Title: Symmetric Informationally Complete POVMs in Prime Dimensions

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Abstract: A symmetric informationally complete positive-operator-valued measure (SIC POVM) is a special POVM that is composed of d^2 subnormalized pure projectors with equal pairwise fidelity. It may be considered a fiducial POVM for reasons of its high symmetry and high tomographic efficiency. Most known SIC POVMs are covariant with respect to the Heisenberg-Weyl (HW) group. We show that in prime dimensions the HW group is the unique group that may generate a SIC POVM. In particular, in prime dimensions not equal to three, each group covariant SIC POVM is covariant with respect to a unique HW group. In addition, the symmetry group of the SIC POVM is a subgroup of the Clifford group, which is the normalizer of the HW group. Hence, two SIC POVMs covariant with respect to the HW group are unitarily equivalent if and only if they are on the same orbit of the Clifford group. In dimension three, each group covariant SIC POVM may be covariant with respect to three or nine HW groups, and the symmetry group of the SIC POVM is a subgroup of at least one of the Clifford groups of these HW groups respectively. There may exist two or three orbits of equivalent SIC POVMs for each group covariant SIC POVM, depending on the order of its symmetry group.

Symmetric informationally complete POVMs in prime dimensions

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Introduction

Main results

Peculiarity in dimension three

Summary

Informationally complete measurements

- A general measurement in quantum mechanics is described by a positive-operator-valued measure (POVM), which is composed of a set of outcomes represented mathematically by positive operators Π_j satisfying $\sum_j \Pi_j = I$, where I is the identity operator.
- Given a state ρ, the probability of obtaining the outcome Π_j is given by the Born rule $p_j = \mathrm{tr}\rho\Pi_j$.
- A POVM is Informationally complete (IC) if we can reconstruct any state according to the statistics of measurement results, that is the set of probabilities of obtaining the outcomes.
- An IC POVM contains at least d^2 outcomes, an IC POVM is minimal if it contains exactly d^2 outcomes.

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Symmetric informationally complete measurements

- A symmetric informationally complete (SIC) POVM [Zauner99] $\sum_{j=1}^{d^2} \Pi_j = I$ is composed of d^2 outcomes that are subnormalized pure projectors, $\Pi_j = |\psi_j\rangle \frac{1}{d} \langle\psi_j|$, such that

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{1 + d\delta_{jk}}{d+1}.$$

- It may be considered as a fiducial POVM for reasons of its high symmetry and high tomographic efficiency.
- Easy state reconstruction

$$\rho = (d+1)\sum_j p_j |\psi_j\rangle\langle\psi_j| - I.$$

- Minimal mean square error given finite number N of measurements [Scott06]

$$e_{\mathrm{MSE}}(\rho) = \frac{1}{N}(d^2 + d - 1 - \mathrm{tr}\rho^2).$$

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Facts about SIC POVMs

- In dimension two, the four outcomes of the SIC POVM form a regular tetrahedron when represented on the Bloch sphere.
- Analytic solutions of SIC POVMs are known for dimension 2 to 15, 19, 24, 35, 48.
- Numerical solutions with high precision are known for dimension 2 to 67.
- SIC POVMs are closely related to mutually unbiased bases, equiangular lines, t-design and Lie algebra.
- SIC POVMs also play an important role in understanding the foundation of quantum mechanics.

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Symmetry group of a SIC POVM

- The symmetry group G_{sym} of a SIC POVM is composed of all unitary operators that leave the SIC POVM invariant. The extended symmetry group may also contain antiunitary operators.
- Its collineation group $\overline{G}_{\mathrm{sym}}$, which is obtained by identifying operators differing only by overall phase factors, is also called the symmetry group.
- Suppose G is a unimodular unitary group (unimodular means all elements have determinant 1), then $|G| \leq d|\overline{G}|$, where $|G|$ ($|\overline{G}|$) denotes the order of G ($\overline{G}$). Moreover, there exists a unique unimodular unitary group G' that satisfies $\overline{G'} = \overline{G}$ and $|G'| = d|\overline{G}|$ [Blichfeldt17].
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Group covariant SIC POVMs

