Title: Cauchy horizon (in)stability in spherically symmetric self-similar gravitational collapse.

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Abstract: Various self-similar spherically symmetric spacetimes admit naked singularities, providing a challenge to the cosmic censorship hypothesis. However, it is not clear if the naked singularities are artefacts of the high degree of symmetry of the spacetimes, or if they are potentially generically present. To address this question, we consider perturbations of (various cases of) these spacetimes, focusing particularly on the behaviour of the perturbations as they impinge on the Cauchy horizon. We describe recent results on self-similar Lemaitre-Tolman-Bondi spacetime, indicating stability of scalar and odd parity perturbations, and instability of even parity perturbations.

Cauchy horizon (in)stability in self-similar spherically symmetric collapse

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Perimeter Institute 7th April 2011

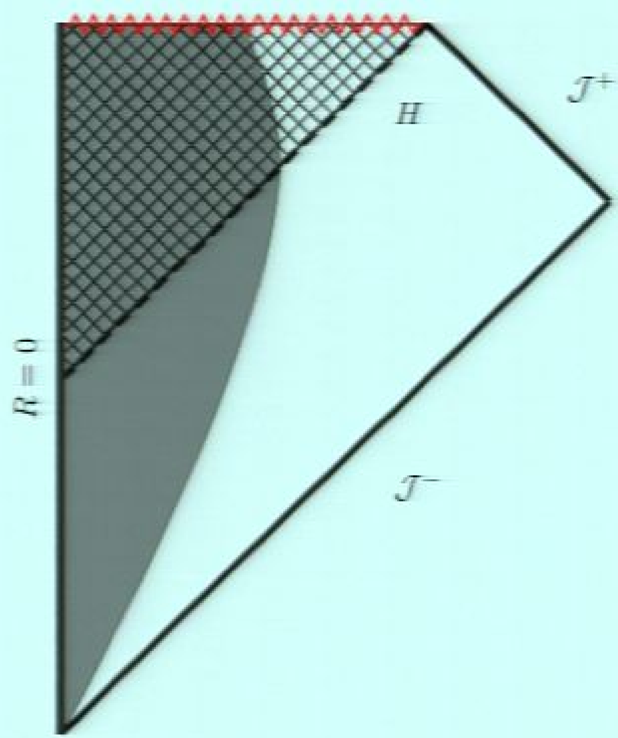
Outline

- Gravitational collapse and the cosmic censorship hypothesis.
- Spherically symmetric self-similar space-times: definition, geometric and physical properties.
    - LTB space-time
- Perturbations:
    - scalar field;
    - odd parity perturbations;
    - even parity perturbations
- Conclusions and comments.

Joint work with Emily Duffy and Thomas Waters.

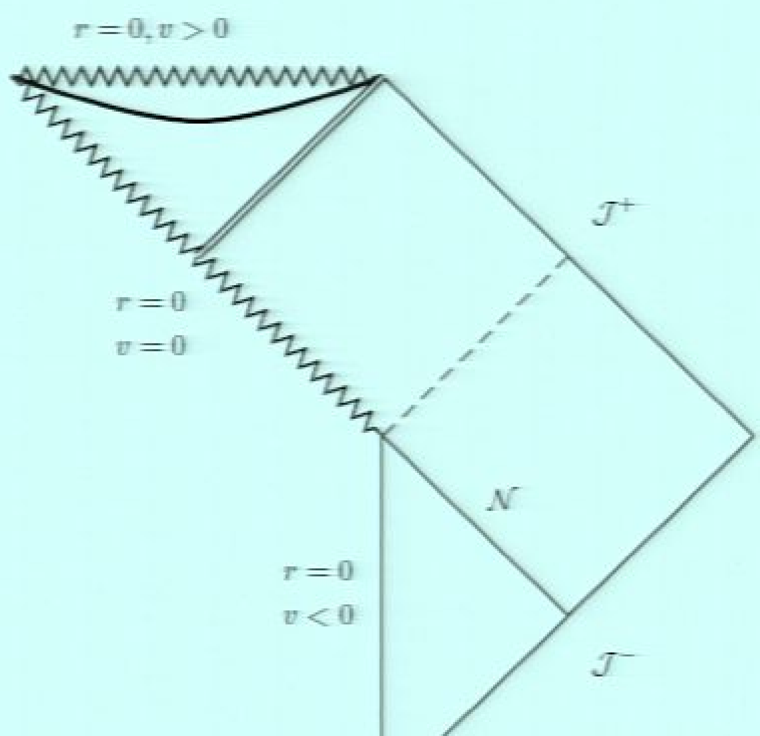
Gravitational collapse and singularities.

- Singularities inevitably form in gravitational collapse.
- Standard picture: Instability, implosion, horizon, singularity.



