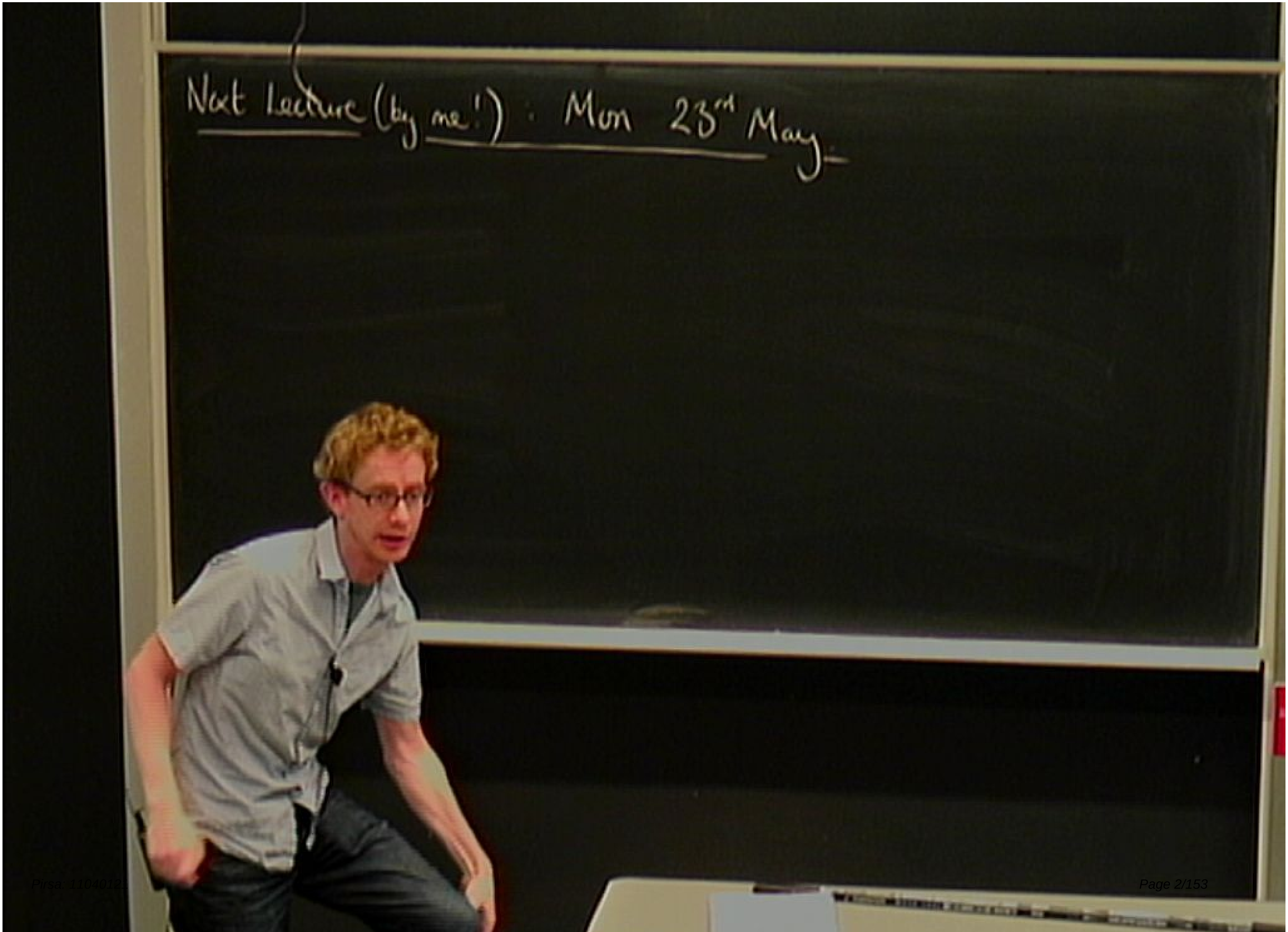
Title: Geometry & Topology for Physics - Lecture 2

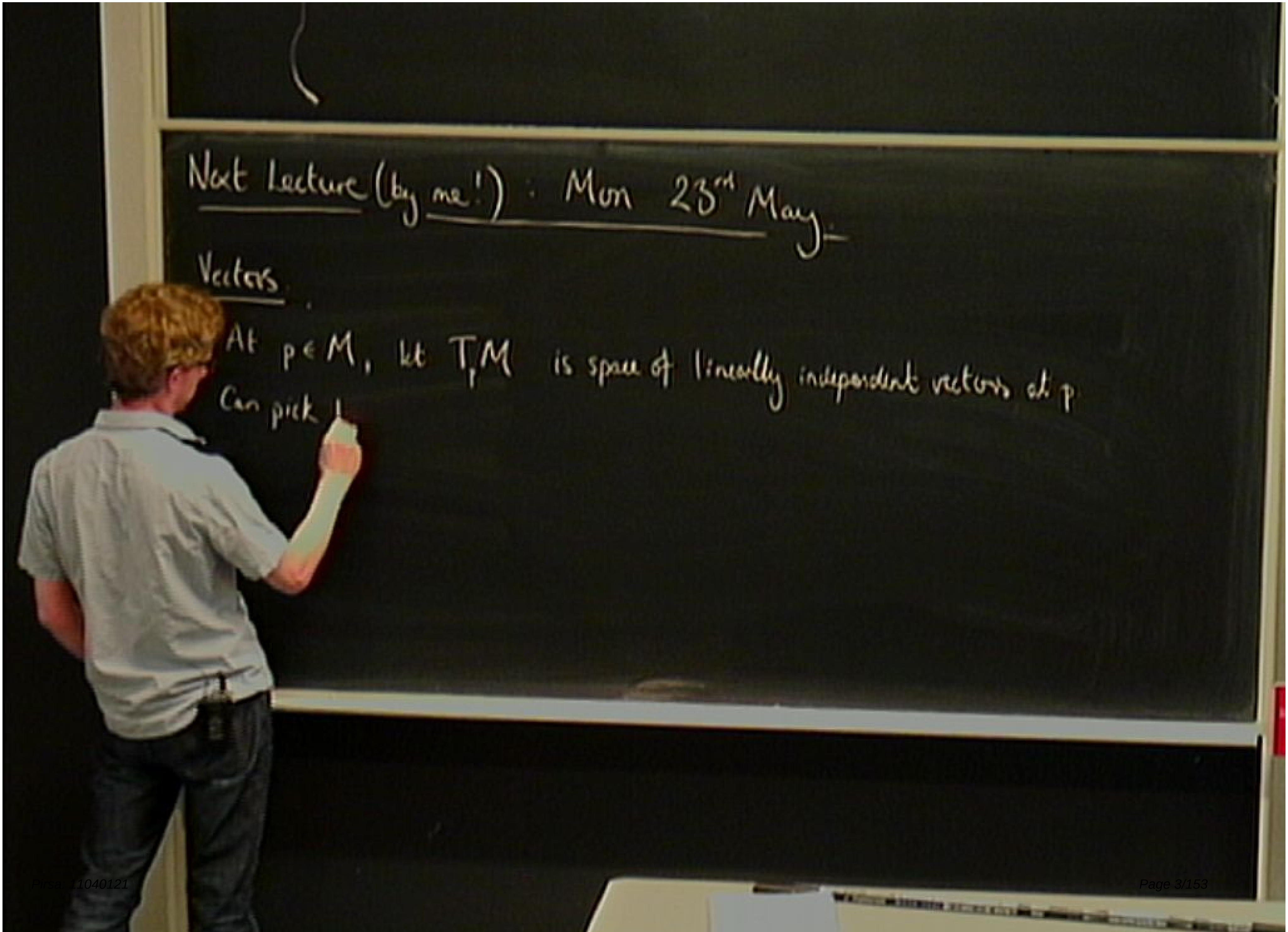
Date: Apr 04, 2011 02:00 PM

URL: http://pirsa.org/11040121

Abstract:



Next lecture (by me!) : Mon 23rd May.

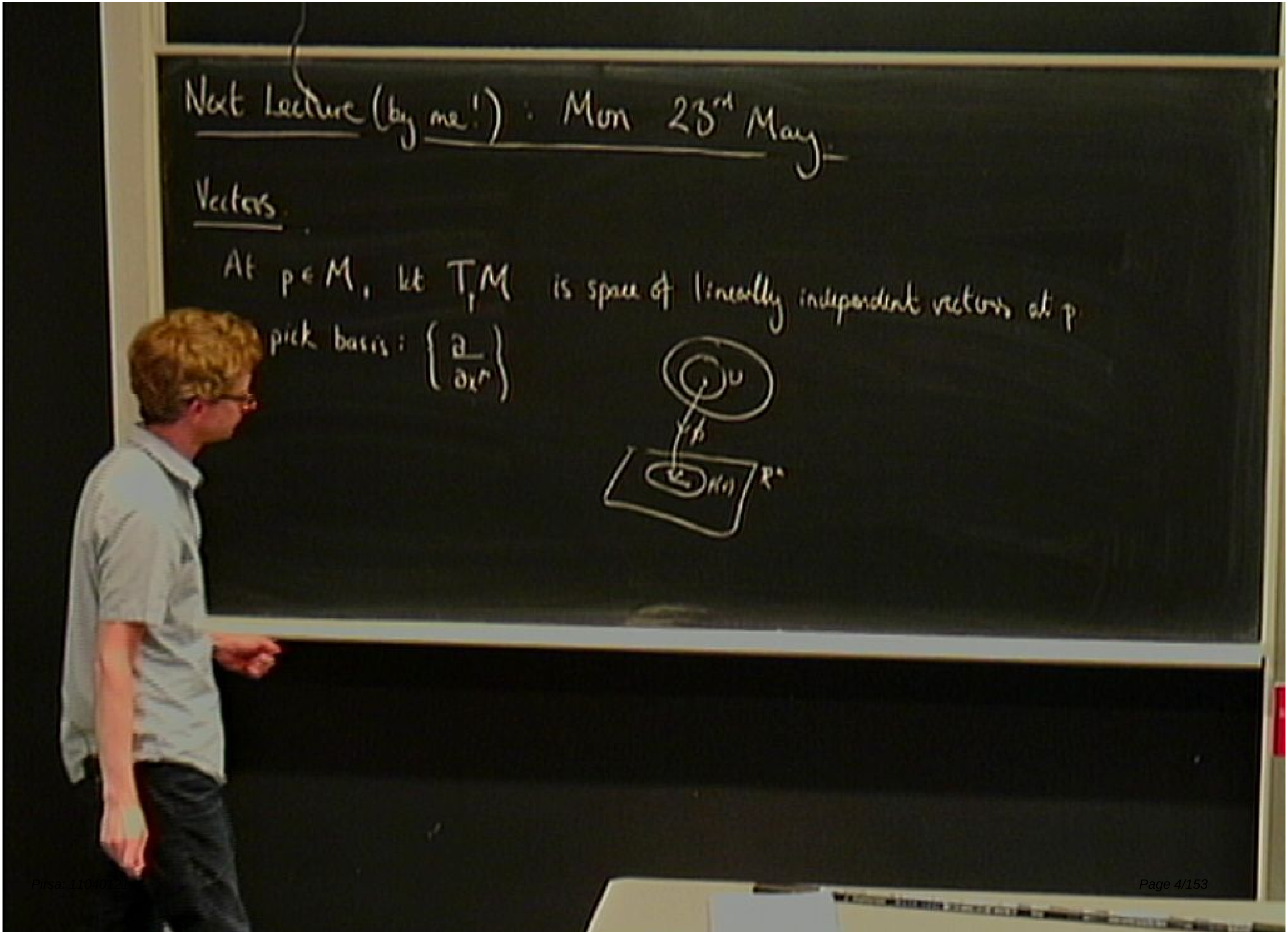


Next lecture (by me!) : Mon 23rd May.

Vectors

At $p \in M$, let T_pM is space of linearly independent vectors at p

Can pick

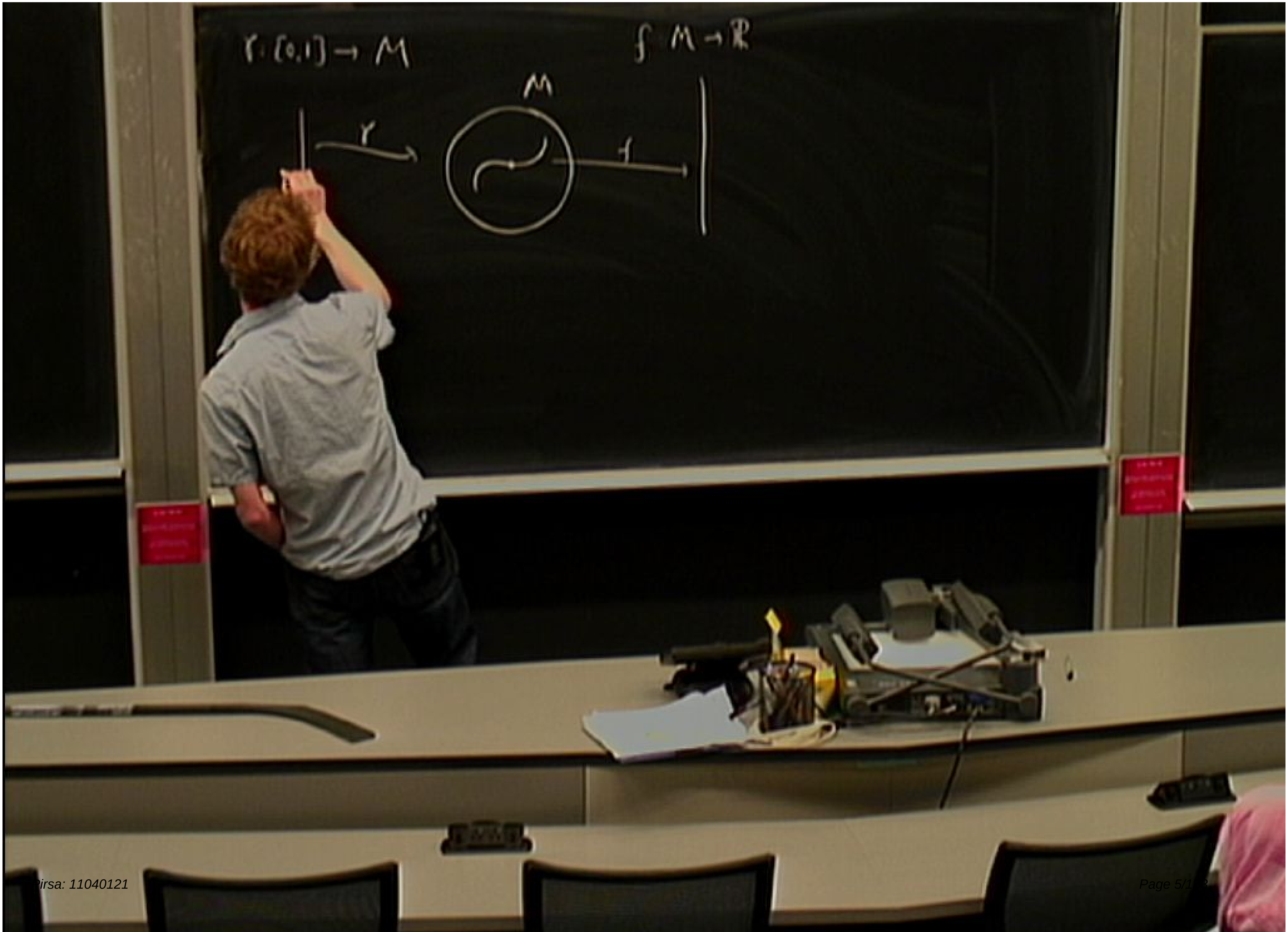


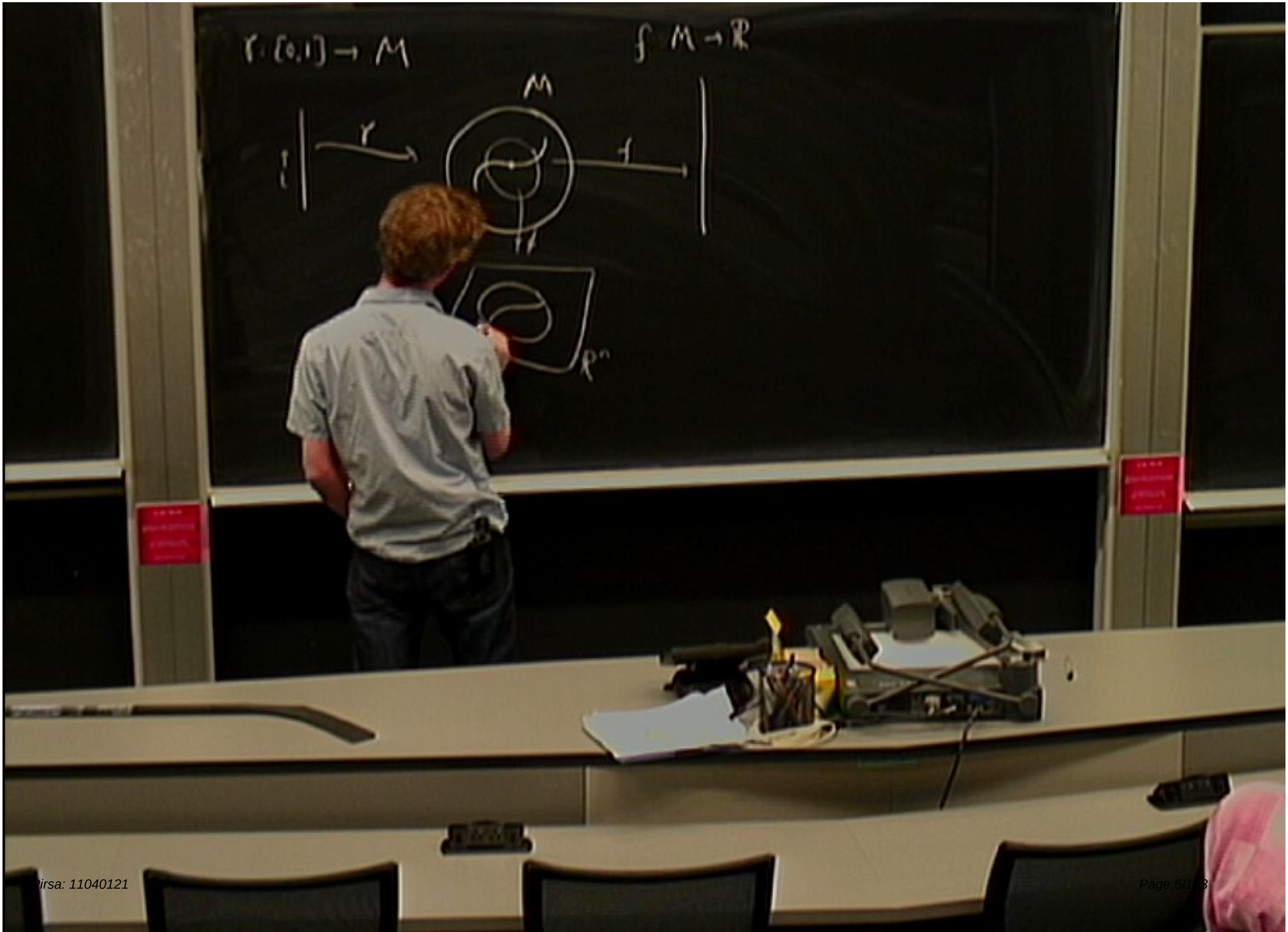
Next Lecture (by me!) : Mon 23rd May.

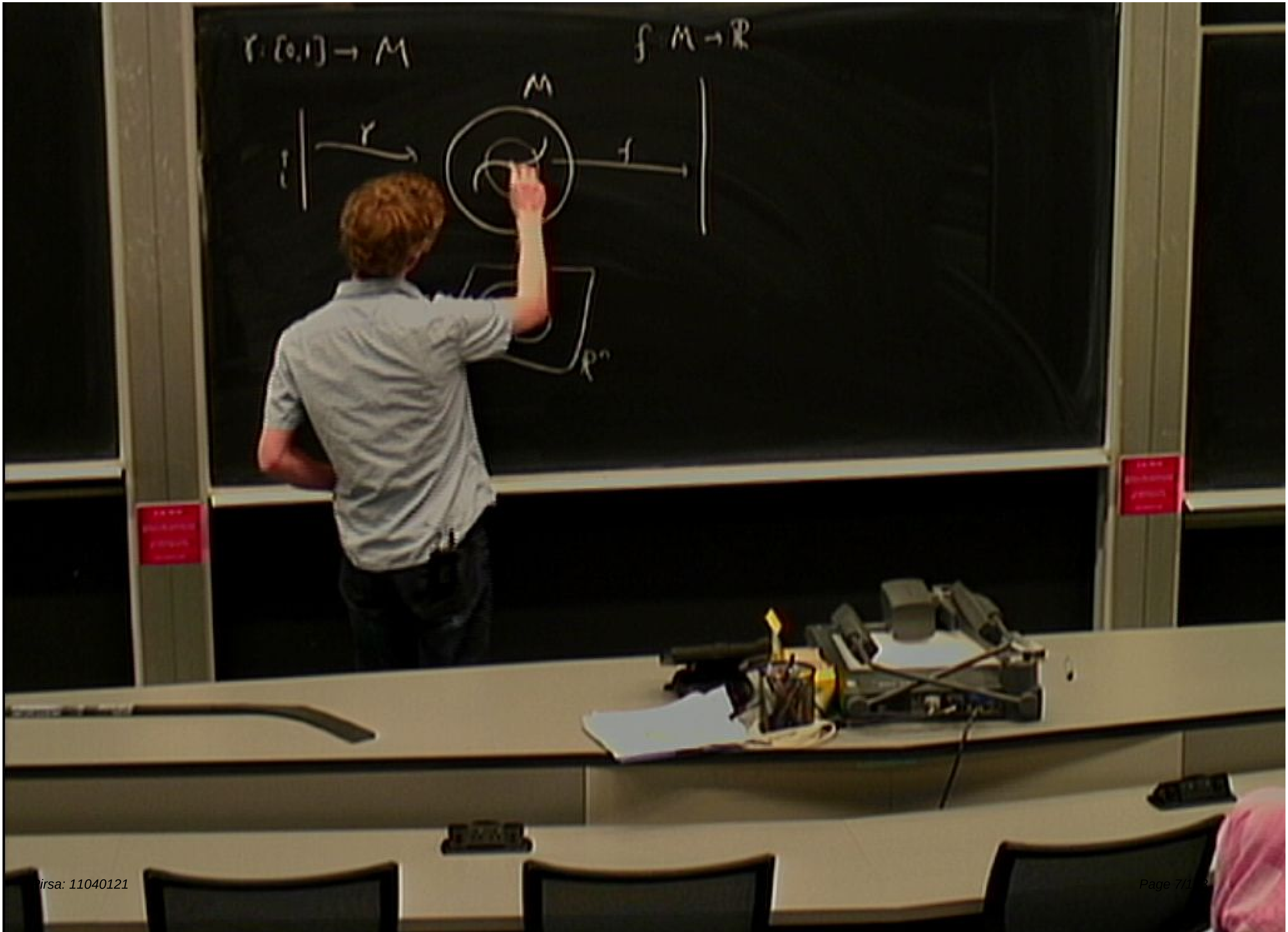
Vectors

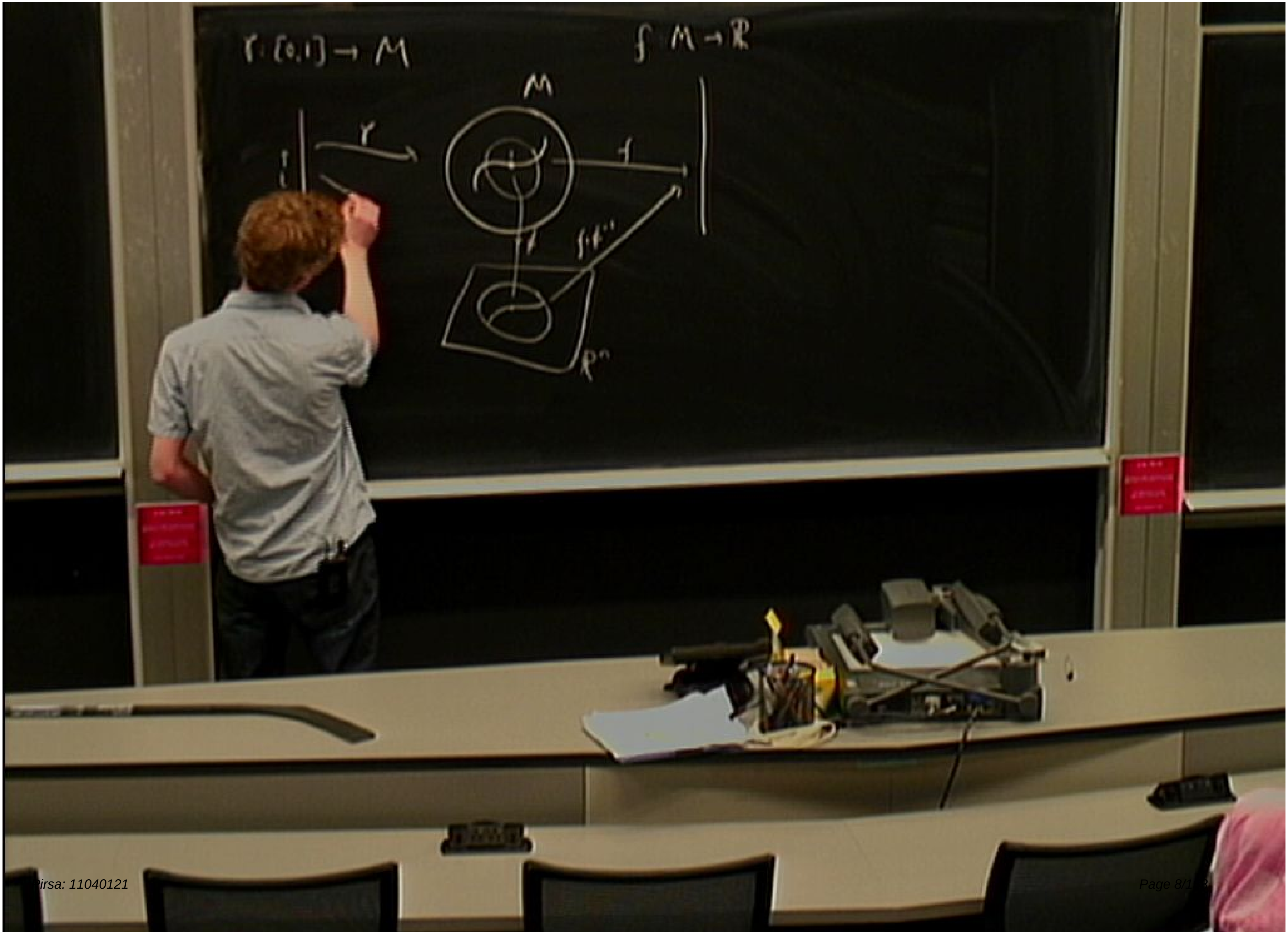
At $p \in M$, let T_pM is space of linearly independent vectors at p

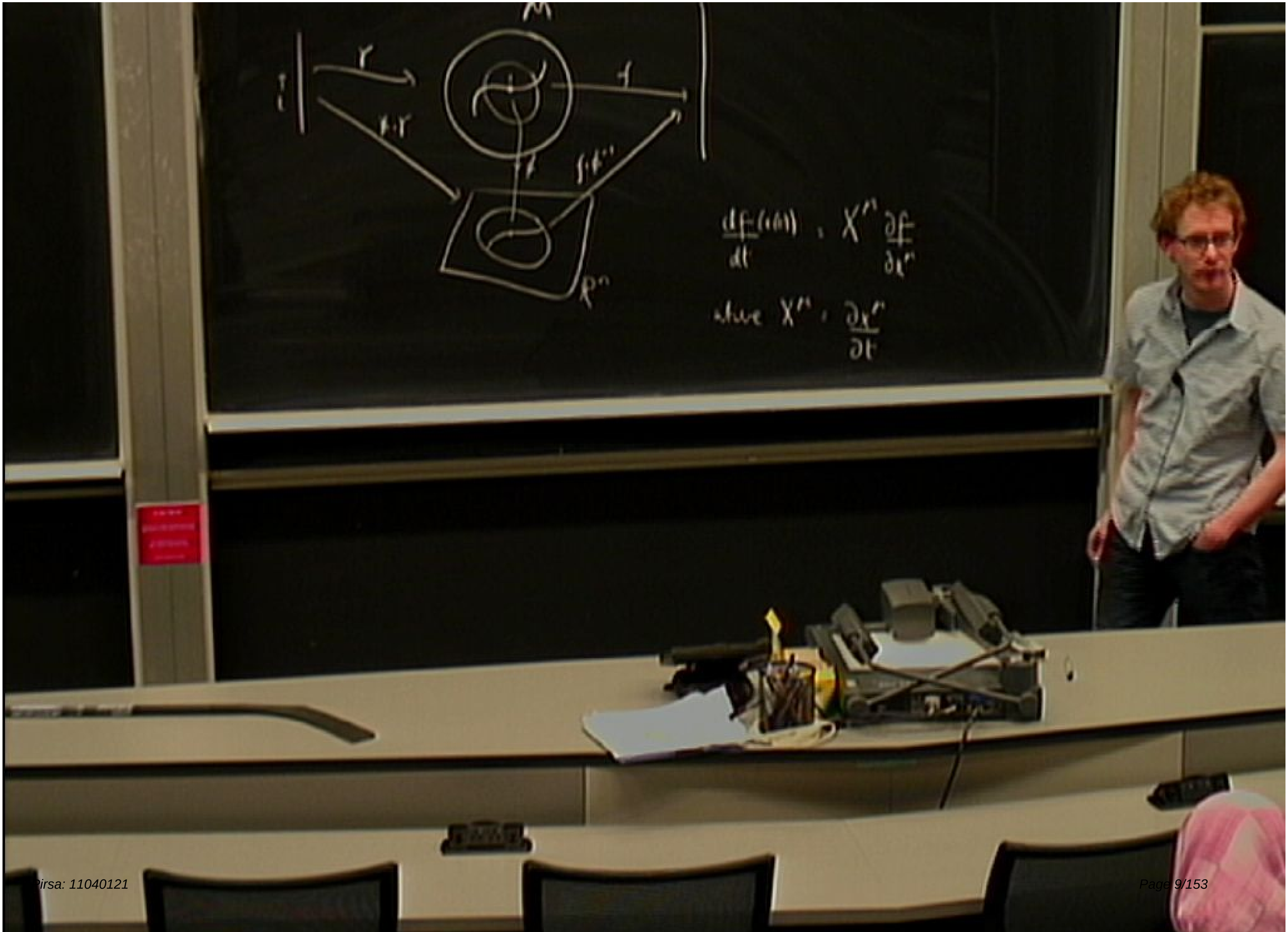
pick basis: $\left\{ \frac{\partial}{\partial x^\mu} \right\}$

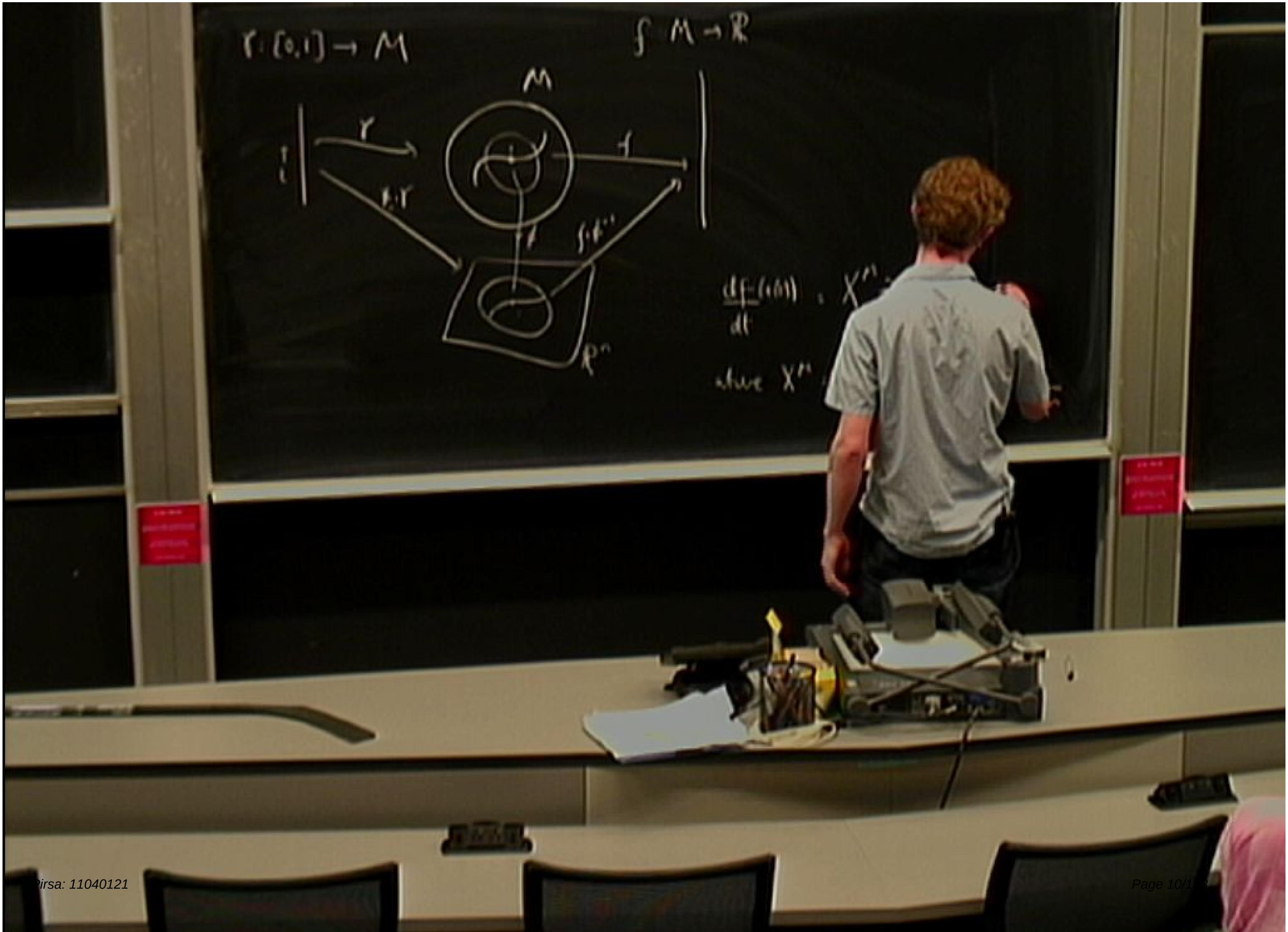

γ: [0,1] → M
f: M → ℝ
M
γ
f

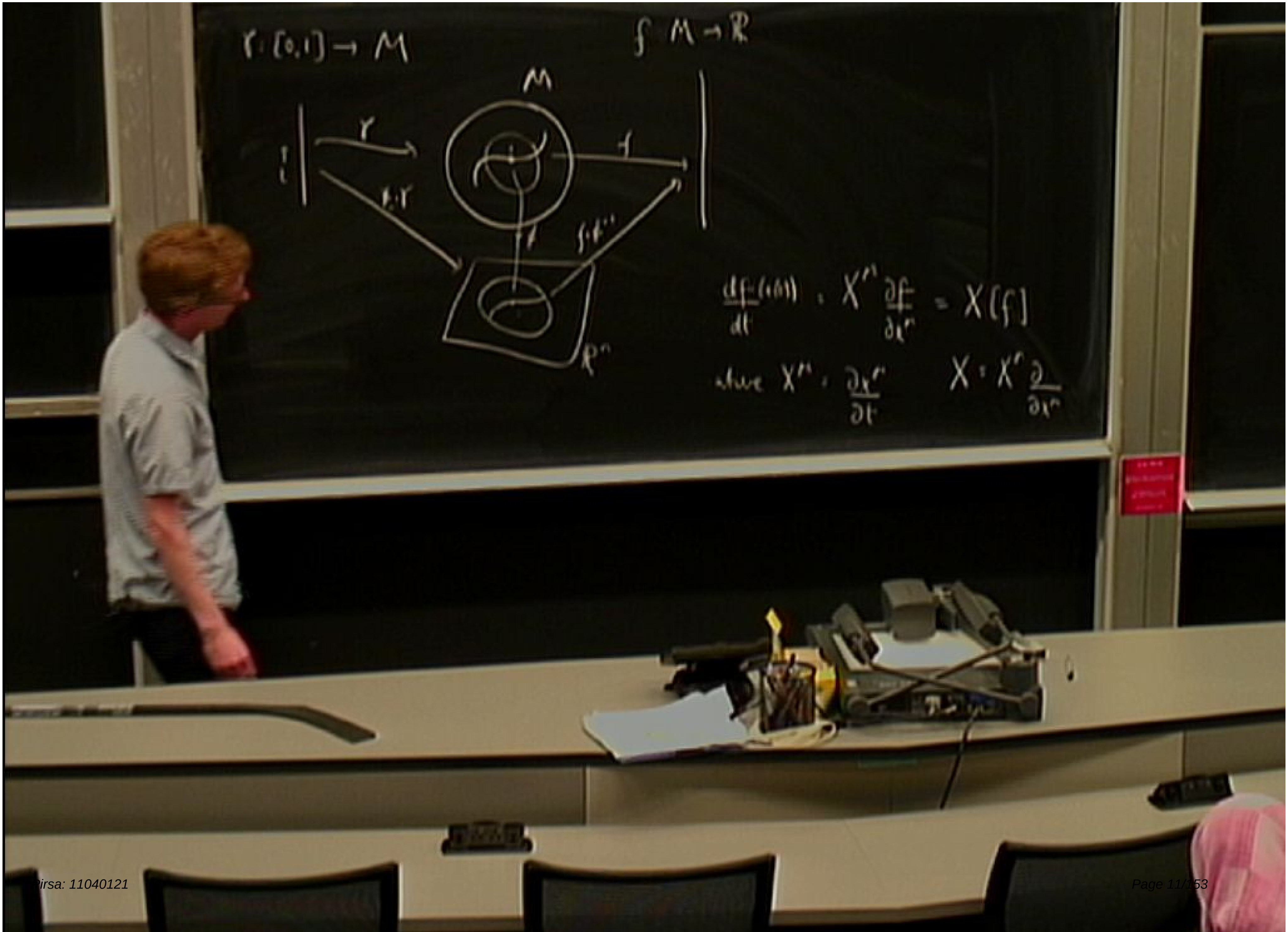

γ: [0,1] → M
f: M → ℝ
M
γ
f
ψ
ℝ^n


γ: [0,1] → M
f: M → ℝ
M
γ
f
ℝⁿ


γ: [0,1] → M
f: M → ℝ
M
γ
f
φ
f∘φ⁻¹
Rⁿ


M
γ
f
f∘γ
φ
f∘φ⁻¹
ℝⁿ
$\frac{df(\gamma(t))}{dt} = X^\mu \frac{\partial f}{\partial x^\mu}$
where $X^\mu = \frac{\partial x^\mu}{\partial t}$


γ : [0,1] → M
f : M → ℝ
M
γ
f
φ∘γ
φ
f∘φ⁻¹
ℝⁿ
df(γ(t))/dt = X^μ
where X^μ

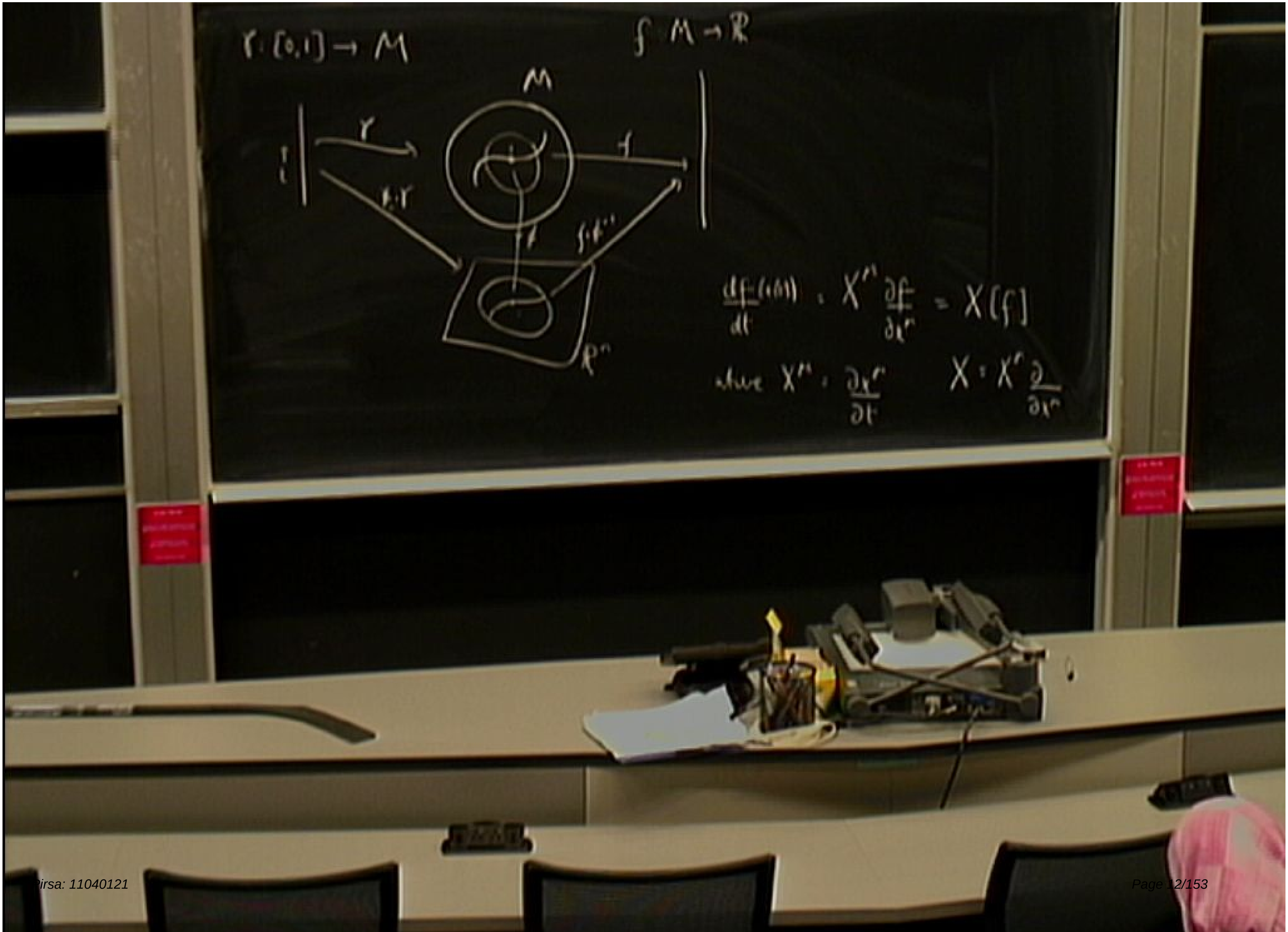


$\gamma : [0,1] \to M$

$f : M \to \mathbb{R}$

$$\frac{d f(\gamma(t))}{dt} = X^\mu \frac{\partial f}{\partial x^\mu} = X(f)$$

where $X^\mu = \frac{\partial x^\mu}{\partial t}$ $\quad X = X^\mu \frac{\partial}{\partial x^\mu}$



$\gamma : [0,1] \to M$

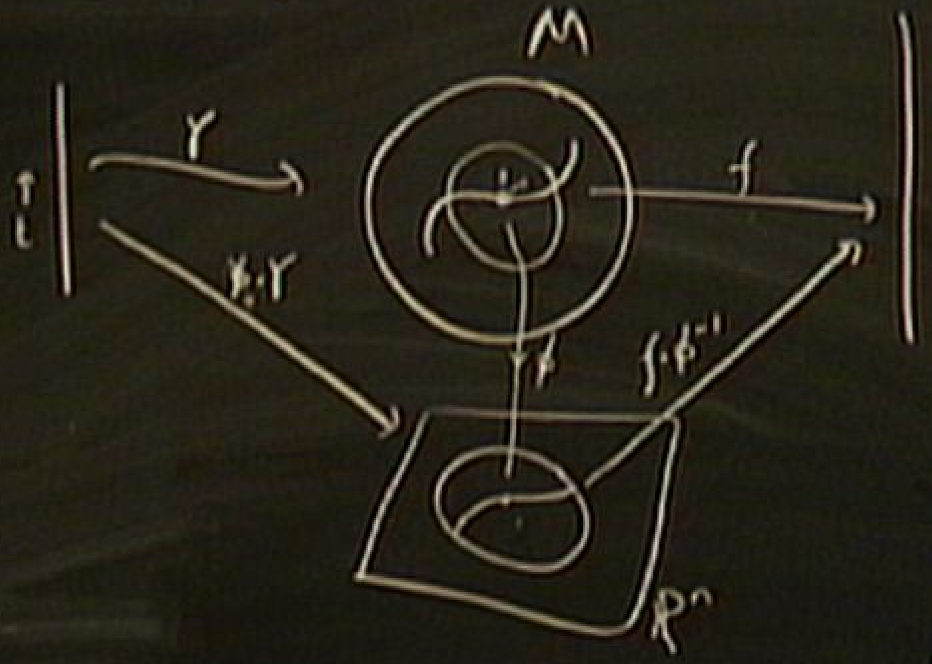
$f : M \to \mathbb{R}$

$$\frac{df(\gamma(t))}{dt} = X^\mu \frac{\partial f}{\partial x^\mu} = X[f]$$

where $X^\mu = \frac{\partial x^\mu}{\partial t}$ $\quad X = X^\mu \frac{\partial}{\partial x^\mu}$

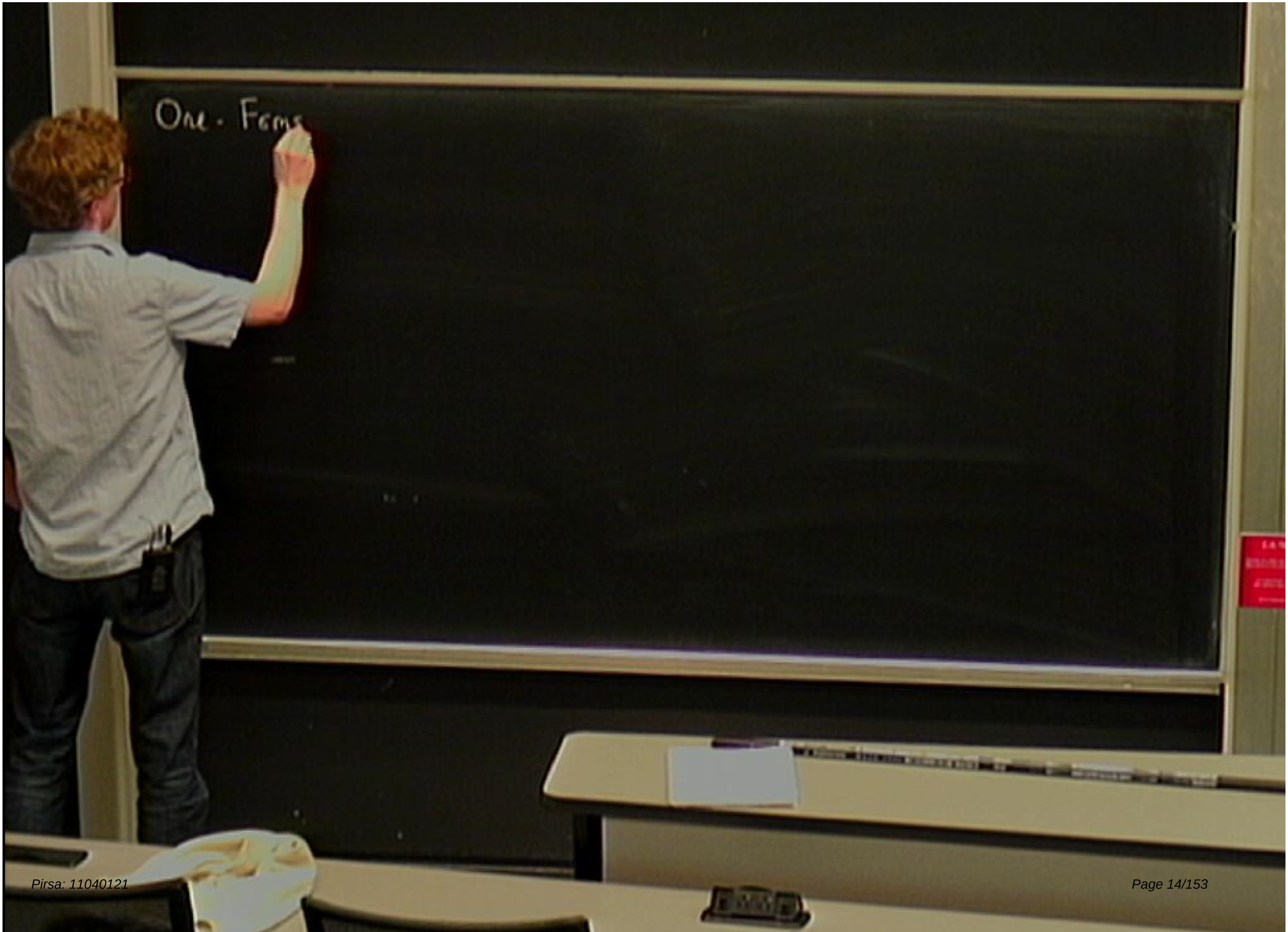
$\gamma : [0,1] \to M$

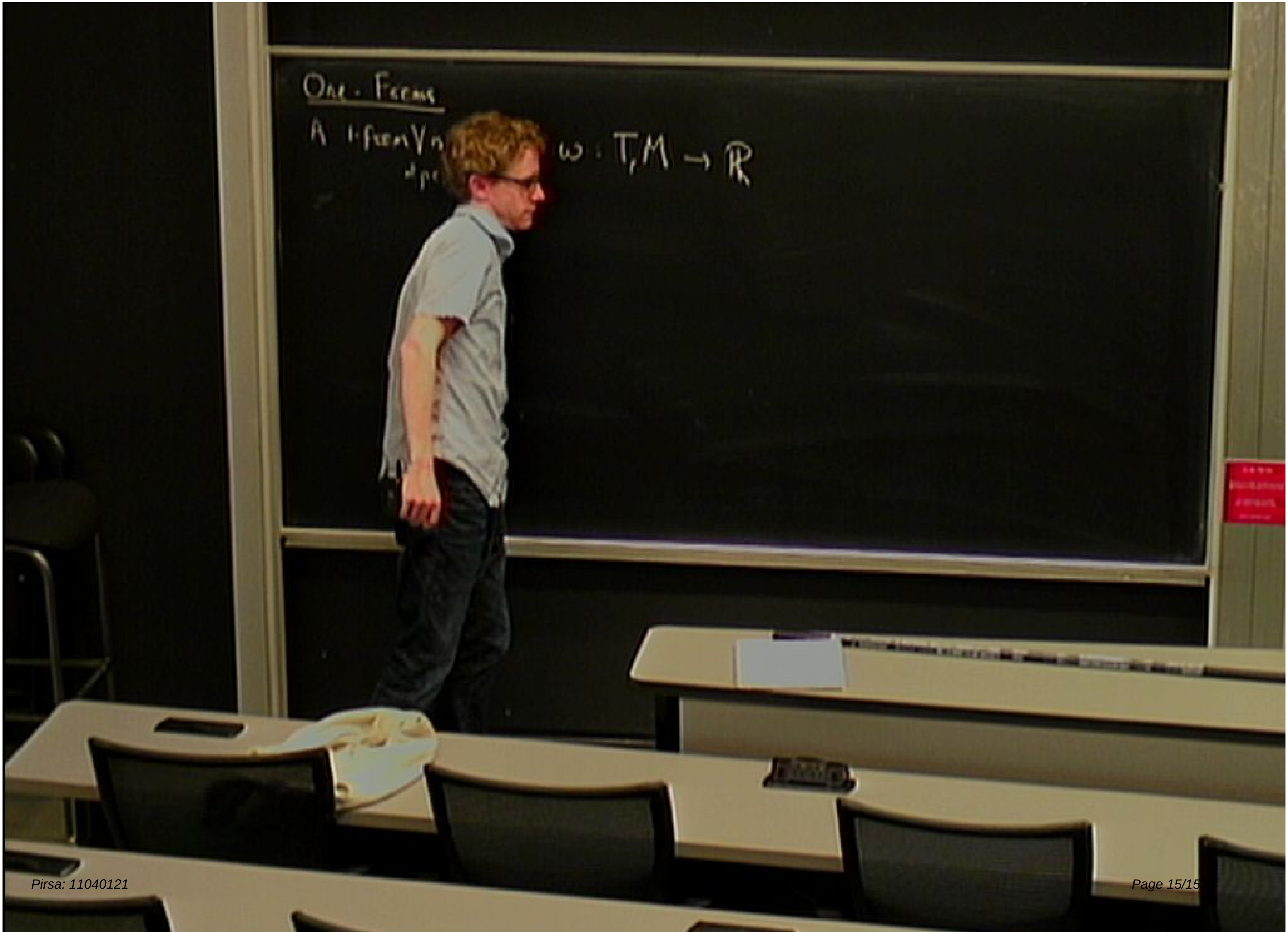
$f : M \to \mathbb{R}$

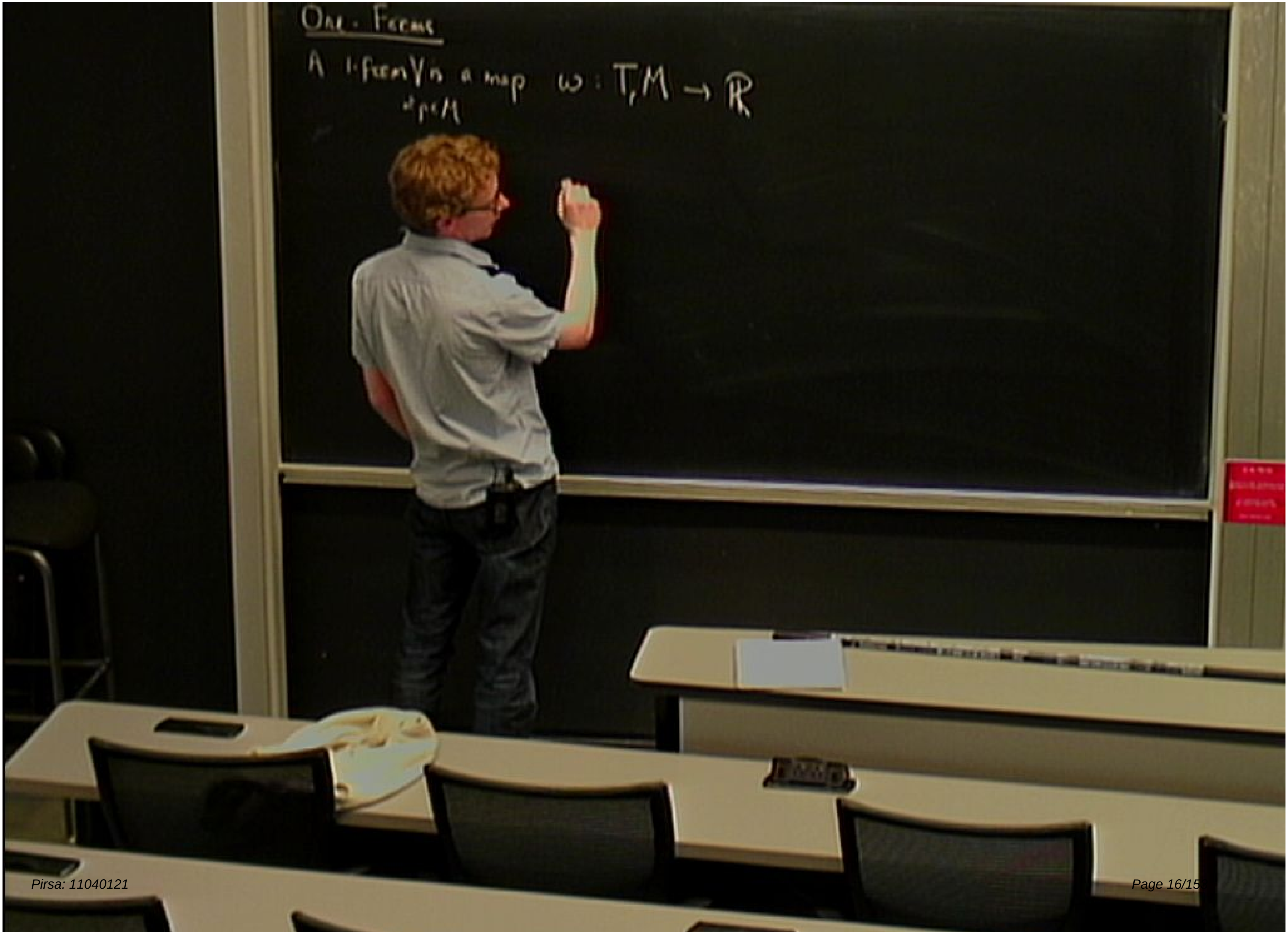


$$\frac{df}{dt}(x(t)) = X^\mu \frac{\partial f}{\partial x^\mu} = X[f]$$

where $X^\mu = \dfrac{\partial x^\mu}{\partial t}$ $\qquad X = X^\mu \dfrac{\partial}{\partial x^\mu}$

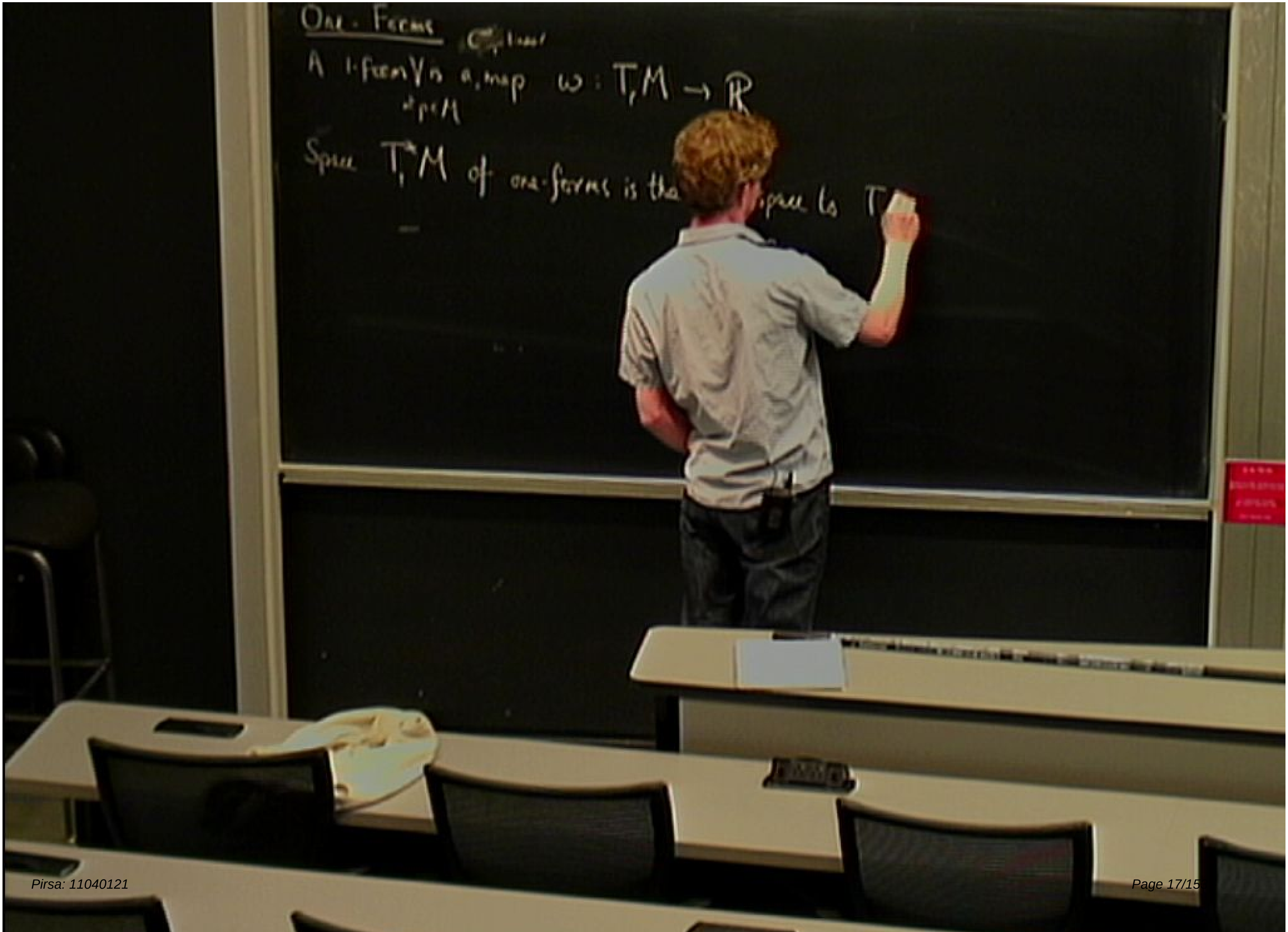

One - Form


One - Forms
A 1-form ∀ n
at p
$\omega : T_pM \to \mathbb{R}$



One-Forms

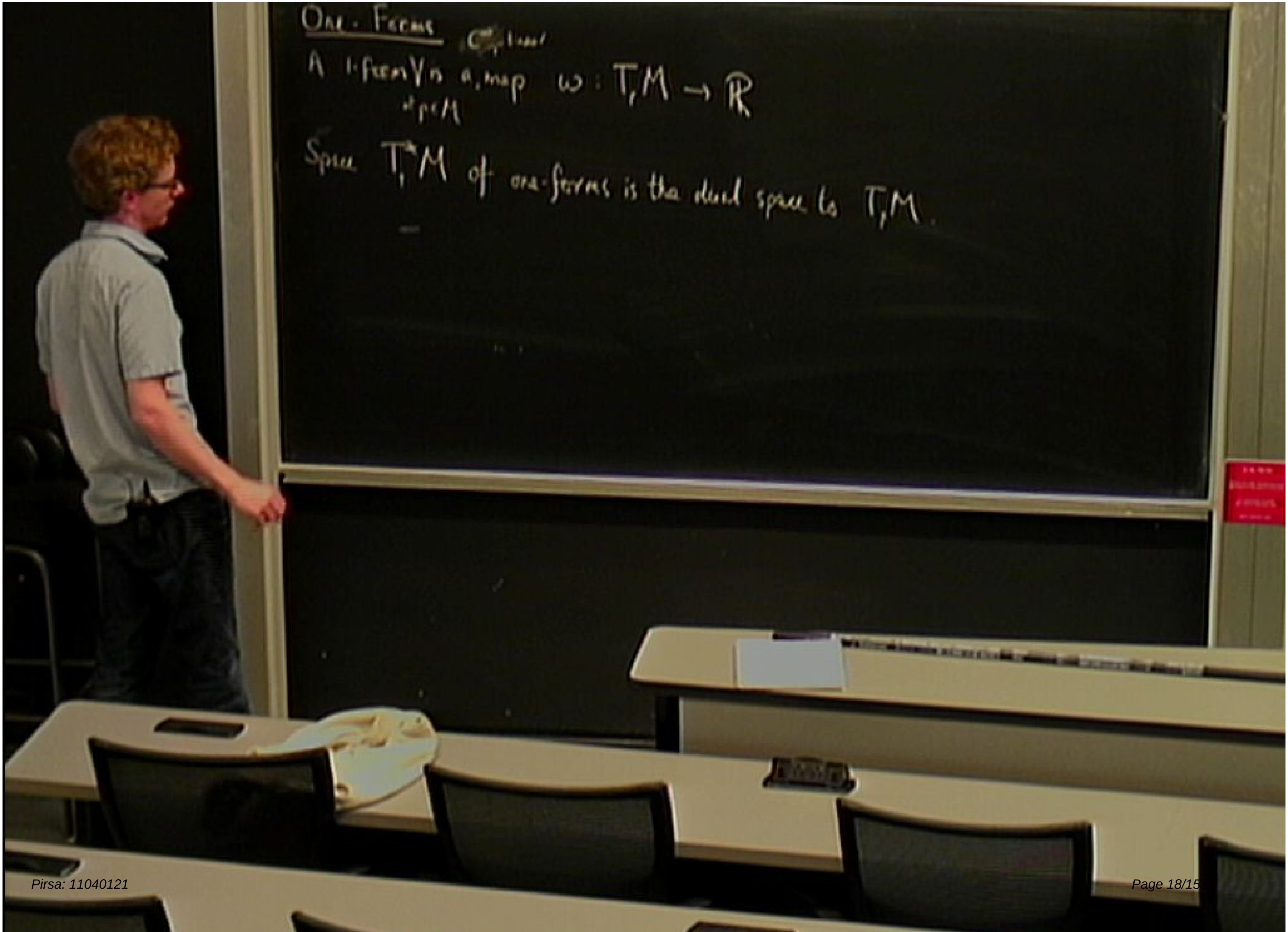
A 1-form at $p \in M$ is a map $\omega : T_pM \to \mathbb{R}$



One - Forms

A 1-form at $p \in M$ is a linear map $\omega : T_pM \to \mathbb{R}$

Space T_p^*M of one-forms is the space to T

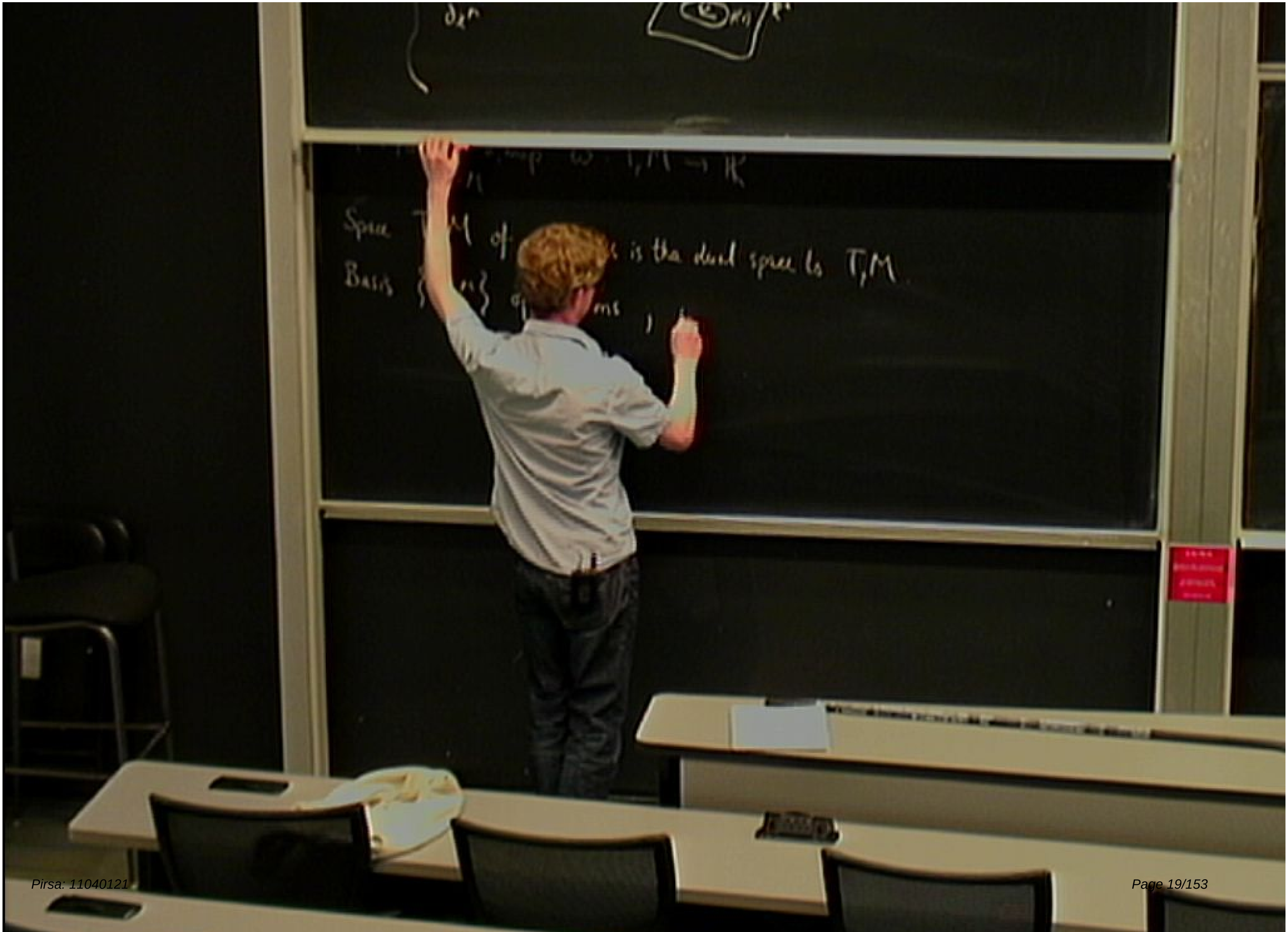


One-Forms

C^∞, local

A 1-form is a map $\omega : T_pM \to \mathbb{R}$
at $p \in M$

Space T_p^*M of one-forms is the dual space to T_pM.



map $\omega: T_pM \to \mathbb{R}$

Space T_p^*M of [illegible] is the dual space to T_pM

Basis $\{ \ldots \}$ of [illegible]ms ,

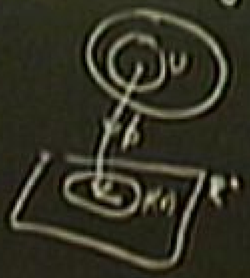
Next Lecture (by me!) Mon 25th May

Vectors

At $p \in M$, let T_pM is space of linearly independent vectors at p

Can pick basis: $\left\{ \frac{\partial}{\partial x^\mu} \right\}$

$V = V^\mu \frac{\partial}{\partial x^\mu}$

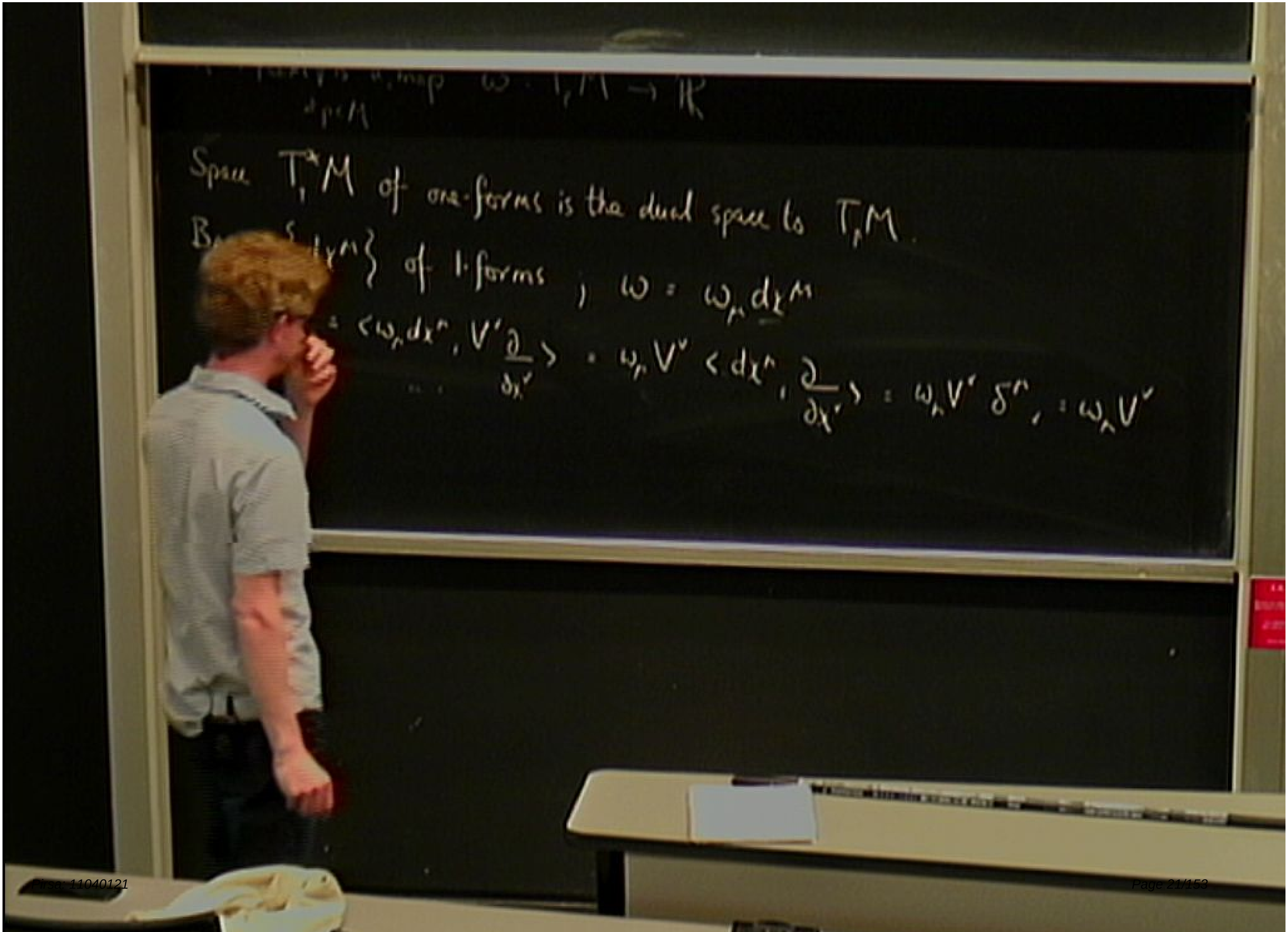


map $\omega : T_pM \to \mathbb{R}$

$p \in M$

Space T_p^*M of one-forms is the dual space to T_pM.

Basis $\{dx^\mu\}$ of 1-forms, $\omega = \omega_\mu dx^\mu$



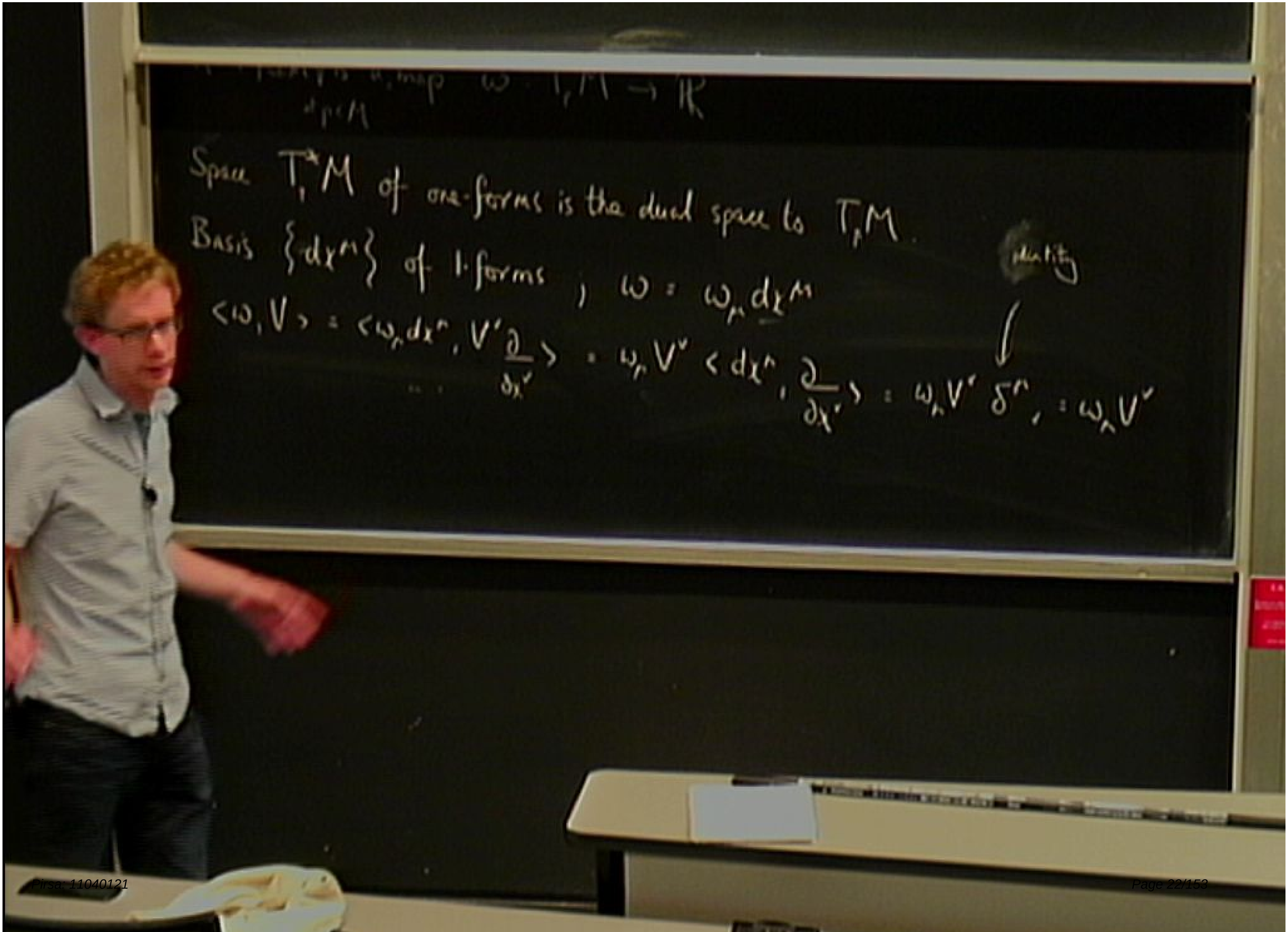
$\omega : T_pM \to \mathbb{R}$

at $p \in M$

Space T_p^*M of one-forms is the dual space to T_pM.

