

Title: Theory of quantum Hall effect in bilayer graphene

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Abstract: Utilizing the Baym-Kadanoff formalism with the polarization function calculated in the random phase approximation, the dynamics of the  $\hat{A}_{\tilde{Z}} \& \frac{1}{2} = 0, \hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4$  quantum Hall states in bilayer graphene is analyzed. In particular, in the undoped graphene, corresponding to the  $\hat{A}_{\tilde{Z}} \& \frac{1}{2} = 0$  state, two phases with nonzero energy gap, the ferromagnetic and layer asymmetric ones, are found. The phase diagram in the plane  $(\hat{A}_{\tilde{Z}}, \hat{B})$ , where  $\hat{A}_{\tilde{Z}} \& \frac{1}{2} = 0$  is a top-bottom gates voltage imbalance, is described. It is shown that the energy gaps in these phases scale linearly,  $\hat{A}_{\tilde{Z}} \& \frac{1}{2} = 0 \sim 10 B [T] K$ , with magnetic field. The ground states of the doped states, with  $\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4$ , are also described. The comparison of these results with recent experiments in bilayer graphene is presented.

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# Theory of Quantum Hall Effect in Bilayer Graphene

Volodya Miransky

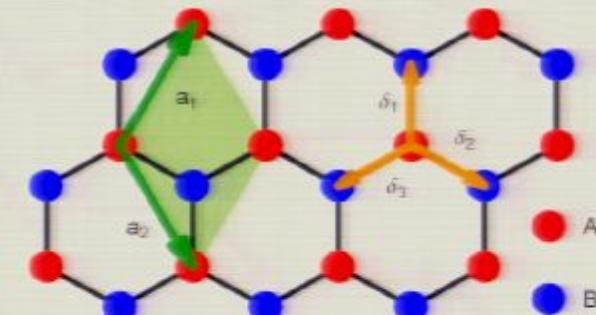
University of Western Ontario

# Lattice in coordinate & reciprocal space

- Translation vectors

$$\mathbf{a}_1 = a \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad \mathbf{a}_2 = a \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right)$$

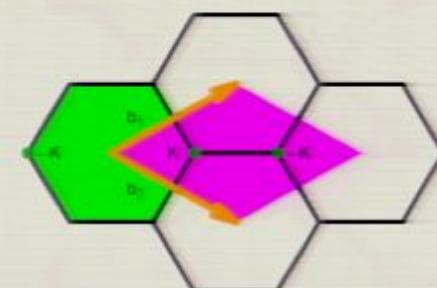
where  $a$  is the lattice constant



- Two carbon atoms per primitive cell

- Reciprocal lattice vectors

$$\mathbf{b}_1 = 2\pi/a(1, 1/\sqrt{3}), \quad \mathbf{b}_2 = 2\pi/a(1, -1/\sqrt{3})$$



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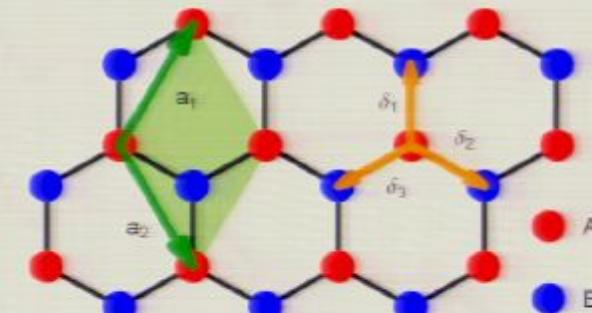
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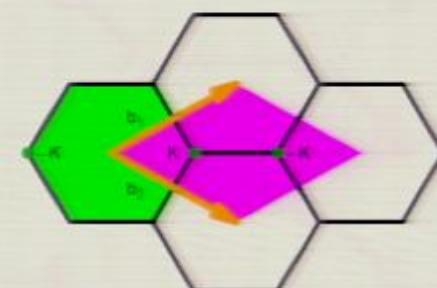
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# Quantum Hall Ferromagnetism (QHF)

Nomura & MacDonald, Phys. Rev. Lett. **96**, [256602](#) (2006)

Goerbig, Moessner & Douçot, Phys. Rev. B **74**, [161407\(R\)](#) (2006)

Alicea & Fisher, Phys. Rev. B **74**, [075422](#) (2006)

- Spin/valley degeneracy of the half-filled Landau level is lifted by the exchange (repulsive Coulomb) interaction
- This is similar to the **Hund's Rule(s)** in atomic physics
- Lowest energy state: the wave function is **antisymmetric** in coordinate space (electrons are as far apart as possible), i.e., it is **symmetric** in spin (or valley) indices

# Quantum Hall effect in graphene

[1] Zheng & Ando, PRB 65, [245420](#) (2002)

[2] Gusynin & Sharapov, PRL 95, [146801](#) (2005)

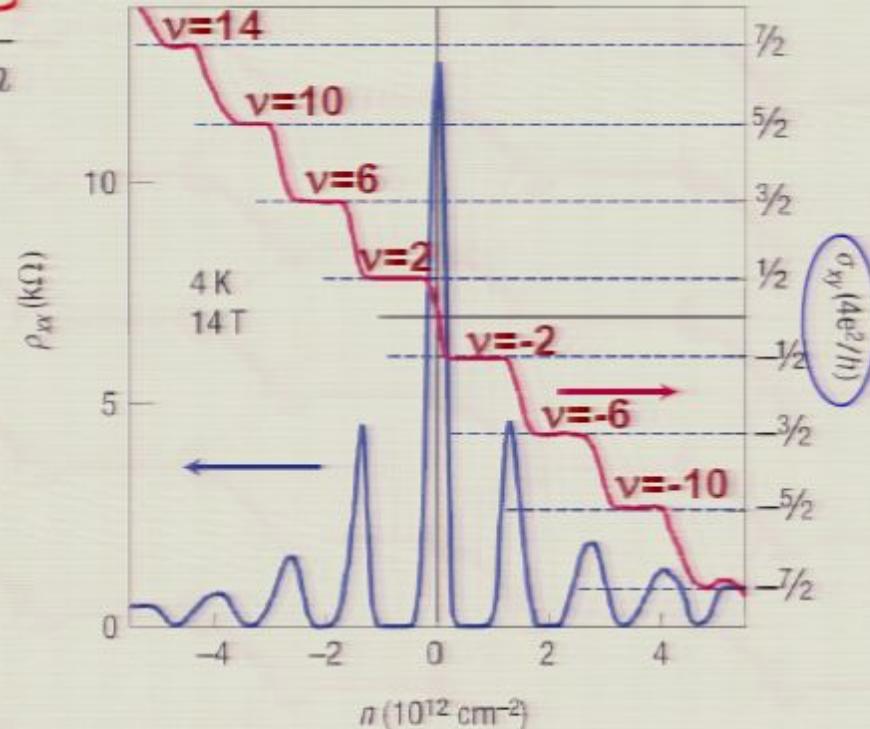
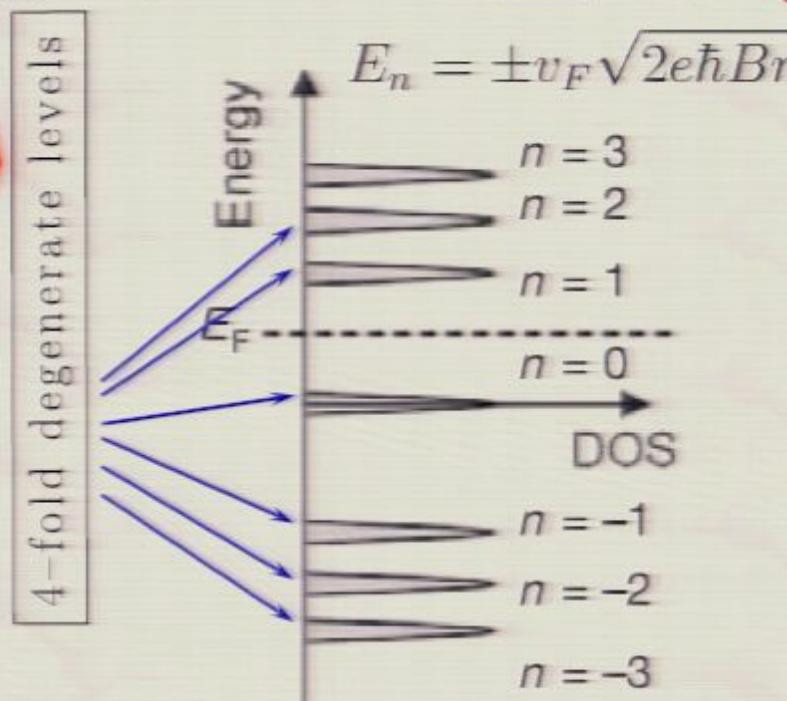
[3] Peres, Guinea, & Castro Neto, PRB 73, [125411](#) (2006)