Collapse to a naked singularity

- However, gravitational collapse may result in a naked singularity, causing problems:
  - Lack of predictability of Einstein equations.
  - Emission of arbitrary information/energy from the singularity.
- Note the presence of a *Cauchy horizon*: the first light ray that emerges from the naked singularity.



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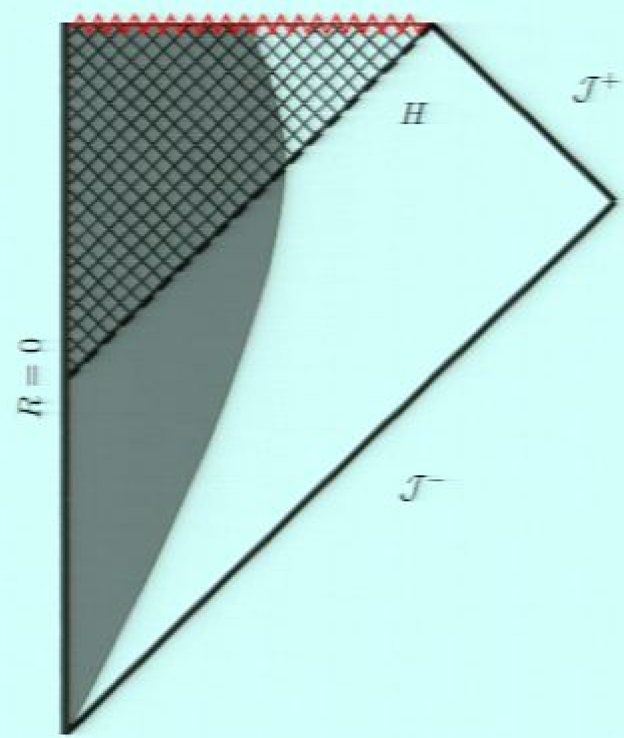

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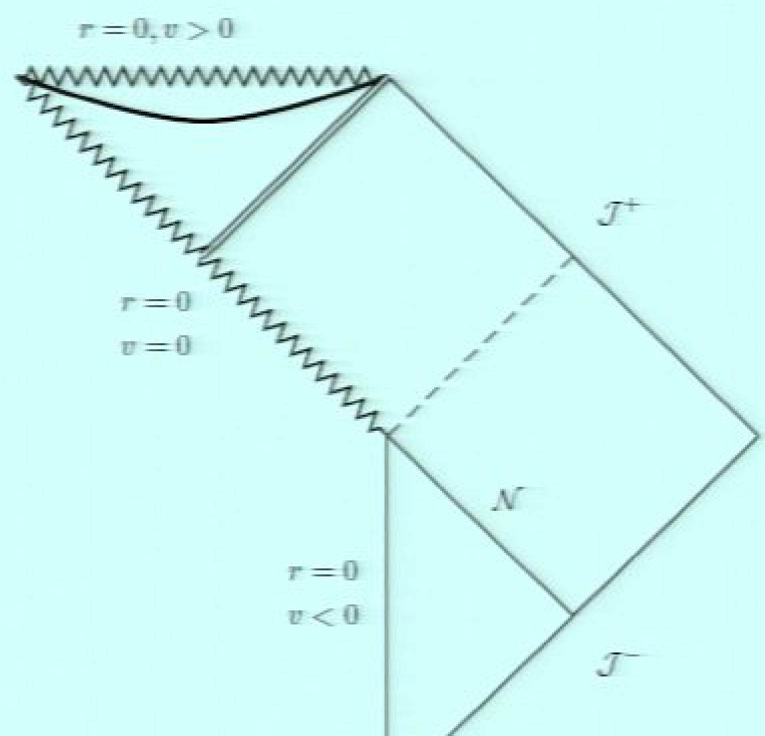


Cosmic censorship

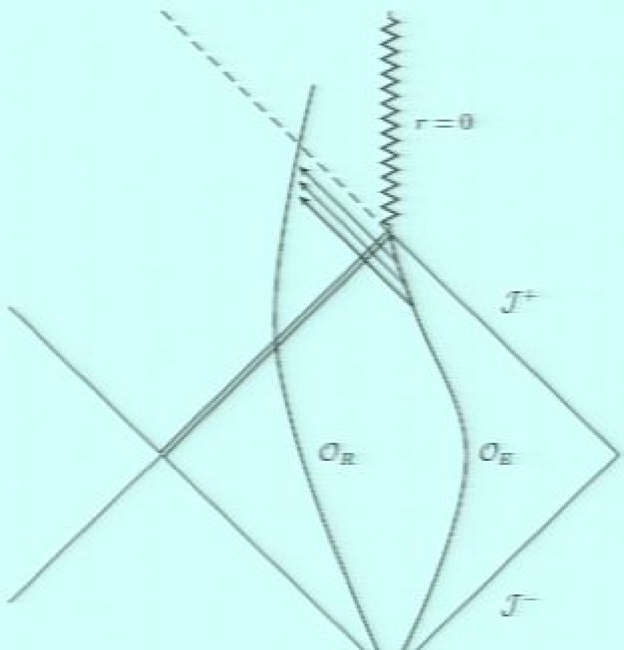
- Penrose 1969: Nature abhors a naked singularity.
- Weak cosmic censorship hypothesis: In generic situations, gravitational collapse from a regular initial configuration leads to the formation of a black hole.
- Strong CCH: In generic situations, gravitational collapse from a regular initial configuration cannot lead to the formation of a naked singularity.
- Conditions on the matter are also assumed (energy condition, Lagrangian fields, singularity-free in flat space-time).

