

Title: Geometry & Topology for Physics - Lecture 2

Date: Apr 04, 2011 02:00 PM

URL: <http://pirsa.org/11040121>

Abstract:

Next lecture (by me!) : Mon 23rd May



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Vectors

At $p \in M$, let $T_p M$ is space of linearly independent vectors at p

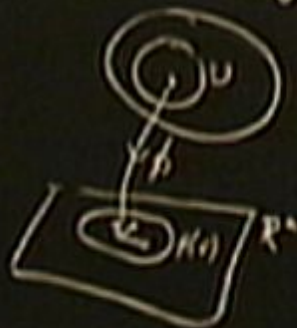
Can pick 1

Next Lecture (by me!) : Mon 23rd May

Vectors

At $p \in M$, let $T_p M$ is space of linearly independent vectors at p

pick basis: $\left\{ \frac{\partial}{\partial x^a} \right\}$



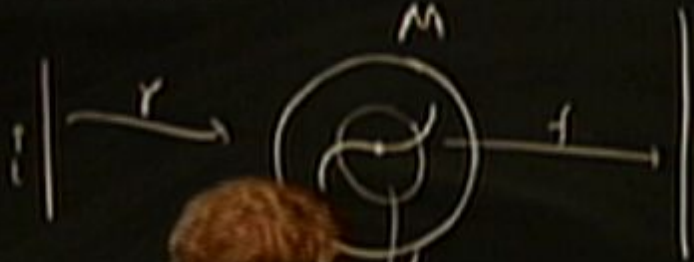
$$\gamma: [0,1] \rightarrow M$$

$$f: M \rightarrow \mathbb{R}$$



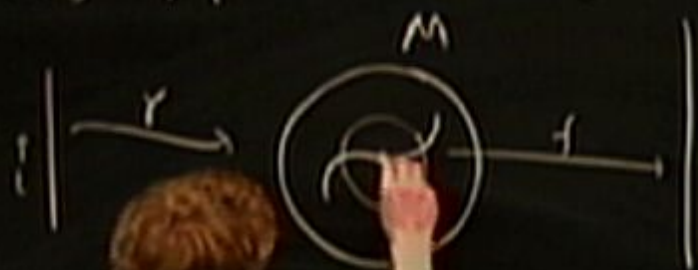
$$\gamma: [0,1] \rightarrow M$$

$$f: M \rightarrow \mathbb{R}$$



$$\gamma: [0,1] \rightarrow M$$

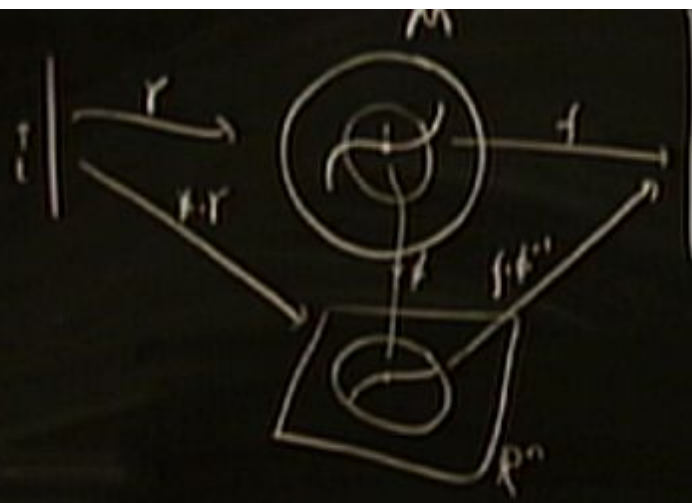
$$f: M \rightarrow \mathbb{R}$$



$$\gamma: [0,1] \rightarrow M$$

$$f: M \rightarrow \mathbb{R}$$





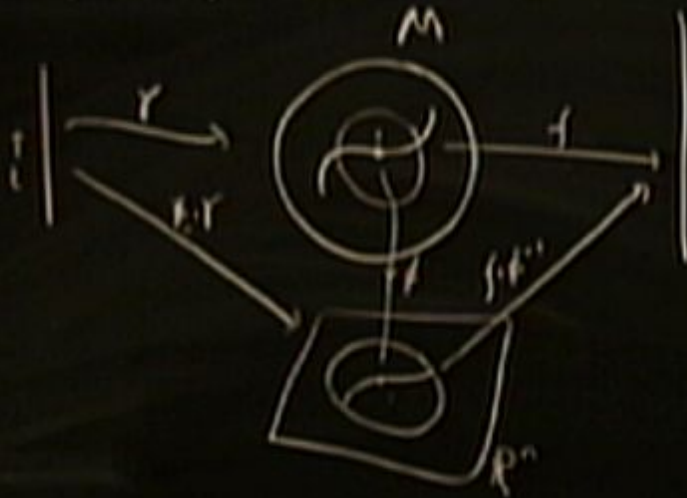
$$\frac{df(t(t))}{dt} = X^m \frac{\partial f}{\partial x^m}$$

where $X^m = \frac{\partial x^m}{\partial t}$



$$\gamma: [0,1] \rightarrow M$$

$$f: M \rightarrow \mathbb{R}$$

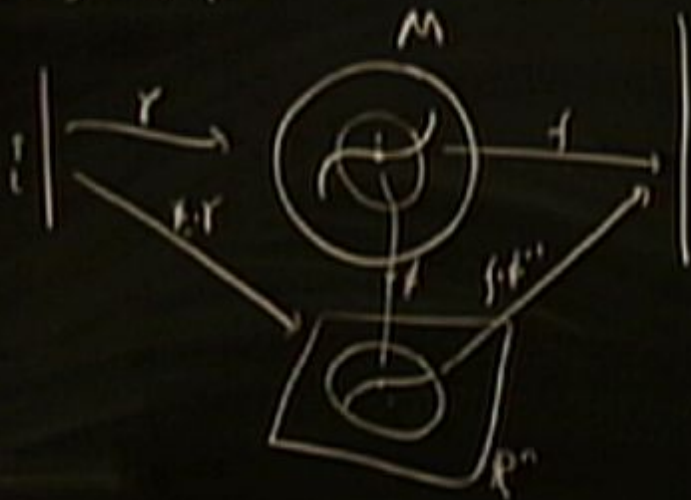


$$\frac{df(\gamma(t))}{dt}, X^n$$

where X^n

$$\gamma: [0,1] \rightarrow M$$

$$f: M \rightarrow \mathbb{R}$$

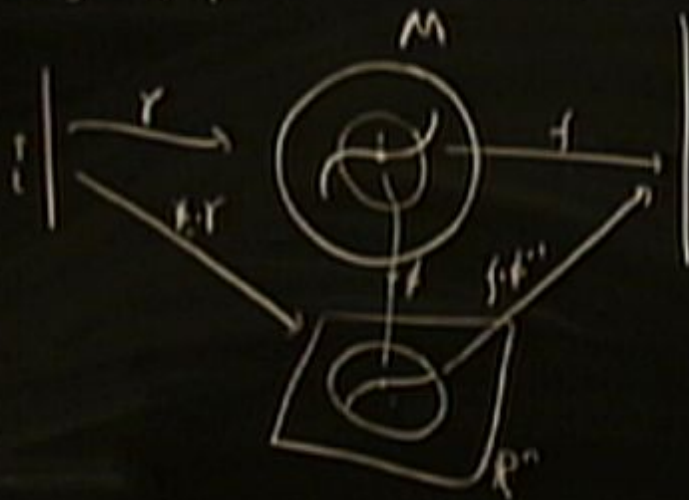


$$\frac{df(\gamma(t))}{dt} = X^{\gamma'} \frac{\partial f}{\partial x^r} = X[f]$$

$$\text{where } X^{\gamma'} = \frac{\partial x^r}{\partial t} \quad X = X^r \frac{\partial}{\partial x^r}$$

$$\gamma: [0,1] \rightarrow M$$

$$f: M \rightarrow \mathbb{R}$$

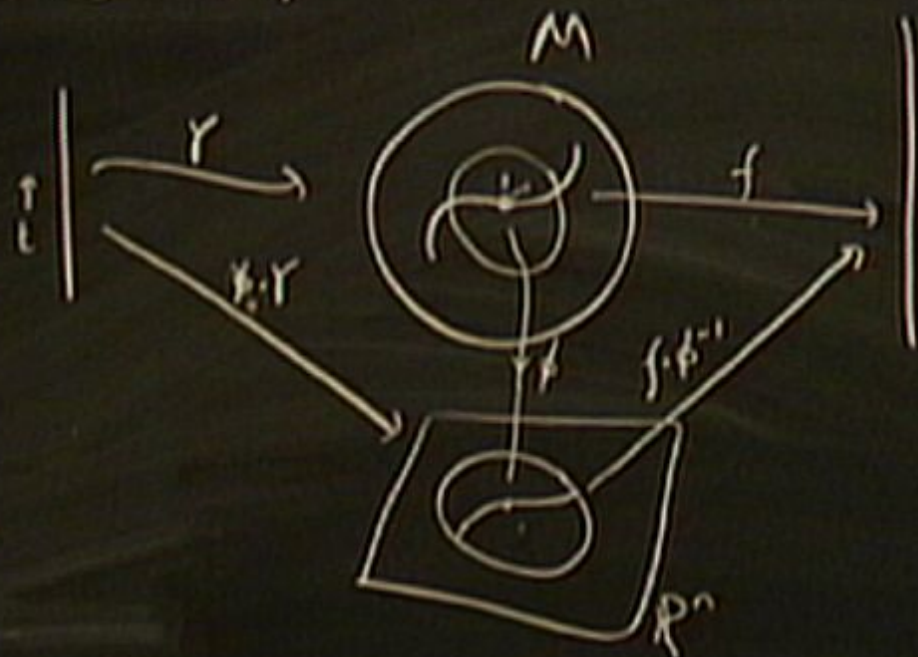


$$\frac{df(\gamma(t))}{dt} = X^{\gamma(t)} \frac{\partial f}{\partial x^r} = X[f]$$

$$\text{where } X^{\gamma(t)} = \frac{\partial \gamma^r}{\partial t} \quad X = X^r \frac{\partial}{\partial x^r}$$

$$\gamma: [0,1] \rightarrow M$$

$$f: M \rightarrow \mathbb{R}$$



$$\frac{df(\gamma(t))}{dt} = X^M \frac{\partial f}{\partial x^r} = X[f]$$

$$\text{where } X^M = \frac{\partial x^r}{\partial t} \quad X = X^r \frac{\partial}{\partial x^r}$$

One - Fem

One-Forms

A 1-form ω on M

$$\omega : T_x M \rightarrow \mathbb{R}$$

One-Forms

A 1-form ω is a map $\omega : T^*M \rightarrow \mathbb{R}$
 $\omega_p \in T^*_p M$



One-Forms C^1 -manif.

A 1-form ω is a map $\omega: T_1M \rightarrow \mathbb{R}$
 $\downarrow p_1^*$

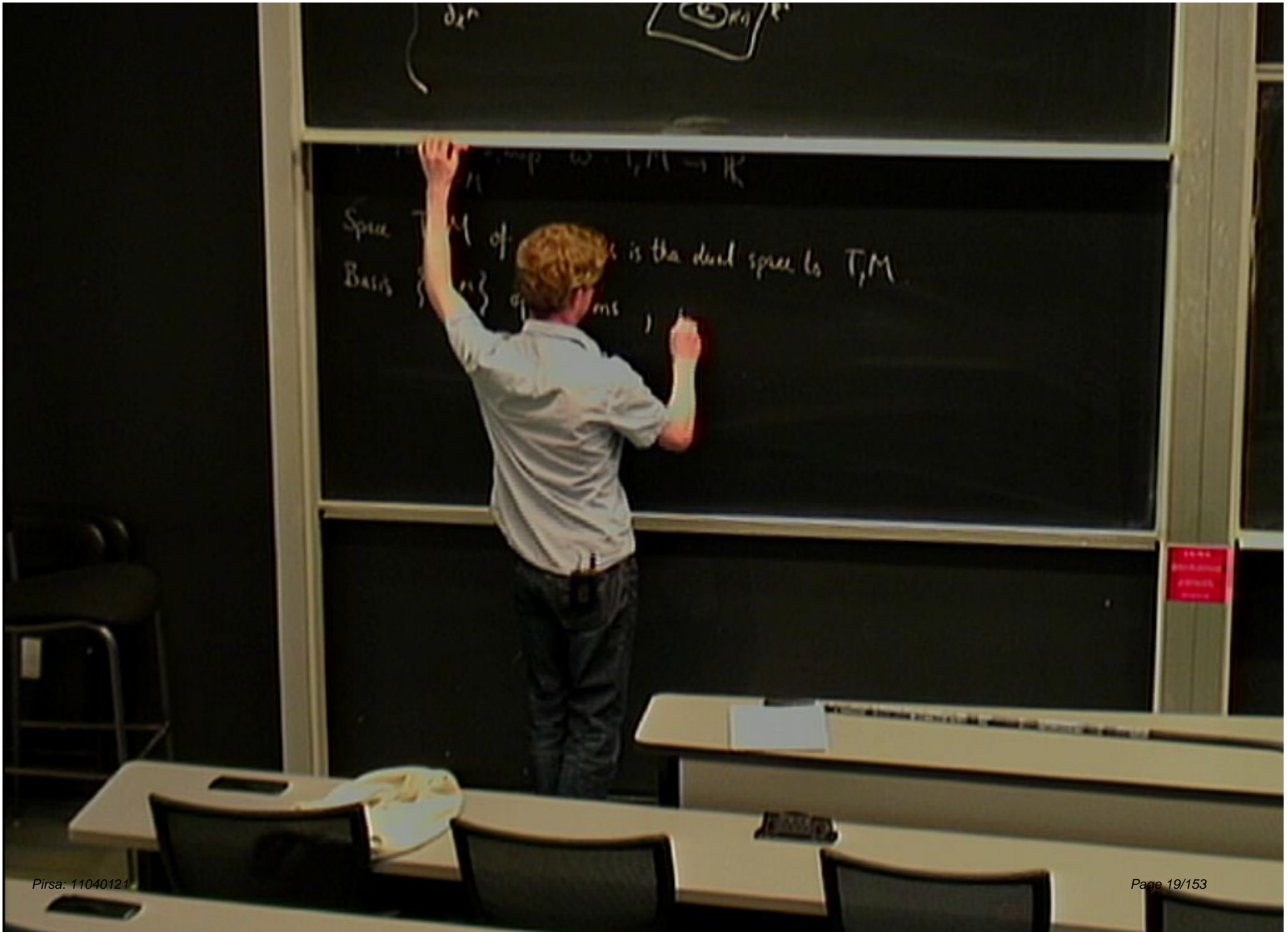
Space T_1^*M of one-forms is the dual space to T_1M

One-Forms \mathbb{C}^{∞} -linear

A 1-form ω is a map $\omega: T_1M \rightarrow \mathbb{R}$
 \downarrow
 π^*M

Space T_1^*M of one-forms is the dual space to T_1M .





Next Lecture (by me!) Mon 25th May

Vectors

At $p \in M$, let $T_p M$ is space of linearly independent vectors at p

Can pick basis: $\left\{ \frac{\partial}{\partial x^i} \right\}$

$$V = V^i \frac{\partial}{\partial x^i}$$



Space $T_p^* M$ of one-forms is the dual space to $T_p M$.

Basis $\{dx^i\}$ of 1-forms, $\omega = \omega_i dx^i$

map $\omega: T_p M \rightarrow \mathbb{R}$

Space $T_p^* M$ of one-forms is the dual space to $T_p M$.

Base $\{dx^a\}$ of 1-forms, $\omega = \omega_a dx^a$

$$= \langle \omega_a dx^a, V^b \frac{\partial}{\partial x^b} \rangle = \omega_a V^b \langle dx^a, \frac{\partial}{\partial x^b} \rangle = \omega_a V^b \delta^a_b = \omega_a V^a$$



map $\omega: T_p M \rightarrow \mathbb{R}$

Space $T_p^* M$ of one-forms is the dual space to $T_p M$.

Basis $\{dx^m\}$ of 1-forms, $\omega = \omega_n dx^n$

$$\langle \omega, V \rangle = \langle \omega_n dx^n, V^v \frac{\partial}{\partial x^v} \rangle = \omega_n V^v \langle dx^n, \frac{\partial}{\partial x^v} \rangle = \omega_n V^v \delta^{nv} = \omega_n V^n$$

identity
↓



Space $T^*_p M$ of one-forms is the dual space to $T_p M$.

Basis $\{dx^m\}$ of 1-forms $= \omega_\mu dx^\mu$

$$\langle \omega, V \rangle = \langle \omega_\mu dx^\mu, V^\nu \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu$$

identity
↓

Smooth Maps

$$f: M \rightarrow N$$

x^n

space of 1-forms is the dual space to $T_p M$

Basis $\{dx^m\}$ of 1-forms, $\omega = \omega_n dx^n$

$$\langle \omega, V \rangle = \langle \omega_n dx^n, V^v \frac{\partial}{\partial x^v} \rangle = \omega_n V^v \langle dx^n, \frac{\partial}{\partial x^v} \rangle = \omega_n V^v \delta^n_v = \omega_n V^n$$

identity



Smooth Maps

$$f: M \rightarrow N$$

\downarrow \downarrow
 \mathbb{R}^m \mathbb{R}^n

is a smooth map
if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

Smooth Maps

$$f: M \rightarrow N$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^m \end{array}$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^n \end{array}$$

is a smooth map
if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

Smooth Maps

$$\begin{array}{ccc} f: M & \rightarrow & N \\ \cup & & \cup \\ U & & V \\ \downarrow & & \downarrow \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

is a smooth map
if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth.