B… $\{dx^\mu\}$ of 1-forms , $\omega = \omega_\mu dx^\mu$

$$= \langle \omega_\mu dx^\mu , V^\nu \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \langle dx^\mu , \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \delta^\mu{}_\nu = \omega_\mu V^\mu$$



is a map $\omega : T_pM \to \mathbb{R}$

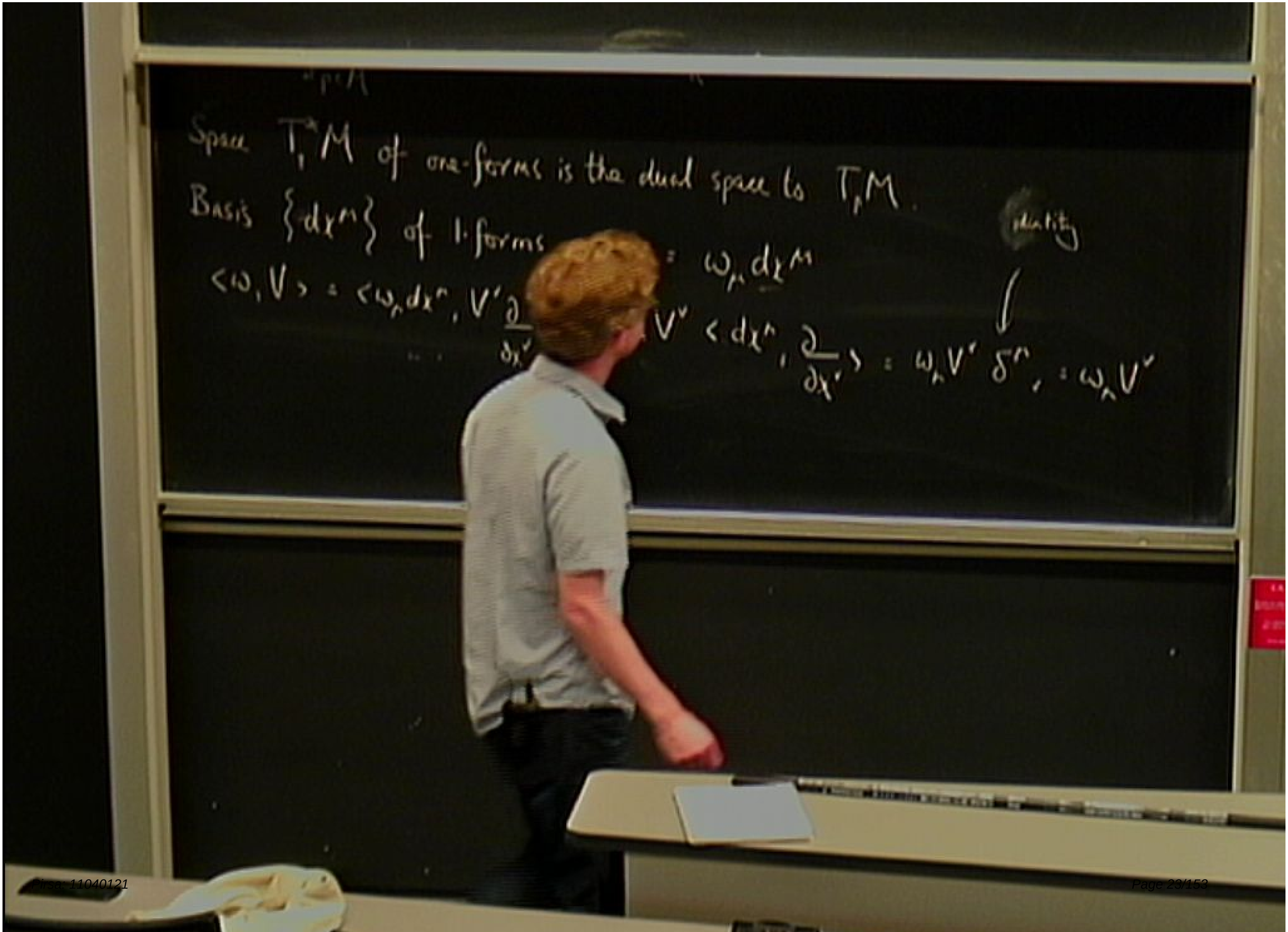
$\forall p \in M$

Space T_p^*M of one-forms is the dual space to T_pM.

Basis $\{dx^\mu\}$ of 1-forms , $\omega = \omega_\mu dx^\mu$

duality

$$\langle \omega, V \rangle = \langle \omega_\mu dx^\mu, V^\nu \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \delta^\mu{}_\nu = \omega_\mu V^\mu$$

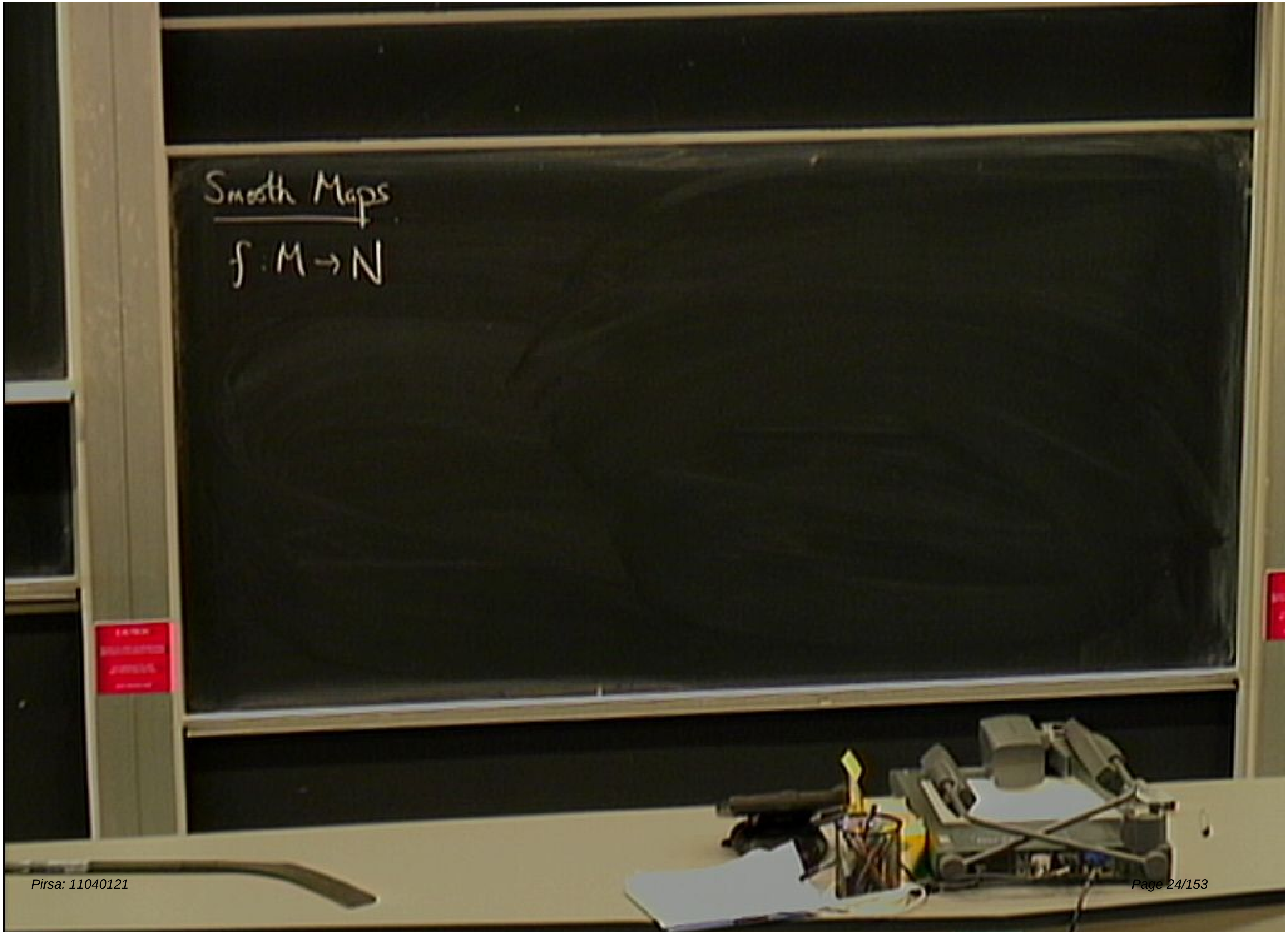


Space T_p^*M of one-forms is the dual space to T_pM.

Basis $\{dx^\mu\}$ of 1-forms $= \omega_\mu dx^\mu$

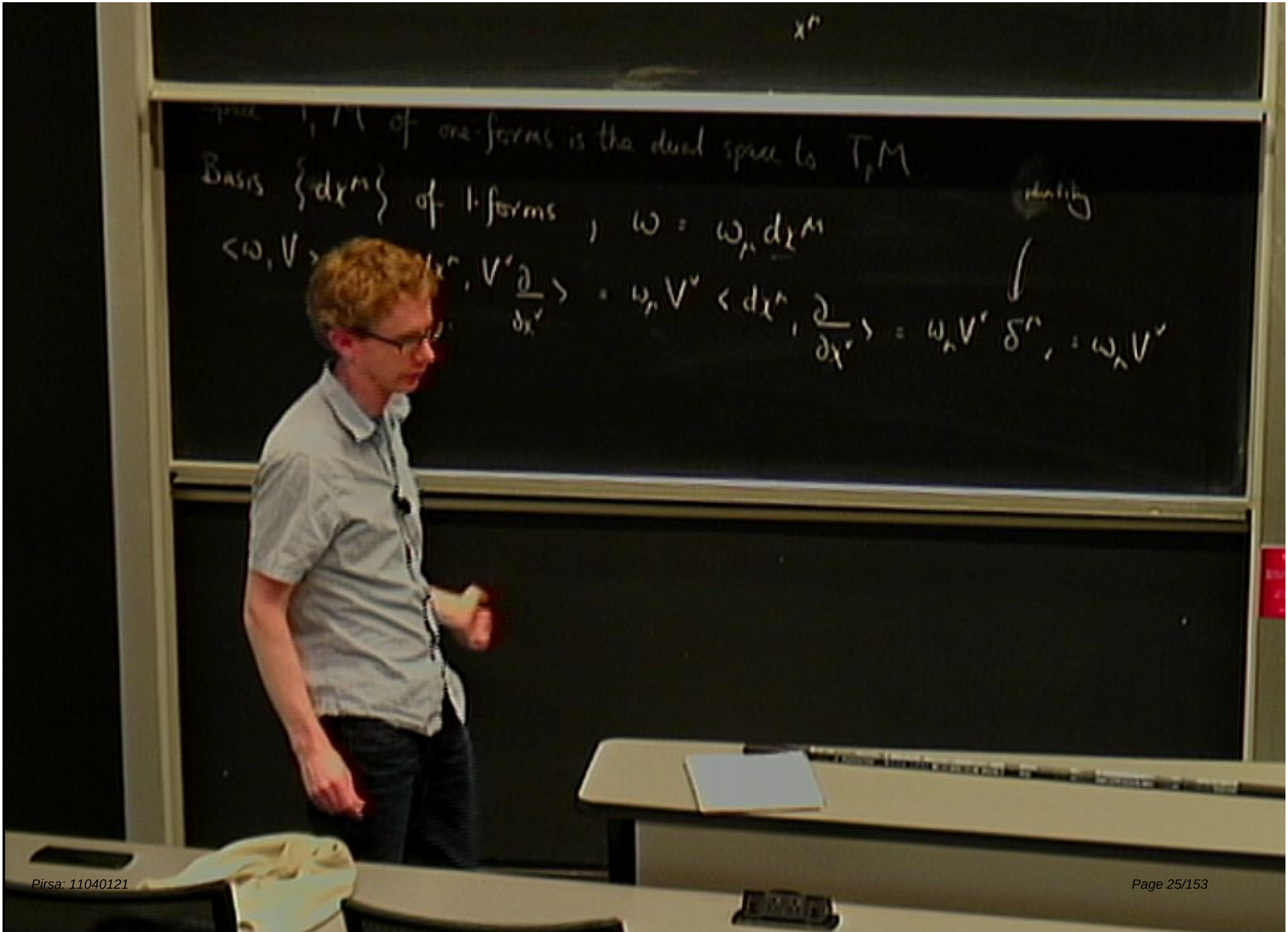
identity

$\langle \omega, V \rangle = \langle \omega_\mu dx^\mu, V^\nu \frac{\partial}{\partial x^\nu}$ $V^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu$



Smooth Maps

$f : M \to N$



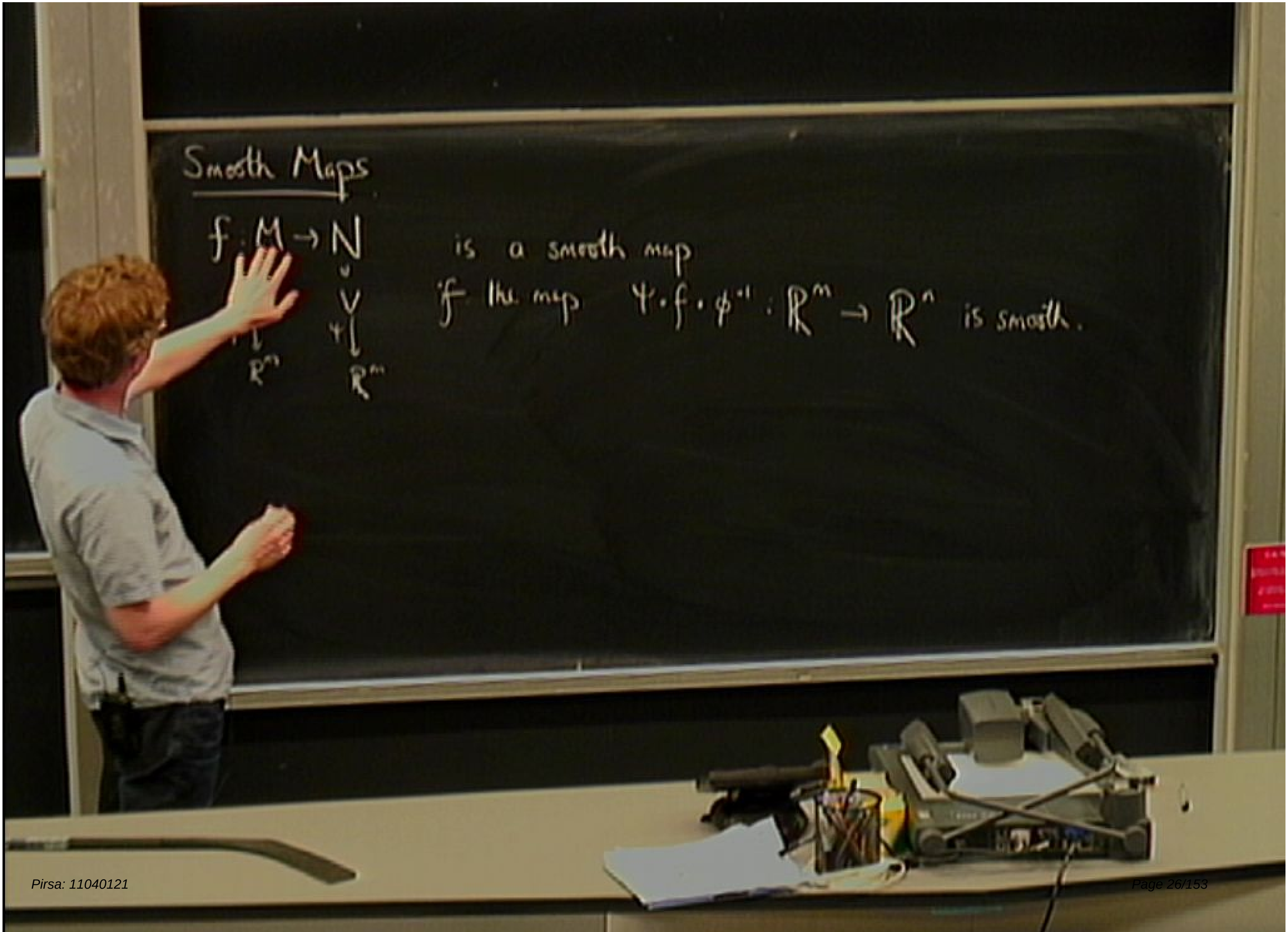
x^μ

space $T_p M$ of one-forms is the dual space to $T_p M$

Basis $\{dx^\mu\}$ of 1-forms , $\omega = \omega_\mu dx^\mu$

identity

$\langle \omega, V \rangle$ … $dx^\mu, V^\nu \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \delta^\mu{}_\nu = \omega_\mu V^\mu$

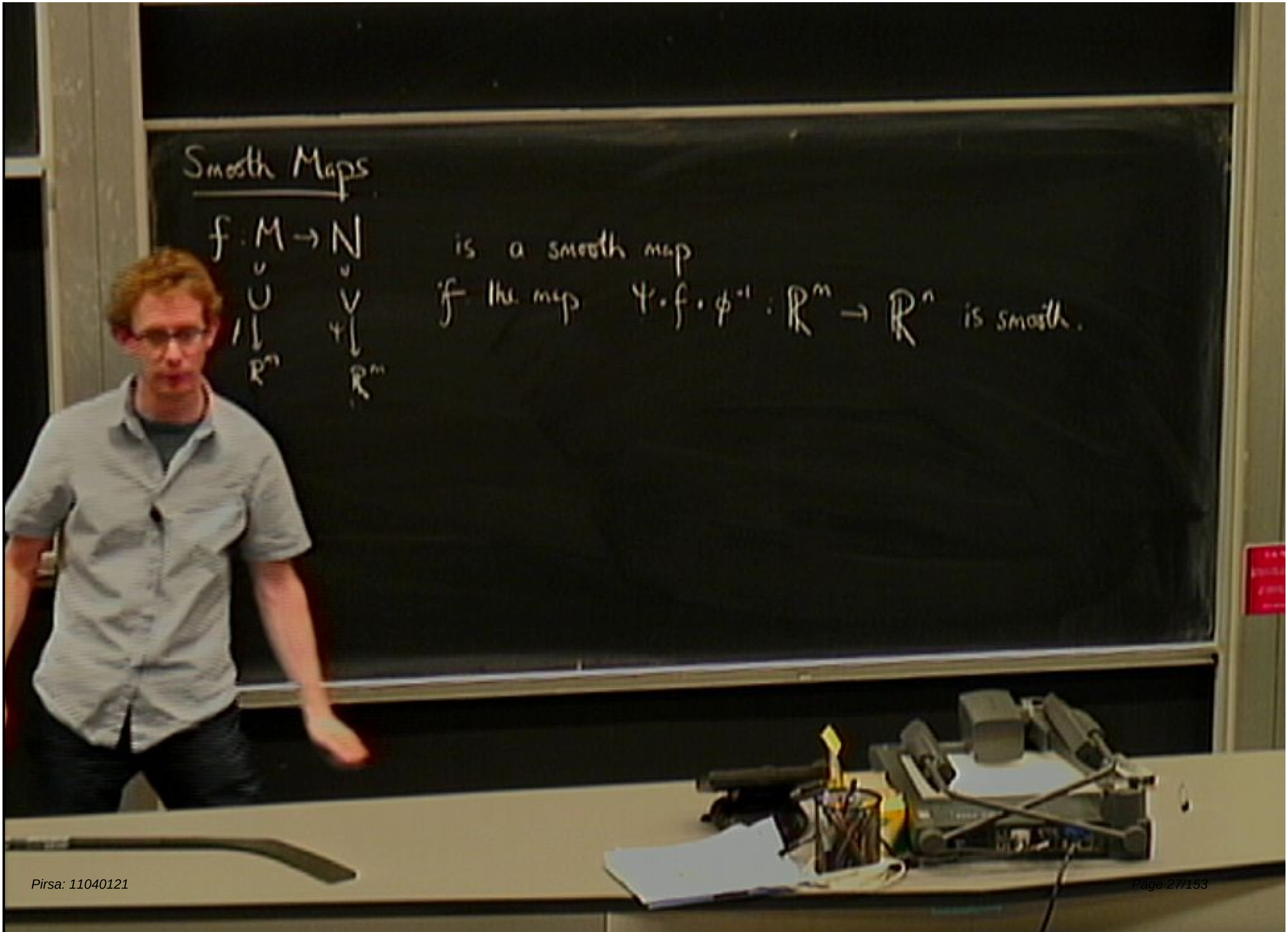


Smooth Maps

$f : M \to N$ is a smooth map

if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \\ & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"'] \\ \mathbb{R}^n & \mathbb{R}^m \end{tikzcd}$$

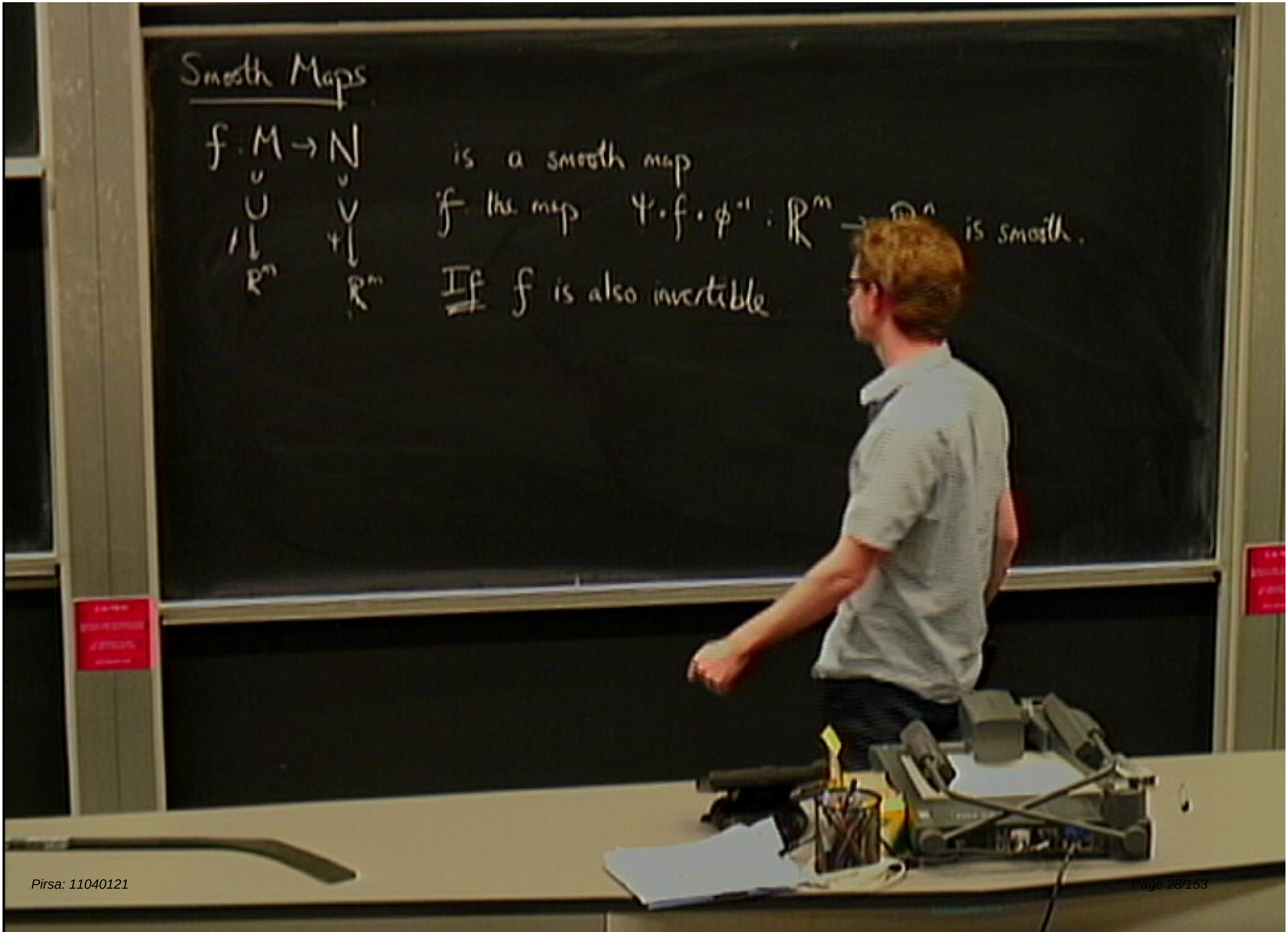


Smooth Maps

$$\begin{tikzcd} f : M \arrow[r] & N \\ U \arrow[u, phantom, "\cup"] \arrow[d, "\phi"'] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"'] \\ \mathbb{R}^m & \mathbb{R}^m \end{tikzcd}$$

is a smooth map

if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.



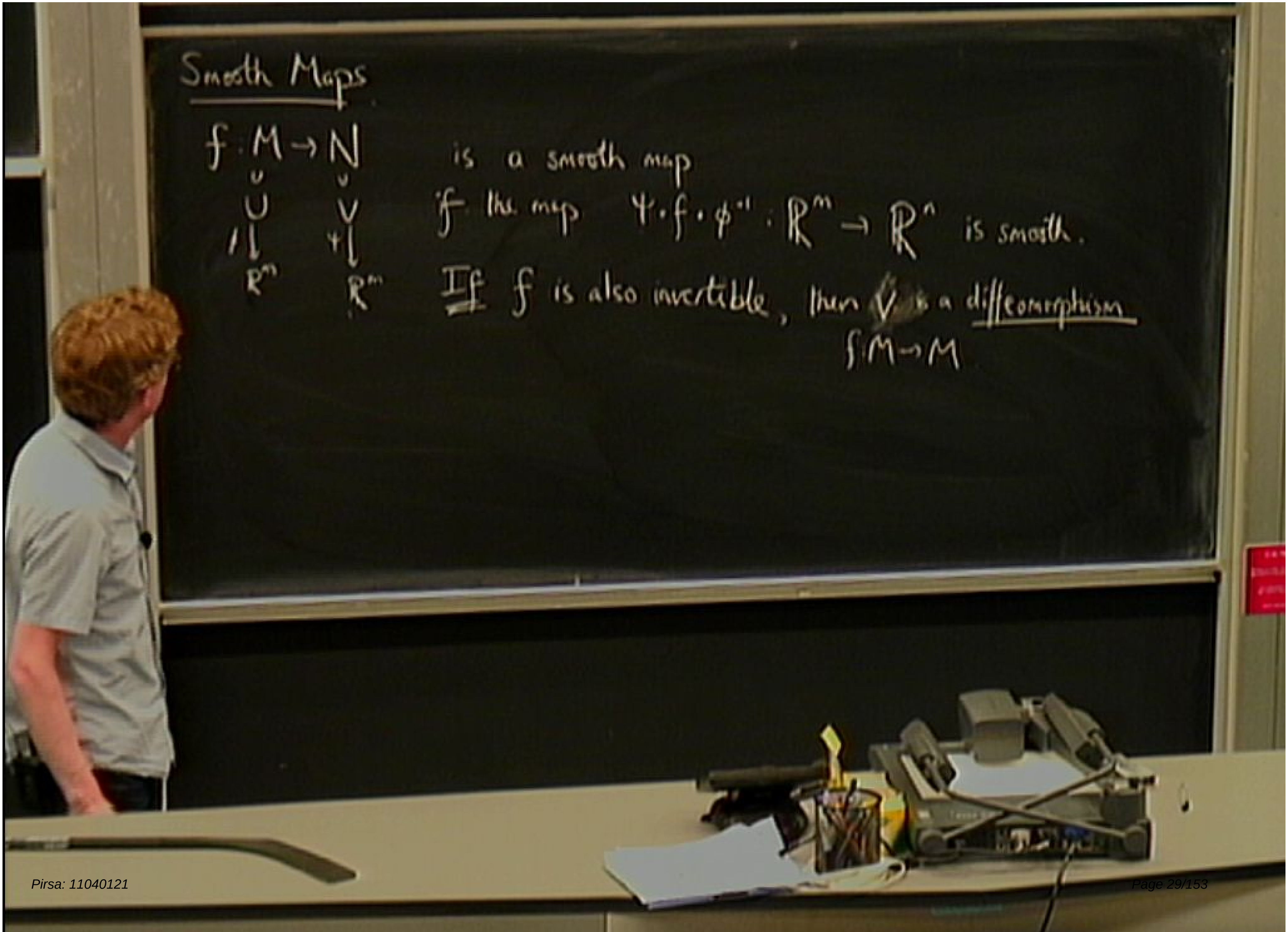
Smooth Maps

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, phantom, "\cup"] \arrow[d, "\phi"'] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"'] \\ \mathbb{R}^m & \mathbb{R}^n \end{tikzcd}$$

is a smooth map

iff the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

If f is also invertible



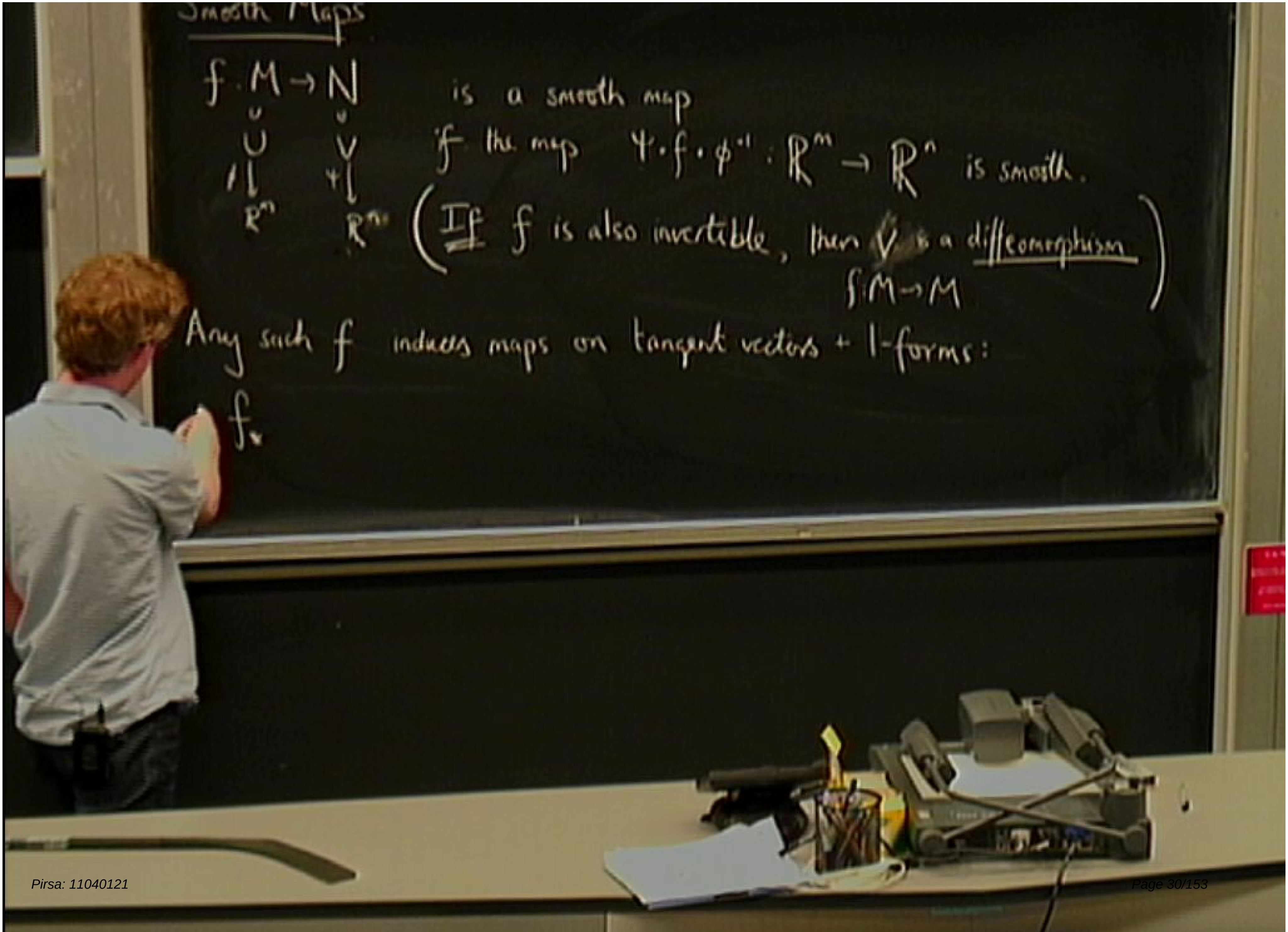
Smooth Maps

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, phantom, "\cup" description] \arrow[d, "\phi"'] & V \arrow[u, phantom, "\cup" description] \arrow[d, "\psi"] \\ \mathbb{R}^m & \mathbb{R}^m \end{tikzcd}$$

$f : M \to N$ is a smooth map if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

If f is also invertible, then V is a diffeomorphism

$f : M \to M$



Smooth Maps

$$\begin{tikzcd} f: M \arrow[r] & N \\ U \arrow[u, phantom, "\cup"] \arrow[d, "\phi"'] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"] \\ \mathbb{R}^m & \mathbb{R}^n \end{tikzcd}$$

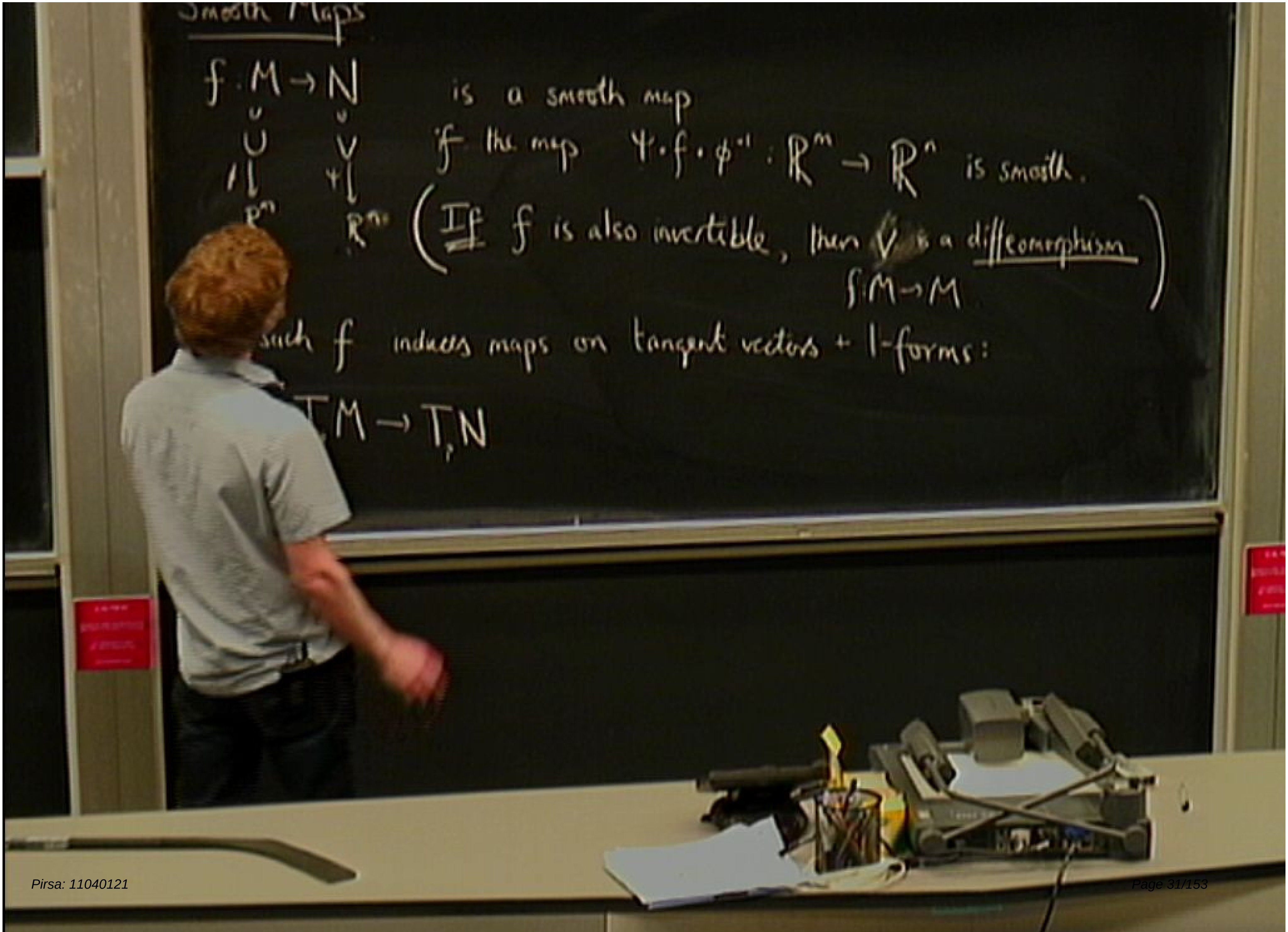
is a smooth map

if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

(**If** f is also invertible, then ψ is a <u>diffeomorphism</u>
$f: M \to M$)

Any such f induces maps on tangent vectors + 1-forms:

f_*



Smooth Maps

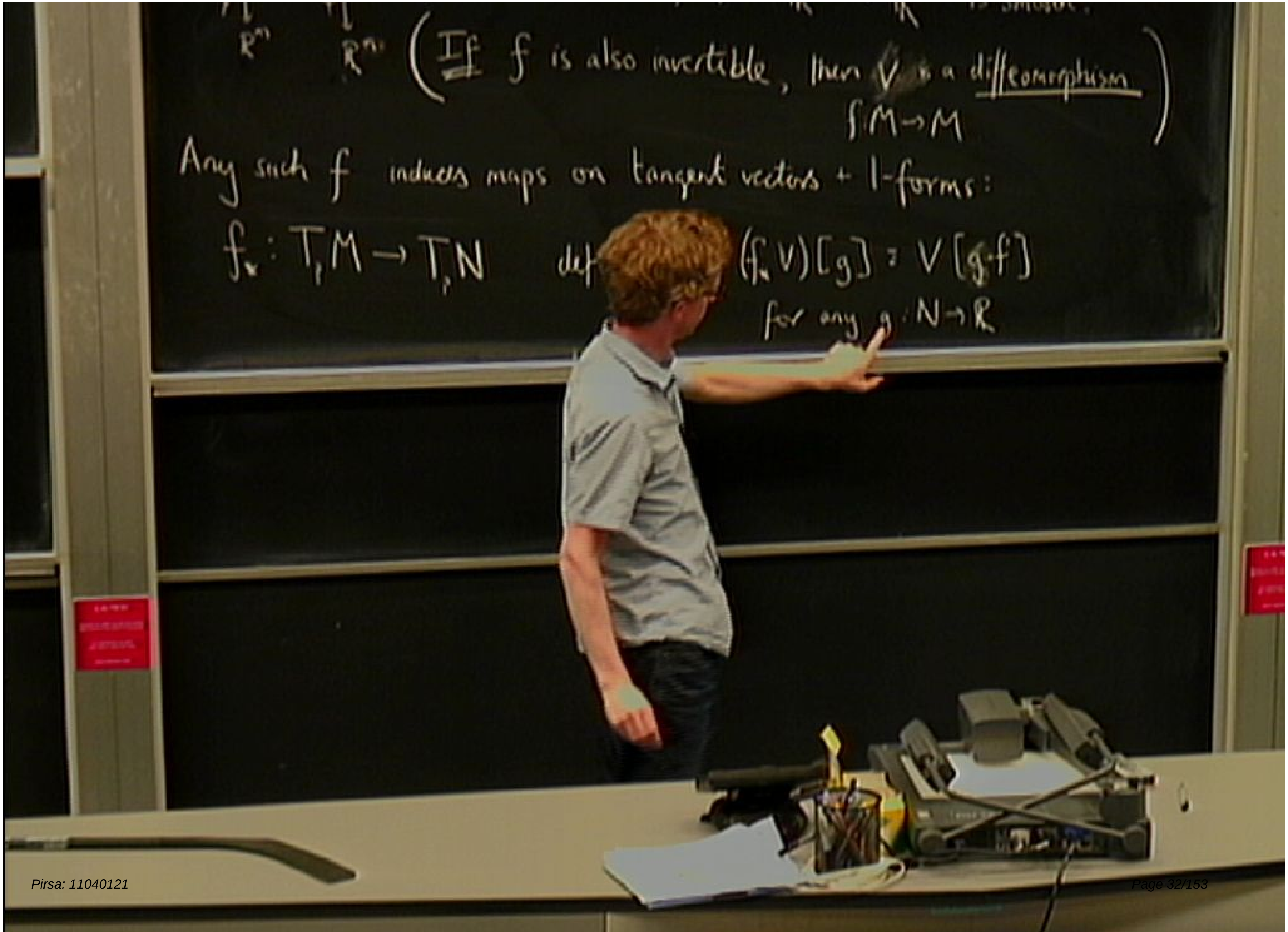
$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, phantom, sloped, "\subset"] \arrow[d, "\phi"'] & V \arrow[u, phantom, sloped, "\subset"] \arrow[d, "\psi"'] \\ \mathbb{R}^m & \mathbb{R}^n \end{tikzcd}$$

is a smooth map if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

(If f is also invertible, then [illegible] is a diffeomorphism $f: M \to M$)

such f induces maps on tangent vectors + 1-forms:

$TM \to T_pN$



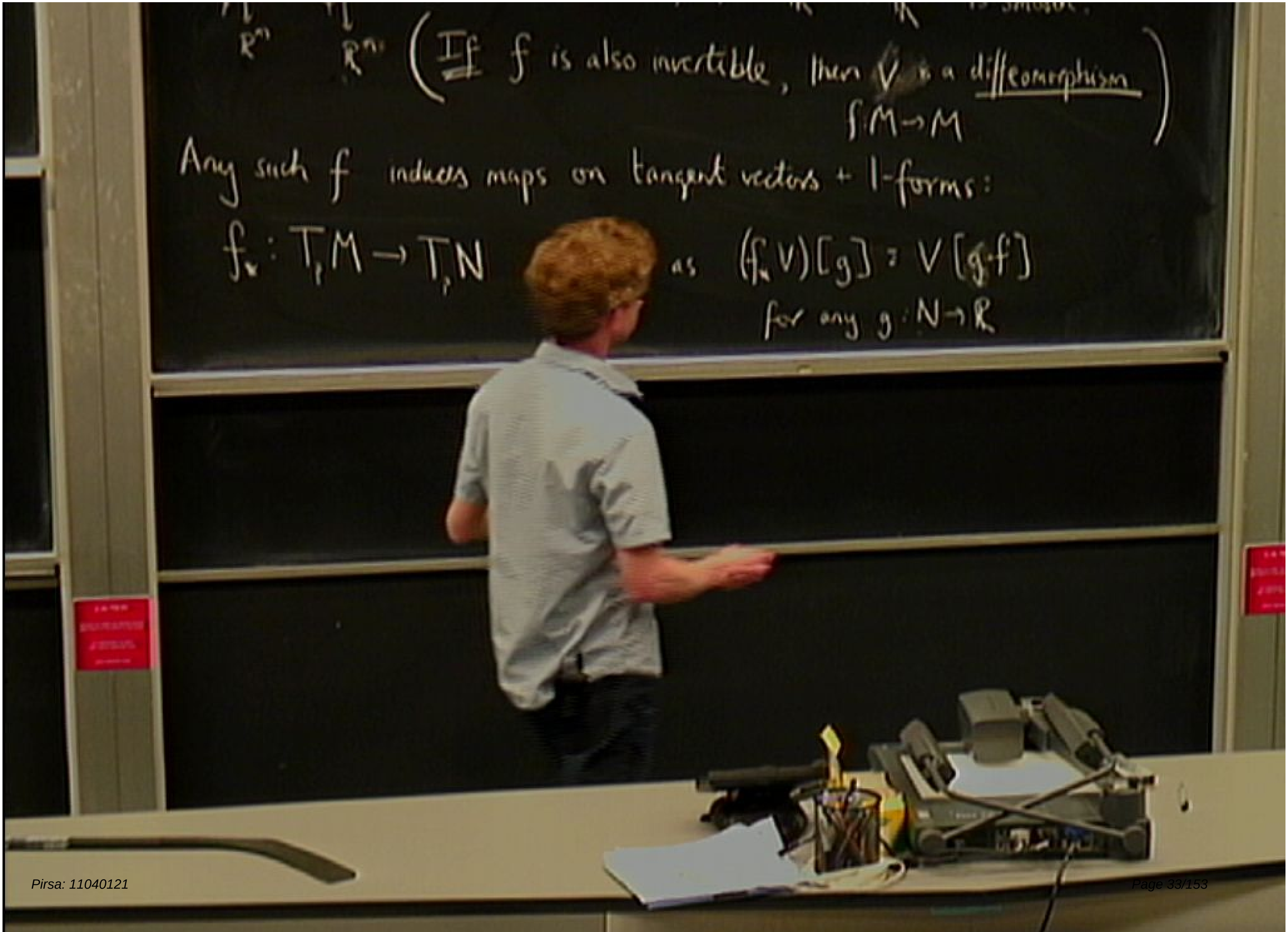
$\mathbb{R}^n$ $\mathbb{R}^m$ (If f is also invertible, then V is a diffeomorphism)

$f: M \to M$

Any such f induces maps on tangent vectors + 1-forms:

$f_*: T_p M \to T_p N$ def $(f_* V)[g] = V[g \cdot f]$

for any $g: N \to \mathbb{R}$



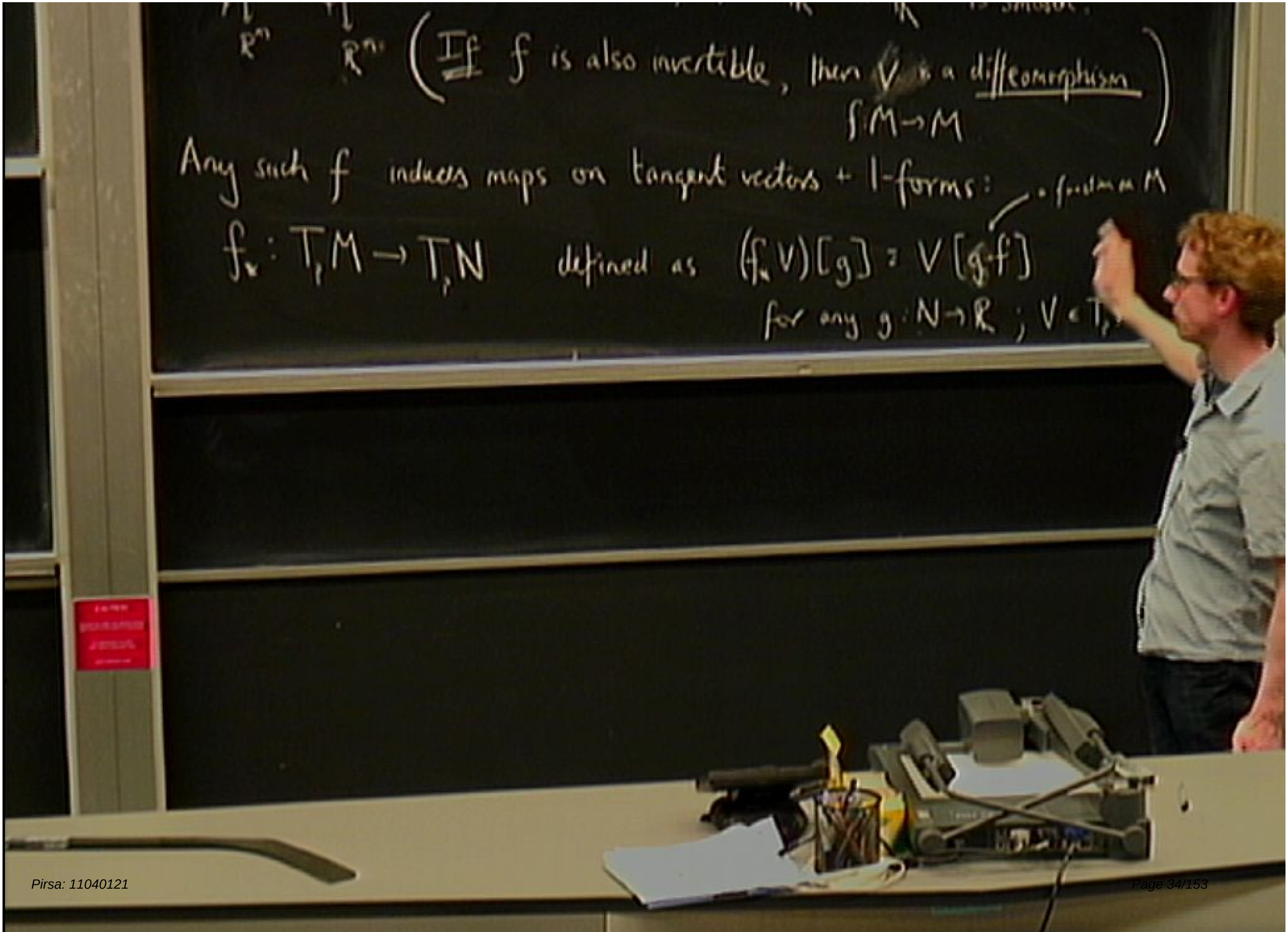
$\mathbb{R}^n$ $\mathbb{R}^m$ ... is smooth.

(If f is also invertible, then V is a <u>diffeomorphism</u> $f: M \to M$)

Any such f induces maps on tangent vectors + 1-forms:

$f_*: T_pM \to T_pN$ ... as $(f_* V)[g] = V[g \cdot f]$

for any $g: N \to \mathbb{R}$



$\mathbb{R}^n$ $\mathbb{R}^{n'}$ (If f is also invertible, then V is a diffeomorphism)

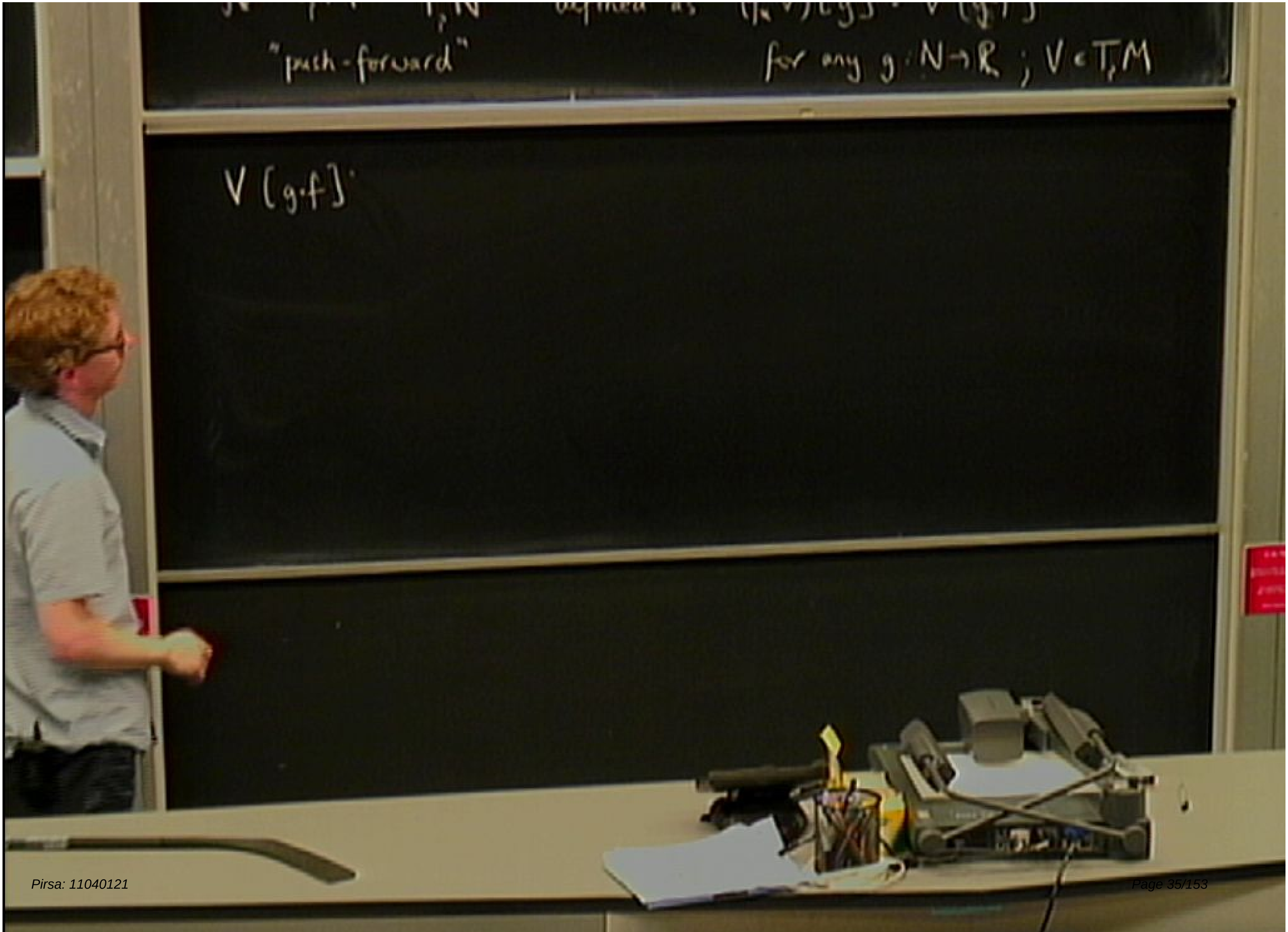
$f: M \to M$

Any such f induces maps on tangent vectors + 1-forms:

$\circ$ function on M

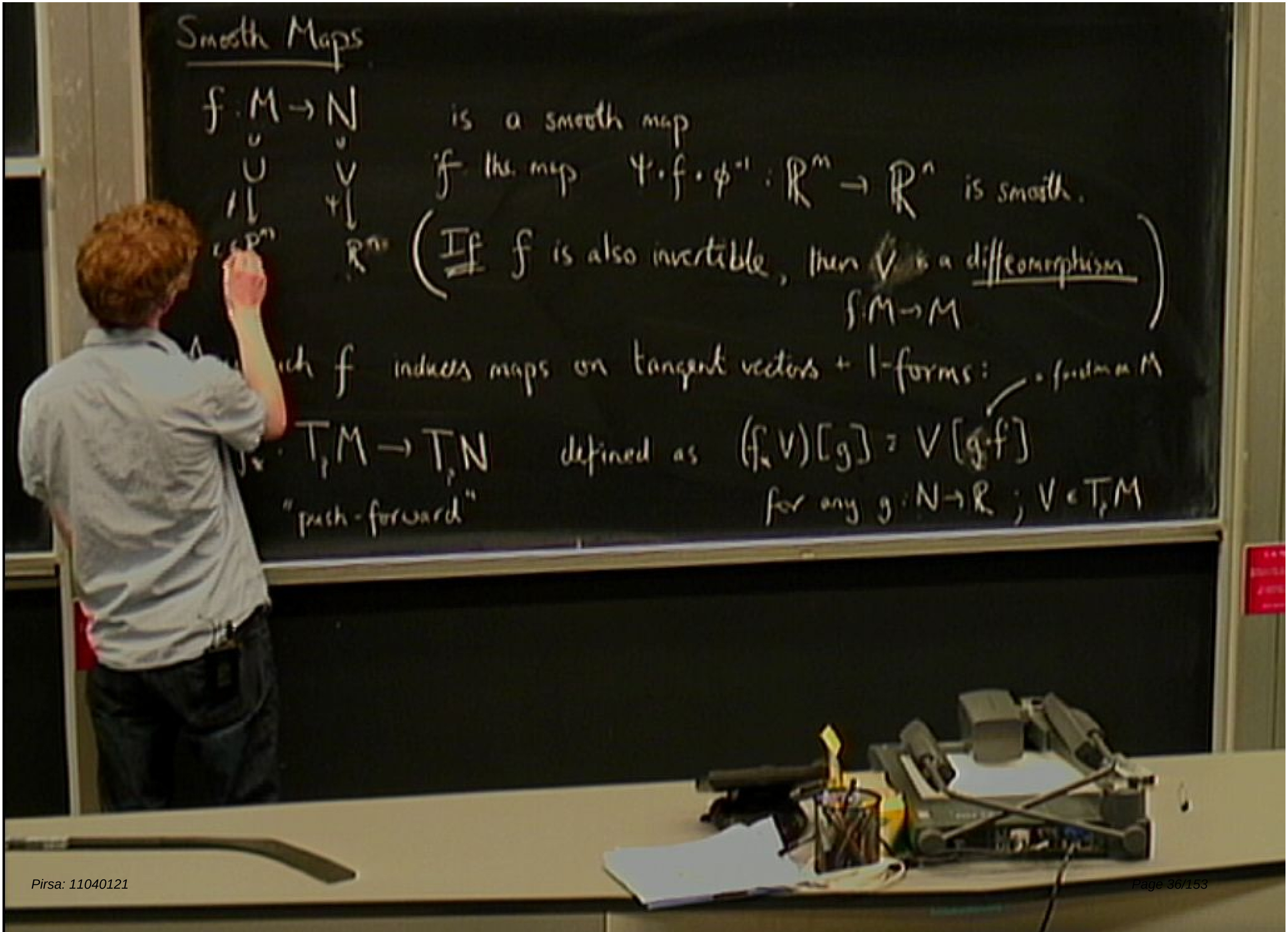
$f_*: T_pM \to T_pN$ defined as $(f_*V)[g] = V[g \circ f]$

for any $g: N \to \mathbb{R}$; $V \in T_p$



"push-forward" for any $g : N \to \mathbb{R}$; $V \in T_pM$

$V[g \circ f]$



Smooth Maps

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, phantom, "\cup"] & N \arrow[d, phantom, "\cup"] \\ U \arrow[d, "\phi"] & V \arrow[d, "\psi"] \\ \mathbb{R}^m & \mathbb{R}^n \end{tikzcd}$$

$f: M \to N$ is a smooth map

if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

(**If** f is also invertible, then [illegible] is a diffeomorphism $f: M \to M$)

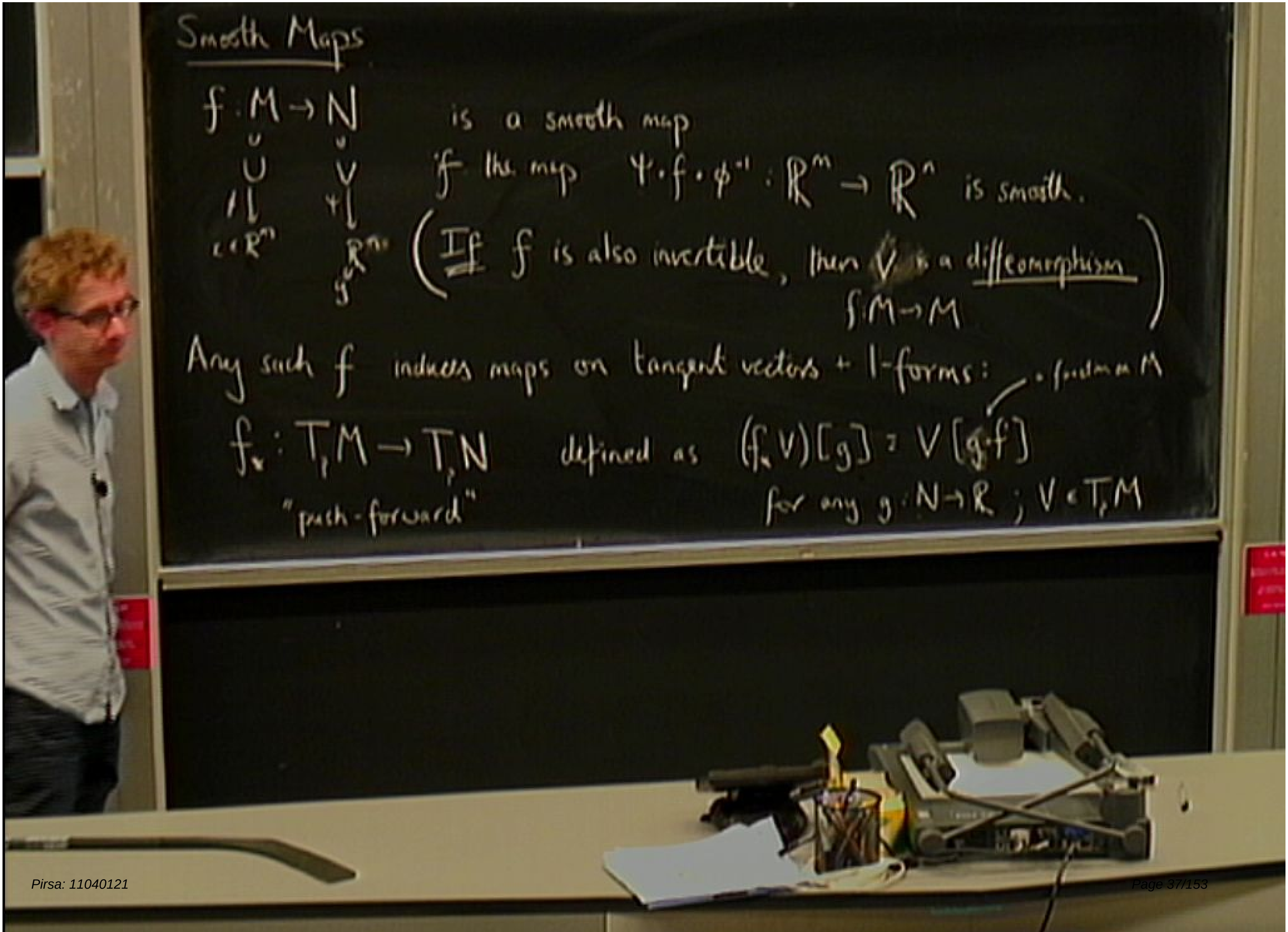
A[illegible]ch f induces maps on tangent vectors + 1-forms:

a fn defn on M

$f_* : T_pM \to T_pN$ defined as $(f_* V)[g] = V[g \circ f]$

"push-forward"

for any $g: N \to \mathbb{R}$; $V \in T_pM$



Smooth Maps

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, phantom, "\cup"] \arrow[d, "\phi"] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"] \\ \subset \mathbb{R}^m & \mathbb{R}^n \arrow[d, "g"] \\ & \end{tikzcd}$$

$f: M \to N$ is a smooth map

if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

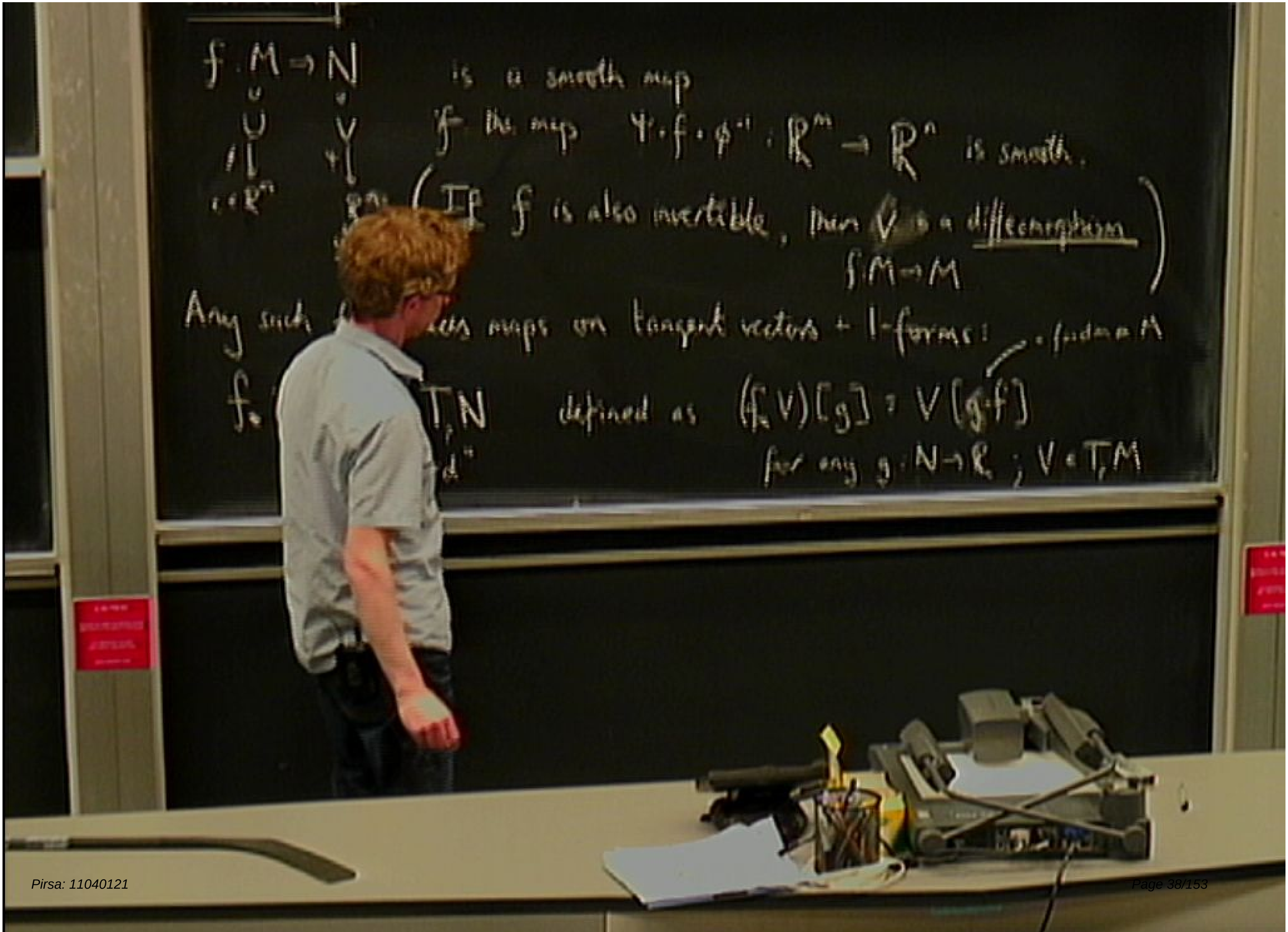
(**If** f is also invertible, then V is a diffeomorphism
$f: M \to M$)

Any such f induces maps on tangent vectors + 1-forms:

• function on M

$f_* : T_p M \to T_p N$ defined as $(f_* V)[g] = V[g \circ f]$

"push-forward" for any $g: N \to R$; $V \in T_p M$



$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, phantom, "\cup"] \arrow[d, "\phi"'] & V \arrow[u, phantom, "\cup"] \arrow[d, "\psi"] \\ \subset \mathbb{R}^m & \mathbb{R}^n \end{tikzcd}$$

$f: M \to N$ is a smooth map

if the map $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

(If f is also invertible, then f is a <u>diffeomorphism</u>)

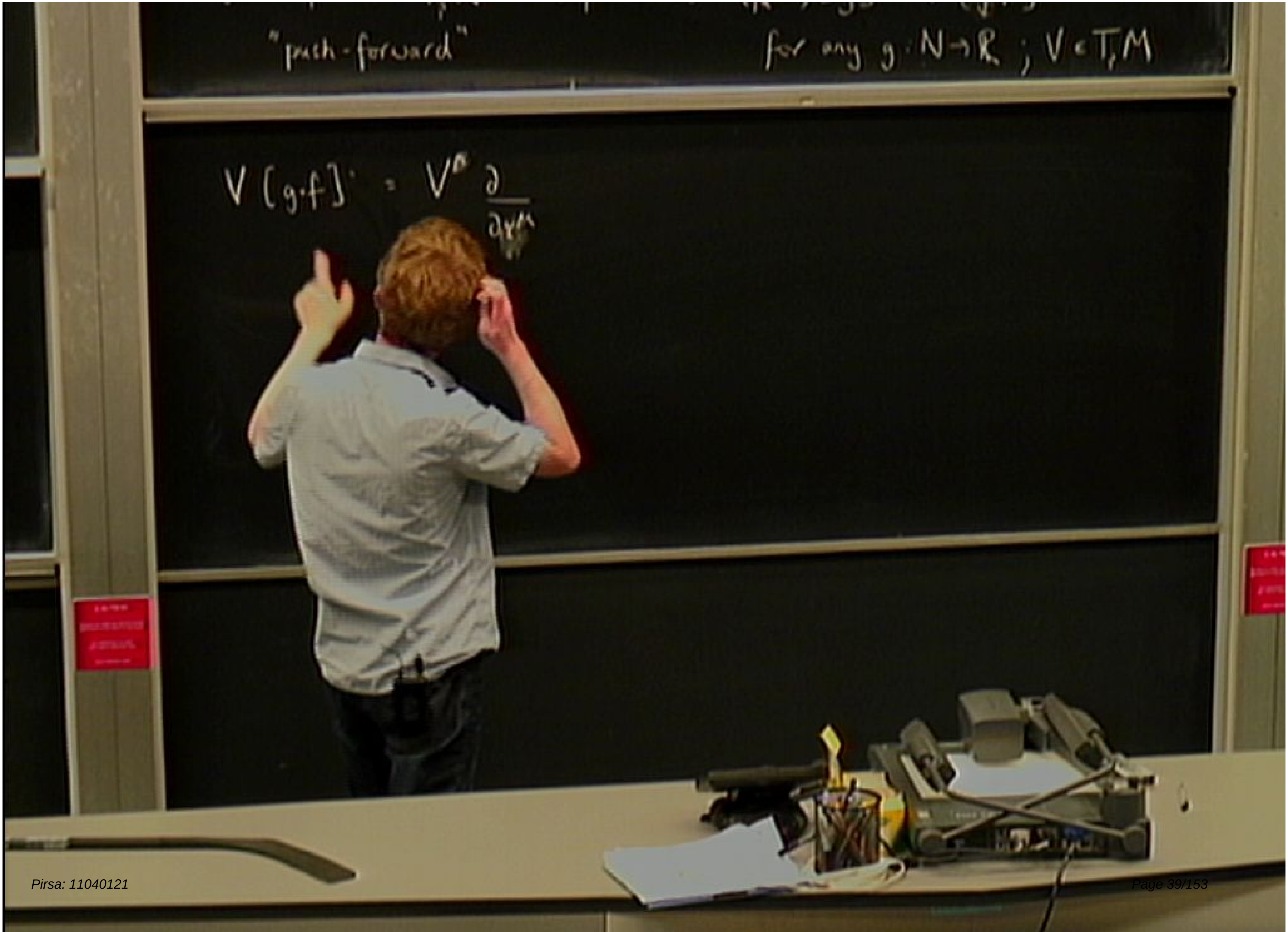
$f: M \to M$

Any such f [illegible]ces maps on tangent vectors + 1-forms: · f:dom = M

$f_* : \ldots T_{p}N$ defined as $(f_* V)[g] = V[g \circ f]$

d"

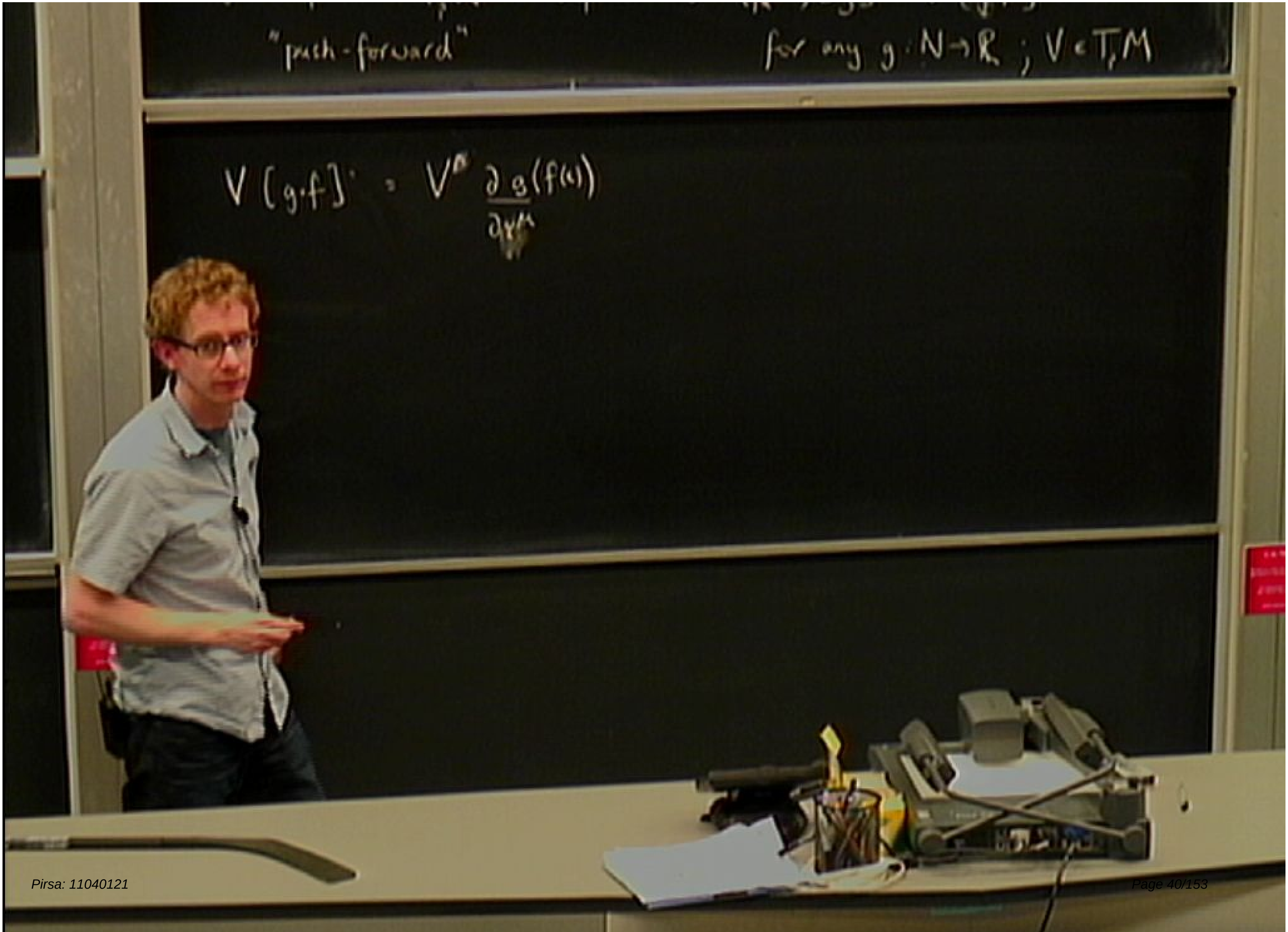
for any $g: N \to \mathbb{R}$; $V \in T_pM$