Examples of naked singularities

- Numerous examples of naked singularity formation have been found:
  - Charged and/or rotating black hole interiors;
  - Self-similar perfect fluid collapse;
  - Self-similar scalar field collapse;
  - Shell-focussing/shell-crossing singularities in dust;
  - Critical solutions at threshold of BH formation.
- Q. How does the censor deal with these?
- A. Look for instabilities, particularly at the Cauchy horizon.

Examples of Cauchy horizon/naked singularity instability

- Charged/rotating black holes
  - Penrose 1969: On crossing the Cauchy horizon, an observer sees, in one final flash, the entire history of the external universe → blue-shift instability.
  - Penrose's initial observation confirmed by increasingly sophisticated and realistic analyses; cf. especially Chandrasekhar and Hartle (1983), Poisson and Israel (1989), Ori (1991) and Dafermos (2001).



- Spherical collapse of a scalar field. Data giving rise to NS are unstable (Christodoulou 1999).
- Black holes in de Sitter background (Brady, Moss and Myers, 1998).
- Plane wave Cauchy horizons (Helliwell and Konkowski, 1997).
- Compact Cauchy horizons are non-generic; Isenberg and Moncrief (1983), Friedrichs, Racz and Wald (1999).

Spherically symmetric self-similar spacetimes

- We consider space-times $(\mathcal{M}, g)$ which
  - are spherically symmetric
  - admit a homothetic Killing vector field

$$\mathcal{L}_{\vec{\xi}} g_{ab} = 2 g_{ab}$$

  - satisfy the dominant energy condition
  - satisfy certain regularity conditions.
- Using advanced Bondi co-ordinates, we can write

$$ds^2 = -2Ge^{\psi} dv^2 + 2e^{\psi} dv dr + r^2 d\Omega^2$$

- $G = G(x), \psi = \psi(x)$ where $x = \frac{v}{r}$; $\vec{\xi} = v\frac{\partial}{\partial v} + r\frac{\partial}{\partial r}$.
- Co-ordinate freedom: $v \rightarrow V(v)$. Remove by taking v to be proper time along the regular center $r = 0$.

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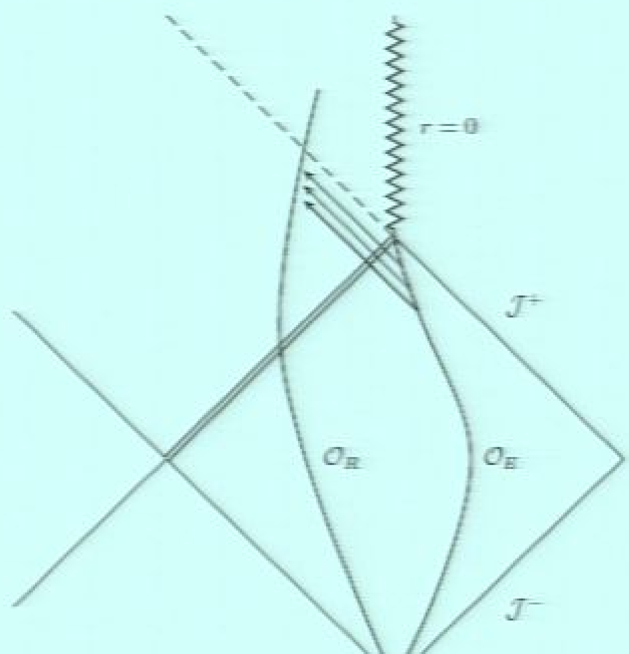

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Dominant energy condition

For every future-pointing timelike v^a,

- $-T^a_b v^b$ is non-spacelike and future-pointing;
- $T_{ab} v^a v^b \geq 0$.
- This results in e.g.

$$\begin{aligned} x\psi' &\leq 0, \\ G' - G(1 - xG)\psi' &\leq 0, \\ 1 - 2(G - xG')e^{-\psi} &\geq 0. \end{aligned}$$

Regularity conditions I - differentiability.