[4] Novoselov et al., Nature 438, [197](#) (2005)

[5] Zhang et al., Nature 438, [201](#) (2005)

$$\sigma_{xy} = \frac{v e^2}{h} = \frac{4e^2}{h} \left( n + \frac{1}{2} \right)$$

} Experiment





# Quantum Hall Ferromagnetism (QHF)

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# Magnetic catalysis in QHE in graphene

- Gusynin, V.M., Sharapov, Shovkovy, PRB **74**, 195429 (2006)
- Herbut, PRL **97**, 146401 (2006)
- Fuchs & Lederer **98**, 016803 (2007)
- Ezawa, J. Phys. Soc. Jpn. **76**, 094701 (2007)

# Magnetic catalysis (MC) scenario

VOLUME 73, NUMBER 26

PHYSICAL REVIEW LETTERS

26 DECEMBER 1994

## Catalysis of Dynamical Flavor Symmetry Breaking by a Magnetic Field in 2 + 1 Dimensions

V. P. Gusynin,<sup>1</sup> V. A. Miransky,<sup>1,2</sup> and I. A. Shovkovy<sup>1</sup>

<sup>1</sup>*Bogolyubov Institute for Theoretical Physics, 252143 Kiev, Ukraine*

<sup>2</sup>*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030*

(Received 11 May 1994)

It is shown that in 2 + 1 dimensions, a constant magnetic field is a strong catalyst of dynamical flavor symmetry breaking, leading to generating a fermion dynamical mass even at the weakest attractive interaction between fermions. The effect is illustrated in the Nambu-Jona-Lasinio model in a magnetic field. The low-energy effective action in this model is derived, and the thermodynamic properties of the model are established.

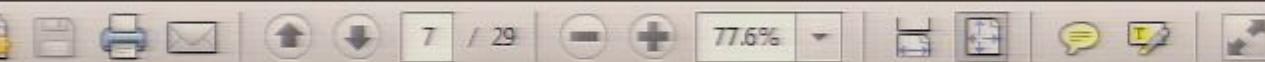
$$E_n = \sqrt{2n|eB|} \Rightarrow E_n = \sqrt{2n|eB| + \Delta_0^2}$$

where  $\Delta_0 \sim \begin{cases} \sqrt{|eB|} \\ \text{or} \\ |eB| \end{cases} \Rightarrow v=0$

In relation to graphene:

Khveshchenko, Phys. Rev. Lett. 87, [206401](#) (2001);

Gorbar, Gusynin, V. M., & Shovkovy, Phys. Rev. B [66](#), [045108](#) (2002)



# General Approach

## Model Hamiltonian

[Gorbar, Gusynin, V. M., Shovkovy, Phys. Rev. B 78 (2008) 085437]

$$H = H_0 + H_C + \int d^2\mathbf{r} [\underbrace{\mu_B B \Psi^\dagger \sigma^3 \Psi}_{\text{Zeeman term}} - \mu_0 \Psi^\dagger \Psi]$$

where

$$H_0 = v_F \int d^2\mathbf{r} \overline{\Psi} (\gamma^1 \pi_x + \gamma^2 \pi_y) \Psi,$$

is the Dirac Hamiltonian, and

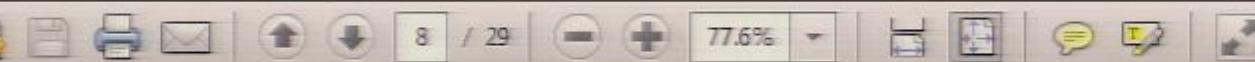
$$H_C = \frac{1}{2} \int d^2\mathbf{r} d^2\mathbf{r}' \Psi^\dagger(\mathbf{r}) \Psi(\mathbf{r}) U_C(\mathbf{r} - \mathbf{r}') \Psi^\dagger(\mathbf{r}') \Psi(\mathbf{r}')$$

is the Coulomb interaction term.

Note that  $\Psi_s^T = (\psi_{KAs}, \psi_{KBs}, \psi_{K'Bs}, \psi_{K'A s}), \quad s = \pm$

*Spin index*

$$v_F \approx 10^6 \text{ m/s}$$



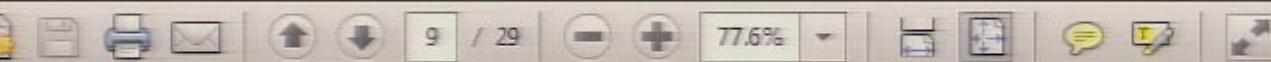
# Symmetry

- The Hamiltonian  $H = H_0 + H_C$  possesses “flavor”  $U(4)$  symmetry (no Zeeman term)
- 16 generators read (*spin*  $\otimes$  *valley*)

$$\frac{\sigma^\alpha}{2} \otimes I_4, \quad \frac{\sigma^\alpha}{2i} \otimes \gamma_3, \quad \frac{\sigma^\alpha}{2} \otimes \gamma_5 \text{ and } \frac{\sigma^\alpha}{2} \otimes \gamma_3 \gamma_5$$

where  $\gamma^3 \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$  – diagonal valley matrix

- A ferromagnetic like order parameter breaks spin degeneracy.  
Thus,  $U(4)$  breaks down to  $U^{(+)}(2)_v \times U^{(-)}(2)_v$
- QHF order parameter and/or Dirac mass breaks  $U(4)$  down to  $U^{(K)}(2)_s \times U^{(K')}(2)_s$



# Full propagator

- One can use the following general ansatz:

$$iG_s = \left[ (i\hbar\partial_t + \mu_s + \tilde{\mu}_s \gamma^3 \gamma^5) \gamma^0 - v_F (\boldsymbol{\pi} \cdot \boldsymbol{\gamma}) - \tilde{\Delta}_s + \Delta_s \gamma^3 \gamma^5 \right]^{-1}$$