- A SIC POVM is group covariant if it can be generated from a single state—fiducial state—by a group of unitary operators.
- Almost all known SIC POVMs are covariant with respect to the Heisenberg-Weyl (HW) group D, which is generated by the phase operator Z and the cyclic shift operator X :

$$\begin{aligned} Z|e_r\rangle &= \omega^r|e_r\rangle, \quad \omega = e^{2\pi i/d}, \\ X|e_r\rangle &= |e_{r-1 \mod d}\rangle, \\ D_{k_1,k_2} &= \tau^{k_1 k_2} X^{k_1} Z^{k_2}, \quad \tau = -e^{\pi i/d}. \end{aligned}$$

X and Z obey the following commutation relation

$$ZXZ^{-1}X^{-1} = \omega I.$$

D is a nonabelian group of order d^3, while $\overline{D}$ is an abelian group of order d^2.

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Clifford group

- The Clifford group $\mathrm{C}(d)$ is the normalizer of the HW group which is composed of unitary operations; in other words, a Clifford unitary $U \in \mathrm{C}(d)$ implements the following transformation:

$$UD_{k_1,k_2}U^\dagger = e^{i\phi}D_{k_1',k_2'}.$$

- The extended Clifford group $\mathrm{EC}(d)$ is the larger group which contains also antiunitary operations. It can be generated by the Clifford group and the complex-conjugation operation.
- Since a fiducial state $|\psi\rangle$ of the HW group satisfies

$$|\langle\psi|D_{k_1,k_2}|\psi\rangle| = \frac{1}{\sqrt{d+1}} \quad (k_1,k_2) \neq (0,0),$$

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Orbits of SIC POVMs

- Fiducial states and SIC POVMs form disjoint orbits under the action of the (extended) Clifford group.
- The stabilizer S (also called stability group) of a fiducial state is composed of all unitary (or antiunitary) operators within the (extended) Clifford group that leave the fiducial state invariant. The number of fiducial states on each orbit is equal to $|\overline{\mathrm{C}}(d)|/|\overline{S}|$ ($|\overline{\mathrm{EC}}(d)|/|\overline{S}|$).
- In **dimension three**, there exists **a continuous family of orbits**. In other dimensions, at most a finite number of orbits are known.

Table: Orbits of SIC POVMS for $d = 2, 3, \ldots 50$ [Scott and Grassl arXiv:0910.5784]. Here # denotes the number of orbits and $|\bar{S}|$ denotes the order of the stabilizer.

d	#	$\lvert\bar{S}\rvert$
2	1	6
	∞	6
3	1	12
	1	48
4	1	6
5	1	3
6	1	3
7	1	3
	1	6
8	1	3
	1	12
9	2	3
10	1	3
11	3	3
12	1	3
	1	6
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d	#	$\lvert\bar{S}\rvert$
14	2	3
15	3	3
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16	2	3
17	3	3
18	2	3
19	3	3
	1	6
	1	18
20	2	3
21	4	3
	1	3
22	1	3
23	6	3
24	2	3
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25	2	3

d	#	$\lvert\bar{S}\rvert$
26	4	3
27	6	3
28	2	3
	1	6
29	4	3
30	3	3
	1	3
31	7	3
32	2	3
33	4	3
34	2	3
	8	3
35	1	6
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43	6	3
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46	3	3
47	8	3
48	4	3
	1	3
	1	6
	1	24
49	7	3
50	2	3

Open questions

1. Are there group covariant SIC POVMs that are not covariant with respect to the HW group?
2. Are there non-Clifford unitaries that can map an HW covariant SIC POVM to itself?
3. Can two SIC POVMs on two different orbits of the Clifford group be equivalent?

   Two SIC POVMs are equivalent if they can be transformed into each other with unitary (or antiunitary) operations.

