_X} P(X = x) \, |x\rangle \langle x|_X$$

Similarly,

$$\rho_{XY} = \sum_{x \in \Omega_X, y \in \Omega_Y} P(X = x, Y = y) \, |xy\rangle \langle xy|_{XY}$$

$$\rho_{Y|X} = \sum_{x \in \Omega_X, y \in \Omega_Y} P(Y = y | X = x) \, |xy\rangle \langle xy|_{XY}$$

Correlations between subsystems

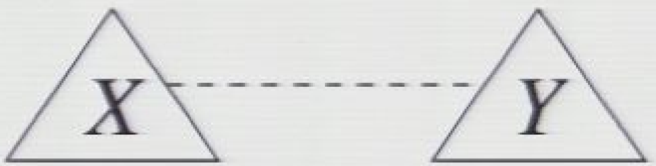

Figure: Classical correlations

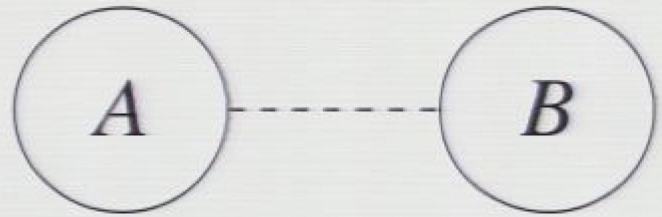

Figure: Quantum correlations

$$P(X, Y) = P(Y|X)P(X)$$

$$\rho_{AB} = \rho_{B|A} \star \rho_A$$

Dynamics

Figure: Classical stochastic map

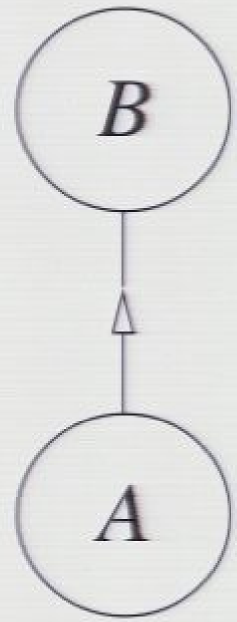

Figure: Quantum CPT map

$$P(Y) = \Gamma_{Y|X}(P(X)) = \sum_X P(Y|X)P(X)$$

$$\rho_A = \mathcal{E}_{B|A}(\rho_A) = \mathrm{Tr}_A\left(\rho_{B|A} \star \rho_A\right) \,?$$

Dynamics

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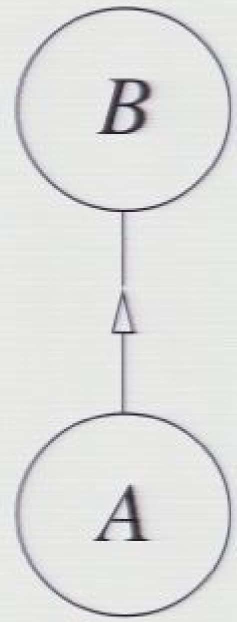

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$$\rho_A = \mathcal{E}_{B|A}(\rho_A) = \mathrm{Tr}_A\left(\rho_{B|A}^{T_A} \star \rho_A\right)$$

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Preparations

Figure: Classical preparation

$$P(Y) = \sum_X P(Y|X)P(X)$$

Figure: Quantum preparation

$$\rho_A = \sum_x P(X = x)\rho_A^{(x)}$$

$$\rho_A = \text{Tr}_X\left(\rho_{A|X} \star \rho_X\right)\ ?$$

What is a Hybrid System?

- Composite of a quantum system and a classical random variable.
- Classical r.v. X has Hilbert space $\mathcal{H}_X$ with preferred basis $\{|1\rangle_X, |2\rangle_X, \ldots, |d_X\rangle_X\}$.
- Quantum system A has Hilbert space $\mathcal{H}_A$.
- Hybrid system has Hilbert space $\mathcal{H}_{XA} = \mathcal{H}_X \otimes \mathcal{H}_A$

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- Quantum system A has Hilbert space $\mathcal{H}_A$.
- Hybrid system has Hilbert space $\mathcal{H}_{XA} = \mathcal{H}_X \otimes \mathcal{H}_A$
- Operators on $\mathcal{H}_{XA}$ restricted to be of the form

$$M_{XA} = \sum_{x \in \Omega_X} |x\rangle \langle x|_X \otimes M_{X=x,A}$$

Quantum|Classical QCS are Sets of States

- A QCS of A given X is of the form

$$\rho_{A|X} = \sum_{x \in \Omega_X} |x\rangle \langle x|_X \otimes \rho_{A|X=x}$$

Proposition

$\rho_{A|X}$ is a QCS of A given X iff each $\rho_{A|X=x}$ is a normalized state on $\mathcal{H}_A$

- Ensemble decomposition: $\rho_A = \sum_x P(X = x)\rho_A^{(x)}$

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- Hybrid joint state: $\rho_{XA} = \sum_{x \in \Omega_X} P(X = x) \, |x\rangle \langle x|_X \otimes \rho_{A|X=x}$