IF f is also invertible

Smooth Maps

$$f: M \rightarrow N$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^n \end{array}$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^m \end{array}$$

is a smooth map

if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

If f is also invertible, then ψ is a diffeomorphism
 $f: M \rightarrow M$

Smooth Maps

$$f: M \rightarrow N$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^m \end{array}$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^n \end{array}$$

is a smooth map

if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

(If f is also invertible, then ψ is a diffeomorphism
 $f: M \rightarrow M$)

Any such f induces maps on tangent vectors + 1-forms:

$$f_*$$

Smooth Maps

$$f: M \rightarrow N$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^m \end{array}$$

$$\begin{array}{c} \cup \\ \downarrow \\ \mathbb{R}^n \end{array}$$

is a smooth map

if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

(If f is also invertible, then ψ is a diffeomorphism)
 $f: M \rightarrow M$

such f induces maps on tangent vectors + 1-forms:

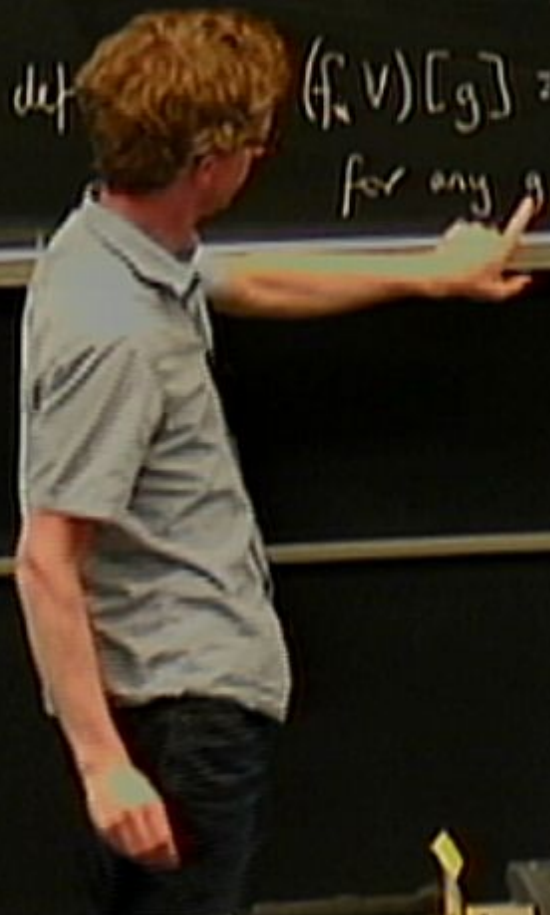
$$T_x M \rightarrow T_x N$$

$\mathbb{R}^n \quad \mathbb{R}^n$ (If f is also invertible, then V is a diffeomorphism)
 $f: M \rightarrow M$

Any such f induces maps on tangent vectors + 1-forms:

$$f_*: T_p M \rightarrow T_p N \quad \text{def } (f_* V)[g] = V[g \circ f]$$

for any $g: N \rightarrow \mathbb{R}$



$\mathbb{R}^n \rightarrow \mathbb{R}^n$ (If f is also invertible, then V is a diffeomorphism)
 $f: M \rightarrow M$

Any such f induces maps on tangent vectors + 1-forms:

$$f_*: T_p M \rightarrow T_p N \quad \text{as} \quad (f_* V)[g] = V[g \circ f]$$

for any $g: N \rightarrow \mathbb{R}$



$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$ (If f is also invertible, then V is a diffeomorphism)
 $f: M \rightarrow M$

Any such f induces maps on tangent vectors + 1-forms: \cdot - pushforward M

$$f_*: T_p M \rightarrow T_p N \quad \text{defined as} \quad (f_* V)[g] = V[g \circ f]$$

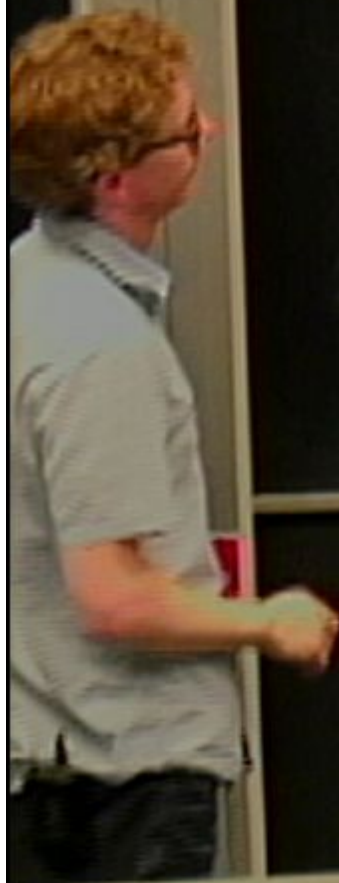
for any $g: N \rightarrow \mathbb{R}$; $V \in T_p M$



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... defined as $(\mathbb{R}^V) \circ g = V \circ f$
"push-forward" for any $g: N \rightarrow \mathbb{R}^V$; $V \in T, M$

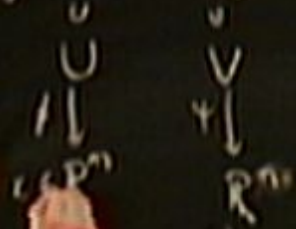
$$V(g \cdot f)$$



Smooth Maps

$$f: M \rightarrow N$$

is a smooth map



if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

(If f is also invertible, then f is a diffeomorphism $f: M \rightarrow M$)

Each f induces maps on tangent vectors + 1-forms:

$$f_*: T_p M \rightarrow T_p N$$

defined as $(f_* V)[g] = V[g \circ f]$

"push-forward"

for any $g: N \rightarrow \mathbb{R}; V \in T_p M$

Smooth Maps

$f: M \rightarrow N$ is a smooth map
if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

$\begin{matrix} \cup & \cup \\ \downarrow & \downarrow \\ \mathbb{R}^m & \mathbb{R}^n \end{matrix}$ (If f is also invertible, then f is a diffeomorphism)
 $f: M \rightarrow M$

Any such f induces maps on tangent vectors + 1-forms:

$f_*: T_p M \rightarrow T_p N$ defined as $(f_* V)[g] = V[g \circ f]$
"push-forward" for any $g: N \rightarrow \mathbb{R}; V \in T_p M$

$$f: M \rightarrow N$$

is a smooth map



if the map $\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is smooth.

(If f is also invertible, then f is a diffeomorphism)
 $f: M \rightarrow M$

Any such f induces maps on tangent vectors + 1-forms:

$$f_*: T_x M \rightarrow T_x N \quad \text{defined as} \quad (f_* V)[g] = V[g \circ f]$$

for any $g: N \rightarrow \mathbb{R}; V \in T_x M$

"push-forward"

for any $g: N \rightarrow R$; $V \in T, M$

$$V(g \circ f)' = V^{\#} \frac{\partial}{\partial x^{\mu}}$$



"push-forward"

for any $g: N \rightarrow \mathbb{R}$; $V \in T_x M$

$$V(g \circ f)' = V^* \frac{\partial g}{\partial x^i}(f(u))$$



"push-forward"

for any $g: N \rightarrow \mathbb{R}$; $V \in T_x M$

$$V [g \circ f]' = V^* \frac{\partial g(f(x))}{\partial x^i} = V^* \frac{\partial y^j}{\partial x^i} \frac{\partial g}{\partial y^j}$$



"push-forward"

for any $g: N \rightarrow \mathbb{R}$; $V \in T_x M$

$$V(g \circ f)' = V^* \frac{\partial g(f(x))}{\partial x^i} = \left(V^* \frac{\partial y^k}{\partial x^i} \right) \frac{\partial g}{\partial y^k} = \left(\frac{f_* V}{f_*} \right)^k \frac{\partial g}{\partial y^k}$$



"push-forward"

for any $g: N \rightarrow \mathbb{R}$; $V \in T_x M$

$$V[g \circ f] = V^* \frac{\partial g(f(x))}{\partial x^i} = \left(V^* \frac{\partial y^j}{\partial x^i} \right) \frac{\partial g}{\partial y^j} = \left(\frac{f_* V}{f_*} \right)^* \frac{\partial g}{\partial y^j}$$

$$(f_* V)^* \cdot V^* \frac{\partial f}{\partial x^i}$$

$f_* : T_p M \rightarrow T_p N$ defined as $(f_* V)[g] = V[g \circ f]$
 "push-forward" for any $g : N \rightarrow \mathbb{R}$; $V \in T_p M$

$$\text{ie. } (f_* V)^{\alpha} = V^{\mu} \frac{\partial x^{\alpha}}{\partial y^{\mu}}$$

$f_* : T_p M \rightarrow T_p N$ defined as $(f_* V)[g] = V[g \circ f]$
 "push-forward" for any $g : N \rightarrow \mathbb{R} ; V \in T_p M$

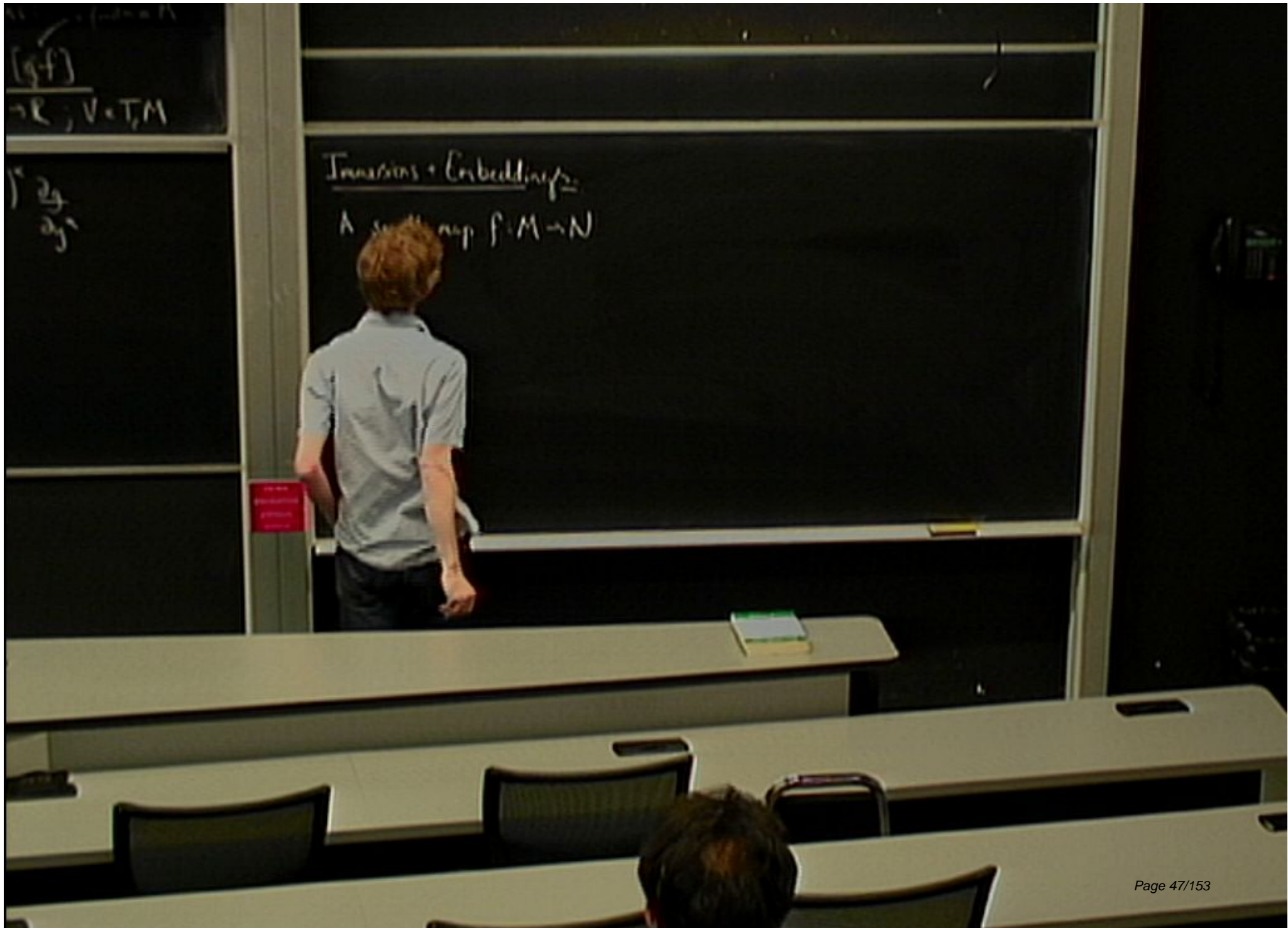
$\frac{\partial}{\partial x^i}$ $\frac{\partial x^i}{\partial y^j}$ $\frac{\partial}{\partial y^j}$
 ie. $(f_* V)^j = V^i \frac{\partial x^i}{\partial y^j}$

$J_x: T_p M \rightarrow T_p N$ defined as $(f_* V)(g) = V(g \circ f)$
 "push-forward" for any $g: N \rightarrow \mathbb{R}$; $V \in T_p M$

$$V(g \circ f) = V^a \frac{\partial (g \circ f)}{\partial x^a} = \left(V^a \frac{\partial y^b}{\partial x^a} \right) \frac{\partial g}{\partial y^b} = (f_* V)^b \frac{\partial g}{\partial y^b}$$

ie. $(f_* V)^a = V^b \frac{\partial y^a}{\partial x^b}$





Immersions + Embeddings
A smooth map $f: M \rightarrow N$

(g, f)
 $\rightarrow R; V \subset T, M$

$\int \frac{dy}{y^2}$

Immersion + Embeddings:

A smooth map $f: M \rightarrow N$ is an

– immersion iff $f_*: T_p M \rightarrow T_p N$ is (injective)

Immersion & Embeddings:

A smooth map $f: M \rightarrow N$ is an

- immersion iff $f_*: T_p M \rightarrow T_p N$ is one-to-one (injective)

- embedding if it's an immersion and f itself is injective.

Immersion & Embeddings:

A smooth map $f: M \rightarrow N$ is an

- immersion iff $f_*: T_p M \rightarrow T_p N$ is one-to-one (injective)

- embedding if it's an immersion and f itself is injective.

eg $f: S^1 \rightarrow \mathbb{R}^2$ i) $f(S^1) = \{pt\}$ $f: \mathbb{O} \rightarrow \boxed{\cdot}$

Immersion & Embeddings:


A smooth map $f: M \rightarrow N$ is an

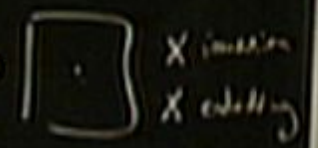
- immersion iff $f_*: T_p M \rightarrow T_p N$ is one-to-one (injective)

- embedding if it's an immersion and f itself is injective.

eg $f: S^1 \rightarrow \mathbb{R}^n$

i) $f(S^1) = \{pt\}$

ii) $f: 0 \mapsto$ 



Immersion & Embeddings:


A smooth map $f: M \rightarrow N$ is an

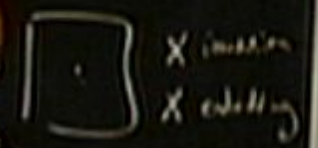
- immersion iff $f_*: T_p M \rightarrow T_p N$ is one-to-one (injective)

- embedding if it's an immersion and f itself is injective

eg $f: S^1 \rightarrow \mathbb{R}^2$

i) $f(S^1) = \{pt\}$

ii) $f: S^1 \rightarrow \mathbb{R}^2$ 



Immersion & Embeddings:

A smooth map $f: M \rightarrow N$ is an

- immersion iff $f_*: T_p M \rightarrow T_p N$ is one-to-one (injective)

- embedding if it's an immersion and f itself is injective.

eg $f: S^1 \rightarrow \mathbb{R}^n$

i) $f(S^1) = \{pt\}$

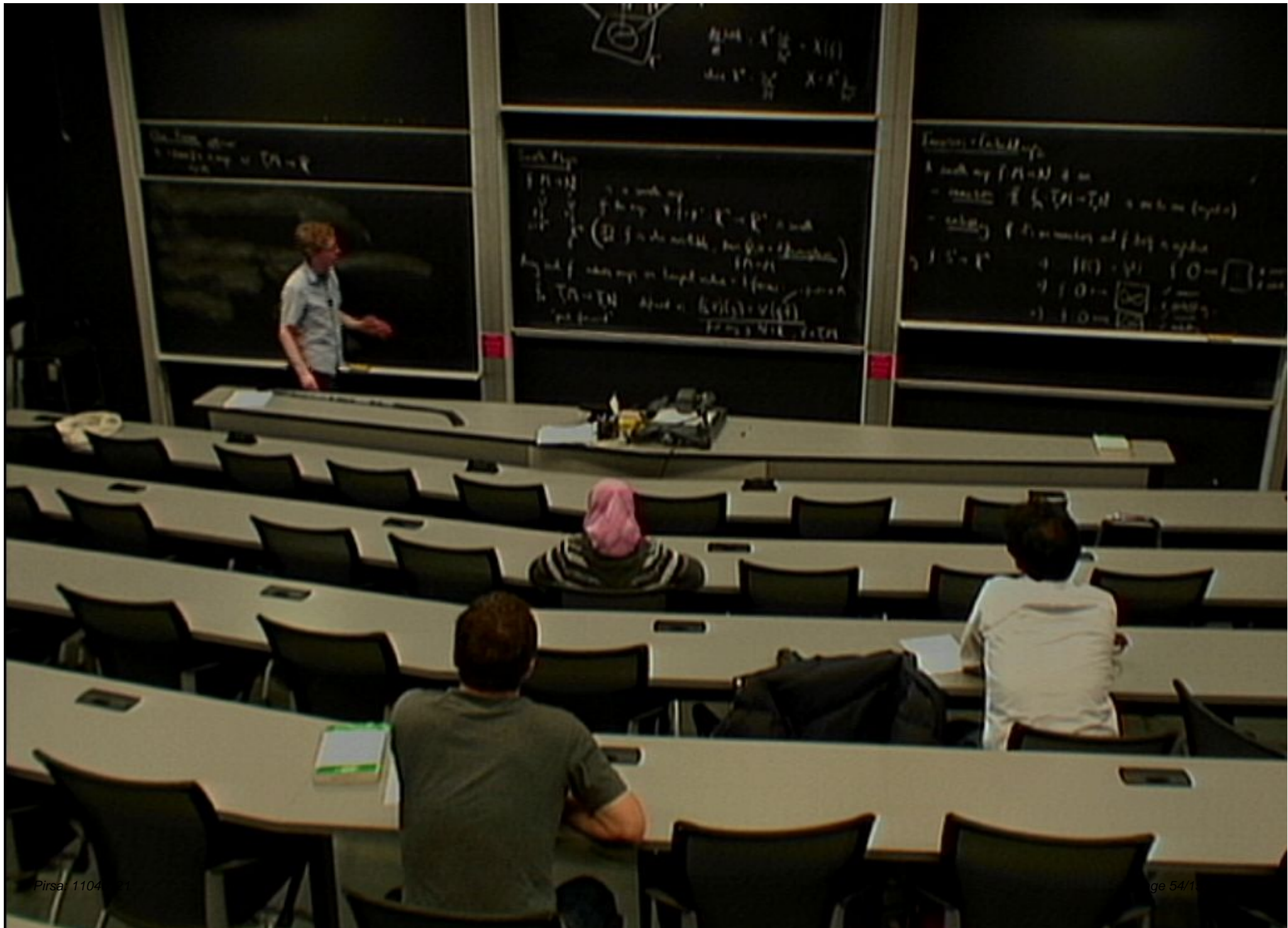
$f: \bigcirc \mapsto \boxed{\cdot}$ \times immersion
 \times embedding

ii) $f: \bigcirc \mapsto \boxed{\infty}$

\checkmark immersion
 \times embedding

iii) $f: \bigcirc \mapsto \boxed{\text{figure 8}}$

\checkmark immersion
 \checkmark embedding



One-Forms C^1 map

A 1-form ω is a map $\omega: T_x M \rightarrow \mathbb{R}$
 $\omega_p \in T_x^* M$