- Metric and curvature are finite along the axis $r = 0$ for $v < 0$. These result in

$$\lim_{x \to -\infty} G = \frac{1}{2}, \lim_{x \to -\infty} \psi = 0.$$

- $\{v = 0\}$ is the past null cone $\mathcal{N}$ of the scaling origin $\mathcal{O} = \{(v, r) = (0, 0)\}$. We take $G, \psi \in C^2(-\infty, 0]$.
- Except in the trivial case (flat space-time), there is a curvature singularity at $\mathcal{O}$.

Regularity conditions II - absence of trapped surfaces.

- Studying collapse from a regular initial configuration, so want to rule out trapped surfaces in early stages.
- The 2-sphere (v, r) is trapped iff $G(v/r) < 0$. Thus we take

$$G(x) > 0 \quad \text{for all} \quad x \in (-\infty, 0].$$

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Naked singularities

A necessary and sufficient condition for $\mathcal{O}$ to be naked is that there exists a future directed outgoing radial null geodesic (ORNG) with past endpoint $\mathcal{O}$.

- $\mathcal{O}$ is naked iff there exists a positive root of

$$xG(x) = 1.$$

- The first positive root (x_c) of $xG(x) = 1$ is the Cauchy horizon of the space-time.
- The level surfaces of x are space-like for $0 < x < x_c$.
- $G'(x) < 0$ for $0 < x < x_c$.

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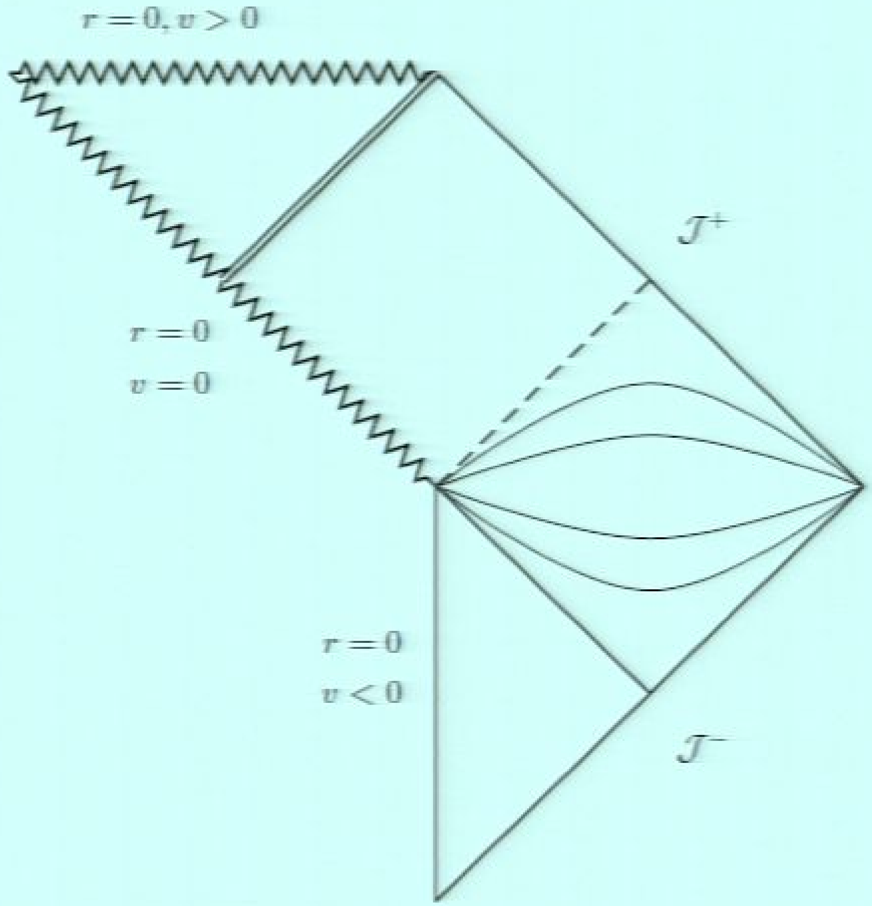

Self-similar Vaidya space-time

- Thick shell of photons; Minkowski space-time inside the inner radius, Schwarzschild space-time outside.
- Matter filled region:

$$ds^2 = -(1 - 2\frac{m(v)}{r})dv^2 + 2dvdr + r^2 d\Omega^2.$$

- Energy-stress-momentum tensor:

$$T_{ab} = \frac{m'(v)}{4\pi r^2} n_a n_b, \qquad n_a = -\nabla_a v.$$

- Self-similar: $m(v) = \lambda v$.
- $\mathcal{O}$ is naked iff $\lambda \in (0, 1/16)$.