We shall give **negative answers** to all three questions when **the dimension is a prime p not equal to three**. Dimension three is quite peculiar and will be discussed at the end of this talk. Since dimension two is trivial, we shall assume the dimension is an **odd prime**.

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Special linear group $\mathrm{SL}(2, p)$

- The structure of the Clifford group in prime dimensions plays a crucial role in answering the above questions. To understand this, we need to introduce the special linear group $\mathrm{SL}(2, p)$, which is composed of 2×2 matrices with entries from the finite field Z_p and with determinant 1,

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

- The extended special linear group $\mathrm{ESL}(2, p)$ is the larger group which contains also 2×2 matrices with determinant -1. It can be written as a union of two cosets, that is $\mathrm{ESL}(2, p) = \mathrm{SL}(2, p) \cup J\mathrm{SL}(2, p)$, where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The structure of the (extended) Clifford group

- Define $\mathrm{ESL}(2,p) \ltimes (Z_p)^2$ as the semidirect product of $\mathrm{ESL}(2,p)$ and $(Z_p)^2$ according to the following product rule:

$$(F_1, \chi_1) \circ (F_2, \chi_2) = (F_1 F_2, \chi_1 + F_1 \chi_2),$$

where $F_1, F_2 \in \mathrm{ESL}(2,p)$, and $\chi_1, \chi_2 \in (Z_p)^2$.

- The structure of the extended Clifford group is best understood from the following isomorphism [Appleby05],

$$\begin{aligned} f_E \quad &: \quad \mathrm{ESL}(p) \ltimes (Z_p)^2 \to \overline{\mathrm{EC}}(p), \\ & \quad UD_{\mathbf{k}}U^\dagger = \omega^{\langle \chi, F\mathbf{k} \rangle} D_{F\mathbf{k}} \qquad \text{for} \quad U = f_E(F, \chi), \end{aligned}$$

where $\langle \mathbf{k}, \mathbf{q} \rangle = k_2 q_1 - k_1 q_2$.

- Explicit correspondence when $\det(F) = 1$:
  $(F, \chi) \to U = D_\chi V_F$

$$V_F = \frac{1}{\sqrt{p}} \sum_{r,s=0}^{p-1} \tau^{\beta^{-1}(\alpha s^2 - 2rs + \delta r^2)} |e_r\rangle\langle e_s|, \quad \beta \neq 0,$$

$$V_F = \sum_{s=0}^{p-1} \tau^{\alpha\gamma s^2} |e_{\alpha s}\rangle\langle e_s|, \quad \beta = 0.$$

- $(J, \mathbf{0})$ ($J = \mathrm{diag}(1, -1)$) is mapped to the complex conjugation operator.
- $[F, \chi]$ will be used to denote the isomorphism image of (F, χ).

p-group

A p-group for a prime p is a nontrivial group whose order is a power of p.

1. Any p-group has a nontrivial centre.
2. The normalizer of any proper subgroup of a p-group is strictly larger than this subgroup.
3. Any irreducible representation of a p-group is monomial.

A unitary matrix is monomial if each row (each column) has exactly one nonzero entry. A monomial matrix can be written as a product of a permutation matrix and a diagonal matrix.

A representation is monomial if the images of all elements in the group are monomial in a suitable basis.

p-group

A p-group for a prime p is a nontrivial group whose order is a power of p.

1. Any p-group has a nontrivial centre.
2. The normalizer of any proper subgroup of a p-group is strictly larger than this subgroup.
3. Any irreducible representation of a p-group is monomial.

A unitary matrix is monomial if each row (each column) has exactly one nonzero entry. A monomial matrix can be written as a product of a permutation matrix and a diagonal matrix.

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Sylow's theorem

Suppose G is a finite group whose order $|G|$ is divisible by p^k but not divisible by p^{k+1} ($k \geq 1$). A Sylow p-subgroup P of G is any subgroup of order p^k.