Space $T_x^* M$ of one-forms at x

On 1-forms, $f: M \rightarrow N$ induces the pull-back $f^*: T_x^* N \rightarrow T_x^* M$

One-Forms \mathbb{R}^n

A 1-form ω is a map $\omega: T_x M \rightarrow \mathbb{R}$
 $d_x f$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T_x^* N \rightarrow T_x^* M$

SAFETY
No Smoking
No Alcohol

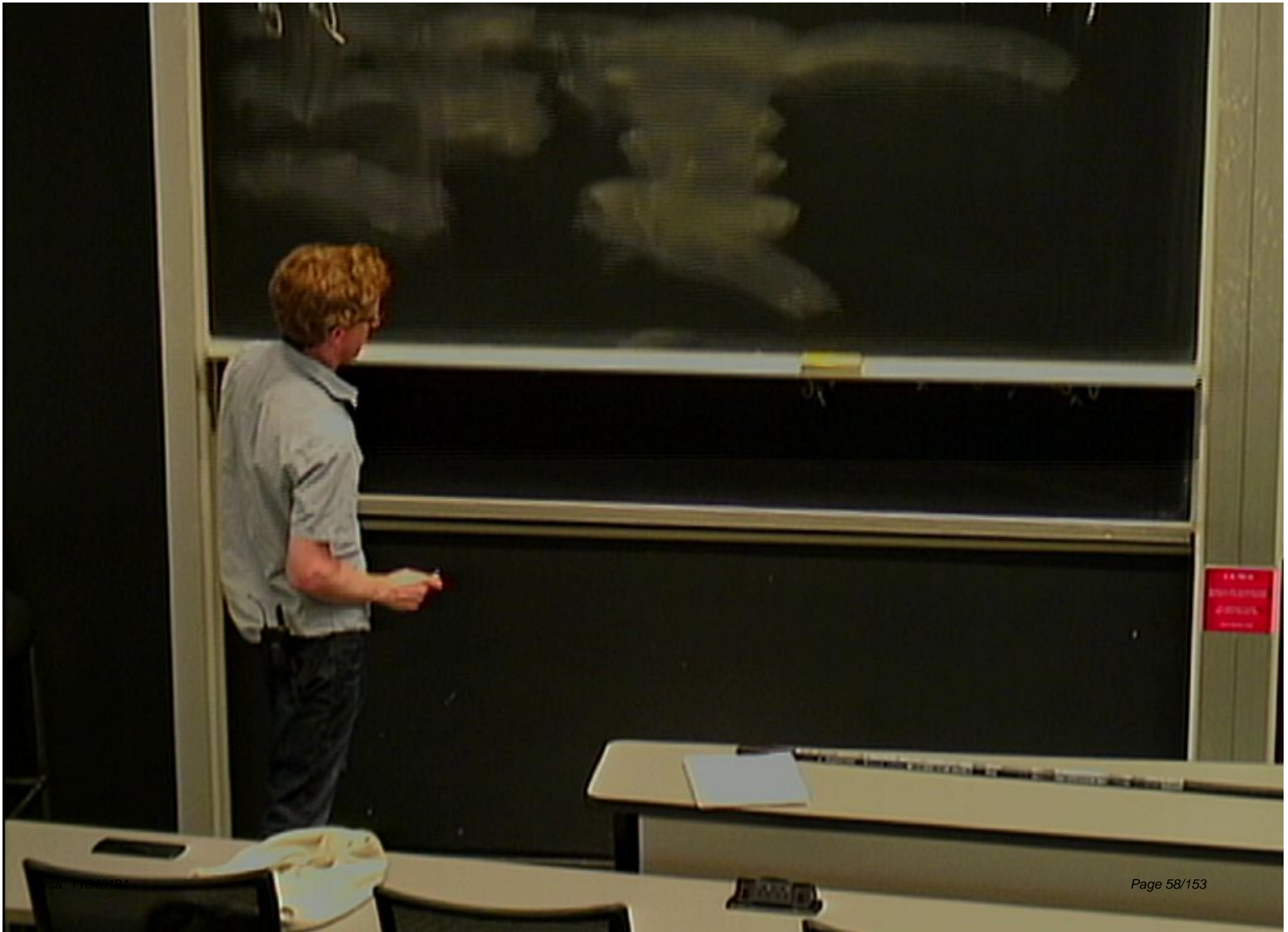
One-Forms \mathbb{C}^1 linear

A 1-form ω is a map $\omega: T_x M \rightarrow \mathbb{R}$
 $\omega_p \in T_x^* M$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T_x^* N \rightarrow T_x^* M$

$V \in T_x M$ (so $f_* V \in T_x N$)

REMEMBER
STUDENT
ATTENDANCE



Def. Form ω

A 1-form ω is a map $\omega: T_x M \rightarrow \mathbb{R}$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T_x^* N \rightarrow T_x^* M$

Suppose $V \in T_x M$ (so $f_* V \in T_x N$); then define $f^* \omega$ ($\omega \in T_x^* N$)

by $\langle f^* \omega, V \rangle = \langle \omega, f_* V \rangle$

One-Forms

A 1-form ω is a map $\omega: T_x M \rightarrow \mathbb{R}$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T_x^* N \rightarrow T_x^* M$

Given $V \in T_x M$ (so $f_* V \in T_x N$); then define $f^* \omega$ ($\omega \in T_x^* N$)
by $\langle f^* \omega, V \rangle = \langle \omega, f_* V \rangle$

One-Forms

A 1-form ω is a map $\omega: T_x M \rightarrow \mathbb{R}$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T_x^* N \rightarrow T_x^* M$

Suppose $V \in T_x M$ (so $f_* V \in T_y N$); then define $f^* \omega$ ($\omega \in T_y^* N$)
by $\langle f^* \omega, V \rangle = \langle \omega, f_* V \rangle$

$$(f^* \omega)_i dx^i$$

$$V = V^i \frac{\partial}{\partial x^i}$$

$$\omega = \omega_j dy^j$$

One-Forms

A 1-form ω is a map $\omega: T^*M \rightarrow \mathbb{R}$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T^*N \rightarrow T^*M$

Suppose $V \in T_x M$ (so $f_* V \in T_y N$); then define $f^* \omega$ ($\omega \in T_y^* N$)
by $\langle f^* \omega, V \rangle = \langle \omega, f_* V \rangle$

$$V = V^i \frac{\partial}{\partial x^i}$$

$$\omega = \omega_j dy^j$$
$$(f^* \omega) = (f^* \omega)_i dx^i$$

One-Forms

A 1-form ω is a map $\omega: T_x M \rightarrow \mathbb{R}$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T_x^* N \rightarrow T_x^* M$

Suppose $V \in T_x M$ (so $f_* V \in T_x N$); then define $f^* \omega$ ($\omega \in T_x^* N$)

$$\text{by } \langle f^* \omega, V \rangle = \langle \omega, f_* V \rangle$$

$$(f^* \omega)_n dx^a = \omega_n (f_* V)^a$$

$$V = V^a \frac{\partial}{\partial x^a}$$

$$\omega = \omega_j dy^j$$

$$(f^* \omega) = (f^* \omega)_n dx^n$$

On 1-forms, $f: M \rightarrow N$ induces the pullback map $f^*: T^*N \rightarrow T^*M$

Suppose $V \in T_p M$ (so $f_* V \in T_{f(p)} N$); then define $f^* \omega$ ($\omega \in T^*_{f(p)} N$)

by $\langle f^* \omega, V \rangle = \langle \omega, f_* V \rangle$

$$(f^* \omega)_n dx^i = \omega_n (f^i_j V^j)$$

$$= \omega_n \frac{\partial y^i}{\partial x^j} V^j$$

$$V = V^j \frac{\partial}{\partial x^j}$$

$$\omega = \omega_n dy^n$$

$$(f^* \omega) = (f^* \omega)_n dx^n$$



Suppose $(V \in T_p M \text{ (so } f_* V \in T_p N))$; then define $f^* \omega \text{ (} \omega \in T_p^* N)$

by $\langle f^* \omega, V \rangle = \langle \omega, f_* V \rangle$

$(f^* \omega)_n dx^n = \omega_n (f_* V)^n$

$= \omega_n \frac{\partial y^s}{\partial x^n} V^s$

$V = V^n \frac{\partial}{\partial x^n}$

$\omega = \omega(y)_s dy^s$

$(f^* \omega) = (f^* \omega)_n dx^n$

$\therefore (f^* \omega)_n = \omega_s \frac{\partial y^s}{\partial x^n}$

if $\omega = \omega_s dy^s$, then $f^* \omega = \omega_s(y(x)) \frac{\partial y^s}{\partial x^n} dx^n$



B. ω is the dual space to $T_p M$. identity
 if 1-forms, $\omega = \omega_\mu dx^\mu$
 $\langle \omega, V^\nu \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle = \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu$

2014
 Department
 of Physics

Lie Derivative

[The rest of the chalkboard is obscured by heavy, dark chalk scribbles.]

2014
Mathematics
Department

Lie Derivative

An integral curve of a vector field X is
a curve whose tangent vector

Lie Derivative

An integral curve of a vector field X is
a curve whose tangent vector obeys dY

Y



Lie Derivative

An integral curve of a vector field X is
a curve whose tangent vector obeys $\frac{dY^a}{dt} = X^a(Y(t))$
 $Y(0) = (0,1)$



Lie Derivative

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 $Y(0,1) \rightarrow M$

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A flow is a map $\sigma: \mathbb{R} \times M \rightarrow M$

σ

Lie Derivative

An integral curve of a vector field X is
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 $Y(0) \rightarrow M$.



A flow is a map $\sigma: \mathbb{R} \times M \rightarrow M$
depending on our vector field X , defined by
 $\sigma(t, x) = \exp(tX)_x$

Lie Derivative

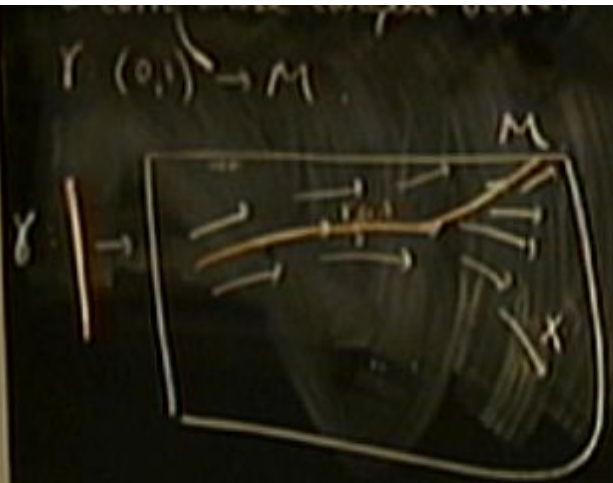
An integral curve of a vector field X is a curve whose tangent vector obeys $\frac{dY^a}{dt} = X^a(Y(t))$
 $Y(0) \rightarrow M$



A flow $\sigma: \mathbb{R} \times M \rightarrow M$
 depending on a vector field X , defined by
 $\frac{d}{dt} \sigma(t, x) = X(\sigma(t, x))$
 $\sigma(0, x) = x$



REMEMBER
 ALWAYS
 WASH YOUR
 HANDS



$\frac{d\gamma}{dt} = X(\gamma(t))$

A flow is a map $\sigma: \mathbb{R} \times M \rightarrow M$
 depending on one vector field X , defined by

$$\sigma(t, x) = \left[x + tX + \frac{t^2}{2}X^2 + \dots \right]_x$$

$\frac{d\sigma(t, x)}{dt} = X[\sigma(t, x)] = X(x) + tX(X) + \frac{t^2}{2}X^2(X) + \dots$
 $= X(x) + tX^2(x) + \dots$



REMEMBER
 ALWAYS
 WRITE
 DOWN

or field X , defined by

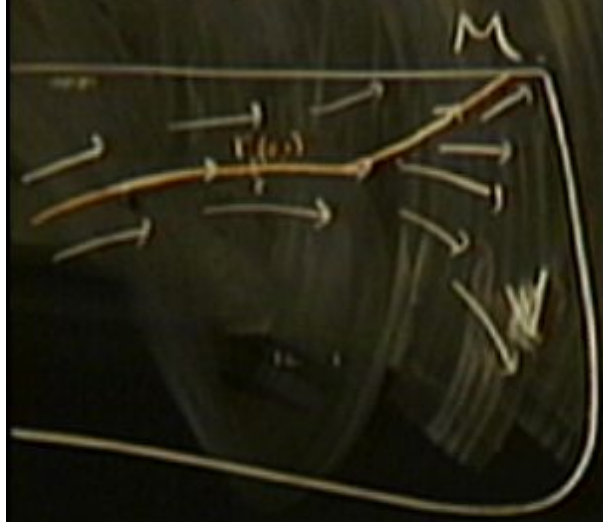
$$\langle \rangle x = \left[1 + tX + \frac{t^2}{2} X^2 + \dots \right] x$$

$$= x^M + tX(x) + \frac{t^2}{2} X[X(x)] + \dots$$

$$= x^M + tX^M(x) + \frac{t^2}{2} X^\nu \frac{\partial X^M}{\partial x^\nu} + \frac{t^3}{3!} X^\lambda \frac{\partial X^\nu}{\partial x^\lambda} \frac{\partial X^M}{\partial x^\nu}$$

Integral curve of a vector field V is

whose tangent vector obeys $\frac{dY^m}{dt} = V^m(Y(t))$
 $\gamma: \mathbb{R} \rightarrow M$.