Electron chemical potential

"Pseudospin" chemical potential

Dirac mass

T-odd mass

- Physical meaning of the order parameters

$$\mu_3 : \langle \Psi^\dagger \sigma^3 \Psi \rangle = \sum_{\kappa=K,K'} \sum_{a=A,B} \langle \psi_{\kappa a+}^\dagger \psi_{\kappa a+} - \psi_{\kappa a-}^\dagger \psi_{\kappa a-} \rangle$$

$$\tilde{\mu}_s : \langle \Psi^\dagger \gamma^3 \gamma^5 P_s \Psi \rangle = \langle \psi_{KAs}^\dagger \psi_{KAs} - \psi_{K'As}^\dagger \psi_{K'As} + \psi_{KBs}^\dagger \psi_{KBs} - \psi_{K'Bs}^\dagger \psi_{K'Bs} \rangle$$

$$\Delta_s : \langle \bar{\Psi} \gamma^3 \gamma^5 P_s \Psi \rangle = \langle \psi_{KAs}^\dagger \psi_{KAs} - \psi_{KBs}^\dagger \psi_{KBs} - (\psi_{K'As}^\dagger \psi_{K'As} - \psi_{K'Bs}^\dagger \psi_{K'Bs}) \rangle$$

$$\tilde{\Delta}_s : \langle \bar{\Psi} P_s \Psi \rangle = \langle \psi_{KAs}^\dagger \psi_{KAs} + \psi_{K'As}^\dagger \psi_{K'As} - \psi_{KBs}^\dagger \psi_{KBs} - \psi_{K'Bs}^\dagger \psi_{K'Bs} \rangle$$

$$P_\pm = \frac{1 \pm \sigma^3}{2}$$



# Energy scales in graphene

- Large Landau energy scale

$$\epsilon_B \equiv \sqrt{2\hbar|eB_{\perp}|v_F^2/c} \simeq 424\sqrt{|B_{\perp}[\text{T}]|} \text{ K}$$

- Small Zeeman energy

$$Z \simeq \mu_B B = 0.67B[\text{T}] \text{ K}$$

- Intermediate dynamical scales

$$Z \ll A \leq M \ll \epsilon_B; \quad A \rightarrow \mu_s, \tilde{\mu}_s; \quad M \rightarrow \Delta_s, \tilde{\Delta}_s$$

$$A \leq M \sim 10^{-2}\epsilon_B$$

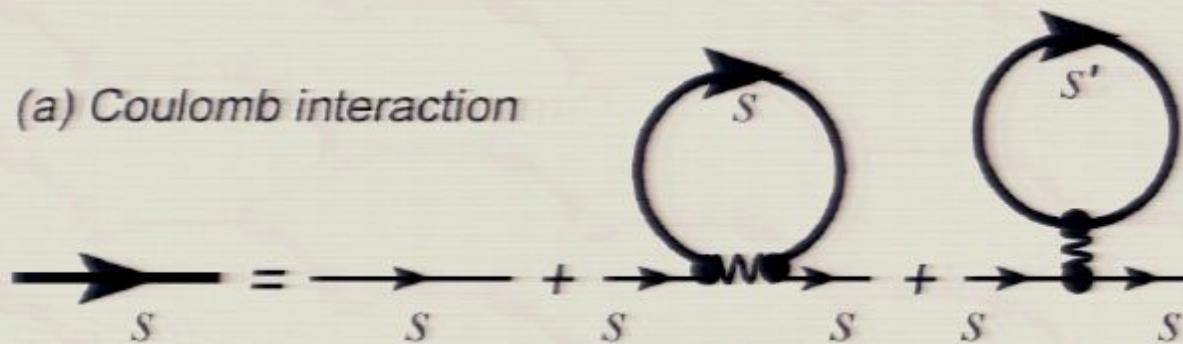
- In a model calculation [Phys. Rev. B **78** (2008) [085437](#)]

$$M = 4.84 \times 10^{-2}\epsilon_B \text{ and } A = 3.90 \times 10^{-2}\epsilon_B$$

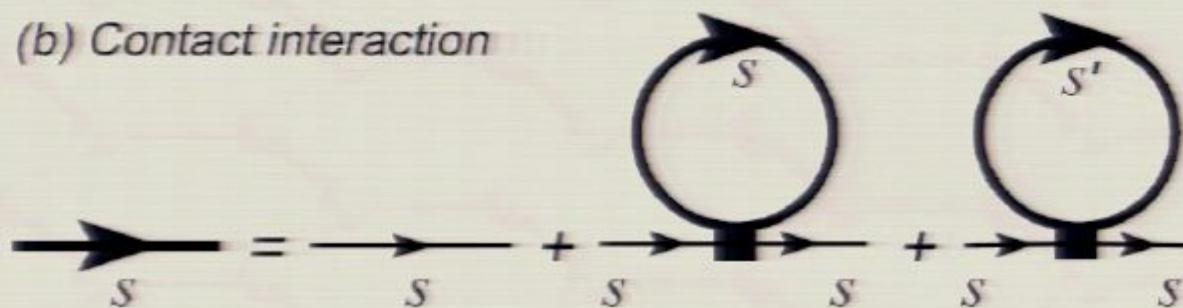
# Schwinger-Dyson (gap) equation

- Hartree-Fock (mean field) approximation:

(a) Coulomb interaction



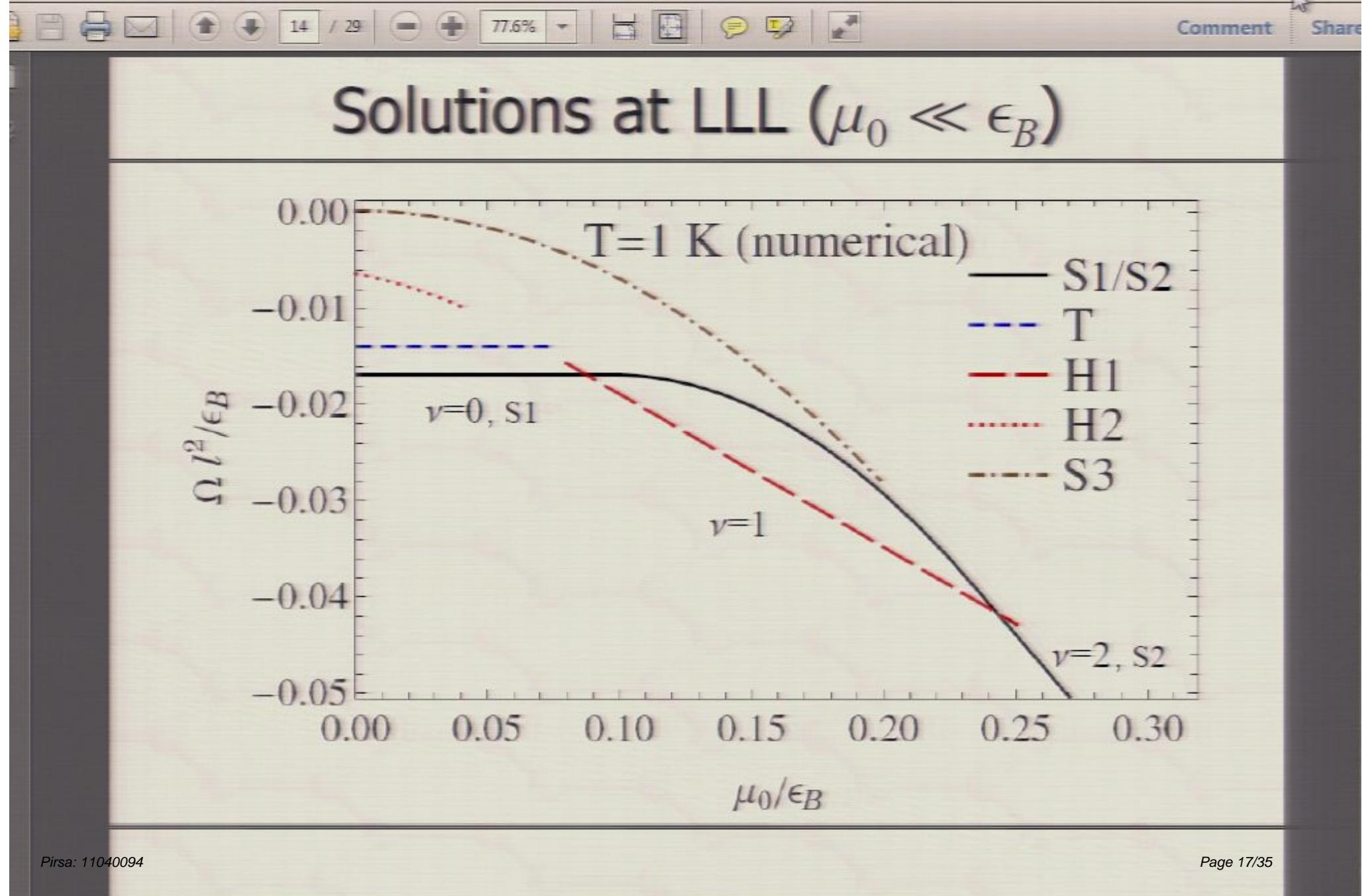
(b) Contact interaction

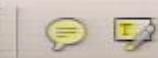




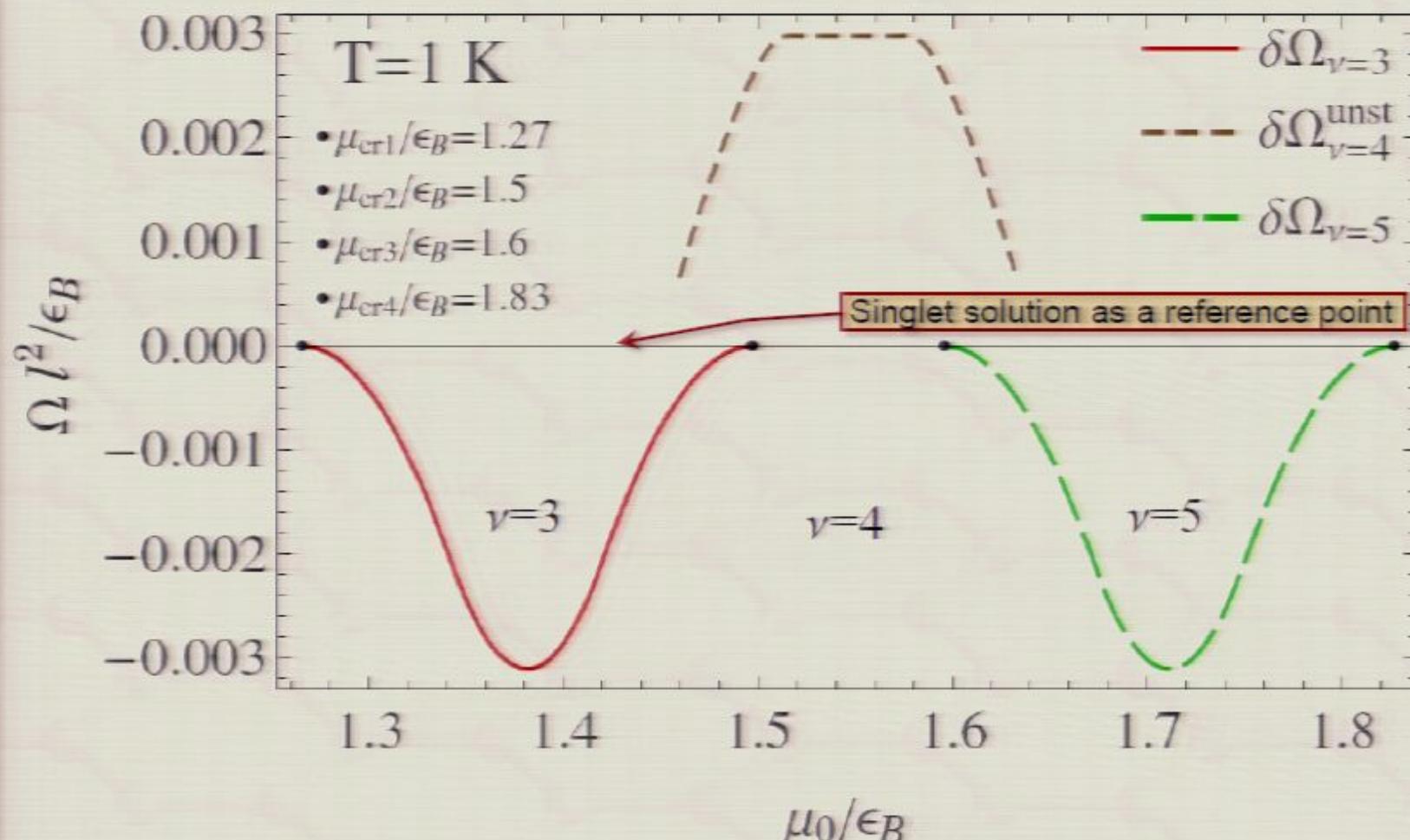
## Three types of solutions

- **S** (*singlet* with respect to  $U^{(+)}(2)_v \times U^{(-)}(2)_v$ );  $v=0$ 
  - Order parameters:  $\mu_s$  and  $\Delta_s$
  - Symmetry:  $U^{(+)}(2)_v \times U^{(-)}(2)_v$
- **T** (*triplet* with respect to  $U^{(\pm)}(2)_v$ );  $v=0$ 
  - Order parameters:  $\tilde{\mu}_s$  and/or  $\tilde{\Delta}_s$
  - Symmetry:  $U^{(K)}(2)_s \times U^{(K')}(2)_s$
- **H** (*hybrid*, i.e., singlet + triplet);  $v=\pm 1$ 
  - Order parameters: mixture of **S** and **T** types
  - Symmetry:  $U^{(+)}(2)_v \times U^{(-)}(1)_K \times U^{(-)}(1)_{K'}$  or  $U^{(+)}(1)_K \times U^{(+)}(1)_{K'} \times U^{(-)}(2)_v$