Preparations

Figure: Classical preparation

$$P(Y) = \sum_X P(Y|X)P(X)$$

Figure: Quantum preparation

$$\rho_A = \sum_x P(X = x)\rho_A^{(x)}$$

$$\rho_A = \mathrm{Tr}_X\left(\rho_{A|X} \star \rho_X\right)$$

Measurements

Figure: Noisy measurement

$$P(Y) = \sum_X P(Y|X)P(X)$$

Figure: POVM measurement

$$P(Y = y) = \mathrm{Tr}_A\left(E_A^{(y)}\rho_A\right)$$

$$\rho_Y = \mathrm{Tr}_A\left(\rho_{Y|A} \star \rho_A\right)\,?$$

Classical|Quantum QCS are POVMs

- A QCS of Y given A is of the form

$$\rho_{Y|A} = \sum_{y \in \Omega_Y} |y\rangle \langle y|_Y \otimes \rho_{Y=y|A}$$

Proposition

$\rho_{Y|A}$ is a QCS of Y given A iff $\rho_{Y=y|A}$ is a POVM on $\mathcal{H}_A$

- Generalized Born rule: $P(Y = y) = \text{Tr}_A \left(E_A^{(y)} \rho_A \right)$

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- Hybrid joint state: $\rho_{YA} = \sum_{y \in \Omega_Y} |y\rangle \langle y|_Y \otimes \sqrt{\rho_A} \rho_{Y=y|A} \sqrt{\rho_A}$

Comparison of notation

Dynamics (CPT Map)	$\mathcal{E}_{B\|A}$ $\rho_B = \mathcal{E}_{B\|A}(\rho_A)$	$\rho_{B\|A}^{T_A}$ $\rho_B = \text{Tr}_A\left(\rho_{B\|A}^{T_A} \star \rho_A\right)$
Preparation (Set of states)	$\left\{\rho_A^{(x)}\right\}$ $\rho_A = \sum_x P(X = x)\rho_A^{(x)}$	$\rho_{A\|X}$ $\rho_A = \text{Tr}_X\left(\rho_{A\|X} \star \rho_X\right)$
Measurement (POVM)	$\left\{E_A^{(y)}\right\}$ $P(Y = y) = \text{Tr}_A\left(E^{(y)}\rho_A\right)$	$\rho_{X\|A}$ $\rho_X = \text{Tr}_A\left(\rho_{X\|A} \star \rho_A\right)$

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Topic

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Classical Bayes' rule

- Two expressions for joint probabilities:

$$
\begin{aligned}
P(X, Y) &= P(Y|X)P(X) \\
&= P(X|Y)P(Y)
\end{aligned}
$$

- Bayes' rule:

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

- Laplacian form of Bayes' rule:

$$P(Y|X) = \frac{P(X|Y)P(Y)}{\sum_Y P(X|Y)P(Y)}$$

Quantum Bayes' rule

- Two expressions for bipartite states:

$$\rho_{AB} = \rho_{B|A} \star \rho_A$$
$$= \rho_{A|B} \star \rho_B$$

- Bayes' rule:

$$\rho_{B|A} = \rho_{A|B} \star \left(\rho_A^{-1} \otimes \rho_B\right)$$

- Laplacian form of Bayes' rule

$$\rho_{B|A} = \rho_{A|B} \star \left(\mathrm{Tr}_B\left(\rho_{A|B} \star \rho_B\right)^{-1} \otimes \rho_B\right)$$

State/POVM duality

- A hybrid joint state can be written two ways:

$$\rho_{XA} = \rho_{A|X} \star \rho_X = \rho_{X|A} \star \rho_A$$

- The two representations are connected via Bayes' rule:

$$\rho_{X|A} = \rho_{A|X} \star \left(\rho_X \otimes \text{Tr}_X \left(\rho_{A|X} \star \rho_X \right)^{-1} \right)$$

$$\rho_{A|X} = \rho_{X|A} \star \left(\text{Tr}_A \left(\rho_{X|A} \star \rho_A \right)^{-1} \otimes \rho_A \right)$$

$$\rho_{X=x|A} = \frac{P(X=x)\rho_{A|X=x}}{\sum_{x' \in \Omega_X} P(X=x')\rho_{A|X=x'}} \qquad \rho_{A|X=x} = \frac{\sqrt{\rho_A}\rho_{X=x|A}\sqrt{\rho_A}}{\text{Tr}_A\left(\rho_{X=x|A}\rho_A\right)}$$

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Topic

Quantum conditional independence

- Quantum Chain Rule:

$$\rho_{ABC} = \rho_{C|AB} \star \left(\rho_{B|A} \star \rho_A\right)$$

Quantum conditional independence

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Definition

If $\rho_{C|AB} = \rho_{C|B}$ then C is conditionally independent of A given B.

Quantum conditional independence

- Quantum Chain Rule:

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Definition

If $\rho_{C|AB} = \rho_{C|B}$ then C is conditionally independent of A given B.