A flow is a map $\sigma: \mathbb{R} \times M \rightarrow M$

depending on our vector field V , defined by

$$\sigma(t, x) \mapsto \exp(tV)x = \left[1 + tV + \frac{t^2}{2}V^2 + \dots \right] x$$

$$= x^m + tV^m(x) + \frac{t^2}{2}V^m(V^i(x)) + \dots$$

$$= x^m + tV^m(x) + \frac{t^2}{2}x^i \frac{\partial V^m}{\partial x^i} + \frac{t^3}{3!}x^i \frac{\partial V^j}{\partial x^i} \frac{\partial V^m}{\partial x^j} + \dots$$



"push-forward"

For any $g: N \rightarrow \mathbb{R}$, $V \in T_x M$

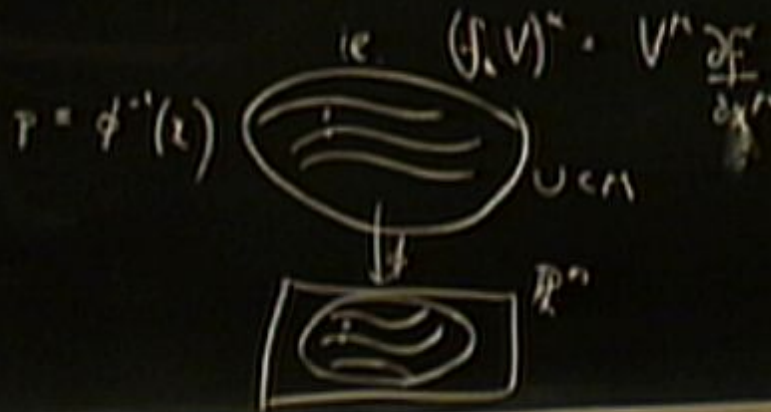
$$V[g \circ f] = V^{\#} \frac{\partial g(f(x))}{\partial y^{\alpha}} = \left(V^{\#} \frac{\partial y^{\alpha}}{\partial x^{\mu}} \right) \frac{\partial g}{\partial y^{\alpha}} = \left(\frac{\partial y^{\alpha}}{\partial x^{\mu}} \right)^{\#} \frac{\partial g}{\partial y^{\alpha}}$$

$$P = \dots \quad (f, V)^{\#} = V^{\#} \frac{\partial f^{\alpha}}{\partial x^{\mu}}$$

"push-forward"

For any $g: N \rightarrow \mathbb{R}$, $V \in T_x M$

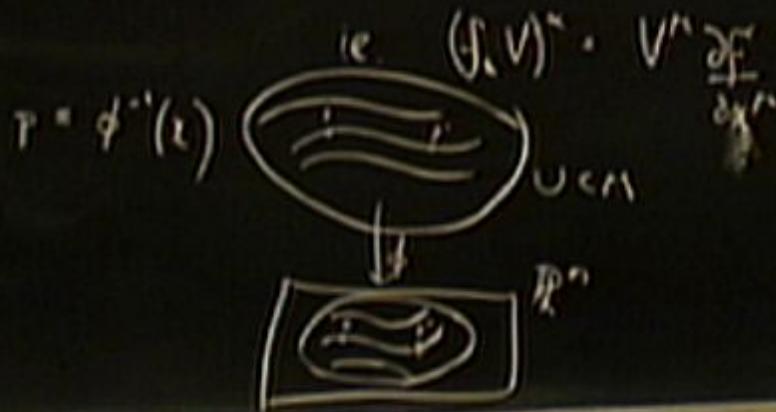
$$V[g \circ f] = V^{\#} \frac{\partial g(f(x))}{\partial x^i} = \left(V^{\#} \frac{\partial y^k}{\partial x^i} \right) \frac{\partial g}{\partial y^k} = \left(\frac{\partial y^k}{\partial x^i} V \right)^{\#} \frac{\partial g}{\partial y^k}$$



"push-forward"

For any $g: N \rightarrow \mathbb{R}$, $V \in T_x M$

$$V[g \circ f] = V^{\#} \frac{\partial g(f(x))}{\partial x^i} = \left(V^{\#} \frac{\partial y^k}{\partial x^i} \right) \frac{\partial g}{\partial y^k} = \left(\frac{\partial y^k}{\partial x^i} V \right)^{\#} \frac{\partial g}{\partial y^k}$$



"push-forward"

For any $g: N \rightarrow \mathbb{R}$; $V \in T_x M$

The Lie derivative \mathcal{L}_Y of a vector



\mathbb{R}^n
 $\mathcal{L}_X(M_0) = \dots$
 $\mathcal{L}_X^2 = \mathcal{L}_X^2 \mathcal{L}_X$

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"push-forward"

For any $g: N \rightarrow \mathbb{R}$; $V \in T_x M$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim_{t \rightarrow 0} \frac{1}{t} (\tau_{-t}^X Y - Y)$$

"push-forward"

For any $g: N \rightarrow \mathbb{R}$; $V \in T_p M$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined

$$\mathcal{L}_X Y = \frac{1}{\epsilon} \left[Y|_{\gamma(\epsilon)} - Y|_x \right]$$

"push-forward"

For any $g: N \rightarrow R$; $V \in T_p M$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim_{t \rightarrow 0} \frac{Y|_{\sigma(t)} - Y|_x}{t}$$

where σ is the flow along X

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[(\sigma_\epsilon)_* Y|_{\sigma_\epsilon^{-1}(p)} - Y|_p \right]$$

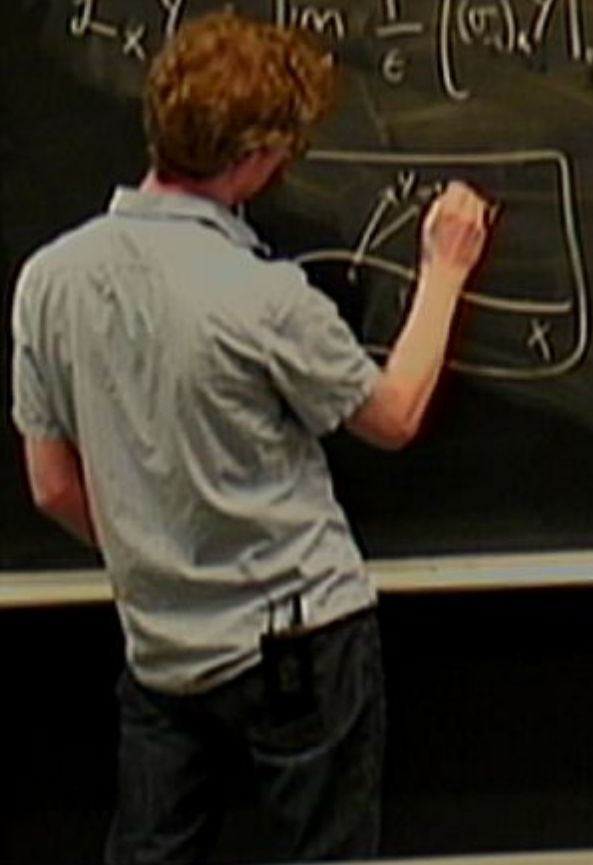
where σ is the flow along X



The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

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The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[(\sigma_\epsilon)_* Y|_{\sigma_\epsilon^{-1}(p)} - Y|_p \right]$$

where σ is the flow along X



$$\frac{d\sigma(t,x)}{dt} = X^{\sigma}[\sigma(t,x)]$$

$$= x^n + tV(x) + \frac{1}{2}X^2(x) + \dots$$
$$= x^n + tX^{\sigma}(x) + \frac{1}{2}X^2(x) + \frac{1}{6}X^3(x) + \dots$$

$$\sigma_i \circ \sigma_r(p) = \sigma_{s+r}(p) = \sigma_r \circ \sigma_i(p)$$

$$\sigma_i \circ \sigma_i(p) = p$$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X

is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[(\sigma_\epsilon)_* Y|_{\sigma_\epsilon^{-1}(x)} - Y|_x \right]$$

where σ is the flow along X



$$Y|_x = Y^i(x) \frac{\partial}{\partial x^i}$$

The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector X is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[(\sigma_\epsilon)_* Y|_{\sigma_\epsilon^{-1}(x)} - Y|_x \right]$$

where σ is the flow along X



$$Y|_x = Y^i(x) \frac{\partial}{\partial x^i}$$

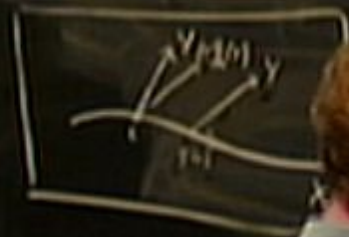
$$Y|_{\sigma_\epsilon^{-1}(x)} = Y^i(x + \epsilon X)$$

The Lie derivative

is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[(\sigma_{-\epsilon})_* Y|_{\sigma(\epsilon)} - Y|_x \right]$$

where σ is the flow along X



$$Y|_x = Y'(x) \frac{\partial}{\partial x^n} \Big|_x$$

$$Y|_{\sigma(\epsilon)} = Y'(x + \epsilon X) \frac{\partial}{\partial x^n} \Big|_{(x + \epsilon X)}$$



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Small red sign on the right side of the chalkboard frame.

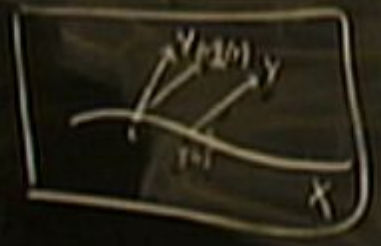


The Lie derivative \mathcal{L}_X of a function f is defined by

is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[(\sigma_{-\epsilon})_* Y|_{\sigma(\epsilon)} - Y|_x \right]$$

where σ is the flow along X



$$Y|_x = Y^i(x) \frac{\partial}{\partial x^i}$$

$$Y|_{\sigma(\epsilon)} = Y^i(x + \epsilon X)$$

$$(\sigma_{-\epsilon})_* Y|_{\sigma(\epsilon)} = Y^i(x) \frac{\partial}{\partial x^i} + \dots$$



Small red sticker on the left window frame.

Small red sticker on the right window frame.

defined as $(f_* V)(g) = V(g \circ f)$
 "push-forward" for any $g: N \rightarrow \mathbb{R}$; $V \in T_x M$

$$V[g \circ f] = V^a \frac{\partial (g \circ f)}{\partial x^a} = \left(V^a \frac{\partial y^i}{\partial x^a} \right) \frac{\partial g}{\partial y^i} = \left(f_* V \right)^i \frac{\partial g}{\partial y^i}$$

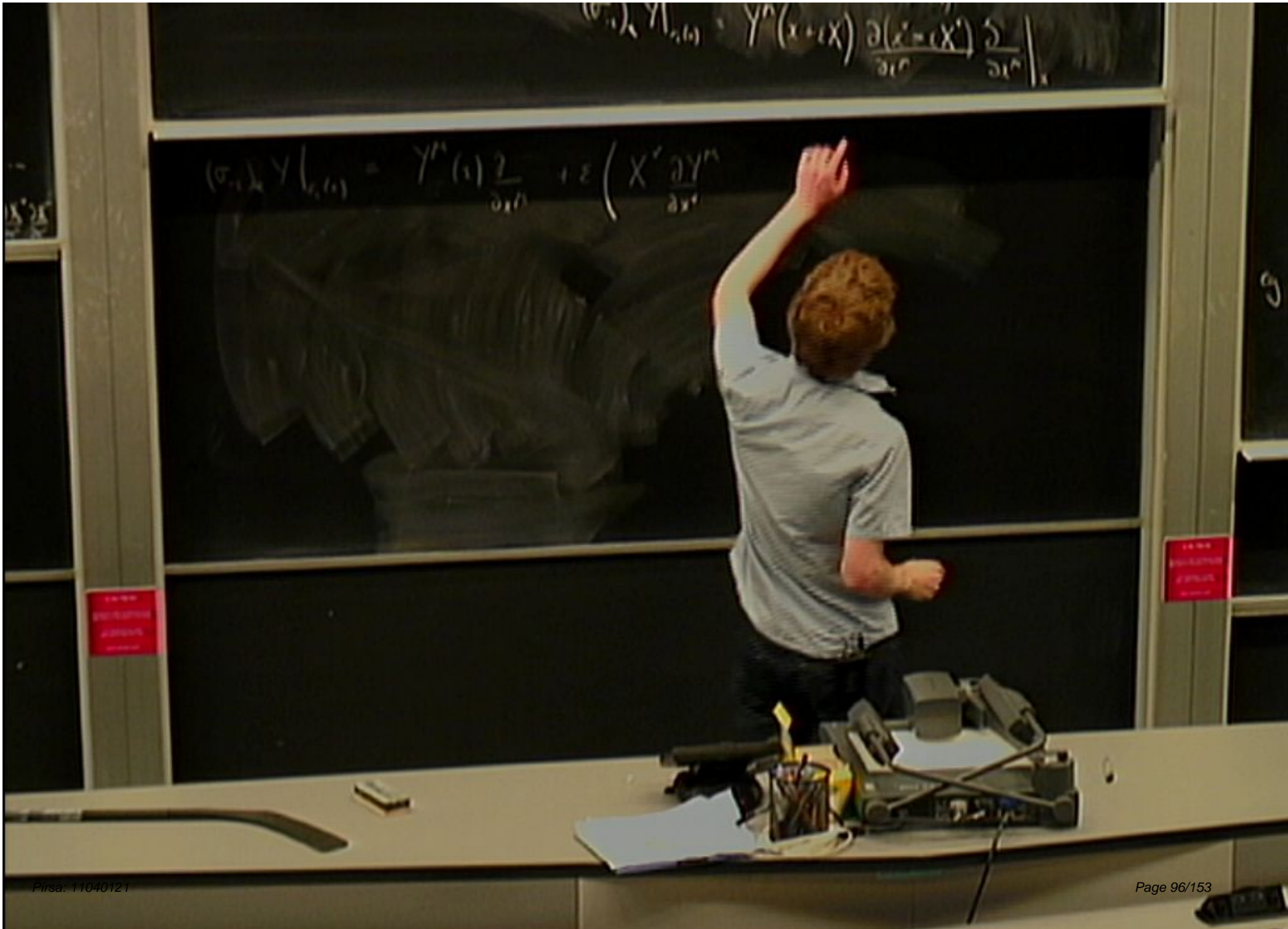
$p = f^{-1}(x)$  $U \subset M$

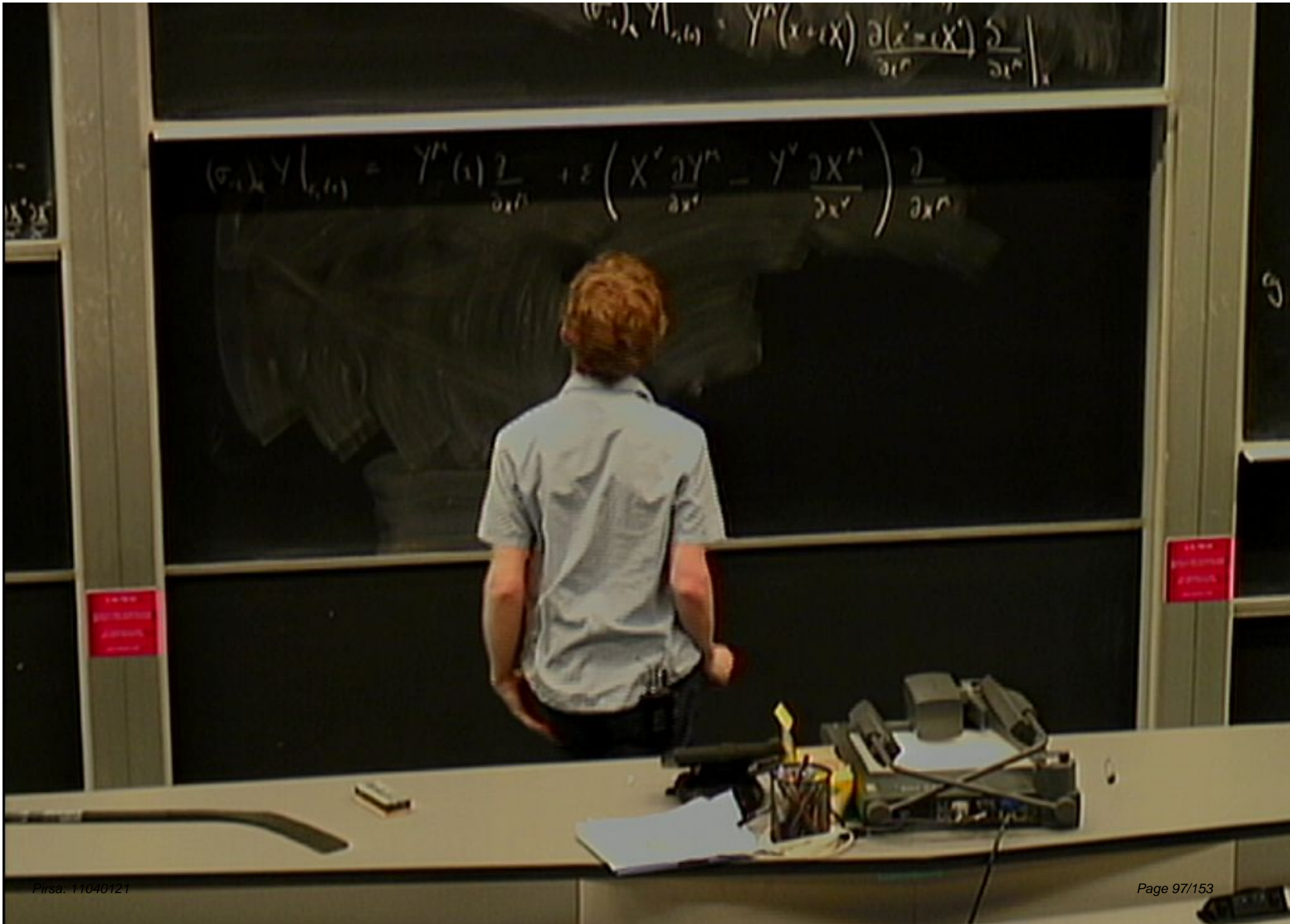
ie $(f_* V)^a = V^b \frac{\partial f^a}{\partial x^b}$

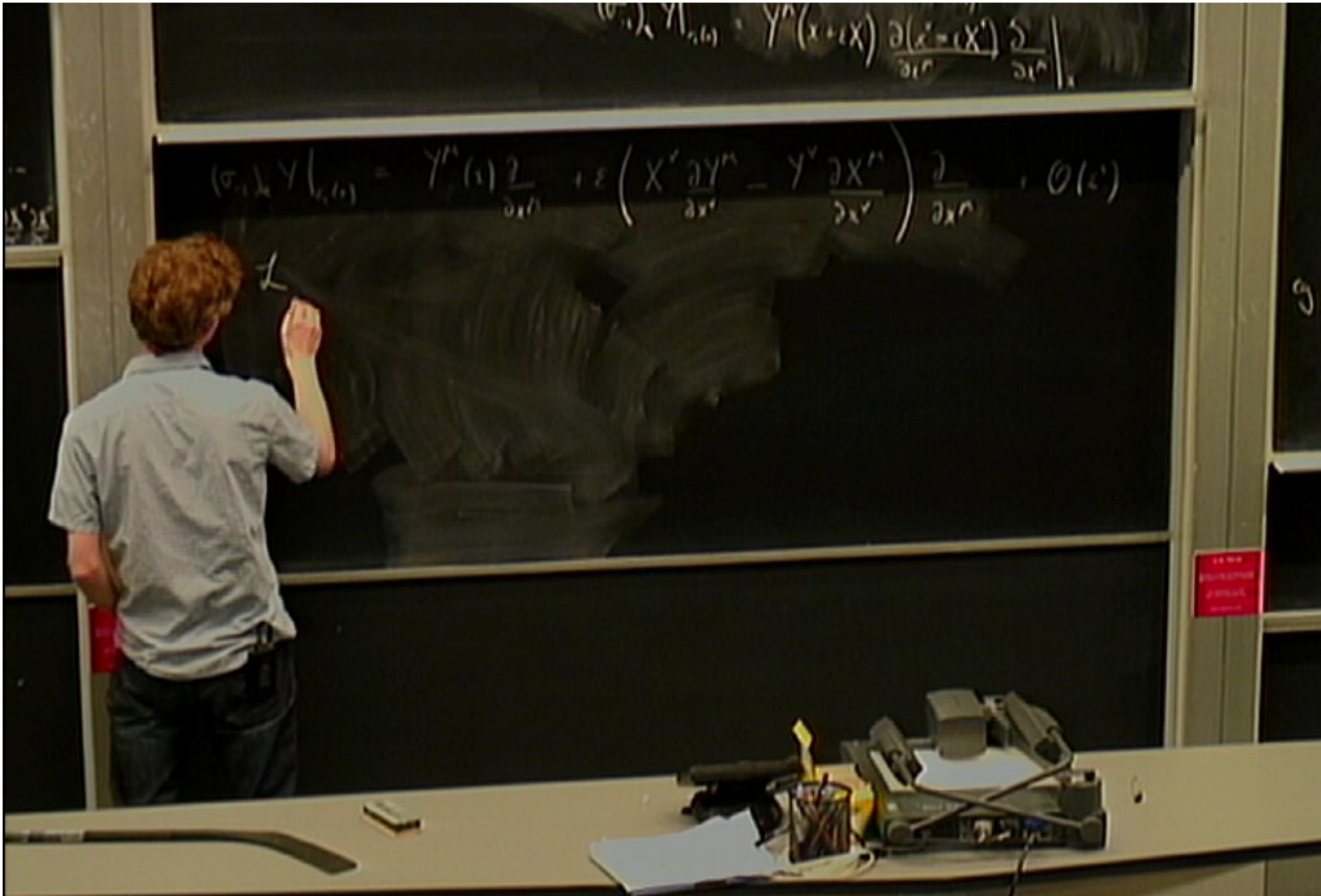
The Lie derivative $\mathcal{L}_X Y$ of a vector Y along a vector
 is defined by

$$\mathcal{L}_X Y = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[(f_\epsilon)_* Y|_{f_\epsilon^{-1}(x)} - Y|_x \right] \quad \text{where } f_\epsilon = \exp(\epsilon X)$$