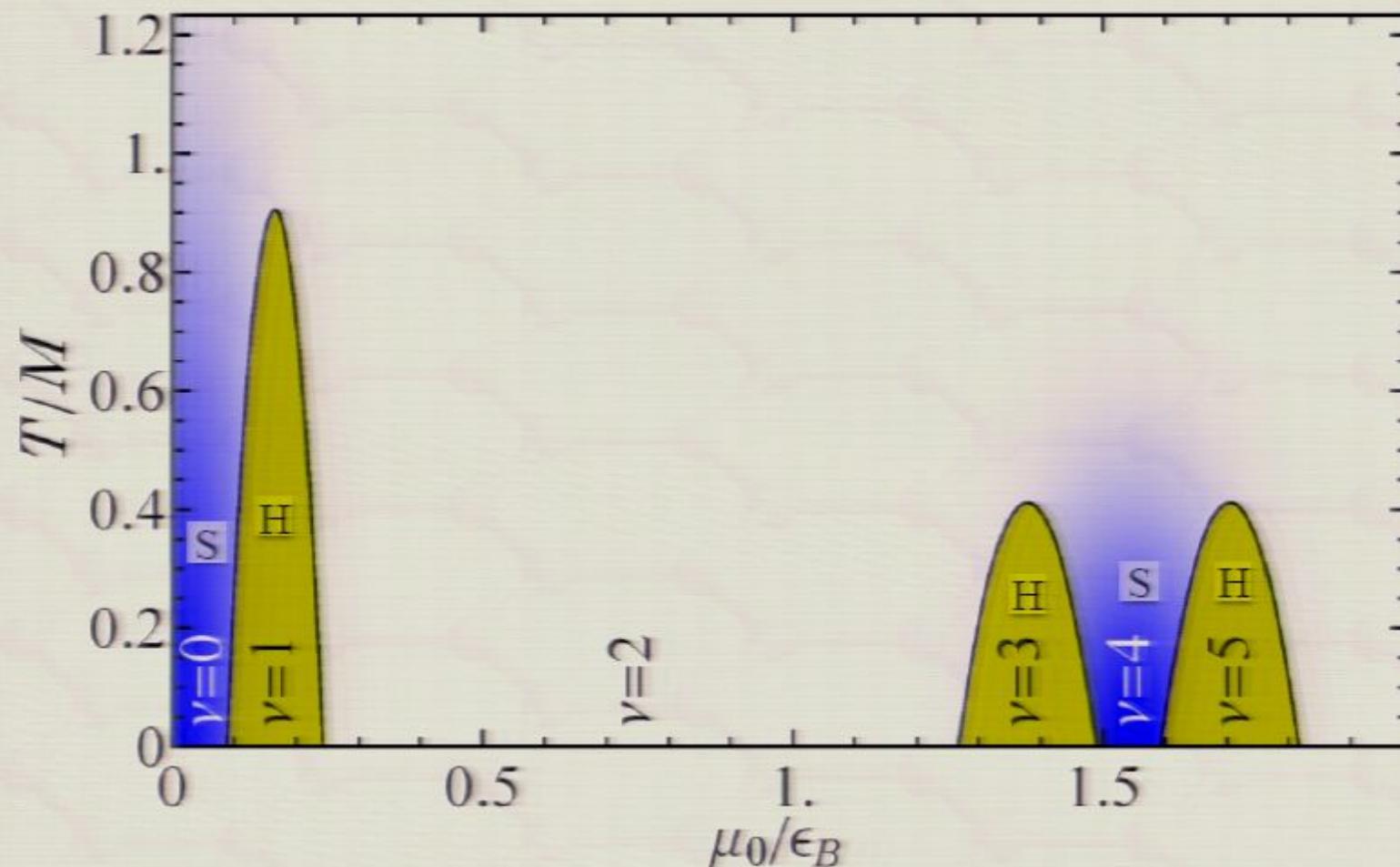




# Hybrid solutions at 1<sup>st</sup> Landau level



# Phase diagram





# Bilayer graphene

The effective low energy Hamiltonian [McCann & Falko, PRL, 96, 086805 (2006)]

Free Hamiltonian:

$$H_0 = -\frac{1}{2m} \int d^2x \Psi_{Vs}^+(x) \begin{pmatrix} 0 & (\pi^\dagger)^2 \\ \pi^2 & 0 \end{pmatrix} \Psi_{Vs}(x)$$

$$m \sim 10^{-2} m_e \sim 10^8 \text{K}/c^2, \pi = \hat{p}_{x_1} + i\hat{p}_{x_2}, \hat{p} = -i\hbar\nabla + e\mathbf{A}/c.$$

Bernal (A<sub>2</sub>-B<sub>1</sub>) stacking:  $\Psi_{Ks}^T = (\psi_{A1}, \psi_{B2})_{Ks}$ , whereas  $\Psi_{K's}^T = (\psi_{B2}, \psi_{A1})_{K's}$ .

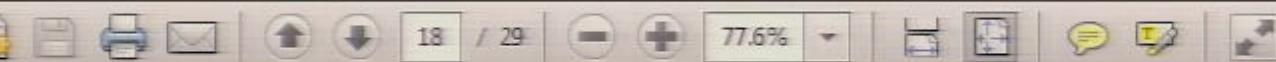
$E = \pm \frac{p^2}{2m}$  without magnetic field  $B$  and  $E_n = \pm \hbar\omega_c \sqrt{n(n-1)}$ ,  $\omega_c = |eB|/mc$  with magnetic field  $B$ .

Interaction Hamiltonian:

$$H_{int} = \mu_B B \int d^2x \Psi^+(x) \sigma^3 \Psi(x) + \frac{e^2}{2\kappa} \int d^3x d^3x' \frac{n(\mathbf{x})n(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \tilde{\Delta}_0 \int d^2x \Psi^+(x) \xi \tau_3 \Psi(x),$$

where  $n(\mathbf{x}) = \delta(z - \frac{d}{2})\rho_1(\mathbf{x}) + \delta(z + \frac{d}{2})\rho_2(\mathbf{x})$  is the three dimensional charge density ( $d \simeq 0.35$  nm is the distance between the layers).

The Pauli matrix  $\tau^3$  in the voltage imbalance term acts on layer components, and  $\xi = \pm 1$  for the valleys  $K$  and  $K'$ , respectively.



# Quantum Hall effect in bilayer graphene

## FIRST EXPERIMENTS:

Novoselov et al., Nature Phys. **2**, 177 (2006)

Henriksen et al., PRL **100**, 087403 (2008)

- Quantum Hall states with the filling factor  $\nu = \pm 4n$ ,  $n = 1, 2, \dots$  predicted in the framework of the one electron problem were revealed.

## RECENT EXPERIMENTS:

Suspended graphene:

Feldman, Martin, Yacoby, Nature Phys. **5**, 889 (2009)

Graphene on  $\text{SiO}_2/\text{Si}$  substrates:

Zhao, Cadden-Zimansky, Jiang, Kim, PRL **104**, 066801 (2010)

- Complete lifting the eightfold degeneracy in the LLL:  $\nu = 0, \pm 1, \pm 2, \pm 3$ .
- The  $\nu = 0$  state is insulating.
- Suspended graphene:  $\Delta E \sim 3.5 - 10.5 B_{\perp} [\text{T}] \text{K}$ ,  $B_{\perp} \lesssim 10 \text{T}$  ;

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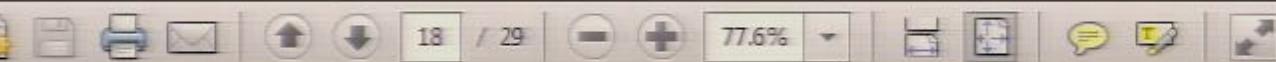
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# QHE in bilayer graphene: theory

Barlas, Cote, Nomura, MacDonald, PRL, **101**, 097601 (2008);

Nandkishore and Levitov, arXiv:0907.5395, arXiv:1002.1966;

Gorbar, Gusynin, V. M., JETP Lett. **91**, 314 (2010); PRB **81**, 155451 (2010).

Gorbar, Gusynin, Jia, V.M. (QHS for  $v = \pm 1, \pm 2, \pm 3$  and  $\pm 4$ , in preparation)

Tőke, Fal'ko, PRB **91**, 115455 (2011)



# Symmetries

$H_{int}$  can be rewritten as

$$H_{int} = \mu_B B \int d^2x \Psi^+(x) \sigma^3 \Psi(x) + \tilde{\Delta}_0 \int d^2x \Psi^+(x) \xi \tau_3 \Psi(x) + \frac{1}{2} \int d^2x d^2x' [V(x-x') (\rho_1(x)\rho_1(x') + \rho_2(x)\rho_2(x')) + 2V_{12}(x-x')\rho_1(x)\rho_2(x')] .$$

$$\text{Intralayer potential } V(x) \xrightarrow{\text{FT}} \tilde{V}(k) = 2\pi e^2 / \kappa k$$

$$\text{interlayer potential } V_{12}(x) \xrightarrow{\text{FT}} \tilde{V}_{12}(k) = (2\pi e^2 / \kappa)(e^{-kd}/k).$$

The two-dimensional charge densities  $\rho_1(x)$  and  $\rho_2(x)$  are:

$$\rho_1(x) = \Psi^+(x) P_1 \Psi(x), \quad \rho_2(x) = \Psi^+(x) P_2 \Psi(x),$$

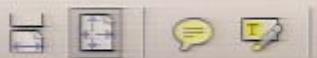
where  $P_1 = \frac{1+\xi\tau^3}{2}$  and  $P_2 = \frac{1-\xi\tau^3}{2}$  are projectors on states in the layers 1 and 2, respectively.