"push-forward"

for any $g : N \to \mathbb{R}$; $V \in T_pM$

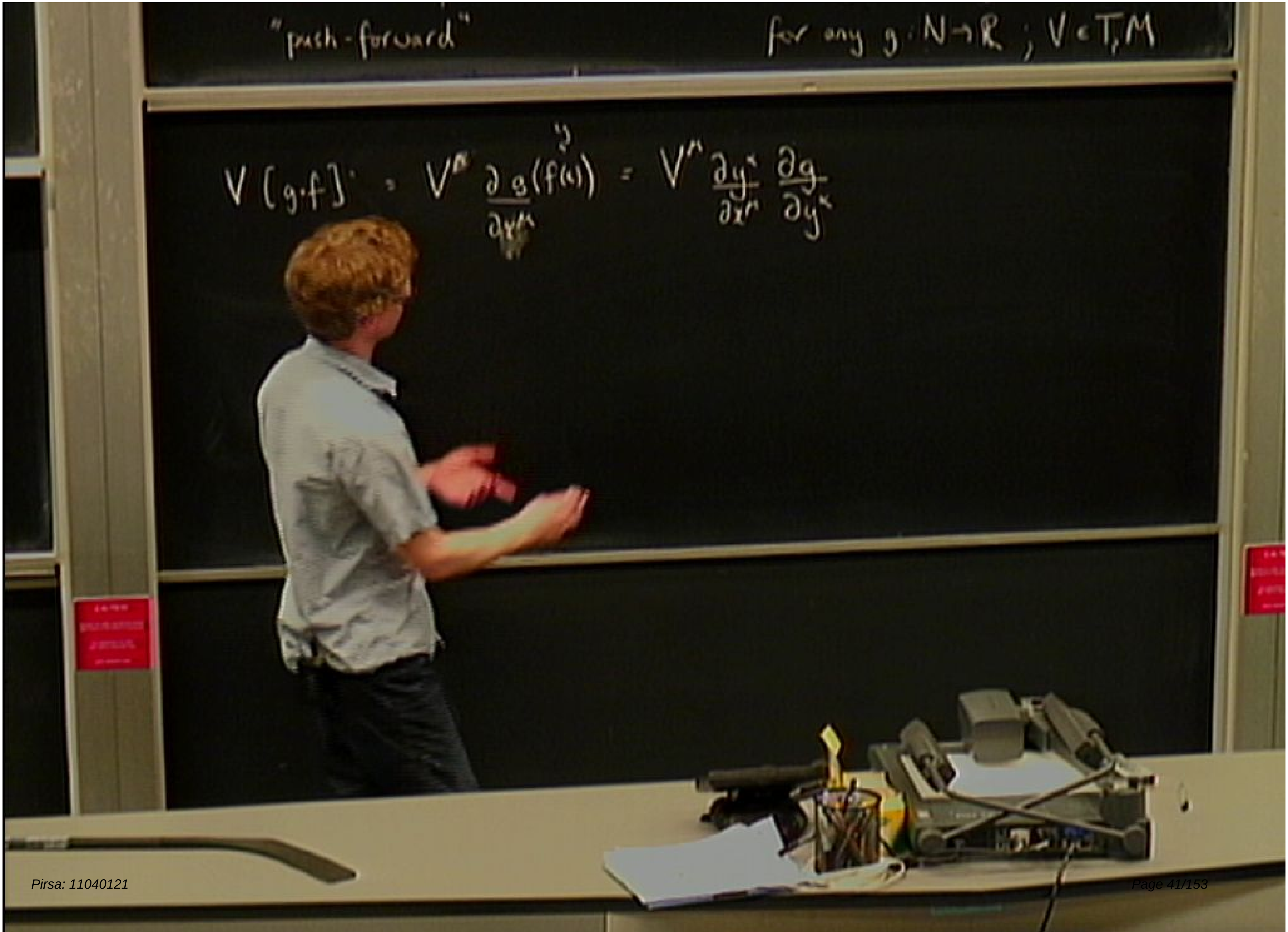
$V[g\circ f] = V^{\mu} \frac{\partial}{\partial x^{\mu}}$



"push-forward"

for any $g : N \to \mathbb{R}$; $V \in T_p M$

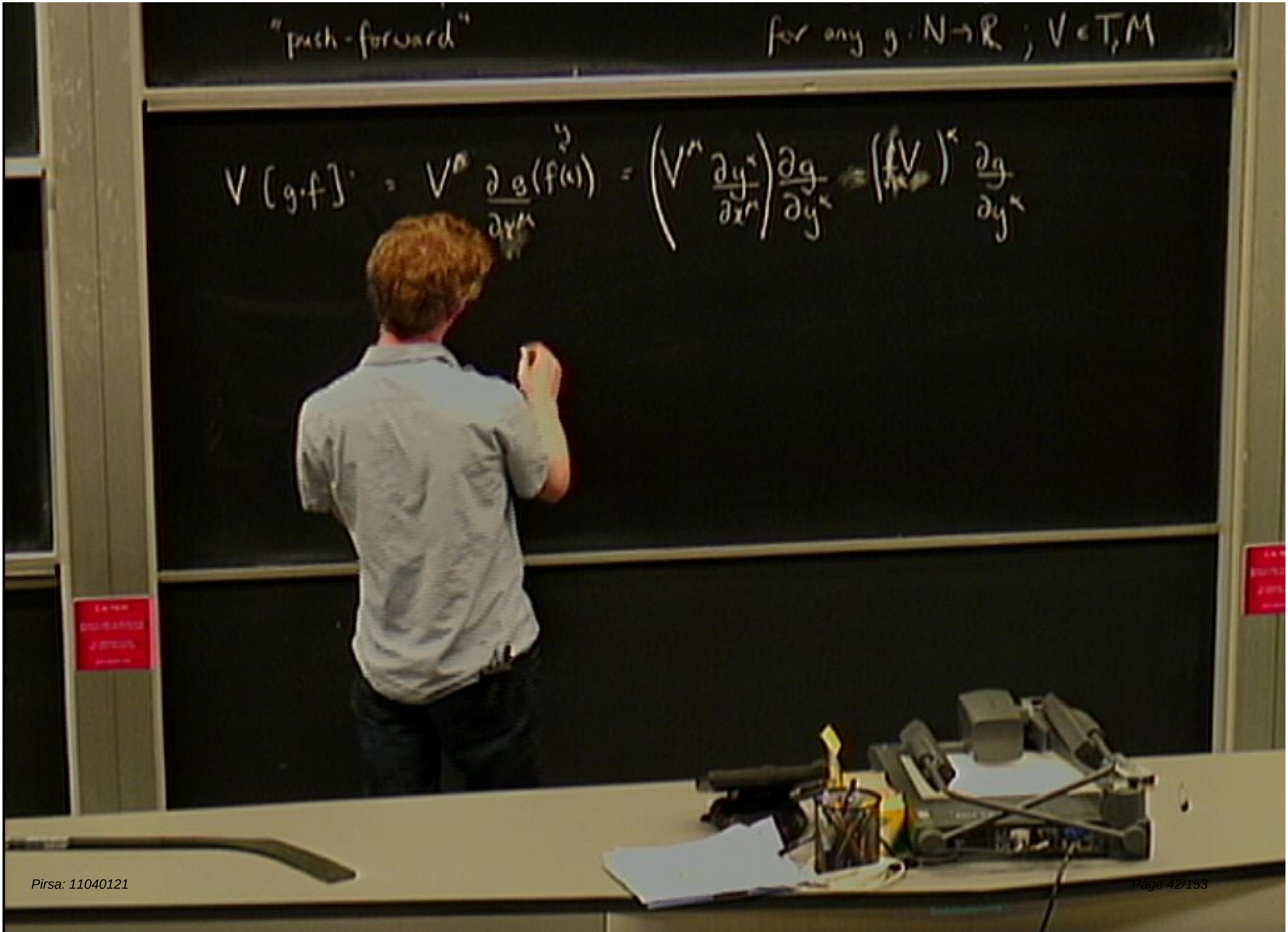
$$V[g \circ f] = V^\mu \frac{\partial g(f(x))}{\partial x^\mu}$$



"push-forward"

for any $g: N \to \mathbb{R}$; $V \in T_p M$

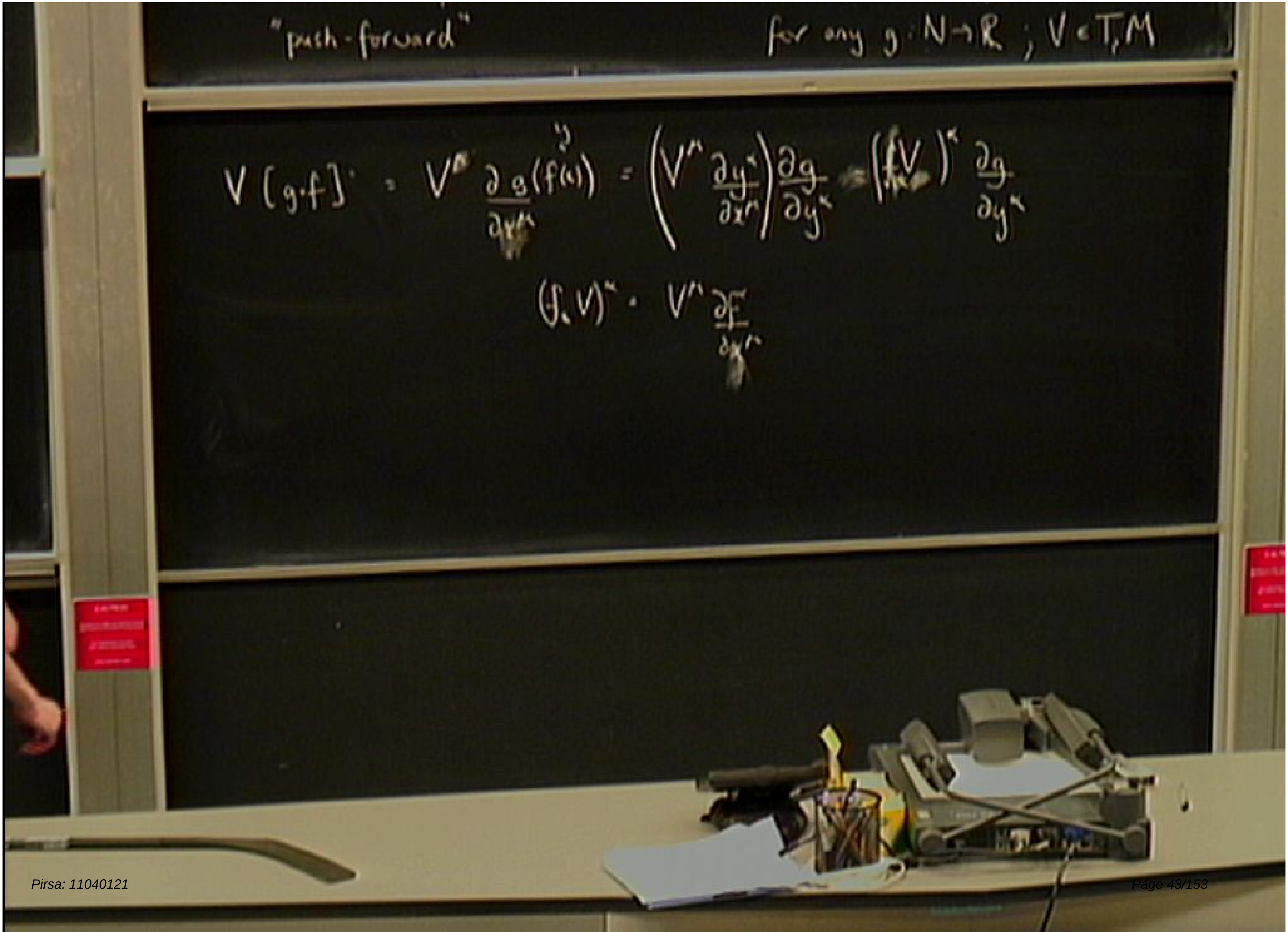
$$V[g \circ f] = V^\mu \frac{\partial g(\overset{y}{f(x)})}{\partial x^\mu} = V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial g}{\partial y^\alpha}$$



"push-forward"

for any $g : N \to \mathbb{R}$; $V \in T_p M$

$$V[g \circ f] = V^\mu \frac{\partial g(f(x))}{\partial x^\mu} \overset{?}{=} \left(V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\right) \frac{\partial g}{\partial y^\alpha} = (f_* V)^\alpha \frac{\partial g}{\partial y^\alpha}$$

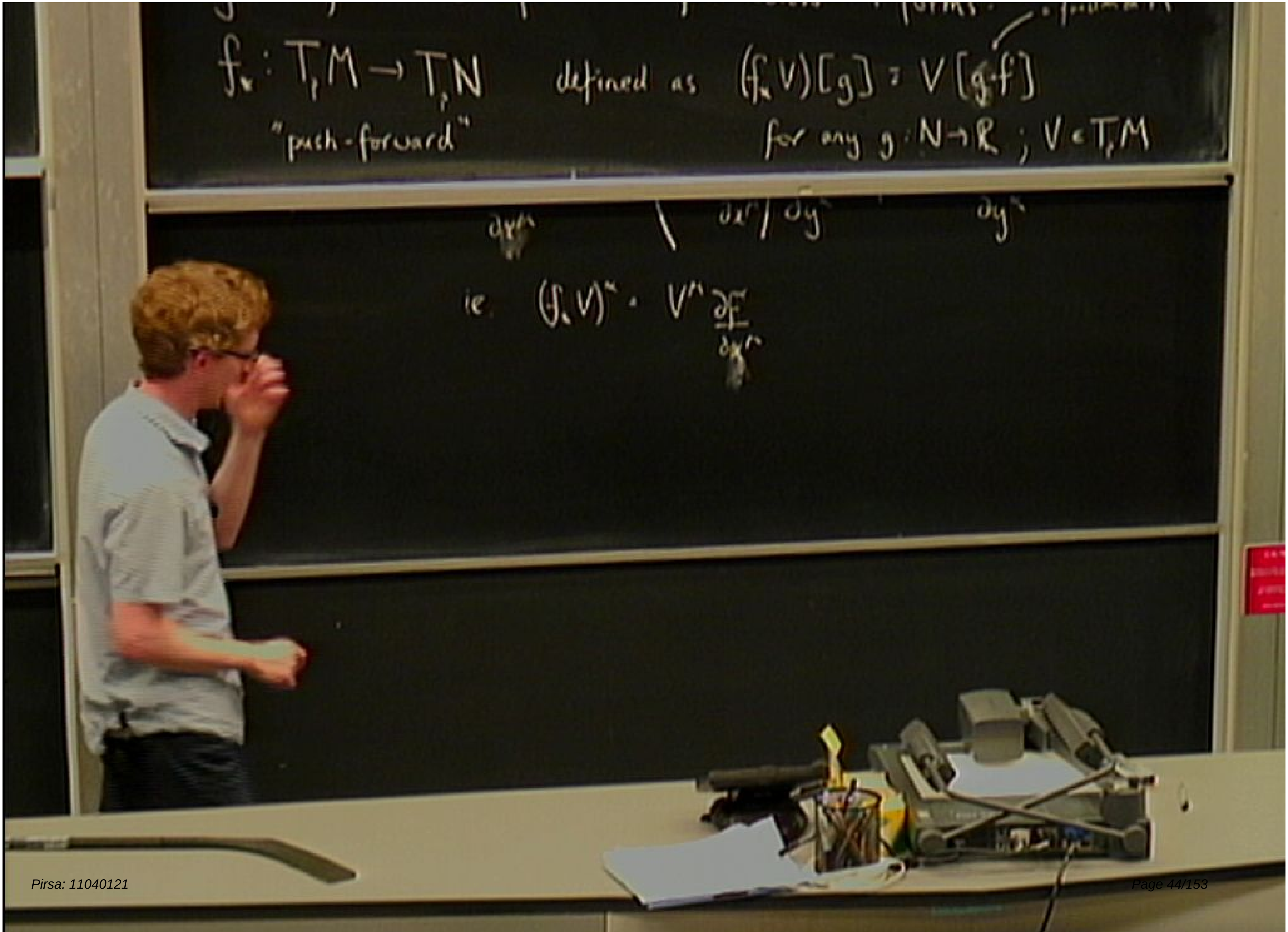


"push-forward"

for any $g : N \to \mathbb{R}$; $V \in T_p M$

$$V[g \circ f] = V^\mu \frac{\partial g(\overset{y}{f(x)})}{\partial x^\mu} = \left(V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\right)\frac{\partial g}{\partial y^\alpha} = (f_* V)^\alpha \frac{\partial g}{\partial y^\alpha}$$

$$(f_* V)^\alpha = V^\mu \frac{\partial f^\alpha}{\partial x^\mu}$$

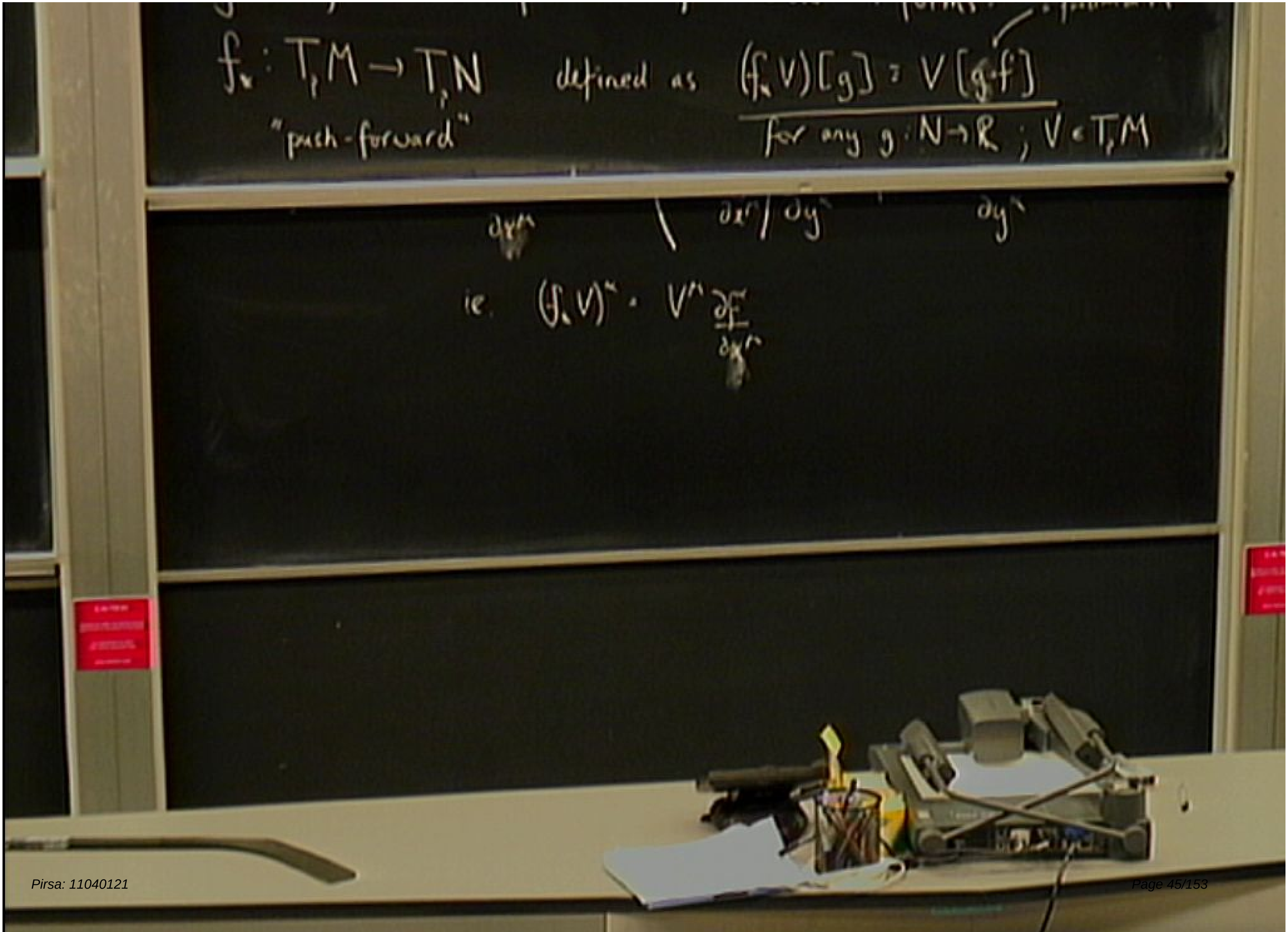


$f_* : T_pM \to T_pN$ defined as $(f_*V)[g] = V[g \circ f]$

"push-forward"

for any $g: N \to \mathbb{R}$; $V \in T_pM$

ie. $(f_*V)^\alpha = V^\mu \frac{\partial f^\alpha}{\partial x^\mu}$



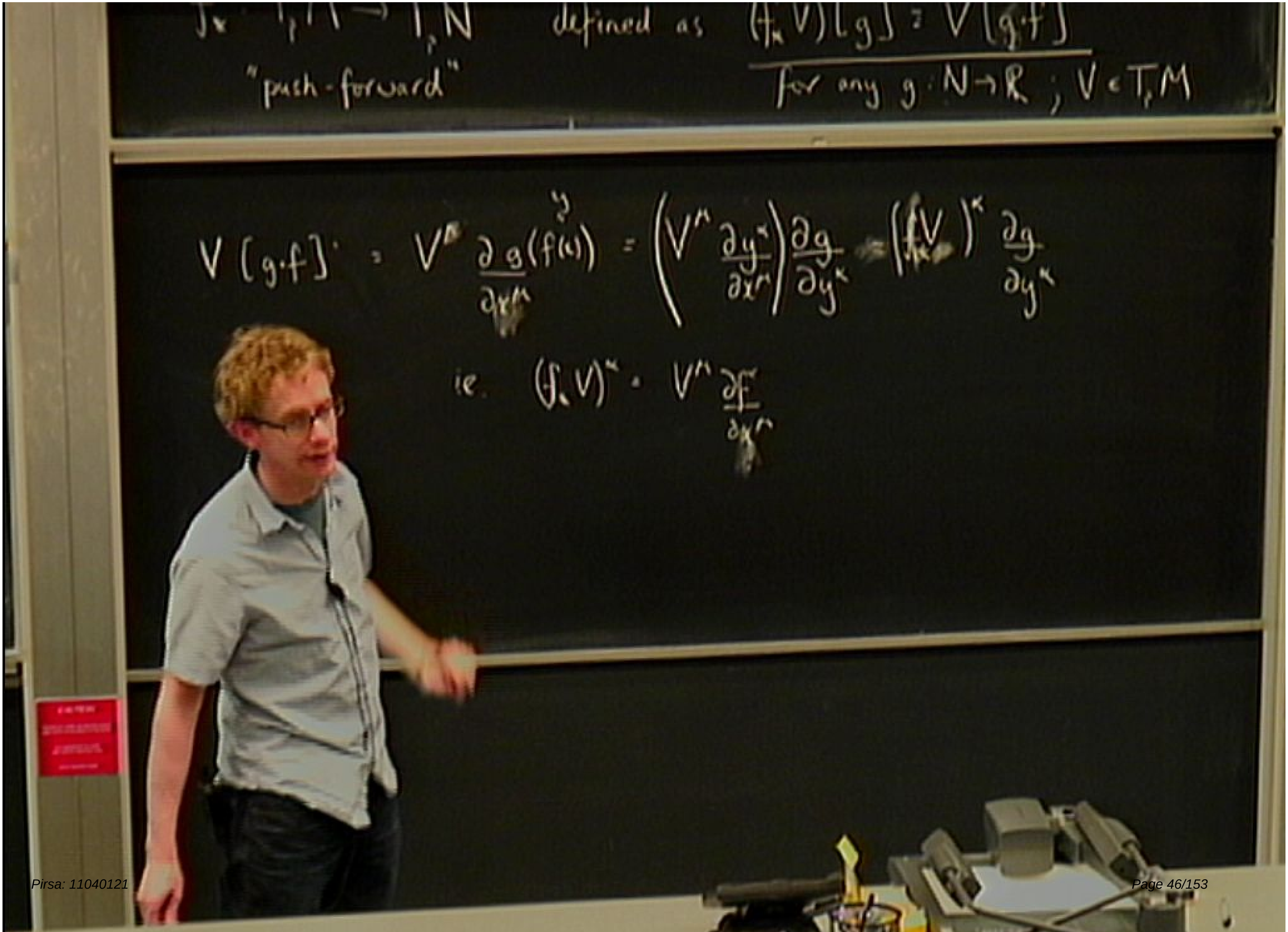
$f_* : T_pM \to T_pN$ defined as $\underline{(f_*V)[g] = V[g \circ f]}$

"push-forward"

for any $g : N \to \mathbb{R}$; $V \in T_pM$

∂x^μ $\Big(\quad \partial x^\mu) \, \partial y^\alpha \qquad \partial y^\alpha$

ie. $(f_*V)^\alpha = V^\mu \dfrac{\partial \xi^\alpha}{\partial x^\mu}$



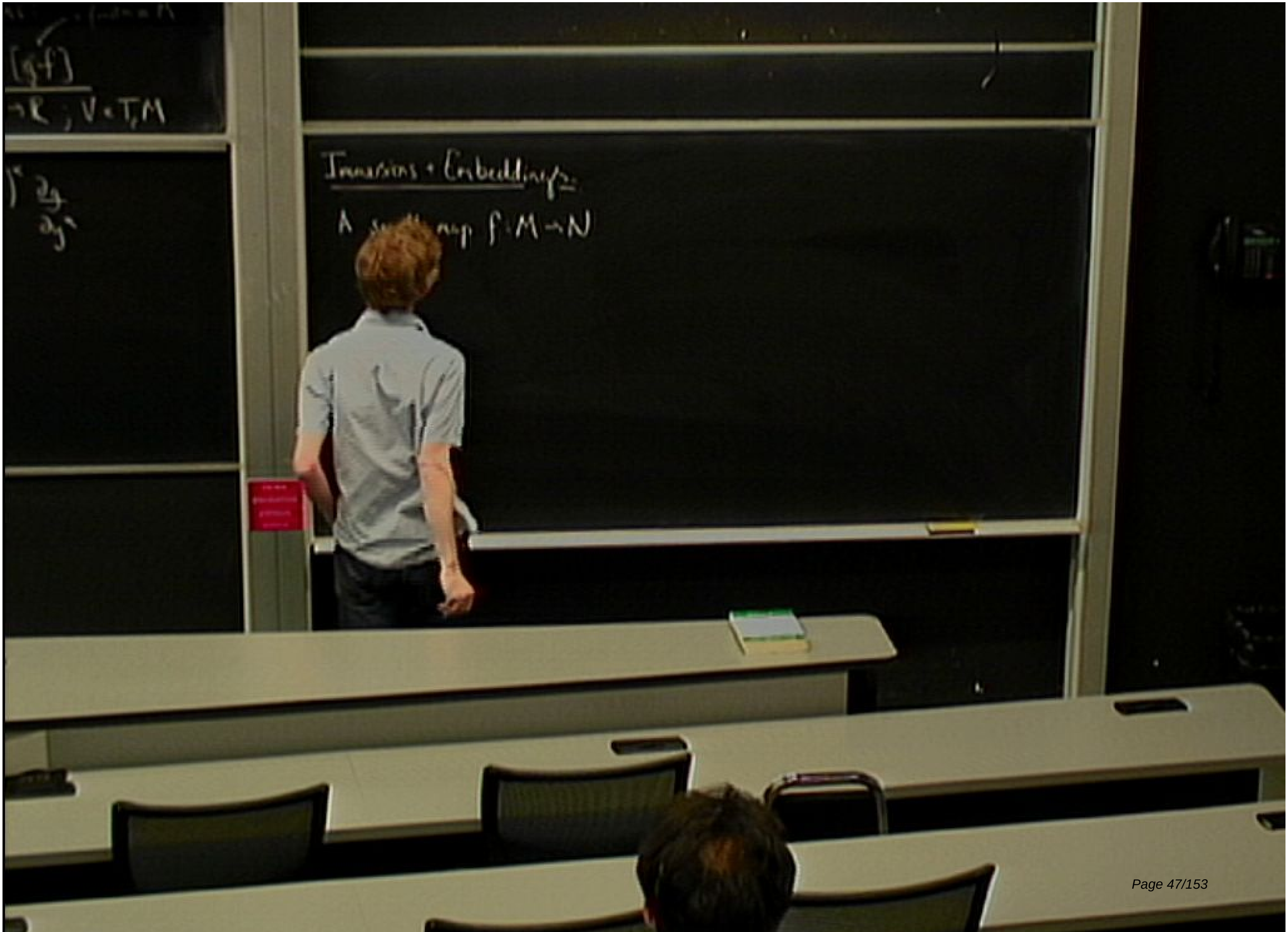
$f_* : T_pM \to T_pN$ defined as $\underline{(f_* V)[g] = V[g \cdot f]}$

"push-forward"

for any $g : N \to \mathbb{R}$; $V \in T_pM$

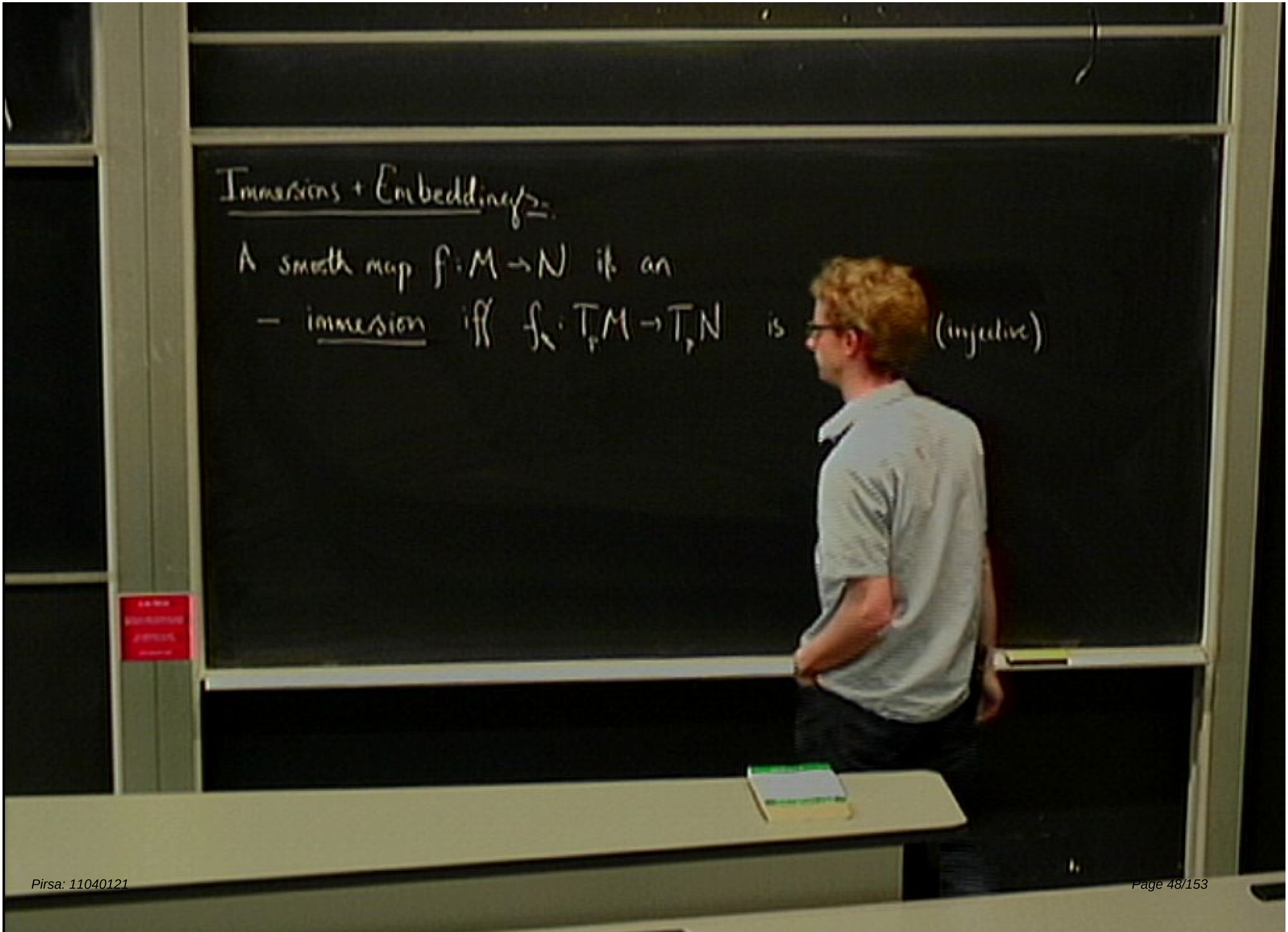
$$V[g \cdot f] = V^\mu \frac{\partial g(\overset{y}{f(x)})}{\partial x^\mu} = \left(V^\mu \frac{\partial y^\kappa}{\partial x^\mu}\right)\frac{\partial g}{\partial y^\kappa} = (f_* V)^\kappa \frac{\partial g}{\partial y^\kappa}$$

ie. $(f_* V)^\kappa = V^\mu \dfrac{\partial f^\kappa}{\partial x^\mu}$



Immersions + Embeddings:

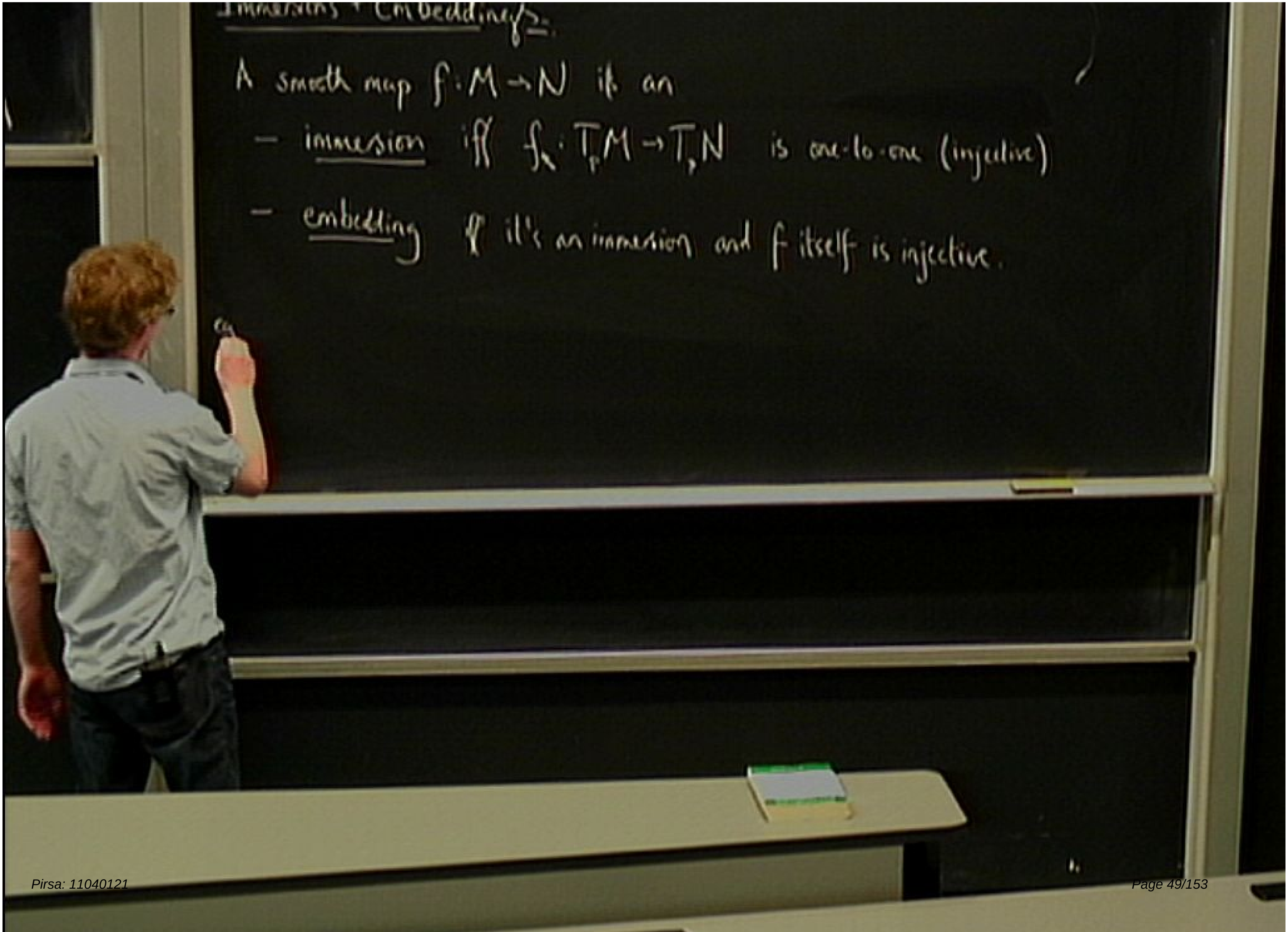
A smooth map $f: M \to N$



Immersions + Embeddings:

A smooth map $f: M \to N$ is an

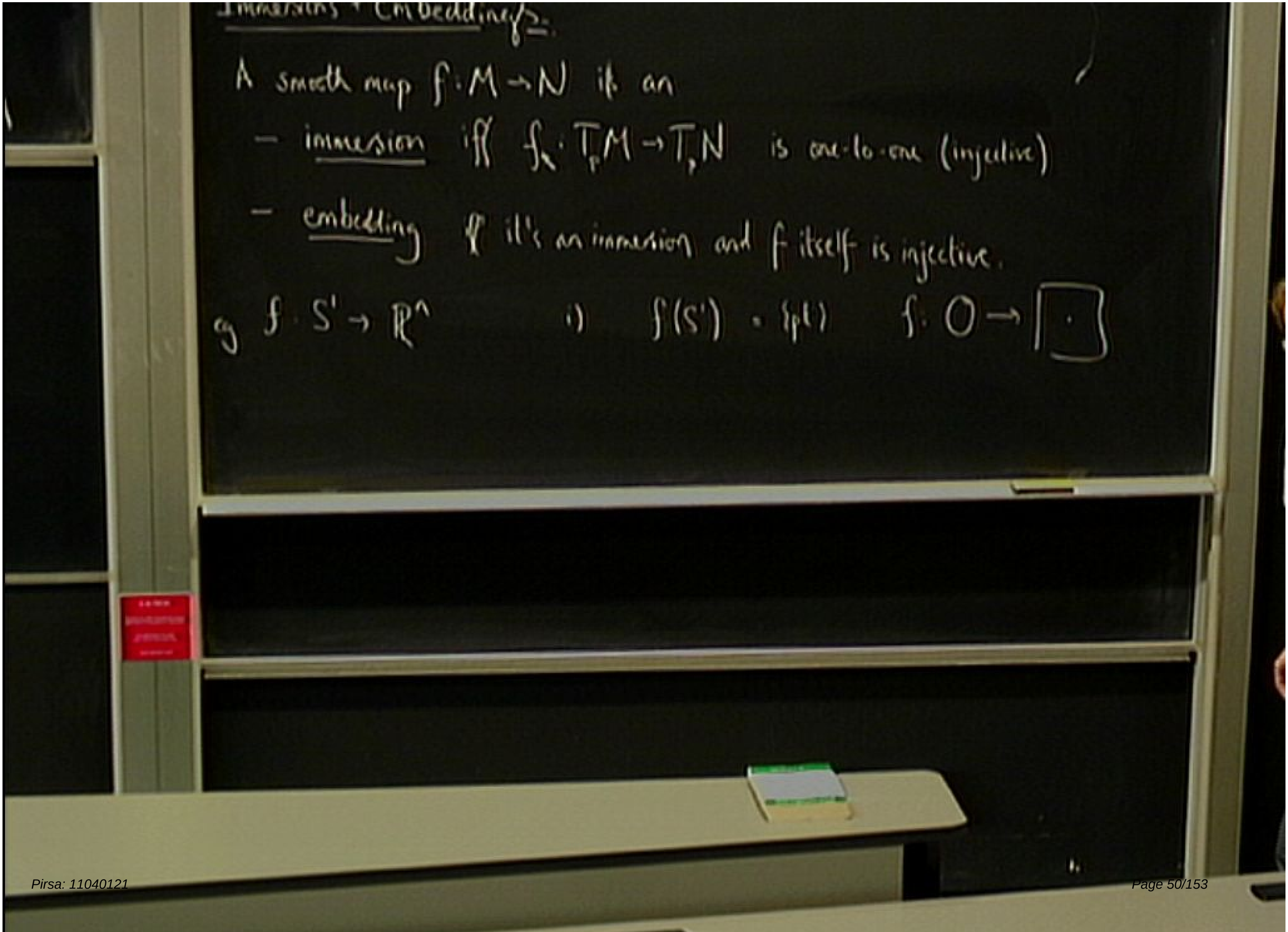
- immersion iff $f_*: T_pM \to T_pN$ is (injective)



Immersions + Embeddings:

A smooth map $f: M \to N$ is an

- immersion iff $f_*: T_pM \to T_pN$ is one-to-one (injective)
- embedding iff it's an immersion and f itself is injective.



Immersions + Embeddings:

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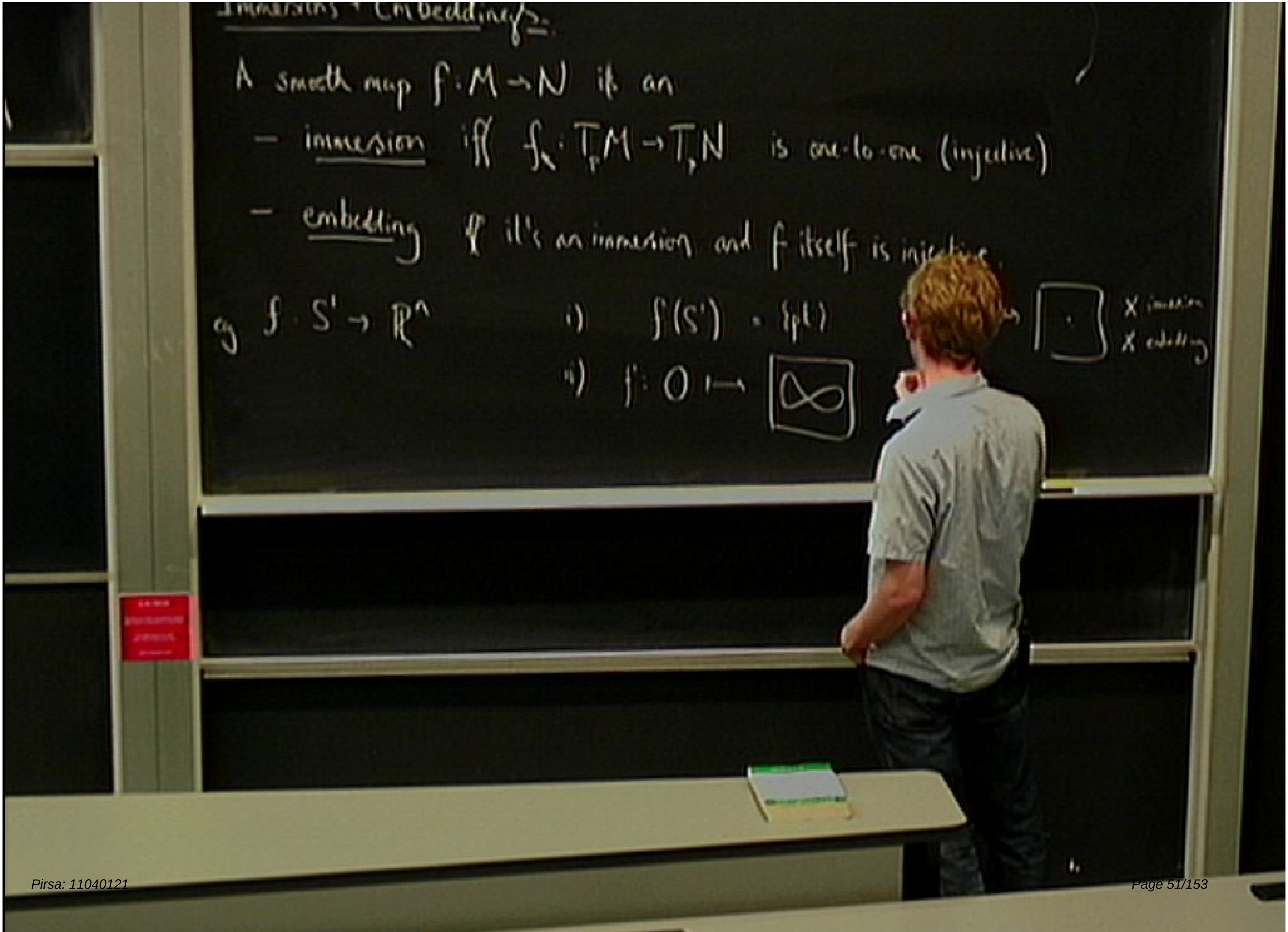
eg $f: S^1 \to \mathbb{R}^n$ i) $f(S^1) = \{pt\}$ $f: \bigcirc \to \boxed{\cdot}$



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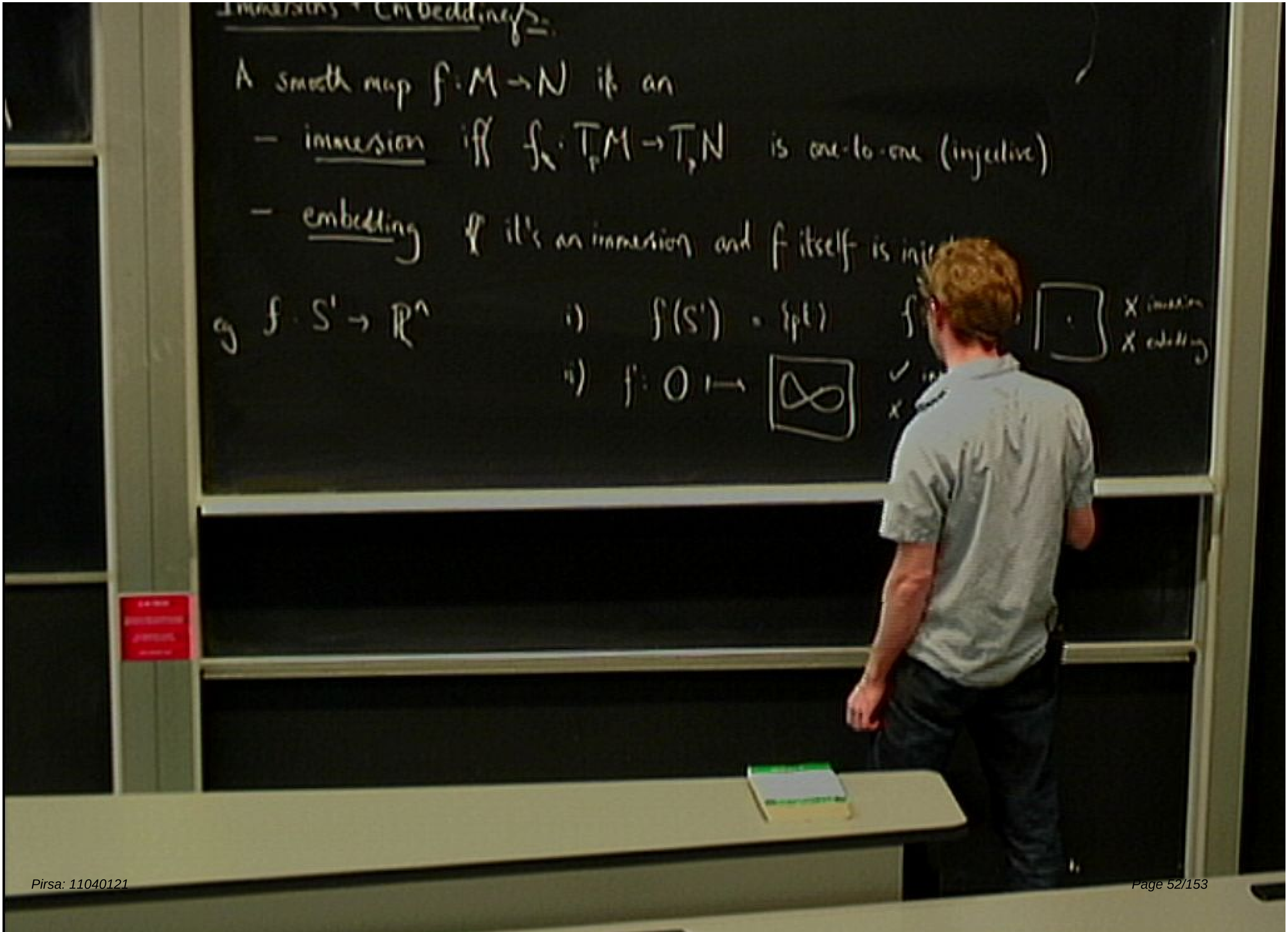
i) $f(S^1) = \{pt\}$ … X immersion X embedding

ii) $f: O \mapsto \infty$



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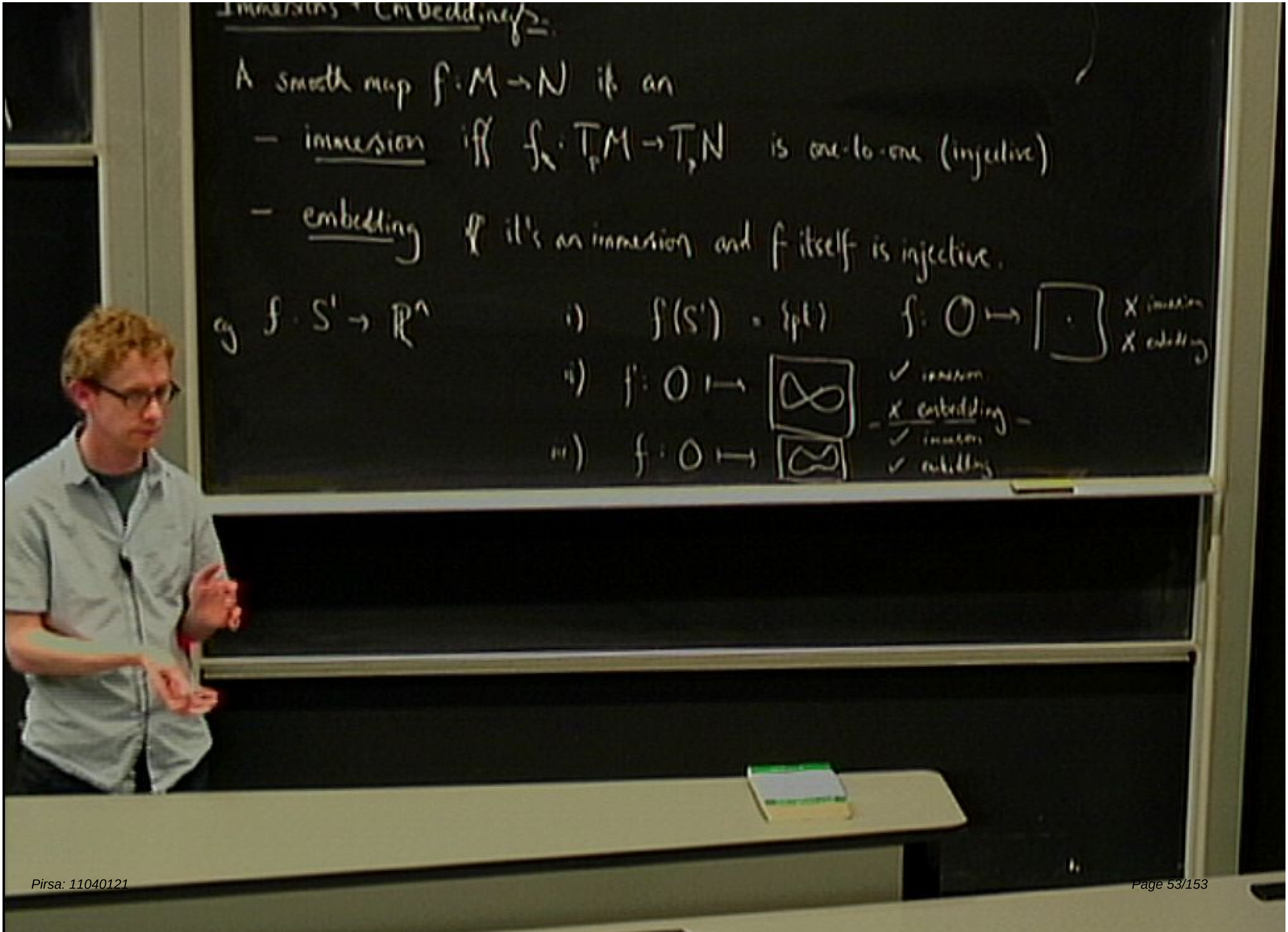
i) $f(S^1) = \{pt\}$ ✗ immersion ✗ embedding

ii) $f: O \longmapsto$ ✓ ✗



Immersions + Embeddings:

A smooth map $f: M \to N$ is an

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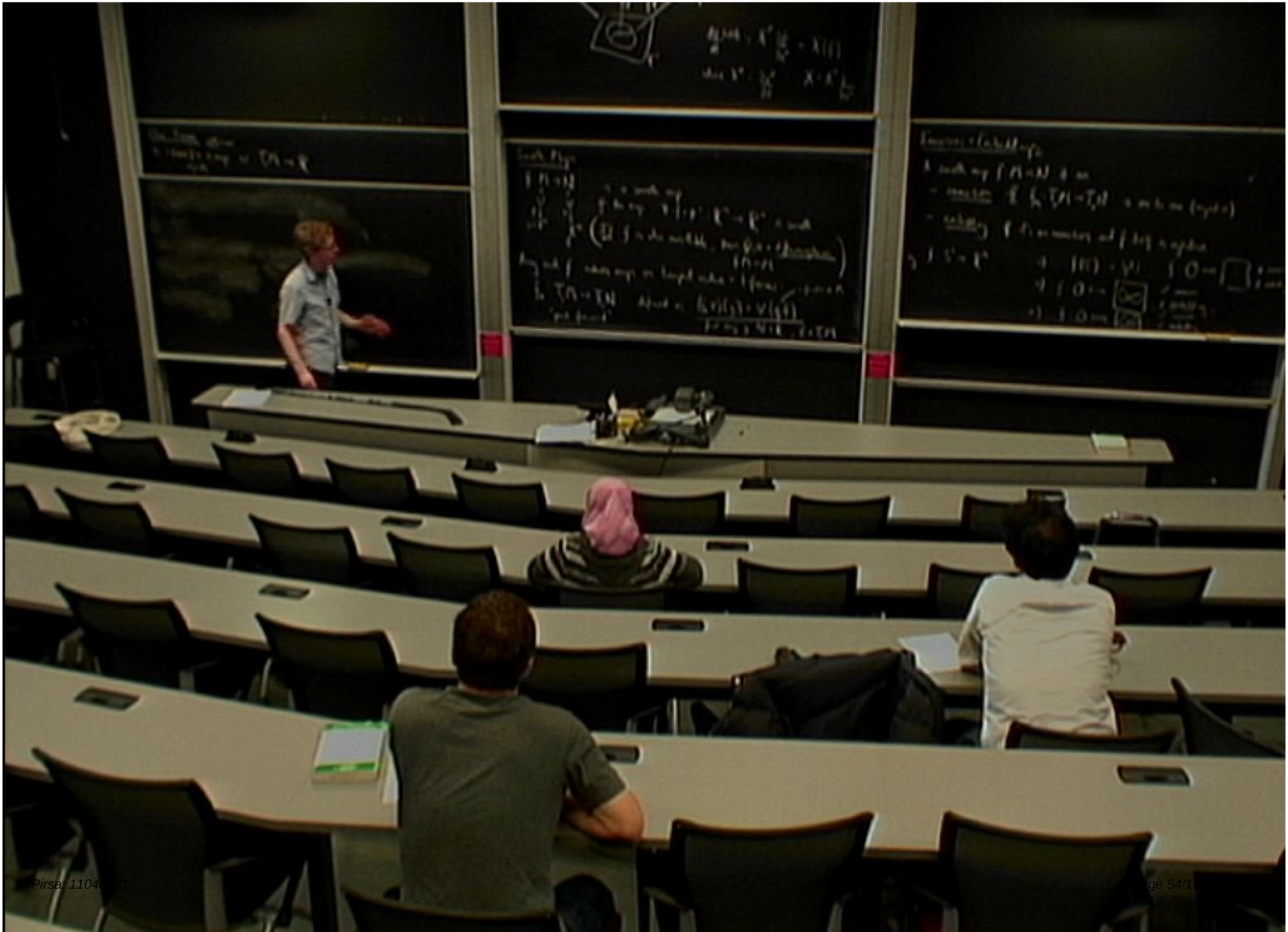
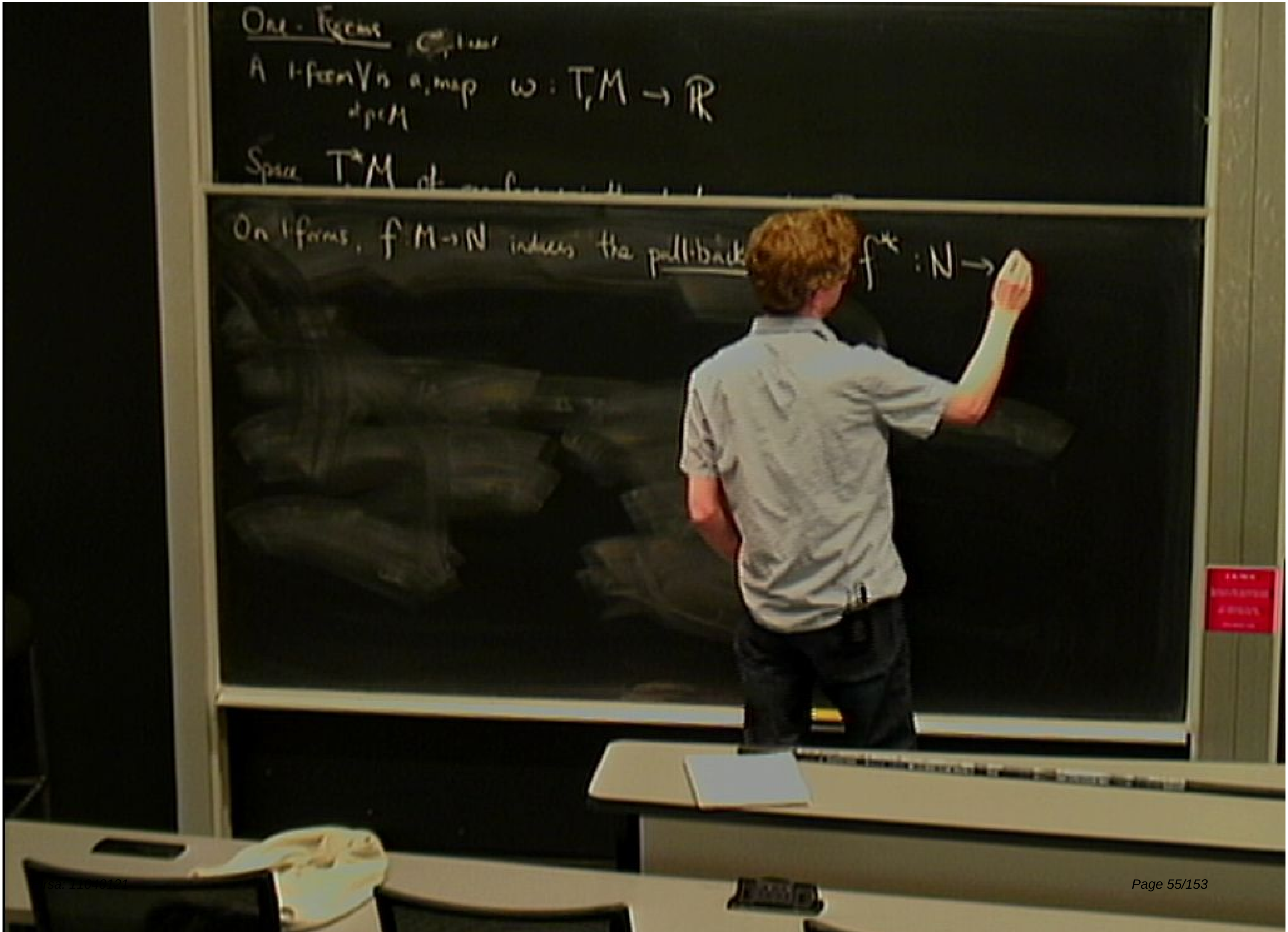
eg $f: S^1 \to \mathbb{R}^n$

i) $f(S^1) = \{pt\}$ $f: O \mapsto$ ✗ immersion ✗ embedding

ii) $f: O \mapsto$ ✓ immersion ✗ embedding

iii) $f: O \mapsto$ ✓ immersion ✓ embedding





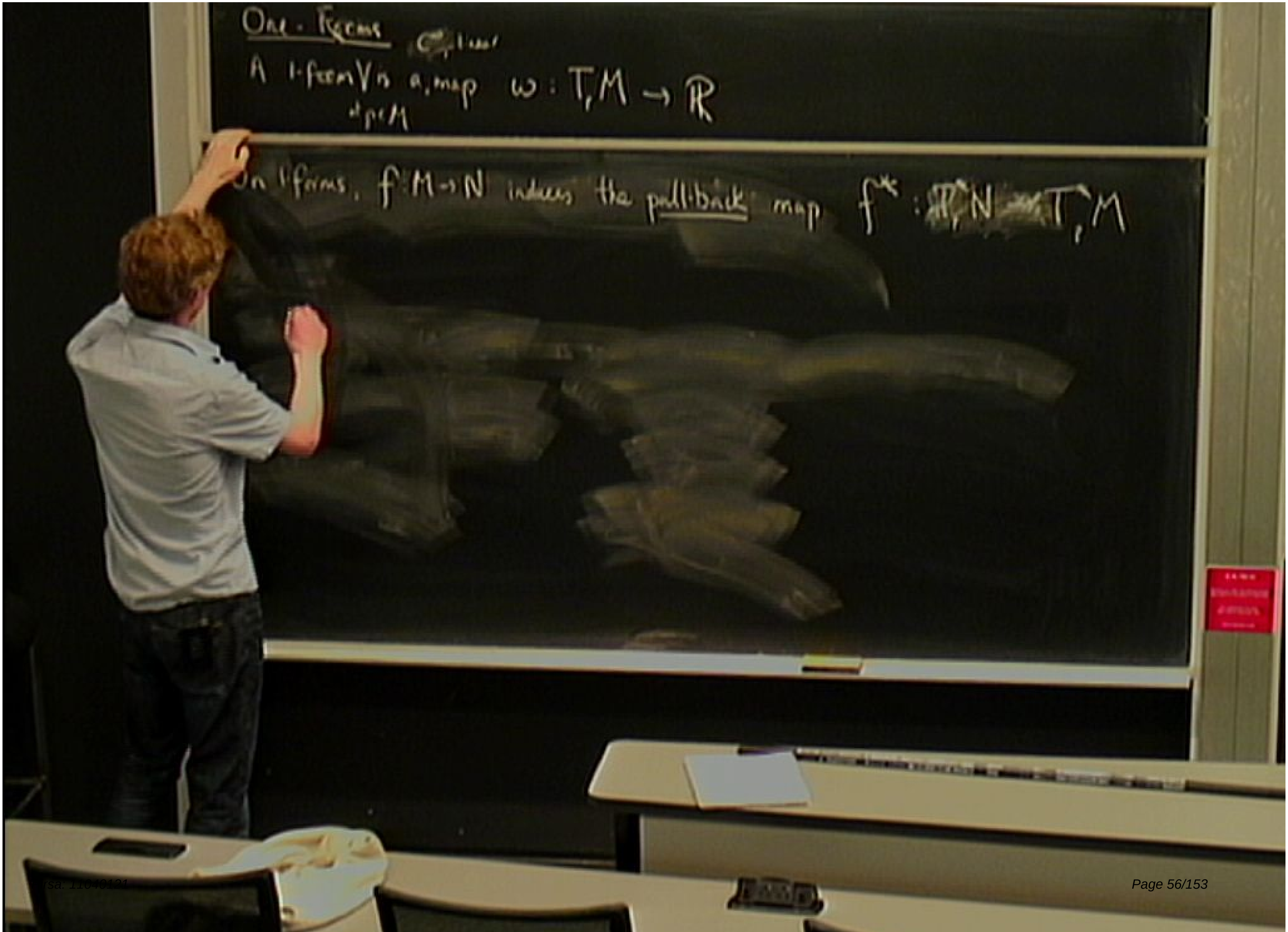
One - Forms

linear

A 1-form at $p \in M$ is a map $\omega : T_pM \to \mathbb{R}$

Space T^*M of

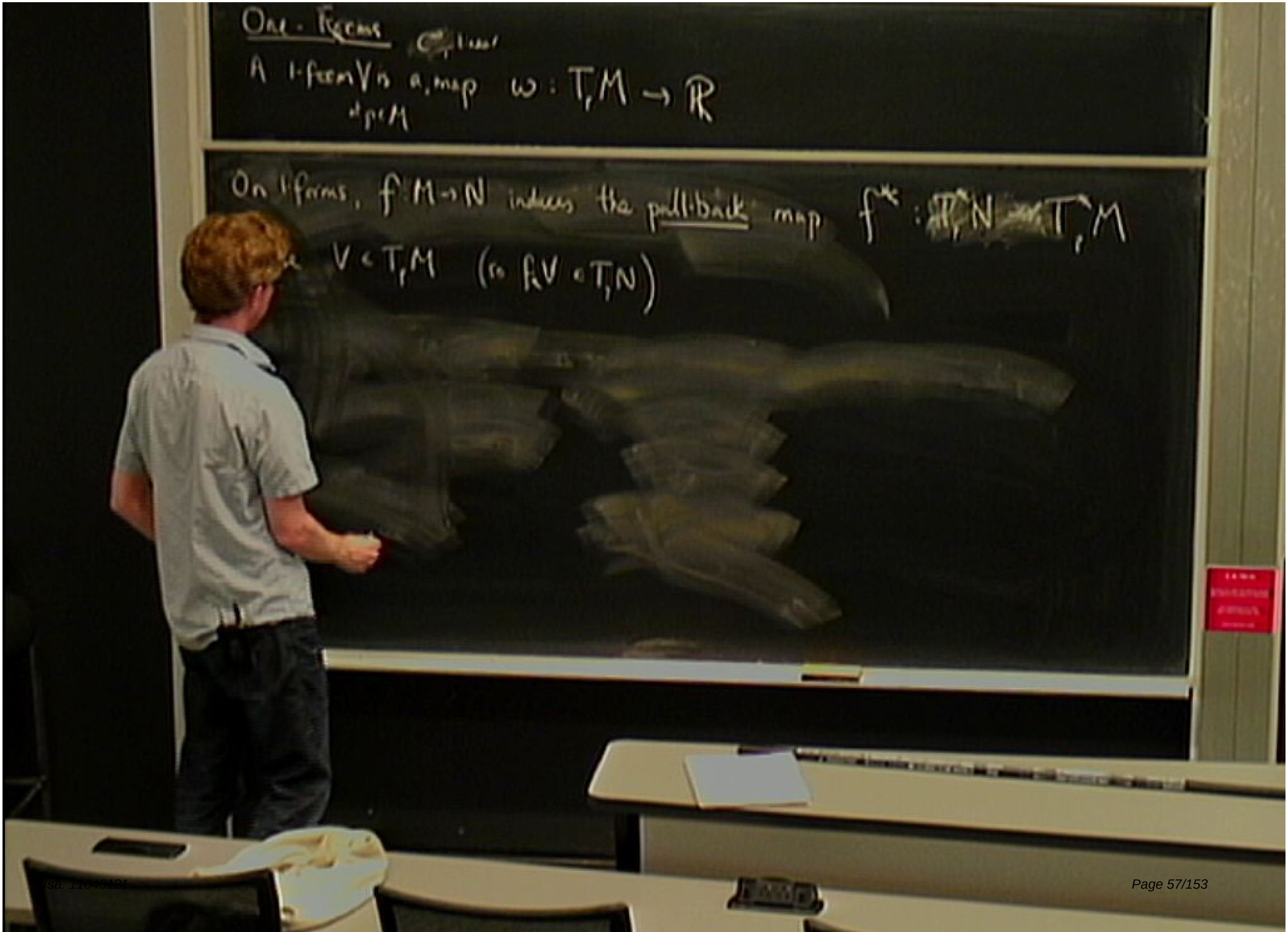
On 1-forms, $f: M \to N$ induces the pullback $f^* : N \to$



One - Forms

A 1-form at $p \in M$ is a linear map $\omega : T_pM \to \mathbb{R}$

On 1-forms, $f: M \to N$ induces the pullback map $f^* : T^*N \to T_p^*M$

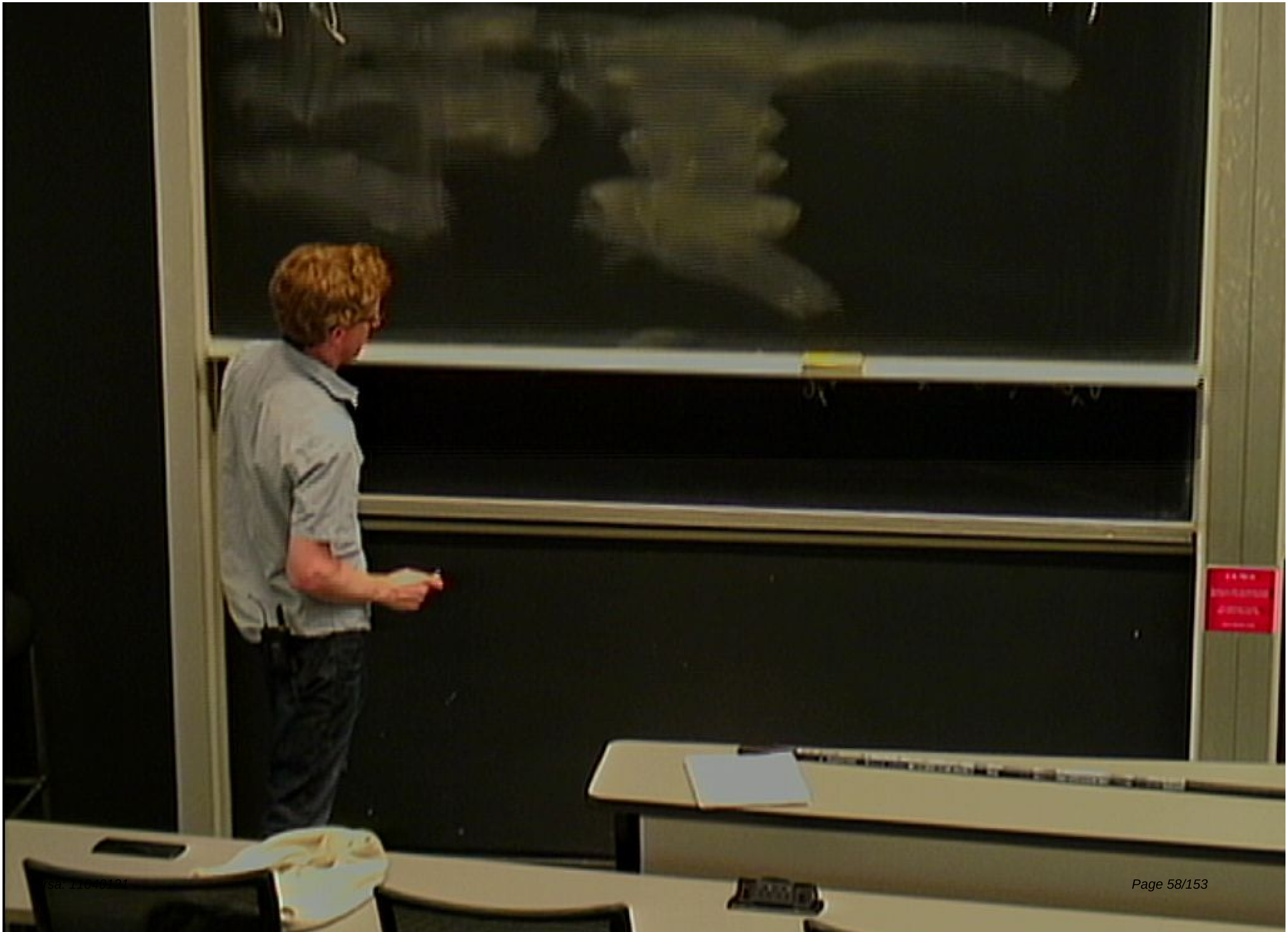


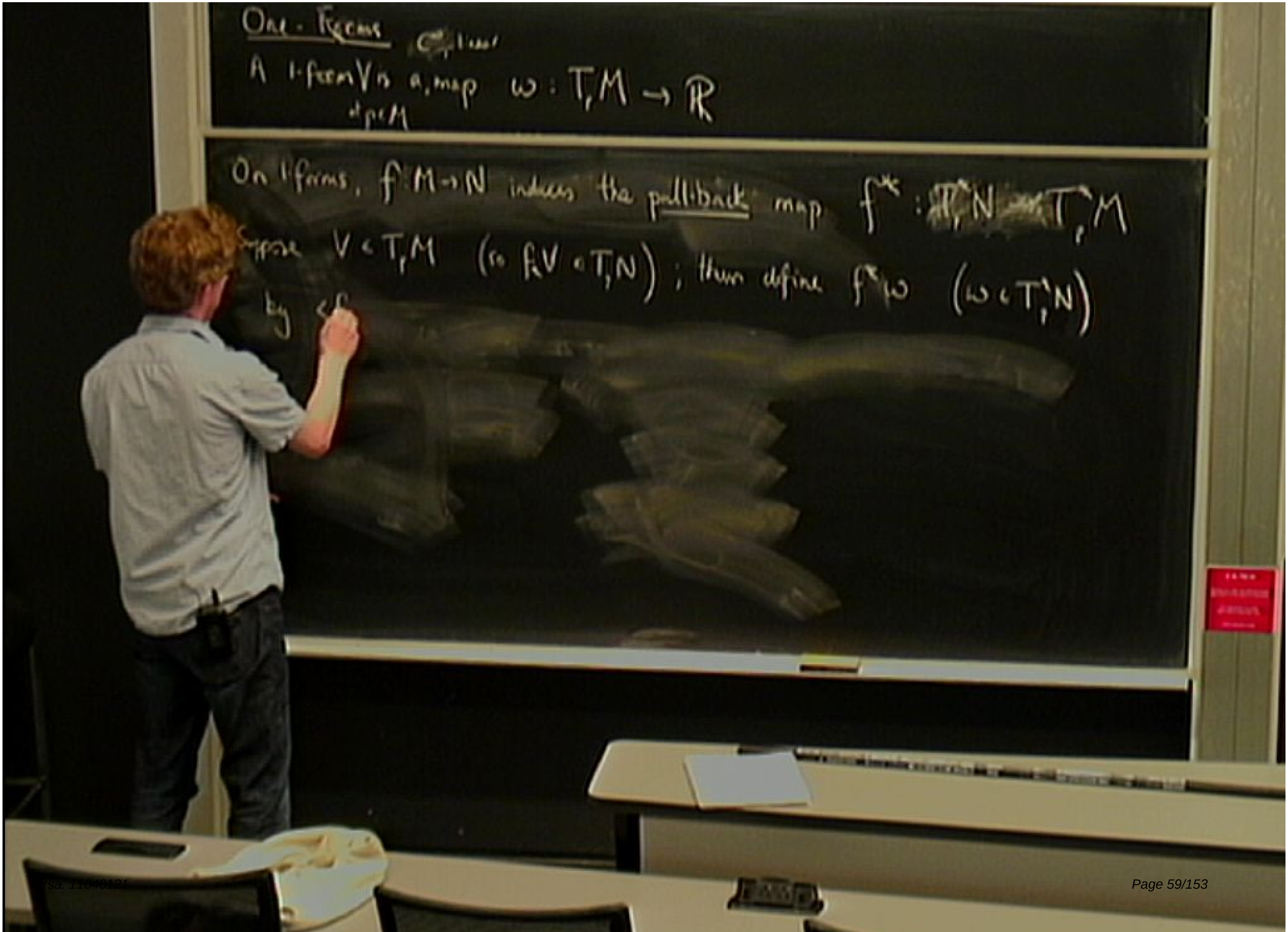
One - Forms

A 1-form at $p \in M$ is a linear map $\omega : T_pM \to \mathbb{R}$

On 1-forms, $f: M \to N$ induces the pullback map $f^* : T_p^*N \to T_p^*M$

$V \in T_pM$ (so $f_*V \in T_pN$)