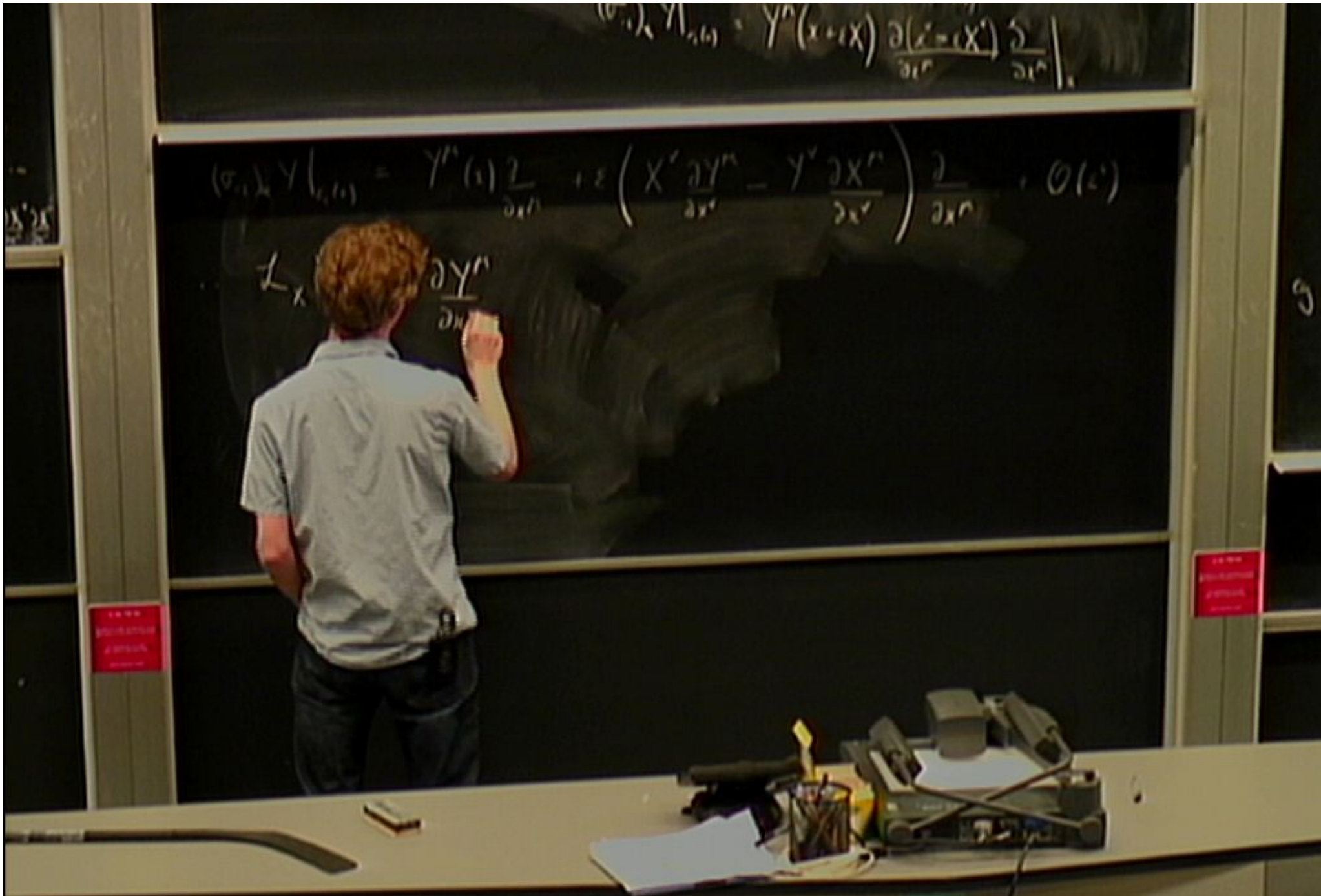






$$(0^{th})_x Y(x_0) = Y(x_0) \frac{\partial(x_0 + \epsilon X)}{\partial x} \Big|_{x_0}$$

$$(1^{st})_x Y(x_0) = Y'(x_0) \frac{\partial}{\partial x} + \epsilon \left(X \frac{\partial Y'}{\partial x} - Y' \frac{\partial X}{\partial x} \right) \frac{\partial}{\partial x} + O(\epsilon^2)$$



$$(0,1)_x Y(1,1) = Y^n(x+\epsilon X) \frac{\partial(x+\epsilon X)}{\partial x^n} \Big|_x$$

$$(0,1)_x Y(1,1) = Y^n(x) \frac{\partial}{\partial x^n} + \epsilon \left(X^v \frac{\partial Y^n}{\partial x^v} - Y^v \frac{\partial X^n}{\partial x^v} \right) \frac{\partial}{\partial x^n} + O(\epsilon^2)$$

$$\mathcal{L}_X Y = \left(X^v \frac{\partial Y^n}{\partial x^v} - Y^v \frac{\partial X^n}{\partial x^v} \right) \frac{\partial}{\partial x^n}$$



Small red rectangular sign with illegible text on the chalkboard frame.

$$(\sigma_{\nu\lambda}) Y_{(\nu,\lambda)} = Y^{\mu}(\lambda) \frac{\partial}{\partial x^{\mu}} + \varepsilon \left(X^{\nu} \frac{\partial Y^{\mu}}{\partial x^{\nu}} - Y^{\nu} \frac{\partial X^{\mu}}{\partial x^{\nu}} \right) \frac{\partial}{\partial x^{\mu}} + O(\varepsilon^2)$$

$$\mathcal{L}_X Y = \left(X^{\nu} \frac{\partial Y^{\mu}}{\partial x^{\nu}} - Y^{\nu} \frac{\partial X^{\mu}}{\partial x^{\nu}} \right) \frac{\partial}{\partial x^{\mu}}$$

$$(\sigma_{\epsilon})_x Y|_{\epsilon=0} = Y''(x) \frac{\partial}{\partial x''} + \epsilon \left(X' \frac{\partial Y''}{\partial x'} - Y' \frac{\partial X''}{\partial x'} \right) \frac{\partial}{\partial x''} + O(\epsilon^2)$$

$$\mathcal{L}_X Y = \left(Y'' - Y' \frac{\partial X''}{\partial x'} \right) \frac{\partial}{\partial x''} \in T_x^* M$$

Define the Lie bracket $[X, Y]f = X(Y[f]) - Y(X[f]) \stackrel{a}{=} (\mathcal{L}_X Y)[f]$

On 1-forms

\mathcal{L}_Y

$$(\sigma_{\epsilon})_* Y|_{\epsilon, \epsilon} = Y^{\mu}(x) \frac{\partial}{\partial x^{\mu}} + \epsilon \left(X^{\nu} \frac{\partial Y^{\mu}}{\partial x^{\nu}} - Y^{\nu} \frac{\partial X^{\mu}}{\partial x^{\nu}} \right) \frac{\partial}{\partial x^{\mu}} + \mathcal{O}(\epsilon^2)$$

$$Y = \left(X^{\nu} \frac{\partial Y^{\mu}}{\partial x^{\nu}} - Y^{\nu} \frac{\partial X^{\mu}}{\partial x^{\nu}} \right) \frac{\partial}{\partial x^{\mu}} \in T_x M$$

Lie bracket $[X, Y]f = X(Y[f]) - Y(X[f]) \stackrel{u}{=} (L_X Y)[f]$

$$\omega \in T_x^* M$$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (f \circ \sigma_{\epsilon}^{-1}) \omega|_{\epsilon, \epsilon} = \omega|_x$$

$$(\sigma_{\varepsilon})_* Y|_{r_\varepsilon(t)} = Y^n(x) \frac{\partial}{\partial x^n} + \varepsilon \left(X^v \frac{\partial Y^n}{\partial x^v} - Y^v \frac{\partial X^n}{\partial x^v} \right) \frac{\partial}{\partial x^n} + O(\varepsilon^2)$$

$$\mathcal{L}_X Y = \left(X^v \frac{\partial Y^n}{\partial x^v} - Y^v \frac{\partial X^n}{\partial x^v} \right) \frac{\partial}{\partial x^n} \in \mathfrak{X} T_x M$$

Define the Lie bracket $[X, Y]f = X(Y[f]) - Y(X[f]) \stackrel{a}{=} (\mathcal{L}_X Y)[f]$

On 1-forms $\omega \in T_x^* M$

$$\mathcal{L}_X \omega = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left((\sigma_{\varepsilon}^*) \omega|_{r_\varepsilon} - \omega \right)$$

Lie Derivative

Differential Forms



Lie Derivative

Differential Forms

[The chalkboard contains several lines of handwritten mathematical text that has been almost entirely obscured by heavy black marker scribbles.]



eg. $L_x(f) = \lim_{\epsilon \rightarrow 0} \sum_{\sigma_i} \epsilon_i |f|_{\sigma_i} - |f|$



eg. $L_x(f) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [\sigma_\epsilon f |_{x+\epsilon} - f |]$
 $= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [f(x+\epsilon) - f(x)] = X^n \frac{\partial f}{\partial x^n} = X[f]$

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Differential Forms

A p-form ω at $x \in M$ is an element of $\bigwedge^p T_x^*M$

if $\omega(x) = \omega_{p_1, \dots, p_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p}$ where $\{dx^{i_j}\}$ is a basis for T_x^*M at $dx^{i_1} \wedge \dots \wedge dx^{i_p}$

Define
On L
 L

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Differential Forms

A p-form ω at $x \in M$ is an element of $\Lambda^p T_x^*M$

if $\omega(x) = \omega_{\mu_1, \dots, \mu_p}(x) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$ where $\{dx^i\}$ basis for $T_x^*M = \mathbb{R}^n$
at x and $\mu_1 < \dots < \mu_p$.

Define
On 1-
2-

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Differential Forms

A p -form ω at $x \in M$ is an element of $\Lambda^p T_x^* M$.

if $\omega(x) = \omega_{i_1, \dots, i_p}(x) \underbrace{dx^{i_1} \wedge \dots \wedge dx^{i_p}}_{\mathbb{R}^{\binom{n}{p}}}$ where $\{dx^i\}$ is a basis for $T_x^* M \cong \mathbb{R}^n$
at $dx^{i_1} \wedge \dots \wedge dx^{i_p} \in \underbrace{\mathbb{R}^{\binom{n}{p}} \times \mathbb{R}^{\binom{n}{p}}}_{\mathbb{R}^{\binom{n}{p}} \times \mathbb{R}^{\binom{n}{p}}}$.

Define
On 1-
2-

SAFETY
INFORMATION
2010/11

Differential Forms

A p-form ω at $x \in M$ is an element of $\Lambda^p T_x^* M$

if $\omega(x) = \omega_{i_1, \dots, i_p}(x) \underbrace{dx^{i_1} \wedge \dots \wedge dx^{i_p}}_{\in \mathbb{R}^{\binom{n}{p}}}$ where $\{dx^{i_j}\}$ is a basis for $T_x^* M \cong \mathbb{R}^n$
at $dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} \in \underbrace{\mathbb{R}^{\binom{n}{p}} \times \mathbb{R}^{\binom{n}{p}}}_{\mathbb{R}^{\binom{n}{p}} \times \mathbb{R}^{\binom{n}{p}}}$

Define
On 1-
2-



Differential Forms

A p-form ω at $x \in M$ is an element of $\Lambda^p T_x^* M$

if $\omega(x) = \omega_{i_1, \dots, i_p}(x) \underbrace{dx^{i_1} \wedge \dots \wedge dx^{i_p}}_{\in \mathbb{R}^{\binom{n}{p}}}$ where $\{dx^{i_j}\}$ is a basis for $T_x^* M \cong \mathbb{R}^n$
 at $dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p} = \underbrace{dx^{i_1} \wedge \dots \wedge dx^{i_p}}_{\in \mathbb{R}^{\binom{n}{p}} \times \mathbb{R}^{\binom{n}{n-p}}}$

Ex: $\alpha = \omega \in \Lambda^r T_x^* M$ and $\eta \in \Lambda^s T_x^* M$

Define
 On 1-
 2-



Exterior derivative d is a map $d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$
space of all r -forms on M

define

$T_p M$

$O(\mathbb{R}^n)$

Exterior derivative d is a map $d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$

defined by

$$d\omega = \sum_i \partial_i \omega_{j_1 \dots j_r} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}$$

space of all r-forms
on M

no smoking
no eating
no drinking

$T_x M$

$O(x)$

Exterior derivative d is a map $d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$

defined by $d: \omega \mapsto d\omega = \partial_i \omega_{j_1 \dots j_r} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}$

$d^2 = 0$

space of all r-forms on M

$T_p M$

$O(z')$

Exterior derivative d is a map $d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$

defined by $d: \omega \mapsto d\omega = ?$

space of all r-forms on M
 $d(x^1 \wedge \dots \wedge x^r)$

$d^2 = 0$

$d(d\omega) = d^2\omega = 0$

$d(x^1 \wedge \dots \wedge x^r)$

$T_p M$

$\mathcal{O}(z)$

\square

γ

defined by $d : \omega \mapsto d\omega = \partial_\alpha \omega_{\mu_1 \dots \mu_r} dx^\alpha \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$

$d^2 = 0$

$d(d\omega) = d\left(\partial_\alpha \omega_{\mu_1 \dots \mu_r} dx^\alpha \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}\right)$

$= \partial_\alpha \partial_\beta \omega_{\mu_1 \dots \mu_r} dx^\alpha \wedge dx^\beta \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$

space of all r-forms on M



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$T_p M$

$\mathcal{O}(U)$

\square
 γ

defined by

$$d : \omega \mapsto d\omega = \partial_r \omega_{r_1 \dots r_n} dx^1 \wedge \dots \wedge dx^{r_1} \wedge \dots \wedge dx^{r_n}$$

space of all n -forms on M

$$\bullet d^2 = 0$$

$$d(d\omega) = d\left(\partial_r \omega_{r_1 \dots r_n} dx^1 \wedge \dots \wedge dx^{r_1} \wedge \dots \wedge dx^{r_n}\right)$$

$$= \partial_r \partial_s \omega_{r_1 \dots r_n} \underbrace{dx^s \wedge dx^1 \wedge \dots \wedge dx^{r_1} \wedge \dots \wedge dx^{r_n}}_{\text{antigenus is zero}}$$

$$= 0$$



$T_x M$

$O(x)$

\square

γ

no smoking
no eating
no drinking

defined by $d : \omega \mapsto d\omega = \partial_i \omega_{j_1 \dots j_r} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}$ on M

$$\begin{aligned}
 \bullet d^2 &= 0 & d(d\omega) &= d\left(\partial_i \omega_{j_1 \dots j_r} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}\right) \\
 & & &= \partial_i \partial_j \omega_{j_1 \dots j_r} dx^i \wedge dx^j \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r} \\
 & & &= 0 \quad \text{aligning is zero}
 \end{aligned}$$