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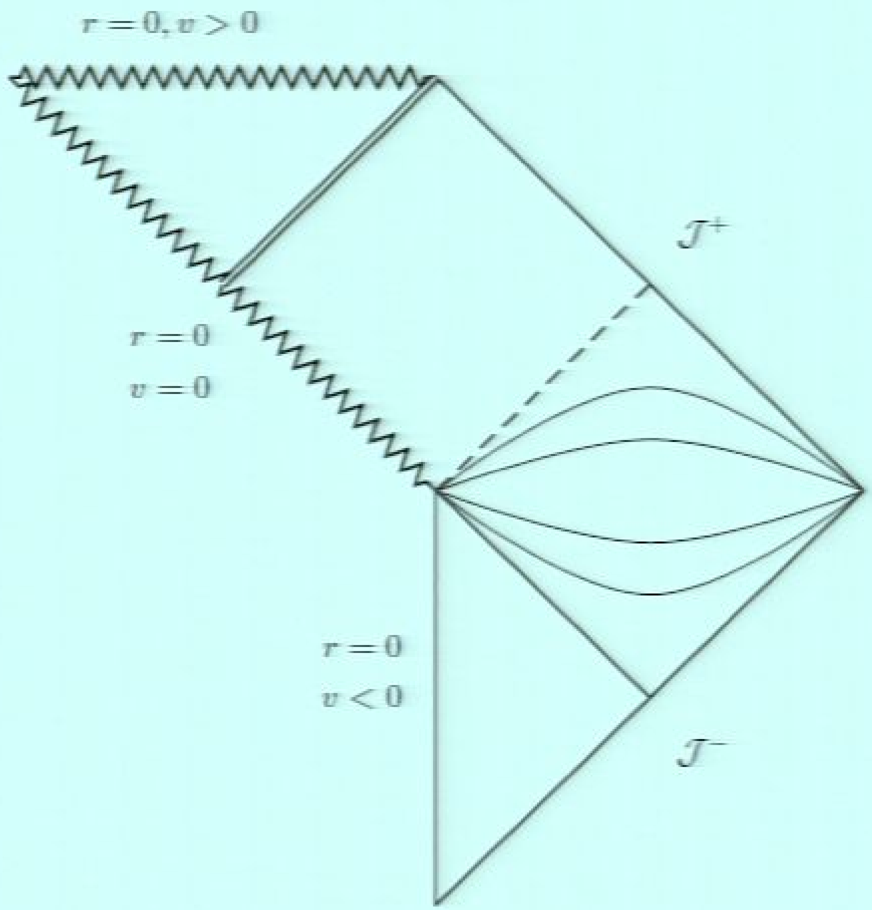

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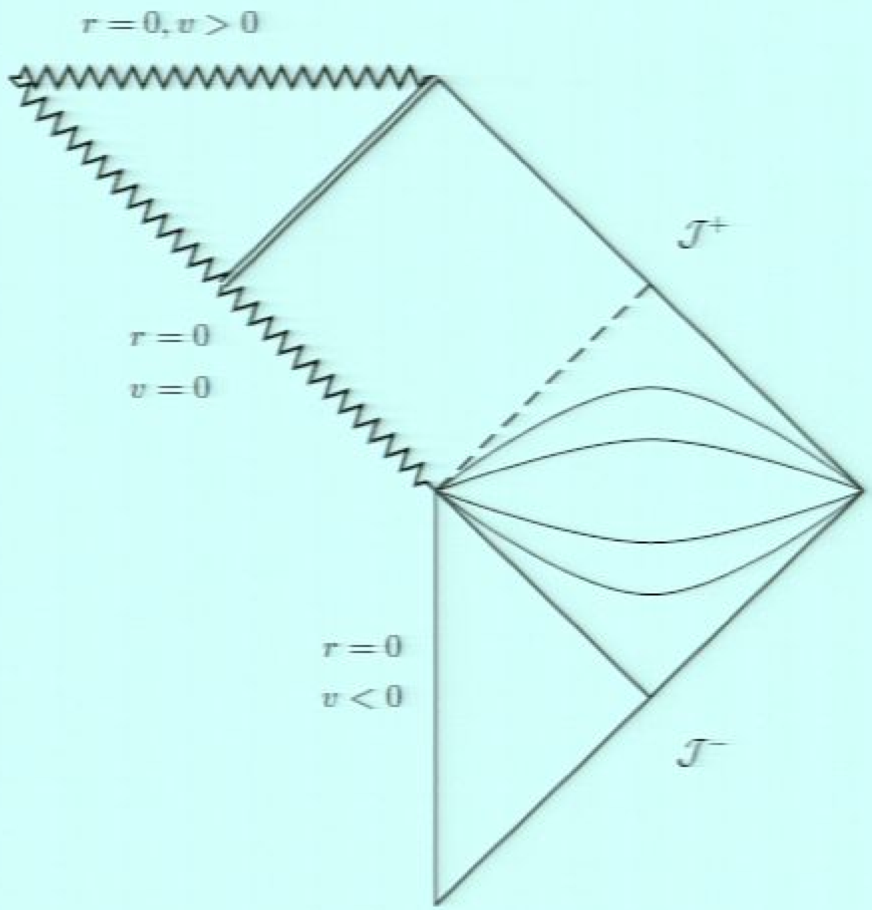