Sylow's theorem For any prime factor p of $|G|$:

1. there exists a Sylow p-subgroup of G;
2. the number of Sylow p-subgroups is equal to $1 \mod p$ and divides $|G/P|$;
3. all Sylow p-subgroups are conjugated to each other;
4. any p-subgroup of G is contained in one of the Sylow p-subgroups.

Algebraic integer

In number theory, an algebraic integer is a root of a monic polynomial with integer coefficents (monic means that the leading coefficient is 1).

- The roots of unity are algebraic integers.
- The set of algebraic integers is closed under addition, multiplication and complex conjugation.
- A fraction is not an algebraic integer if it is not an integer.

Question 1

Are there group covariant SIC POVMs that are not covariant with respect to the HW group?

Group covariant SIC POVMs are HW covariant in prime dimensions

Proposition If a SIC POVM is covariant with respect to a unimodular unitary group G, then G is necessarily nonabelian.

Proof.

1. Suppose otherwise, we can assume G is diagonal.
2. Let $|\psi_j\rangle = (a_{j1}, a_{j2}, \ldots, a_{jd})^T$ for $j = 1, 2, \ldots, d^2$ be the d^2 fiducial vectors, then $|a_{jk}|$ is independent of j.
3. The condition $\sum_{j=1}^{d^2} |\psi_j\rangle\langle\psi_j| = dI$ further implies that $|a_{jk}| = 1/\sqrt{d}$.
4. Suppose $U = \mathrm{diag}(u_1, \ldots, u_d) \in G$ does not stabilize $|\psi_1\rangle$,

$$d^2 |\langle\psi_1|U|\psi_1\rangle|^2 = \left|\sum_{j=1}^{d} u_j\right|^2 = \frac{d^2}{d+1}.$$

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Proposition In a prime dimension p, suppose a SIC POVM is covariant with respect to a unimodular unitary group G, then it is also covariant with respect to any Sylow p-subgroup, say P, of G. Moreover, P has order at least p^3 and it is irreducible when taken as a representation of itself.

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1. Since $|G/P|$ is not divisible by p and the number of fiducial vectors in the SIC POVM is a power of p, the SIC POVM is also covariant with respect to P.
2. P is nonabelian according to the previous proposition.
3. The order of P is at least p^3, since all p-groups of order p or p^2 are abelian.
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Proof. Suppose the SIC POVM is covariant with respect to the unimodular unitary group G and P is a Sylow p-subgroup of G. We shall show that P contains the HW group (possibly in a different basis).

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Question 2

Are there non-Clifford unitaries that can map an HW covariant SIC POVM to itself?

Proposition Suppose G_{sym} is the symmetry of a group covariant SIC POVM in a prime dimension not equal to three, then G_{sym} contains the HW group as a Sylow p-subgroup.

Proof.

1. Suppose otherwise, then D is a proper subgroup of one of the Sylow p-subgroups of G_{sym}, say P.
2. The normalizer $N_P(D)$ of D in P must be strictly larger than D and thus contains a p-group P' of order p^4 which in turn contains D as a normal subgroup.
3. The group P' is also a Sylow p-subgroup of the Clifford group, which can be identified with the Sylow p-subgroup generated by X, Z and

$$V = \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right] = \mathrm{diag}(1, \tau, \tau^4, \ldots, \tau^{(p-1)^2}).$$

5. Let H be the order-p^3 subgroup of P' generated by $\omega I, Z$ and V. Since the abelian group H cannot generate all p^2 fiducial vectors from a given fiducial vector, say $|\psi\rangle$, the stabilizer of $|\psi\rangle$ in H has order p^2 and thus contains an element of the form $V^j Z^k$ with $1 \leq j \leq p-1$ and $0 \leq k \leq p-1$.
6. Since $V^j Z^k$ is diagonal, it stabilizes all p fiducial vectors of the form $Z^l|\psi\rangle$ for $l = 0, \ldots, p-1$.
7. The p fiducial vectors must belong to a same eigenspace of $V^j Z^k$, otherwise they would be orthogonal.
8. However, each eigenspace of $V^j Z^k$ has dimention at most two and cannot admit more than two fiducial vectors when $\boldsymbol{p \geq 5}$. This contradiction completes the proof that the HW group is a Sylow-p subgroup of the symmetry group.