$$\bullet d(f^* \omega) = f^*(d\omega) \quad \text{where } f : M \rightarrow N$$



$T_x M$

$O(x)$

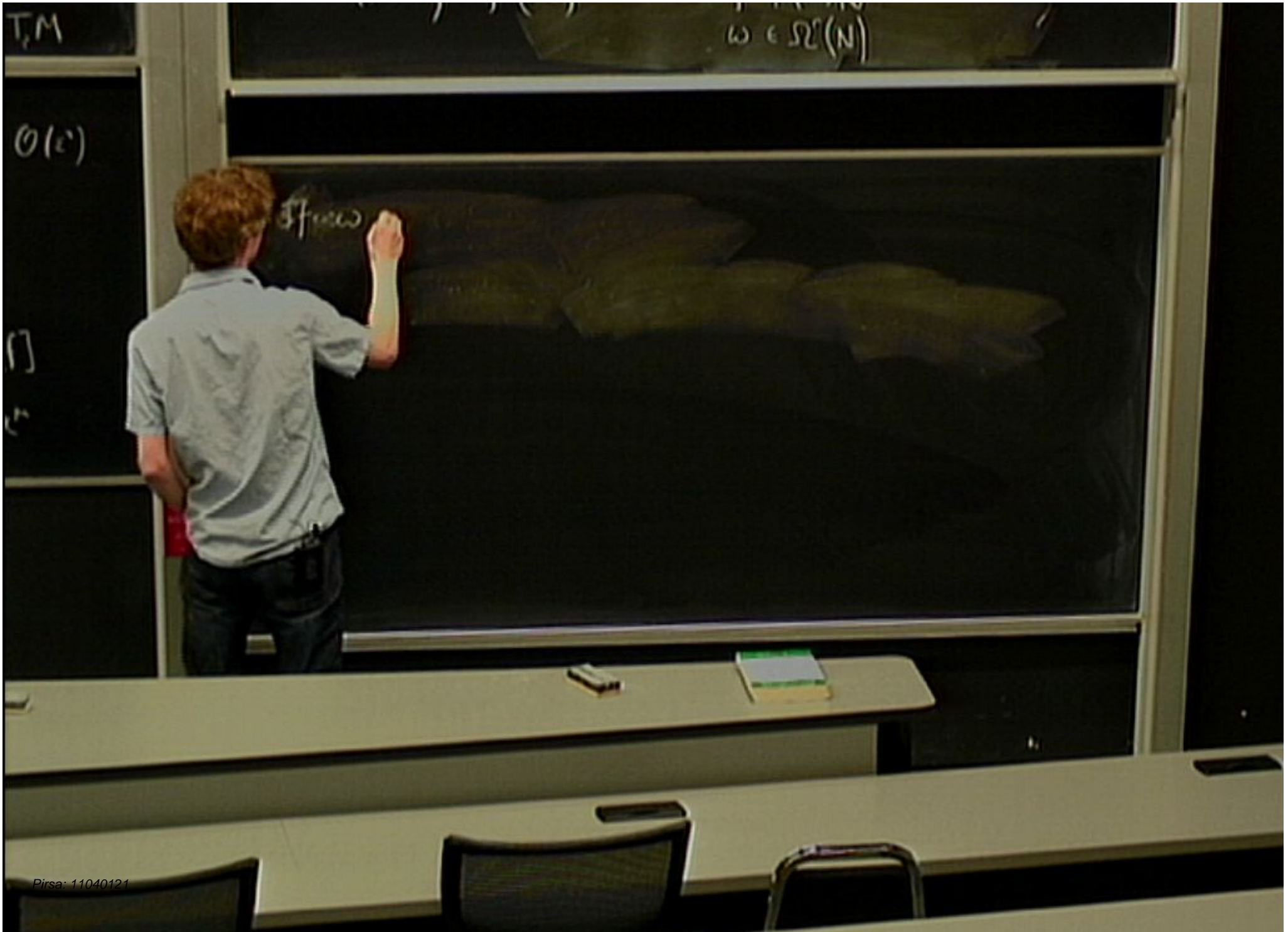
defined by $d : \omega \mapsto d\omega = \partial_i \omega_{j_1 \dots j_r} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}$ on M

$d^2 = 0$

$$\begin{aligned} d(d\omega) &= d\left(\partial_i \omega_{j_1 \dots j_r} dx^i \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}\right) \\ &= \partial_i \partial_j \omega_{j_1 \dots j_r} \underbrace{dx^i \wedge dx^j \wedge dx^{j_1} \wedge \dots \wedge dx^{j_r}}_{\text{aligns to zero}} \\ &= 0 \end{aligned}$$

$f^*(\omega) = f^*(d\omega)$ where $f : M \rightarrow N$
 $\omega \in \Omega^r(N)$

no smoking
no alcohol



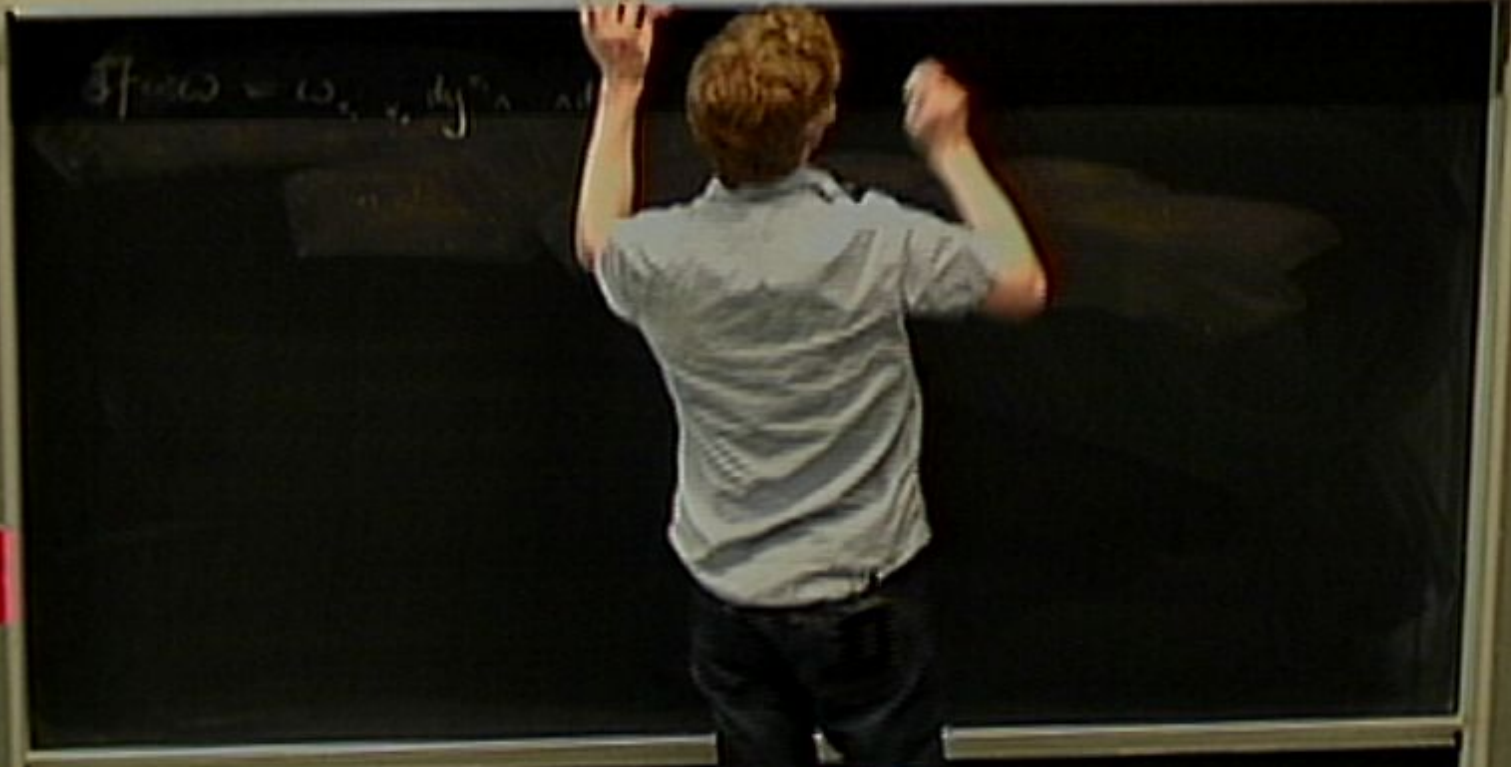
T, M

$O(z)$

\square

γ

$d(f^*\omega) = f^*(d\omega) \quad \text{for } f: M \rightarrow N$
 $\omega \in \Omega^k(N)$



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T, M

$O(z')$

[]

7

$$\begin{aligned} \mathcal{L}\{f(x)\} &= \omega \\ f'(x) &= \omega \end{aligned}$$

$\frac{\partial y}{\partial x} = dx^{m-1} \cdot \dots \cdot dx^{m_0}$

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T, M

$O(z')$

$$f'(z) = \omega_1 \dots \omega_n \cdot dy^{r_1} \wedge \dots \wedge dy^{r_n}$$

$$f''(z) = \omega_1 \dots \omega_n \frac{\partial y^{r_1}}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial y^{r_n}}{\partial x^{j_n}} dx^{j_1} \wedge \dots \wedge dx^{j_n}$$

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T, M

$O(z)$

$$\mathcal{F}\{\omega\} = \omega_{x_1} \dots \omega_{x_n} dy^{n-1} \wedge dy^{n-2} \dots \wedge dy^1$$

$$= \omega_{x_1} \dots \omega_{x_n} \frac{\partial y^1}{\partial x^1} \dots \frac{\partial y^{n-1}}{\partial x^{n-1}} dx^{n-1} \wedge dx^{n-2} \dots \wedge dx^1$$

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T, M

$O(\epsilon^2)$

$$f(\omega) = \omega_{i_1 \dots i_n} dy^{i_1} \wedge \dots \wedge dy^{i_n}$$

$$f^* \omega = \omega_{i_1 \dots i_n} \frac{\partial y^{i_1}}{\partial x^j} \dots \frac{\partial y^{i_n}}{\partial x^k} dx^{j_1} \wedge \dots \wedge dx^{j_n}$$

$$d(f^* \omega) = \frac{\partial}{\partial x^r} \left(\omega_{i_1 \dots i_n} \frac{\partial y^{i_1}}{\partial x^j} \dots \frac{\partial y^{i_n}}{\partial x^k} \right) dx^r \wedge dx^{j_1} \wedge \dots \wedge dx^{j_n}$$



Bitte nicht rauchen
Bitte nicht trinken

T, M

$\mathcal{O}(z)$

$$f^{\mu} \omega = \omega_{\nu_1 \dots \nu_n} dy^{\nu_1} \wedge \dots \wedge dy^{\nu_n}$$

$$f^{\mu} \omega = \omega_{\nu_1 \dots \nu_n} \frac{\partial y^{\nu_1}}{\partial x^{\mu}} \dots \frac{\partial y^{\nu_n}}{\partial x^{\mu}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_n}$$

$$d(f^{\mu} \omega) = \frac{\partial}{\partial x^{\nu}} \left(\omega_{\nu_1 \dots \nu_n} \frac{\partial y^{\nu_1}}{\partial x^{\mu}} \dots \frac{\partial y^{\nu_n}}{\partial x^{\mu}} \right) dx^{\nu} \wedge dx^{\nu_1} \wedge \dots \wedge dx^{\nu_n}$$



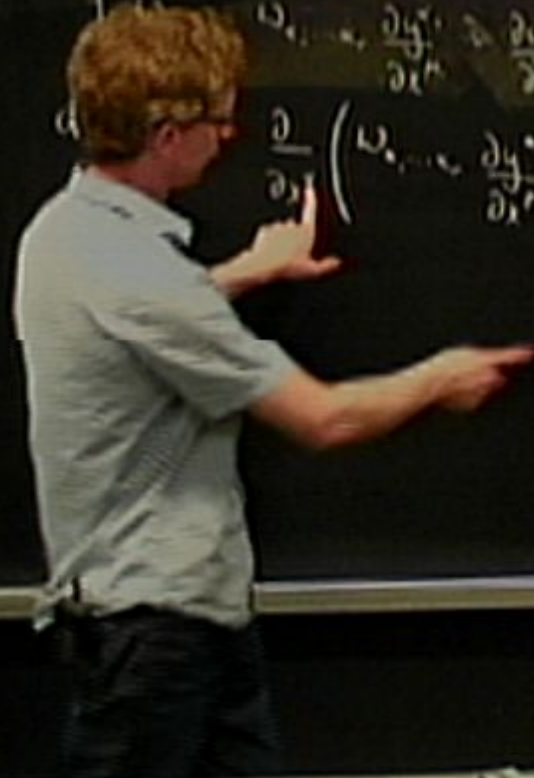
T, M

$O(z)$

$$\mathcal{F}(\omega) = \omega_{\nu_1 \dots \nu_n} dy^{\nu_1} \wedge \dots \wedge dy^{\nu_n}$$

$$\omega_{\nu_1 \dots \nu_n} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\nu_n}}{\partial x^{\mu_n}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$\frac{\partial}{\partial x^{\mu_1}} \left(\omega_{\nu_1 \dots \nu_n} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\nu_n}}{\partial x^{\mu_n}} \right) dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$



NO
SMOKING
HERE

T, M

$\mathcal{O}(z')$

$$f^{\alpha} \omega = \omega_{\alpha} \dots \wedge dy^{\alpha_1} \wedge \dots \wedge dy^{\alpha_r}$$

$$f^{\alpha} \omega = \omega_{\alpha} \dots \wedge \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \wedge \dots \wedge \frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$$

$$d(f^{\alpha} \omega) = \frac{\partial}{\partial x^{\nu}} \left(\omega_{\alpha} \dots \wedge \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \wedge \dots \wedge \frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}} \right) dx^{\nu} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r}$$



T, M

$O(z')$

$$df(x) = \omega_1 \dots \wedge dy^{i_1} \wedge \dots \wedge dy^{i_r}$$

$$f^* \omega = \omega_1 \dots \wedge \frac{\partial y^{i_1}}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial y^{i_r}}{\partial x^{j_r}} dx^{j_1} \wedge \dots \wedge dx^{j_r}$$

$$df = \frac{\partial}{\partial x^{j_1}} \left(\omega_1 \dots \wedge \frac{\partial y^{i_1}}{\partial x^{j_1}} \wedge \dots \wedge \frac{\partial y^{i_r}}{\partial x^{j_r}} \right) dx^{j_1} \wedge \dots \wedge dx^{j_r}$$

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T, M

$\mathcal{O}(z')$

$$d(\omega) = \omega_{\alpha_1 \dots \alpha_n} dy^{\alpha_1} \wedge \dots \wedge dy^{\alpha_n}$$

$$f^* \omega = \omega_{\alpha_1 \dots \alpha_n} \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\alpha_n}}{\partial x^{\mu_n}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$d(f^* \omega) = \frac{\partial}{\partial x^{\nu}} \left(\omega_{\alpha_1 \dots \alpha_n} \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \dots \frac{\partial y^{\alpha_n}}{\partial x^{\mu_n}} \right) dx^{\nu} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

$$= \omega$$

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T, M

$\mathcal{O}(z)$

\square

Σ

$$f^{\mu}(\omega) = \omega_{\nu} \cdot \frac{\partial y^{\nu}}{\partial x^{\mu}} \cdot dx^{\mu}$$
$$f^{\mu}(\omega) = \omega_{\nu} \cdot \frac{\partial y^{\nu}}{\partial x^{\mu}} \cdot dx^{\mu} = \omega_{\nu} \cdot \frac{\partial y^{\nu}}{\partial x^{\mu}} \cdot dx^{\mu}$$
$$f^{\mu}(\omega) = \frac{\partial}{\partial x^{\nu}} \left(\omega_{\nu} \cdot \left(\frac{\partial y^{\nu}}{\partial x^{\mu}} \cdot dx^{\mu} \right) \right) dx^{\nu} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

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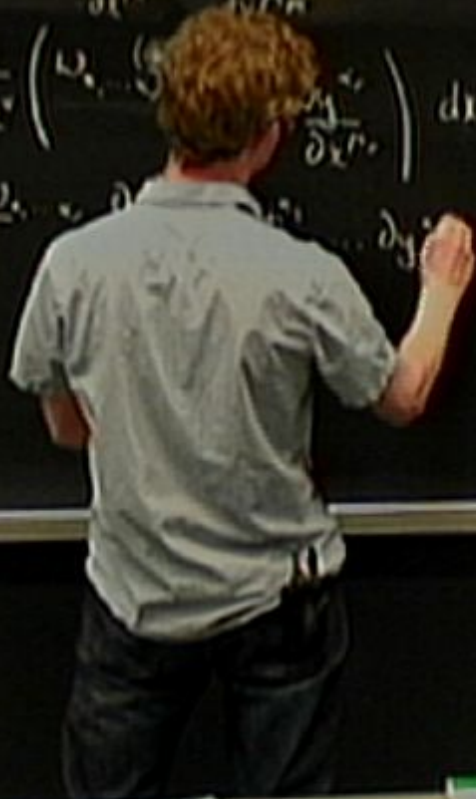


T, M

$O(z)$

\square
 τ

$$\begin{aligned}
 \partial^{\mu} \omega &= \omega_{\nu} \dots \partial y^{\nu} \wedge \dots \wedge dy^{\mu} \\
 f^{\mu} \omega &= \omega_{\nu} \dots \frac{\partial y^{\nu}}{\partial x^{\mu}} \dots \frac{\partial y^{\mu}}{\partial x^{\mu}} dx^{\mu} \wedge \dots \wedge dx^{\mu} \\
 d(f^{\mu} \omega) &= \frac{\partial}{\partial x^{\nu}} \left(\omega_{\nu} \dots \frac{\partial y^{\nu}}{\partial x^{\mu}} \dots \frac{\partial y^{\mu}}{\partial x^{\mu}} \right) dx^{\nu} \wedge dx^{\mu} \wedge \dots \wedge dx^{\mu} \\
 &= \frac{\partial \omega_{\nu}}{\partial x^{\nu}} \dots \frac{\partial y^{\nu}}{\partial x^{\mu}} \dots \frac{\partial y^{\mu}}{\partial x^{\mu}} \dots dy^{\nu} \wedge \dots
 \end{aligned}$$



NO OPEN
SMOKING
PERMITTED
HERE

T, M

$O(z)$

$$\begin{aligned} \int^x \omega &= \omega_{i_1 \dots i_n} dy^{i_1} \wedge \dots \wedge dy^{i_n} \\ \int^x \omega &= \omega_{i_1 \dots i_n} \frac{\partial y^{i_1}}{\partial x^{j_1}} \dots \frac{\partial y^{i_n}}{\partial x^{j_n}} dx^{j_1} \wedge \dots \wedge dx^{j_n} \\ d(\int^x \omega) &= \frac{\partial}{\partial x^{j_1}} \left(\omega_{i_1 \dots i_n} \frac{\partial y^{i_1}}{\partial x^{j_1}} \dots \frac{\partial y^{i_n}}{\partial x^{j_n}} \right) dx^{j_1} \wedge \dots \wedge dx^{j_n} \\ &= \frac{\partial \omega_{i_1 \dots i_n}}{\partial x^{j_1}} dx^{j_1} \wedge \dots \wedge dx^{j_n} + \omega_{i_1 \dots i_n} \frac{\partial^2 y^{i_1}}{\partial x^{j_1} \partial x^{k_1}} dx^{k_1} \wedge \dots \wedge dx^{j_n} + \dots \end{aligned}$$