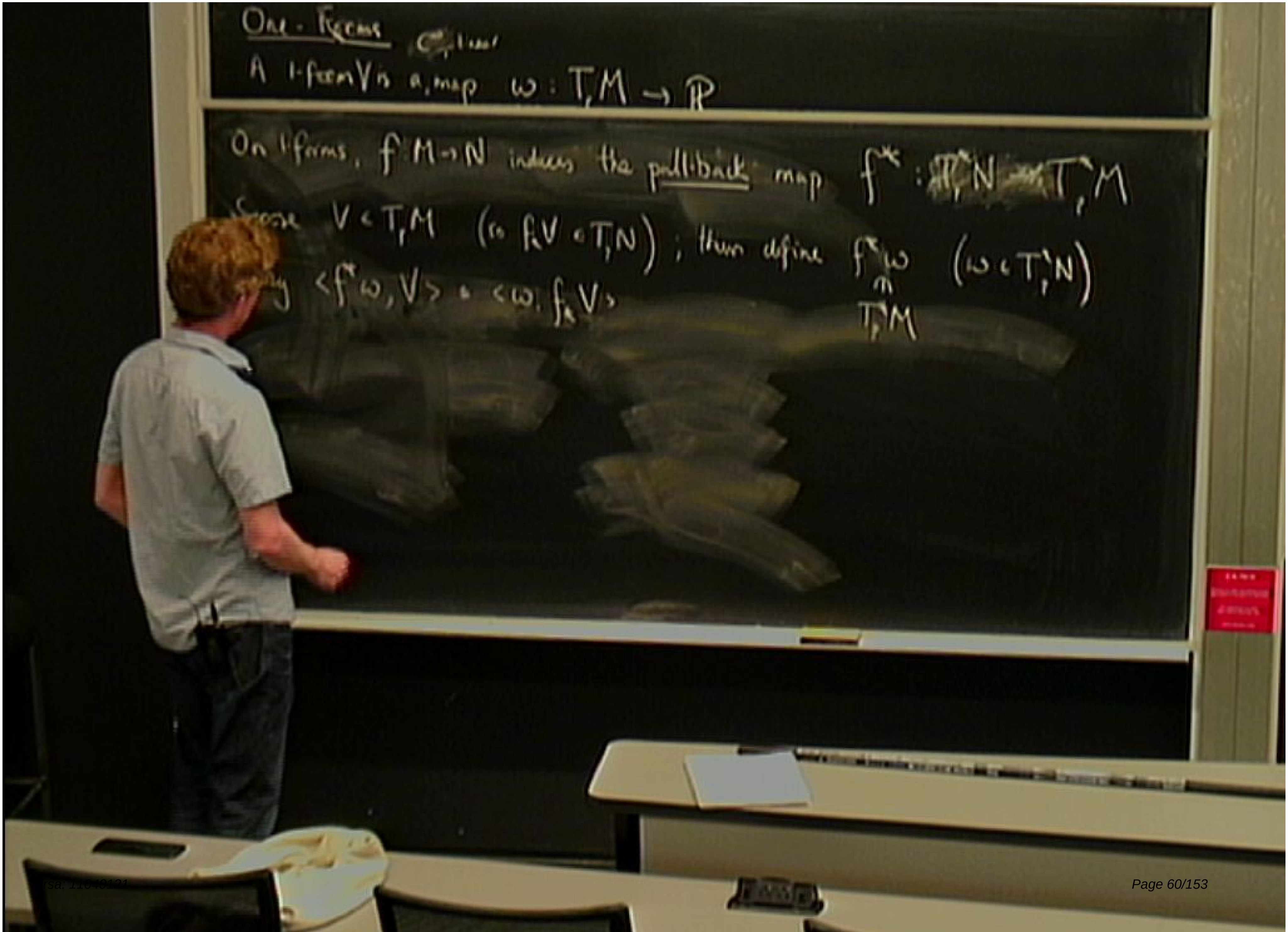
One - Forms

A 1-form at $p \in M$ is a linear map $\omega : T_pM \to \mathbb{R}$

On 1-forms, $f: M \to N$ induces the pullback map $f^* : T_p^*N \to T_p^*M$

Suppose $V \in T_pM$ (so $f_*V \in T_pN$); then define $f^*\omega$ $(\omega \in T_p^*N)$

by $\langle f$



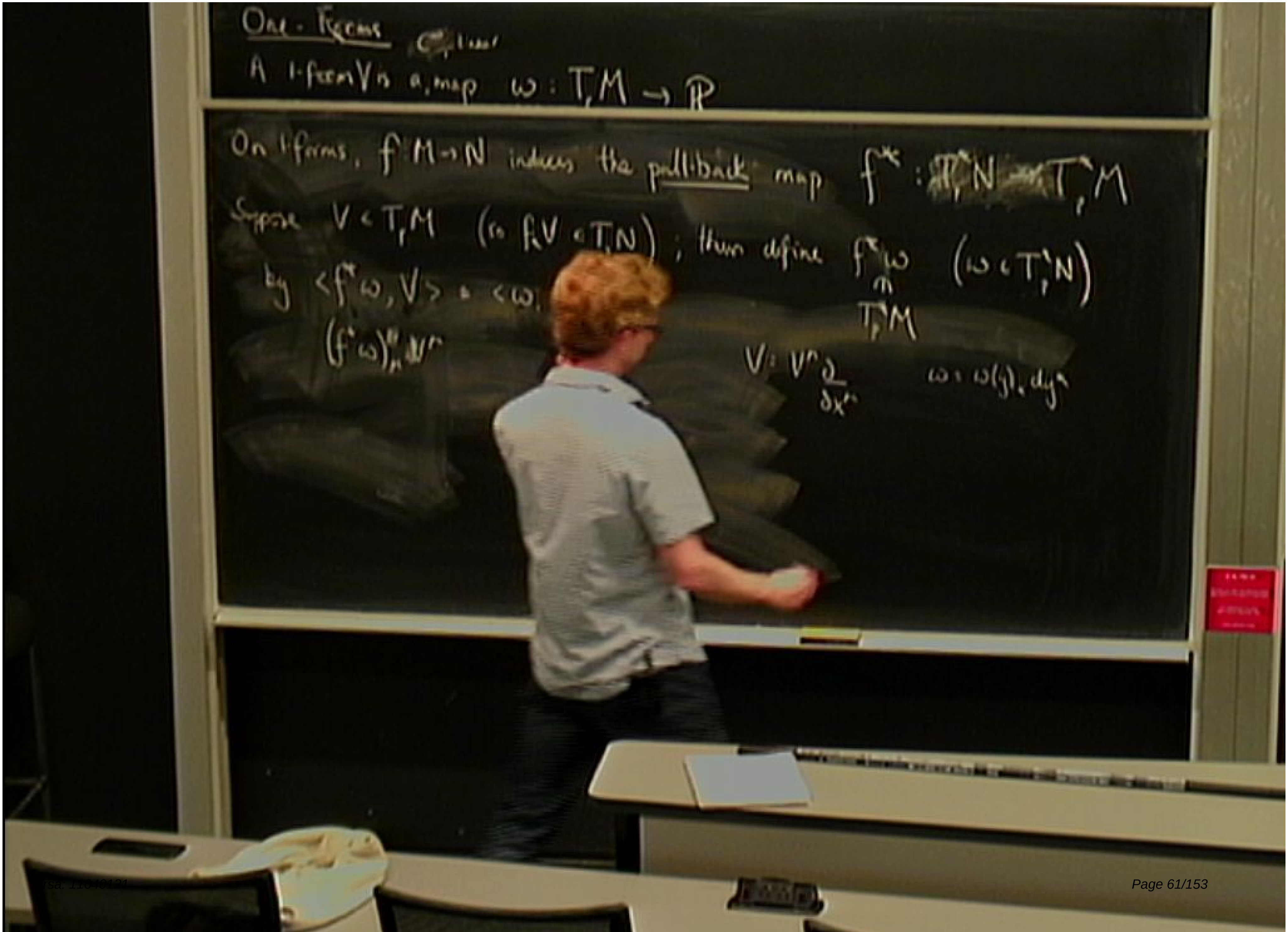
<u>One - Forms</u>

A 1-form is a linear map $\omega : T_pM \to \mathbb{R}$

On 1-forms, $f: M \to N$ induces the <u>pullback</u> map $f^* : T_p^*N \to T_p^*M$

Suppose $V \in T_pM$ (so $f_*V \in T_pN$); then define $\underset{T_p^*M}{\underset{\cap}{f^*\omega}}$ $(\omega \in T_p^*N)$

by $\langle f^*\omega, V\rangle = \langle \omega, f_*V\rangle$



One - Forms C^∞, linear

A 1-form V is a map $\omega : T_pM \to \mathbb{R}$

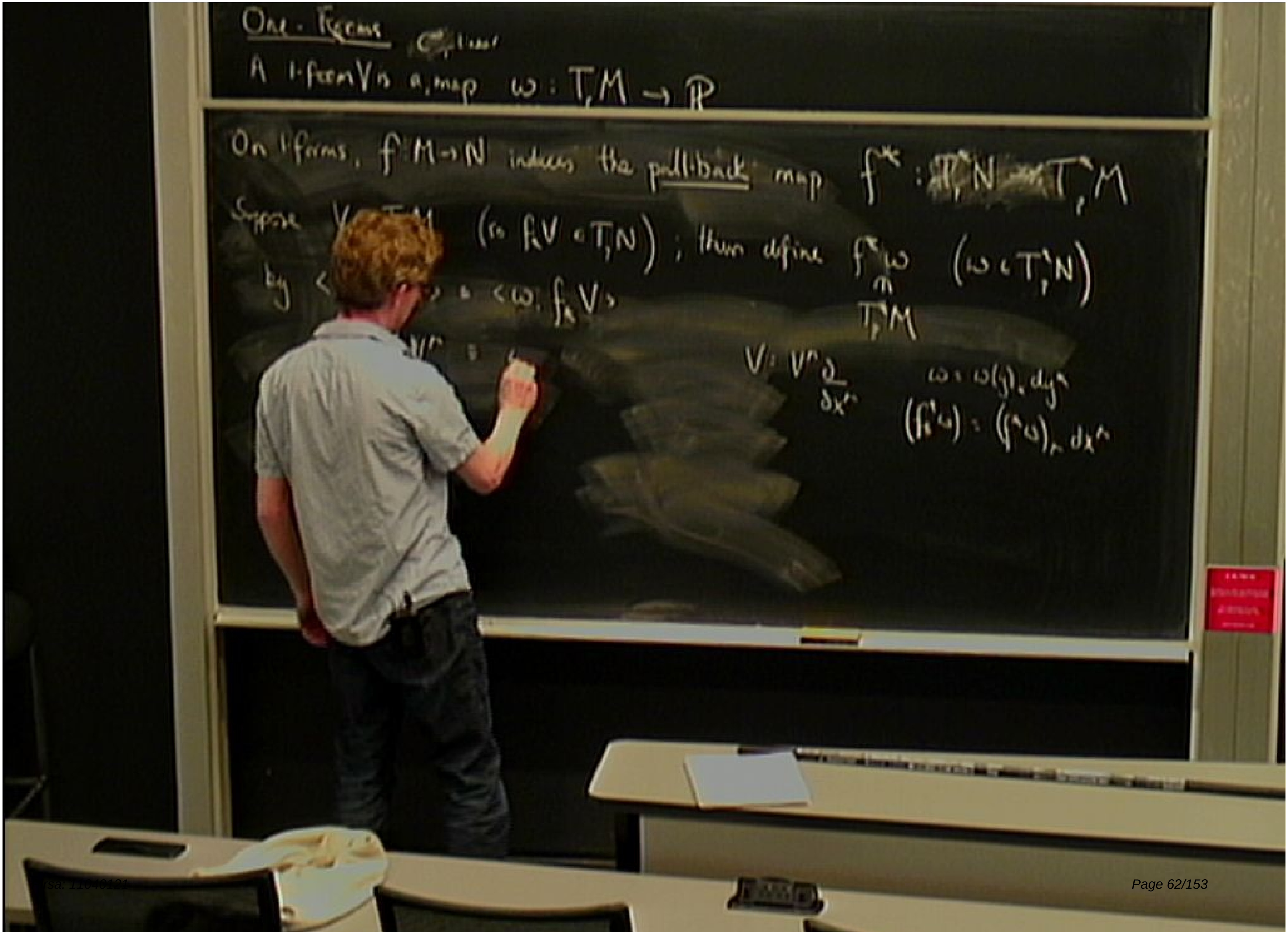
On 1-forms, $f: M \to N$ induces the pullback map $f^* : T_p^*N \to T_p^*M$

Suppose $V \in T_pM$ (so $f_*V \in T_pN$); then define $\underset{\substack{\cap \\ T_p^*M}}{f^*\omega}$ $(\omega \in T_p^*N)$

by $\langle f^*\omega, V \rangle = \langle \omega,$

$(f^*\omega)_\mu V^\mu$

$V = V^\mu \frac{\partial}{\partial x^\mu}$ $\omega = \omega(y)_\alpha dy^\alpha$



One - Forms

C^∞, linear

A 1-form is a map $\omega : T_pM \to \mathbb{R}$

On 1-forms, $f: M \to N$ induces the pullback map $f^* : T_p^*N \to T_p^*M$

Suppose $V \in T_pM$ (so $f_*V \in T_pN$); then define $\underset{T_p^*M}{\overset{}{\underset{\cap}{f^*\omega}}}$ $(\omega \in T_p^*N)$

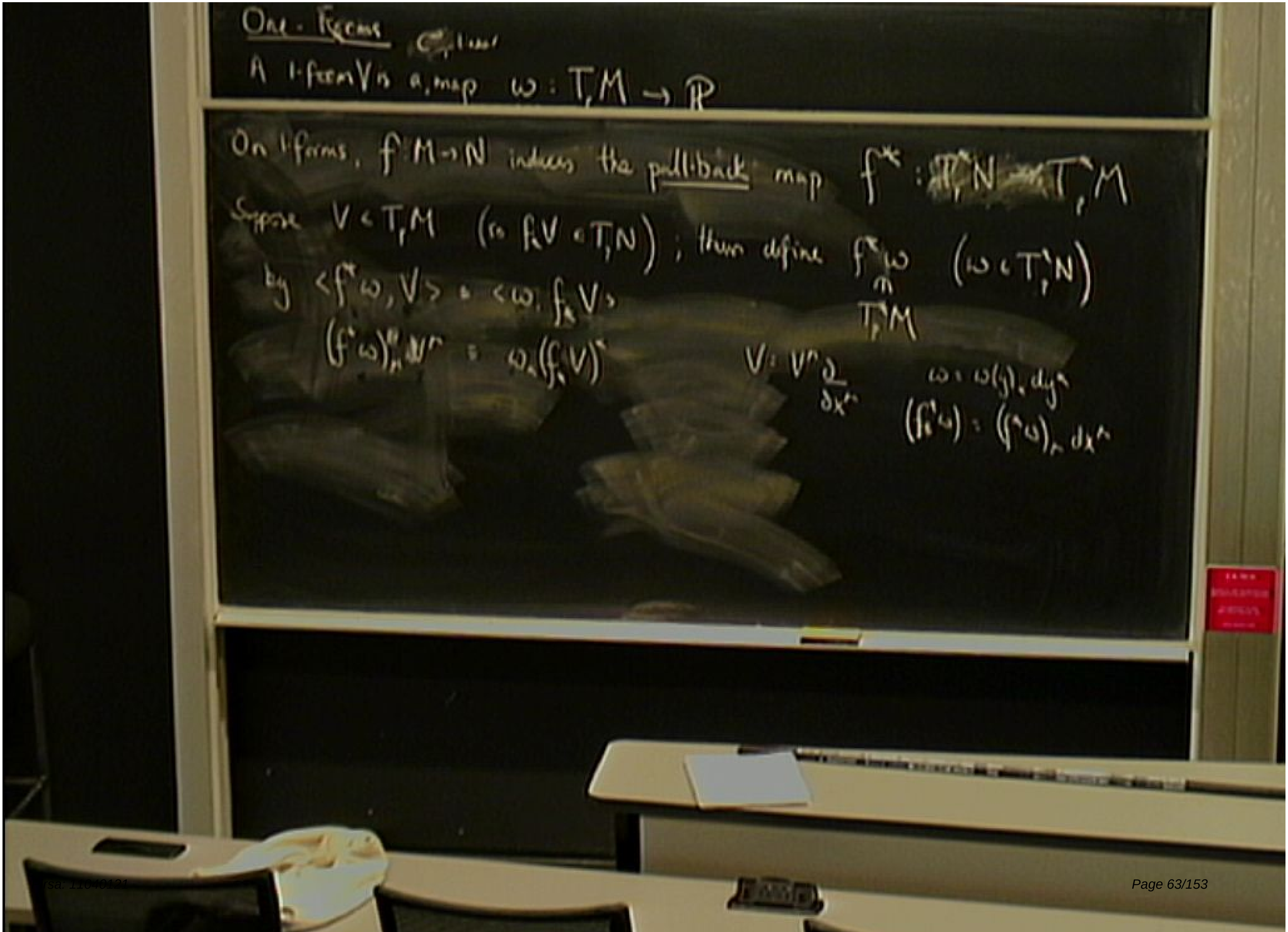
by $\langle$ [illegible] $\rangle = \langle \omega, f_*V \rangle$

$V^\mu =$ [illegible]

$V = V^\mu \frac{\partial}{\partial x^\mu}$

$\omega = \omega(y)_\nu dy^\nu$

$(f^*\omega) = (f^*\omega)_\mu dx^\mu$



One - Forms

C^∞, linear

A 1-form V is a map $\omega : T_pM \to \mathbb{R}$

On 1-forms, $f: M \to N$ induces the pullback map $f^* : T^*_pN \to T^*_pM$

Suppose $V \in T_pM$ (so $f_*V \in T_pN$) ; then define $\underset{\underset{T^*_pM}{\cap}}{f^*\omega}$ $(\omega \in T^*_pN)$

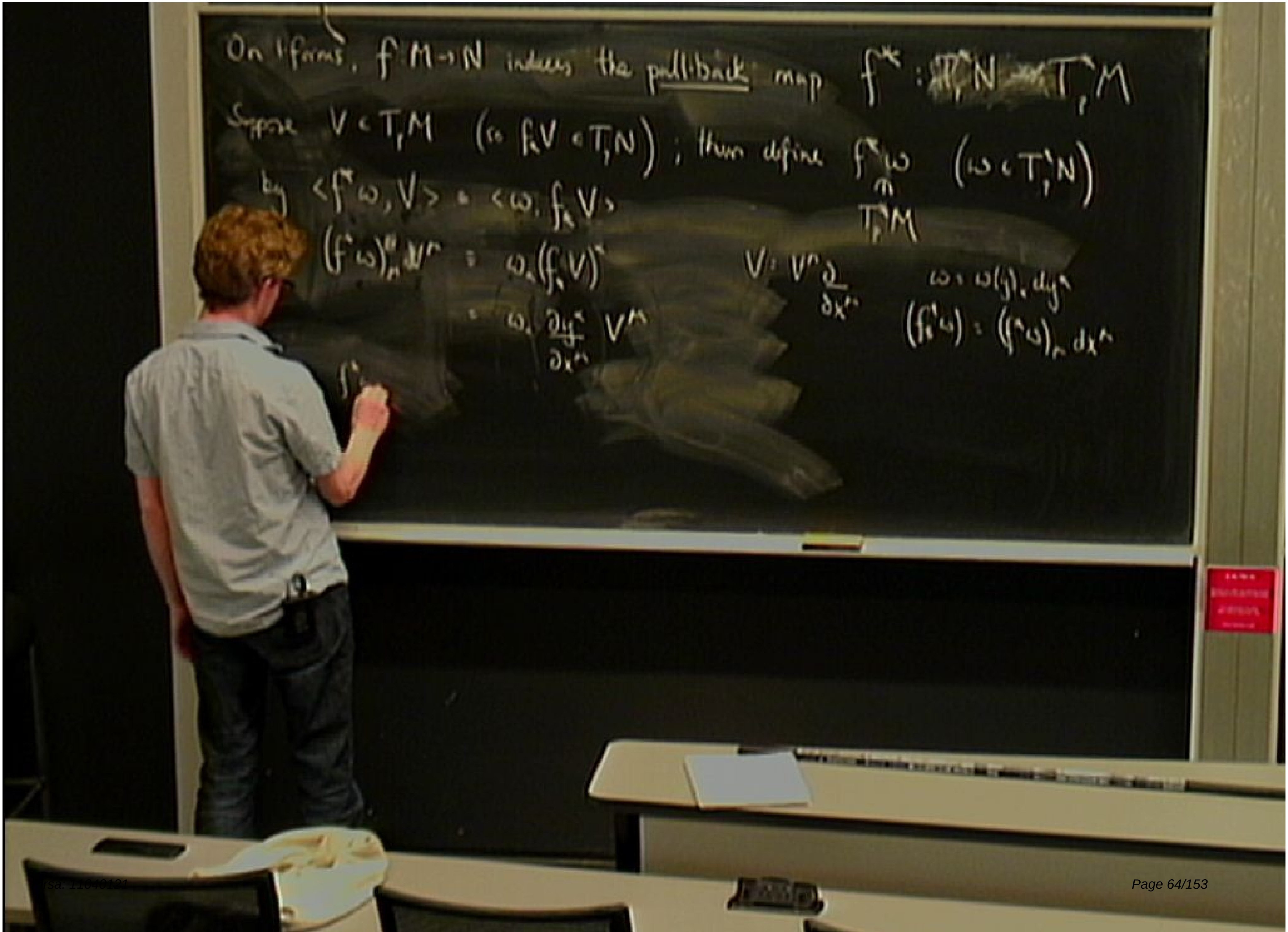
by $\langle f^*\omega, V \rangle = \langle \omega, f_*V \rangle$

$(f^*\omega)_\mu V^\mu = \omega_\alpha (f_*V)^\alpha$

$V = V^\mu \frac{\partial}{\partial x^\mu}$

$\omega = \omega(y)_\alpha dy^\alpha$

$(f^*\omega) = (f^*\omega)_\mu dx^\mu$



On 1-forms, $f: M \to N$ induces the pullback map $f^*: T_p^*N \to T_p^*M$

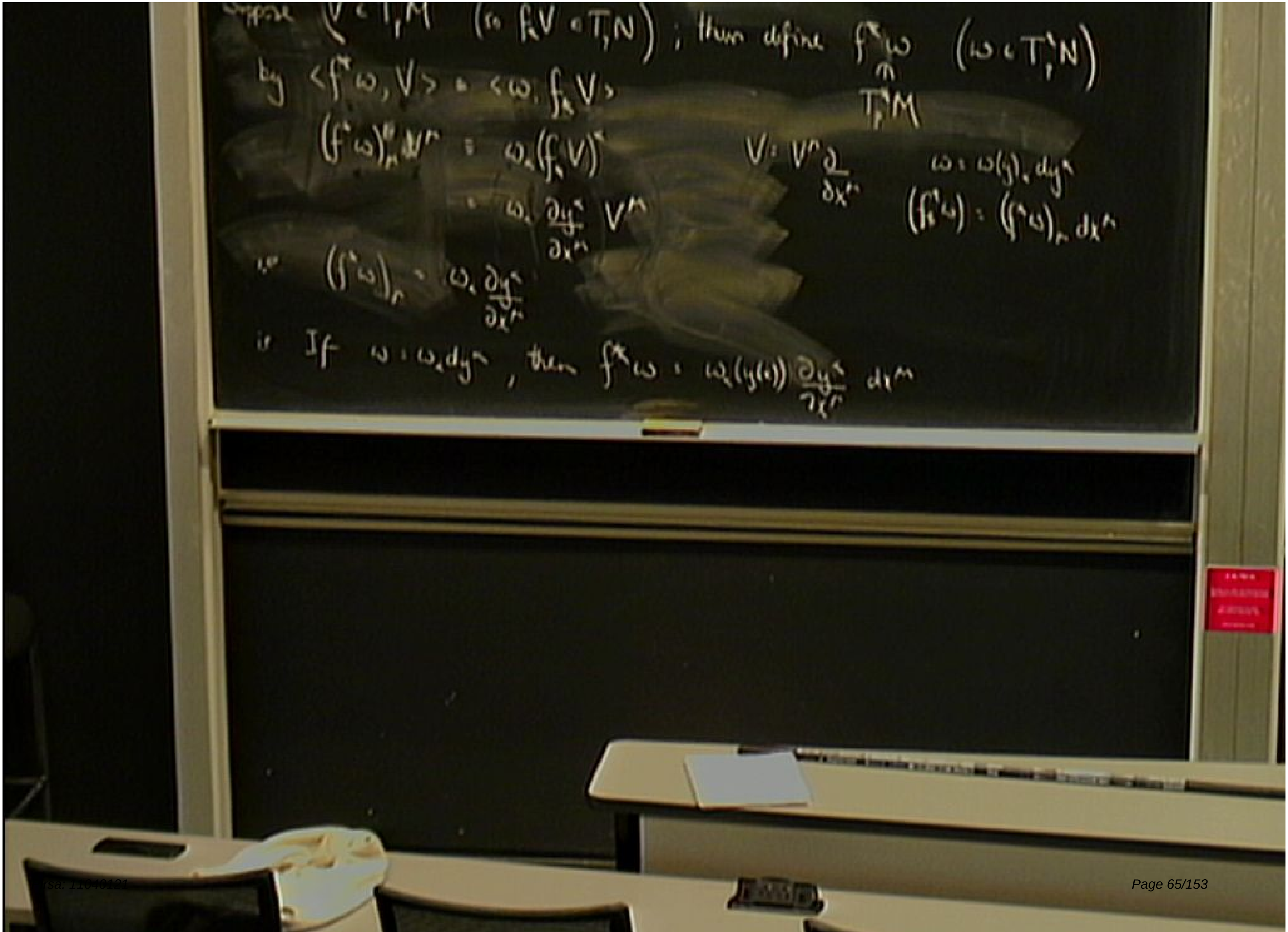
Suppose $V \in T_pM$ (so $f_*V \in T_pN$); then define $\underset{T_p^*M}{\underset{\in}{f^*\omega}}$ $(\omega \in T_p^*N)$

by $\langle f^*\omega, V \rangle = \langle \omega, f_*V \rangle$

$$(f^*\omega)_\mu V^\mu = \omega_\nu (f_*V)^\nu \qquad V = V^\mu \frac{\partial}{\partial x^\mu} \qquad \omega = \omega(y)_\nu dy^\nu$$

$$= \omega_\nu \frac{\partial y^\nu}{\partial x^\mu} V^\mu \qquad (f^*\omega) = (f^*\omega)_\mu dx^\mu$$

f^*



Suppose $V \in T_pM$ (so $f_*V \in T_qN$); then define $\underset{\substack{\cap \\ T^*_pM}}{f^*\omega}$ $(\omega \in T^*_qN)$

by $\langle f^*\omega, V \rangle = \langle \omega, f_*V \rangle$

$$(f^*\omega)_\mu V^\mu = \omega_\alpha (f_*V)^\alpha$$

$$= \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu} V^\mu$$

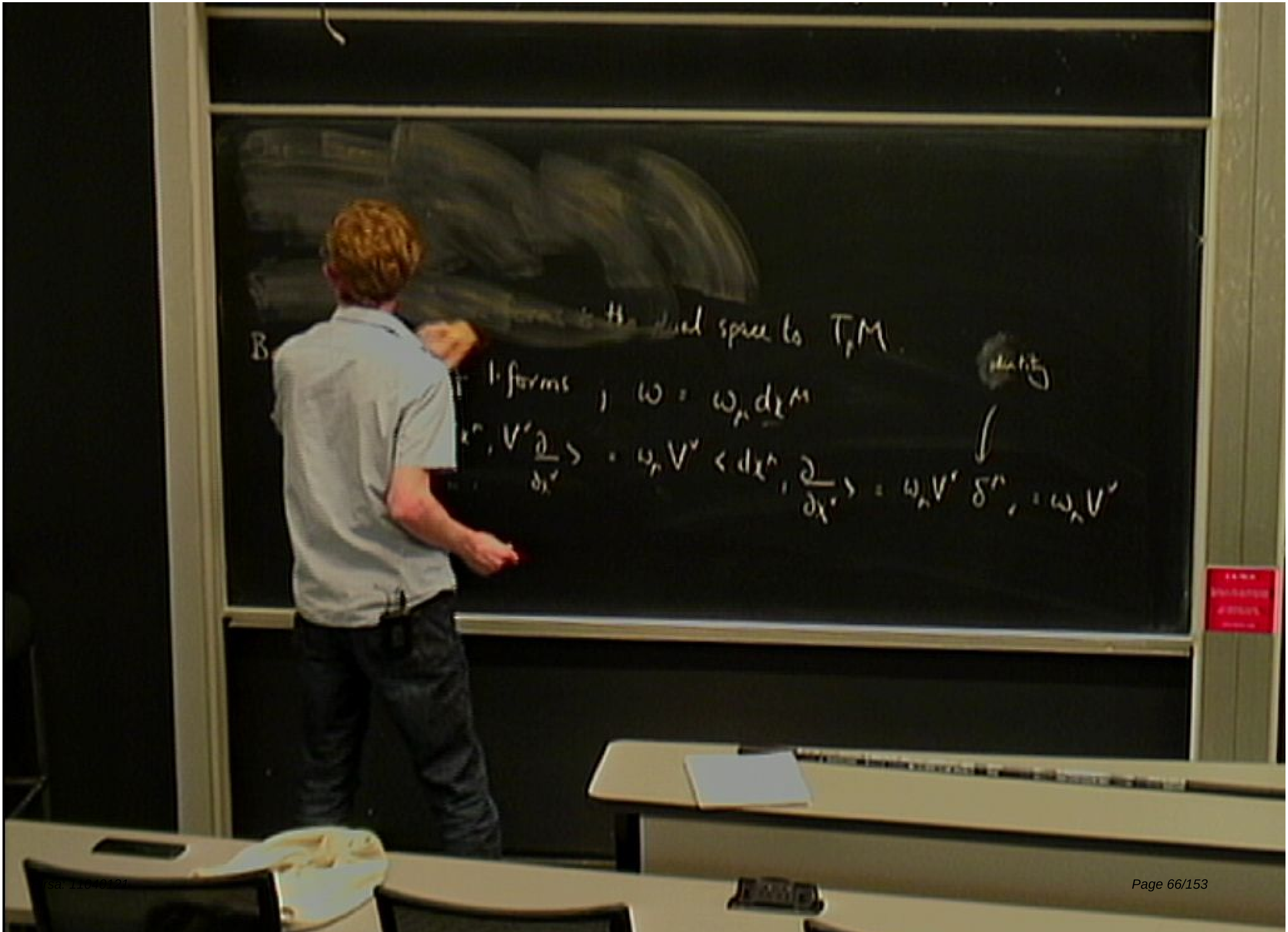
$V = V^\mu \frac{\partial}{\partial x^\mu}$

$\omega = \omega(y)_\alpha dy^\alpha$

$(f^*\omega) = (f^*\omega)_\mu dx^\mu$

$\Rightarrow (f^*\omega)_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}$

$\therefore$ If $\omega = \omega_\alpha dy^\alpha$, then $f^*\omega = \omega_\alpha(y(x)) \frac{\partial y^\alpha}{\partial x^\mu} dx^\mu$



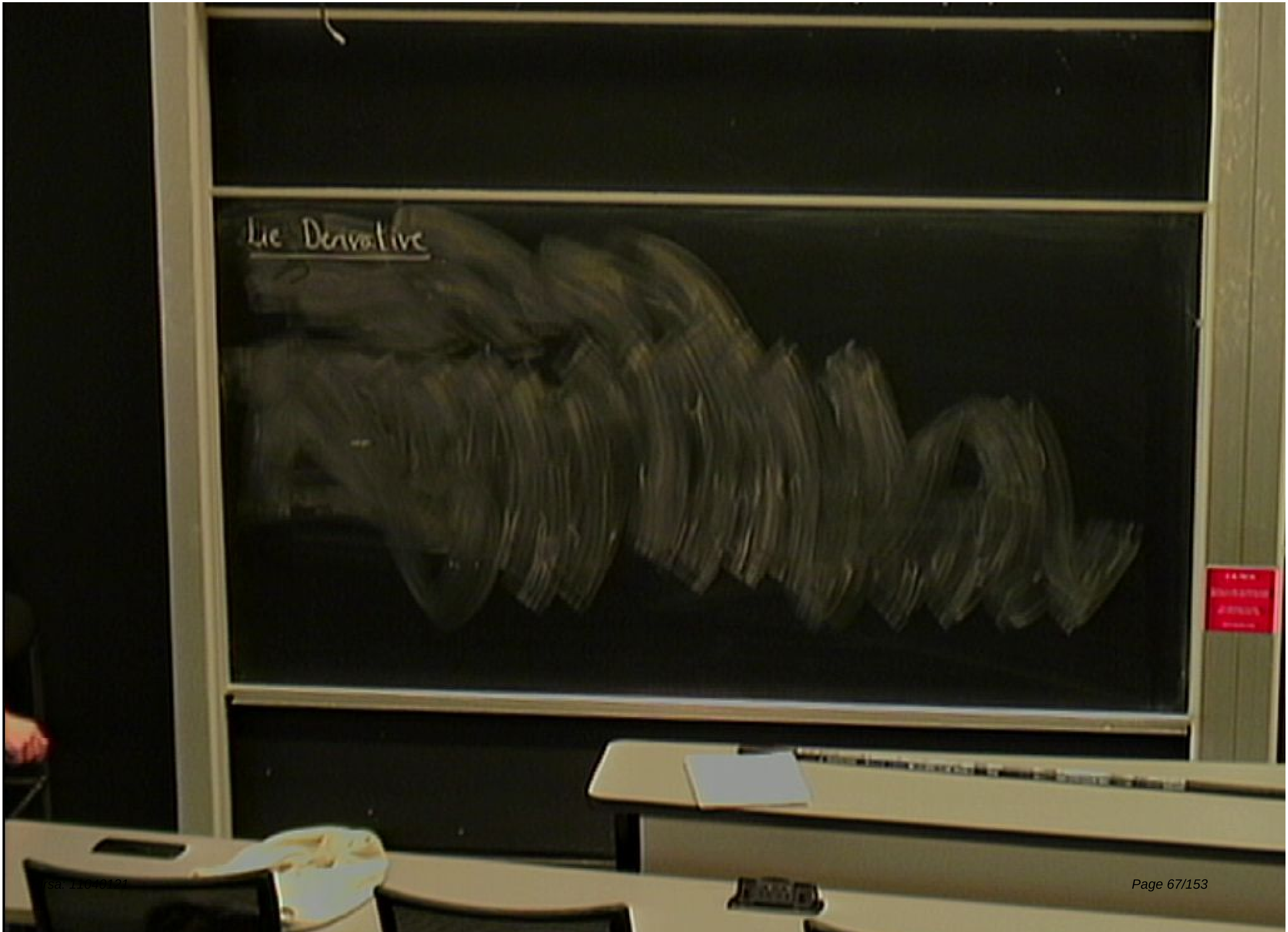
B

the dual space to T_pM.

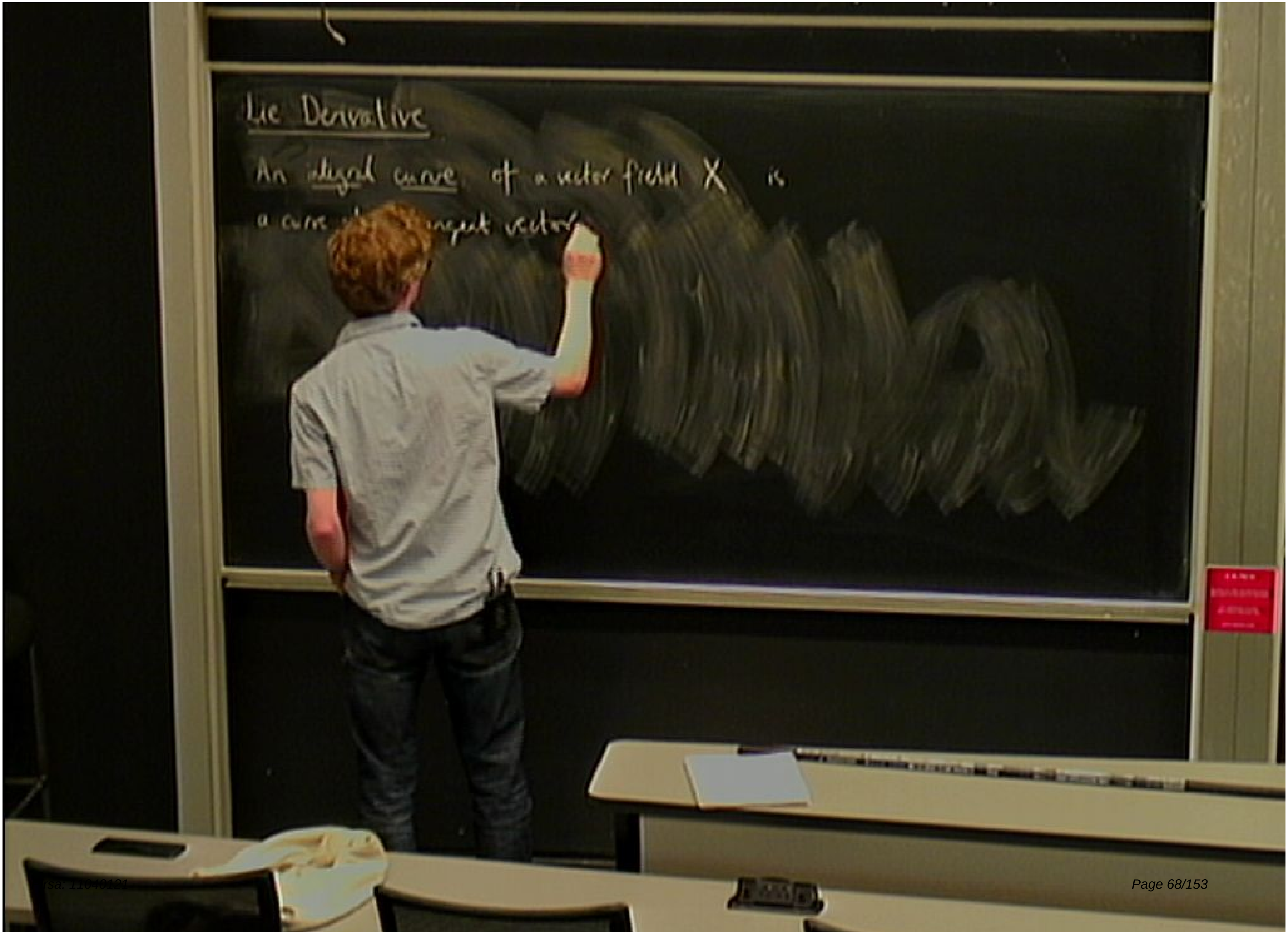
1-forms ; $\omega = \omega_\mu dx^\mu$

duality

$$x^\mu, V^\nu \frac{\partial}{\partial x^\nu}\rangle = \omega_\mu V^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu}\rangle = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu$$

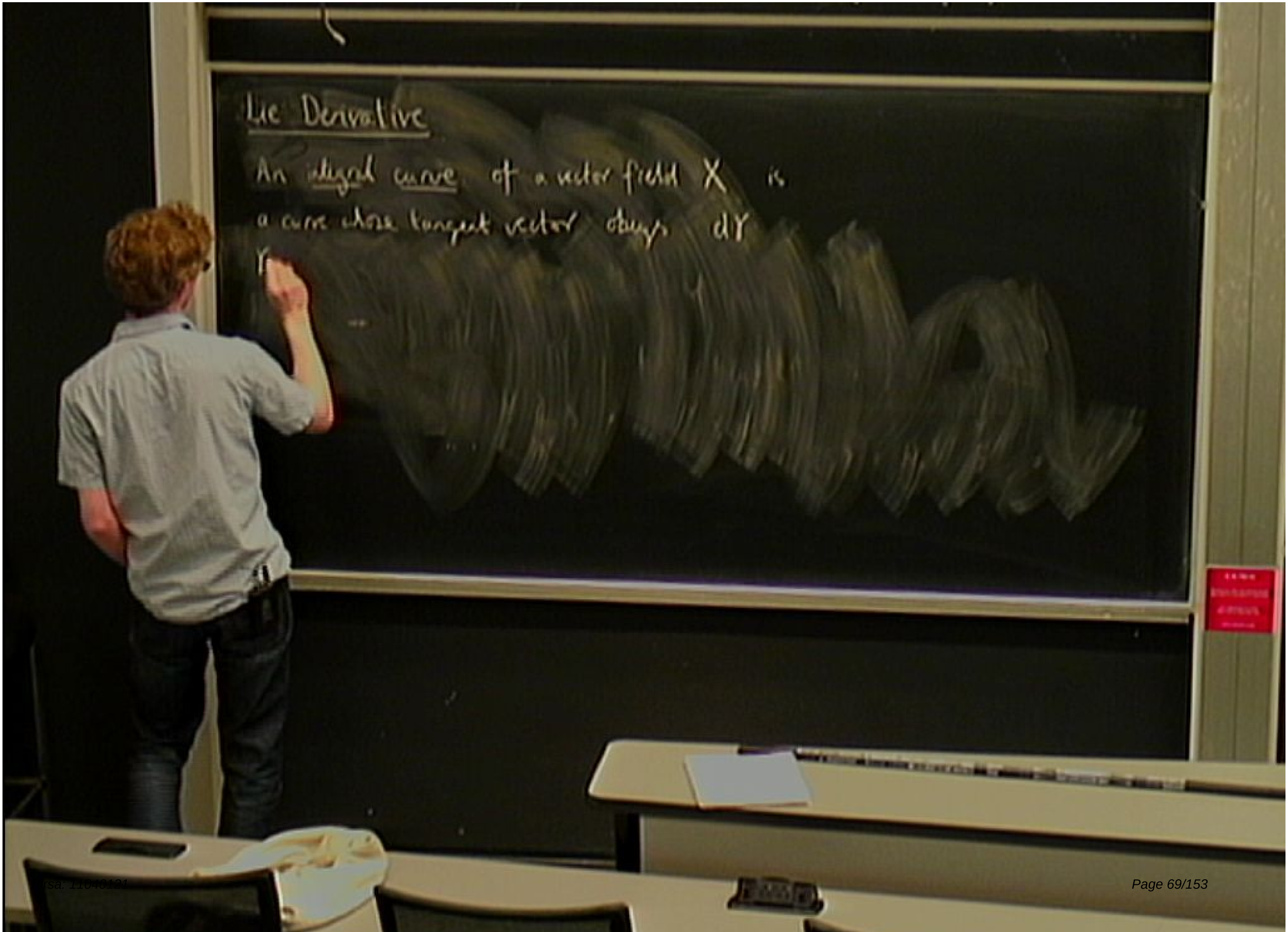


Lie Derivative



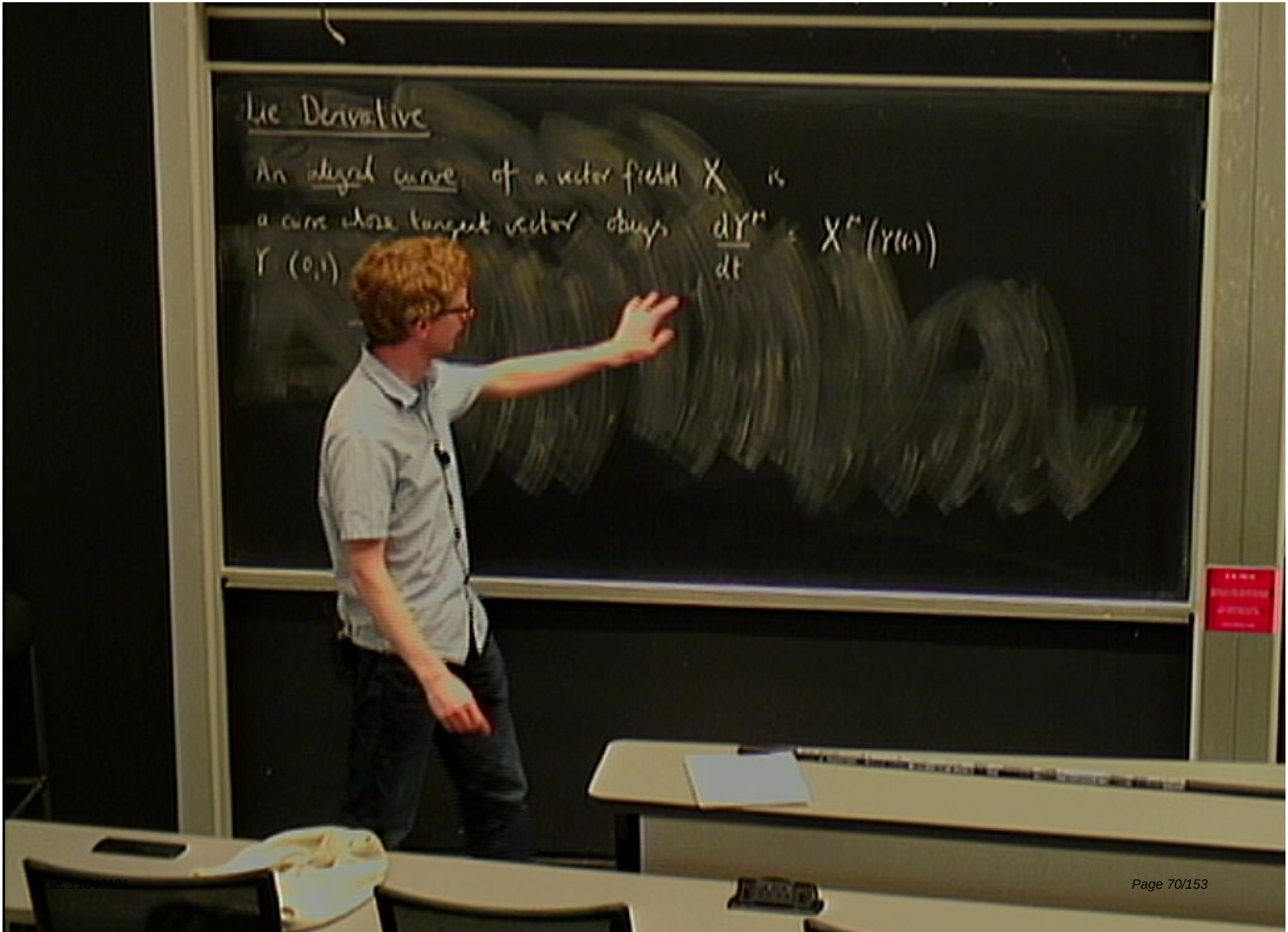
Lie Derivative

An integral curve of a vector field X is a curve [illegible] ngent vector



Lie Derivative

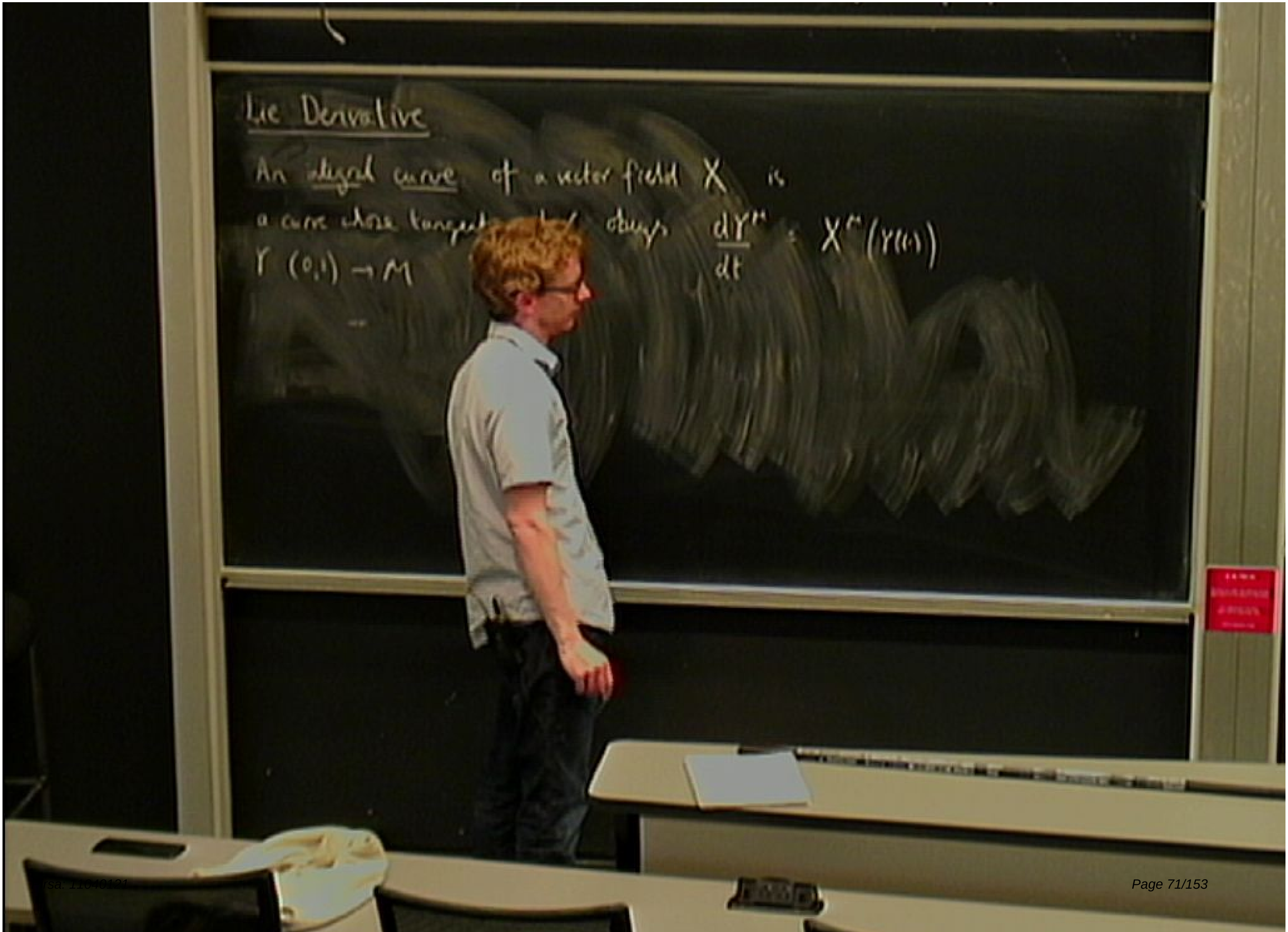
An integral curve of a vector field X is
a curve whose tangent vector obeys dY
Y



Lie Derivative

An integral curve of a vector field X is a curve whose tangent vector obeys $\frac{d\gamma^\mu}{dt} = X^\mu(\gamma(t))$

$\gamma\ (0,1)$



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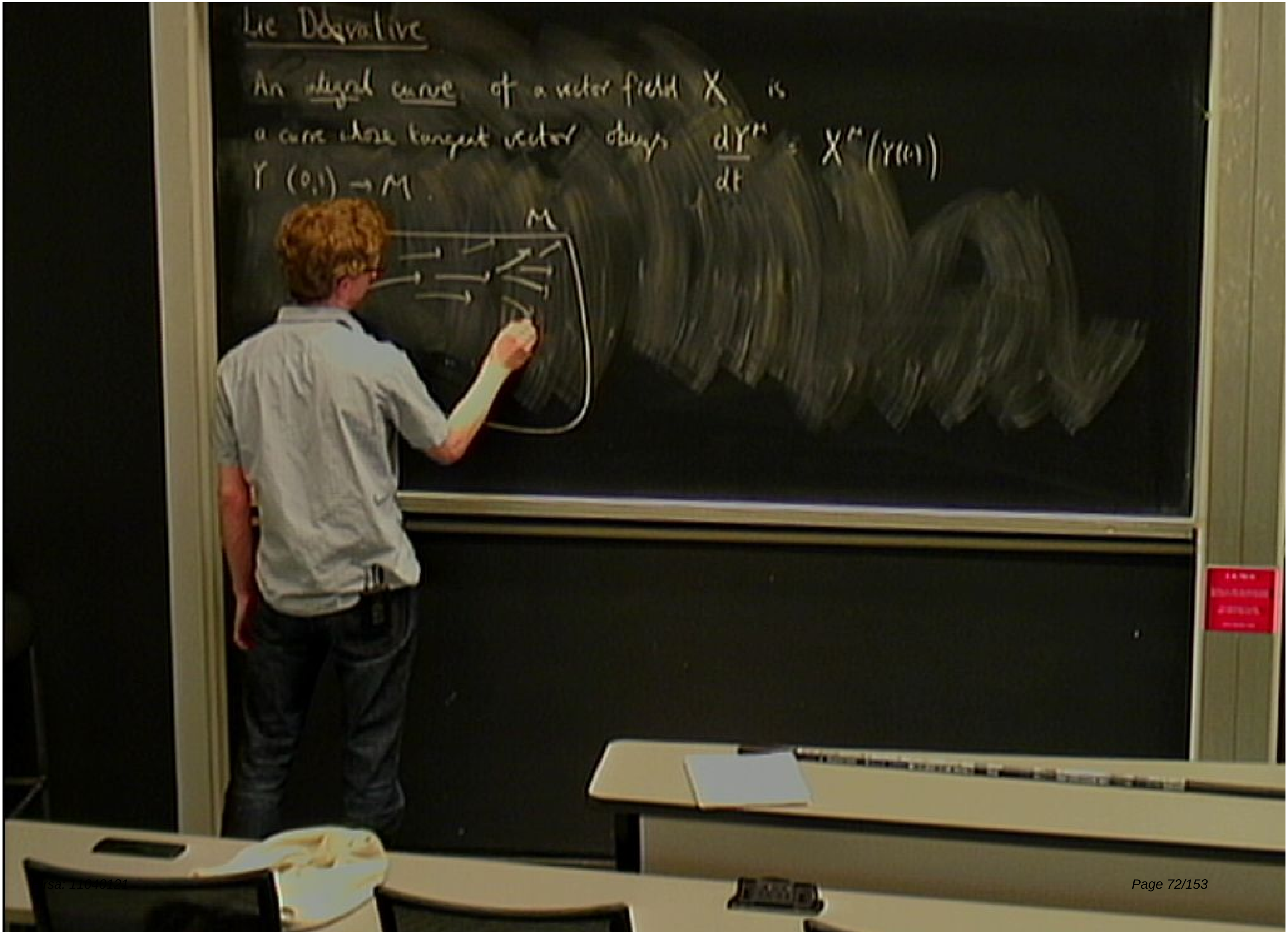
An integral curve of a vector field X is a curve whose tangent [illegible] always $\frac{d\gamma^\mu}{dt} = X^\mu(\gamma(t))$

$\gamma\ (0,1) \to M$



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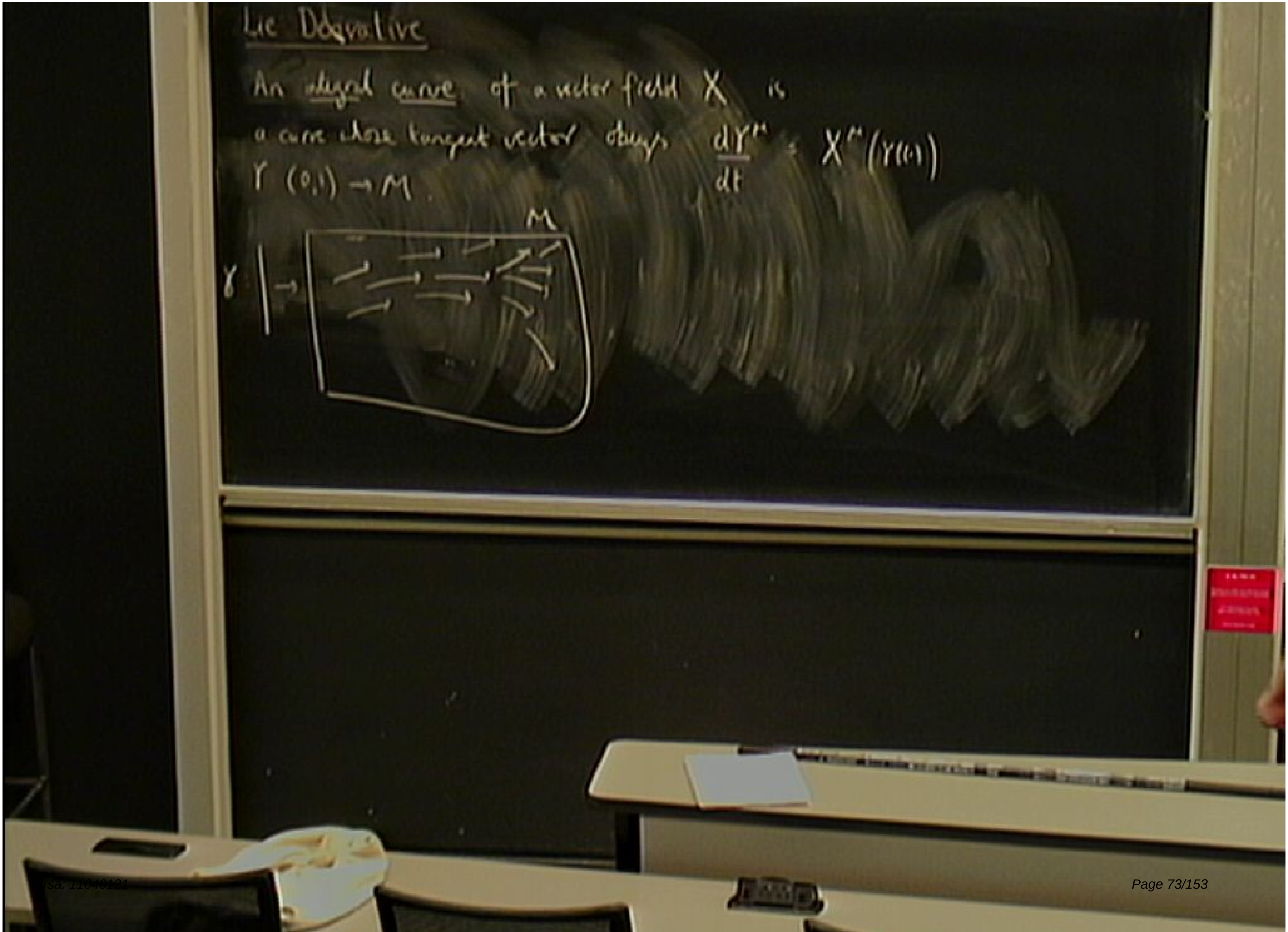

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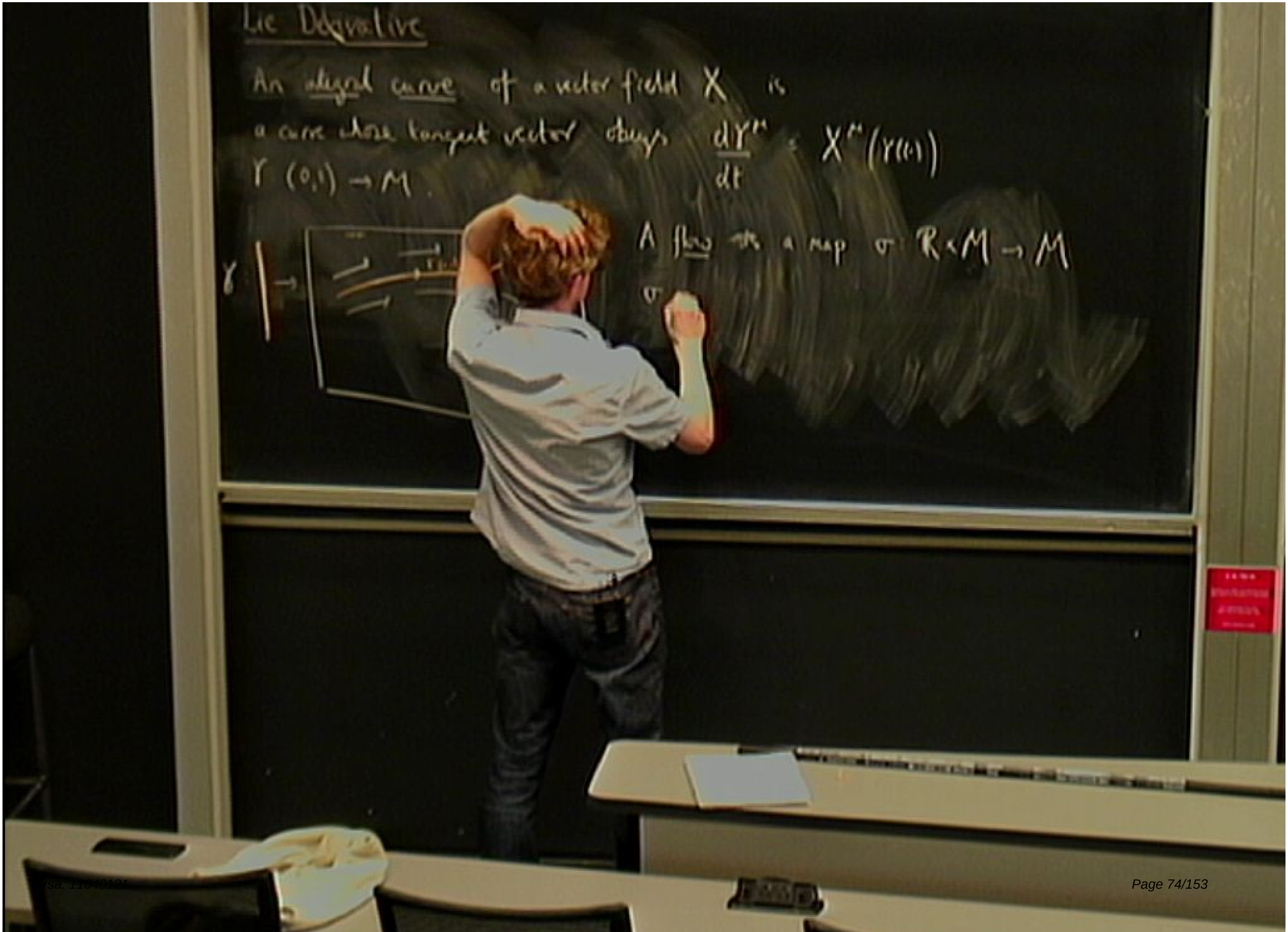


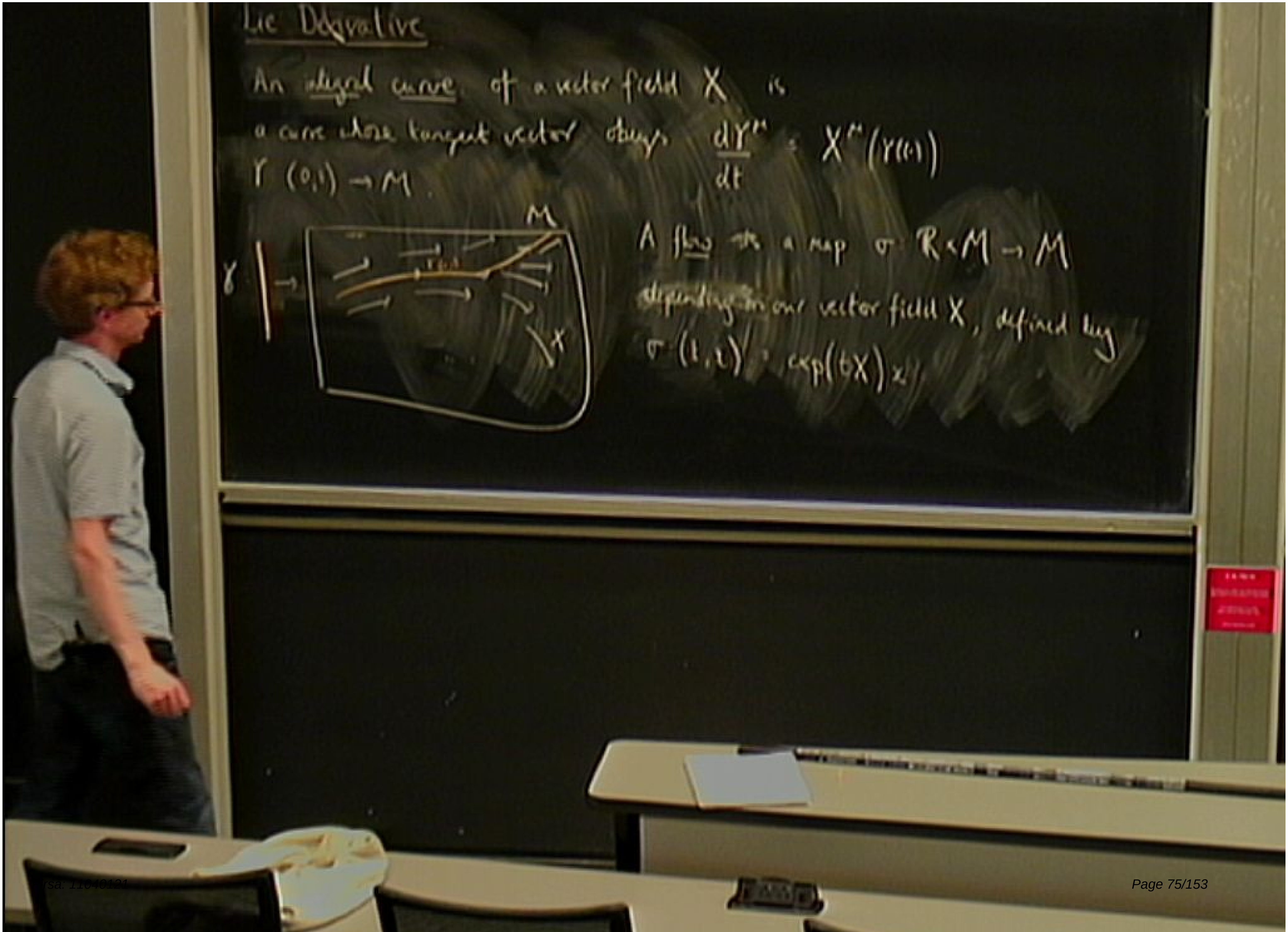
Lie Derivative

An integral curve of a vector field X is a curve whose tangent vector obeys $\frac{d\gamma^\mu}{dt} = X^\mu(\gamma(t))$

$\gamma: (0,1) \to M$

A flow is a map $\sigma: \mathbb{R}\times M \to M$

σ



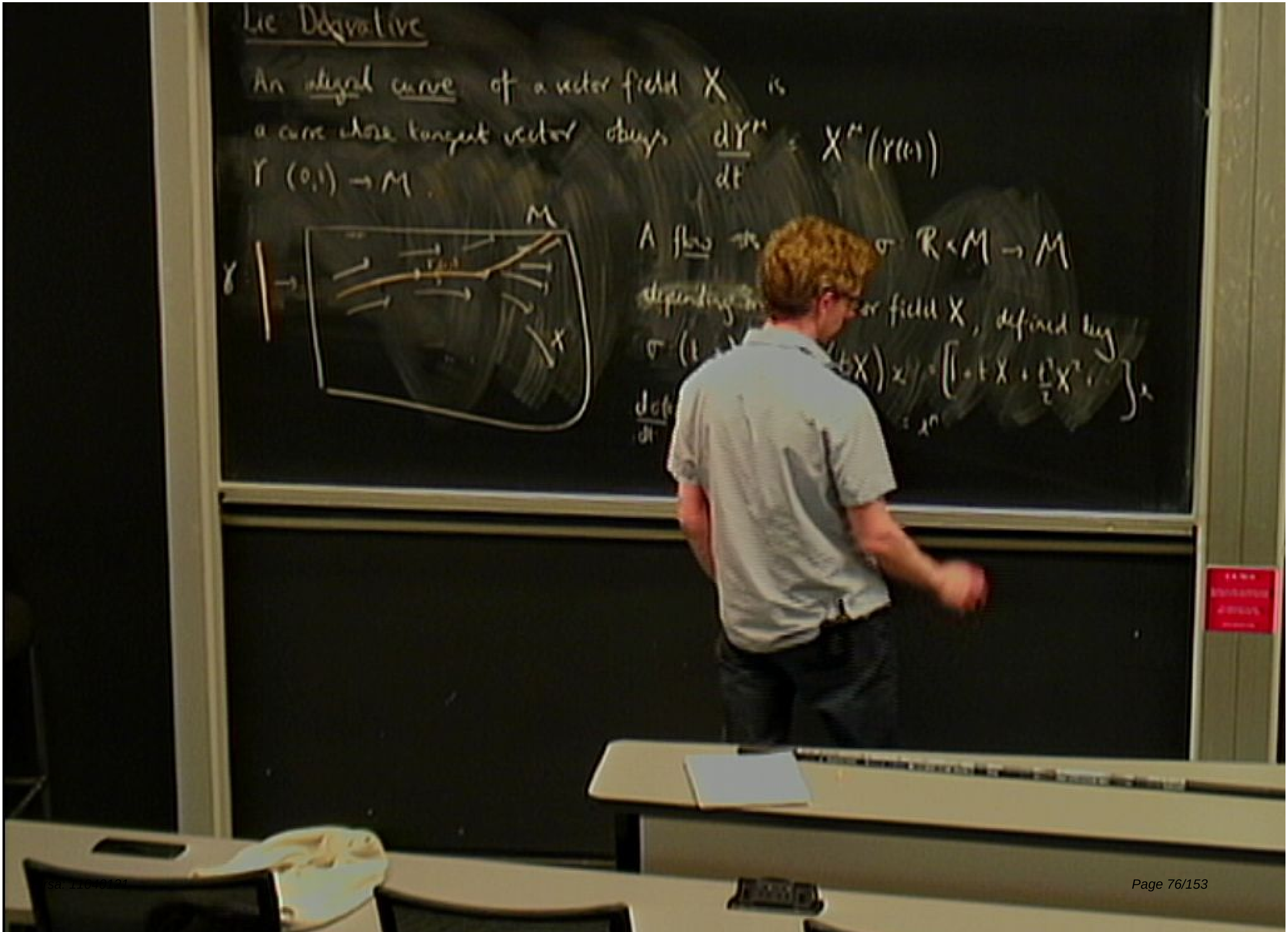
Lie Derivative

An integral curve of a vector field X is a curve whose tangent vector obeys $\frac{d\gamma^\mu}{dt} = X^\mu(\gamma(t))$

$\gamma\ (0,1) \to M$.

A flow is a map $\sigma: \mathbb{R}\times M \to M$ depending on our vector field X, defined by

$\sigma(t,x) = \exp(tX)x$



Lie Derivative

An integral curve of a vector field X is a curve whose tangent vector obeys $\frac{d\gamma^\mu}{dt} = X^\mu(\gamma(t))$

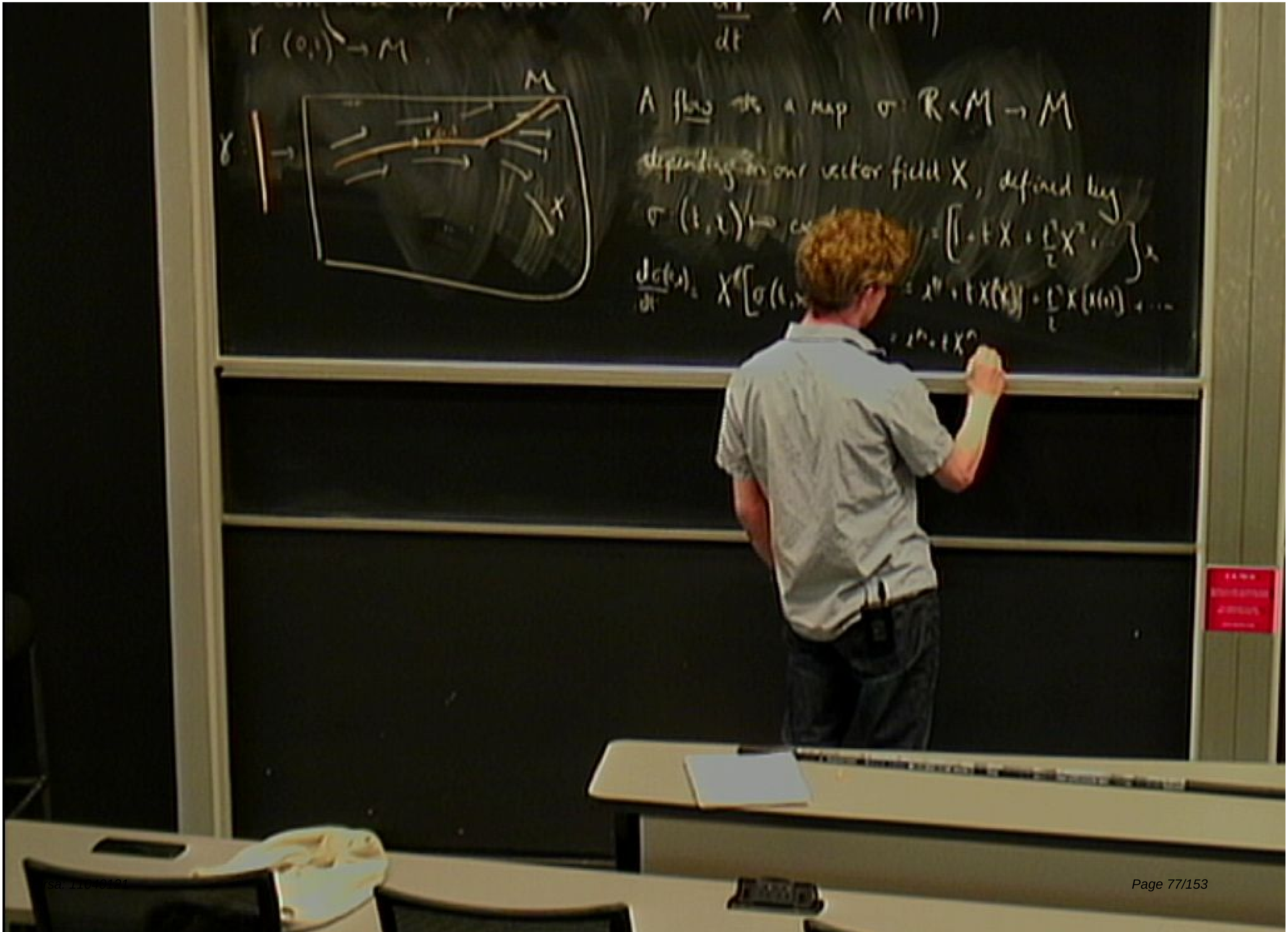
$\gamma\ (0,1) \to M$.

A flow ... $\sigma: \mathbb{R} \times M \to M$

depending ... vector field X, defined by

$\sigma(t, \ldots tX)x = [1 + tX + \frac{t^2}{2}X^2 + \ldots]x$

$\frac{d\sigma}{dt} \ldots = x^\mu$



$\gamma : (0,1) \to M$ $\qquad \frac{d\gamma}{dt} = X(\gamma(t))$

A flow is a map $\sigma : \mathbb{R}\times M \to M$

depending on one vector field X, defined by

$\sigma(t,x) \mapsto$ ex $\dots = \left\{1 + tX + \frac{t^2}{2}X^2 + \dots\right\}x$

$\frac{d\sigma(t,x)}{dt} = X^\mu[\sigma(t,x \dots = x^\mu + tX(x^\mu) + \frac{t^2}{2}X(X(x)) + \dots$

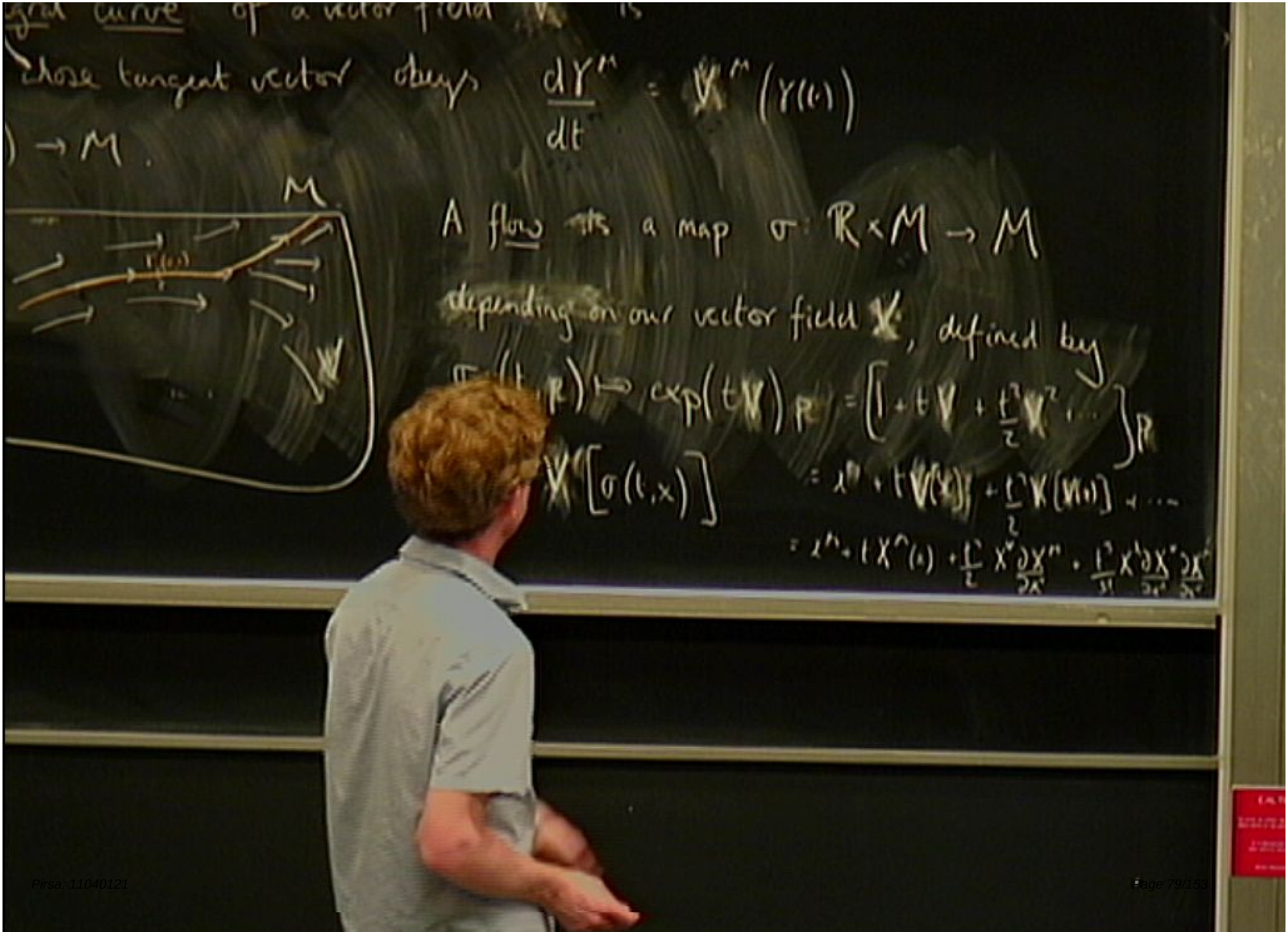
$= x^\mu + tX^\mu$

or field X, defined by

$$) x = \left\{ 1 + tX + \frac{t^2}{2} X^2 + \ldots \right\} x$$

$$= x^\mu + tX[x] + \frac{t^2}{2} X[X(x)] + \ldots$$

$$= x^\mu + tX^\mu(x) + \frac{t^2}{2} X^\nu \frac{\partial X^\mu}{\partial x^\nu} + \frac{t^3}{3!} X^\lambda \frac{\partial X^\nu}{\partial x^\lambda} \frac{\partial X^\mu}{\partial x^\nu}$$



gral curve of a vector field V is

whose tangent vector obeys $\frac{d\gamma^\mu}{dt} = V^\mu(\gamma(t))$

$) \to M$.