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Sibley's Theorem Suppose G is a finite group with a faithful, irreducible, unimodular and quasiprimitive representation of prime degree $p \geq 5$. If a Sylow p-subgroup P of G has order p^3, then P is normal in G, and G/P is isomorphic to a subgroup of $\mathrm{SL}(2, p)$.

- A quasiprimitive representation is one whose restriction to every subgroup is homogeneous, that is a multiple of one irreducible representation of the subgroup.
- An irreducible representation of a prime degree that is not quasiprimitive is monomial

Theorem In any prime dimension not equal to three, each group covariant SIC POVM is covariant with respect to a unique HW group. Furthermore, its symmetry group is a subgroup of the Clifford group.

Proof.

1. According to Sibley's theorem, if G_{sym} is quasiprimitive, then D is a normal Sylow p-subgroup of G_{sym}, in other words, G_{sym} is a subgroup of the Clifford group of D. In addition, G_{sym} contains only one HW group according to Sylow's theorem.

2. If G_{sym} is monomial, to show that D is a normal subgroup of G_{sym}, it suffices to show that $UZU^\dagger, UXU^\dagger \in D$ for any $U \in G_{\mathrm{sym}}$.
   - 2.1 Let T be the normal subgroup of G_{sym} that is composed of of its diagonal elements. One can show that $T = \langle \omega I, Z \rangle$.
   - 2.2 Since $T = \langle \omega I, Z \rangle$ is a normal subgroup of G_{sym}, $UZU^\dagger = \omega^{k_1} Z^{k_2}$ for some integers k_1, k_2.
   - 2.3 The following equality

$$U^\dagger XUZ(U^\dagger XU)^\dagger = \omega^{-k_2} Z$$

   implies that $X^{-k_2} U^\dagger XU \in T \in D$, hence $U^\dagger XU \in D$.
3. In conclusion, G_{sym} is a subgroup of the Clifford group and contains only one HW group.

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Corollary For any HW covariant SIC POVM in a prime dimension p not equal to three, the stabilizer of each fiducial vector within the Clifford group is either a cyclic group or a dicyclic group; its order is no larger than $2(p+1)$.

A dicyclic group is a group of order $4n$ (n is a positive integer) that is characterized by the presentation

$$\langle a, b | a^{2n} = 1,\ b^2 = a^n,\ bab^{-1} = a^{-1} \rangle.$$

Question 3

Can two SIC POVMs on two different orbits of the (extended) Clifford group be equivalent?

Theorem In any prime dimension not equal to three, two SIC POVMs covariant with respect to the HW group are unitarily (unitarily or antiunitarily) equivalent if and only if they are on the same orbit of the (extended) Clifford group.

Since each SIC POVM is covariant with respect to a unique HW group, any unitary (antiunitary) operation that maps one SIC POVM to the other must leave the HW group invariant and hence belongs to the (extended) Clifford group.

Peculiarity of SIC POVMs in dimension three

SIC POVMs in dimension three

- There is a one-parameter family of fiducial vectors in dimension three [Appleby05],

$$|\psi_{\mathrm{f}}(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -\mathrm{e}^{\mathrm{i}t} \end{pmatrix}.$$

  They are in one-to-one correspondence with distinct orbits for $0 \leq t \leq \pi/3$.
- There are three kinds of orbits, two exceptional orbits corresponding to $t = 0$ and $t = \frac{\pi}{3}$, respectively; infinitely many generic orbits corresponding to $0 < t < \frac{\pi}{3}$.
- The number of SIC POVMs on each of the three kinds of orbits is 1, 4, 8, respectively.