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T, M

$\mathcal{O}(z)$

\square

Σ

$$f^{\mu}(x) = \omega_{\mu} + \frac{\partial y^{\nu}}{\partial x^{\mu}} \wedge \dots \wedge \frac{\partial y^{\rho}}{\partial x^{\mu}} \wedge \dots \wedge \frac{\partial y^{\sigma}}{\partial x^{\mu}} \wedge \dots$$

$$f^{\mu}(x) = \omega_{\mu} + \frac{\partial y^{\nu}}{\partial x^{\mu}} \wedge \frac{\partial y^{\rho}}{\partial x^{\mu}} \wedge \dots \wedge \frac{\partial y^{\sigma}}{\partial x^{\mu}} \wedge \dots$$

$$d(f^{\mu}(x)) = \left(\omega_{\mu} + \frac{\partial y^{\nu}}{\partial x^{\mu}} \wedge \frac{\partial y^{\rho}}{\partial x^{\mu}} \wedge \dots \wedge \frac{\partial y^{\sigma}}{\partial x^{\mu}} \wedge \dots \right) dx^{\nu} \wedge dx^{\rho} \wedge \dots \wedge dx^{\sigma} \wedge \dots$$

$$\dots \wedge \frac{\partial y^{\nu}}{\partial x^{\mu}} \wedge \frac{\partial y^{\rho}}{\partial x^{\mu}} \wedge \dots \wedge \frac{\partial y^{\sigma}}{\partial x^{\mu}} \wedge \dots dx^{\nu} \wedge dx^{\rho} \wedge \dots \wedge dx^{\sigma} \wedge \dots$$

DO NOT
write on
blackboard

T, M

$\mathcal{O}(z')$

$$\partial f^{\alpha} \omega = \omega_{\alpha} \wedge \frac{\partial y^{\alpha}}{\partial x^{\nu}} dx^{\nu}$$

$$f^{\alpha} \omega = \omega_{\alpha} \wedge \frac{\partial y^{\alpha}}{\partial x^{\nu}} dx^{\nu} = \frac{\partial y^{\alpha}}{\partial x^{\nu}} dx^{\nu} \wedge \dots \wedge dx^{\alpha}$$

$$d(f^{\alpha} \omega) = \frac{\partial}{\partial x^{\nu}} \left(\omega_{\alpha} \frac{\partial y^{\alpha}}{\partial x^{\nu}} \right) dx^{\nu} \wedge dx^{\alpha} \wedge \dots \wedge dx^{\alpha}$$

$$= \frac{\partial y^{\alpha}}{\partial x^{\nu}} \omega_{\alpha} \wedge \frac{\partial y^{\nu}}{\partial x^{\nu}} dx^{\nu} \wedge \dots \wedge dx^{\alpha}$$

$$= f^{\alpha}(d\omega)$$



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$\mathcal{O}(z')$

$$f^* \omega = \omega_{i_1 \dots i_n} dy^{i_1} \wedge \dots \wedge dy^{i_n}$$

$$f^* \omega = \omega_{i_1 \dots i_n} \frac{\partial y^{i_1}}{\partial x^{j_1}} \dots \frac{\partial y^{i_n}}{\partial x^{j_n}} dx^{j_1} \wedge \dots \wedge dx^{j_n}$$

$$d(f^* \omega) = \frac{\partial}{\partial x^{j_1}} \left(\omega_{i_1 \dots i_n} \frac{\partial y^{i_1}}{\partial x^{j_1}} \dots \frac{\partial y^{i_n}}{\partial x^{j_n}} \right) dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_n}$$

$$= \frac{\partial \omega_{i_1 \dots i_n}}{\partial x^{j_1}} \dots \frac{\partial y^{i_1}}{\partial x^{j_1}} \wedge \frac{\partial y^{i_2}}{\partial x^{j_2}} \dots \frac{\partial y^{i_n}}{\partial x^{j_n}} dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_n}$$

$$= f^*(d\omega)$$

Note we need f to be smooth



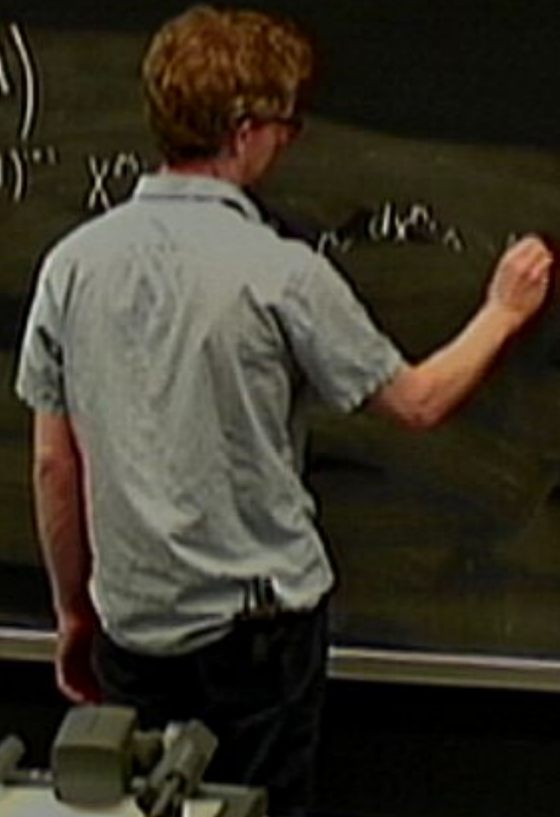
Contraction

C_x

Contraction

$$L_x : \Omega^r(M) \rightarrow \Omega^{r-1}(M)$$

$$L_x \cdot \omega = \sum_{i=1}^n (-1)^{i-1} x^i \omega_i$$



NO MOBILE PHONES
OR OTHER ELECTRONIC
DEVICES

NO MOBILE PHONES
OR OTHER ELECTRONIC
DEVICES

Contraction

$$\Omega^r(M) \rightarrow \Omega^{r-1}(M)$$

$$\omega \mapsto L_X \omega = \sum_{i=1}^n (-1)^{i+1} X^i \omega_{i,1} \wedge \dots \wedge \omega_{i,r} \wedge dx^1 \wedge \dots \wedge dx^r$$

Contraction

$$L_X : \Omega^r(M) \rightarrow \Omega^{r-1}(M)$$

$$L_X \omega = \sum_{i=1}^n (-1)^{i-1} X^i \omega_{i,1} \dots \omega_{i,r-1} dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n$$

$$X^i \omega$$

NO
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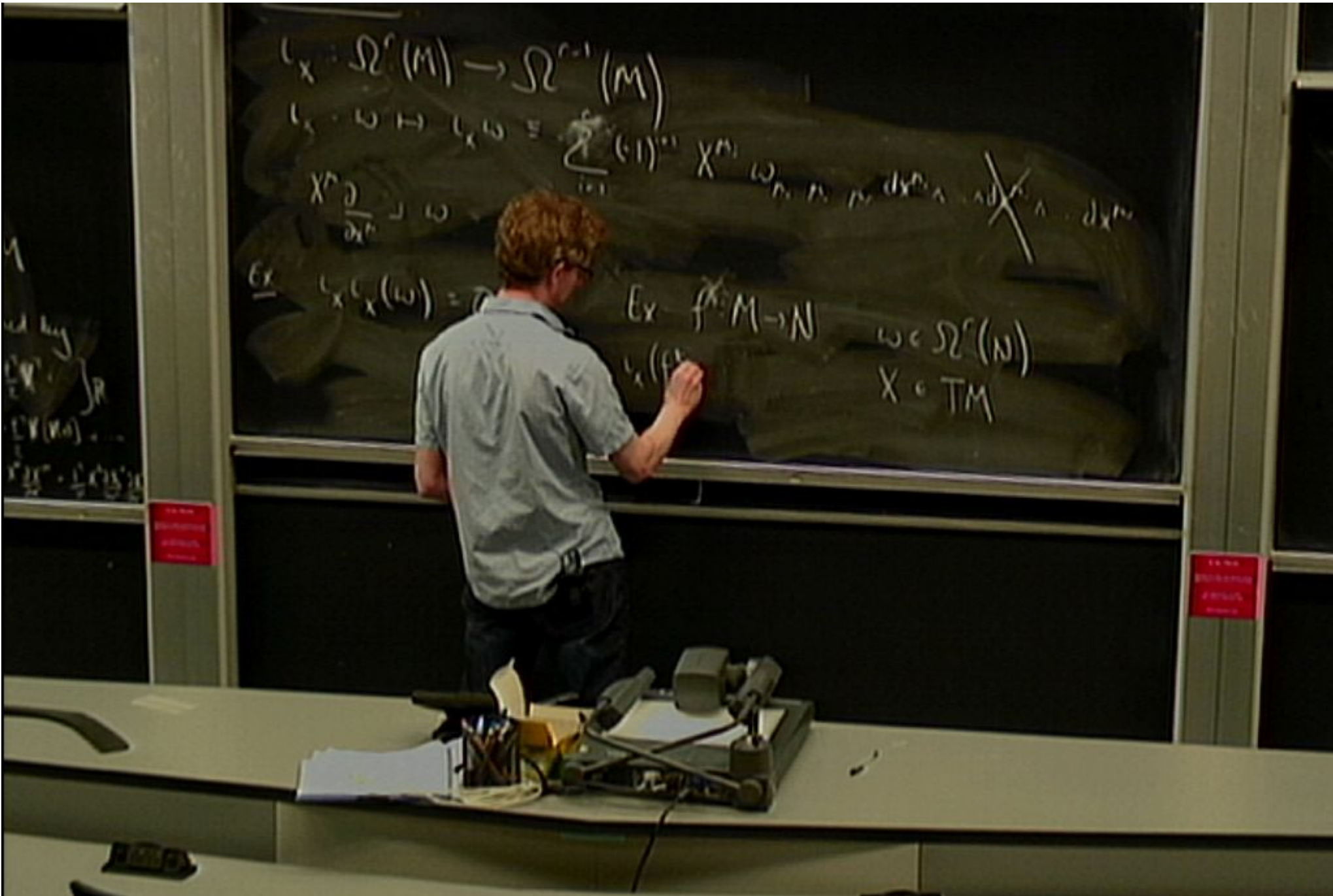
Contraction

$$L_X : \Omega^r(M) \rightarrow \Omega^{r-1}(M)$$

$$L_X : \omega \mapsto$$

$$X^r \frac{\partial}{\partial x^r}$$

$$\sum_{i_1 < \dots < i_r} (-1)^{i_1} X^{i_1} \omega_{i_1, \dots, i_r} dx^{i_2} \wedge \dots \wedge dx^{i_r}$$



$$L_X: \Omega^r(M) \rightarrow \Omega^{r-1}(M)$$

$$L_X \omega = \sum_{i=1}^n (-1)^{i-1} X^i \omega_{i,1} \dots \omega_{i,r}$$

$$X^i \frac{\partial}{\partial x^i} \omega$$

$$L_X(L_X \omega) = 0$$

Ex. $f: M \rightarrow N$

$$\omega \in \Omega^r(N)$$

$$X \in TM$$

$$L_x: \Omega^r(M) \rightarrow \Omega^{r-1}(M)$$

$$L_x \cdot \omega \mapsto L_x \omega = \sum_{i=1}^n (-1)^{i-1} X^i \omega_{i,1} \dots \omega_{i,r-1} dx^1 \wedge \dots \wedge dx^i \wedge \dots \wedge dx^n$$

$$X^i \frac{\partial}{\partial x^i}$$

Ex: $f^*: M \rightarrow N$ $\omega \in \Omega^r(N)$

$$L_x(f^*\omega) = f^*(L_{f(x)}\omega) \quad X \in TM$$



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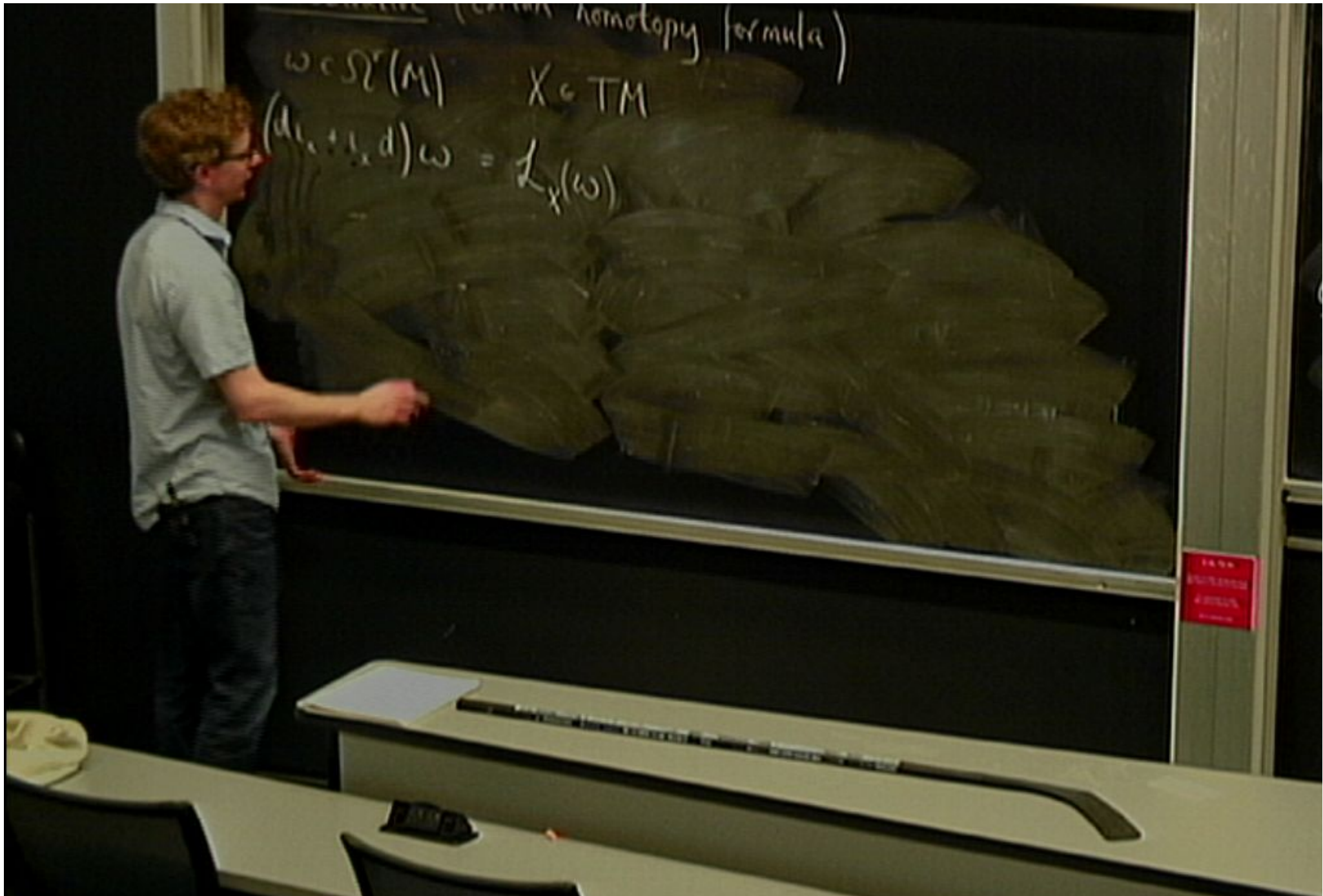


Lie Derivative (Cartan homotopy formula)

$$\omega \in \Omega^r(M) \quad X \in TM$$

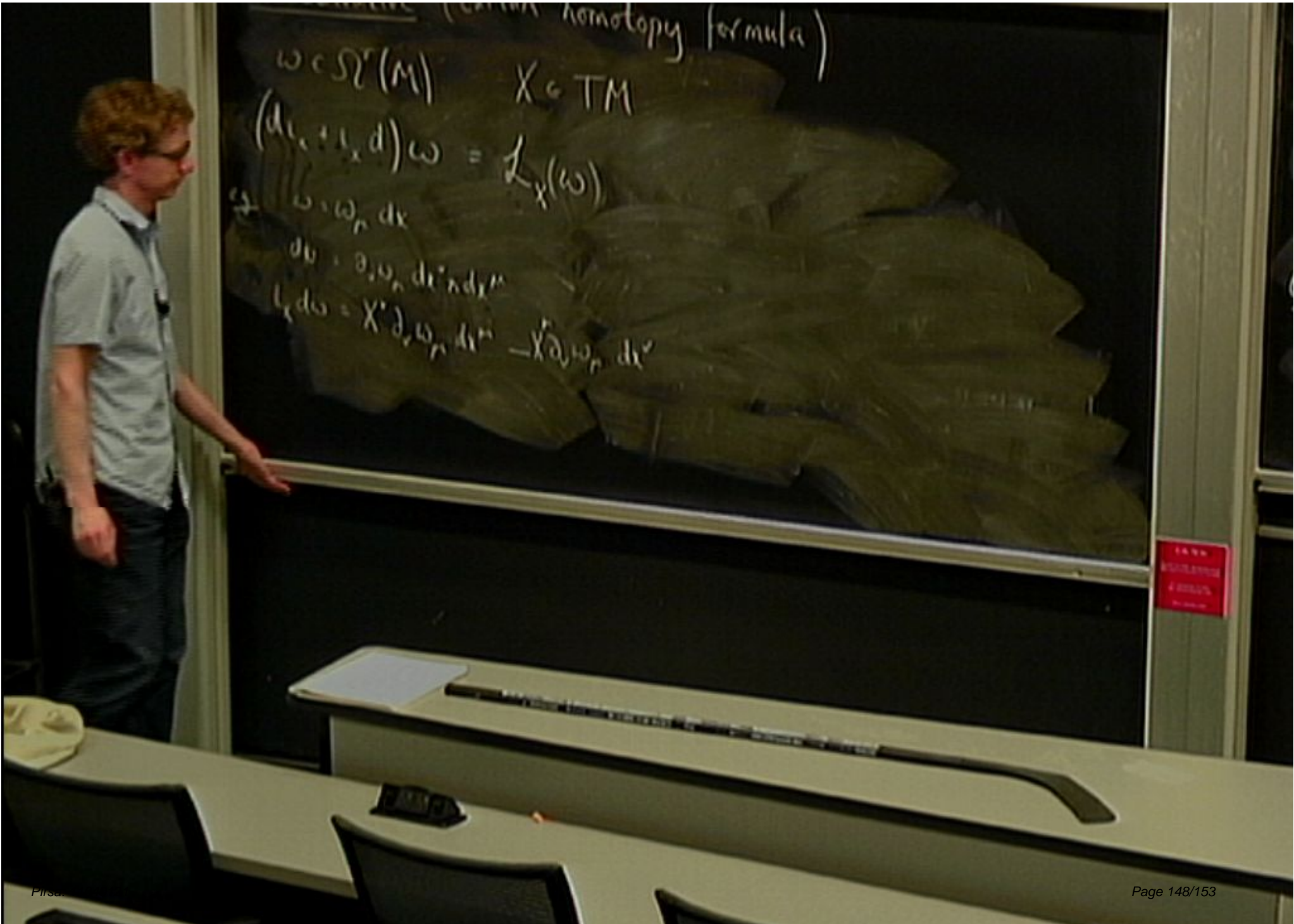
$$(d_X + i_X d)\omega$$

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Cartan homotopy formula
 $\omega \in \Omega^r(M)$ $X \in TM$
 $(d + L_X)d\omega = L_X(\omega)$