A flow is a map $\sigma : \mathbb{R} \times M \to M$

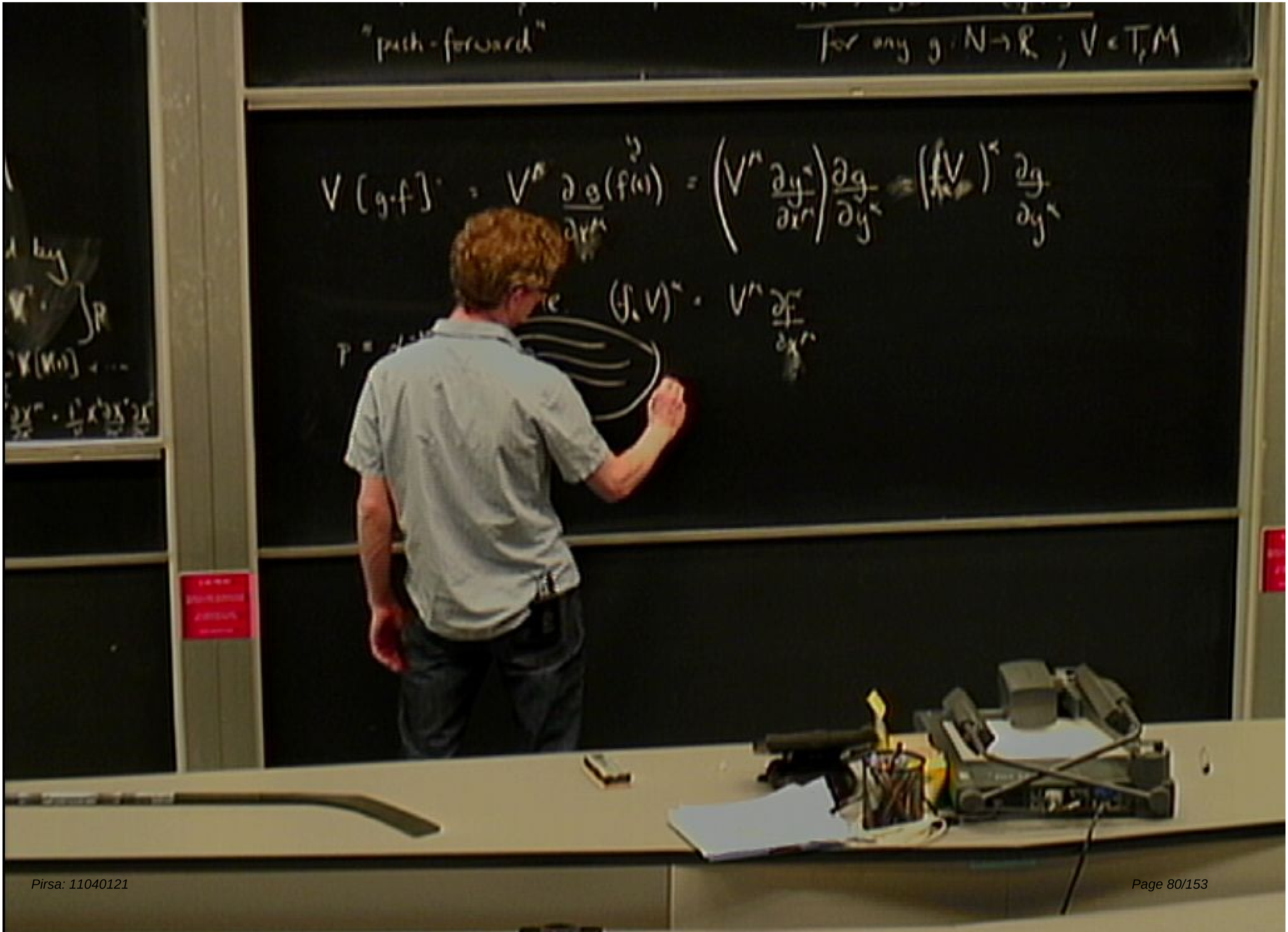
depending on our vector field V, defined by

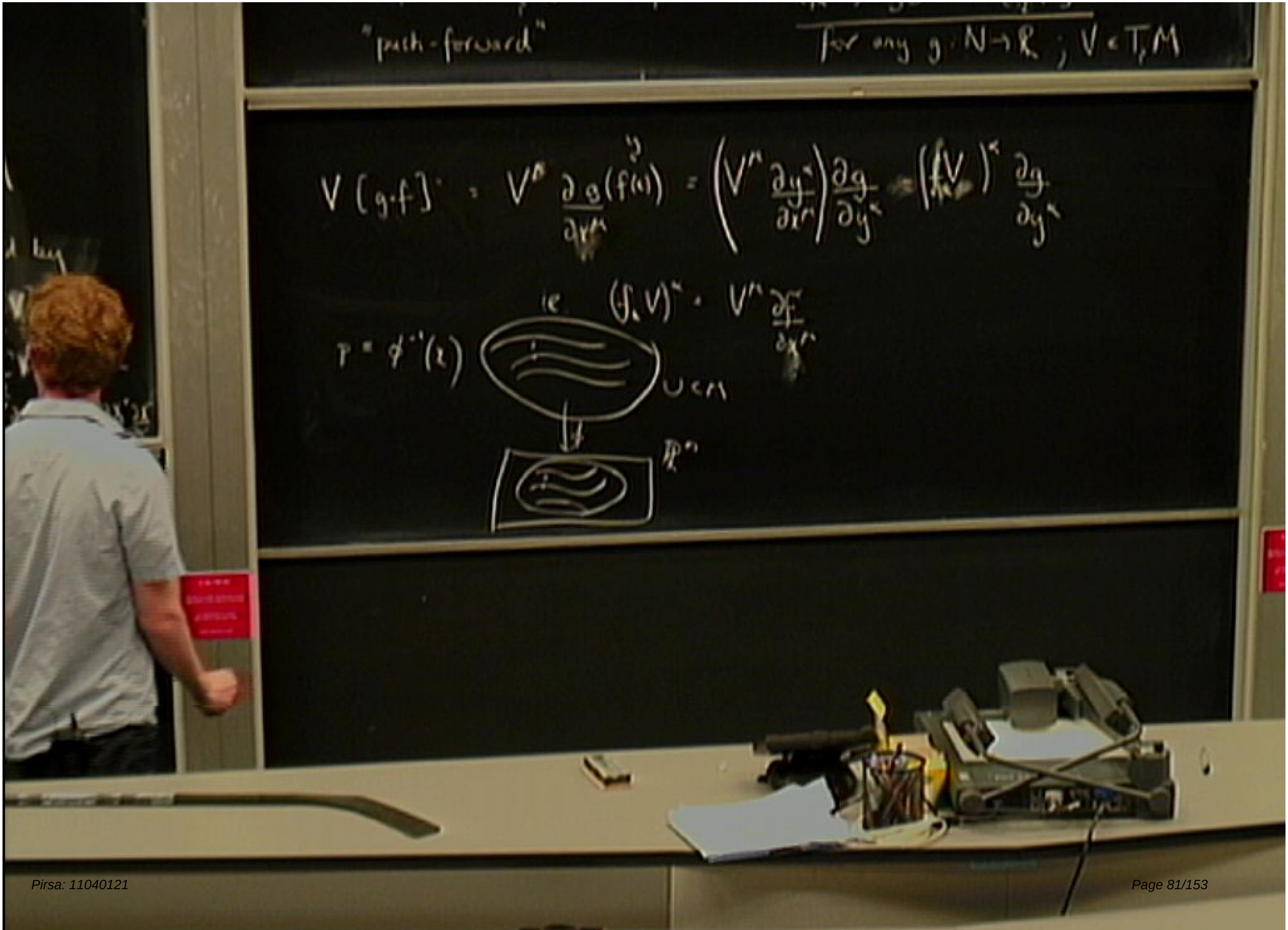
$\sigma(t,p) \mapsto \exp(tV)p = \{1 + tV + \frac{t^2}{2}V^2 \dots\}p$

$V^\mu[\sigma(t,x)]$

$= x^\mu + tV(x^\mu) + \frac{t^2}{2}V(V(x^\mu)) + \dots$

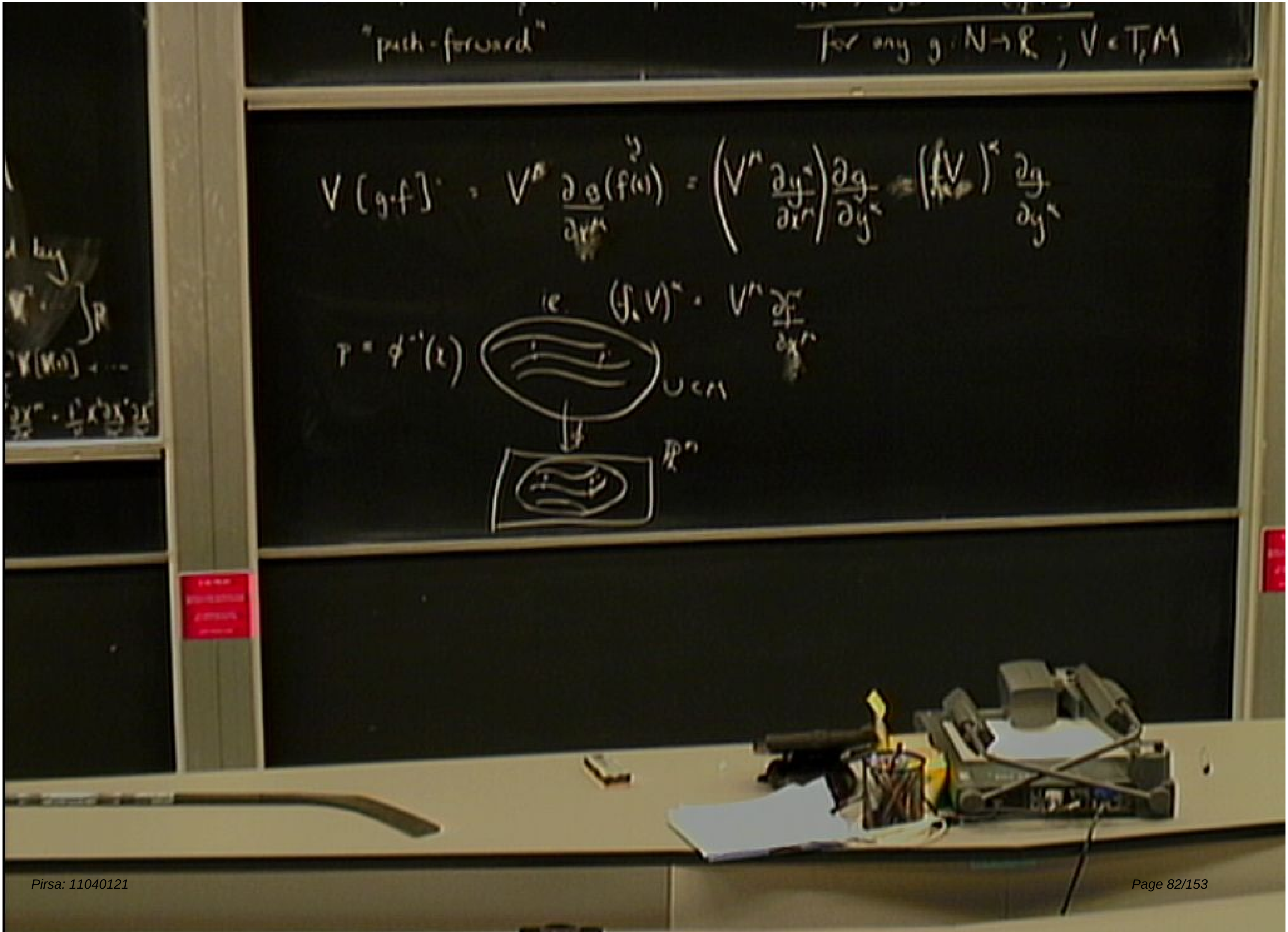
$= x^\mu + tX^\mu(x) + \frac{t^2}{2}X^\nu\frac{\partial X^\mu}{\partial x^\nu} + \frac{t^3}{3!}X^\lambda\frac{\partial X^\nu}{\partial x^\lambda}\frac{\partial X^\mu}{\partial x^\nu}$


"push-forward"
for any g : N→ℝ ; V ∈ T_p M
V [g∘f] = V^μ ∂g(f(x))/∂x^μ = (V^μ ∂y^α/∂x^μ) ∂g/∂y^α = (f_* V)^α ∂g/∂y^α
(f_* V)^α = V^μ ∂f^α/∂x^μ
R



$$V[g \circ f] = V^\mu \frac{\partial \overset{y}{g(f(x))}}{\partial x^\mu} = \left(V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\right)\frac{\partial g}{\partial y^\alpha} = (f_* V)^\alpha \frac{\partial g}{\partial y^\alpha}$$

ie. $(f_* V)^\alpha = V^\mu \frac{\partial f^\alpha}{\partial x^\mu}$



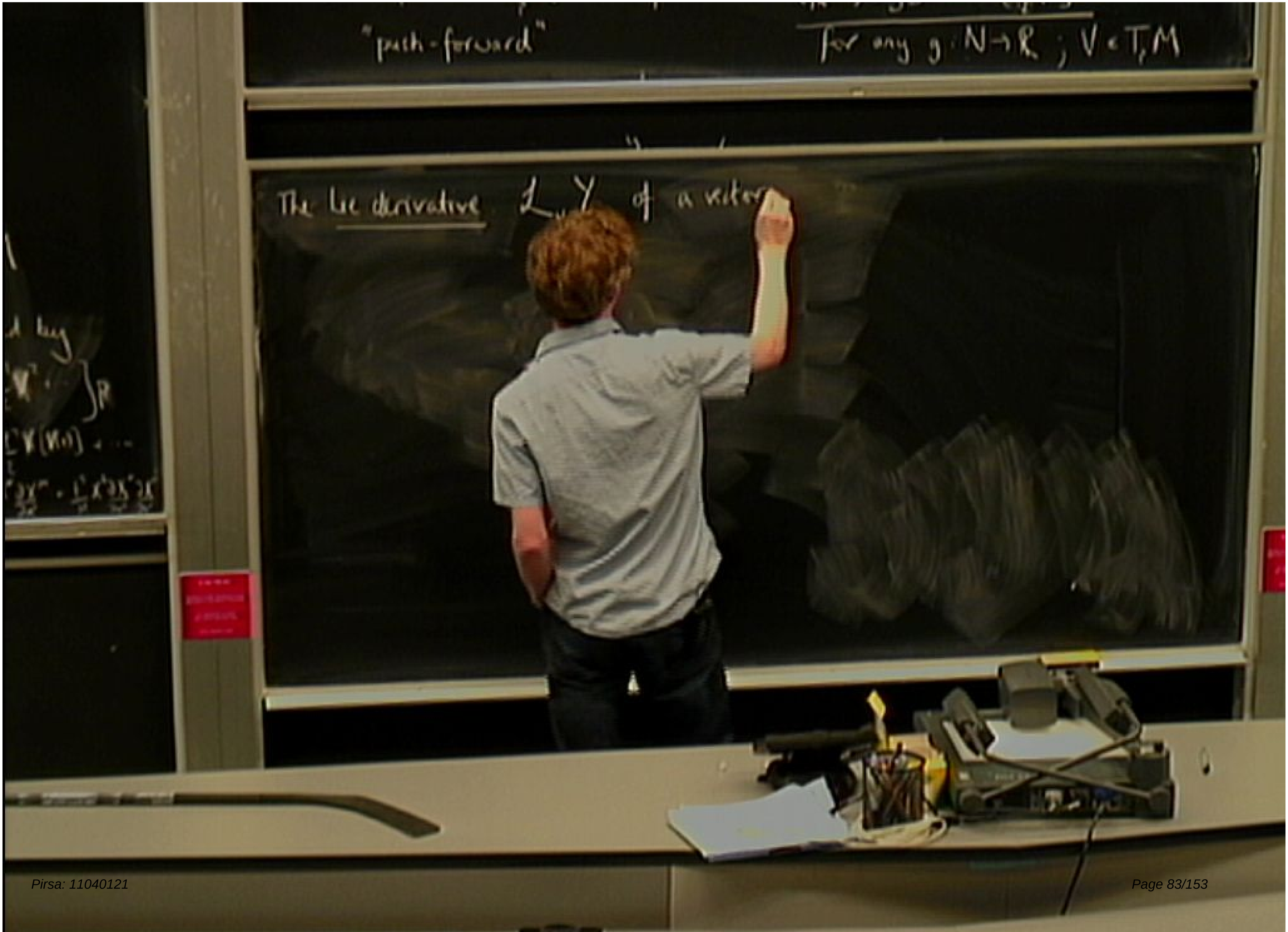
"push-forward"

for any $g : N \to \mathbb{R}$; $V \in T_p M$

$$V[g \circ f] = V^\mu \frac{\partial g(f(x))}{\partial x^\mu} = \left(V^\mu \frac{\partial y^\alpha}{\partial x^\mu}\right) \frac{\partial g}{\partial y^\alpha} = (f_* V)^\alpha \frac{\partial g}{\partial y^\alpha}$$

i.e. $(f_* V)^\alpha = V^\mu \frac{\partial f^\alpha}{\partial x^\mu}$

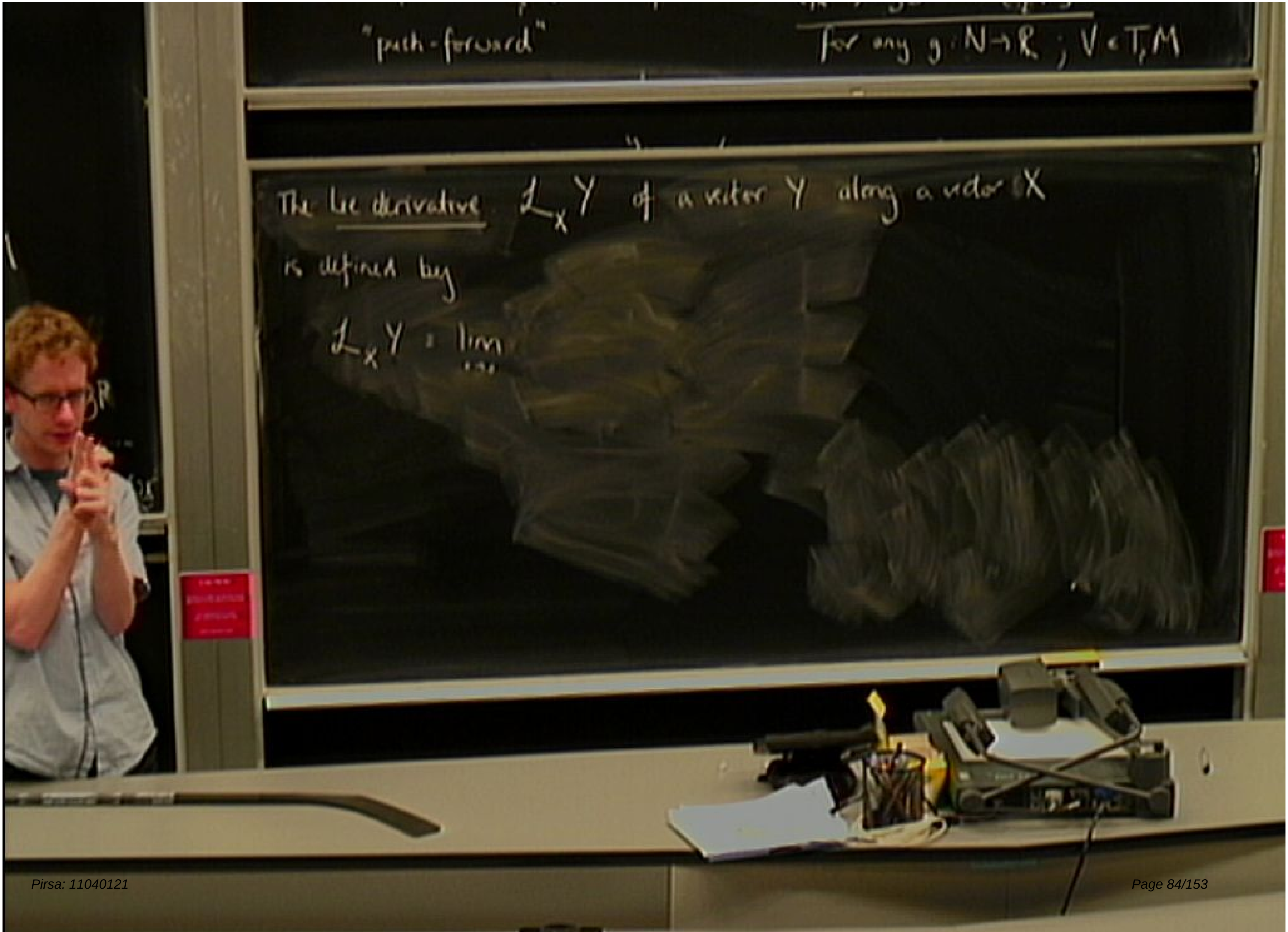
$p = \phi^{-1}(x)$



"push-forward"

for any $g: N \to \mathbb{R}$; $V \in T_pM$

The Lie derivative $\mathcal{L}_X Y$ of a vector

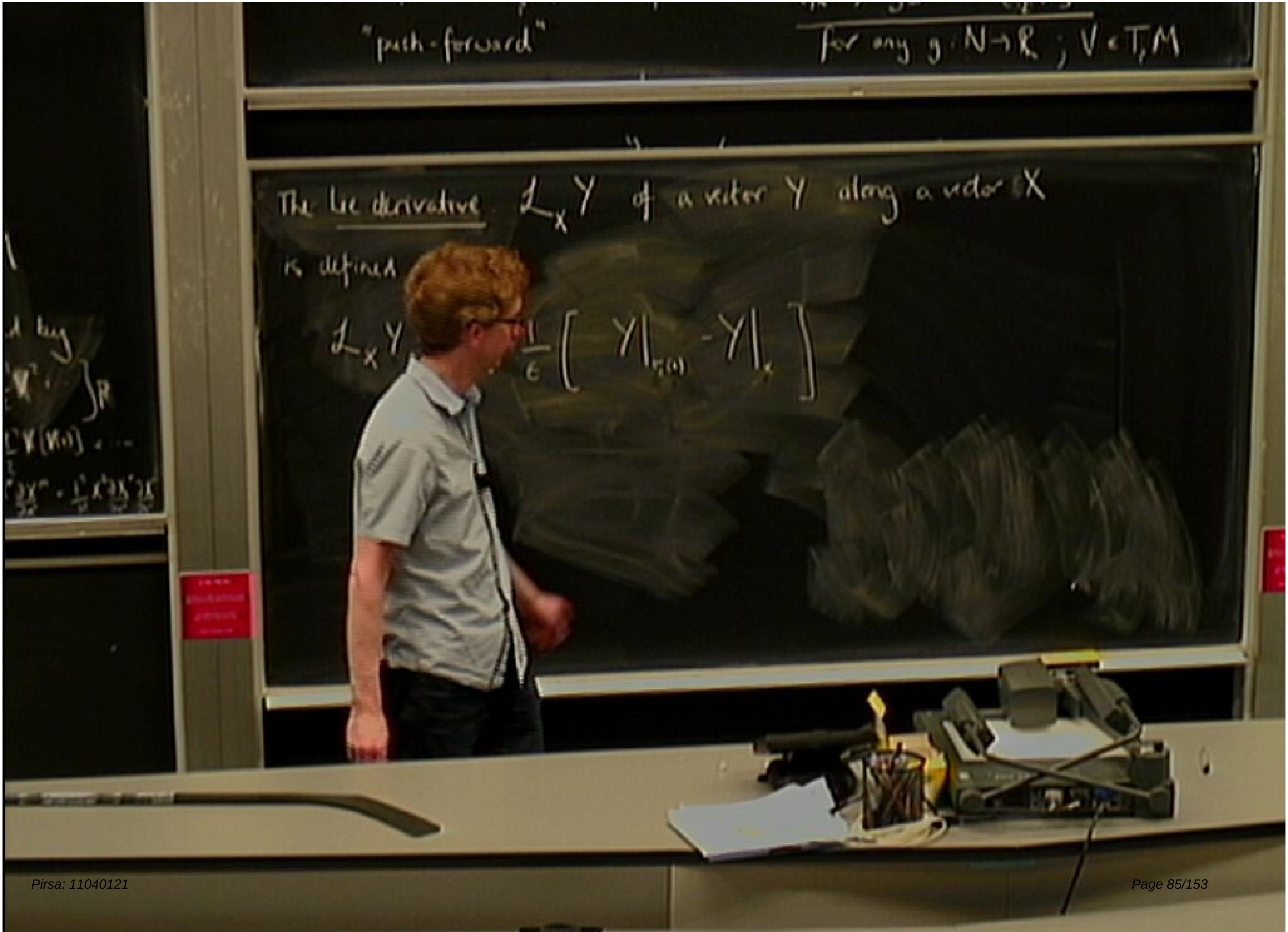


"push-forward"

For any $g: N \to \mathbb{R}$; $V \in T_p M$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim$$

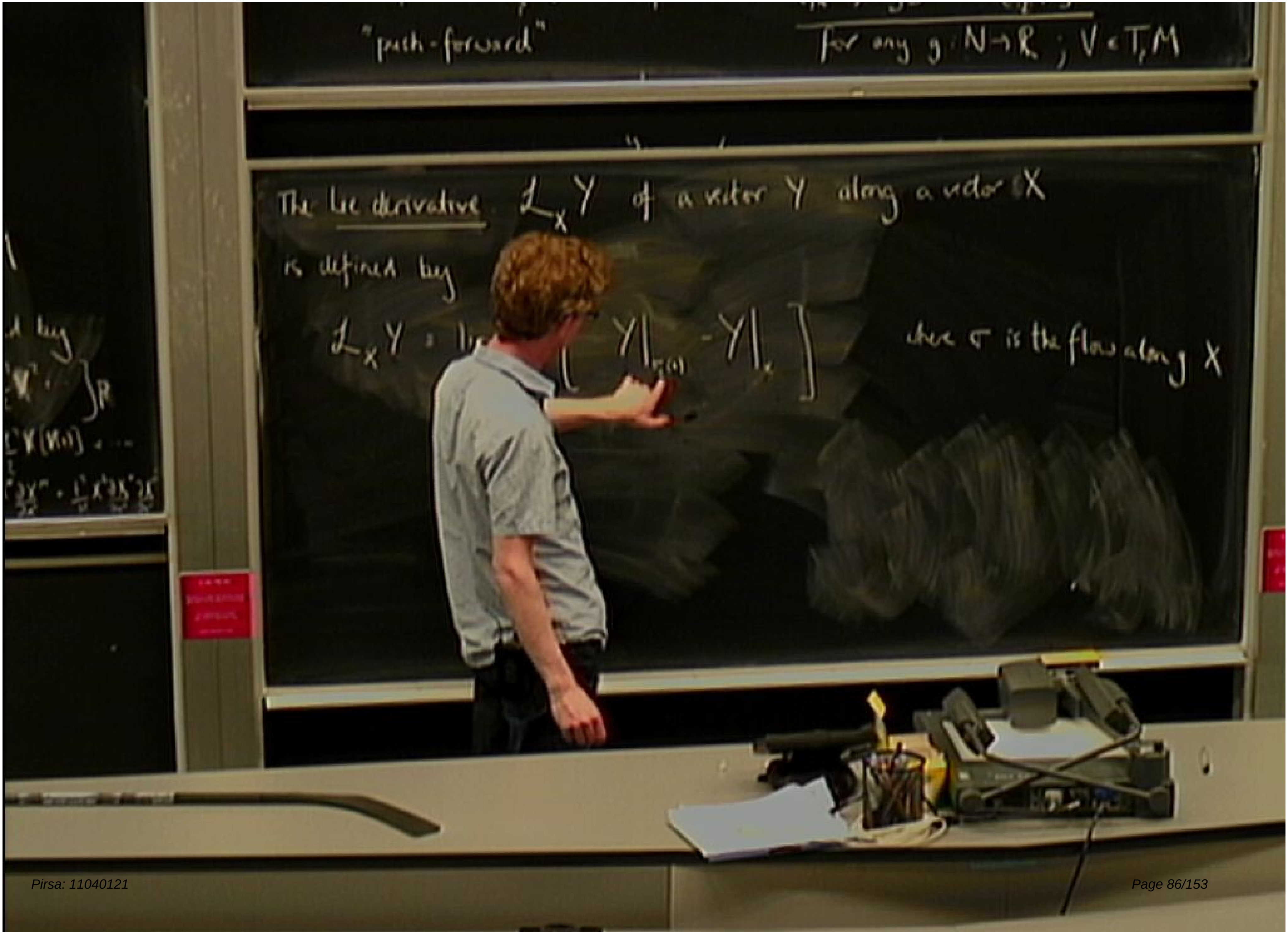


"push-forward"

For any $g : N \to \mathbb{R}$; $V \in T_pM$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined

$$\mathcal{L}_X Y \quad \frac{1}{\epsilon}\left[Y|_{\gamma(\epsilon)} - Y|_x \right]$$



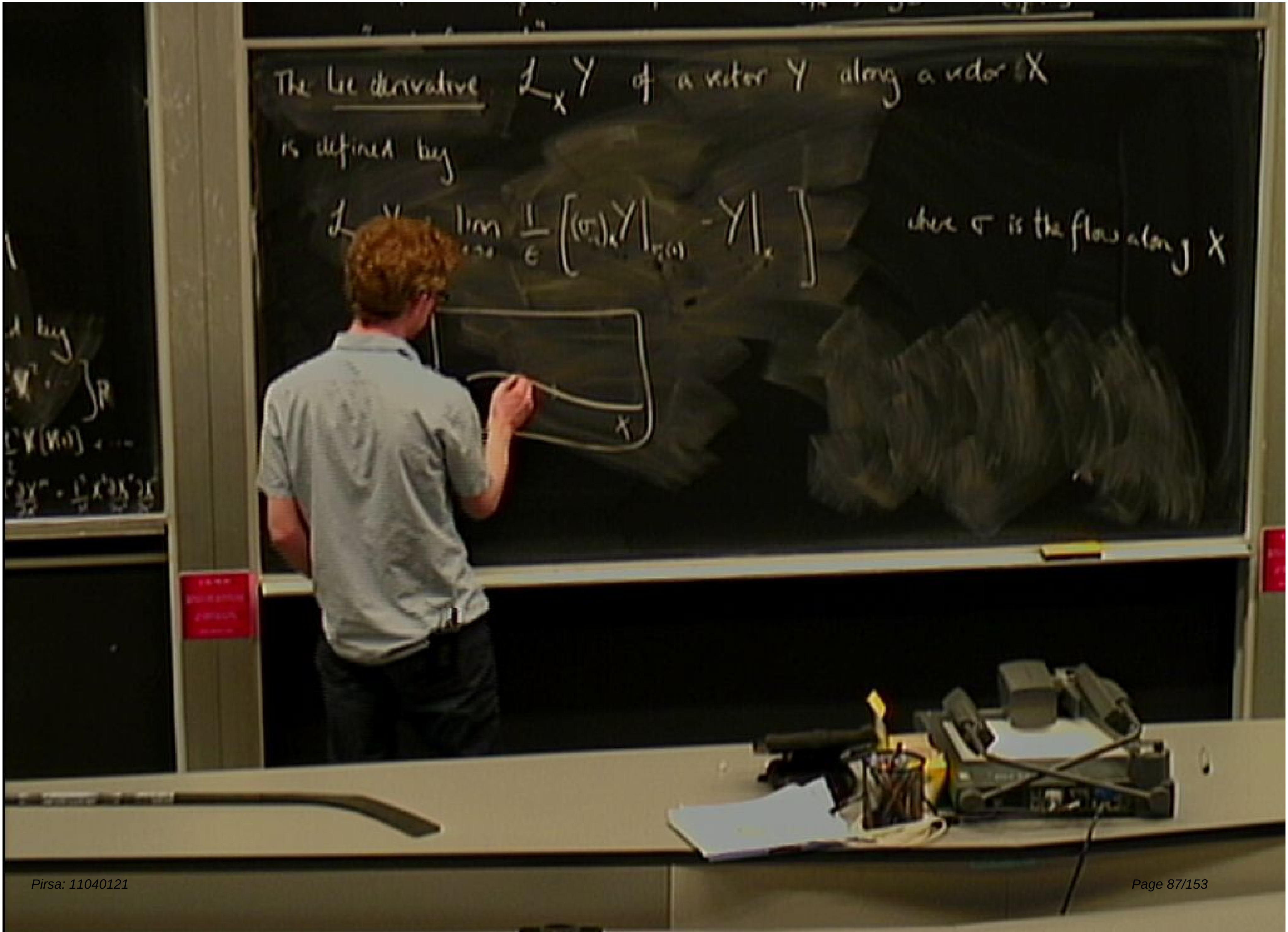
"push-forward"

For any $g : N \to \mathbb{R}$; $V \in T_p M$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim \left[Y|_{\sigma(t)} - Y|_x \right]$$

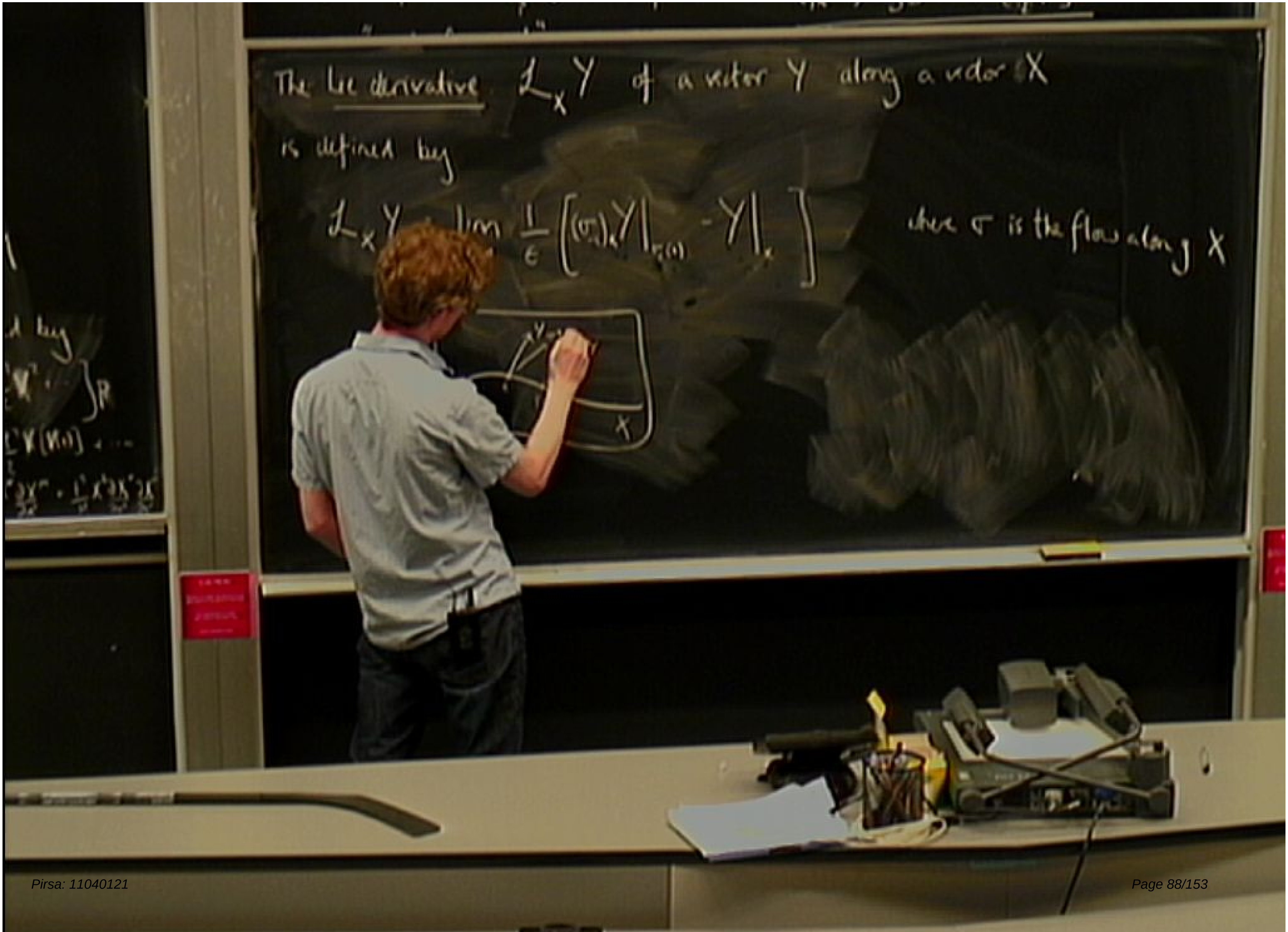
where σ is the flow along X



The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_* Y\big|_{\sigma_\epsilon(x)} - Y\big|_x\right]$$

where σ is the flow along X



The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

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The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

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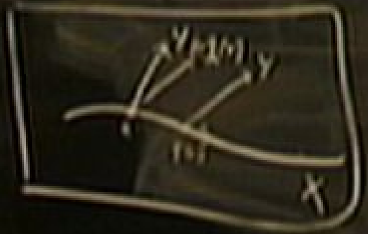
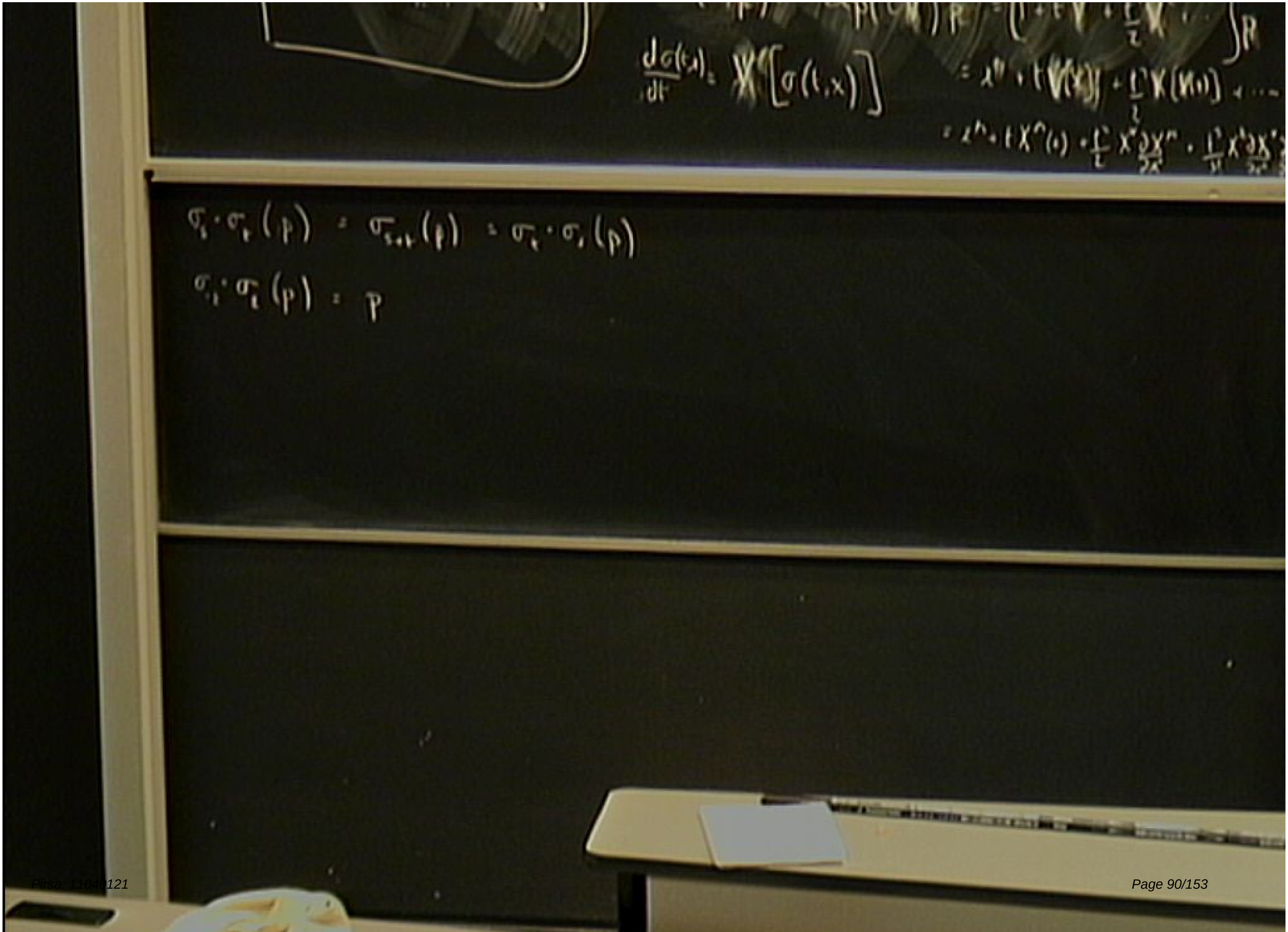


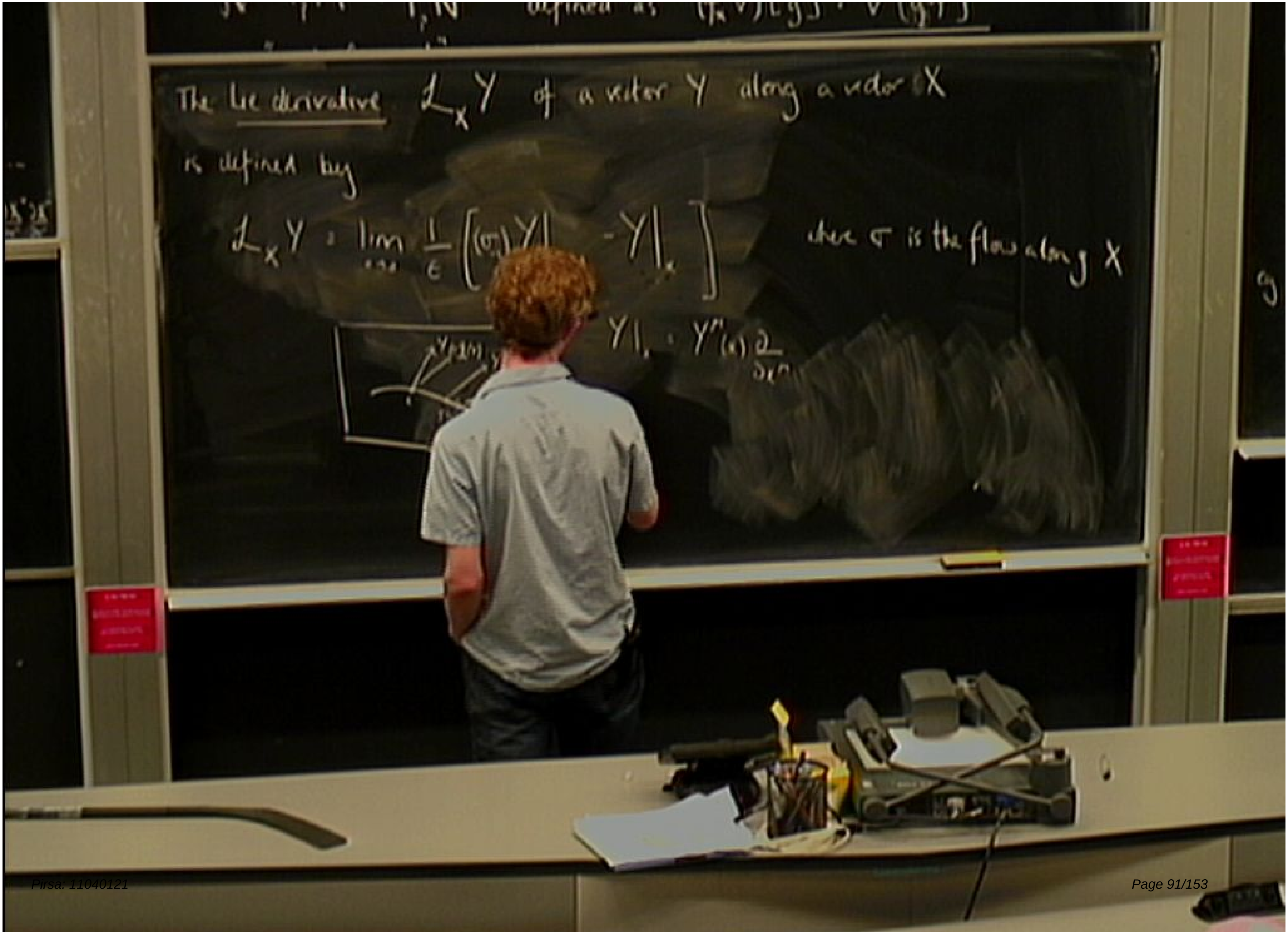


$$\frac{d\sigma(t,x)}{dt} = X\left[\sigma(t,x)\right]$$

$$= x^\mu + t X^\mu(x) + \frac{t^2}{2} X^\nu \partial_\nu X^\mu + \frac{t^3}{3!} X^\rho \partial_\rho X^\nu$$

$$\sigma_s \cdot \sigma_t (p) = \sigma_{s+t}(p) = \sigma_t \cdot \sigma_s (p)$$

$$\sigma_{-t} \cdot \sigma_t (p) = p$$



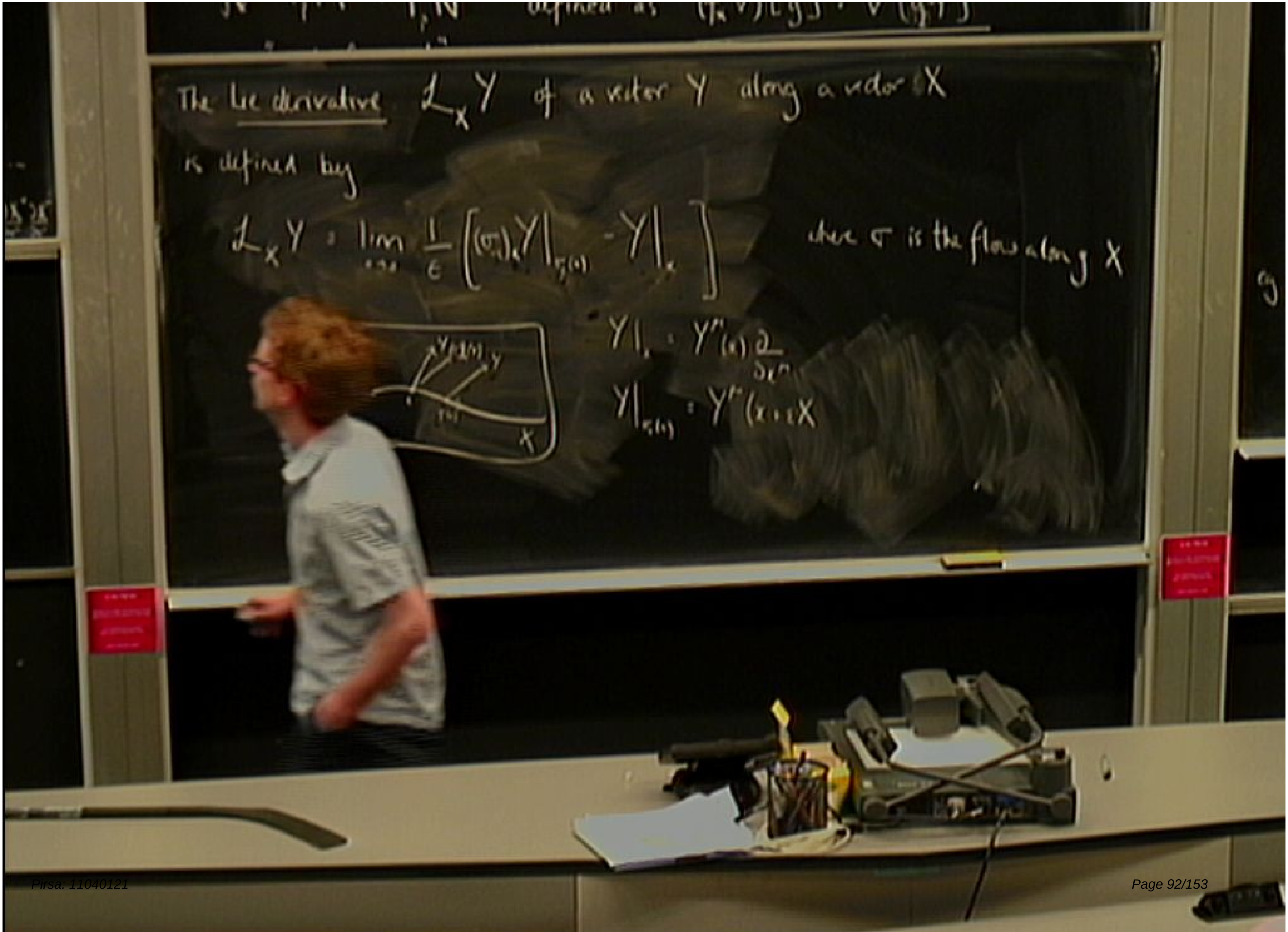
defined as $(\phi_* v)[g] = v(g \circ \phi)$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[(\sigma_{\epsilon})_* Y| \quad - Y|_x\right]$$

where σ is the flow along X

$$Y|_x = Y^\mu(x)\frac{\partial}{\partial x^\mu}$$



The <u>Lie derivative</u> $\mathcal{L}_X Y$ of a vector Y along a vector X

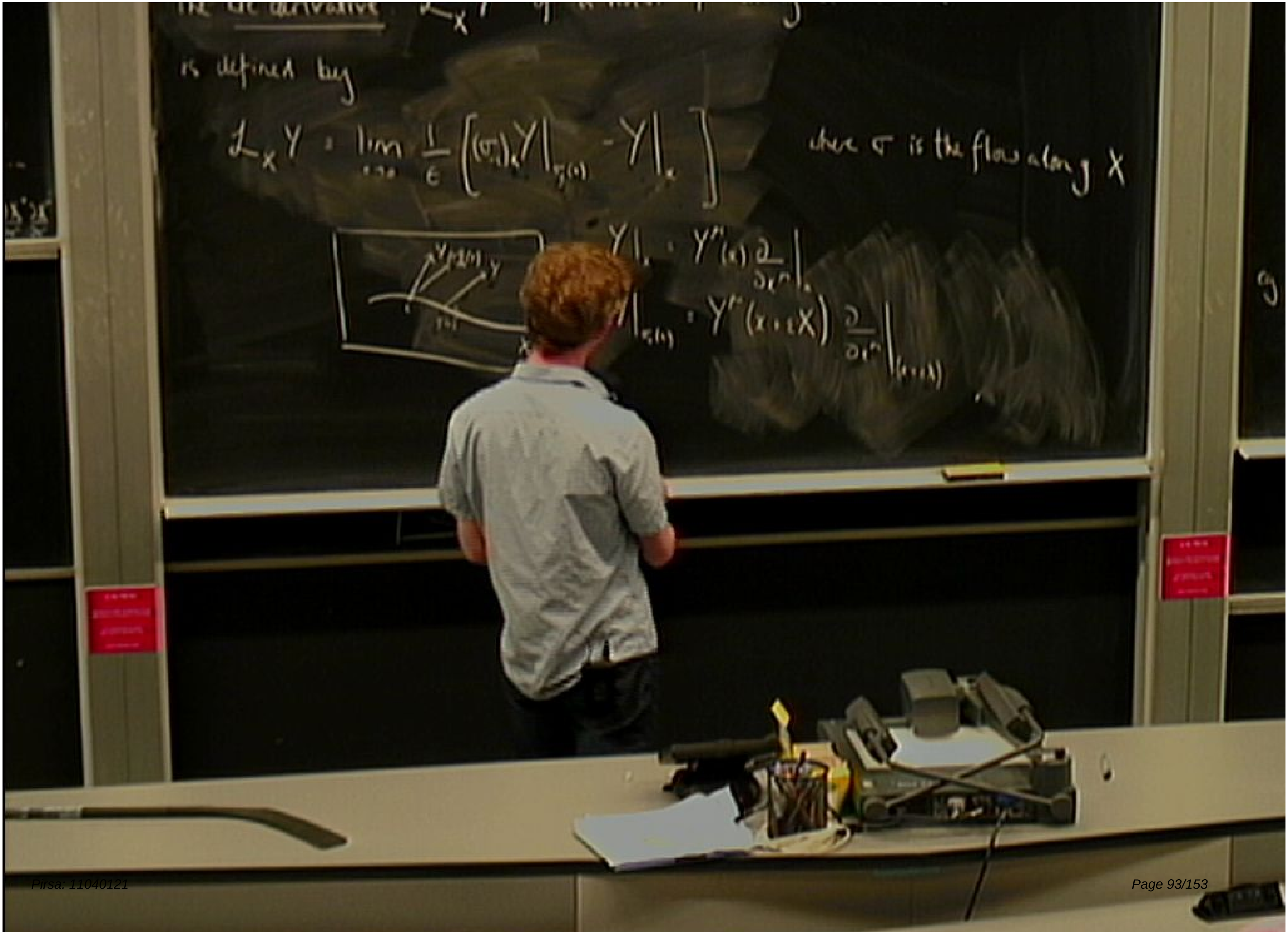
is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} - Y|_x\right]$$

where σ is the flow along X

$$Y|_x = Y^\mu(x)\frac{\partial}{\partial x^\mu}$$

$$Y|_{\sigma_\epsilon(x)} = Y^\mu(x+\epsilon X$$



The Lie derivative $\mathcal{L}_X Y$

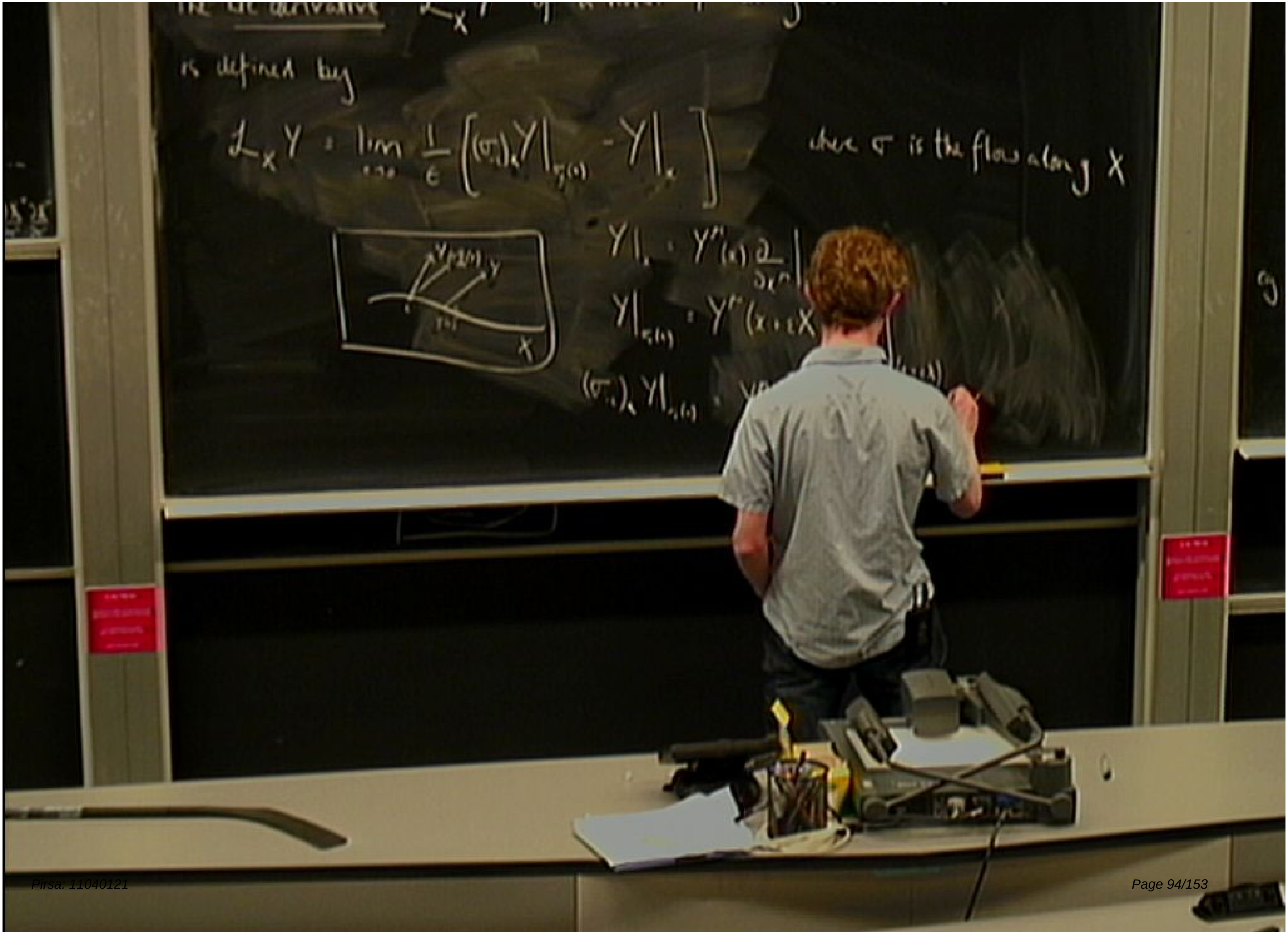
is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_* Y\big|_{\sigma_\epsilon(x)} - Y\big|_x \right]$$

where σ is the flow along X

$$Y\big|_x = Y^\mu(x) \frac{\partial}{\partial x^\mu}\Big|_x$$

$$Y\big|_{\sigma_\epsilon(x)} = Y^\mu(x + \epsilon X) \frac{\partial}{\partial x^\mu}\Big|_{(x+\epsilon X)}$$



the Lie derivative $\mathcal{L}_X$

is defined by

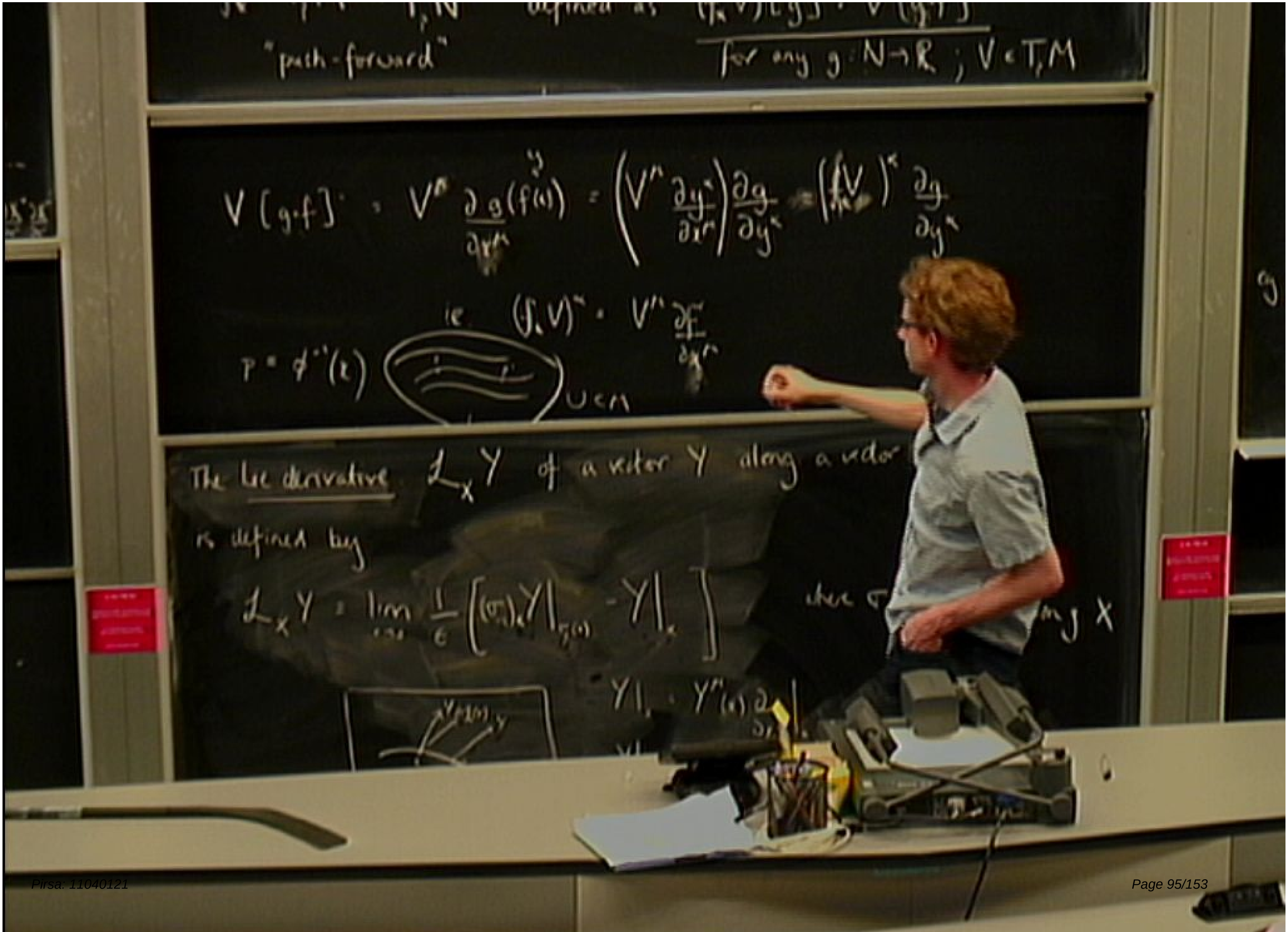
$$\mathcal{L}_X Y = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} - Y|_x \right]$$

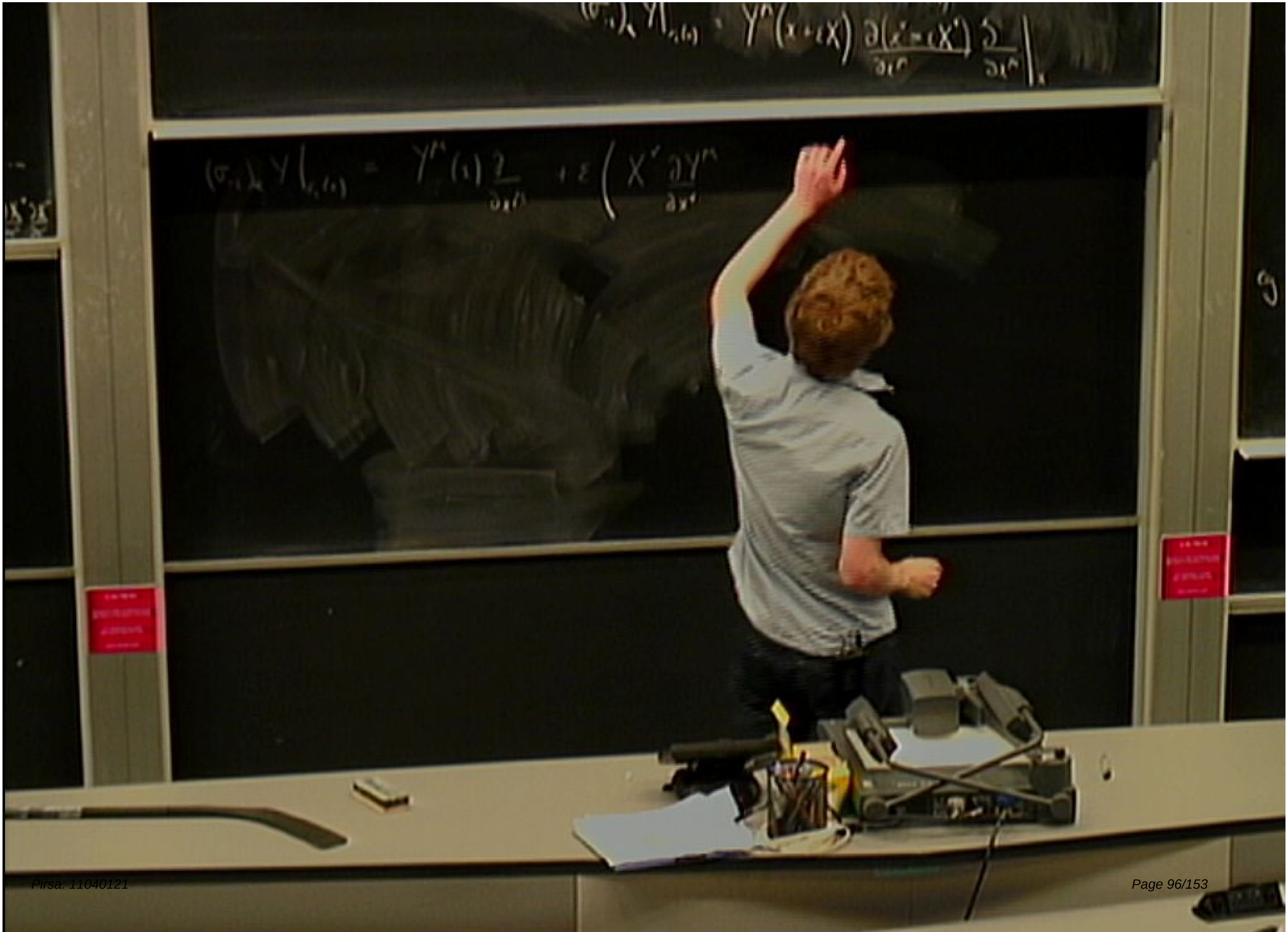
where σ is the flow along X

$$Y|_x = Y^\mu(x) \frac{\partial}{\partial x^\mu}\Big|$$

$$Y|_{\sigma_\epsilon(x)} = Y^\mu(x + \epsilon X$$

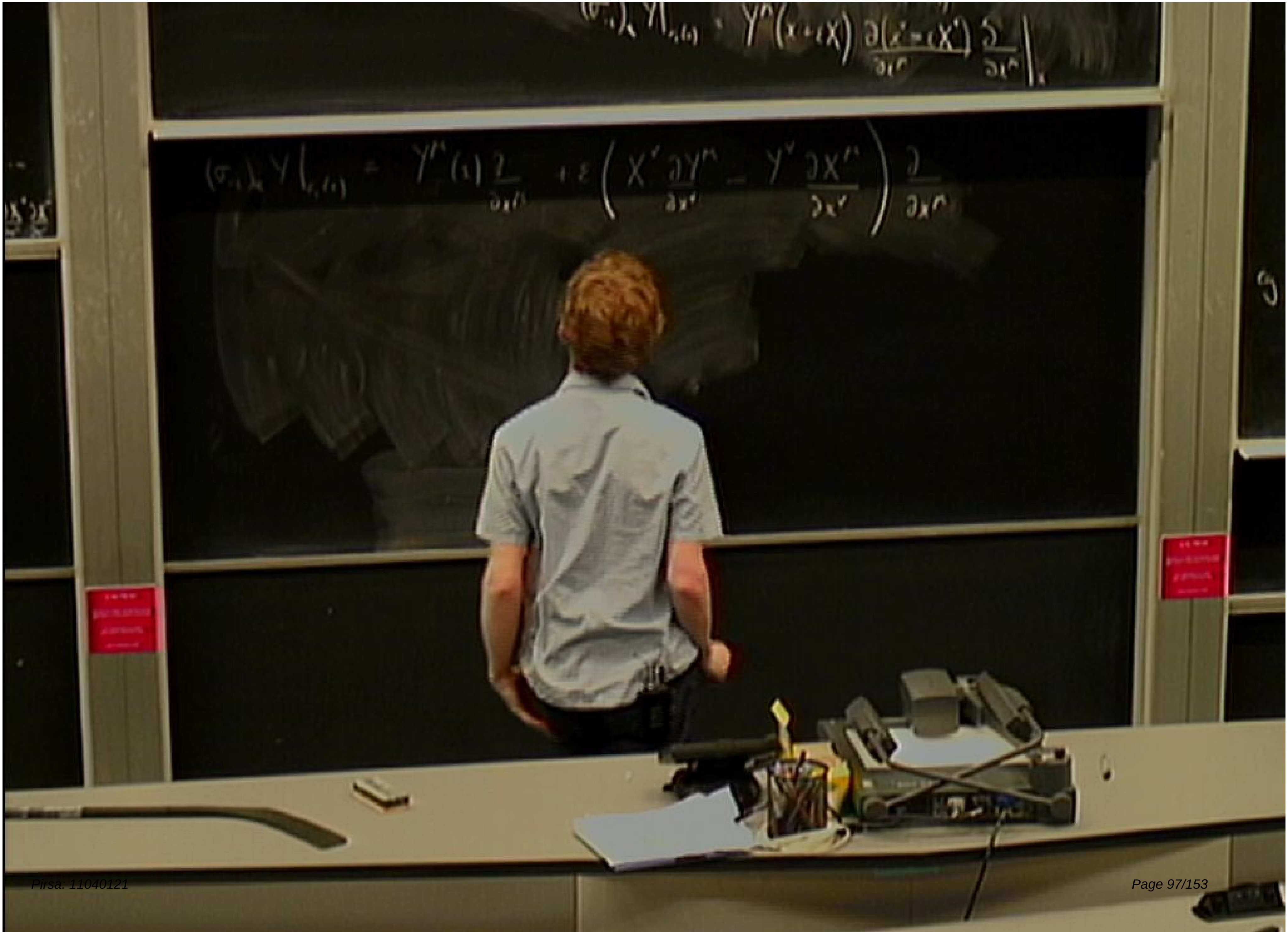
$$(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} = Y$$


"push-forward"
for any g: N → ℝ ; V ∈ T_pM
V[g∘f] = V^μ ∂g(f(x))/∂x^μ = (V^μ ∂y^α/∂x^μ) ∂g/∂y^α = (f_*V)^α ∂g/∂y^α
i.e. (f_*V)^α = V^μ ∂f^α/∂x^μ
p = φ^{-1}(t)
U ⊂ M
The Lie derivative L_X Y of a vector Y along a vector
is defined by
L_X Y = lim 1/t [(σ_{-t})_* Y|_{σ_t(x)} − Y|_x]
where σ
X
Y|_x = Y^μ(x) ∂/∂x^μ



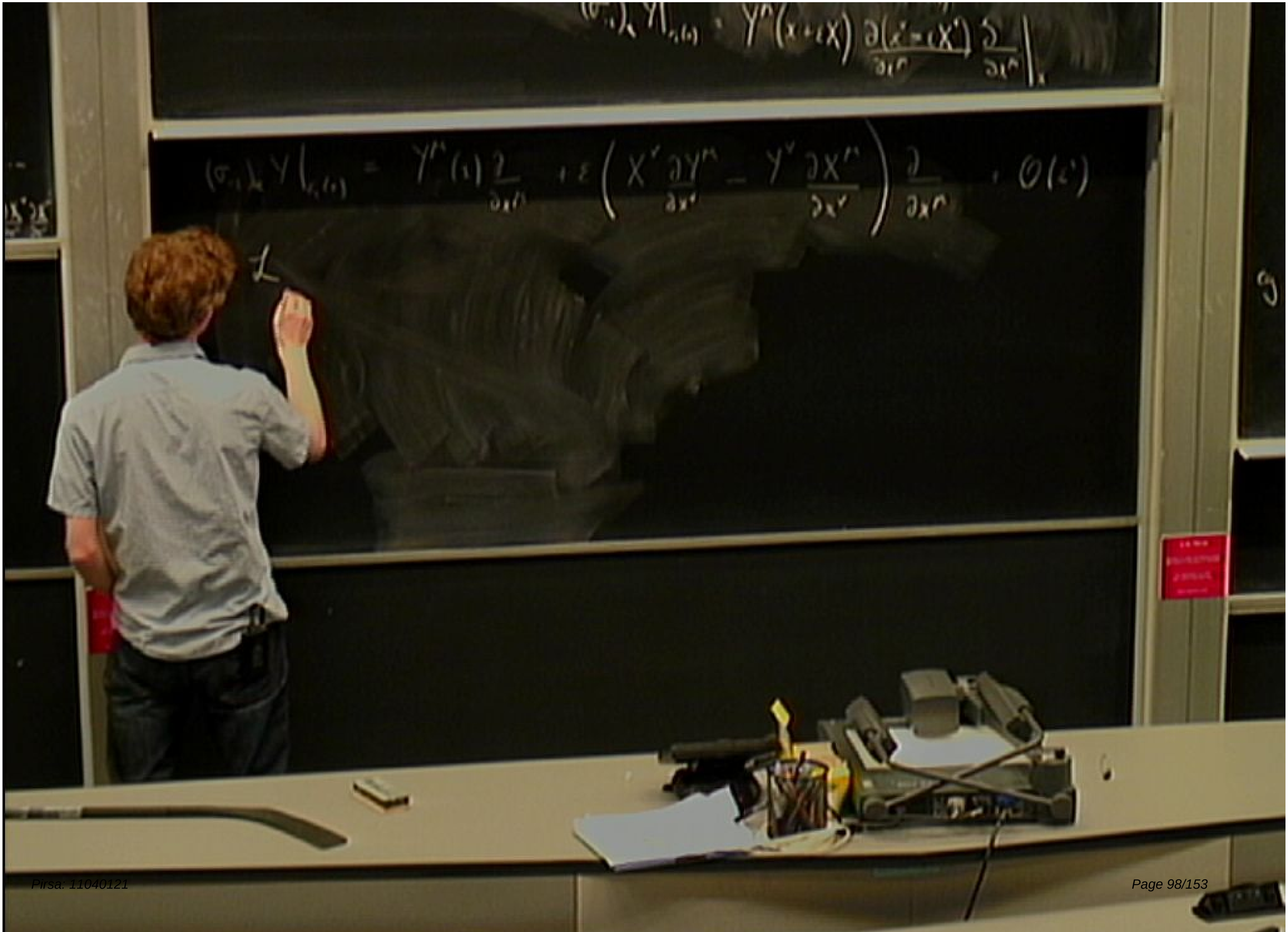
$$(\sigma_{-\varepsilon})_* Y|_{x(\varepsilon)} = Y^\mu(x+\varepsilon X) \frac{\partial(x^\nu - \varepsilon X^\nu)}{\partial x^\mu} \frac{\partial}{\partial x^\nu}\Big|_x$$

$$(\sigma_{-\varepsilon})_* Y|_{x(\varepsilon)} = Y^\mu(x) \frac{\partial}{\partial x^\mu} + \varepsilon \Big(X^\nu \frac{\partial Y^\mu}{\partial x^\nu}$$



$$(\sigma_\varepsilon)_* Y|_{\sigma_\varepsilon(x)} = Y^\mu(x+\varepsilon X)\,\frac{\partial(x^\nu+\varepsilon X^\nu)}{\partial x^\mu}\,\frac{\partial}{\partial x^\nu}\Big|_x$$