Additional HW groups in the Clifford group

- In the Clifford group, there are other subgroups that are unitarily equivalent to the standard HW group. They will also be referred to as HW groups.
- There are $p+1$ Sylow p-subgroups in the Clifford group $\mathrm{C}(p)$, each one has order p^4.
- Each Sylow p-subgroup contains p HW groups, including the standard HW group. The total number of HW groups in the Clifford group is p^2.

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Peculiarity in dimension three

Theorem In dimension three, each group covariant SIC POVM may be covariant with respect to 3 or 9 HW groups. Its (extended) symmetry group is a subgroup of at least one of the (extended) Clifford groups associated with these HW groups respectively. There may exist two or three orbits of equivalent SIC POVMs depending on the order of their symmetry group.

Suppose the symmetry group G_{sym} is **not monomial**. According to the classification of finite linear groups of degree 3 by Blichfeldt, $|G_{\mathrm{sym}}|$ is not divisible by p^5 (it is divisible by p^3).

1. If $|G_{\mathrm{sym}}|$ is not divisible by p^4, there is a counterexample to Sibley's theorem—a unimodular unitary group of order 1080 whose collineation group (of order 360) is isomorphic to the alternating group of six letters. However, no SIC POVM can be covariant with respect to this group.
2. Otherwise, G_{sym} is a subgroup of the Clifford group.
3. Since all proper subgroups of the Clifford group are monomial when $p = 3$, G_{sym} is exactly the Clifford group itself, which contains nine HW groups.
4. Only the two orbits of SIC POVMs generated from the fiducial vectors $(0, 1, -1)^T/\sqrt{2}$ and $(0, 1, -e^{2\pi i/9})^T/\sqrt{2}$, respectively, have this symmetry.

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Suppose the symmetry group G_{sym} is **monomial**.

1. According to a similar reasoning as in the case $p \geq 5$, G_{sym} is also a subgroup of certain Clifford group (not necessarily the Clifford group of the standard HW group).
2. The symmetry group $\overline{G}_{\mathrm{sym}}$ has order either 27 or 54; it contains three HW groups in either case.
3. Only the two orbits of SIC POVMs generated from the fiducial vectors $(0, 1, -e^{\pi i/9})^T/\sqrt{2}$ and $(0, 1, -e^{\pi i/3})^T/\sqrt{2}$, respectively, have symmetry group of order 54.
4. When $0 < t < \frac{\pi}{9}$, SIC POVMs on the three orbits corresponding to the parameters t, $\frac{2\pi}{9} + t$, $\frac{2\pi}{9} - t$, respectively, are unitarily equivalent. All these SIC POVMs have symmetry group of order 27.

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2. The symmetry group $\overline{G}_{\mathrm{sym}}$ has order either 27 or 54; it contains three HW groups in either case.
3. Only the two orbits of SIC POVMs generated from the fiducial vectors $(0, 1, -\mathrm{e}^{\pi \mathrm{i}/9})^T/\sqrt{2}$ and $(0, 1, -\mathrm{e}^{\pi \mathrm{i}/3})^T/\sqrt{2}$, respectively, have symmetry group of order 54.
4. When $0 < t < \frac{\pi}{9}$, SIC POVMs on the three orbits corresponding to the parameters t, $\frac{2\pi}{9} + t$, $\frac{2\pi}{9} - t$, respectively, are unitarily equivalent. All these SIC POVMs have symmetry group of order 27.

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Summary

- We prove that each group covariant SIC POVM is covariant with respect to the HW group if the dimension is a prime.
- In any prime dimension not equal to three, each group covariant SIC POVM is covariant with respect to a unique HW group; its symmetry group is a subgroup of the Clifford group. SIC POVMs on different orbits are not equivalent.
- In dimension three, each group covariant SIC POVM may be covariant with respect to 3 or 9 HW groups; its symmetry group is a subgroup of at least one of the Clifford groups associated with these HW groups, respectively. There may exist 2 or 3 orbits of equivalent SIC POVMs, depending on the order of its symmetry group.

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Thank You!