Theorem

The following conditions are equivalent:

- $\rho_{C|AB} = \rho_{C|B}$
- $\rho_{A|BC} = \rho_{A|B}$
- $I(A : C|B) = 0.$

Further, conditional independence implies that

- $\rho_{AC|B} = \rho_{A|B}\rho_{C|B}.$

Quantum Conditional Independence

- Quantum Chain Rule:

$$\rho_{ABC} = \rho_{C|AB} \star \left(\rho_{B|A} \star \rho_A\right)$$

Definition

If $\rho_{C|AB} = \rho_{C|B}$ then C is conditionally independent of A given B.

Corollary

$$\rho_{ABC} = \rho_{C|B} \star \left(\rho_{B|A} \star \rho_A\right) \quad \textit{iff} \quad \rho_{ABC} = \rho_{A|B} \star \left(\rho_{B|C} \star \rho_C\right)$$

- The two decompositions are related by Bayes' rule.

Topic

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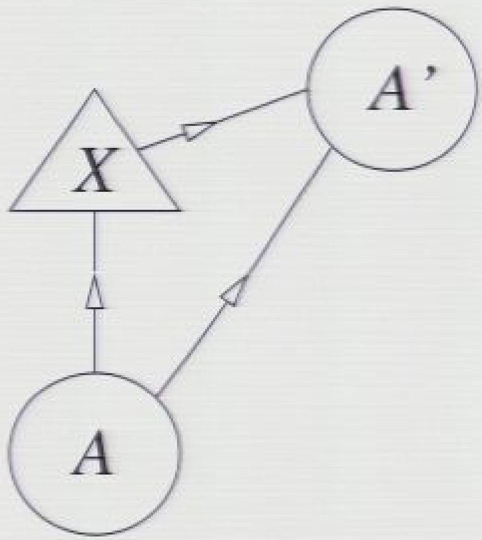
Quantum (Hybrid) Bayesian Conditioning

- Classically, upon learning $X = x$:

$$P(Y) \to P(Y|X = x)$$

- Quantumly: $\rho_A \to \rho_{A|X=x}$?

Projection postulate vs. Bayes' rule

Projection Postulate	Bayesian Conditioning
$\rho_A \to \frac{\sqrt{\rho_{X=x\|A}}\,\rho_A\,\sqrt{\rho_{X=x\|A}}}{\mathrm{Tr}_A\left(\rho_{X=x\|A}\,\rho_A\right)}$	$\rho_A \to \frac{\sqrt{\rho_A}\,\rho_{X=x\|A}\,\sqrt{\rho_A}}{\mathrm{Tr}_A\left(\rho_{X=x\|A}\,\rho_A\right)}$
	

Conditioning on a Preparation Variable

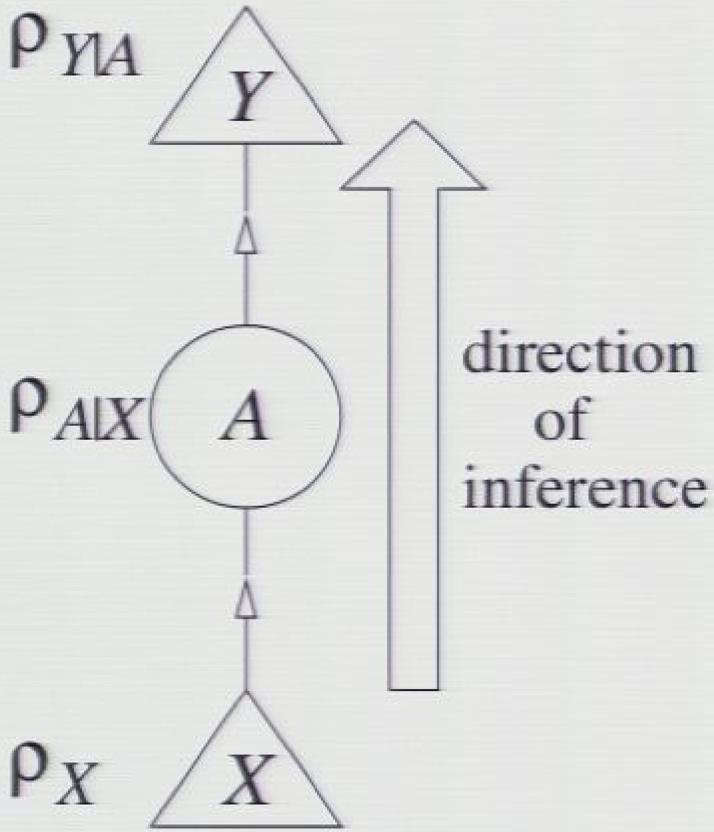


Figure: Prep. & meas. experiment

- Tripartite CI state:

$$\rho_{XAY} = \rho_{Y|A} \star \left(\rho_{A|X} \star \rho_X\right)$$

- Joint probabilities:

$$\rho_{XY} = \mathrm{Tr}_A\left(\rho_{XAY}\right)$$

- Marginal for Y:

$$\rho_Y = \mathrm{Tr}_A\left(\rho_{Y|A} \star \rho_A\right)$$

- Conditional probabilities:

$$\rho_{Y|X} = \mathrm{Tr}_A\left(\rho_{Y|A} \star \rho_{A|X}\right)$$

- Bayesian update:

$$\rho_A \to \rho_{A|X=x}$$

Conditioning on a Direct Measurement

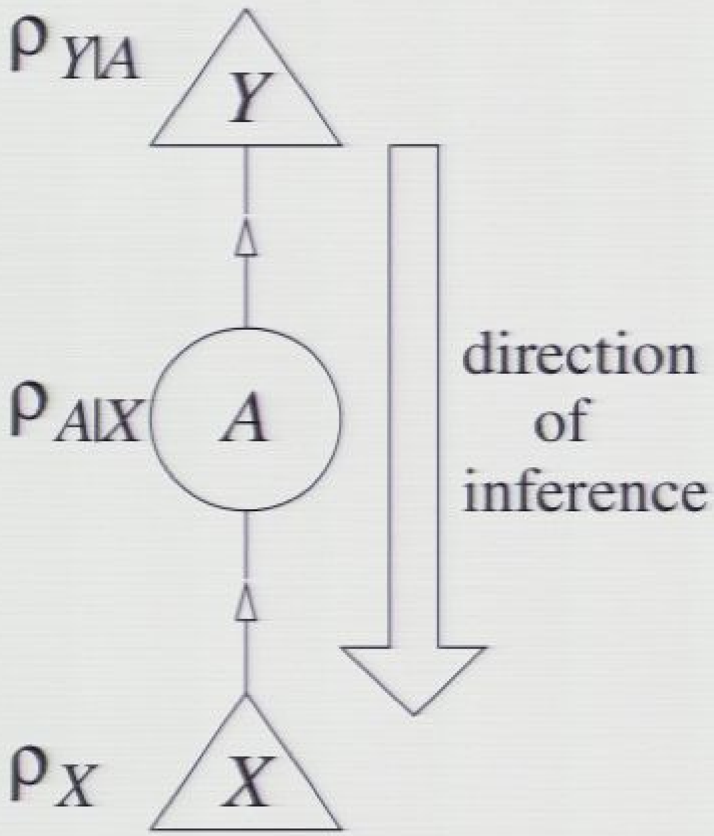


Figure: Prep. & meas. experiment

- Due to symmetry of CI:

$$\rho_{XAY} = \rho_{X|A} \star \left(\rho_{A|Y} \star \rho_Y\right)$$

- Marginal for X:

$$\rho_X = \text{Tr}_A \left(\rho_{X|A} \star \rho_A\right)$$

- Conditional probabilities:

$$\rho_{X|Y} = \text{Tr}_A \left(\rho_{X|A} \star \rho_{A|Y}\right)$$

- Bayesian update:

$$\rho_A \to \rho_{A|Y=y}$$

- c.f. Barnett, Pegg & Jeffers, J. Mod. Opt. 47:1779 (2000).

Conditioning on a Preparation Variable

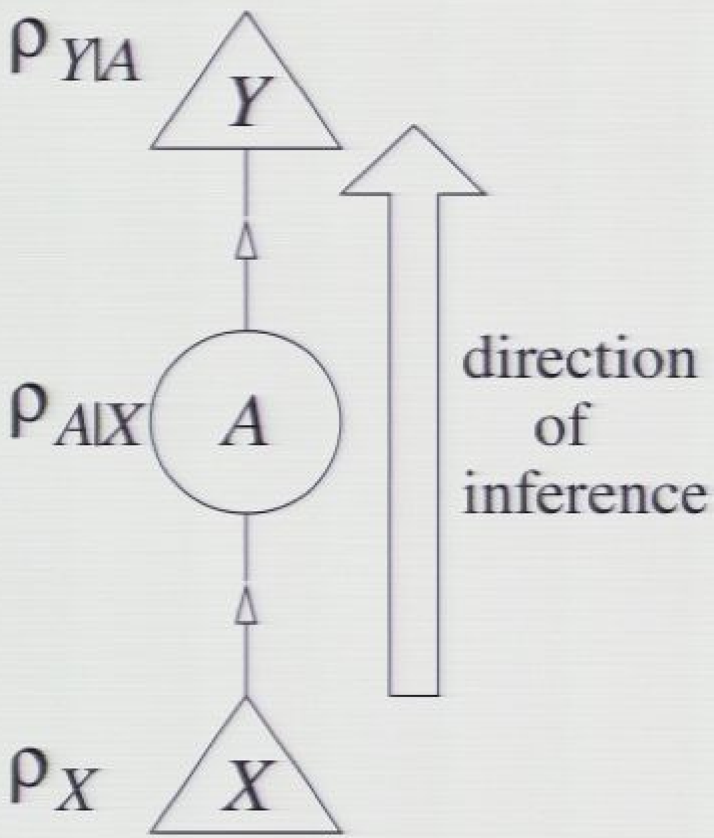


Figure: Prep. & meas. experiment

- Tripartite CI state:

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- Marginal for Y:

$$\rho_Y = \text{Tr}_A \left(\rho_{Y|A} \star \rho_A\right)$$

- Conditional probabilities:

$$\rho_{Y|X} = \text{Tr}_A \left(\rho_{Y|A} \star \rho_{A|X}\right)$$

- Bayesian update:

$$\rho_A \rightarrow \rho_{A|X=x}$$

Conditioning on a Direct Measurement

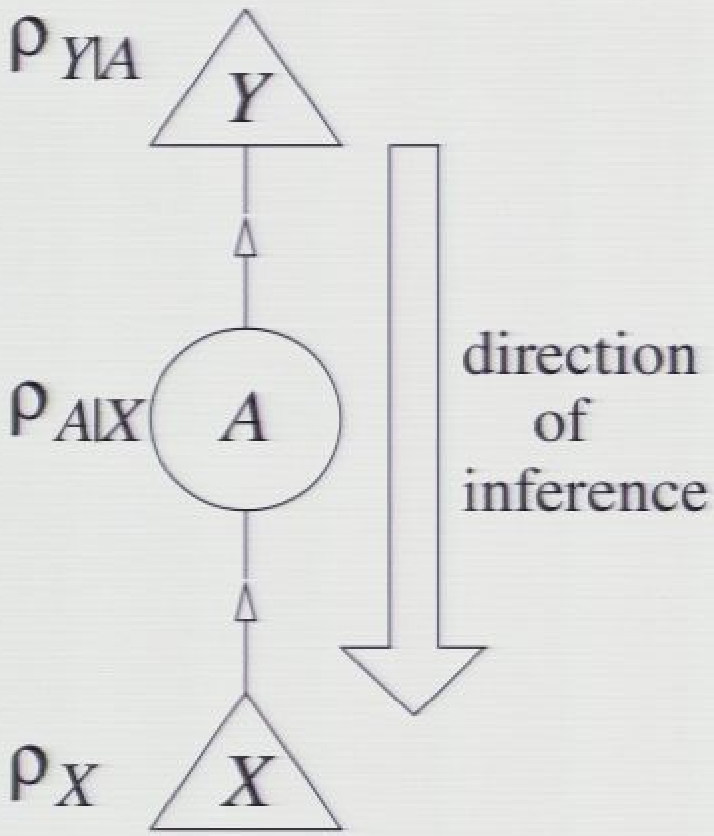

Figure: Prep. & meas. experiment

- Due to symmetry of CI:

$$\rho_{XAY} = \rho_{X|A} \star \left(\rho_{A|Y} \star \rho_Y\right)$$

- Marginal for X:

$$\rho_X = \text{Tr}_A \left(\rho_{X|A} \star \rho_A\right)$$

- Conditional probabilities:

$$\rho_{X|Y} = \text{Tr}_A \left(\rho_{X|A} \star \rho_{A|Y}\right)$$

- Bayesian update:

$$\rho_A \to \rho_{A|Y=y}$$

- c.f. Barnett, Pegg & Jeffers, J. Mod. Opt. 47:1779 (2000).

Conditioning on a Remote Measurement

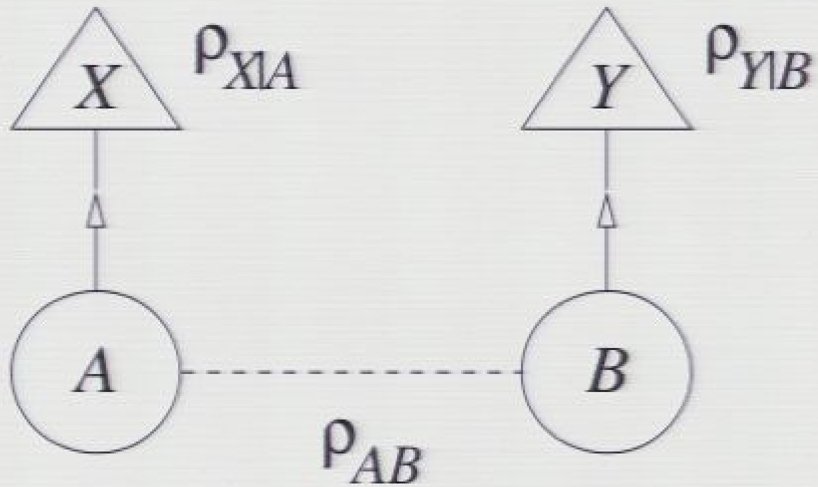

Figure: Bipartite experiment

- Joint probability: $\rho_{XY} = \mathrm{Tr}_{AB}\left(\left(\rho_{X|A} \otimes \rho_{Y|B}\right) \star \rho_{AB}\right)$
- B can be factored out: $\rho_{XY} = \mathrm{Tr}_{A}\left(\rho_{Y|A} \star \left(\rho_{A|X} \star \rho_{X}\right)\right)$
- where $\rho_{Y|A} = \mathrm{Tr}_{B}\left(\rho_{Y|B}\rho_{B|A}\right)$

Summary of Conditioning

Table: Which states update via Bayesian conditioning?