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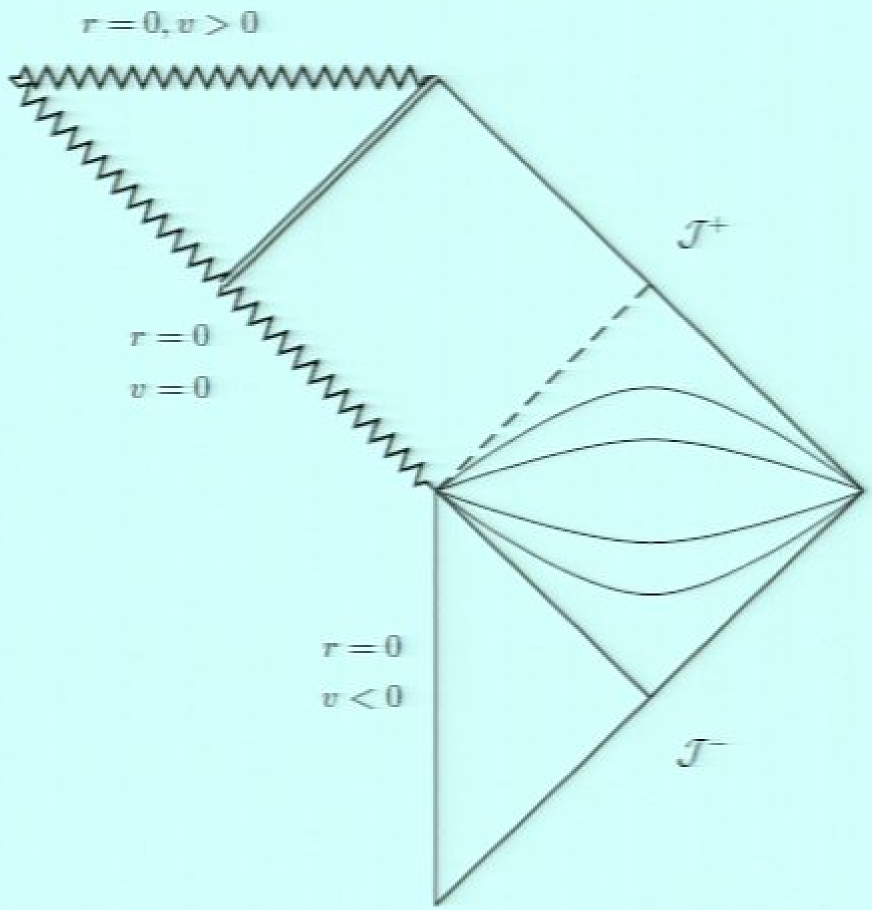

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Lemaître-Tolman-Bondi space-time

- Spherically symmetric dust filled space-time ($T^{ab} = \rho u^a u^b$).

$$ds^2 = -dt^2 + \frac{R'(t,r)}{1+f(r)} dr^2 + R^2(t,r) d\Omega^2.$$

- Self-similar: $f(r) = 0, R(t,r) = r(1+\kappa z)^{2/3}, z = -t/r.$
- $\mathcal{O}$ is naked iff $0 < \kappa < \kappa_* = \frac{3}{(2(26+15\sqrt{3}))^{1/3}} \simeq 0.638.$

Scalar waves

- We consider minimally coupled, massless scalar field: $\Box\phi = 0$. After usual angular mode decomposition we get ($\rho = \log r$)

$$\alpha\phi_{,\tau\tau} + 2\beta\phi_{,\tau\rho} + \gamma\phi_{,\rho\rho} + (\alpha' + 2\beta)\phi_{,\tau} + (\beta' + 2\gamma)\phi_{,\rho} - \ell(\ell+1)\phi = 0 \quad (1)$$