EXIT
EXIT
EXIT



(Cartan homotopy formula)

$$\omega \in \Omega^r(M) \quad X \in TM$$

$$(d_L + L_X d)\omega = L_X(\omega)$$

$$\text{eg } \omega = \omega_n dx$$

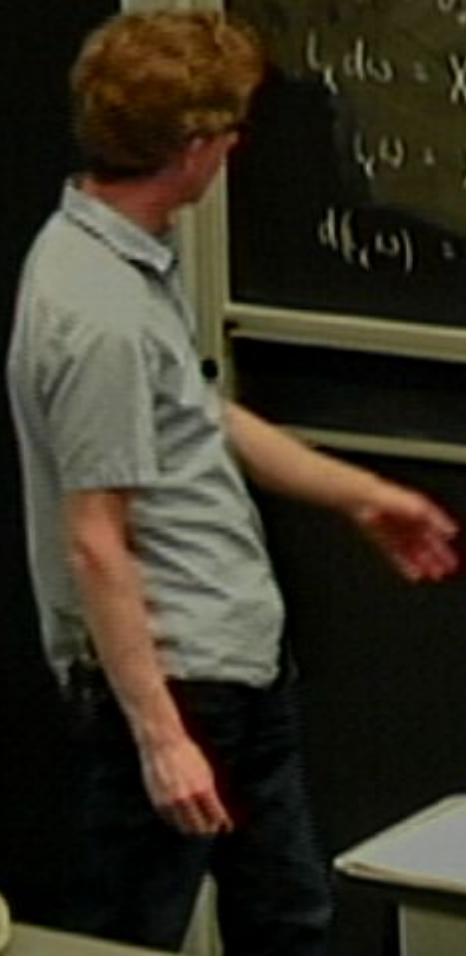
$$d\omega = \partial_r \omega_n dx^r dx^n$$

$$L_X d\omega = X^r \partial_r \omega_n dx^n - X^r \partial_r \omega_n dx^r$$

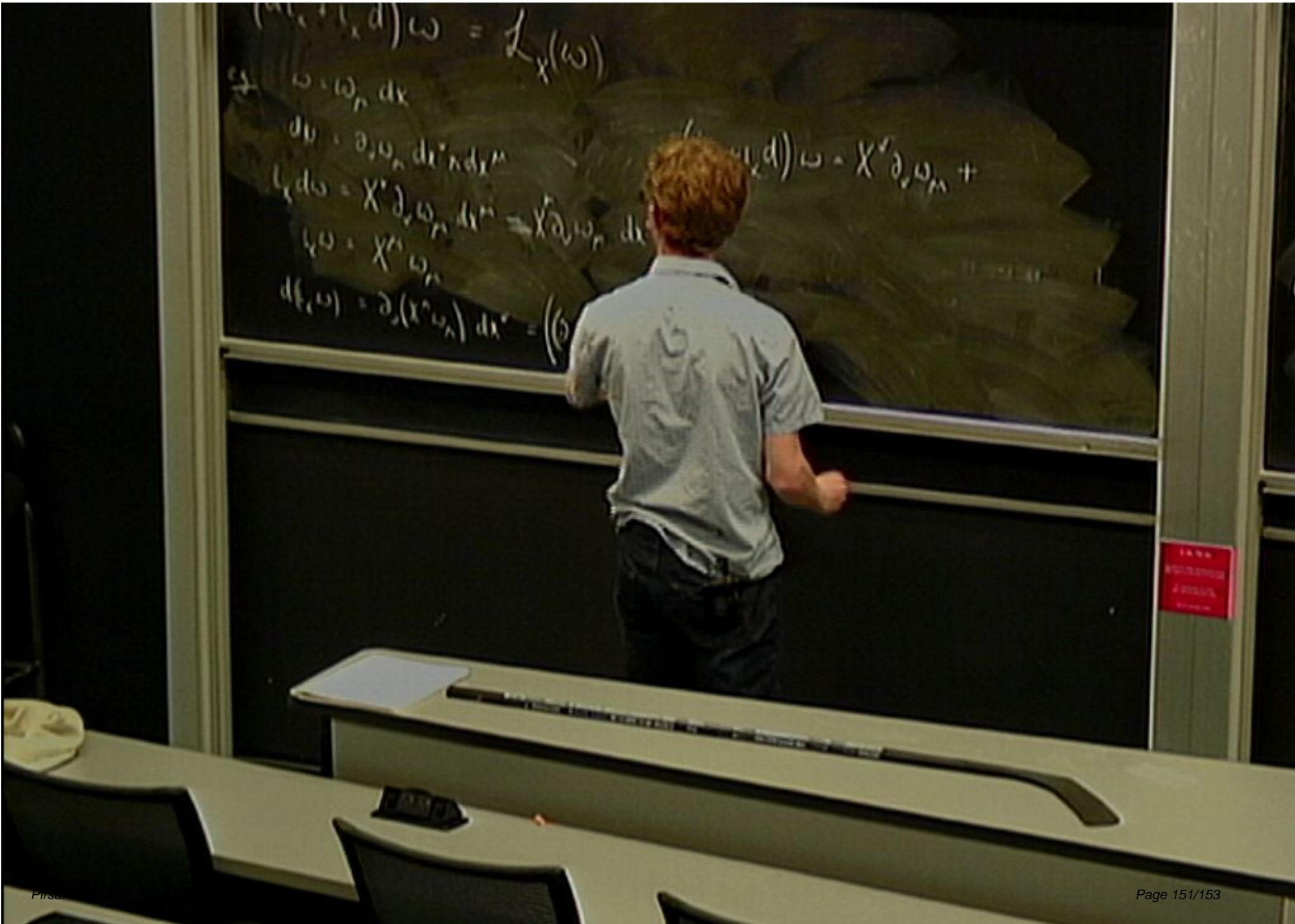
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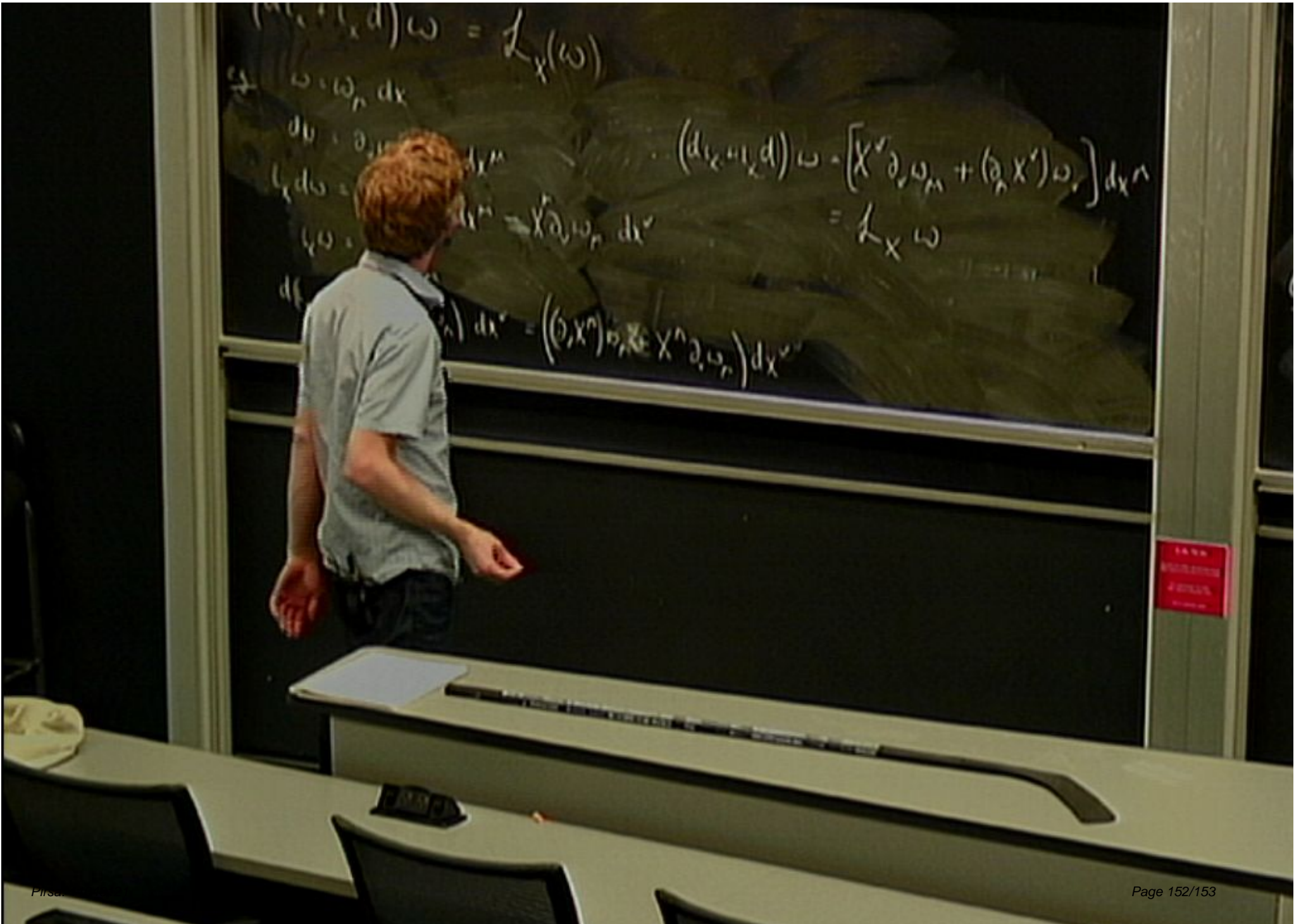


$$\begin{aligned}
 (x^m, \dots, x^d) \omega &= \int_{X^d} \omega \\
 \text{if } \omega &= \omega_n dx \\
 d\omega &= \partial_r \omega_n dx^r dx^1 \dots dx^n \\
 L_x d\omega &= X^r \partial_r \omega_n dx^1 \dots dx^n = X^r \partial_r \omega_n dx^1 \dots dx^n \\
 \psi \omega &= X^r \omega_n \\
 d(\psi \omega) &= \partial_r (X^r \omega_n) dx^1 \dots dx^n = \left((\partial_r X^r) \omega_n + X^r \partial_r \omega_n \right) dx^1 \dots dx^n
 \end{aligned}$$



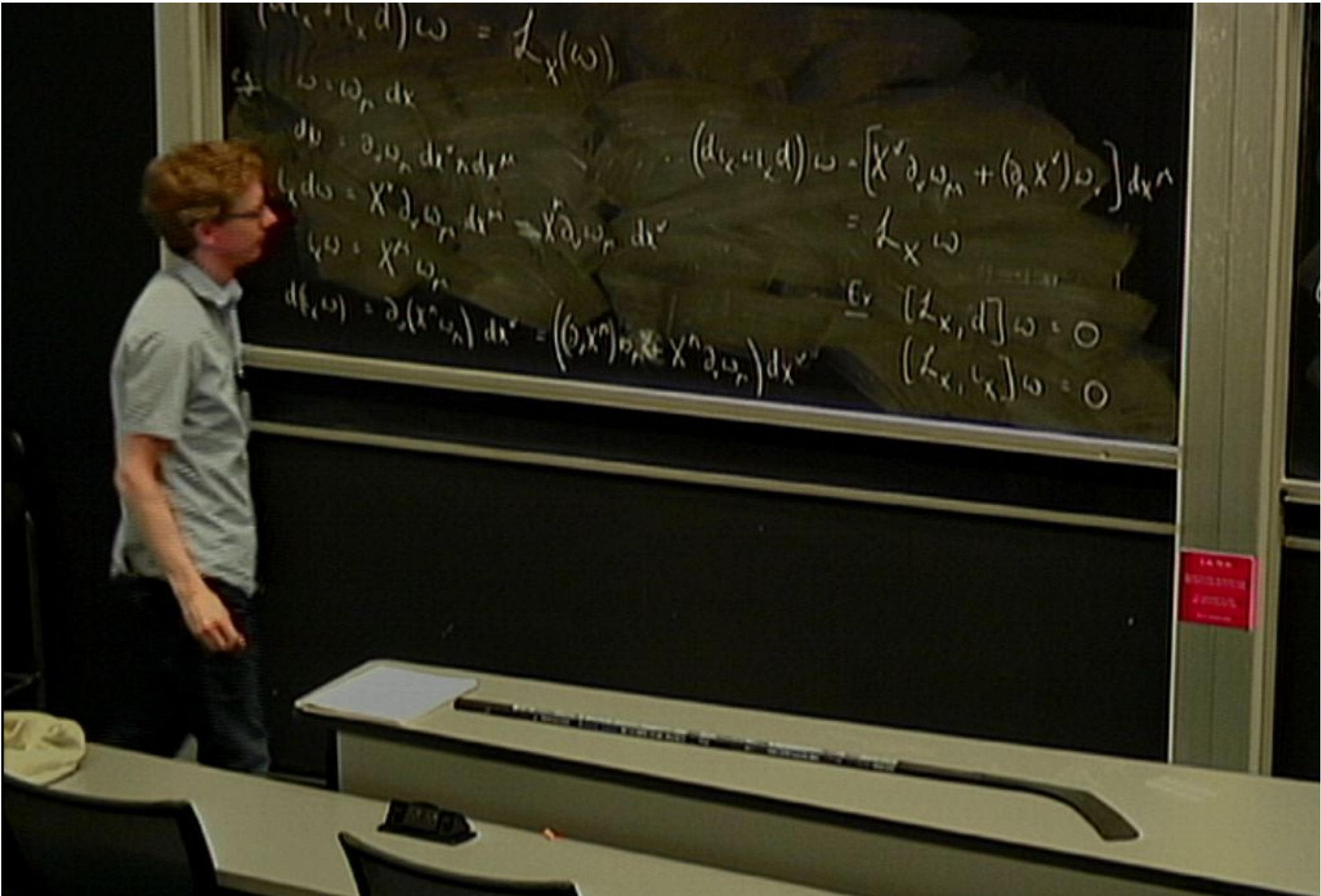
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$(dL_X + L_X d)\omega = L_X(\omega)$
 $\int \omega = \int \omega_n dx$
 $d\omega = \partial_r \omega_n dx^r dx^n$
 $L_X d\omega = \partial_r \omega_n dx^r dx^n + X^r \partial_r \omega_n dx^r dx^n$
 $L_X \omega = X^r \partial_r \omega_n dx^n$
 $(dL_X + L_X d)\omega = [X^r \partial_r \omega_n + (\partial_r X^r) \omega_n] dx^n = L_X \omega$
 $(\partial_r X^n) \omega_n + X^n \partial_r \omega_n$

NO
 NO
 NO
 NO



$(d_{L_X} + L_X d)\omega = \mathcal{L}_X(\omega)$
 cf. $\omega = \omega_n dx^1 \wedge \dots \wedge dx^n$
 $d\omega = \partial_\nu \omega_n dx^\nu \wedge dx^1 \wedge \dots \wedge dx^n$
 $L_X d\omega = X^\nu \partial_\nu \omega_n dx^1 \wedge \dots \wedge dx^n = X^\nu \partial_\nu \omega_n dx^\nu$
 $L_X \omega = X^\nu \omega_n dx^1 \wedge \dots \wedge dx^n$
 $d(L_X \omega) = \partial_\nu (X^\nu \omega_n) dx^\nu \wedge dx^1 \wedge \dots \wedge dx^n = \left((\partial_\nu X^\nu) \omega_n + X^\nu \partial_\nu \omega_n \right) dx^\nu \wedge dx^1 \wedge \dots \wedge dx^n$
 $(d_{L_X} + L_X d)\omega = \left[X^\nu \partial_\nu \omega_n + (\partial_\nu X^\nu) \omega_n \right] dx^1 \wedge \dots \wedge dx^n = \mathcal{L}_X \omega$
Ex $[\mathcal{L}_X, d]\omega = 0$
 $[\mathcal{L}_X, L_X]\omega = 0$

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