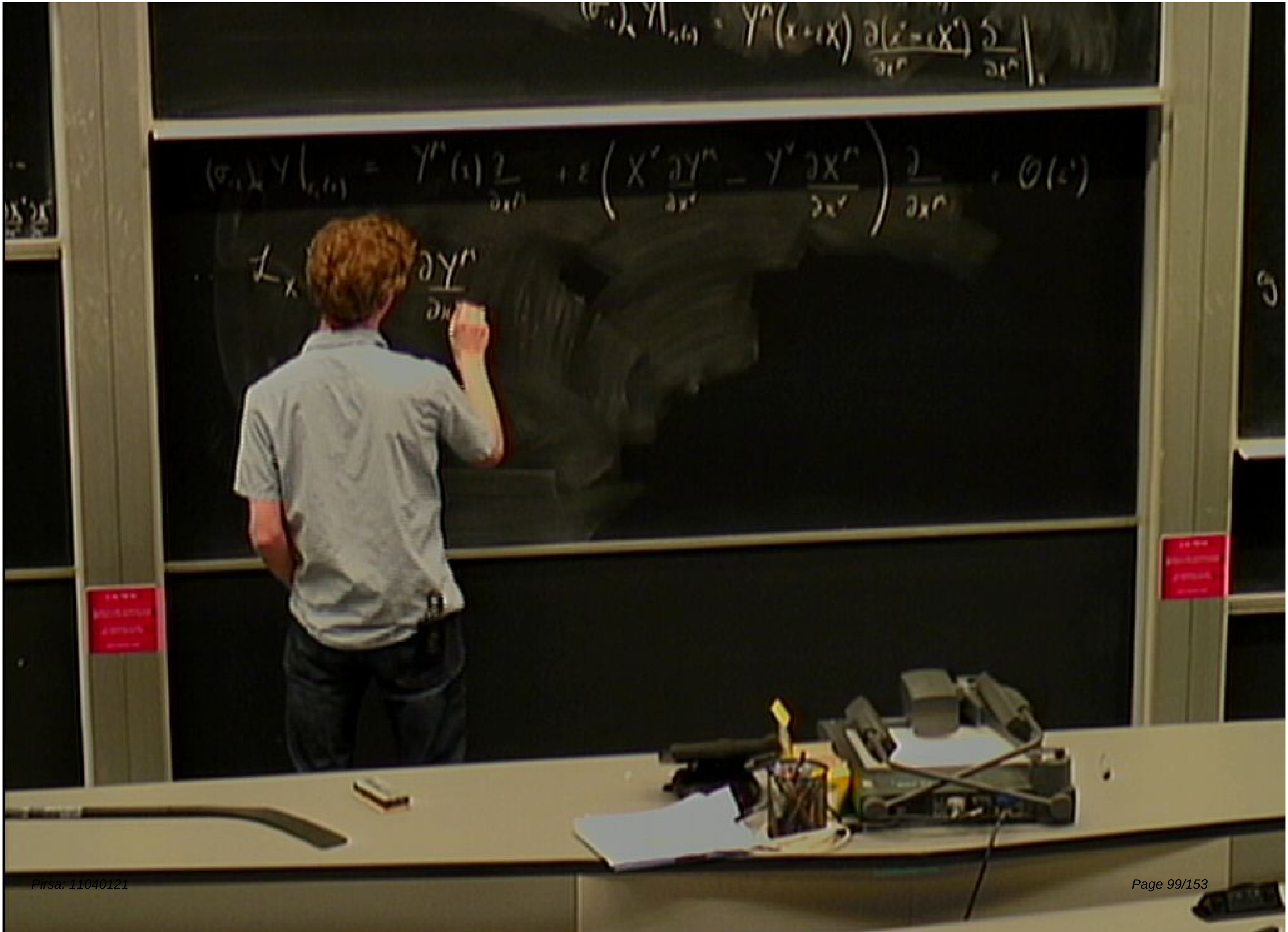
$$(\sigma_\varepsilon)_* Y|_{\sigma_\varepsilon(x)} = Y^\mu(x)\,\frac{\partial}{\partial x^\mu} + \varepsilon\left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu}$$

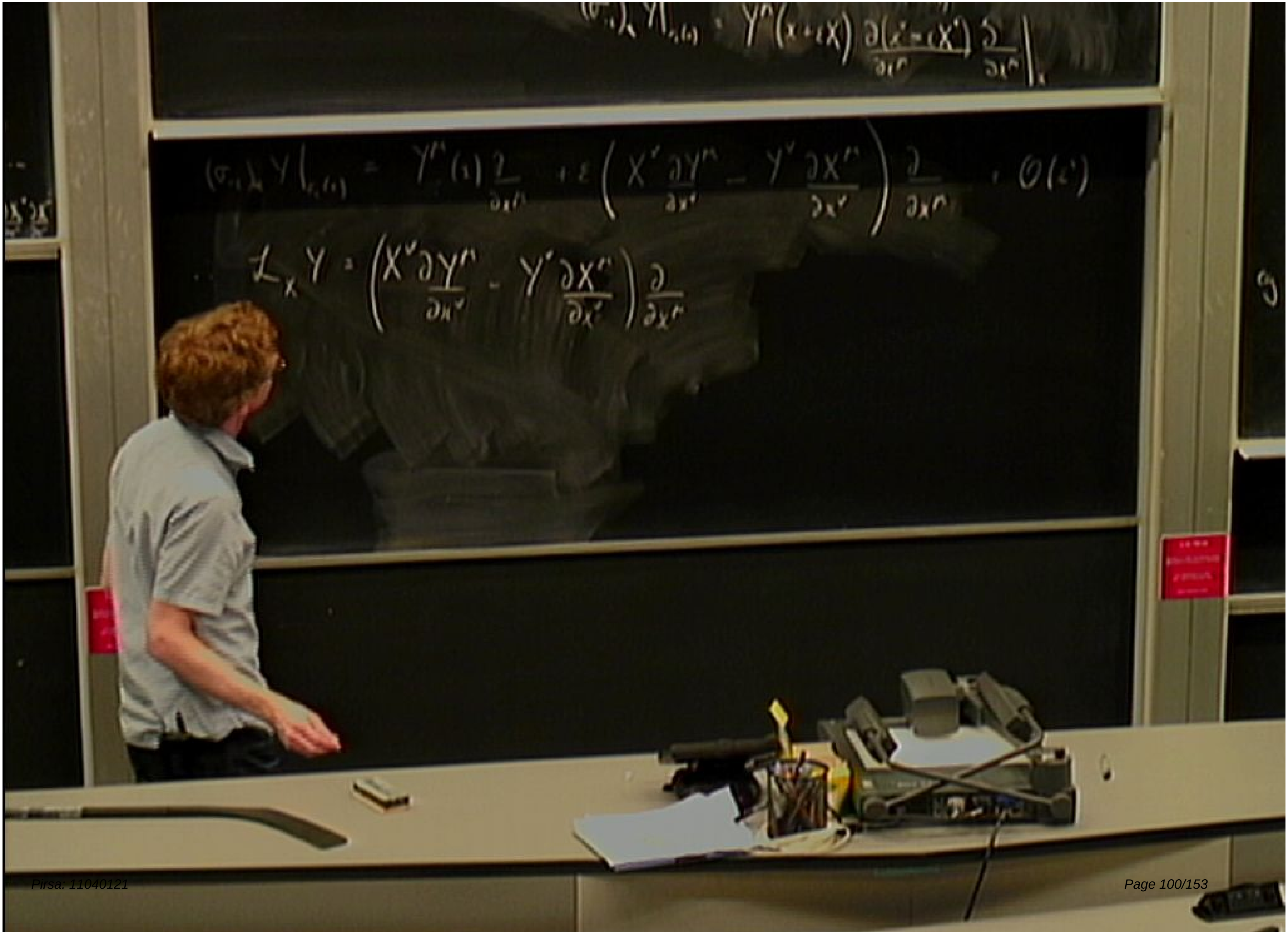


$$(\sigma_{-\epsilon})_* Y\big|_{x(\epsilon)} = Y^\mu(x+\epsilon X)\,\frac{\partial(x^\nu - \epsilon X^\nu)}{\partial x^\mu}\,\frac{\partial}{\partial x^\nu}\Big|_x$$

$$(\sigma_{-\epsilon})_* Y\big|_{x(\epsilon)} = Y^\mu(x)\frac{\partial}{\partial x^\mu} + \epsilon\left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} + \mathcal{O}(\epsilon^2)$$

$\mathcal{L}$

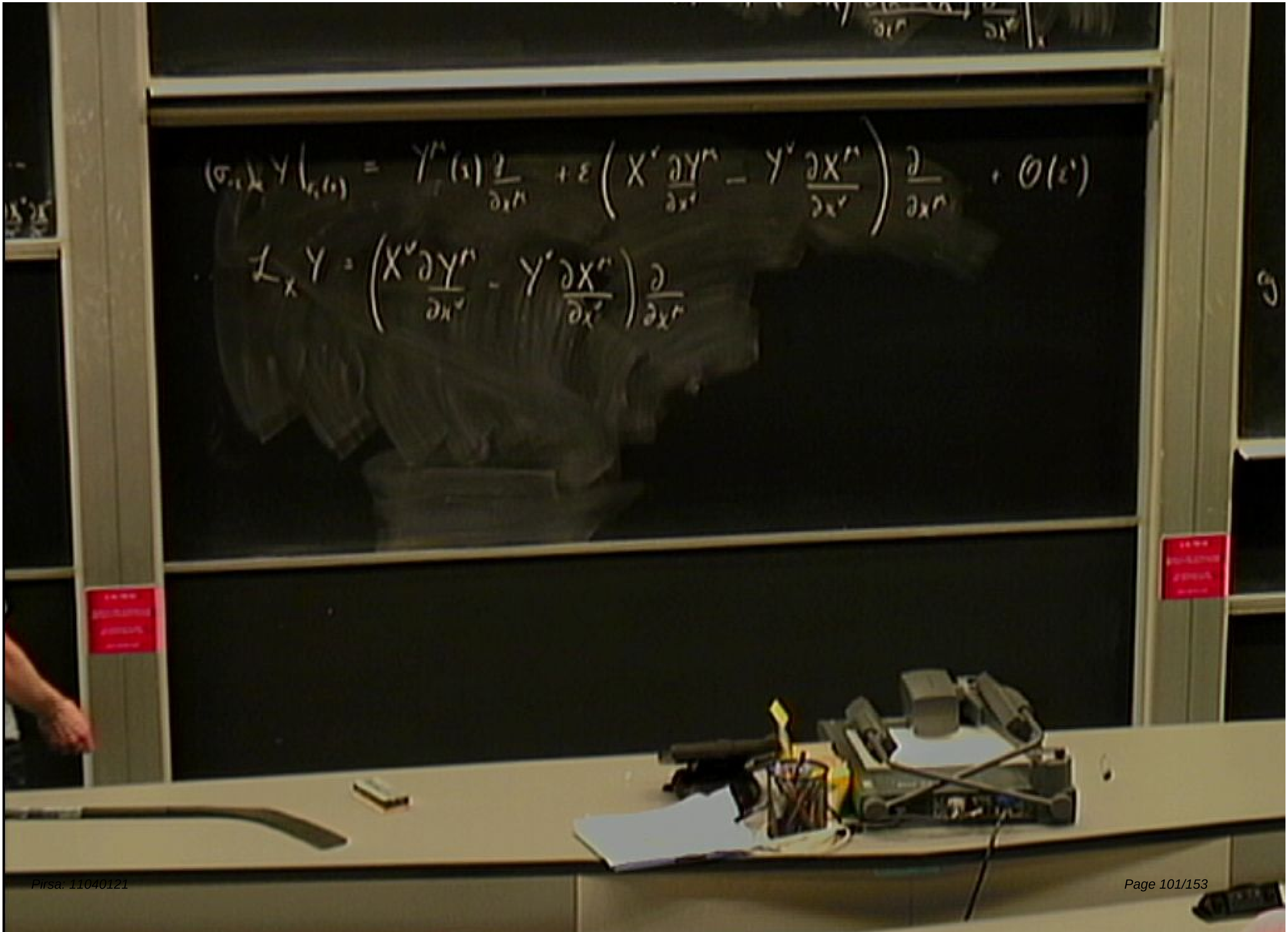

$(\sigma_{-t})_* Y|_{\sigma_t(x)} = Y^\mu(x+\epsilon X)\,\frac{\partial(x^\nu - \epsilon X^\nu)}{\partial x^\mu}\,\frac{\partial}{\partial x^\nu}\Big|_x$
$(\sigma_{-t})_* Y|_{\sigma_t(x)} = Y^\mu(x)\frac{\partial}{\partial x^\mu} + \epsilon\left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} + O(\epsilon^2)$
$\mathcal{L}_X \quad \frac{\partial Y^\mu}{\partial x}$



$$(\sigma_{-\epsilon})_* Y\Big|_{\sigma_\epsilon(x)} = Y^\mu(x+\epsilon X)\,\frac{\partial(x^\nu - \epsilon X^\nu)}{\partial x^\mu}\,\frac{\partial}{\partial x^\nu}\Big|_x$$

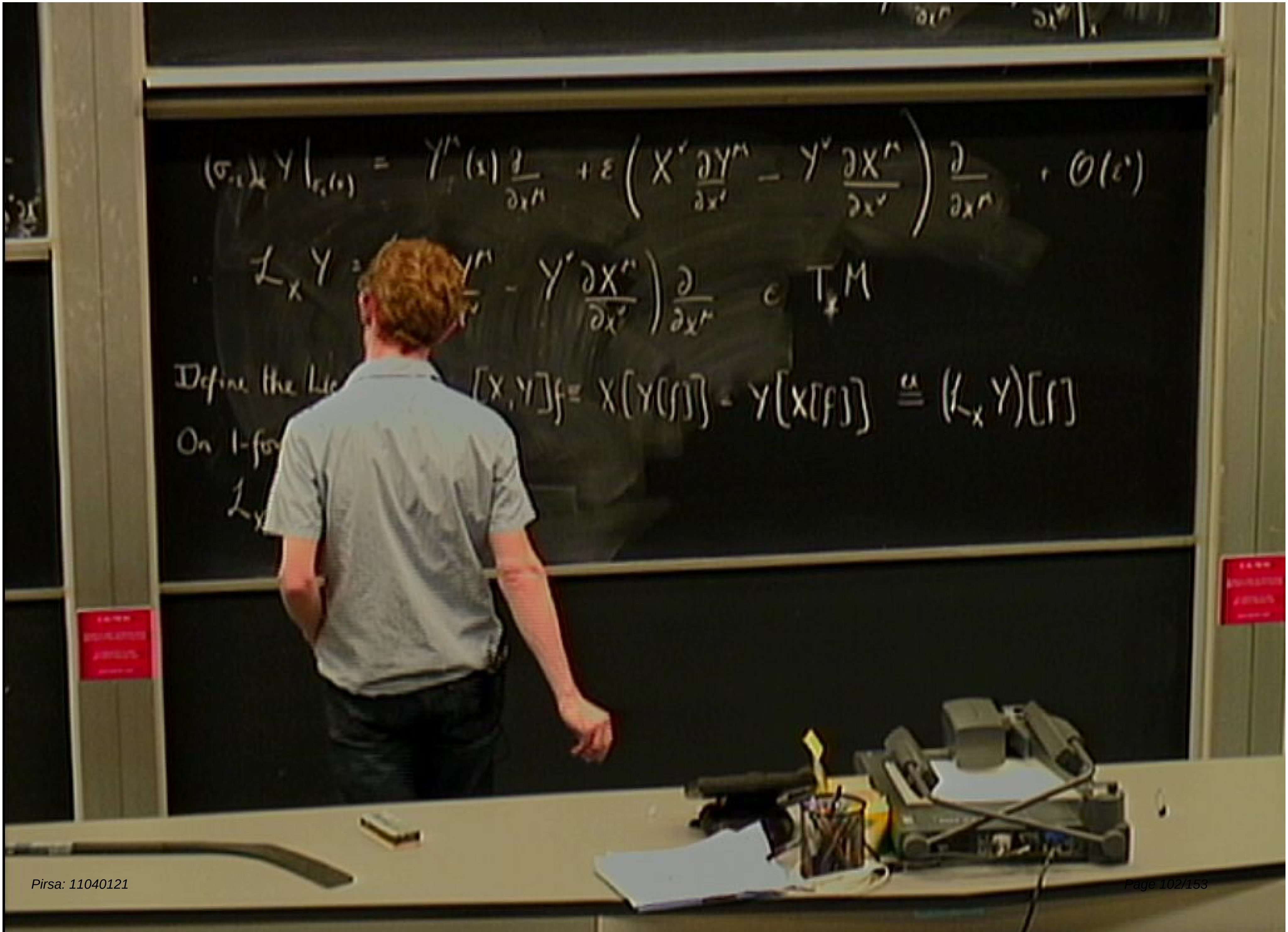
$$(\sigma_{-\epsilon})_* Y\Big|_{\sigma_\epsilon(x)} = Y^\mu(x)\,\frac{\partial}{\partial x^\mu} + \epsilon\left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} + O(\epsilon^2)$$

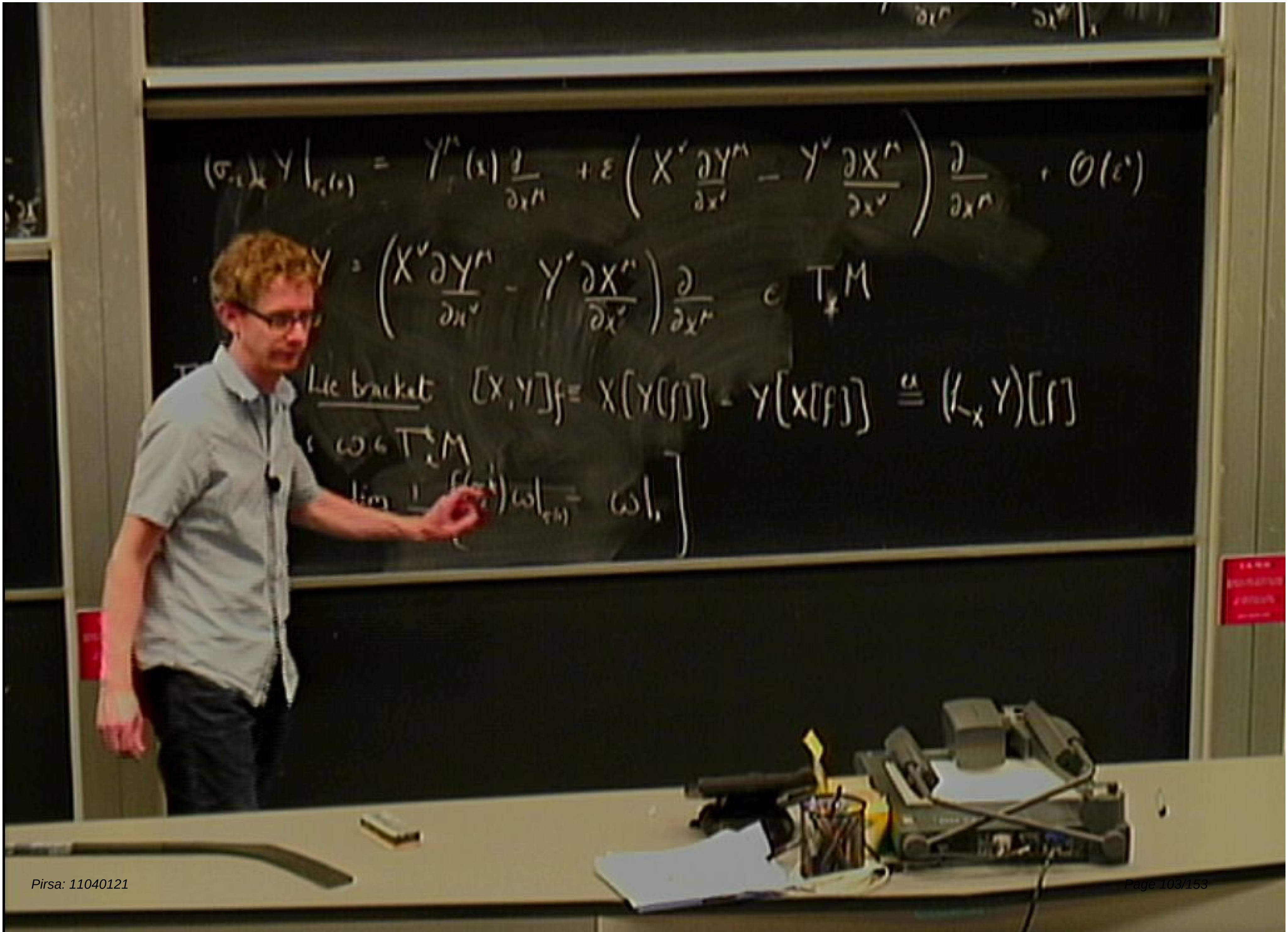
$$\therefore\ \mathcal{L}_X Y = \left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu}$$



$$(\sigma_{-\varepsilon})_* Y\big|_{\sigma_\varepsilon(x)} = Y^\mu(x)\frac{\partial}{\partial x^\mu} + \varepsilon\left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} + \mathcal{O}(\varepsilon^2)$$

$$\mathcal{L}_X Y = \left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu}$$





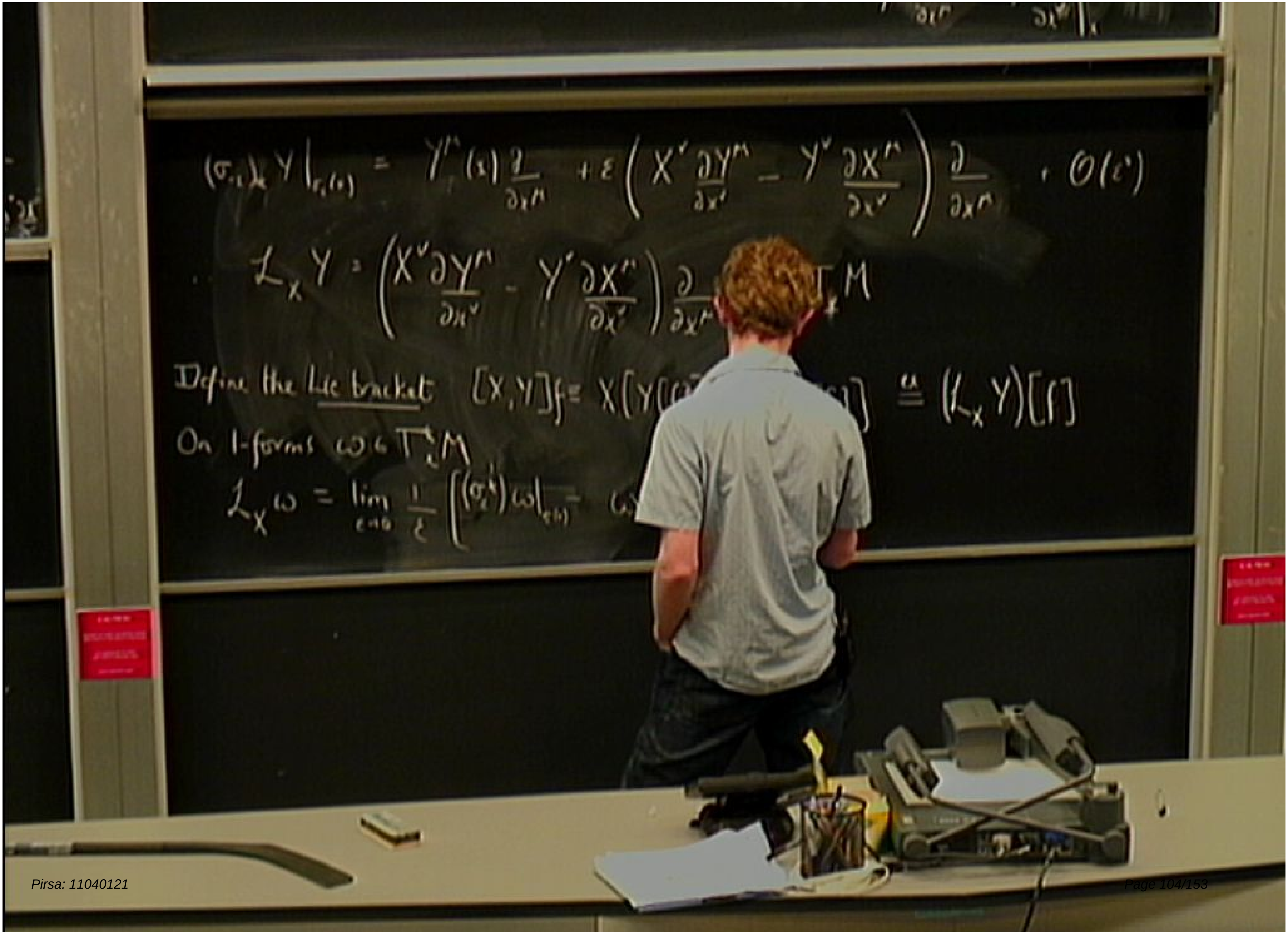
$$(\sigma_{-\varepsilon})_* Y\big|_{\sigma_\varepsilon(x)} = Y^\mu(x)\frac{\partial}{\partial x^\mu} + \varepsilon\left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} + \mathcal{O}(\varepsilon^2)$$

$$Y = \left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} \in T_x M$$

Lie bracket $[X,Y]f = X[Y[f]] - Y[X[f]] \overset{\text{def}}{=} (\mathcal{L}_X Y)[f]$

$\omega \in T^*_x M$

$$\lim \frac{1}{\varepsilon}\left[(\sigma_\varepsilon^*)\,\omega\big|_{\sigma_\varepsilon(x)} - \omega\big|_x\right]$$



$$(\sigma_{-\epsilon})_* Y|_{\sigma_\epsilon(x)} = Y^\mu(x)\frac{\partial}{\partial x^\mu} + \epsilon\left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} + \mathcal{O}(\epsilon^2)$$

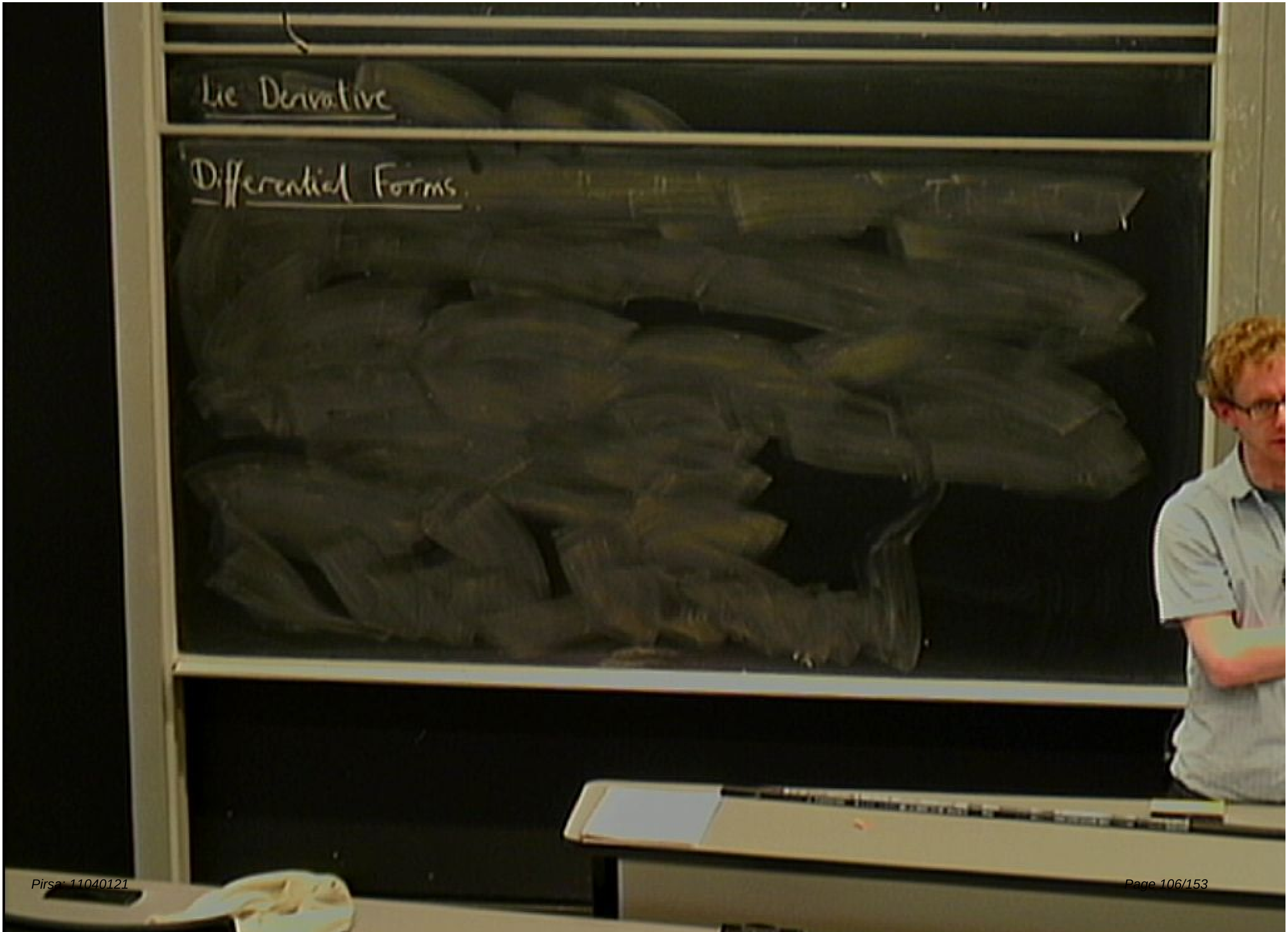
$$\mathcal{L}_X Y = \left(X^\nu\frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu\frac{\partial X^\mu}{\partial x^\nu}\right)\frac{\partial}{\partial x^\mu} \quad T_x M$$

Define the Lie bracket $[X,Y]f = X(Y(f \quad f)] \overset{ex}{=} (\mathcal{L}_X Y)[f]$

On 1-forms $\omega \in T^*_x M$

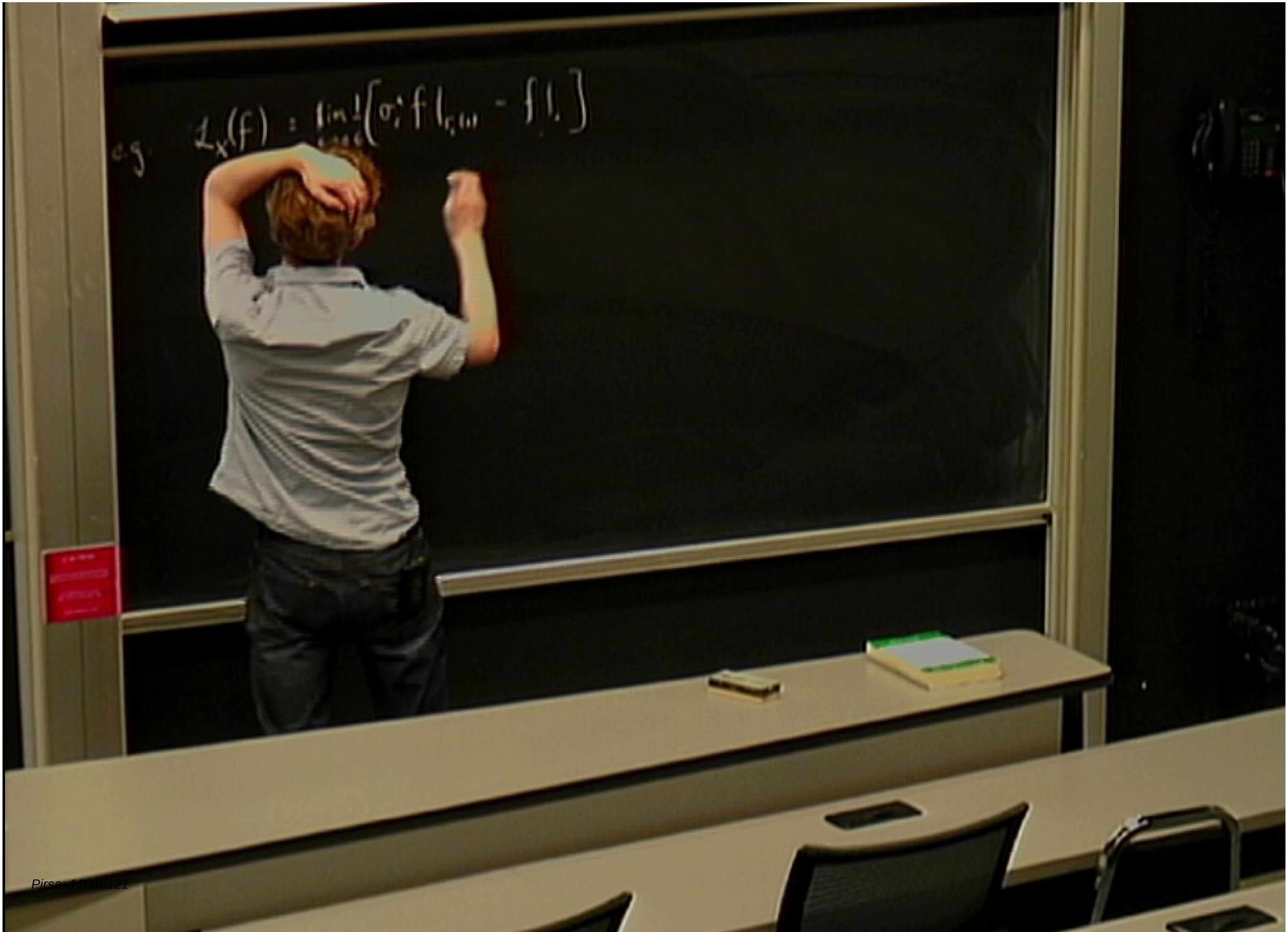
$$\mathcal{L}_X \omega = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((\sigma_\epsilon^*)\omega|_{\sigma_\epsilon(x)} - \omega\right.$$


Lie Derivative
Differential Forms

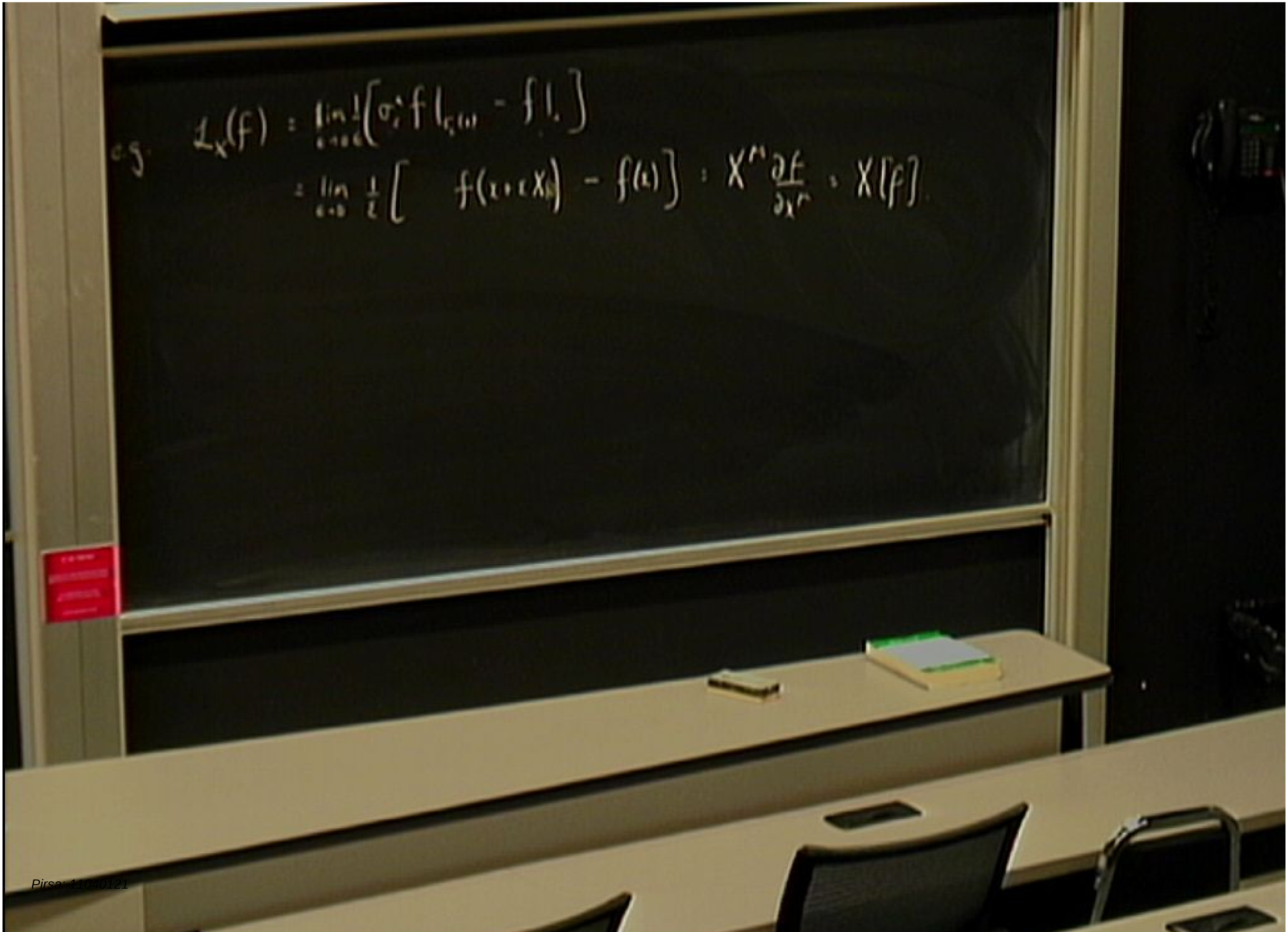


Lie Derivative

Differential Forms

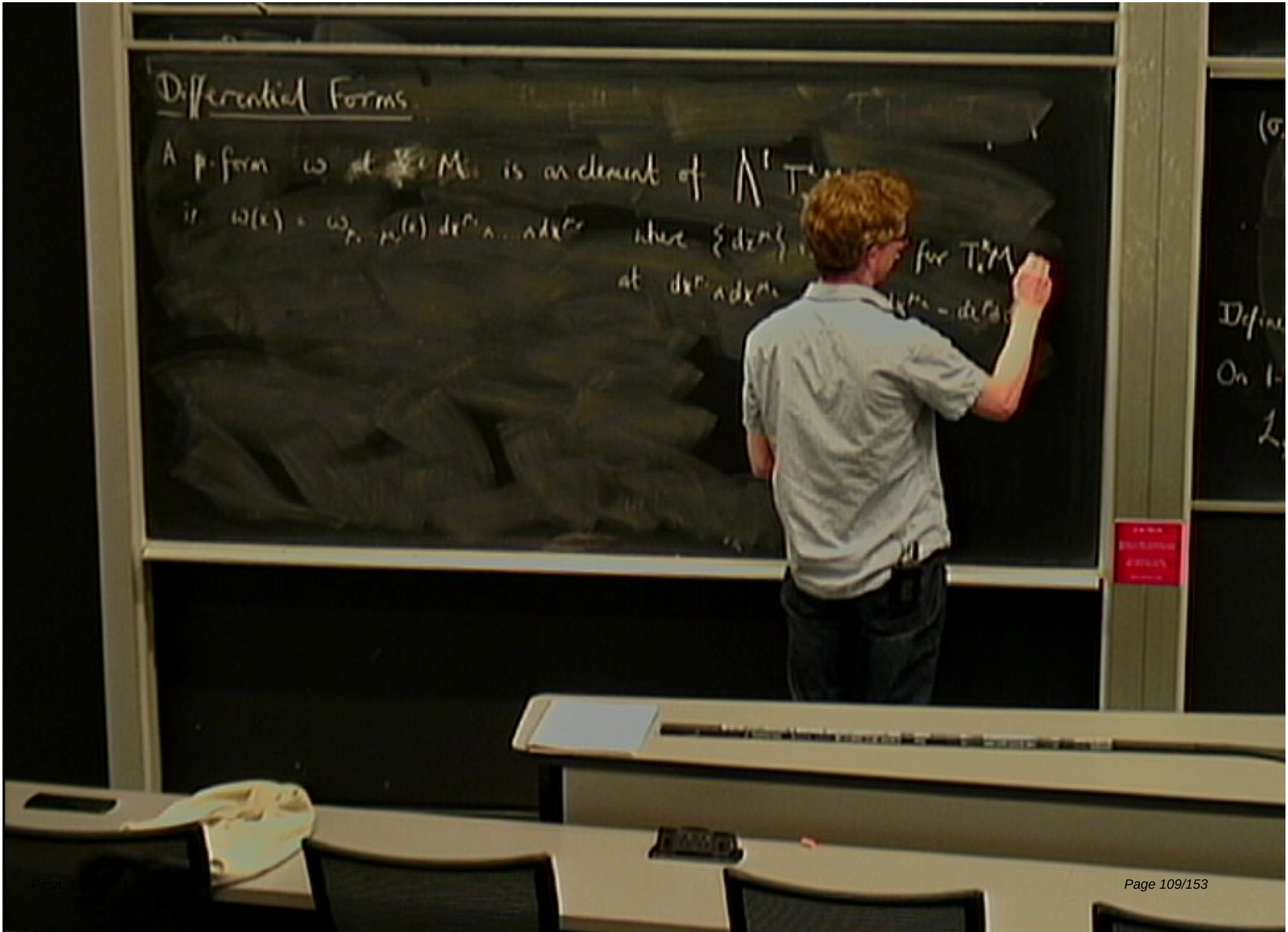


e.g. $\mathcal{L}_X(f) = \lim_{t \to 0} \frac{1}{t} \left[\sigma_t^* f|_{\sigma_t(x)} - f|_x \right]$



e.g. $\mathcal{L}_X(f) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\sigma_\epsilon^* f|_{\sigma_\epsilon(x)} - f|_x\right]$

$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[f(x+\epsilon X) - f(x)\right] = X^\mu \frac{\partial f}{\partial x^\mu} = X[f]$



Differential Forms

A p-form ω at x ∈ M is an element of $\Lambda^p T^*_x M$

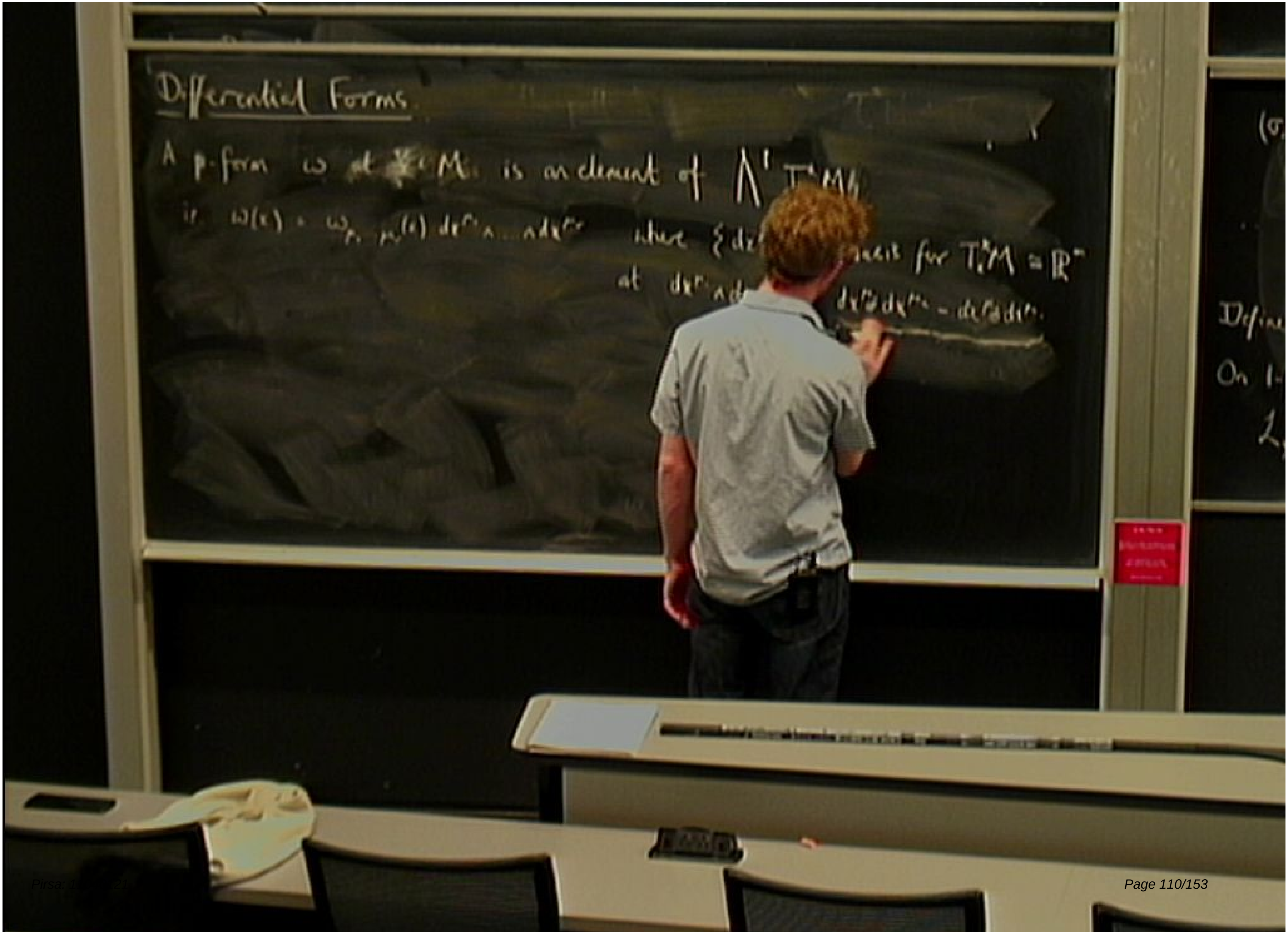
if $\omega(x) = \omega_{\mu_1\ldots\mu_p}(x)\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$ where $\{dx^{\mu}\}$ [illegible] for $T^*_x M$

at $dx^{\mu_1}\wedge dx^{\mu_2}$ [illegible] $dx^{\mu_2} - dx^{\mu_1}$

(σ,

Define

On 1-

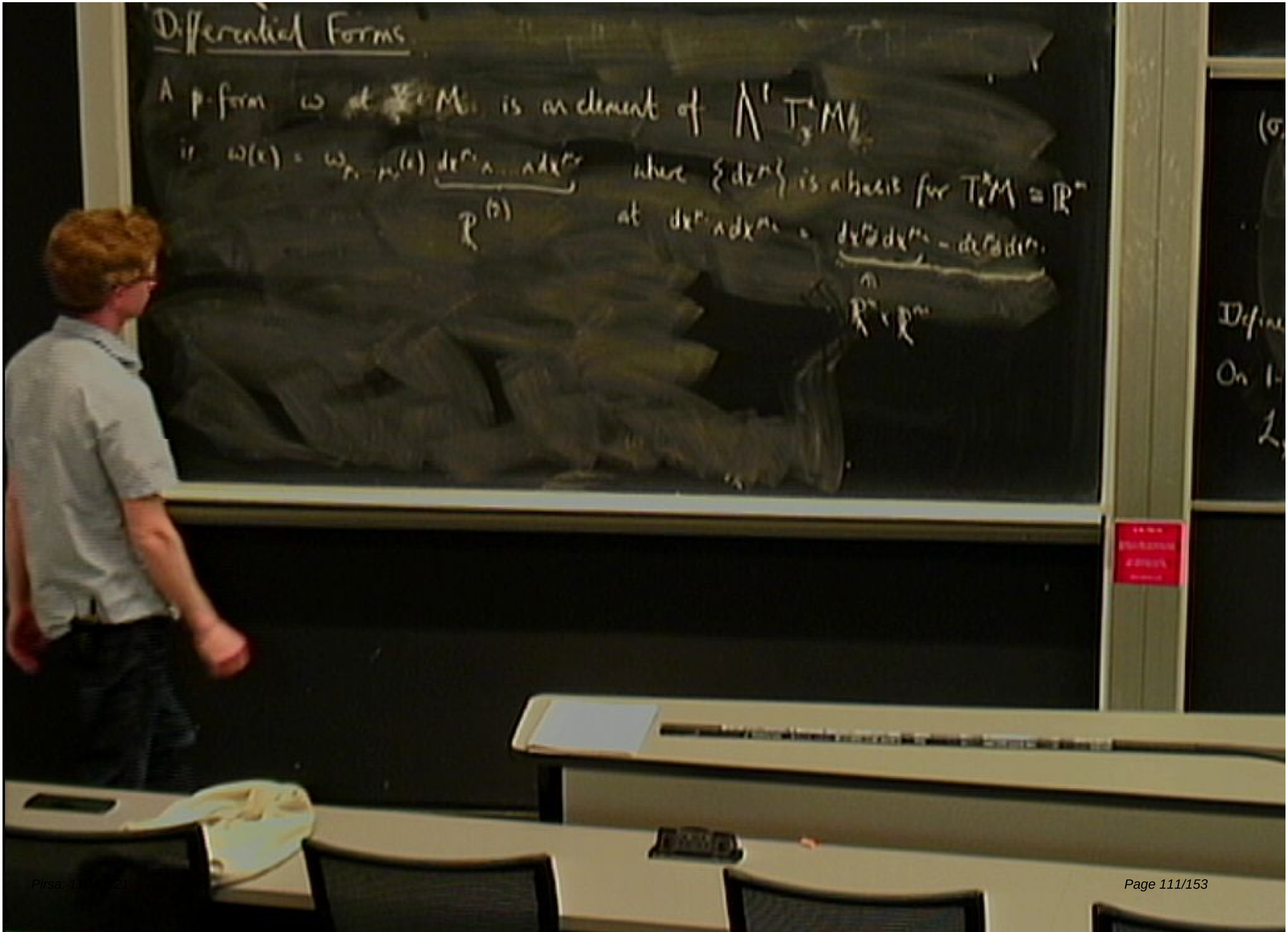


Differential Forms

A p-form ω at $x \in M$ is an element of $\Lambda^p T_x^* M$

i.e. $\omega(x) = \omega_{\mu_1 \ldots \mu_p}(x)\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$ where $\{dx^{\mu}$ [illegible] basis for $T_x^* M \cong \mathbb{R}^n$

st $dx^{\mu_1} \wedge d$ [illegible] $dx^{\mu_1} \otimes dx^{\mu_2} - dx^{\mu_2} \otimes dx^{\mu_1}$



Differential Forms

A p-form ω at [illegible] M is an element of $\Lambda^p T_x^* M$

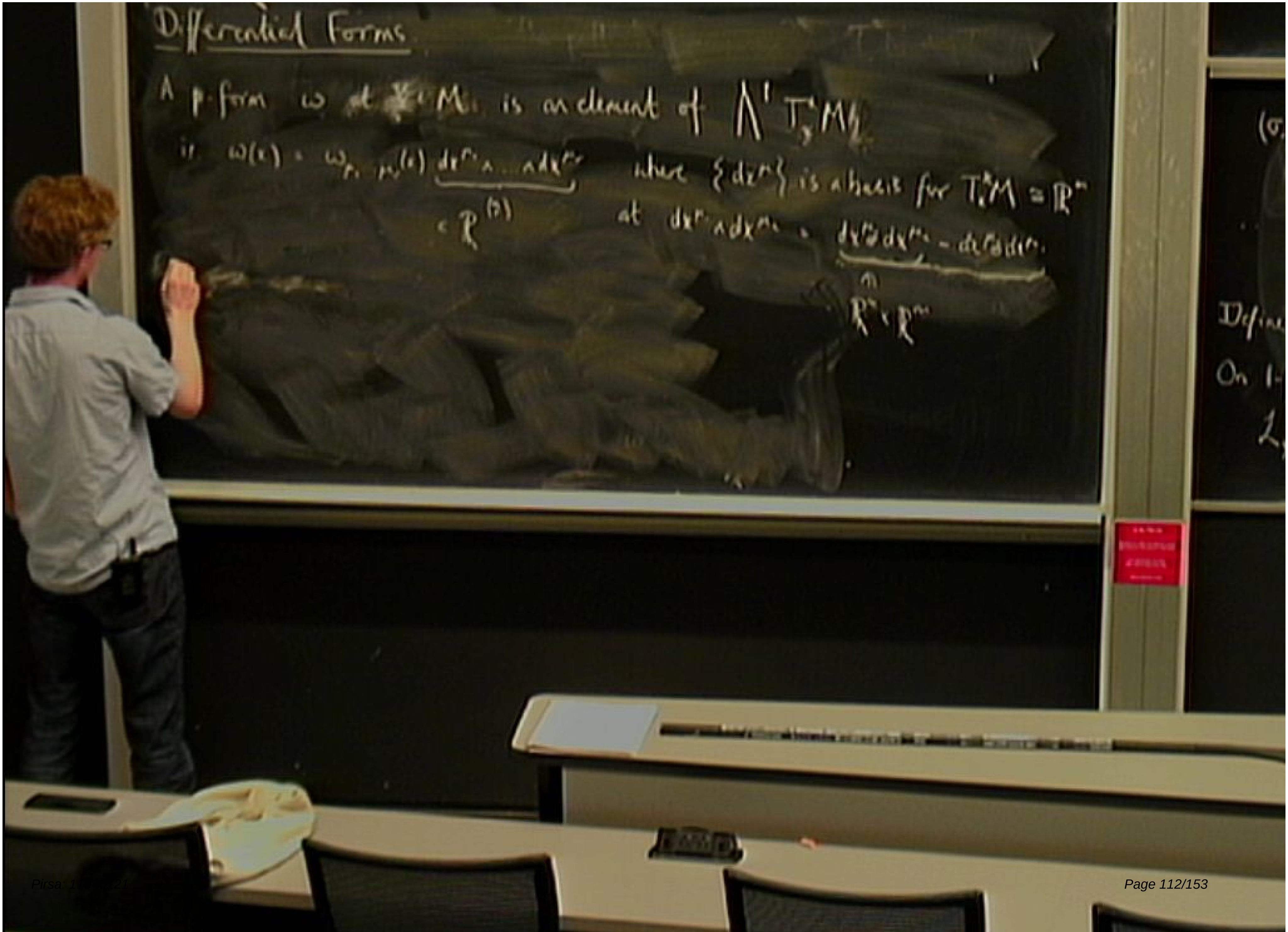
if $\omega(x) = \omega_{\mu_1 \ldots \mu_p}(x) \underbrace{dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}}_{\mathbb{R}^{\binom{n}{p}}}$ where $\{dx^{\mu}\}$ is a basis for $T_x^* M \cong \mathbb{R}^n$

at $dx^{\mu_1} \wedge dx^{\mu_2} = \underbrace{dx^{\mu_1} \otimes dx^{\mu_2} - dx^{\mu_2} \otimes dx^{\mu_1}}_{\overset{\cap}{\mathbb{R}^n \otimes \mathbb{R}^n}}$

(σ

Defin

On 1-

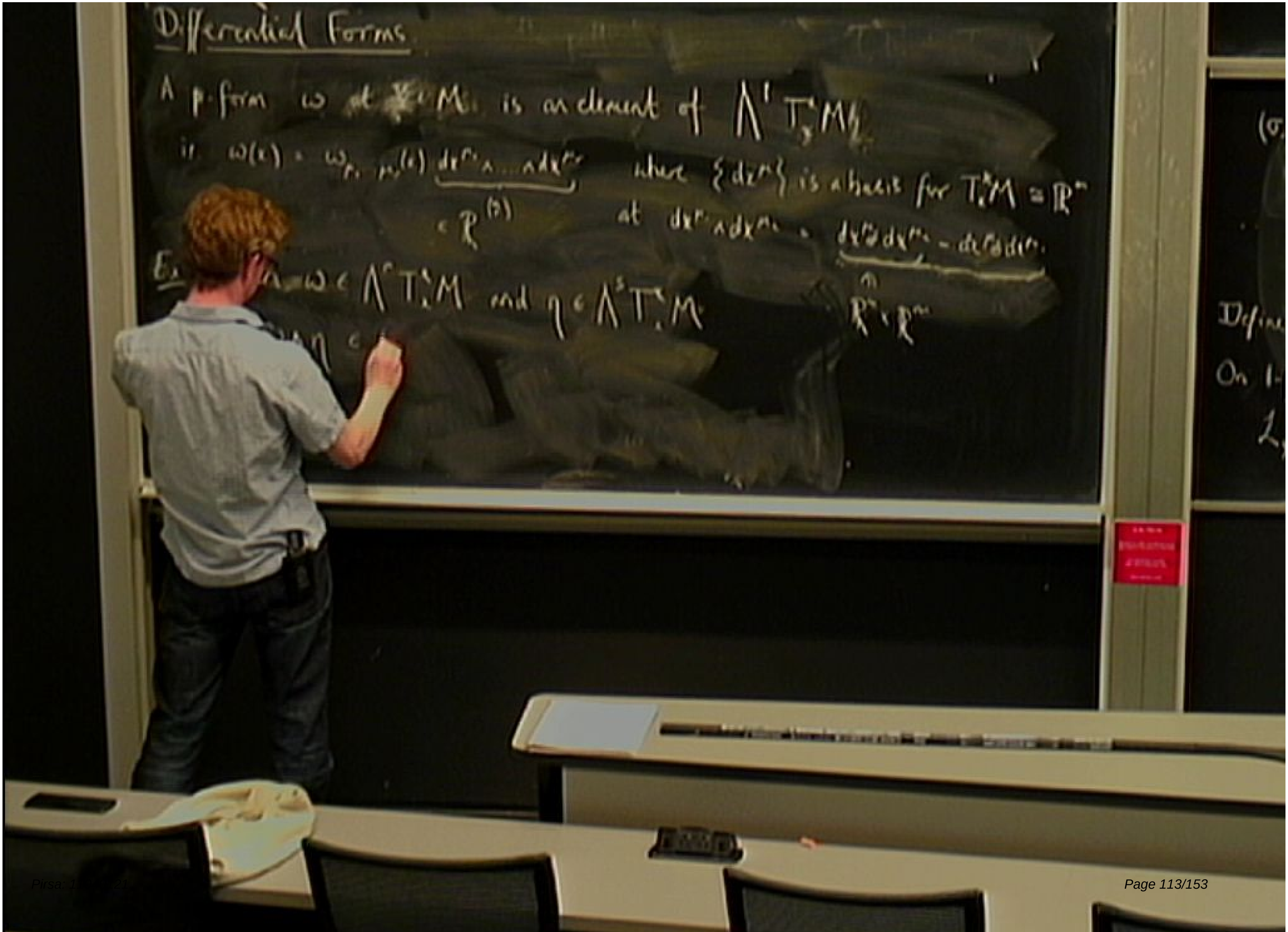


Differential Forms

A p-form ω at x ∈ M is an element of $\Lambda^p T^*_x M$

if $\omega(x) = \omega_{\mu_1 \ldots \mu_p}(x) \underbrace{dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}}_{\in \mathbb{R}^{\binom{n}{p}}}$ where $\{dx^\mu\}$ is a basis for $T^*_x M \cong \mathbb{R}^n$

at $dx^{\mu_1} \wedge dx^{\mu_2} = \underbrace{dx^{\mu_1} \otimes dx^{\mu_2} - dx^{\mu_2} \otimes dx^{\mu_1}}_{\subset \, \mathbb{R}^n \otimes \mathbb{R}^n}$



Differential Forms

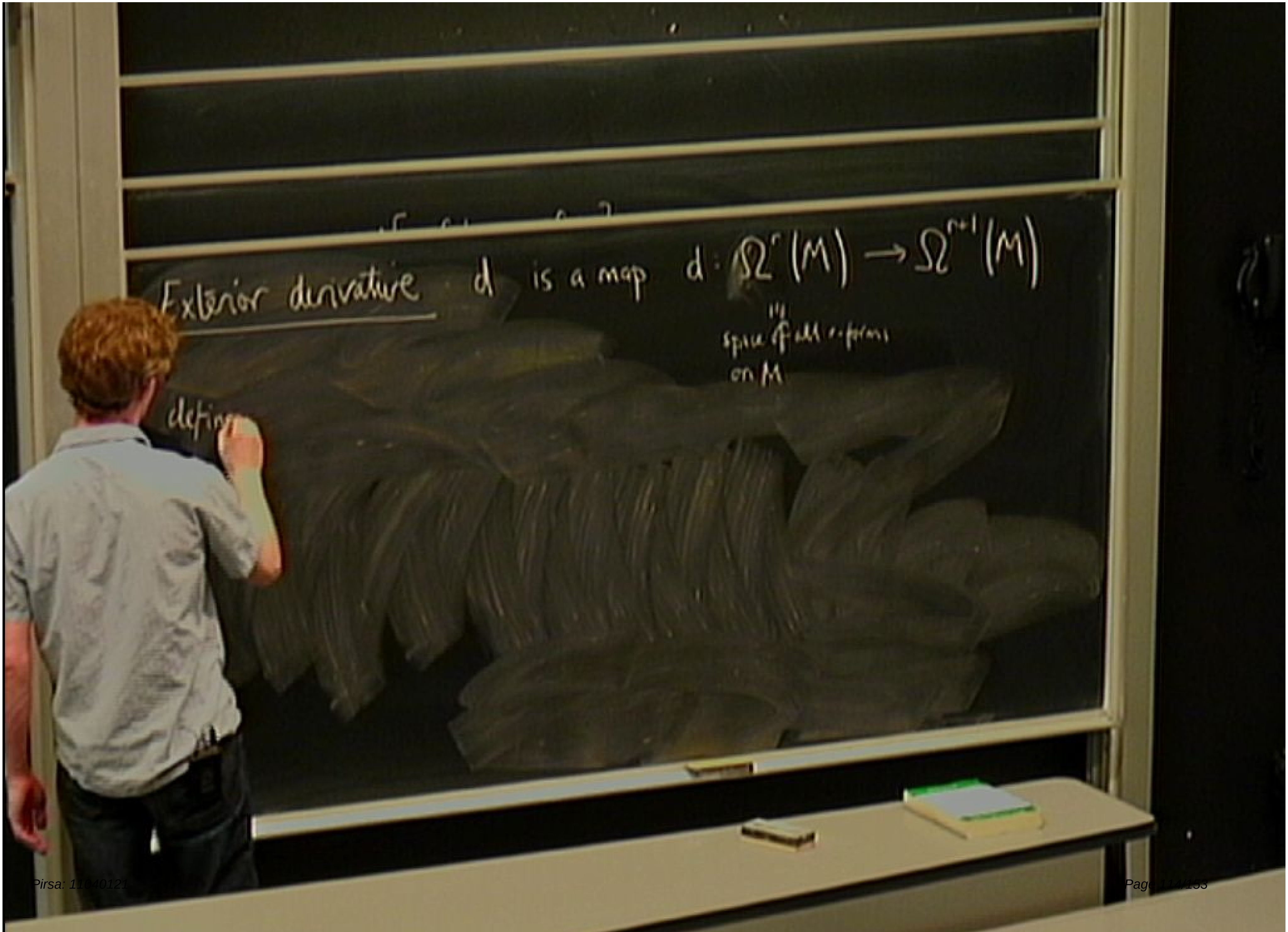
A p-form ω at $x \in M$ is an element of $\Lambda^p T_x^* M$

if $\omega(x) = \omega_{\mu_1 \dots \mu_p}(x)\, \underbrace{dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}}_{\in \mathbb{R}^{\binom{n}{p}}}$ where $\{dx^\mu\}$ is a basis for $T_x^* M \cong \mathbb{R}^n$

st $dx^{\mu_1} \wedge dx^{\mu_2} = \underbrace{dx^{\mu_1} \otimes dx^{\mu_2} - dx^{\mu_2} \otimes dx^{\mu_1}}_{\in \mathbb{R}^n \otimes \mathbb{R}^n}$

Ex: $\omega \in \Lambda^r T_x^* M$ and $\eta \in \Lambda^s T_x^* M$

$\omega \wedge \eta \in$



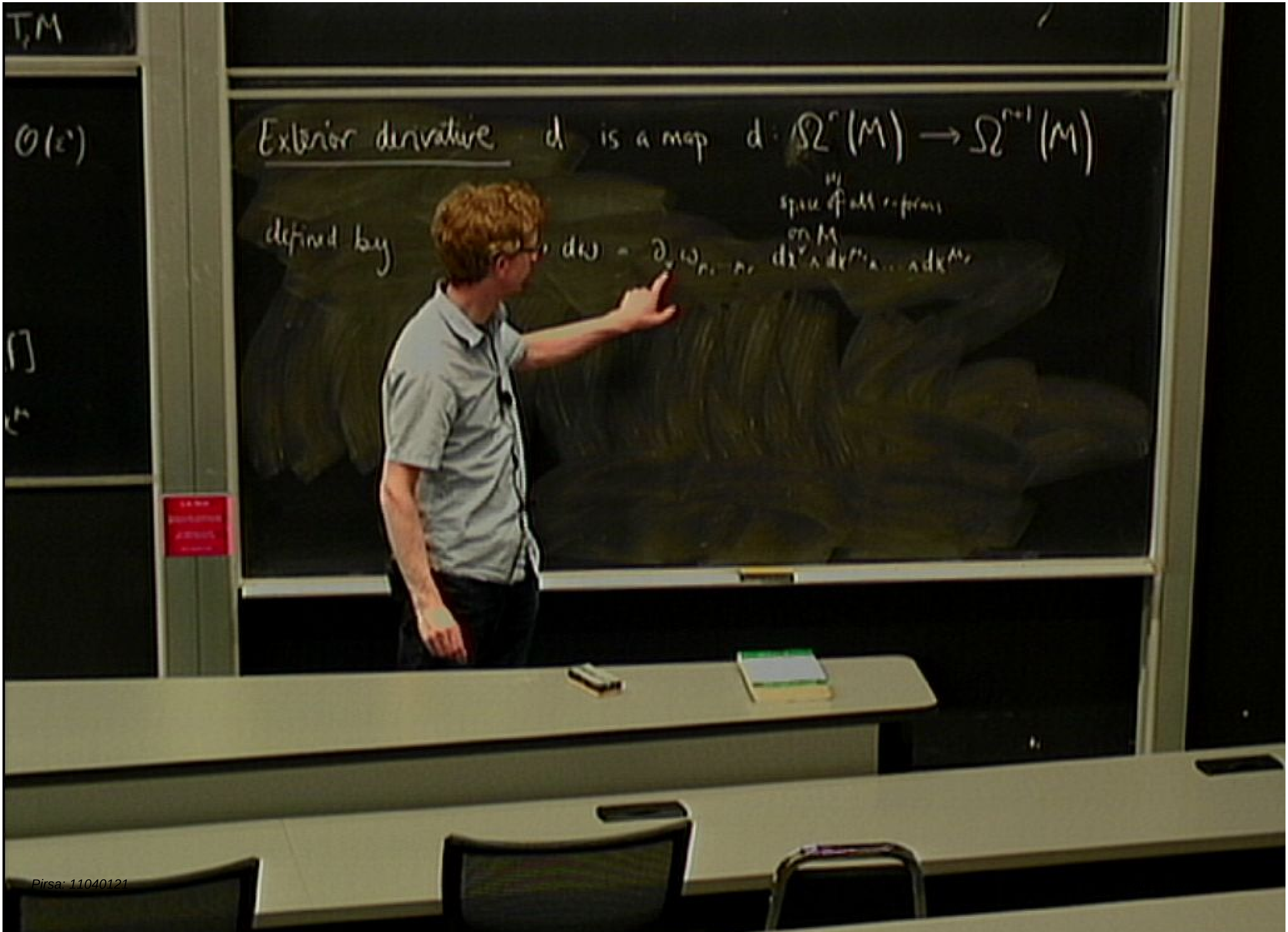
Exterior derivative   d is a map $d : \Omega^r(M) \to \Omega^{r+1}(M)$

$\parallel$

space of all r-forms

on M

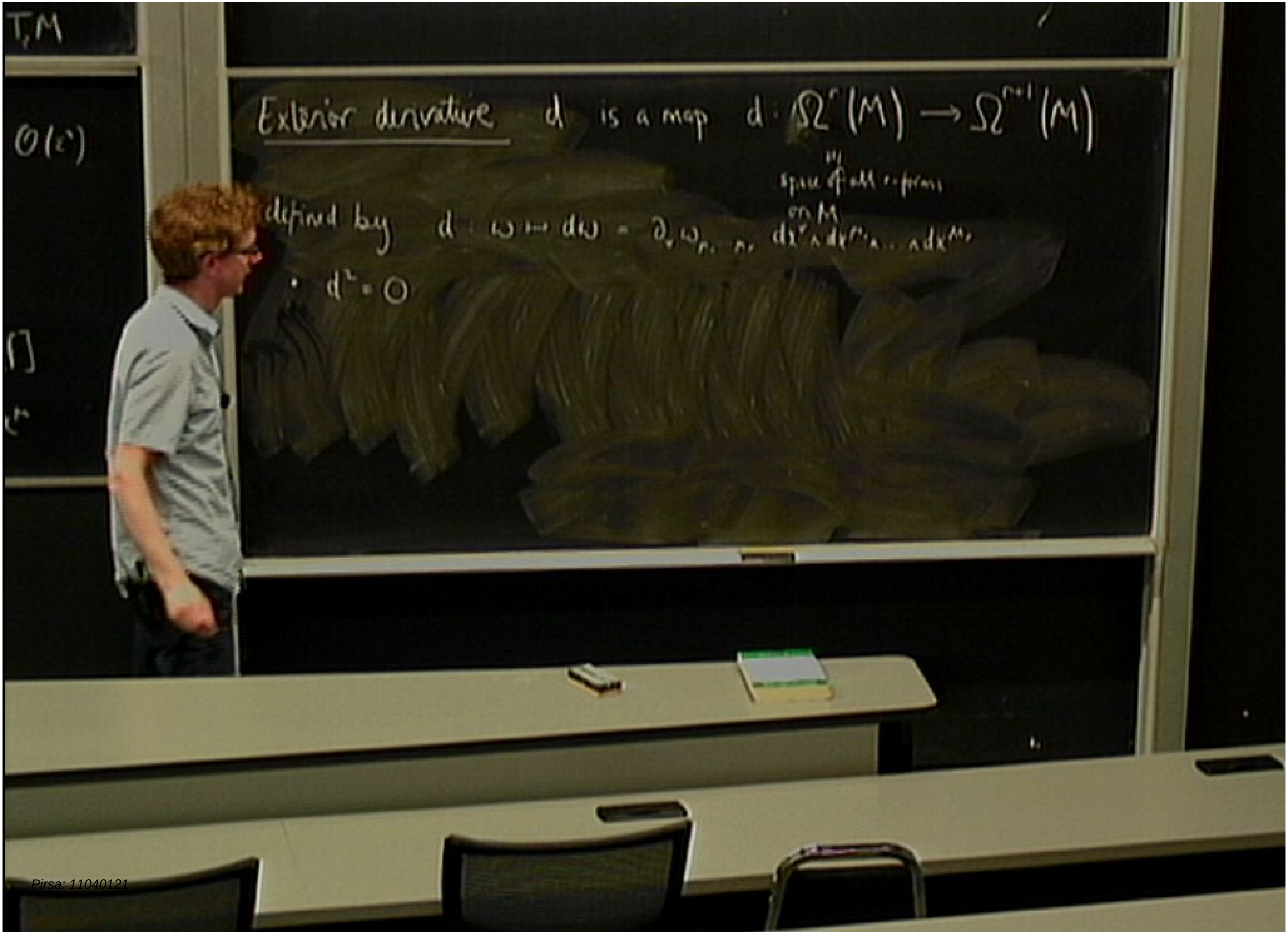
defin



Exterior derivative d is a map $d : \Omega^r(M) \rightarrow \Omega^{r+1}(M)$

space of all r-forms on M

defined by $d\omega = \partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$



T_pM

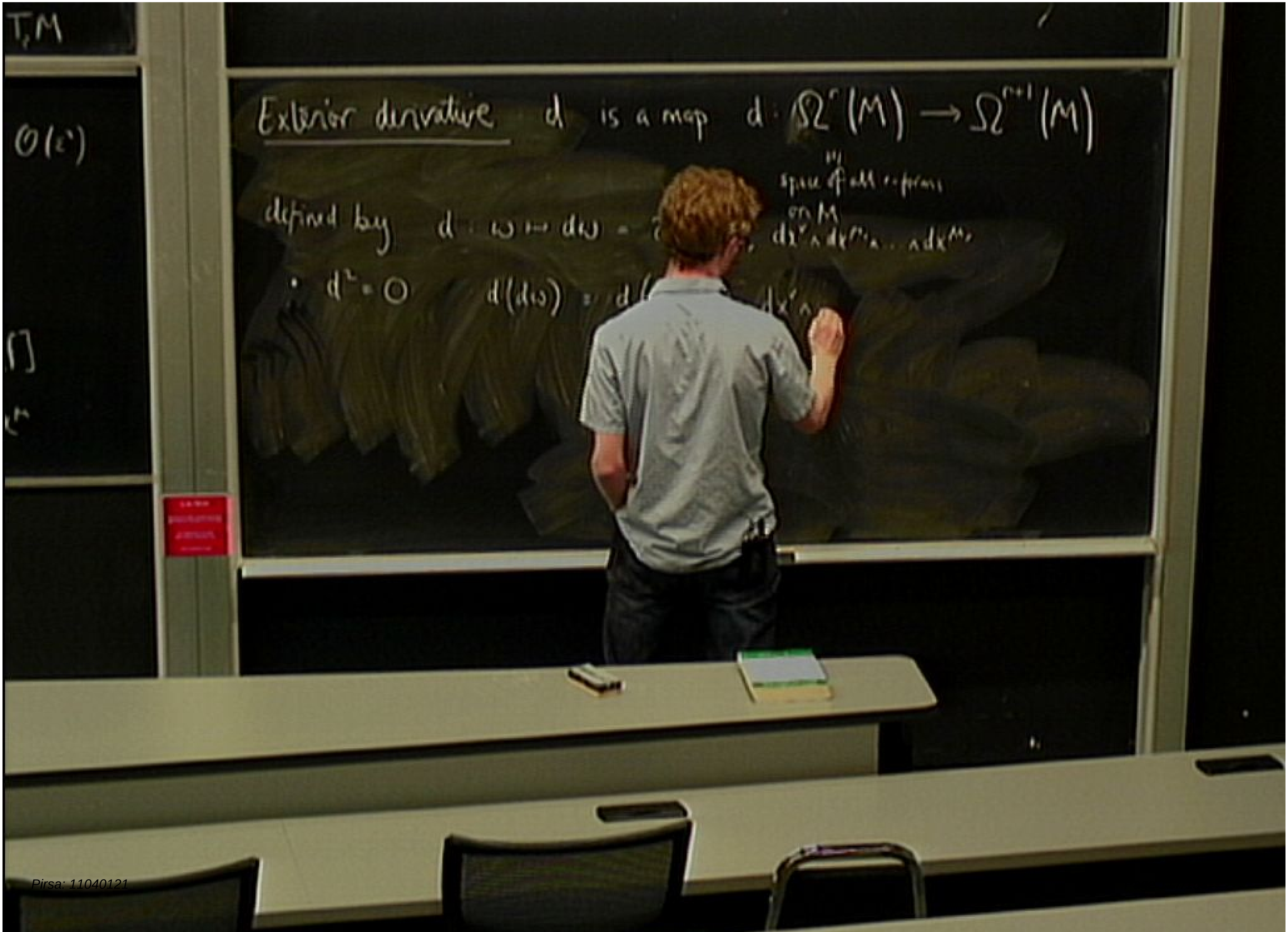
$O(\epsilon^2)$

<u>Exterior derivative</u> d is a map $d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M)$

space of all r-forms on M

defined by $d : \omega \mapsto d\omega = \partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$