- Cauchy horizon is at $\tau = \tau_c$; $\alpha = -2\tau(1 - \tau G) < 0$ on $(0, \tau_c)$; $\alpha(\tau_c) = 0$.
- Note that (1) is also satisfied by $\phi_{,\rho}, \phi_{,\rho\rho}, \ldots$ (self-similarity of the PDE).
- Strategy: study evolution in τ of different energy norms for ϕ to obtain $H^{1,2}$ bounds, then apply Sobolev-type inequality to obtain pointwise bounds.
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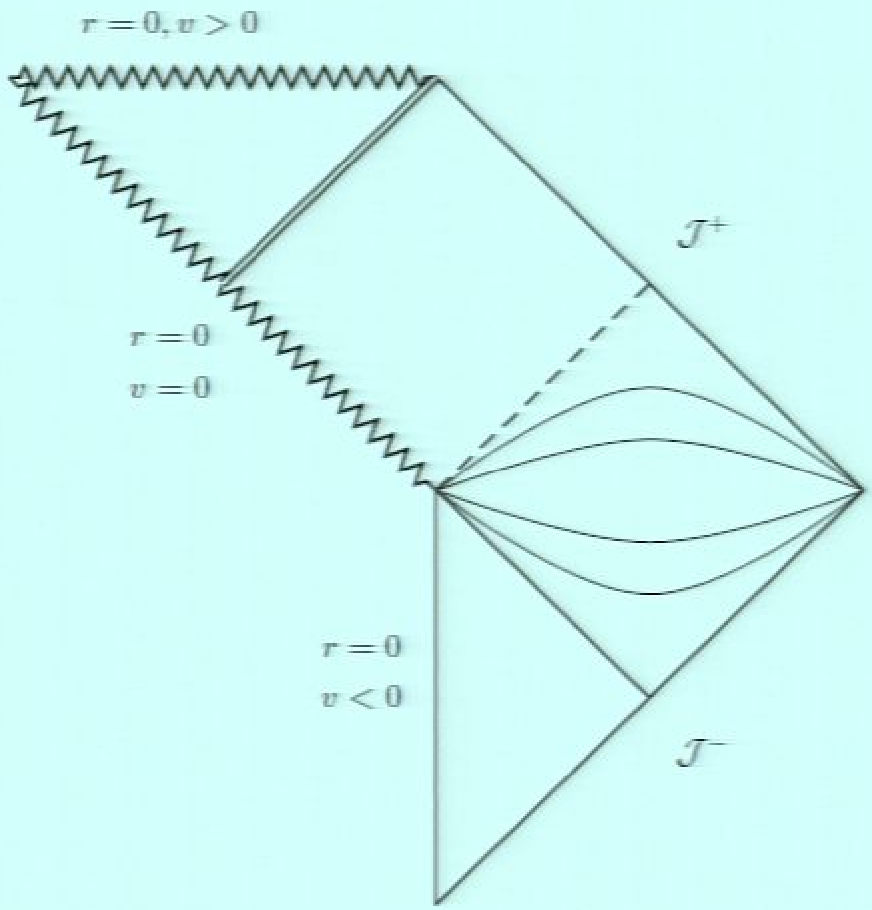

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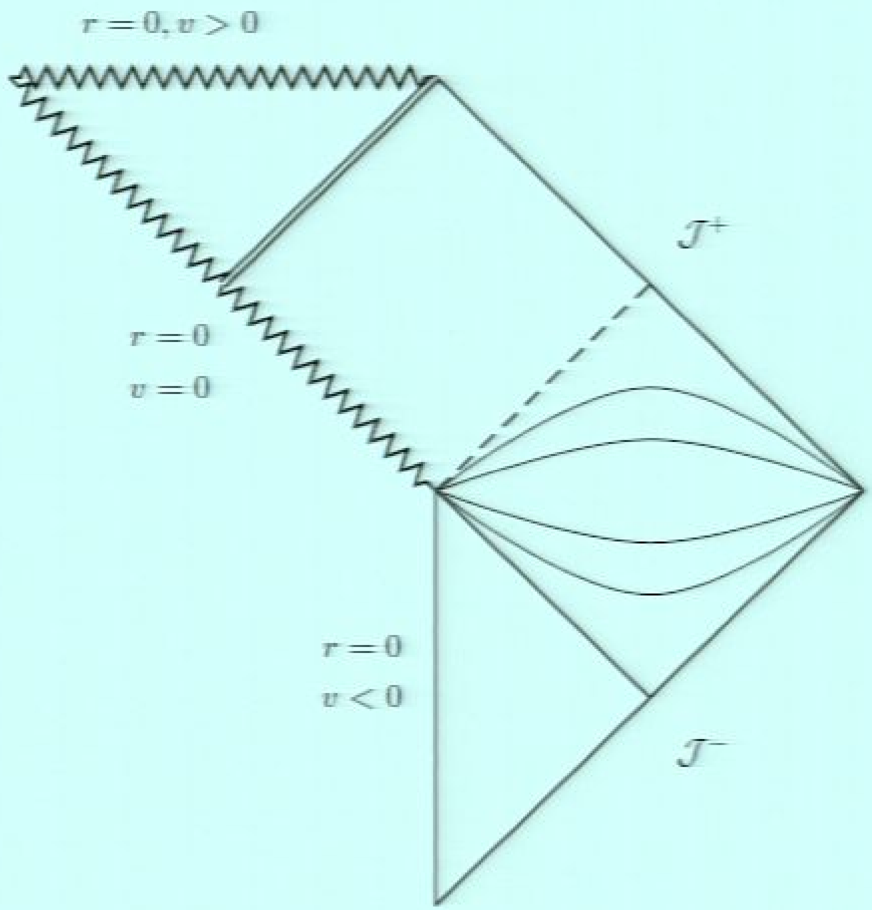