If both the Zeeman and  $\tilde{\Delta}_0$  terms are ignored, the bilayer symmetry is

$$G_2 = U^{(K)}(2)_S \times U^{(K')}(2)_S \times Z_{2V}^{(+)} \times Z_{2V}^{(-)},$$

where  $Z_{2V}^{(s)}$  describes the valley transformation  $\xi \rightarrow -\xi$  for a fixed spin  $s = \pm$ .

$G_2$  is much lower than  $G_1 = U(4)$  in monolayer graphene.



# Order parameters

Although the  $G_1$  and  $G_2$  symmetries are quite different, it is noticeable that their spontaneous breakdowns can be described by the same QHF and MC order parameters.  $G_1$  and  $G_2$  define the same four conserved commuting currents whose charge densities (and four corresponding chemical potentials) span the QHF order parameters:

$$\mu_3 : \langle \Psi^\dagger \sigma^3 \Psi \rangle = \sum_{\kappa=K,K'} \sum_{a=A_1,B_2} \langle \psi_{\kappa a+}^\dagger \psi_{\kappa a+} - \psi_{\kappa a-}^\dagger \psi_{\kappa a-} \rangle,$$

$$\tilde{\mu}_s : \langle \Psi_s^\dagger \xi \Psi_s \rangle = \langle \psi_{KA_1s}^\dagger \psi_{KA_1s} - \psi_{K'A_1s}^\dagger \psi_{K'A_1s} + \psi_{KB_2s}^\dagger \psi_{KB_2s} - \psi_{K'B_2s}^\dagger \psi_{K'B_2s} \rangle.$$

While the first order parameter describes  $SU^{(K)}(2)_s \times SU^{(K')}(2)_s$  spin symmetry breakdown, the second one breaks the discrete subgroup  $Z_{2V}^{(s)}$ . Their MC cousins are

$$\Delta_s : \langle \Psi_s^\dagger \tau_3 \Psi_s \rangle = \langle \psi_{KA_1s}^\dagger \psi_{KA_1s} - \psi_{KB_2s}^\dagger \psi_{KB_2s} - (\psi_{K'A_1s}^\dagger \psi_{K'A_1s} - \psi_{K'B_2s}^\dagger \psi_{K'B_2s}) \rangle,$$

$$\tilde{\Delta}_s : \langle \Psi_s^\dagger \xi \tau_3 \Psi_s \rangle = \langle \psi_{KA_1s}^\dagger \psi_{KA_1s} + \psi_{K'A_1s}^\dagger \psi_{K'A_1s} - \psi_{KB_2s}^\dagger \psi_{KB_2s} - \psi_{K'B_2s}^\dagger \psi_{K'B_2s} \rangle.$$

# LLL quasiparticle propagator

Bare LLL propagator

$$\tilde{S}_{\xi s}(\mathbf{r}; \omega) = \frac{1}{2\pi l^2} \exp\left(-\frac{\mathbf{r}^2}{4l^2}\right) \left[ L_0\left(\frac{\mathbf{r}^2}{2l^2}\right) + L_1\left(\frac{\mathbf{r}^2}{2l^2}\right) \right] S_{\xi s}(\omega) P_-,$$

where

$$S_{\xi s}(\omega) = \frac{1}{\omega + \bar{\mu}_s + \xi \tilde{\Delta}_0 + i\delta \operatorname{sgn} \omega}, \quad \bar{\mu} = \mu_0 - sZ, \quad P_{\pm} = \frac{1 \pm \tau_3}{2}, \quad l = \sqrt{\frac{\hbar c}{|eB|}}.$$

Full LLL propagator

$$\tilde{G}_{\xi s}(\mathbf{r}; \omega) = \frac{1}{2\pi l^2} \exp\left(-\frac{\mathbf{r}^2}{4l^2}\right) \left[ G_{\xi s0}(\omega) L_0\left(\frac{\mathbf{r}^2}{2l^2}\right) + G_{\xi s1}(\omega) L_1\left(\frac{\mathbf{r}^2}{2l^2}\right) \right] P_-,$$

where

$$G_{\xi sn}(\omega) = \frac{1}{\omega - E_{\xi ns} + i\delta \operatorname{sgn} \omega}, \quad \text{and} \quad E_{\xi ns} = -(\mu_s(n) + \Delta_s(n)) + \xi(\bar{\mu}_s(n) - \tilde{\Delta}_s(n)), \quad n = 0, 1,$$

are the energies of the LLL states depending on the order parameters  $\mu_s(n), \bar{\mu}_s(n), \Delta_s(n), \tilde{\Delta}_s(n)$ .



## Exchange (FOCK) interactions:

$$\tilde{V}_{eff}(\omega, k) = \frac{2\pi e^2}{\kappa} \frac{1}{k + \frac{4\pi e^2}{\kappa} \Pi(\omega, k^2)}$$

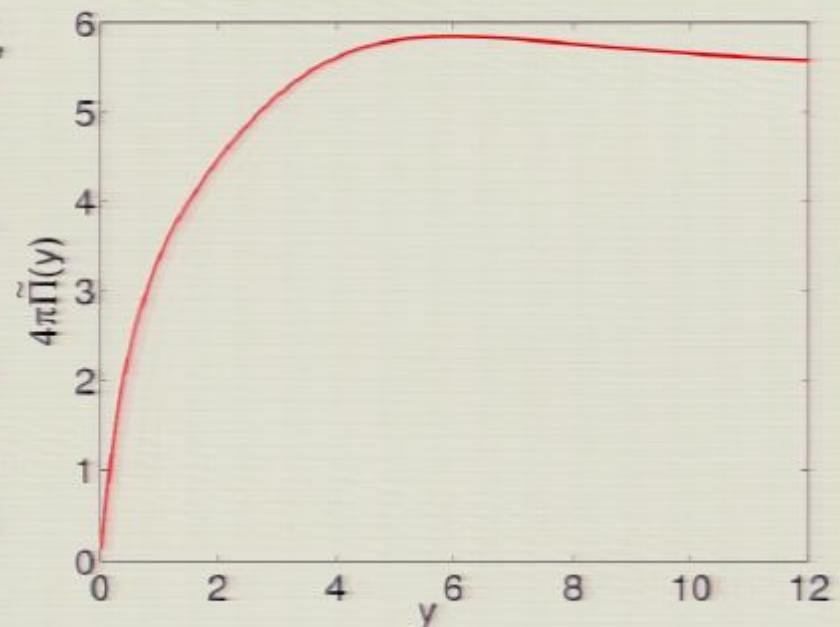
The static polarization function  $\Pi(0, k^2) \equiv \frac{m}{\hbar^2} \tilde{\Pi}(y)$ ,  $y \equiv \frac{k^2 l^2}{2}$ .