- $d^2 = 0$



T_pM

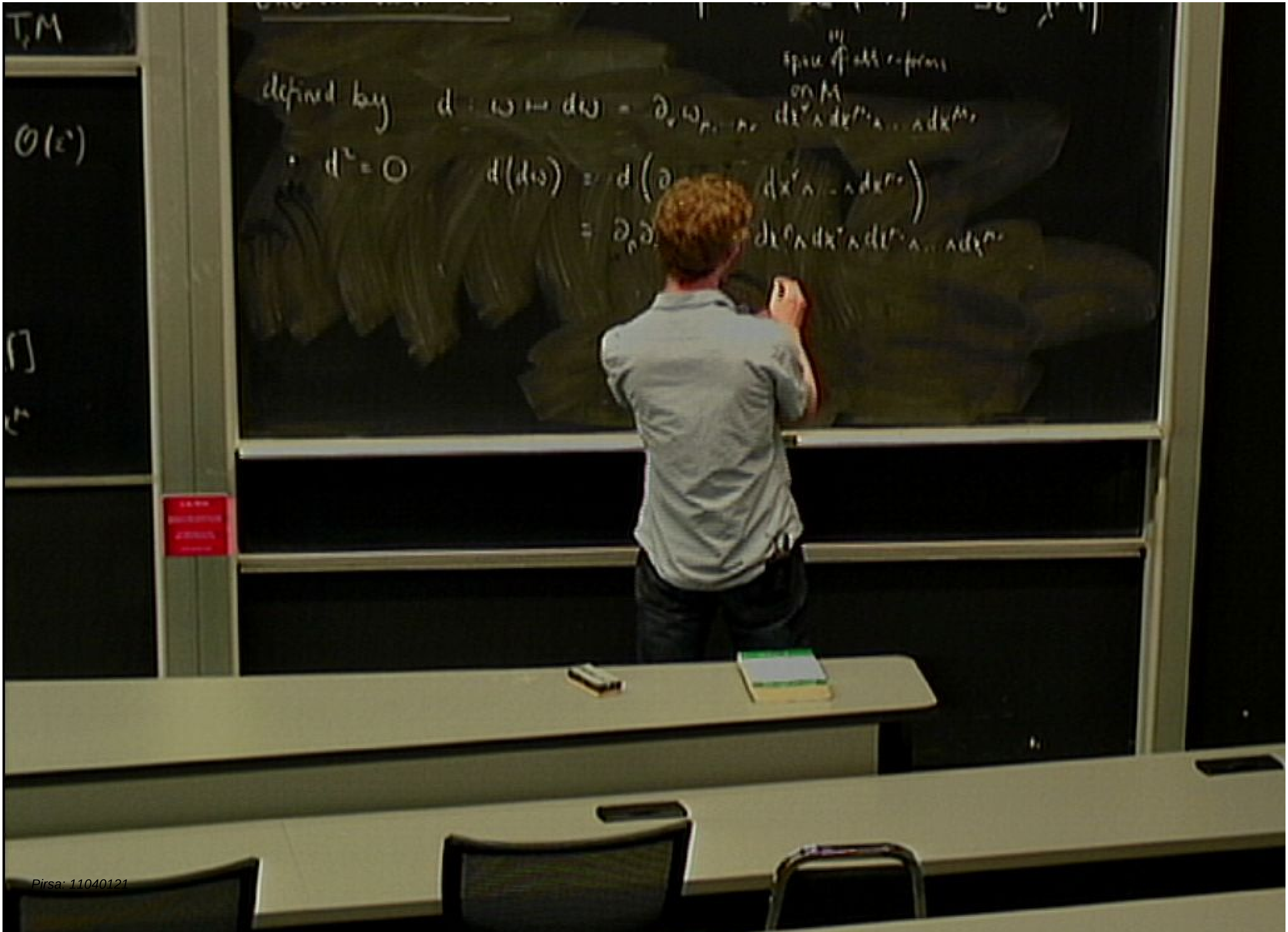
$O(\epsilon^2)$

Exterior derivative   d is a map $d : \Omega^r(M) \to \Omega^{r+1}(M)$

space of all r-forms on M

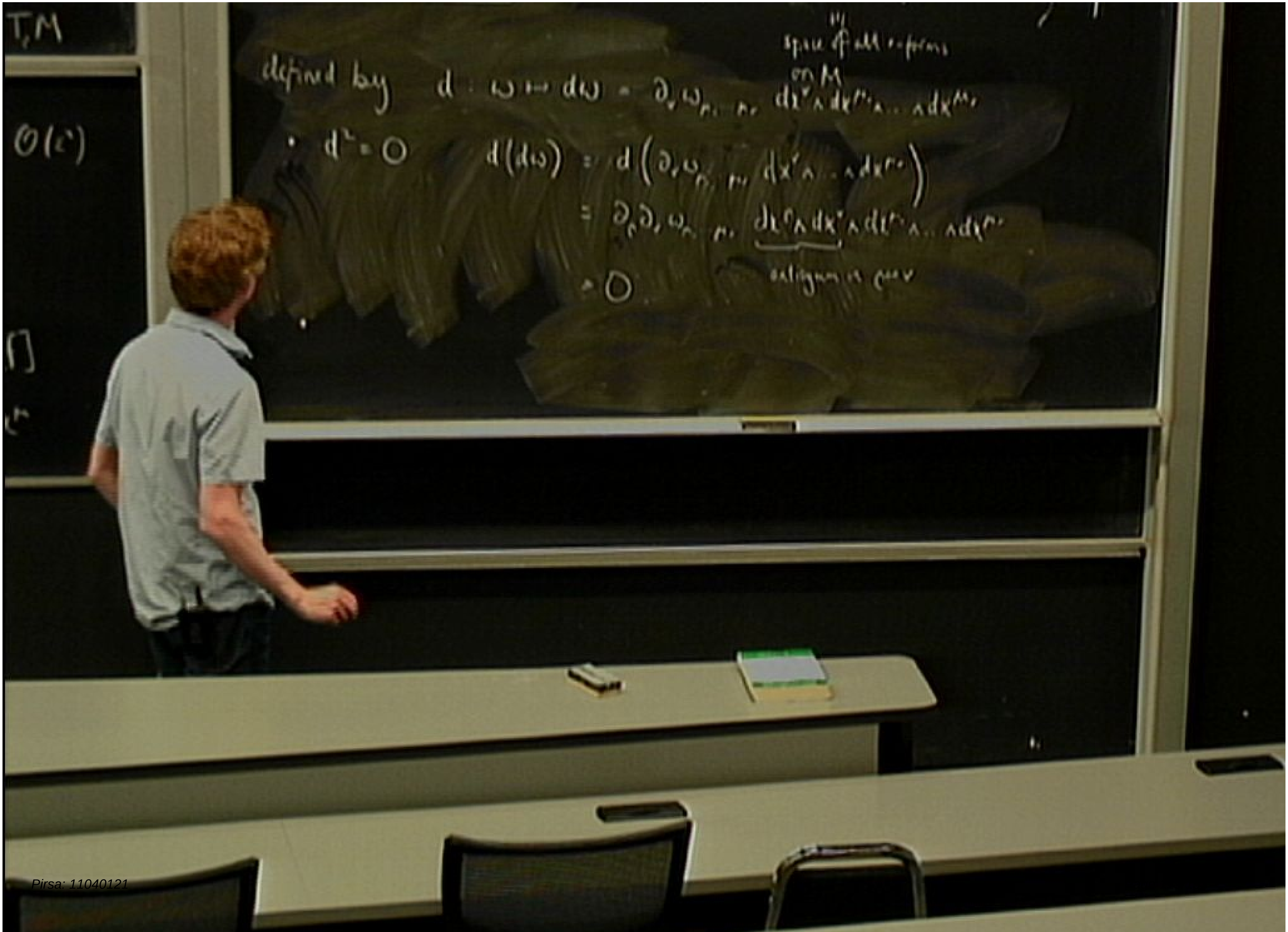
defined by $d : \omega \mapsto d\omega =$ … $dx^{\nu} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$

- $d^2 = 0$ $d(d\omega) = d($ … $dx^{\nu} \wedge$



defined by $d : \omega \mapsto d\omega = \partial_\nu \omega_{\mu_1 \dots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$

- $d^2 = 0$ $\quad d(d\omega) = d\left(\partial \dots dx^\nu \wedge \dots \wedge dx^{\mu_r}\right)$
$\quad = \partial_\rho \partial \dots dx^\rho \wedge dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$



T_pM

$O(\varepsilon^2)$

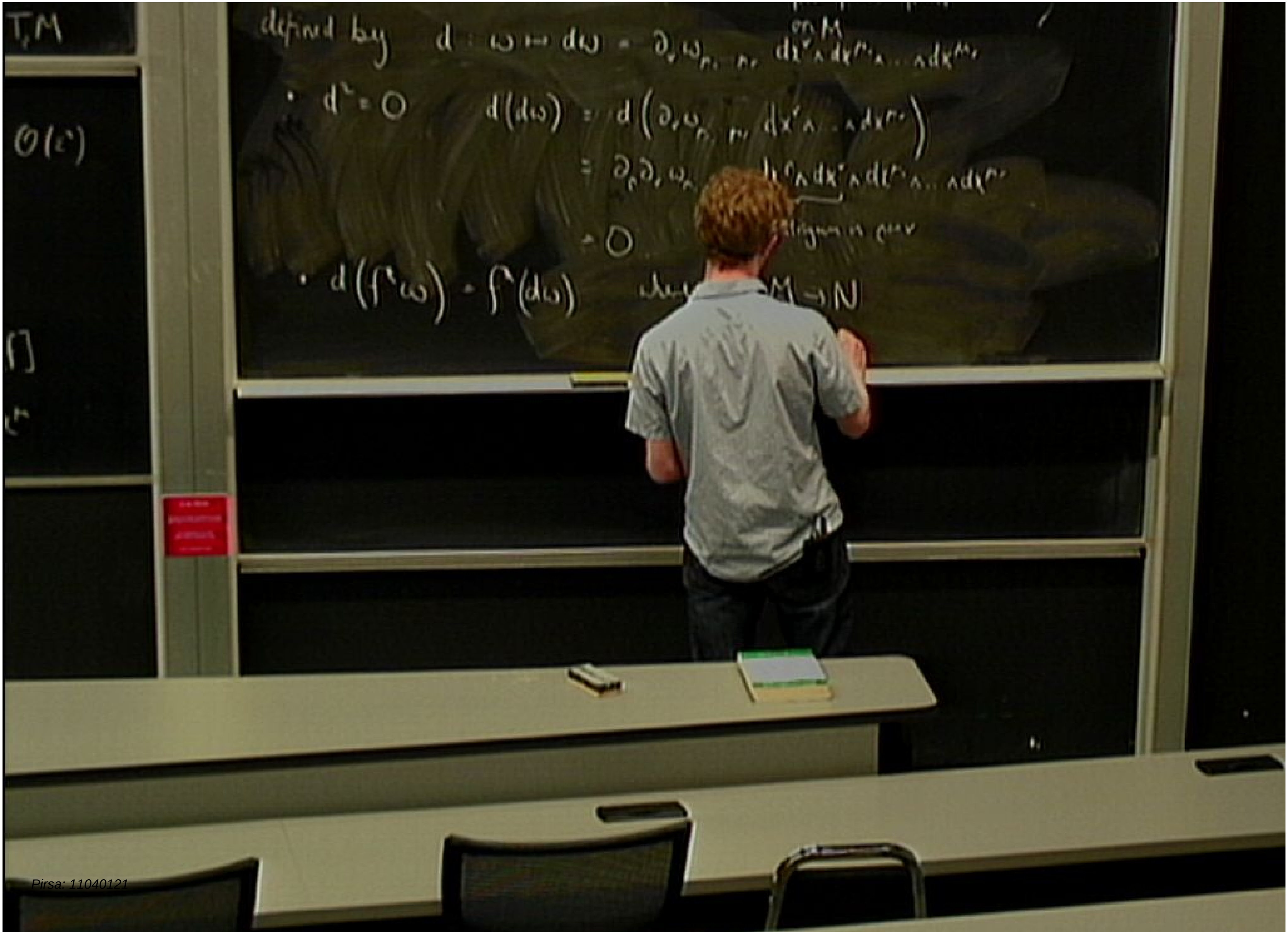
space of all n-forms on M

defined by $d : \omega \mapsto d\omega = \partial_\nu \omega_{\mu_1 \dots \mu_n} dx^\nu \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$

- $d^2 = 0$ $\quad d(d\omega) = d\left(\partial_\nu \omega_{\mu_1 \dots \mu_n} dx^\nu \wedge \dots \wedge dx^{\mu_n}\right)$

$$= \partial_\rho \partial_\nu \omega_{\mu_1 \dots \mu_n} \underbrace{dx^\rho \wedge dx^\nu}_{\text{antisymmetric in } \rho \leftrightarrow \nu} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$= 0$$



T_pM

$O(\varepsilon^2)$

on M

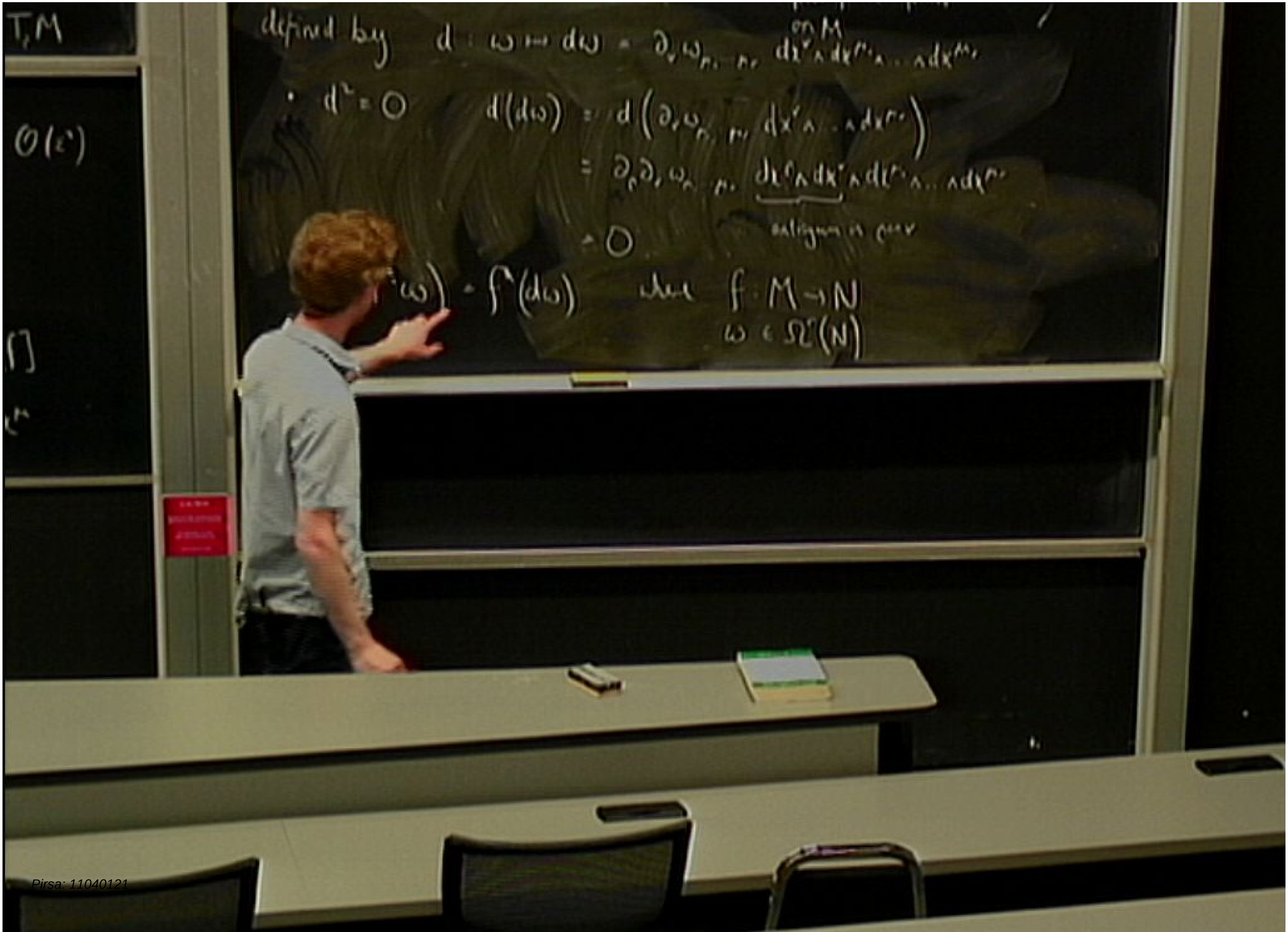
defined by $d : \omega \mapsto d\omega = \partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$

- $d^2 = 0$ $\quad d(d\omega) = d\left(\partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^\nu \wedge \ldots \wedge dx^{\mu_r}\right)$

$$= \partial_\rho \partial_\nu \omega_{\mu_1 \ldots \mu_r} \underbrace{dx^\rho \wedge dx^\nu}_{\text{antisym in } \rho\nu} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$$

$$= 0$$

- $d(f^*\omega) = f^*(d\omega)$ where $f: M \to N$



defined by $d : \omega \mapsto d\omega = \partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$

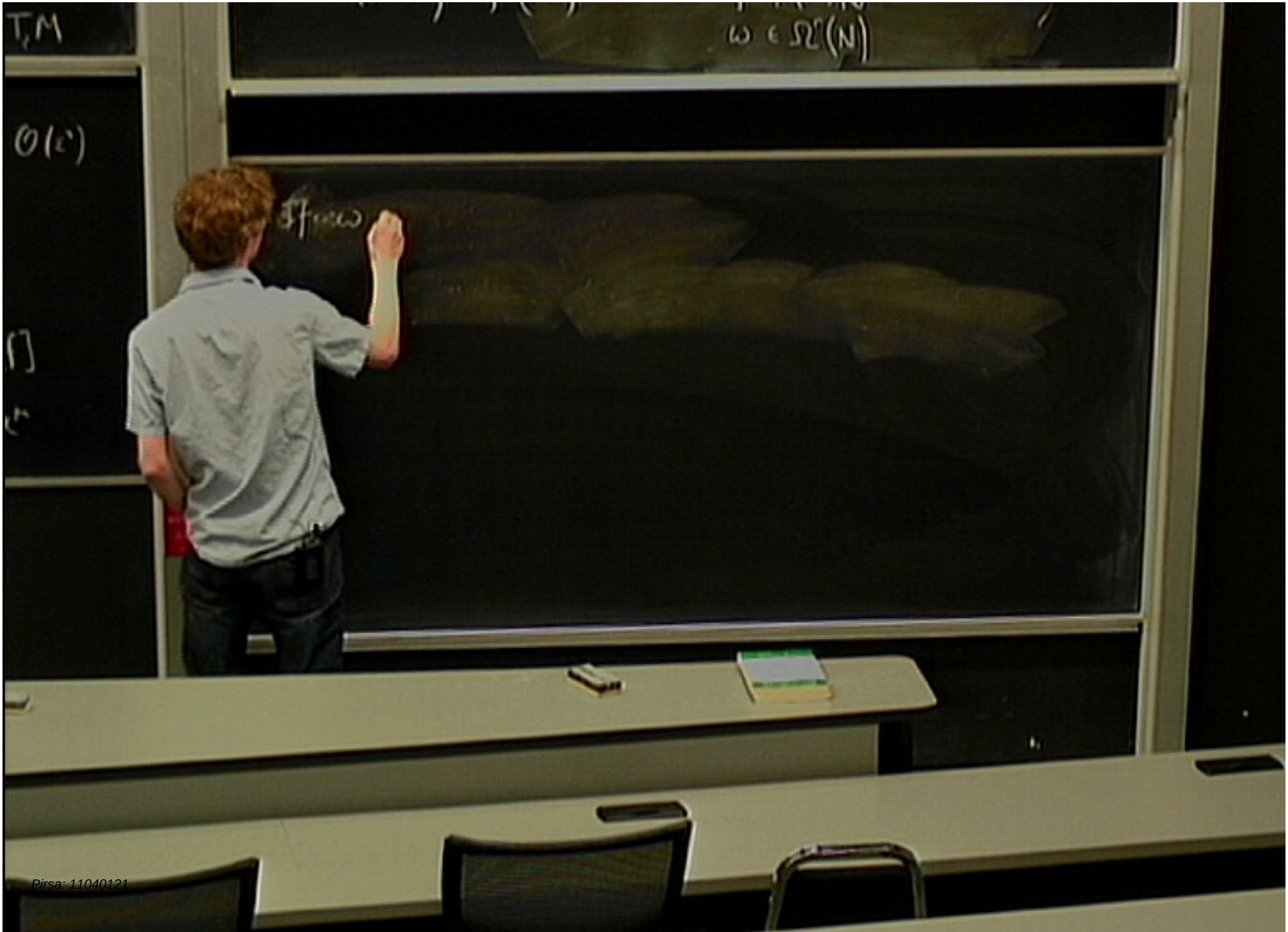
- $d^2 = 0$ $\quad d(d\omega) = d\left(\partial_\nu \omega_{\mu_1 \ldots \mu_r} dx^\nu \wedge \ldots \wedge dx^{\mu_r}\right)$

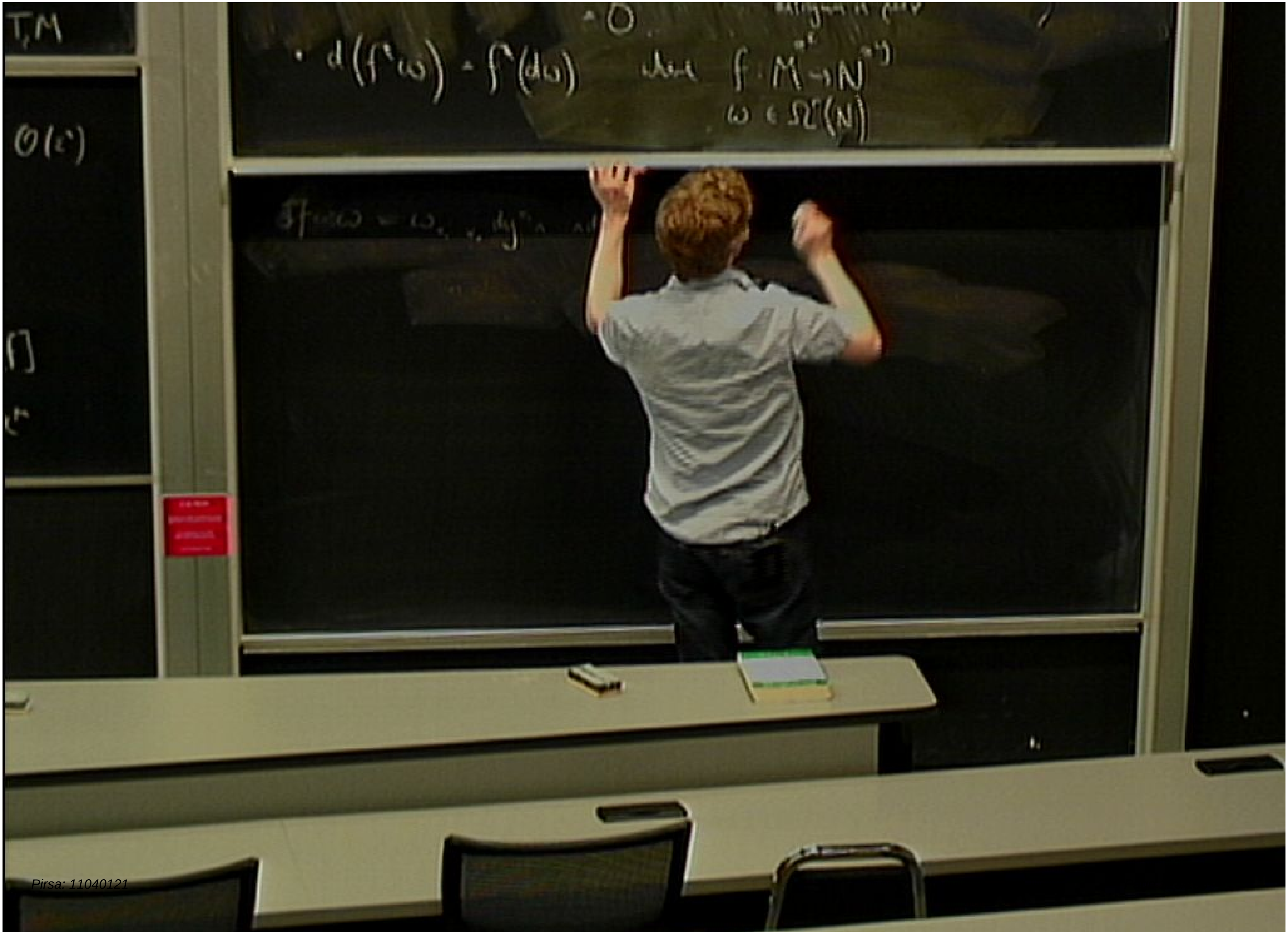
$$= \partial_\rho \partial_\nu \omega_{\mu_1 \ldots \mu_r} \underbrace{dx^\rho \wedge dx^\nu}_{\text{antisym in } \rho\nu} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$$

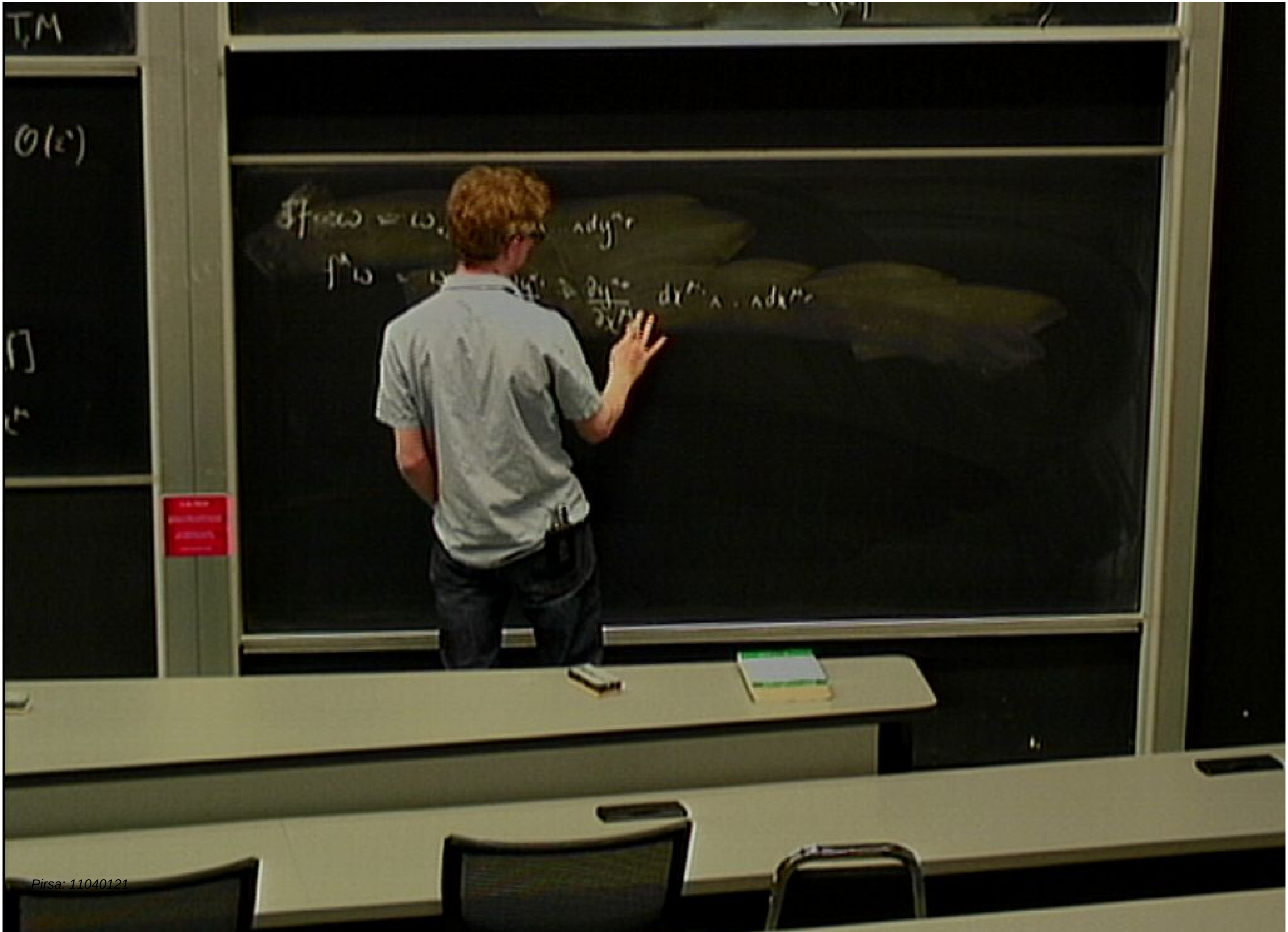
$$= 0$$

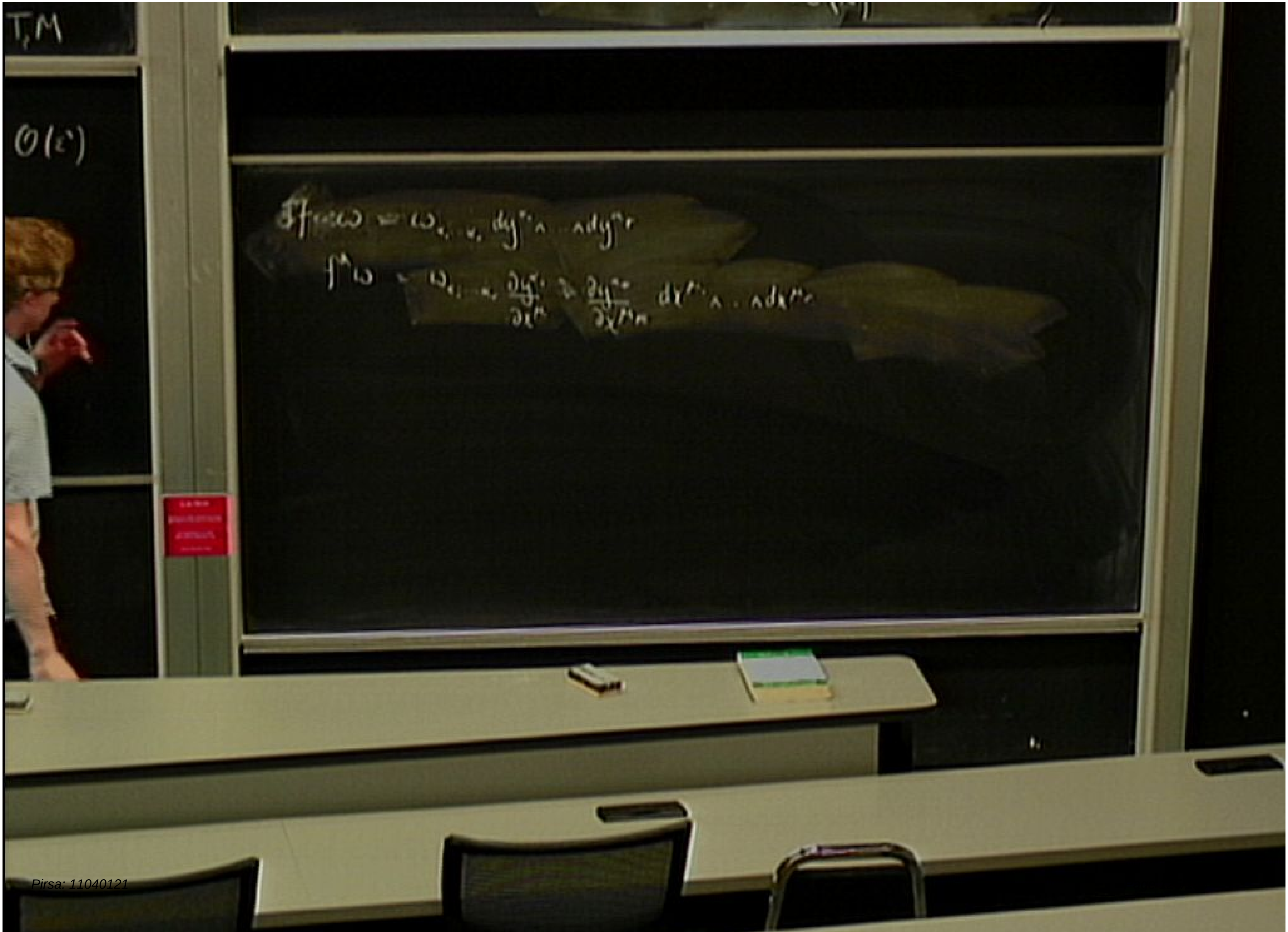
$\ldots \omega) = f^*(d\omega)$ where $f : M \to N$

$\omega \in \Omega^r(N)$


T_pM
$\omega \in \Omega^1(N)$
$\mathcal{O}(\varepsilon^2)$
$\oint f_\gamma^* \omega$

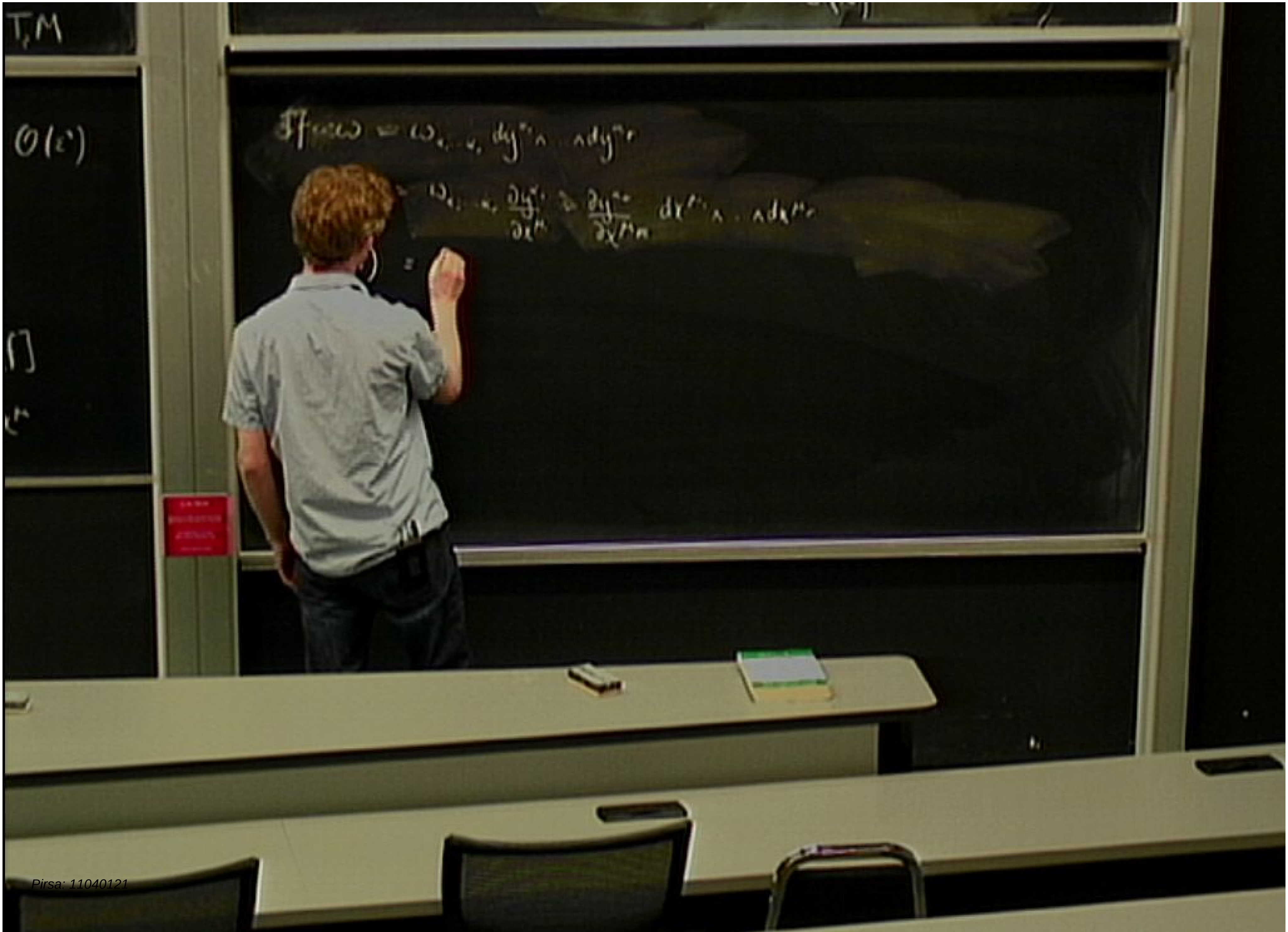

T_pM
$\theta(\varepsilon^2)$
$= 0$
$\cdot\ d(f^*\omega) = f^*(d\omega)$ where $f: M^m \to N^n$
$\omega \in \Omega^r(N)$
$f^*\omega = \omega_{\ldots}\, dy^{\ldots} \wedge \ldots$


T_pM
$\mathcal{O}(\epsilon^2)$
$\omega = \omega \ldots \wedge dy^{\mu_n}$
$f^*\omega = \ldots \frac{\partial y^{\mu_n}}{\partial x^{\nu_n}} \, dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_n}$



If $\omega = \omega_{a_1 \ldots a_r} dy^{a_1} \wedge \ldots \wedge dy^{a_r}$

$f^* \omega = \omega_{a_1 \ldots a_r} \frac{\partial y^{a_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{a_r}}{\partial x^{\mu_r}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$



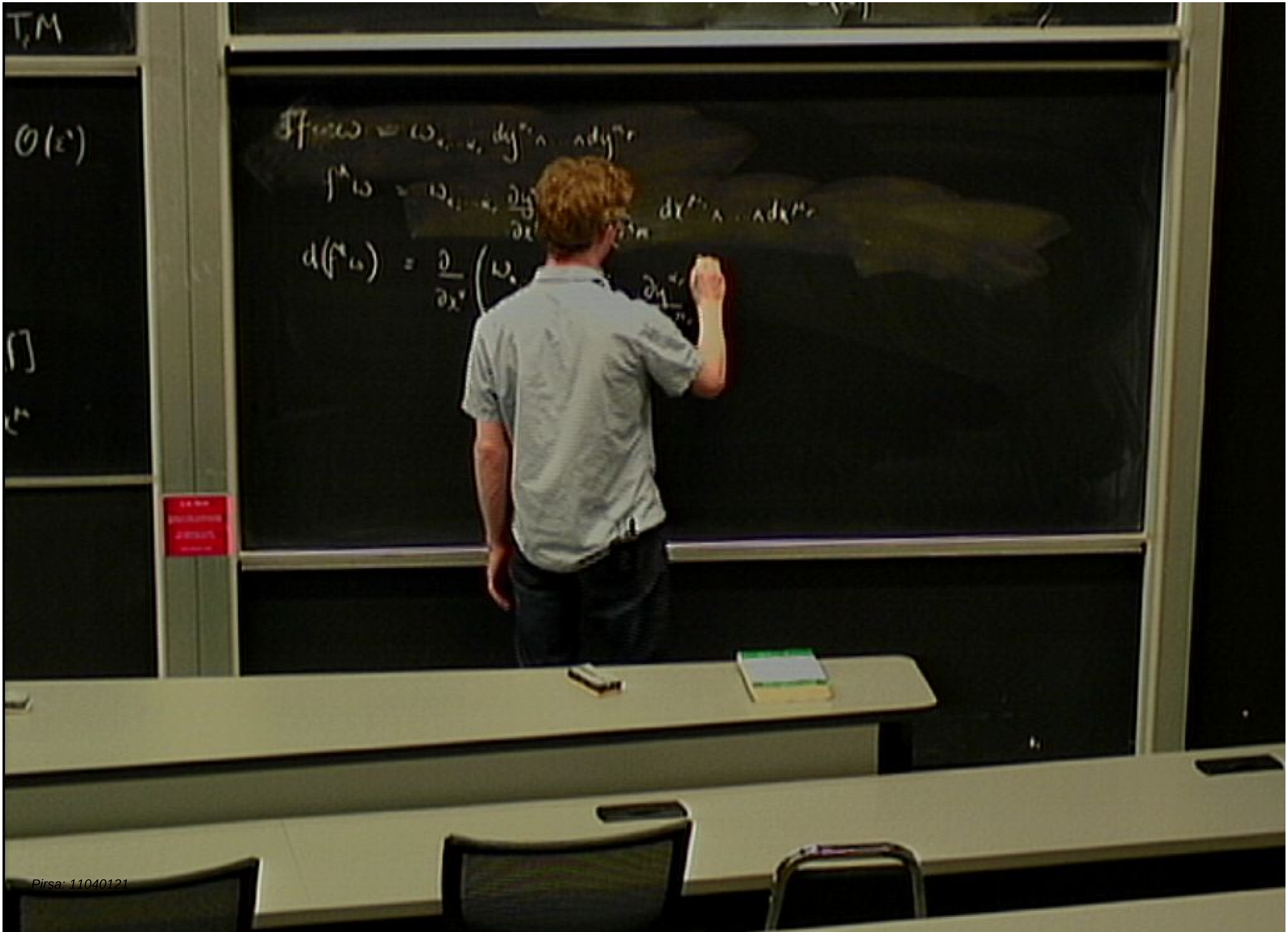
T_pM

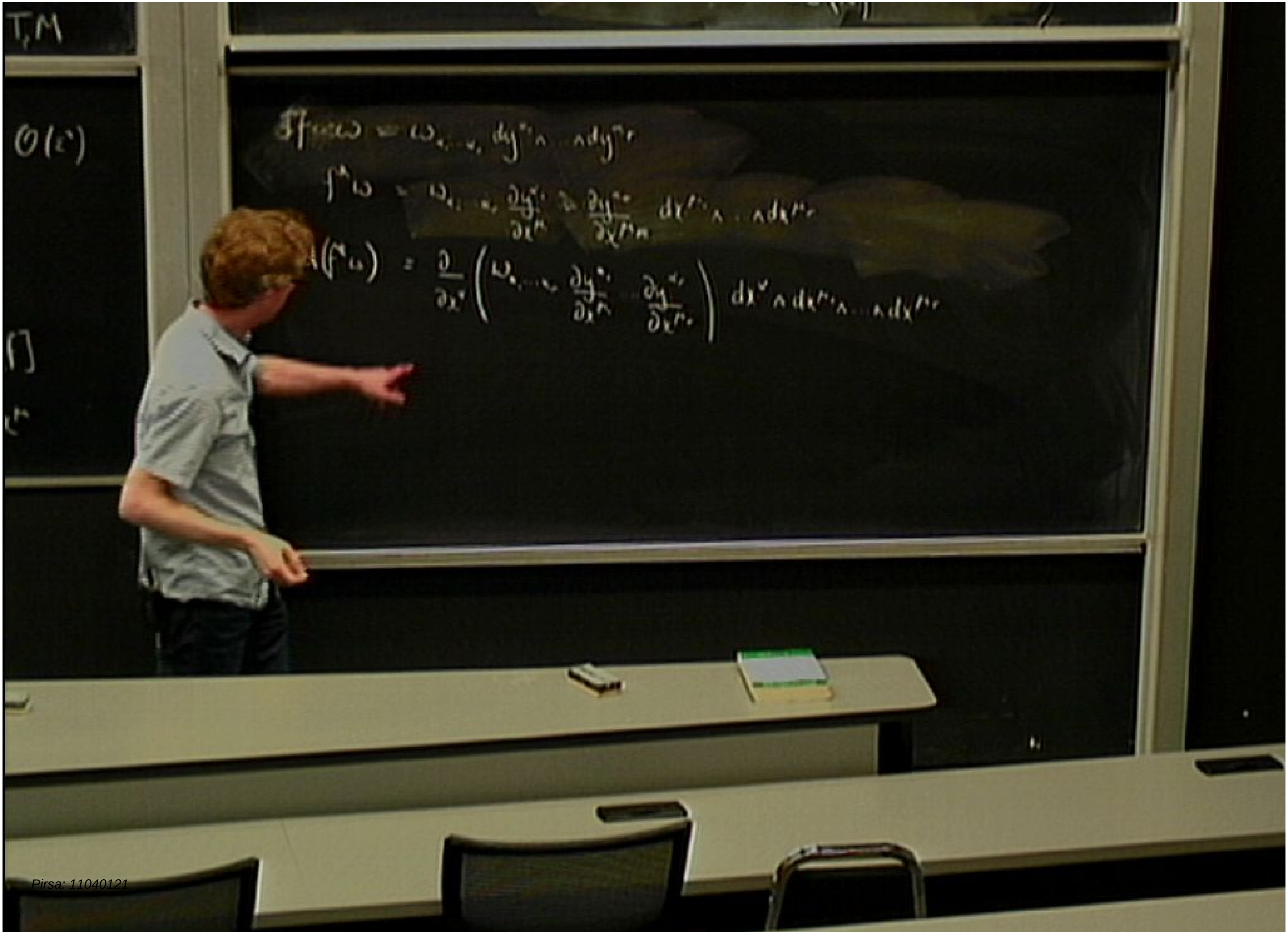
$O(\varepsilon^2)$

$$f^*\omega = \omega_{\nu_1\ldots\nu_p}\, dy^{\nu_1}\wedge\ldots\wedge dy^{\nu_p}$$

$$= \omega_{\nu_1\ldots\nu_p}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\ldots\frac{\partial y^{\nu_p}}{\partial x^{\mu_p}}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$$

$$=$$


T_pM
$O(\epsilon^2)$
If $\omega = \omega_{\nu_1 \dots \nu_n} dy^{\nu_1} \wedge \dots \wedge dy^{\nu_n}$
$f^* \omega = \omega_{\nu_1 \dots \nu_n} \frac{\partial y}{\partial x} \dots \, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$
$d(f^* \omega) = \frac{\partial}{\partial x^\mu} \Big(\omega_{\nu} \dots \frac{\partial y^{\nu_n}}{\partial x^{\mu_n}}$



T_pM

$\mathcal{O}(\varepsilon^2)$

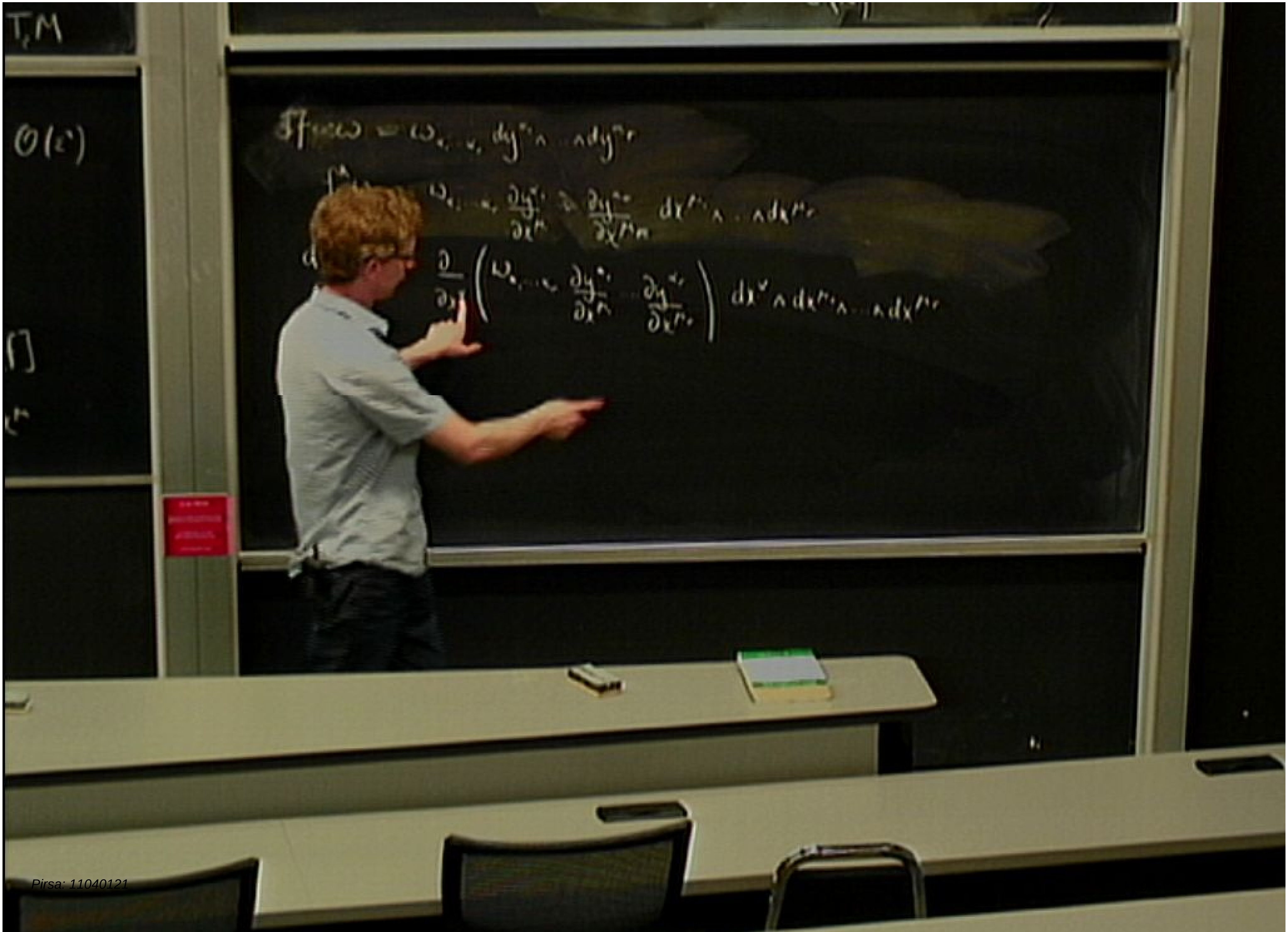
$f]$

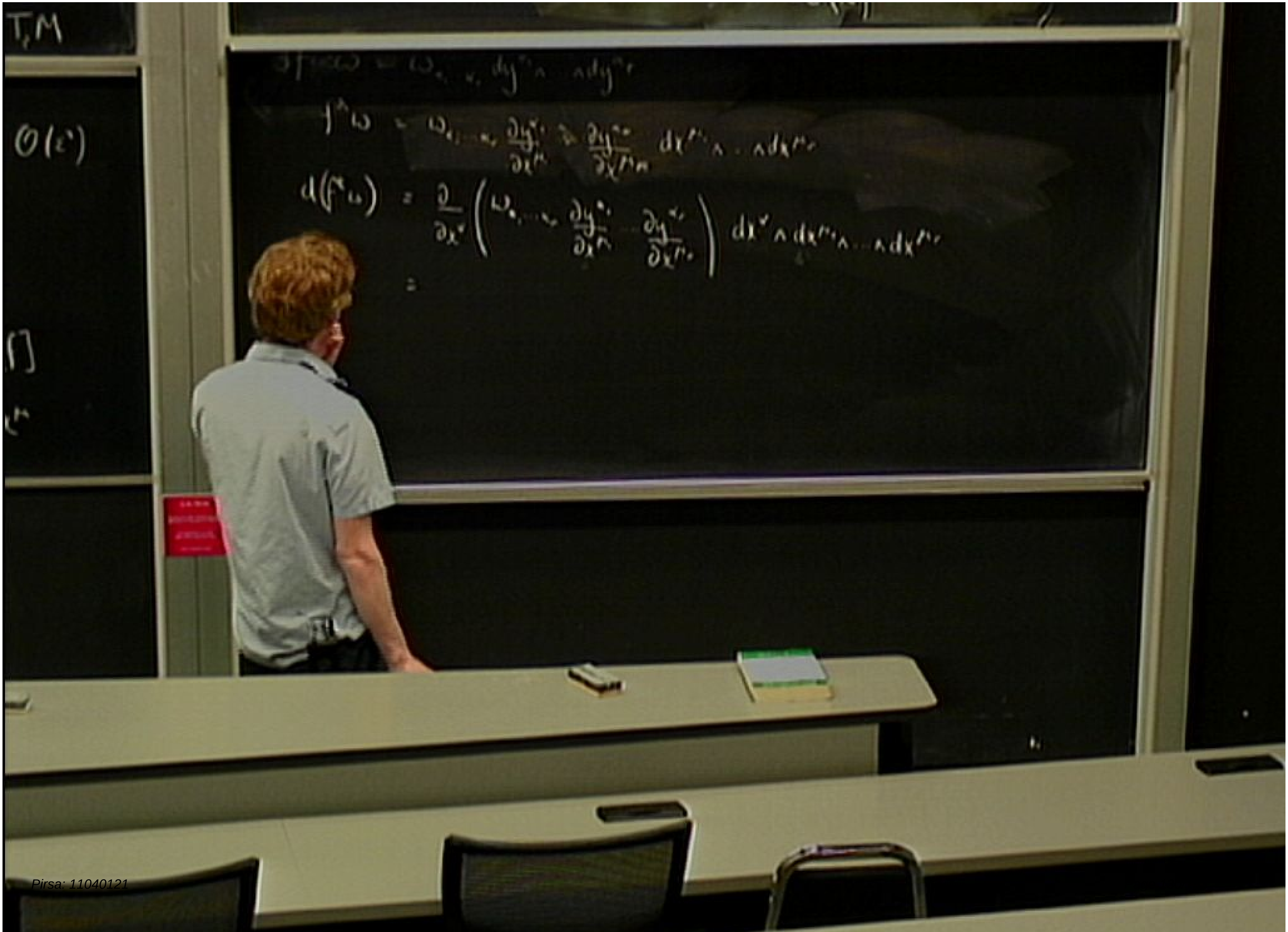
x^μ

$$f(x)\omega = \omega_{\alpha_1\dots\alpha_r} dy^{\alpha_1}\wedge\dots\wedge dy^{\alpha_r}$$

$$f^*\omega = \omega_{\alpha_1\dots\alpha_r}\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}}\, dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}$$

$$(f^*\omega) = \frac{\partial}{\partial x^\nu}\left(\omega_{\alpha_1\dots\alpha_r}\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}}\right) dx^\nu\wedge dx^{\mu_1}\wedge\dots\wedge dx^{\mu_r}$$


T_pM
$O(\varepsilon^2)$
$\omega = \omega_{\nu_1\ldots\nu_r}\, dy^{\nu_1}\wedge\ldots\wedge dy^{\nu_r}$
$\omega_{\nu_1\ldots\nu_r}\,\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\ldots\frac{\partial y^{\nu_r}}{\partial x^{\mu_r}}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}$
$\frac{\partial}{\partial x^{\nu}}\left(\omega_{\nu_1\ldots\nu_r}\,\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\ldots\frac{\partial y^{\nu_r}}{\partial x^{\mu_r}}\right) dx^{\nu}\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}$



T_pM

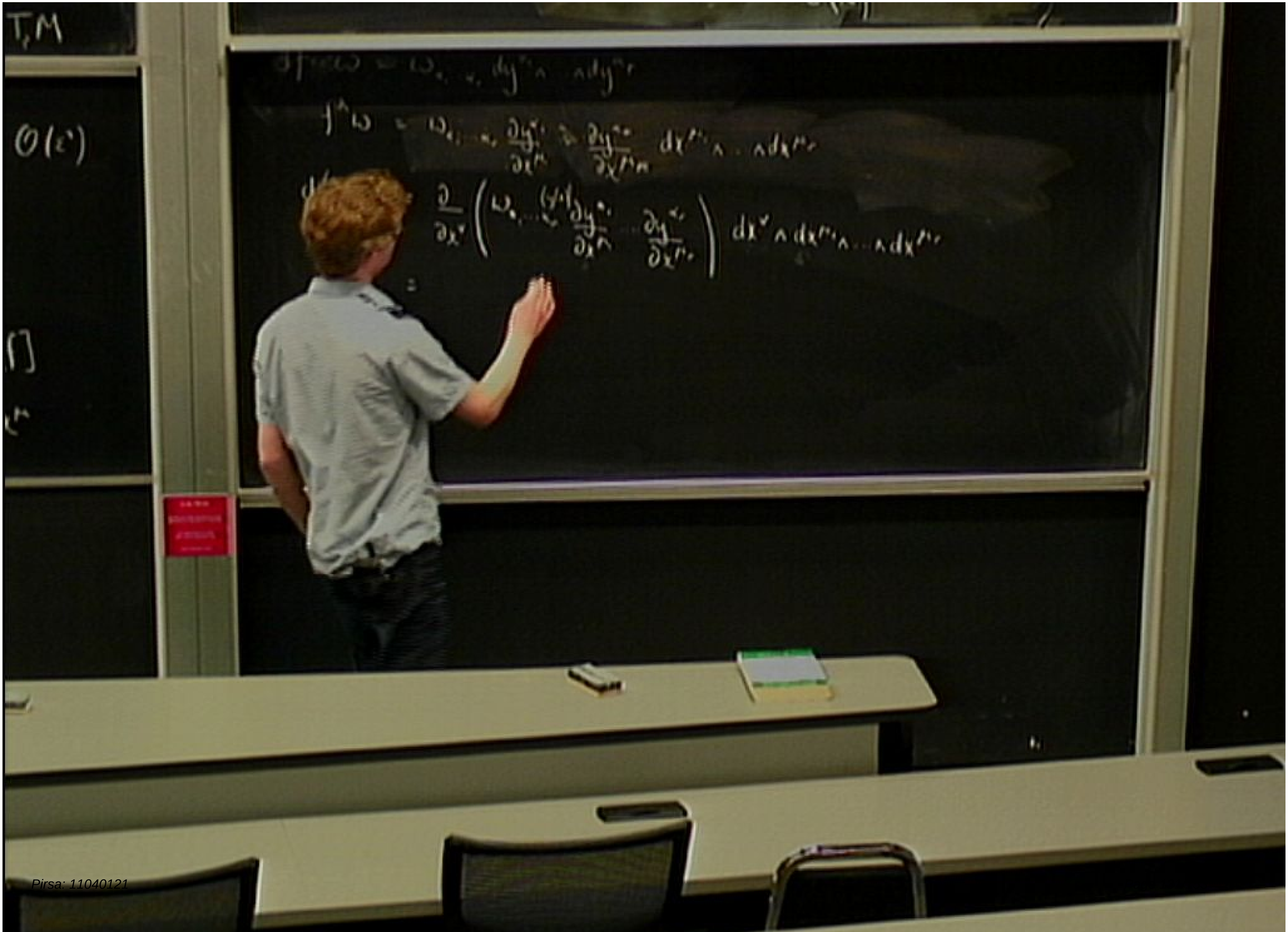
$\mathcal{O}(\varepsilon^2)$

[illegible] $= \omega_{\nu_1\ldots\nu_n}\, dy^{\nu_1}\wedge\ldots\wedge dy^{\nu_n}$

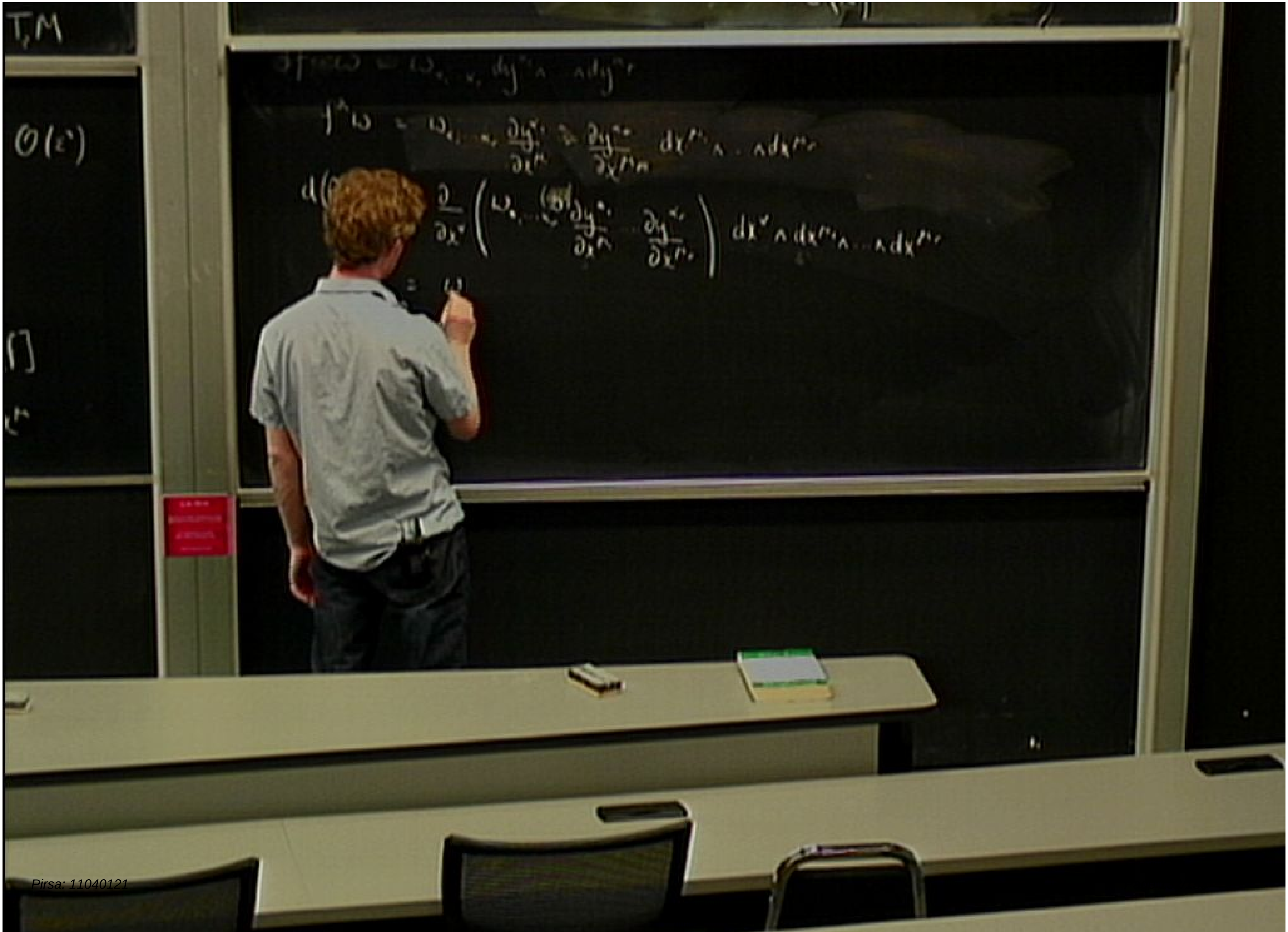
$$f^*\omega = \omega_{\nu_1\ldots\nu_n} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \ldots \frac{\partial y^{\nu_n}}{\partial x^{\mu_n}}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$$

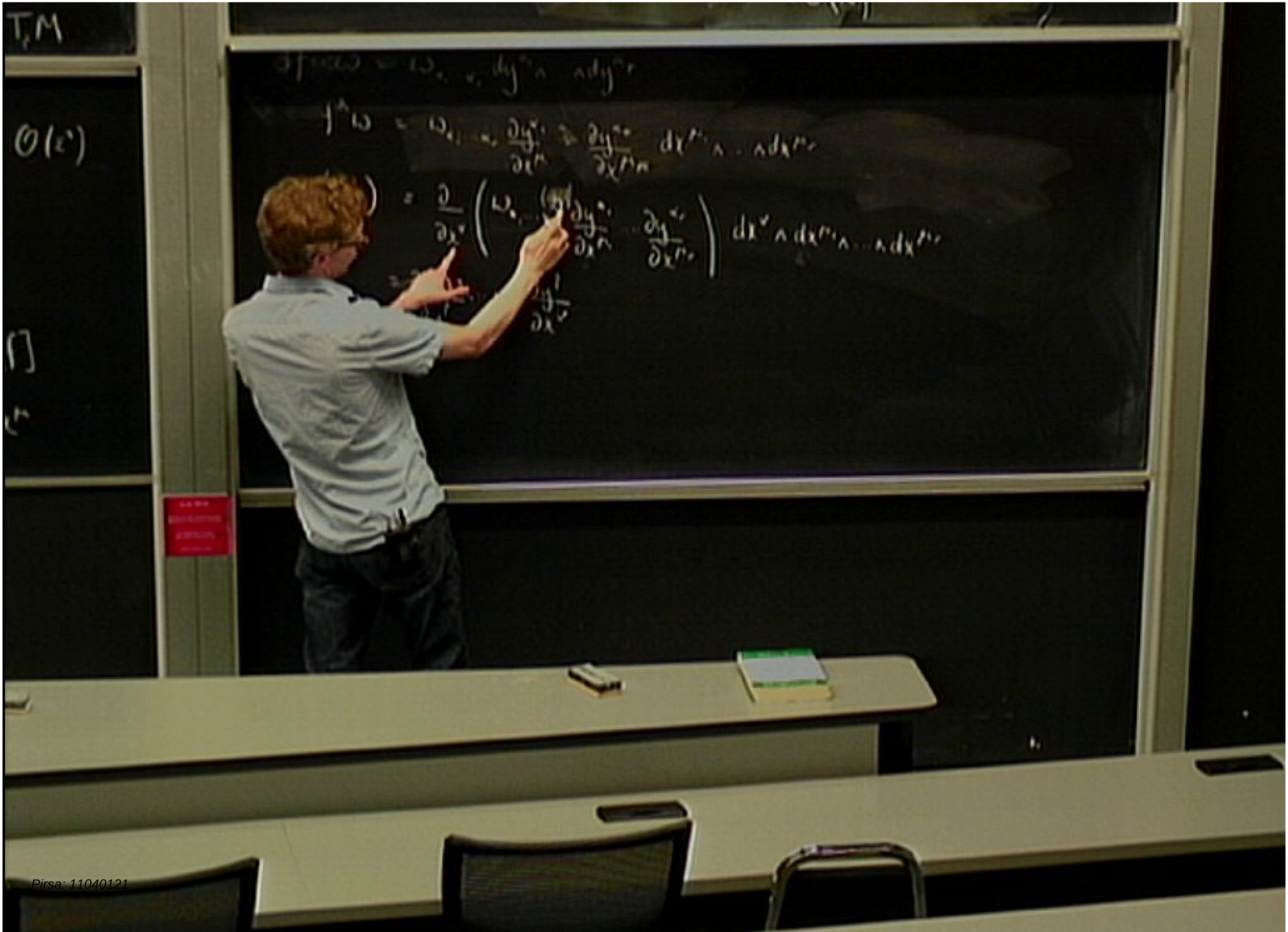
$$d(f^*\omega) = \frac{\partial}{\partial x^{\sigma}}\left(\omega_{\nu_1\ldots\nu_n} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \ldots \frac{\partial y^{\nu_n}}{\partial x^{\mu_n}}\right) dx^{\sigma}\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$$

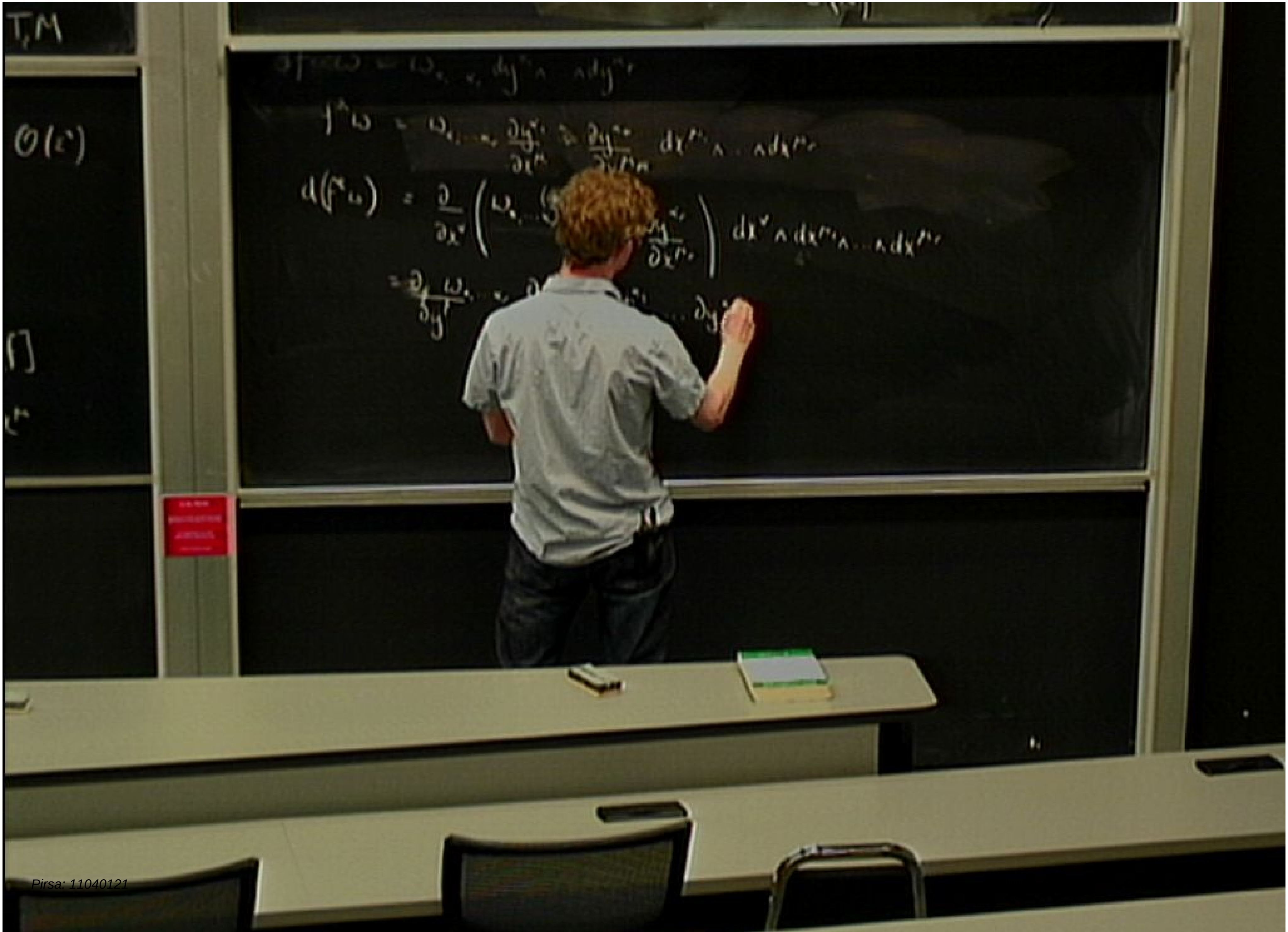
$$=$$

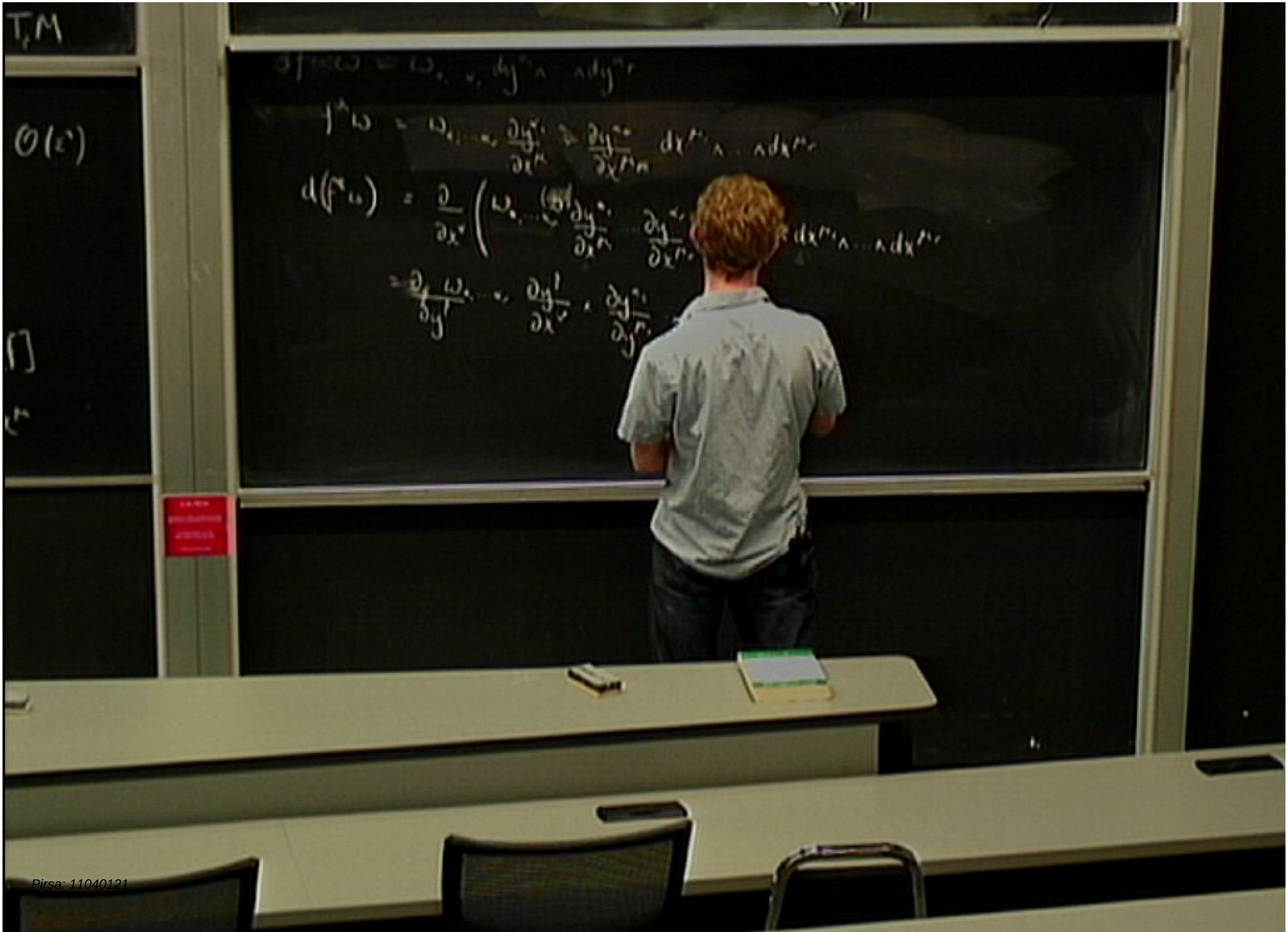


T_pM
O(ε²)
f*ω = ω_{ν1…νk} ∂y^{ν1}/∂x^{μ1} … ∂y^{νk}/∂x^{μk} dx^{μ1} ∧ … ∧ dx^{μk}
∂/∂x^ν (ω_{ν1…νk}(y(x)) ∂y^{ν1}/∂x^{μ1} … ∂y^{νk}/∂x^{μk}) dx^ν ∧ dx^{μ1} ∧ … ∧ dx^{μk}
=


T_pM
$\mathcal{O}(\epsilon^2)$
$f^*\omega = \omega_{\nu_1\ldots\nu_r} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\nu_r}}{\partial x^{\mu_r}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$
$\frac{\partial}{\partial x^\nu} \left(\omega_{\nu_1\ldots} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\nu_r}}{\partial x^{\mu_r}} \right) dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}$
$= \omega$


T_pM
Θ(ε²)
f^*ω = ω_{ν_1…ν_n} ∂y^{ν_1}/∂x^{μ_1} … ∂y^{ν_n}/∂x^{μ_n} dx^{μ_1} ∧ … ∧ dx^{μ_n}
= ∂/∂x^ν (ω_{ν_1…} ∂y^{ν_1}/∂x^{μ_1} … ∂y^{ν_n}/∂x^{μ_n}) dx^ν ∧ dx^{μ_1} ∧ … ∧ dx^{μ_n}
∂y/∂x^ν


T_pM
Θ(c)
f^*ω = ω_{ν_1…ν_n} ∂y^{ν_1}/∂x^{μ_1} … ∂y^{ν_n}/∂x^{μ_n} dx^{μ_1} ∧ … ∧ dx^{μ_n}
d(f^*ω) = ∂/∂x^ν (ω_{ν_1…} … ∂y^{ν_n}/∂x^{μ_n}) dx^ν ∧ dx^{μ_1} ∧ … ∧ dx^{μ_n}
= ∂/∂y^σ ω_{ν_1…ν_n} … ∂y …