Updating on:	Predictive state	Retrodictive state
Preparation variable	✓	X
Direct measurement outcome	X	✓
Remote measurement outcome	✓	It's complicated

Topic

Quantum State Pooling

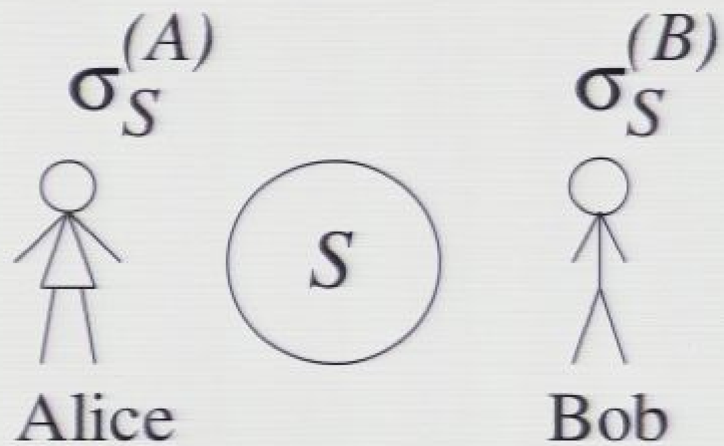

Figure: Initial State Assignments

Quantum State Pooling

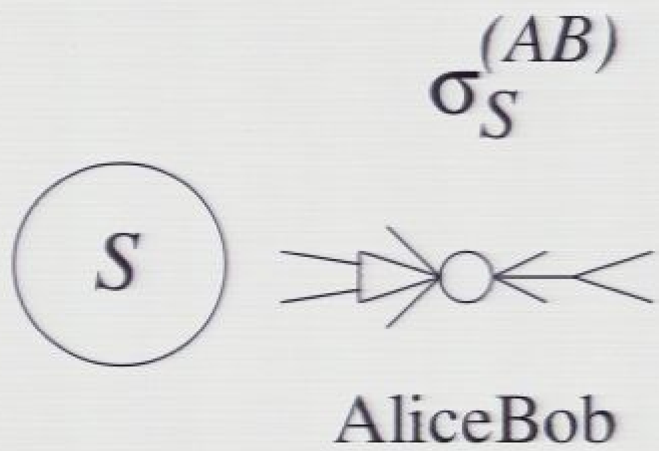

Figure: Final State Assignments

$$\sigma_S^{(AB)} = f(\sigma_S^{(A)}, \sigma_S^{(B)})$$

Expert Advice

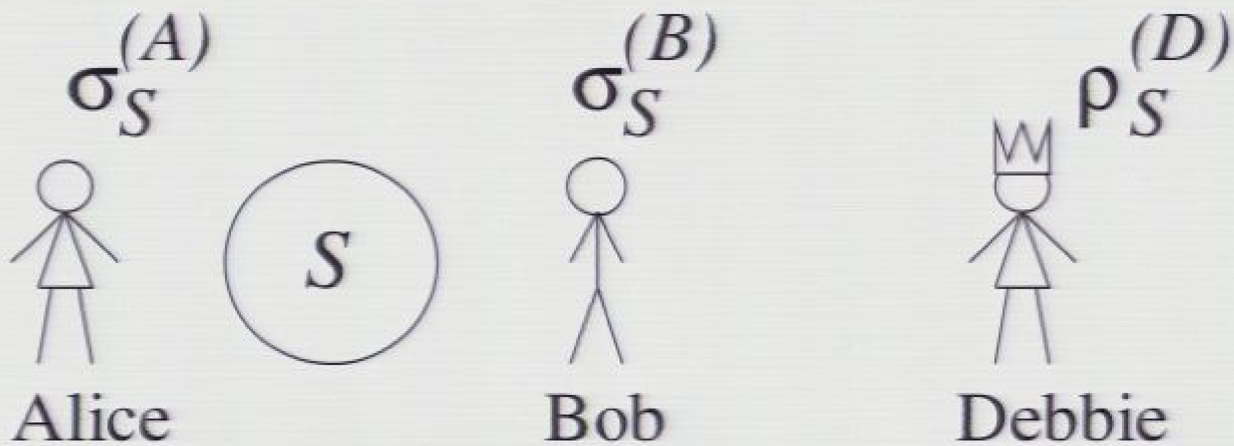

Figure: Initial State Assignments

Expert Advice

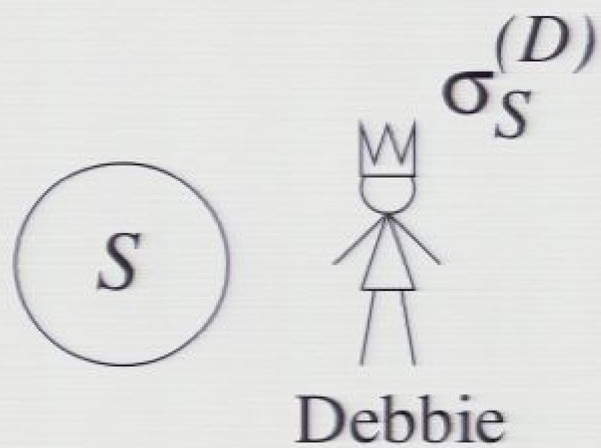

Figure: Final State Assignment

$$\sigma_S^{(D)} = f(\sigma_S^{(A)}, \sigma_S^{(B)}, \rho_S^{(D)})$$