Because  
 $m \sim 10^{-2} m_e \sim 10^8 K/c^2 \gg \hbar^2/e^2 l$ ,  
the polarization function dominates  
in  $\tilde{V}_{eff}$ :

$$\tilde{V}_{eff}(k) = C(y) \hbar^2 / ml^2 k^2.$$

## HARTREE interactions:

$$\tilde{V}_H = -\frac{2\pi e^2 d}{\kappa_{eff}}, \quad \kappa_{eff} - \kappa \simeq 1 - 4.$$





# $\nu = 0$ QH state

Two competing solutions of the gap equation at the neutrality point  $\nu = 0$

(I) spin polarized solution

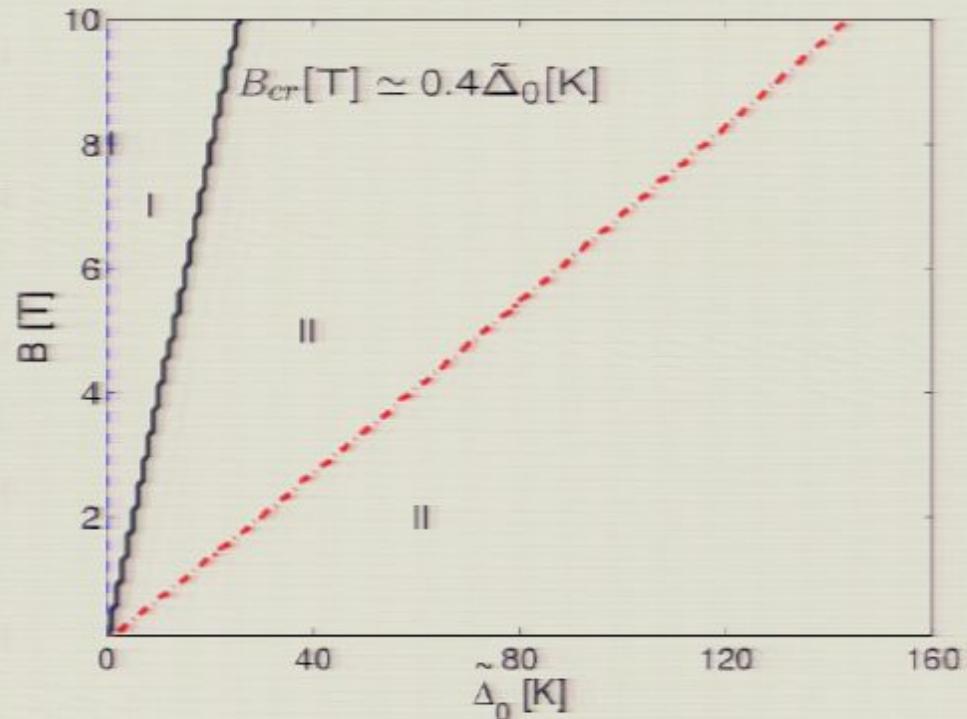
$$\begin{aligned}\Delta E^{(I)} &= (E_{\xi 1+}^{(I)} - E_{\xi 1-}^{(I)})/2 \\ &\simeq 14.4B[\text{T}] \text{K}\end{aligned}$$

(II) layer polarized solution

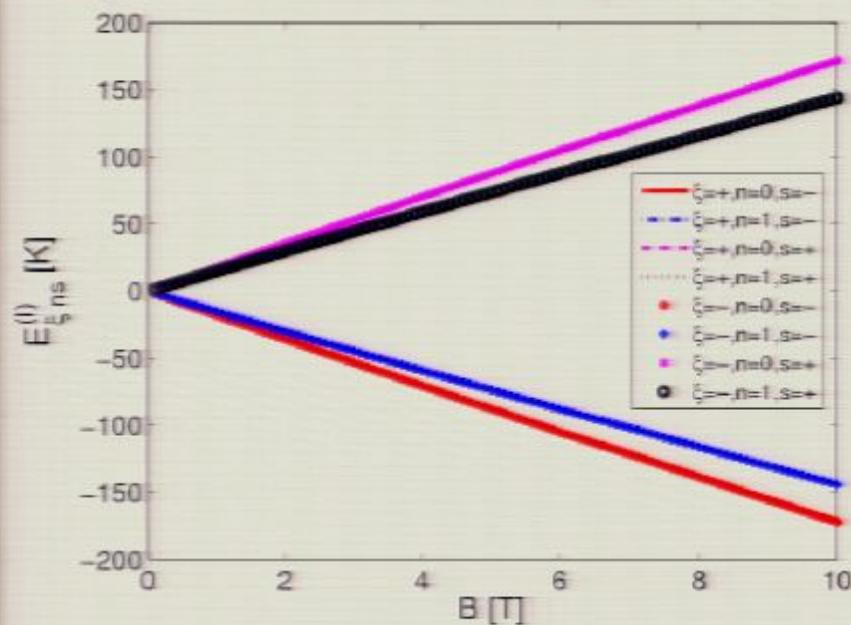
$$\begin{aligned}\Delta E^{(II)} &= (E_{-1-}^{(II)} - E_{+1+}^{(II)})/2 \\ &\simeq 9.3B[\text{T}] \text{K} + \tilde{\Delta}_0\end{aligned}$$

$$\tilde{\Delta}_0 \sim e\mathcal{E}_{\perp}d$$

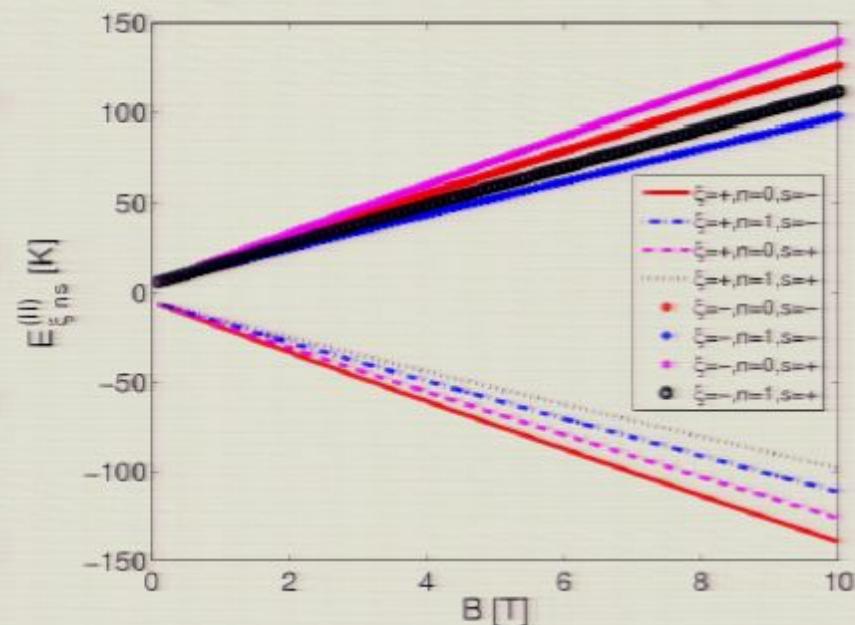
$$\mathcal{E}_{\perp}^{(cr)} \sim 5 \frac{\text{mV}}{\text{nm}} B[\text{T}]$$



# Energy spectra



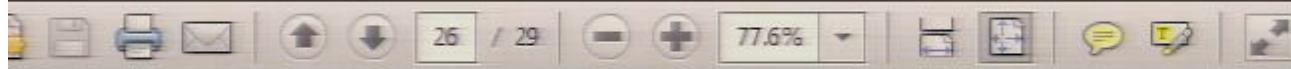
$$\Delta E^{(I)} \simeq 14.4 B[T] \text{ K}$$