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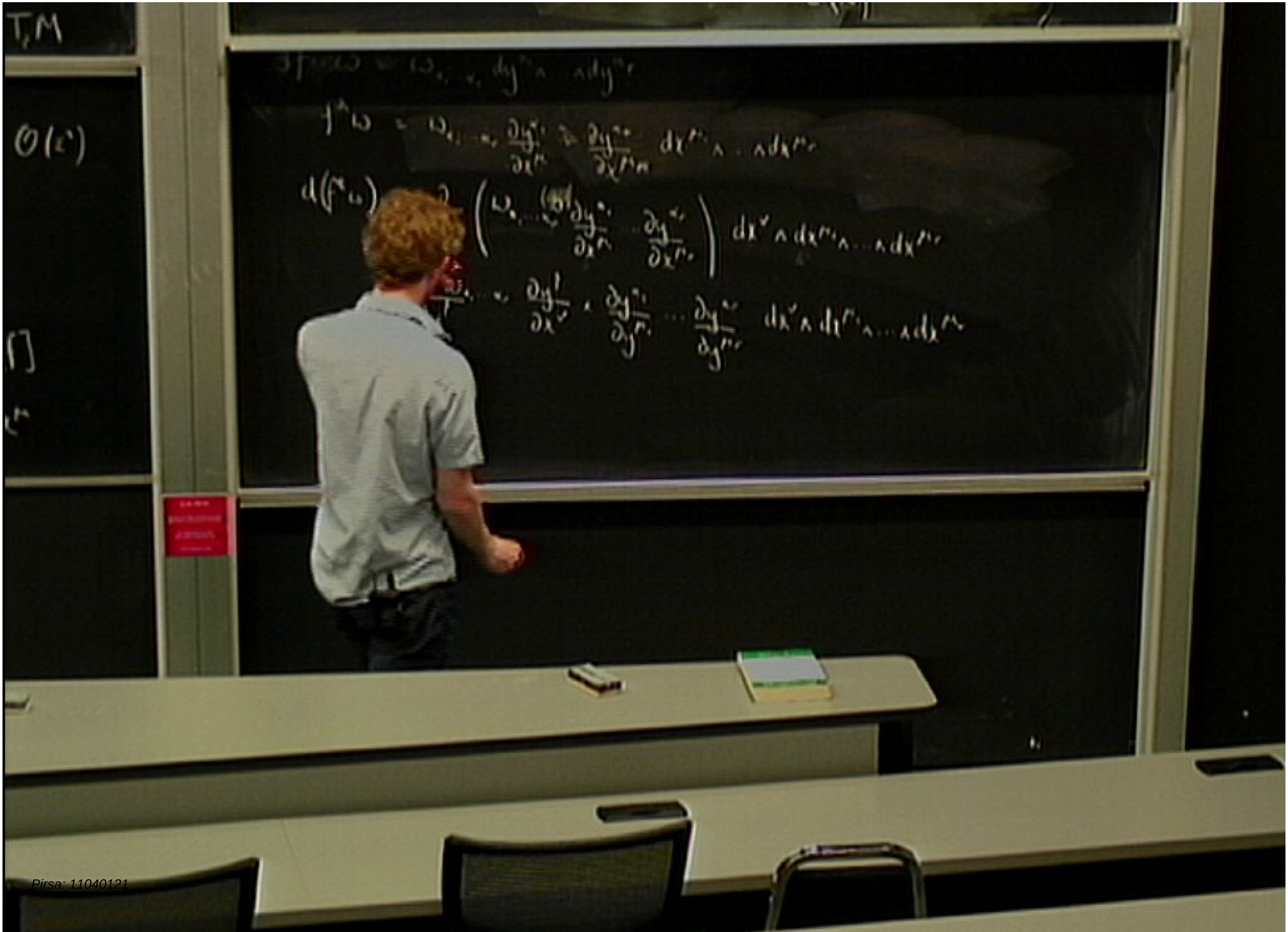
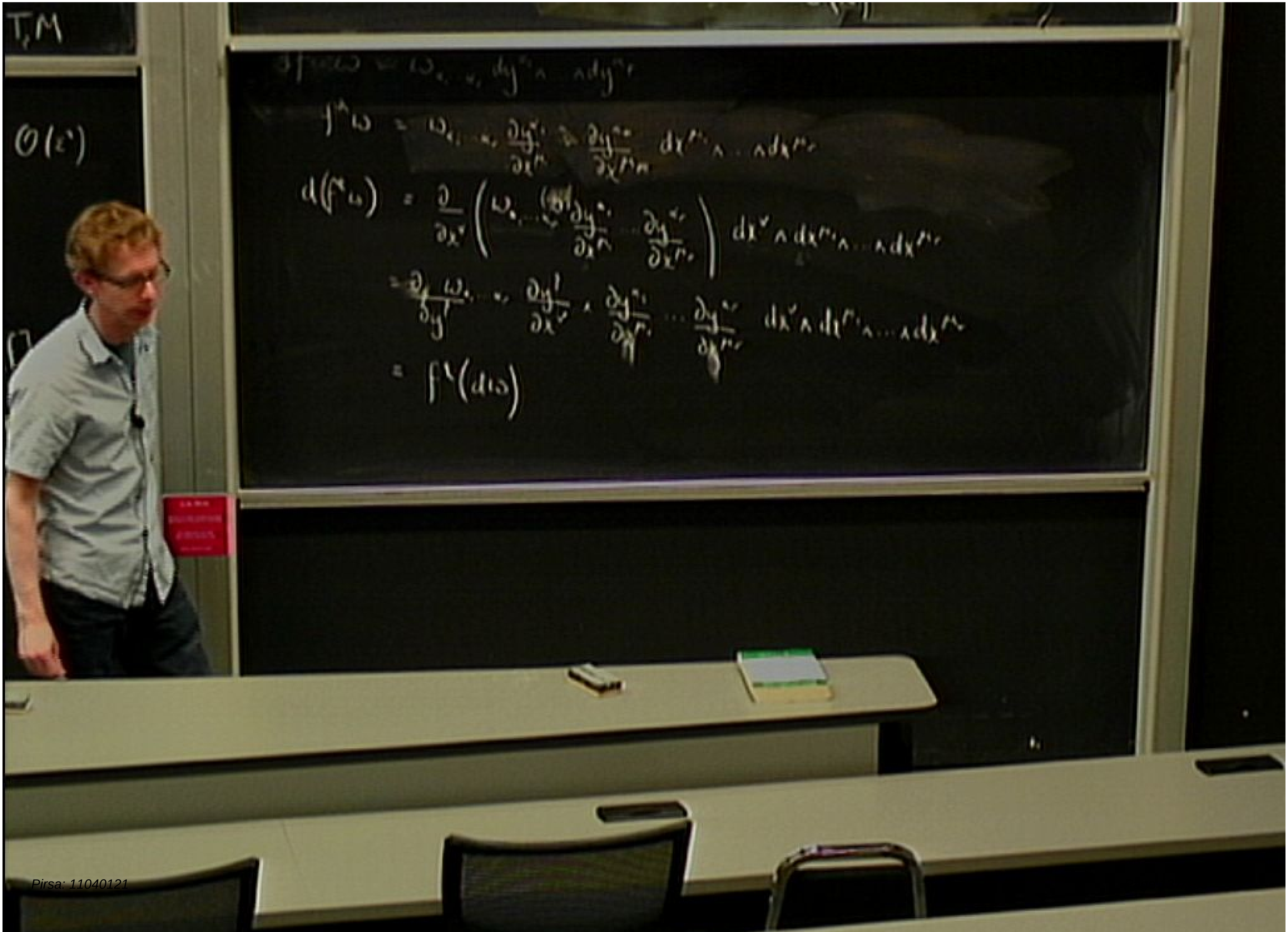

T_pM
$\mathcal{O}(\varepsilon^2)$
$\omega = \omega_{\nu_1\ldots\nu_r}\, dy^{\nu_1}\wedge\ldots\wedge dy^{\nu_r}$
$f^*\omega = \omega_{\nu_1\ldots\nu_r} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \ldots \frac{\partial y^{\nu_r}}{\partial x^{\mu_r}}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}$
$d(f^*\omega) = \partial_\nu \left(\omega_{\nu_1\ldots\nu_r}(f(x)) \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \ldots \frac{\partial y^{\nu_r}}{\partial x^{\mu_r}}\right) dx^\nu\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}$
$\frac{\partial\omega_{\nu_1\ldots\nu_r}}{\partial y^\lambda} \frac{\partial y^\lambda}{\partial x^\nu} \cdot \frac{\partial y^{\nu_1}}{\partial y^{\mu_1}} \ldots \frac{\partial y^{\nu_r}}{\partial y^{\mu_r}}\, dx^\nu\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_r}$



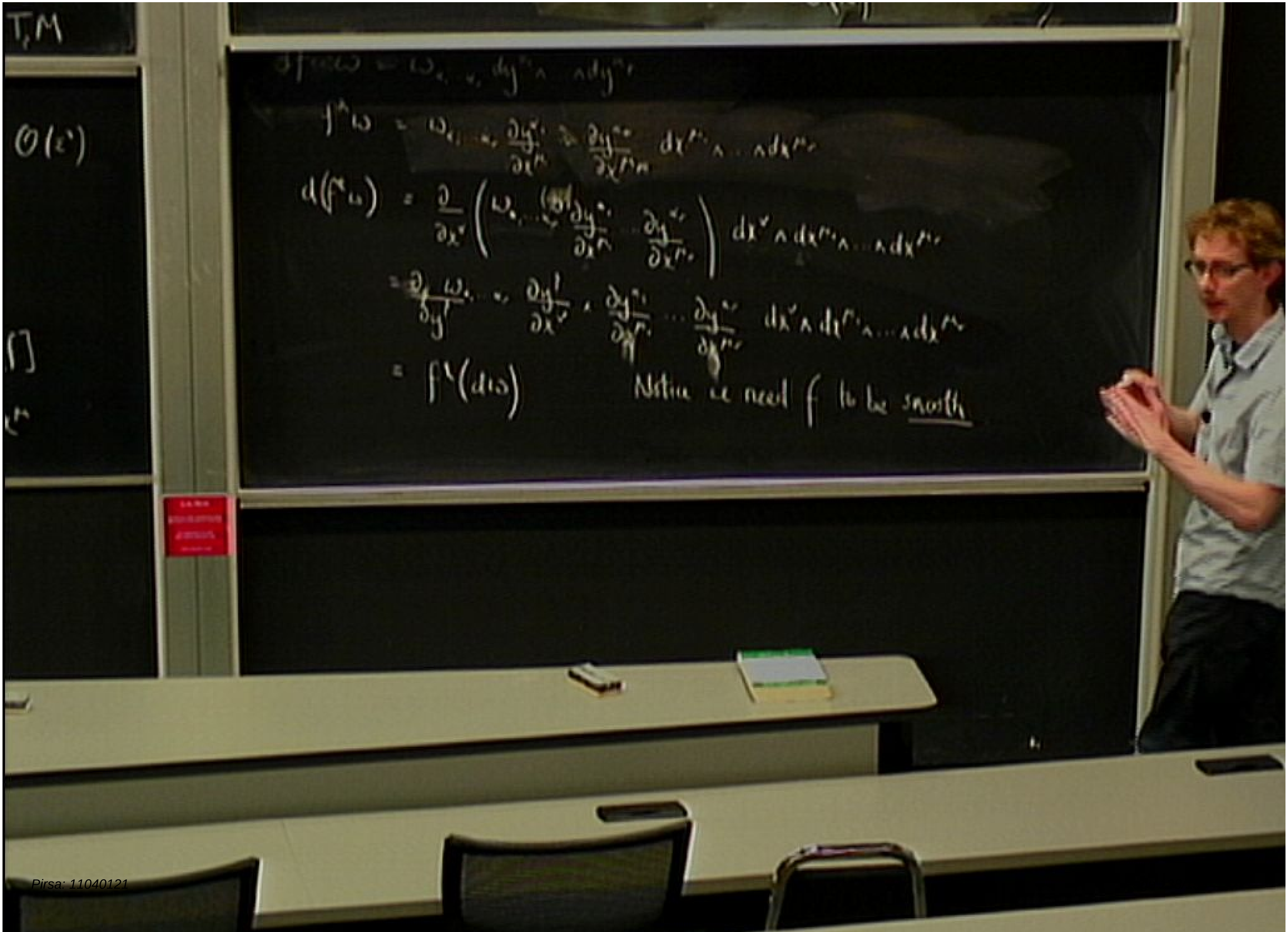
$\omega = \omega_{\nu_1\ldots\nu_k}\, dy^{\nu_1}\wedge\ldots\wedge dy^{\nu_k}$

$$f^*\omega = \omega_{\nu_1\ldots\nu_k}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\nu_k}}{\partial x^{\mu_k}}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_k}$$

$$d(f^*\omega) = \frac{\partial}{\partial x^\nu}\left(\omega_{\nu_1\ldots\nu_k}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\nu_k}}{\partial x^{\mu_k}}\right) dx^\nu\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_k}$$

$$= \frac{\partial\,\omega_{\nu_1\ldots\nu_k}}{\partial y^\lambda}\frac{\partial y^\lambda}{\partial x^\nu}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\nu_k}}{\partial x^{\mu_k}}\, dx^\nu\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_k}$$

$$= f^*(d\omega)$$



T_pM

$\mathcal{O}(\varepsilon^2)$

$$f^*\omega = \omega_{\nu_1\ldots\nu_n} dy^{\nu_1}\wedge\ldots\wedge dy^{\nu_n}$$

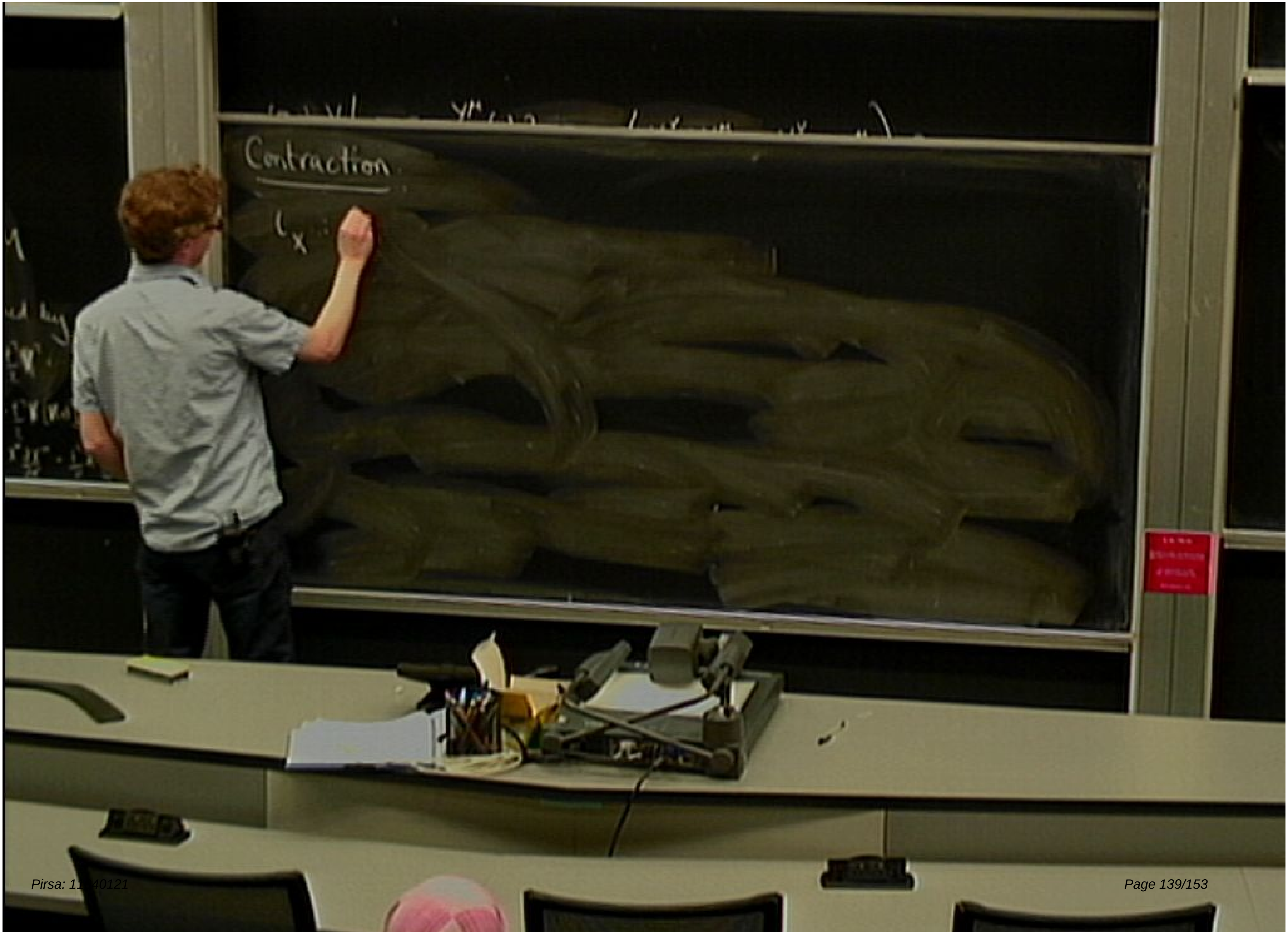
$$f^*\omega = \omega_{\nu_1\ldots\nu_n}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\nu_n}}{\partial x^{\mu_n}}\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$$

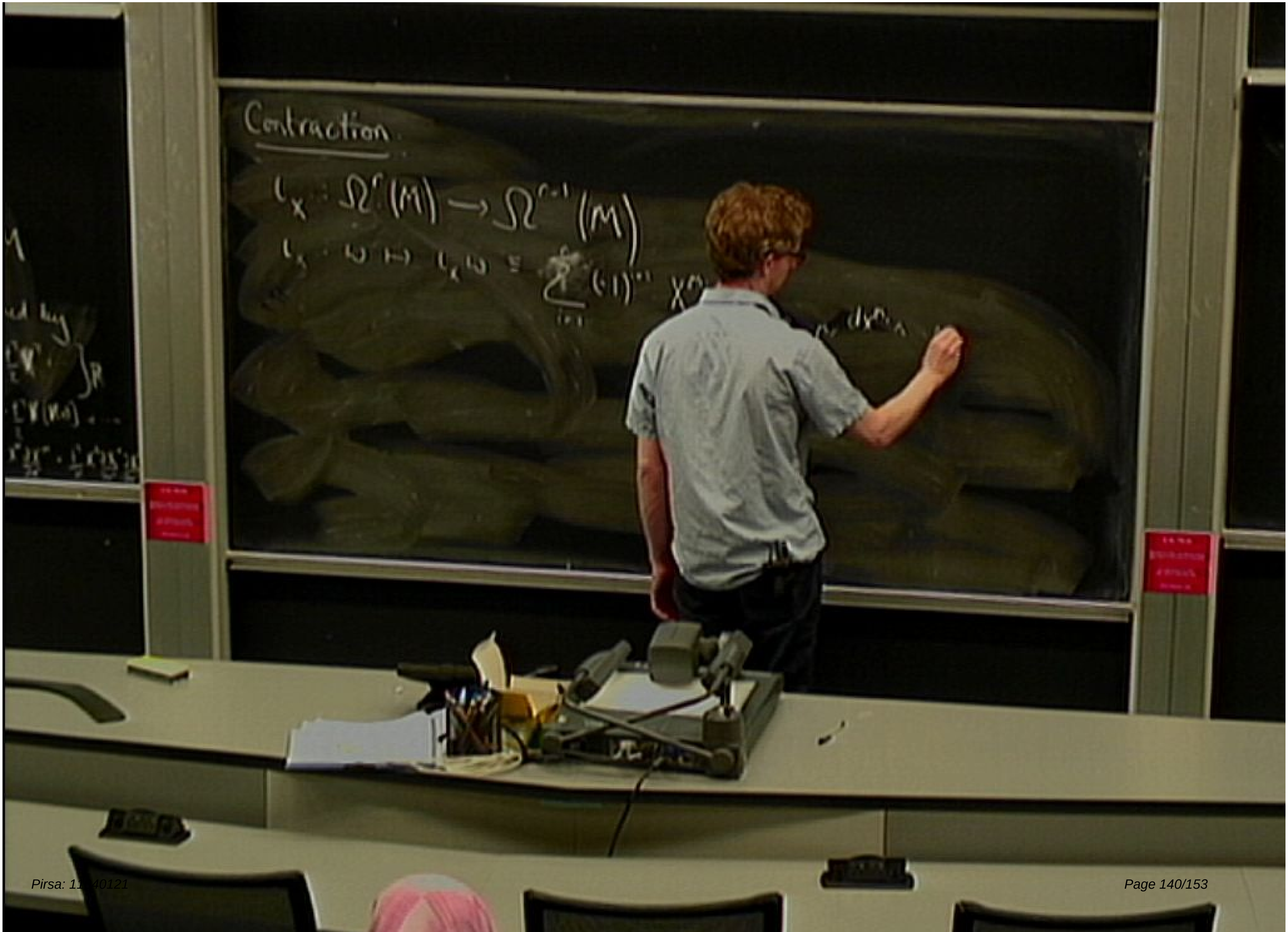
$$d(f^*\omega) = \frac{\partial}{\partial x^\nu}\left(\omega_{\nu_1\ldots\nu_n}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\nu_n}}{\partial x^{\mu_n}}\right) dx^\nu\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$$

$$= \frac{\partial\,\omega_{\nu_1\ldots\nu_n}}{\partial y^l}\frac{\partial y^l}{\partial x^\nu}\,\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\nu_n}}{\partial x^{\mu_n}}\, dx^\nu\wedge dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_n}$$

$$= f^*(d\omega)$$

Notice we need f to be smooth

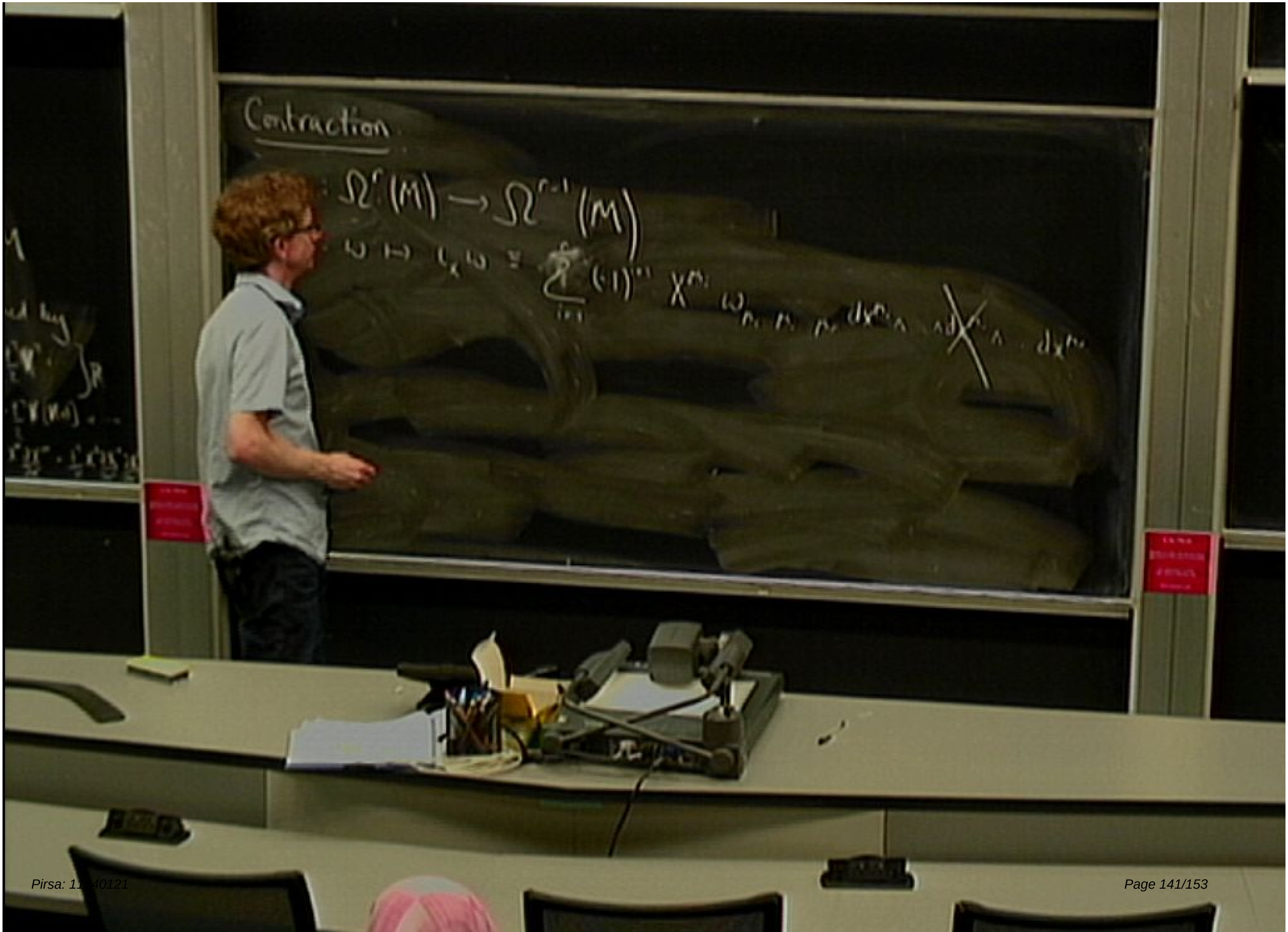

Contraction
ι_X :

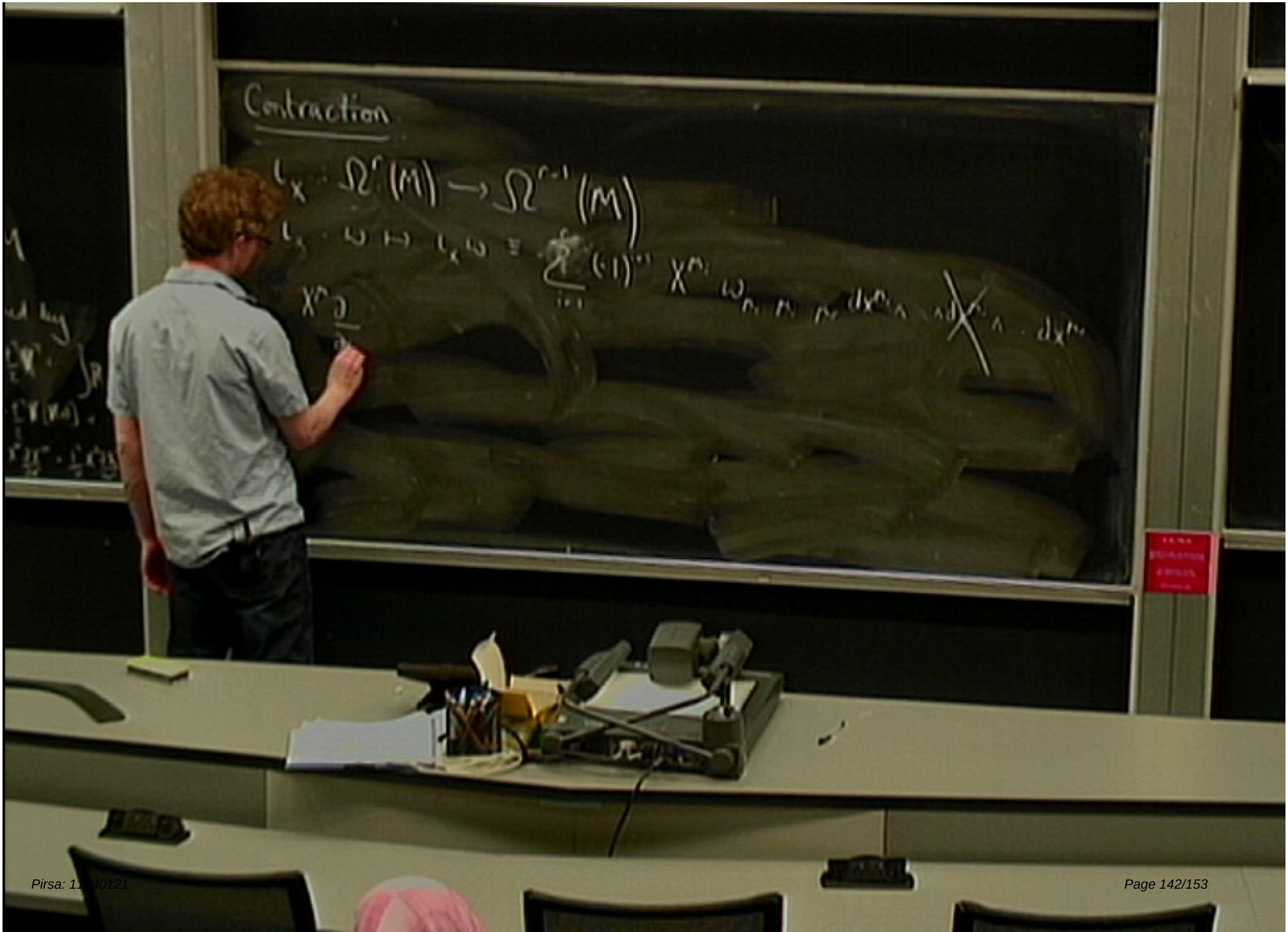


Contraction

$\iota_X : \Omega^r(M) \to \Omega^{r-1}(M)$

$\iota_X : \omega \mapsto \iota_X \omega = \sum_{i=1}^{r} (-1)^{i+1} X^{\mu_i} \ldots dx^{\mu_r}$


Contraction
$\Omega^r(M) \to \Omega^{r-1}(M)$
$\omega \mapsto \iota_X \omega \overset{\text{def}}{=} \sum_{i=1}^{r} (-1)^{i+1} X^{\mu_i} \omega_{\mu_1 \mu_2 \ldots \mu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_i} \wedge \ldots \wedge dx^{\mu_r}$

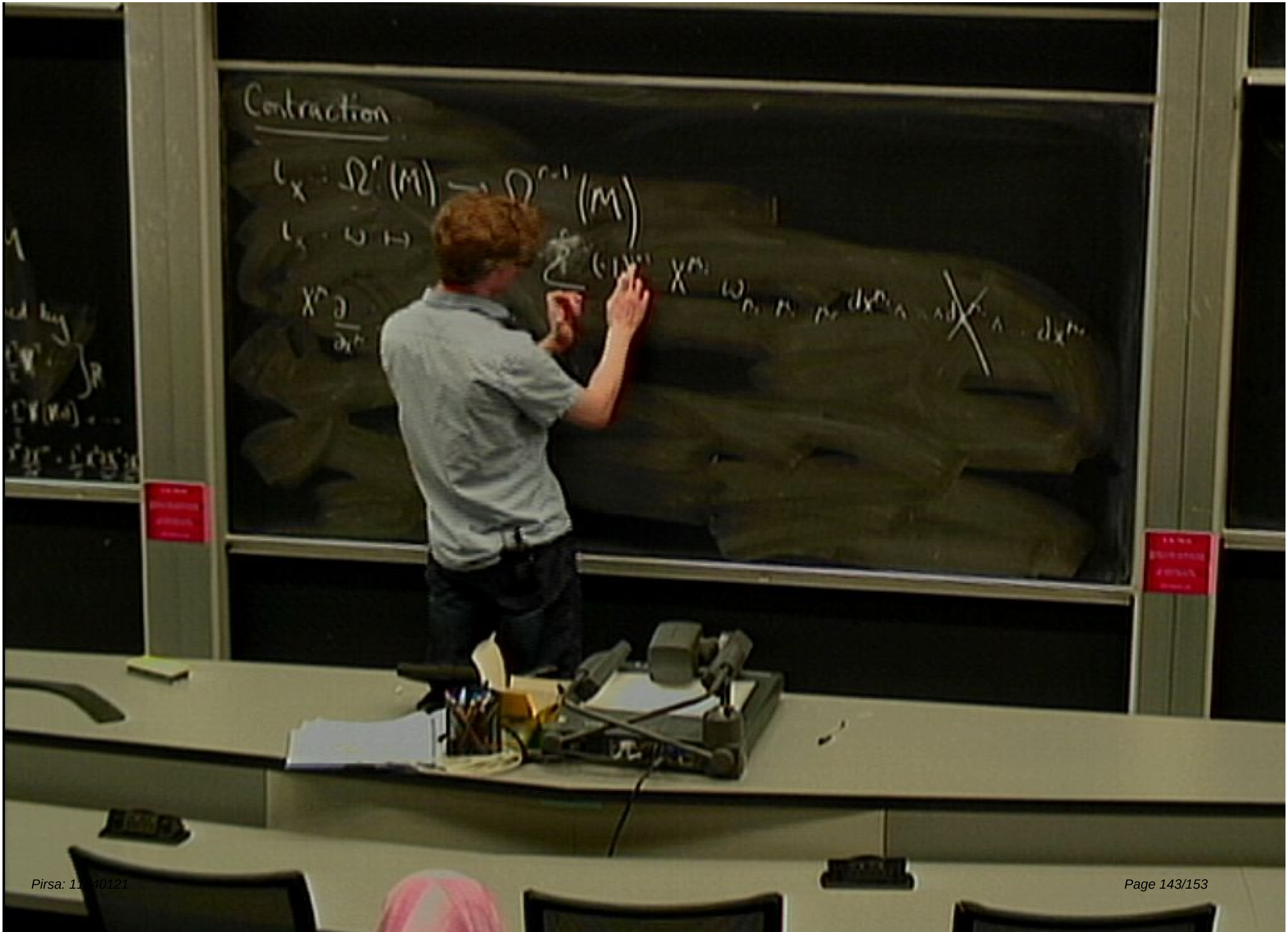


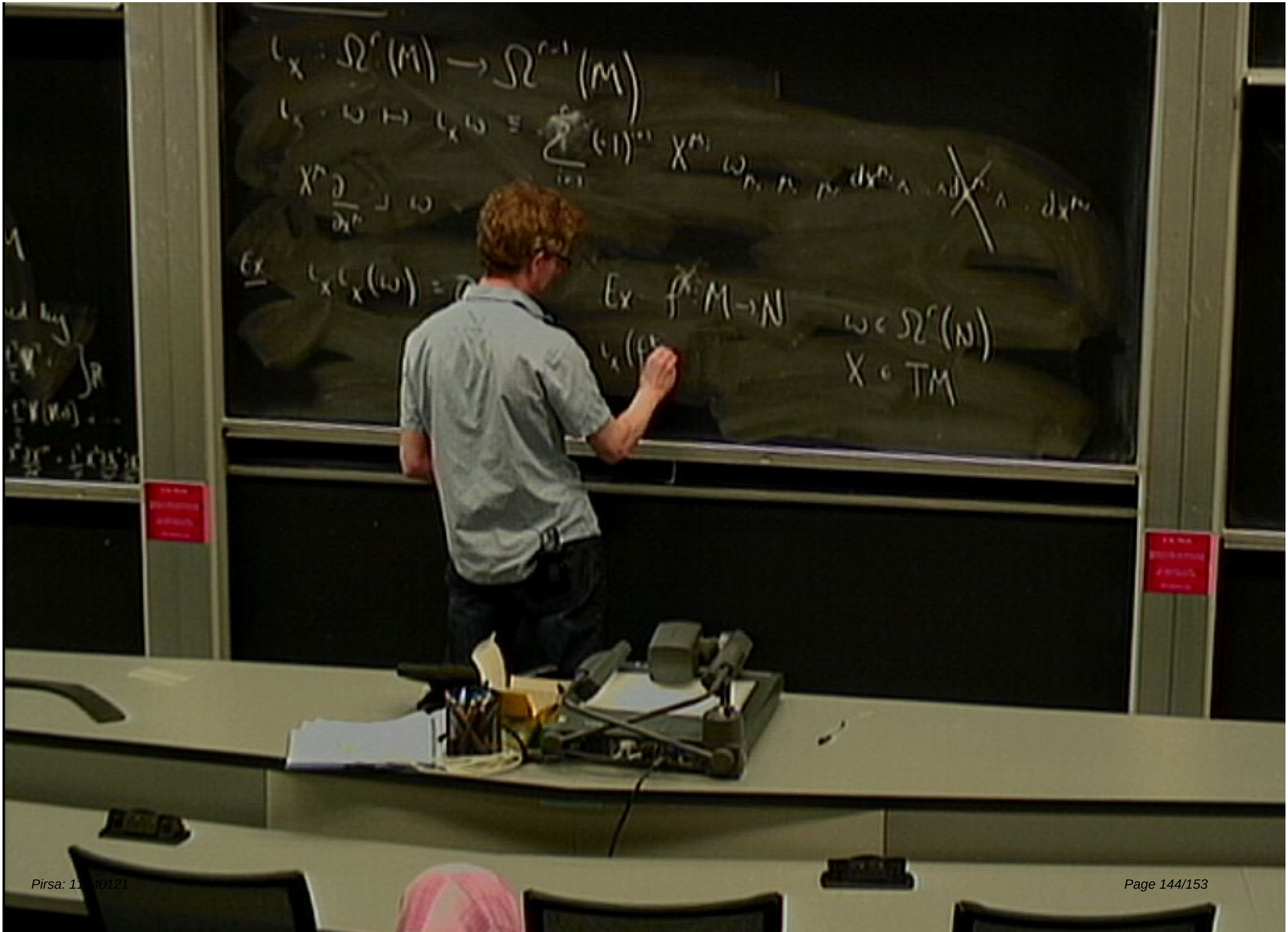
Contraction

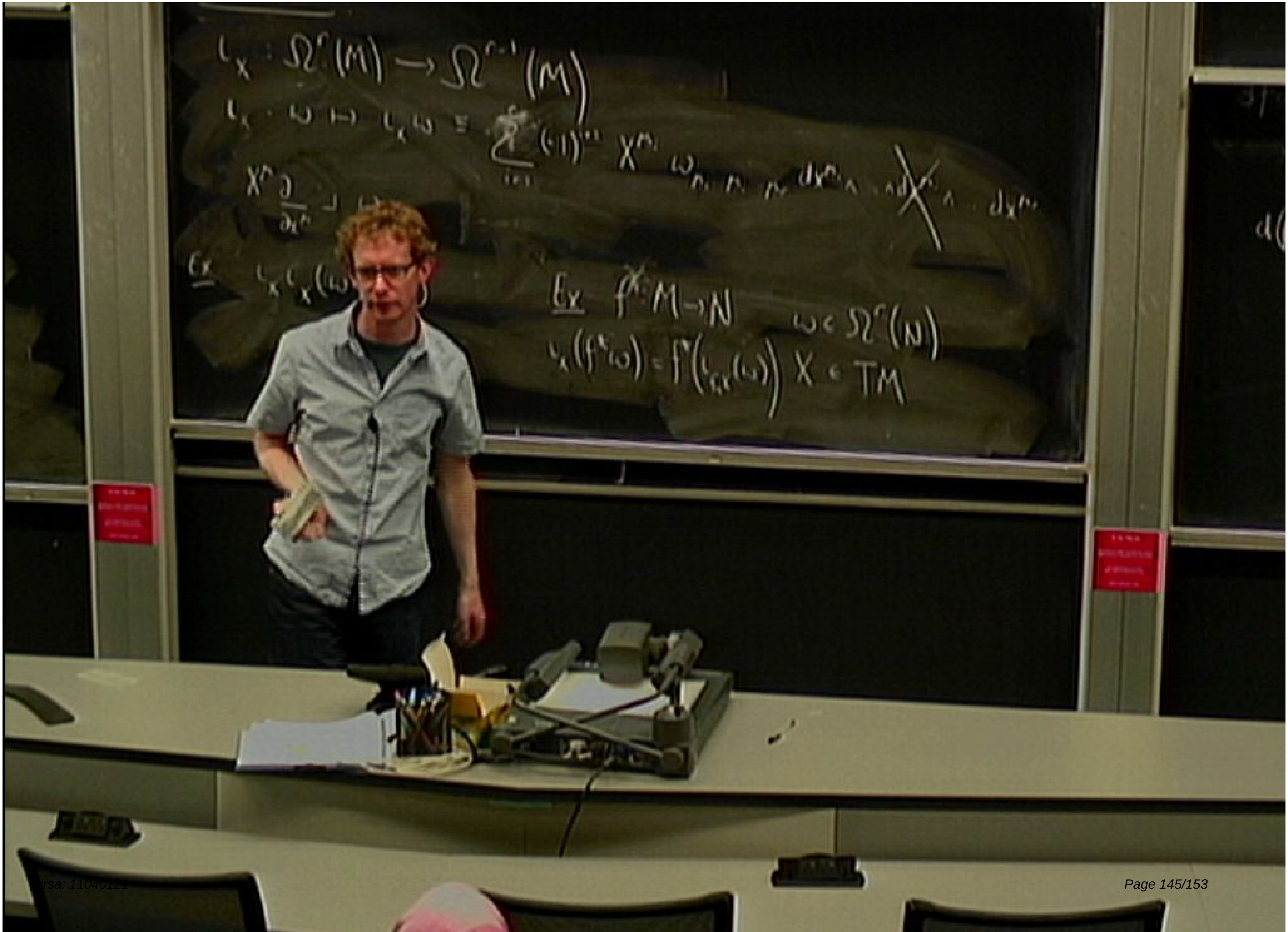
$\iota_X : \Omega^r(M) \to \Omega^{r-1}(M)$

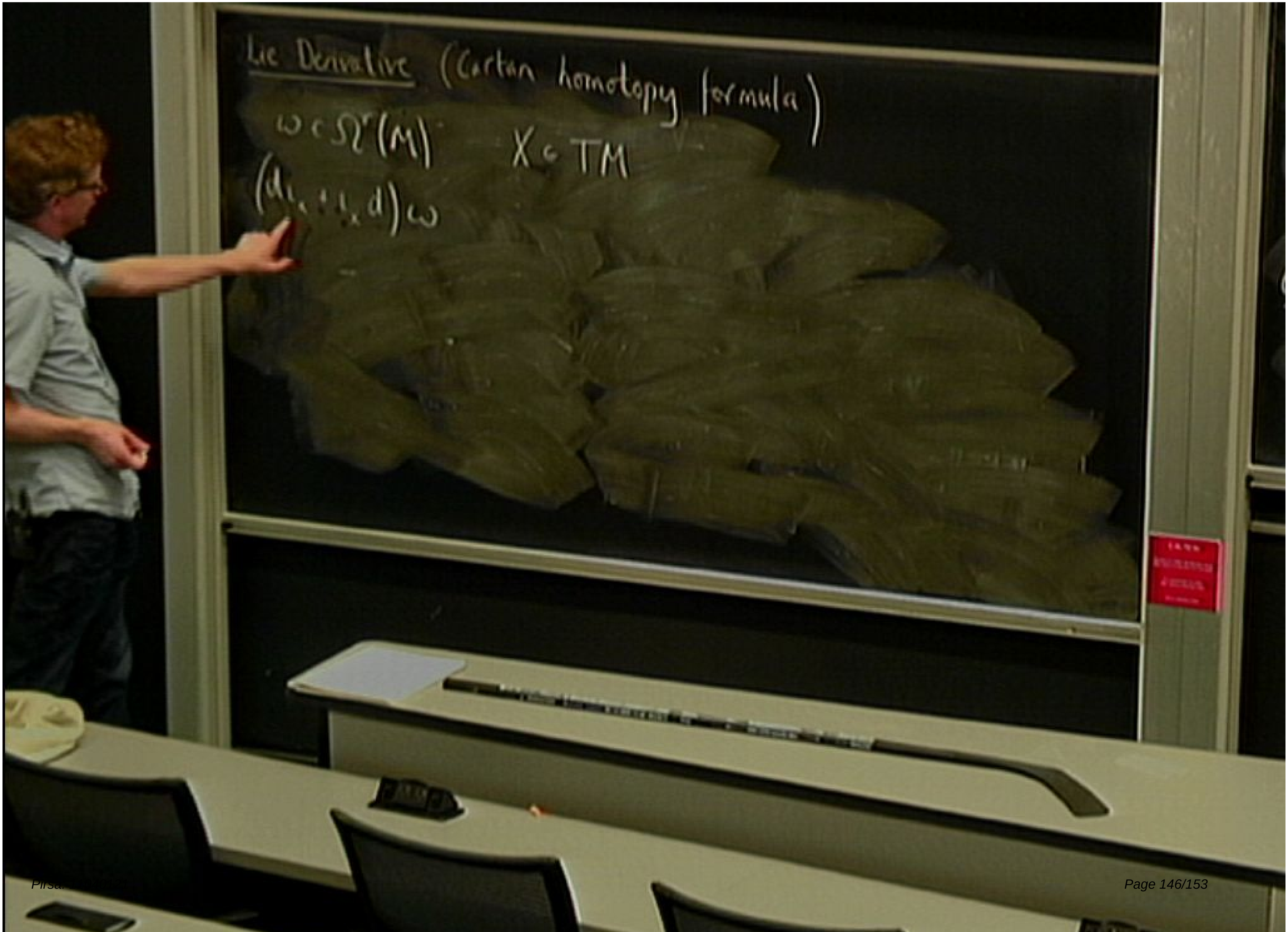
$\iota_X : \omega \mapsto \iota_X \omega \equiv \sum_{i=1}^{r} (-1)^{i-1} X^{\mu_i} \omega_{\mu_1 \mu_2 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge \cancel{dx^{\mu_i}} \wedge \dots \wedge dx^{\mu_r}$

$X^\mu \frac{\partial}{\partial}$


Contraction
ι_X : Ω^r(M) → Ω^{r-1}(M)
ι_X : ω ↦
(-1)^{s-1} X^{μ_s} ω_{μ_1 μ_2 ... μ_r} dx^{μ_1} ∧ ... ∧ dx^{μ_s} ∧ ... ∧ dx^{μ_r}
X^μ ∂/∂x^μ


ι_X : Ω^r(M) → Ω^{r-1}(M)
ι_X : ω ↦ ι_X ω = Σ_{i=1}^r (-1)^{i-1} X^{μ_i} ω_{μ_1 μ_2 … μ_r} dx^{μ_1} ∧ … ∧ dx^{μ_i} ∧ … ∧ dx^{μ_r}
X^μ ∂/∂x^μ ⌟ ω
Ex
ι_X ι_X(ω) = 0
Ex f : M → N
ω ∈ Ω^r(N)
X ∈ TM
ι_X((f

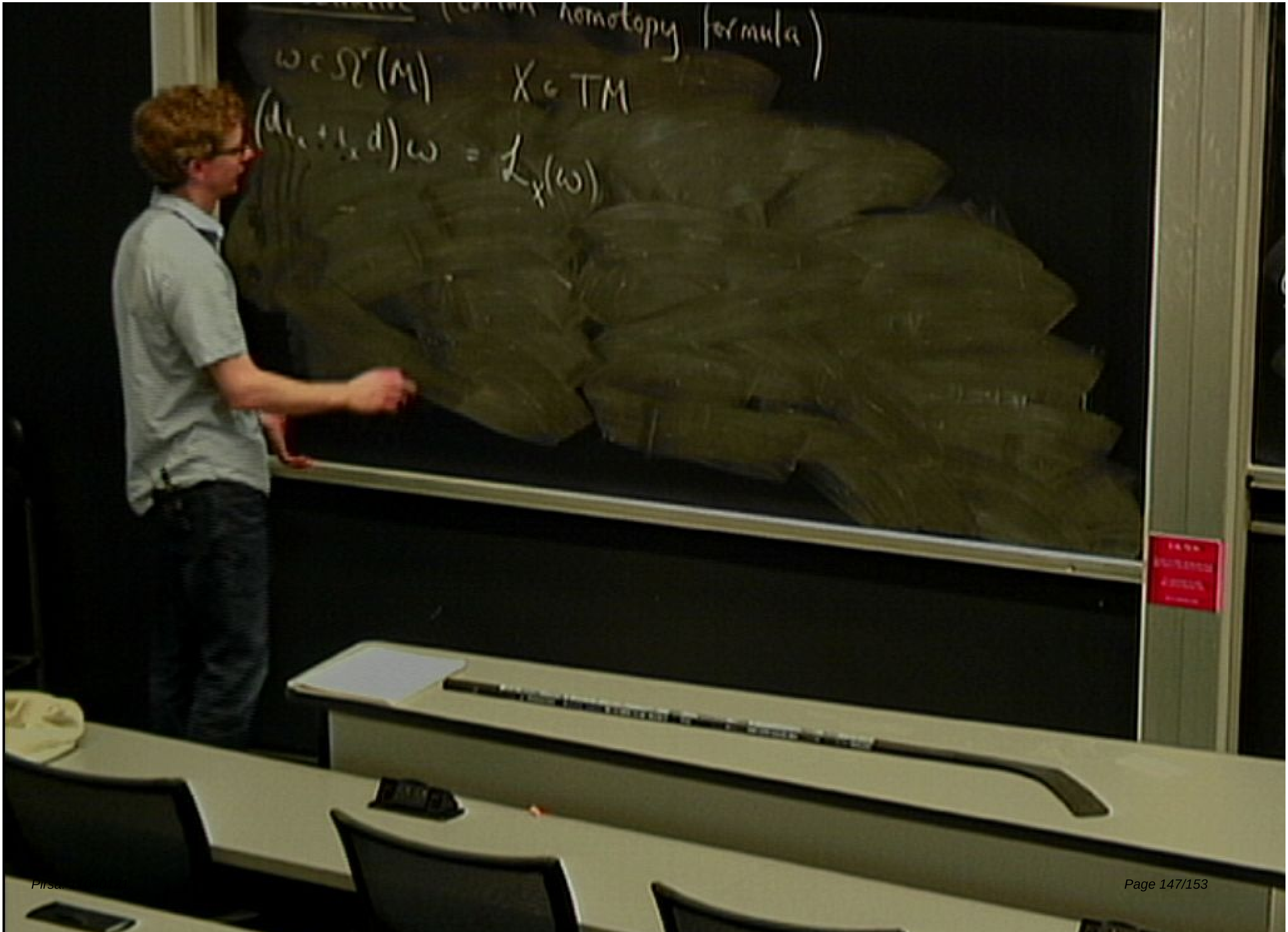

$\iota_X : \Omega^r(M) \to \Omega^{r-1}(M)$
$\iota_X : \omega \mapsto \iota_X \omega = \sum_{i=1}^{r} (-1)^{i-1} X^{\mu_i} \omega_{\mu_1 \mu_2 \dots \mu_r} dx^{\mu_1} \wedge \dots \wedge \cancel{dx^{\mu_i}} \wedge \dots dx^{\mu_r}$
$X^\mu \frac{\partial}{\partial x^\mu} \lrcorner \omega$
Ex $\iota_X \iota_X(\omega$
Ex $f: M \to N$ $\omega \in \Omega^r(N)$
$\iota_X(f^*\omega) = f^*(\iota_{f_*X}(\omega))$ $X \in TM$

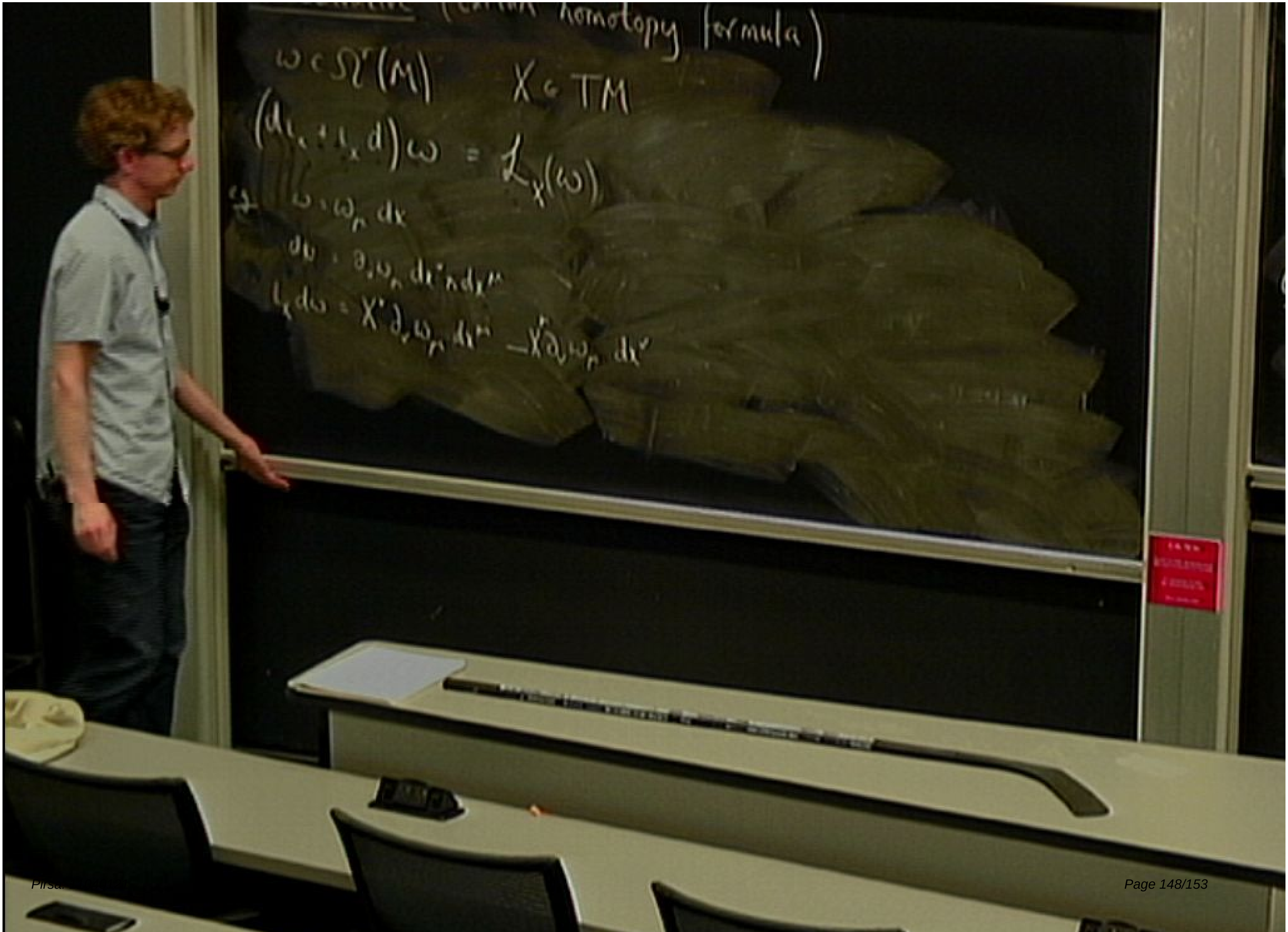


Lie Derivative (Cartan homotopy formula)

$\omega \in \Omega^r(M)$ $X \in TM$

$(d\iota_X + \iota_X d)\omega$


(Cartan homotopy formula)
$\omega \in \Omega^r(M)$ $X \in TM$
$(d\iota_X + \iota_X d)\omega = \mathcal{L}_X(\omega)$



(Cartan homotopy formula)

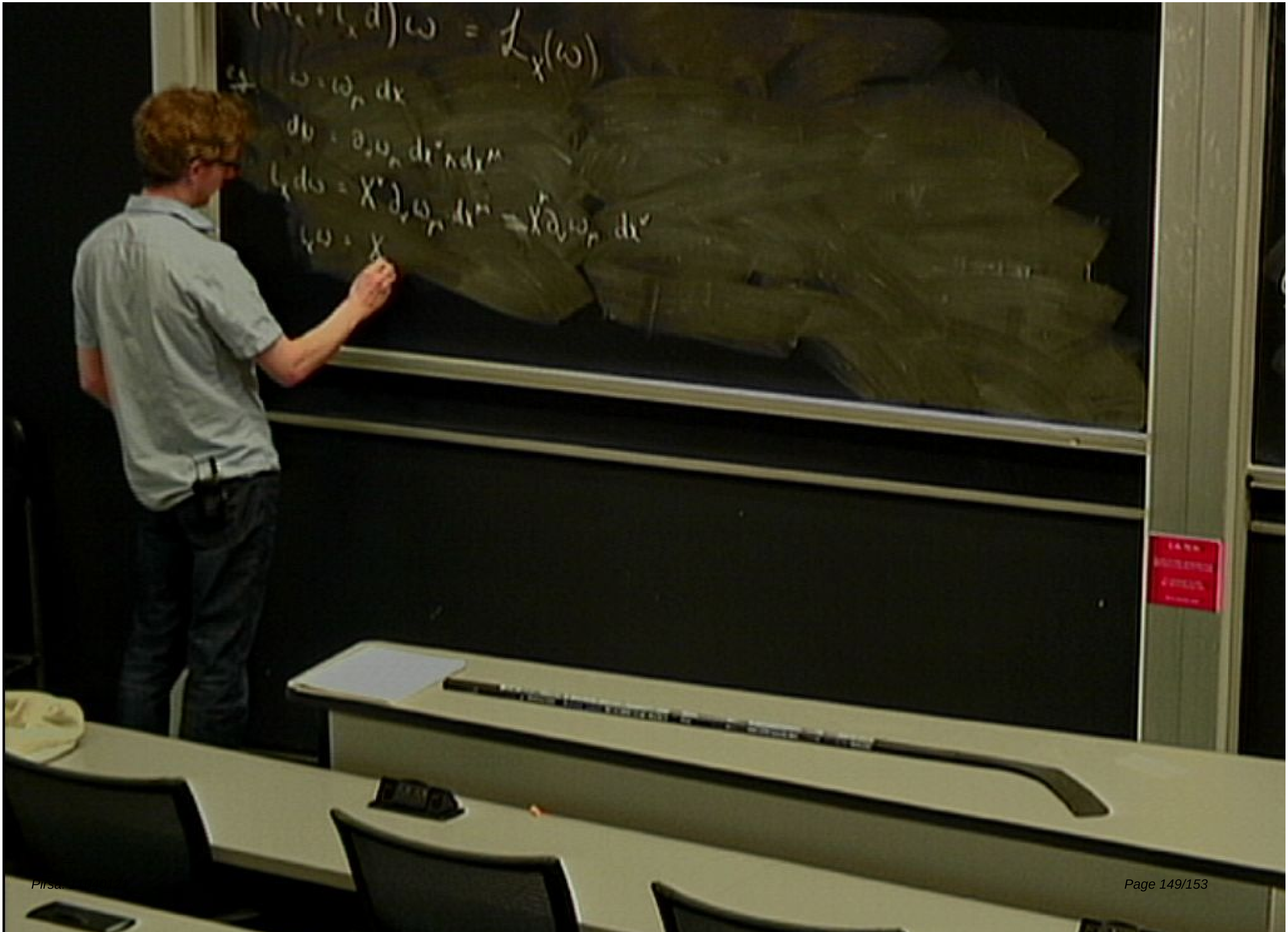
$\omega \in \Omega^r(M)$ $X \in TM$

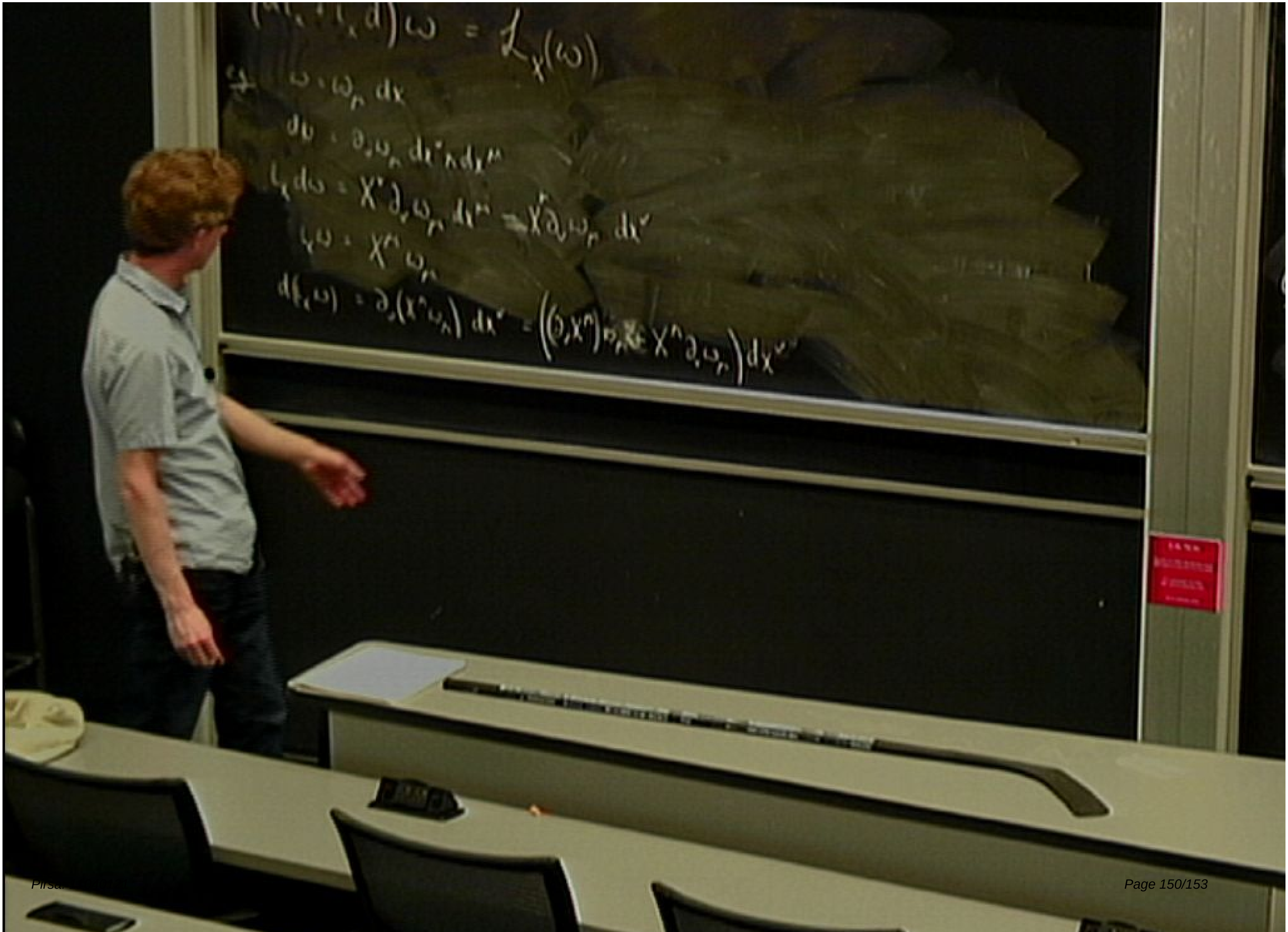
$(d\iota_X + \iota_X d)\omega = \mathcal{L}_X(\omega)$

eg. $\omega = \omega_\mu dx$

$d\omega = \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu$

$\iota_X d\omega = X^\nu \partial_\nu \omega_\mu dx^\mu - X^\mu \partial_\nu \omega_\mu dx^\nu$


$\iota_X d)\omega = \mathcal{L}_X(\omega)$
$\omega = \omega_\mu dx$
$d\omega = \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu$
$\iota_X d\omega = X^\nu \partial_\nu \omega_\mu dx^\mu - X^\mu \partial_\nu \omega_\mu dx^\nu$
$\iota_X \omega = X$



$$(d\iota_X + \iota_X d)\omega = \mathcal{L}_X(\omega)$$

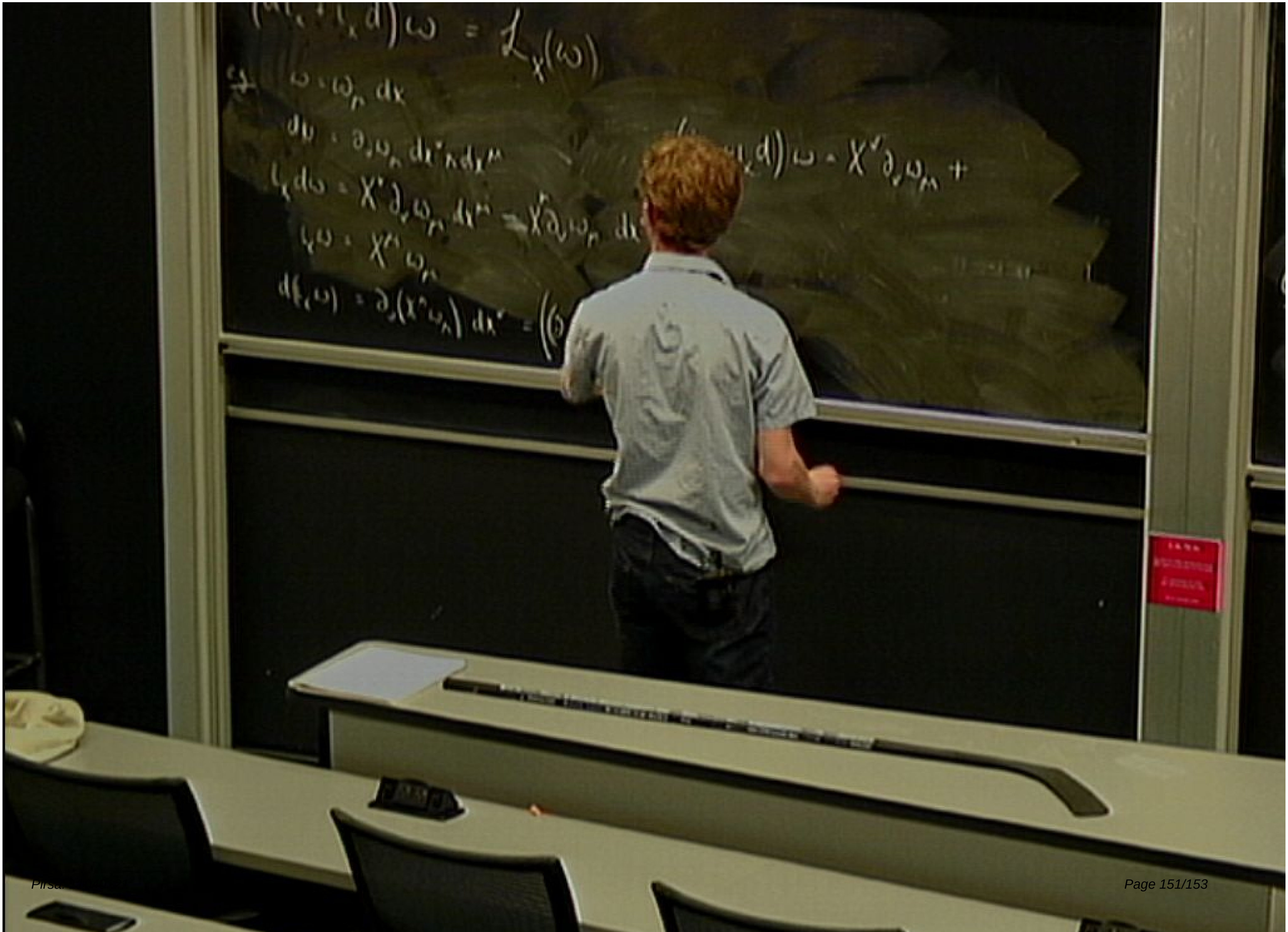
$$\omega = \omega_\mu dx^\mu$$

$$d\omega = \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu$$

$$\iota_X d\omega = X^\nu \partial_\nu \omega_\mu dx^\mu - X^\mu \partial_\nu \omega_\mu dx^\nu$$

$$\iota_X \omega = X^\mu \omega_\mu$$

$$d(\iota_X \omega) = \partial_\nu (X^\mu \omega_\mu) dx^\nu = \left((\partial_\nu X^\mu)\omega_\mu + X^\mu \partial_\nu \omega_\mu\right) dx^\nu$$



$(d\iota_X + \iota_X d)\omega = \mathcal{L}_X(\omega)$

$\omega = \omega_\mu dx^\mu$

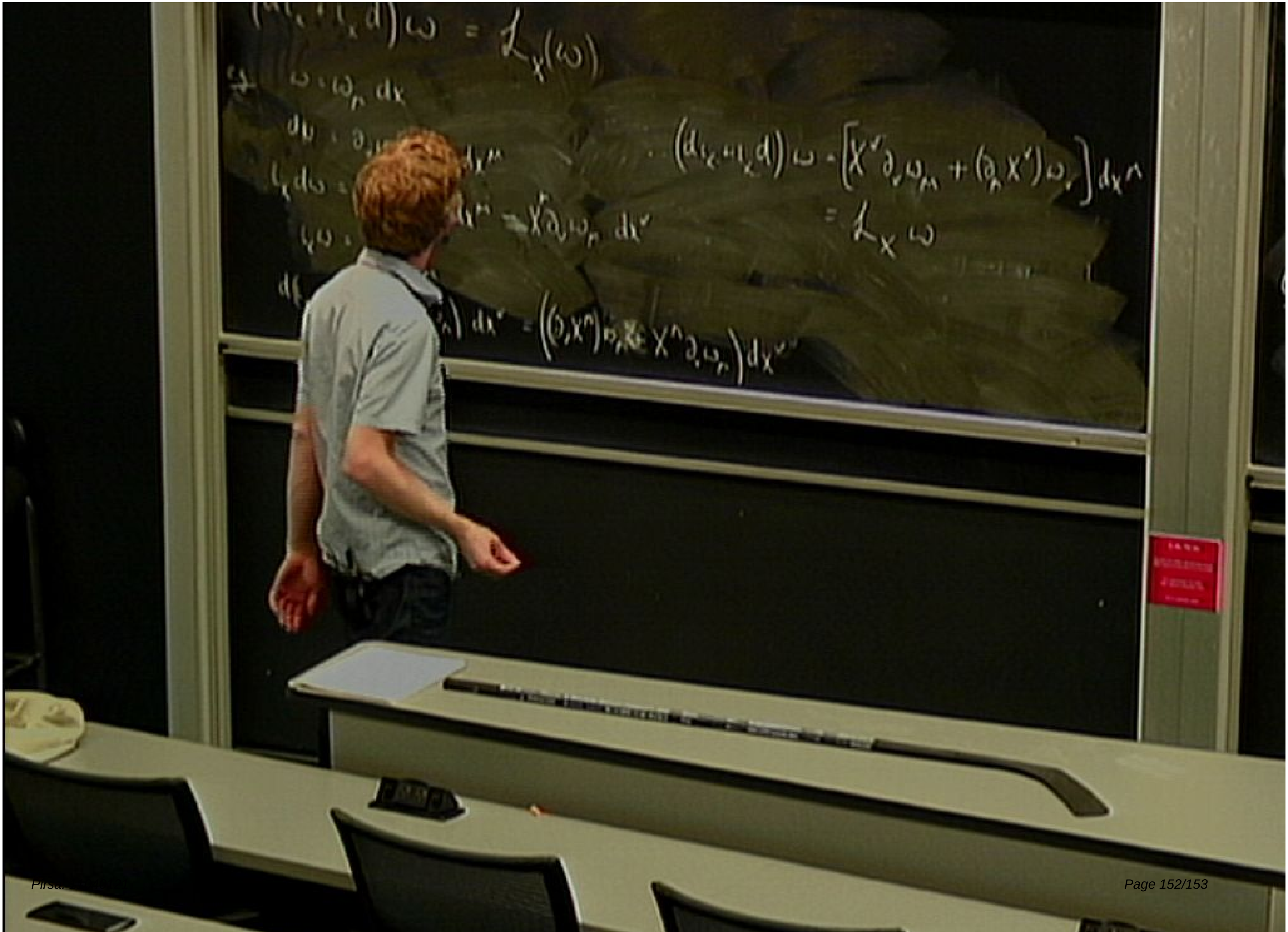
$d\omega = \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu$

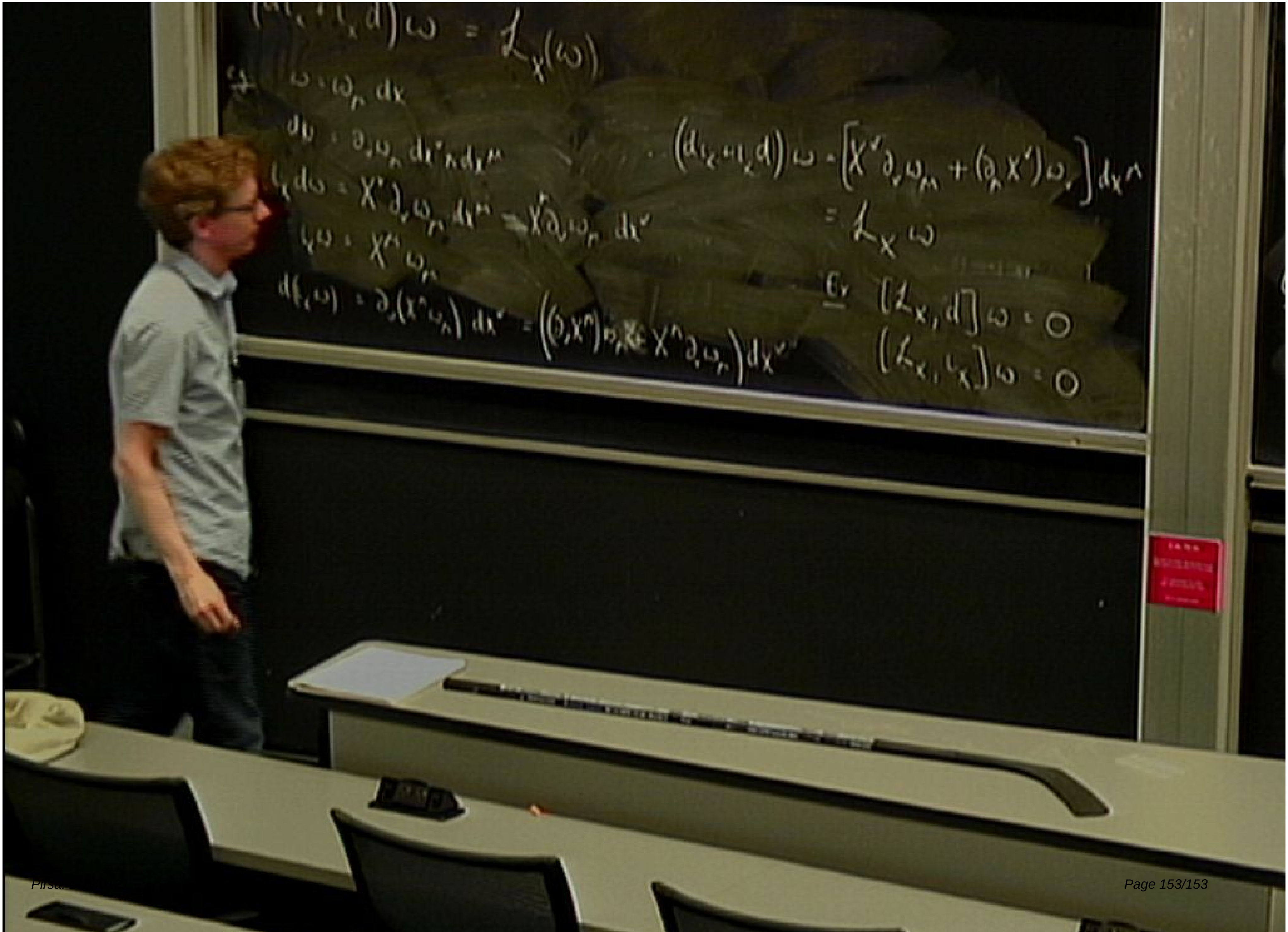
$\iota_X d\omega = X^\nu \partial_\nu \omega_\mu dx^\mu - X^\mu \partial_\nu \omega_\mu dx$

$\iota_X \omega = X^\mu \omega_\mu$

$d(\iota_X \omega) = \partial_\nu (X^\mu \omega_\mu) dx^\nu = ((\partial$

$(d\iota_X + \iota_X d)\omega = X^\nu \partial_\nu \omega_\mu +$


$\omega = \omega_\mu dx$
$(d\iota_X + \iota_X d)\omega = \mathcal{L}_X(\omega)$
$\iota_X d\omega$
$X^\nu \partial_\nu \omega_\mu dx^\nu$
$(d\iota_X + \iota_X d)\omega = [X^\nu \partial_\nu \omega_\mu + (\partial_\mu X^\nu)\omega_\nu] dx^\mu$
$= \mathcal{L}_X \omega$
$= ((\partial_\nu X^\mu)\omega_\mu + X^\mu \partial_\nu \omega_\mu) dx^\nu$



$(d\iota_X + \iota_X d)\omega = \mathcal{L}_X(\omega)$

Ex. $\omega = \omega_\mu dx^\mu$

$d\omega = \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu$

$\iota_X d\omega = X^\nu \partial_\nu \omega_\mu dx^\mu - X^\mu \partial_\nu \omega_\mu dx^\nu$

$\iota_X \omega = X^\mu \omega_\mu$

$d(\iota_X \omega) = \partial_\nu (X^\mu \omega_\mu) dx^\nu = ((\partial_\nu X^\mu)\omega_\mu + X^\mu \partial_\nu \omega_\mu) dx^\nu$

$(d\iota_X + \iota_X d)\omega = [X^\nu \partial_\nu \omega_\mu + (\partial_\mu X^\nu)\omega_\nu] dx^\mu$

$= \mathcal{L}_X \omega$

Ex. $[\mathcal{L}_X, d]\omega = 0$

$[\mathcal{L}_X, \iota_X]\omega = 0$