Supra-Bayesian Approach

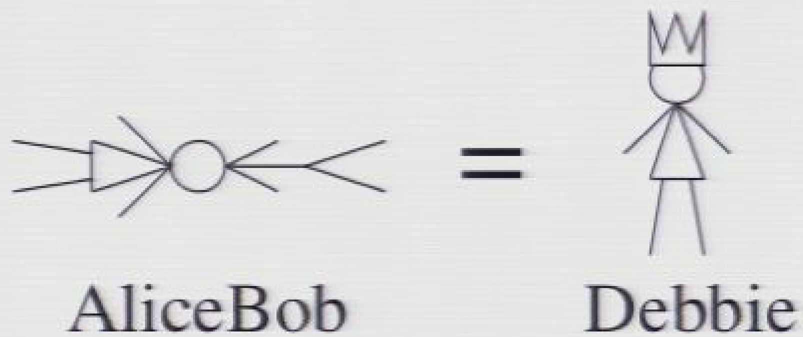

Figure: Supra-Bayesian Pooling

$$\sigma_S^{(AB)} = \sigma_S^{(D)}$$

Classical Expert Advice

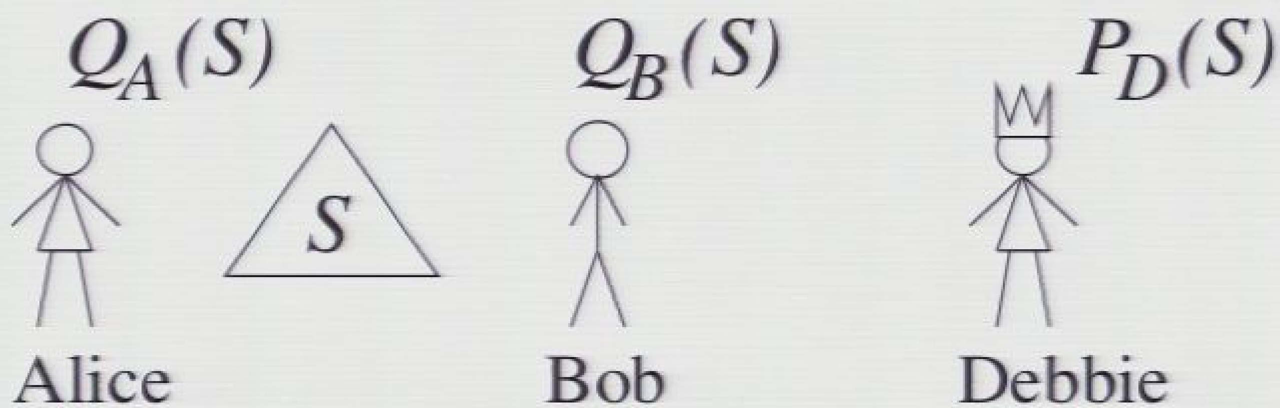

Figure: Initial State Assignments

Classical Expert Advice

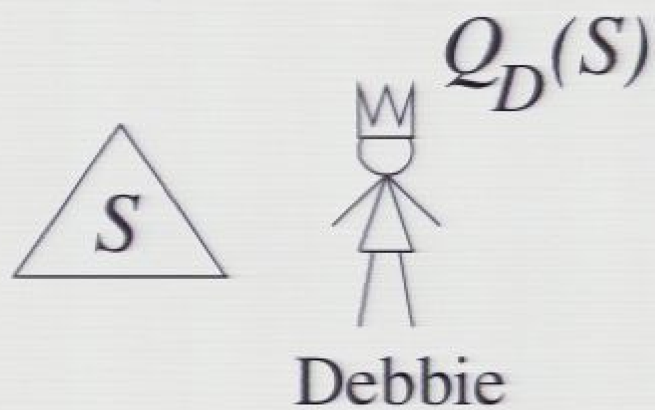

Figure: Final State Assignment

$$Q_D(S) = f(Q_A(S), Q_B(S), P_D(S))$$

Diplomatic Pooling

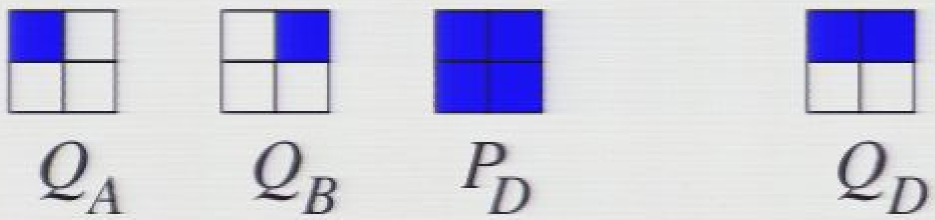

Figure: Pooling Incompatible Assignments

- Linear pool:

$$Q_D(S) = w_A Q_A(S) + w_B Q_B(S) + w_D P_D(S)$$

Scientific Pooling

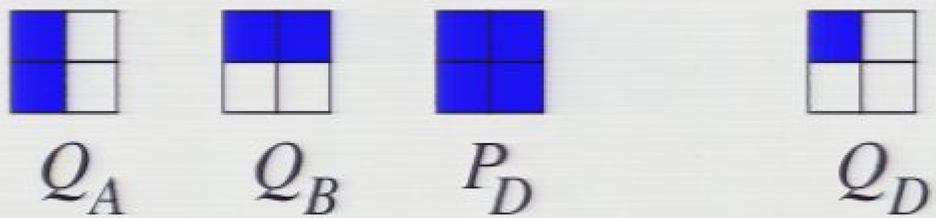

Figure: Pooling Independent Evidence

- Multiplicative (log-linear) pool:

$$Q_D(S) \propto \frac{Q_A(S)Q_B(S)}{P_D(S)}$$

Supra-Bayesian Pooling

- Bayesian inference says that:

$$Q_D(S) = P_D(S|R_A = Q_A(S), R_B = Q_B(S))$$
$$= \frac{P_D(R_A = Q_A(S), R_B = Q_B(S)|S)P_D(S)}{\sum_S P_D(R_A = Q_A(S), R_B = Q_B(S)|S)P_D(S)}$$

- Similarly:

$$\sigma_S^{(D)} = \rho^{(D)}_{S|R_A=\sigma_S^{(A)}, R_B=\sigma_S^{(B)}}$$
$$= \rho^{(D)}_{R_A=\sigma_S^{(A)}, R_B=\sigma_S^{(B)}|S} \star \left(\rho_S^{(D)} \otimes \mathrm{Tr}_S \left(\rho^{(D)}_{R_A=\sigma_S^{(A)}, R_B=\sigma_S^{(B)}|S} \star \rho_S^{(D)} \right) \right)$$

Shared priors

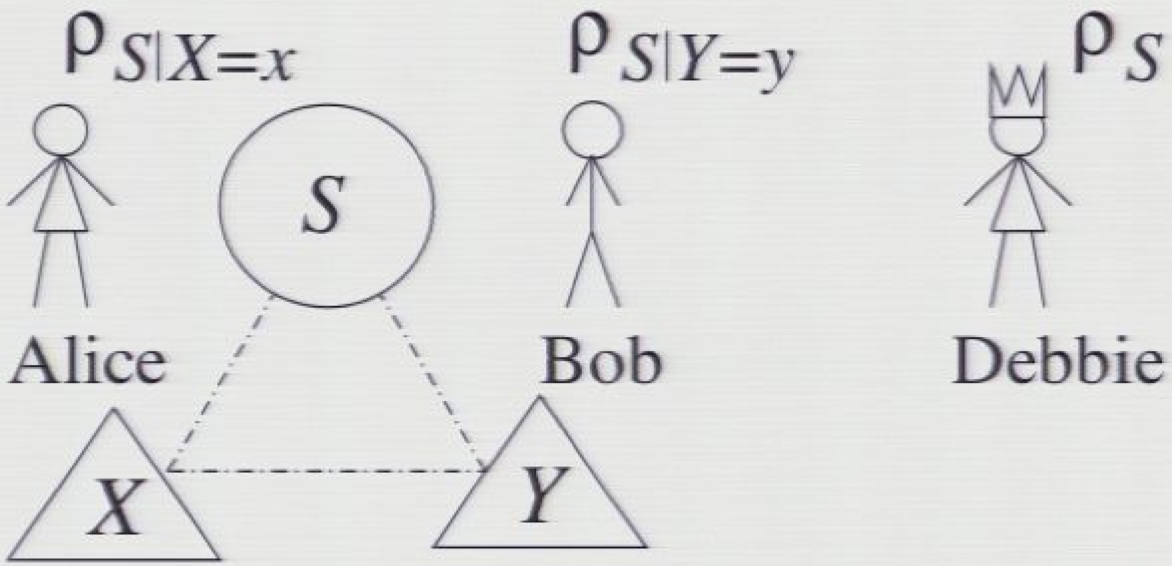

Figure: The Case of Shared Priors

Theorem

If X and Y are conditionally independent given S then

$$\rho^{(D)}_{S|R_A=\rho_{S|X=x},R_B=\rho_{S|Y=y}} \propto \rho_{S|X=x}\rho_S^{-1}\rho_{S|Y=y}$$

Shared Priors

Theorem (Stronger Version)

If the minimal sufficient statistics for X and Y with respect to S are conditionally independent given S then

$$\rho^{(D)}_{S|R_A=\rho_{S|X=x},R_B=\rho_{S|Y=y}} \propto \rho_{S|X=x}\rho_S^{-1}\rho_{S|Y=y}$$

Further results

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Forthcoming paper(s) with R. W. Spekkens also include:

- Dynamics (CPT maps, instruments)
- Temporal joint states
- Quantum state compatibility

Earlier papers with related ideas:

- M. Asorey et. al., Open.Syst.Info.Dyn. 12:319–329 (2006).
- M. S. Leifer, Phys. Rev. A 74:042310 (2006).
- M. S. Leifer, AIP Conference Proceedings 889:172–186 (2007).
- M. S. Leifer & D. Poulin, Ann. Phys. 323:1899 (2008).

Open question

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What is the meaning of fully quantum Bayesian conditioning?

$$\rho_B \to \rho_{B|A} = \rho_{A|B} \star \left(\mathrm{Tr}_B \left(\rho_{A|B} \star \rho_B \right)^{-1} \otimes \rho_B \right)$$

Thanks for your attention!

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- University College London