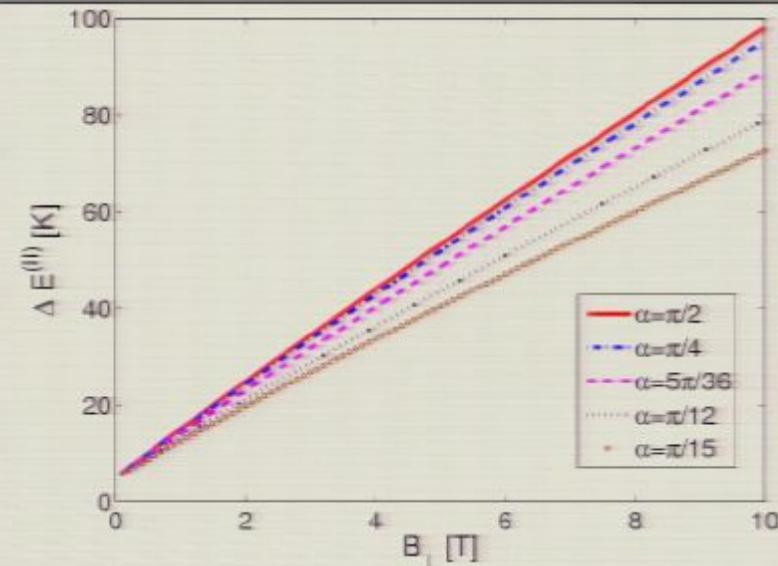
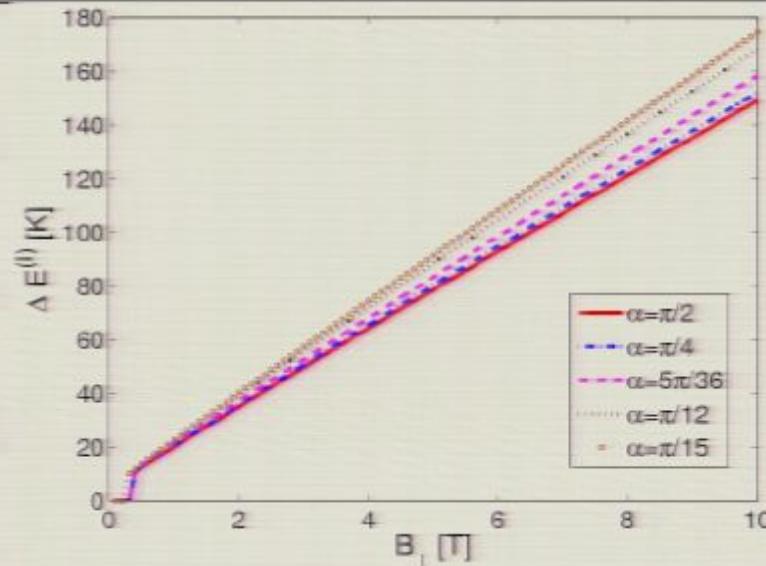
$$\Delta E^{(II)} \simeq 9.3 B[T] \text{ K} + \tilde{\Delta}_0$$

The LLL energies of the solutions I (left panel) and II (right panel) as functions of  $B$  with  $B_{\parallel} = 0$ .

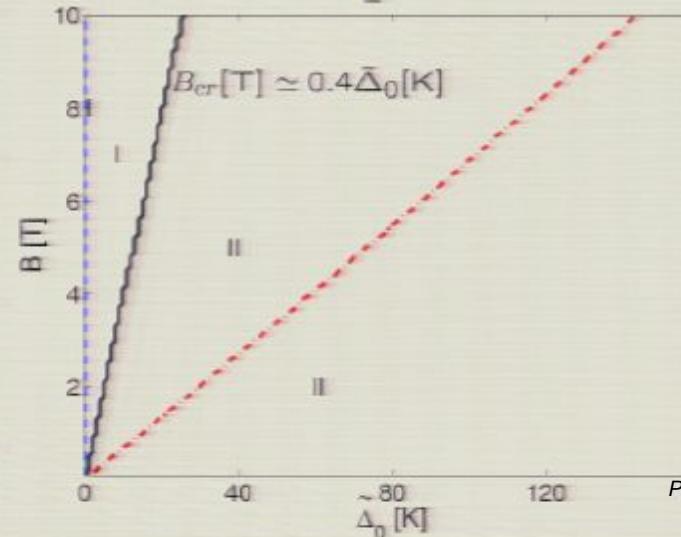
Here  $\tilde{\Delta}_0 = 0$  and  $\tilde{\Delta}_0 = 5 \text{ K}$  for solution I & solution II, respectively.



# Energy gaps



The dependence of the energy gaps of solutions I (left panel) and II (right panel) on the field  $B_{\perp}$  for different angles. The parameter  $\tilde{\Delta}_0 = 5\text{K}$  for both solutions I,II.





## Latest Experiments

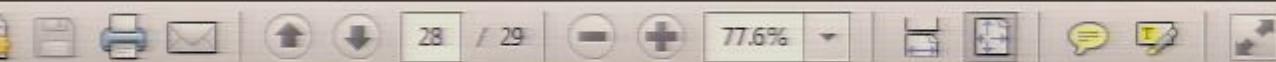
Weitz, Allen, Feldman, Martin, Yacoby, Science, **330**, 812 (2010)

Martin, Feldman, Weitz, Allen, Yacoby, PRL, **105**, 256806 (2010)

Kim, Lee, Tutuc, arxiv:1102.0265 [cond-mat]

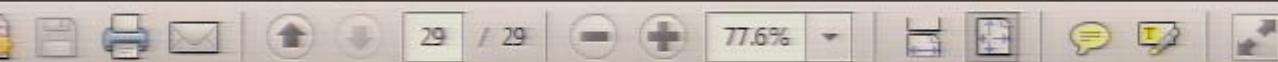
The confirmation of the phase transition between the ferromagnetic (spin polarized) and layer asymmetric (layer polarized) QH states

$$\mathcal{E}_{\perp}^{(cr)} \sim 10 \frac{\text{mV}}{\text{nm}} B [\text{T}]$$



# Outlook

- (i) Analysis of the higher,  $v = \pm 1, \pm 2$  and  $\pm 3$ , LLL plateaus [Gorbar, Gusynin, Jia, V.M., in preparation].
- (ii) The present ansatz with the sixteen order parameters is the minimal one for describing the breakdown of the  $U^{(K)}(2)_S \times U^{(K')}(2)_S \times Z_{2V}^{(+)} \times Z_{2V}^{(-)}$  symmetry in bilayer graphene.  
It could be extended in order to look for other solutions of the gap equation. A natural extension would be to include order parameters that mix the  $n=0$  and  $n=1$  LLL states.
- (iii) It would be important to analyze the gap equation with a non-static polarization function.
- (iv) It would be interesting to describe explicitly the dynamics around the threshold value  $B_{thr}$ , when the crossover between the regimes with the nonrelativistic-like scaling  $\Delta E \sim |eB|$  and the relativistic-like one  $\Delta E \sim \sqrt{|eB|}$  should take place.



# Conclusion

- It seems that a dynamical Dirac mass (masses) is (are) necessarily produced in graphene in a strong magnetic field
- The set of order parameters which describes the QHE in graphene is quite large
- Feedback of QHE in graphene for particle physics: dynamics in dense Quantum Chromodynamics in a strong magnetic field [Gorbar, V.M., Shovkovy, PRD 91, 085003 (2001)]  
In 3+1 dimensions, the analog of the Haldane mass term,  $\Delta \bar{\Psi} \gamma^3 \gamma^5 \Psi$ , describes an axial-vector current density, rather than a mass.



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