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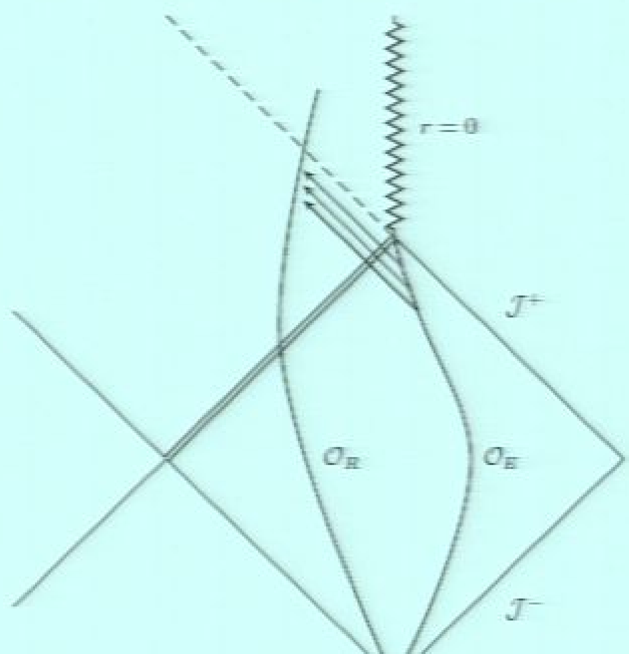

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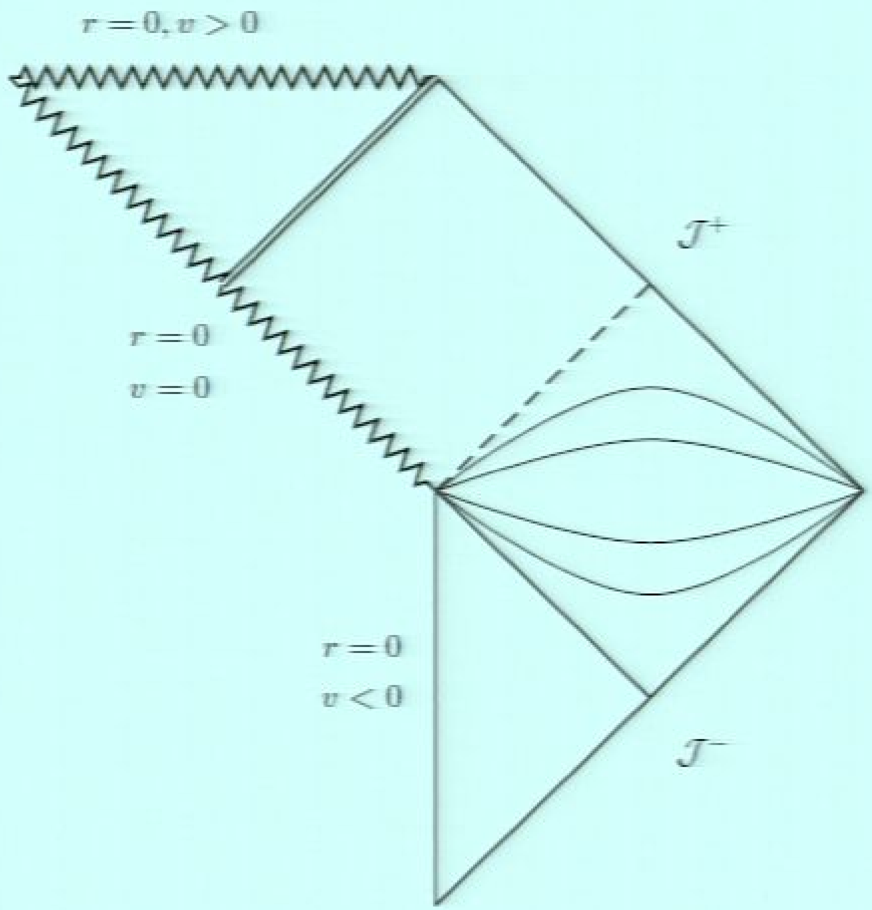

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Step One

Rewrite (1) as a first order symmetric hyperbolic system, with the Cauchy horizon moved out to $t = \infty$:

$$t = -\int_{\tau_i}^{\tau} \frac{ds}{\alpha(s)}, \quad \vec{\varphi} = \begin{pmatrix} \phi \\ \alpha\phi_{,\tau} + \beta\phi_{,\rho} \\ \phi_{,\rho} \end{pmatrix}.$$

$$\vec{\varphi}_{,t} + A\vec{\varphi}_{,\rho} + B\vec{\varphi} = 0.$$

Standard argument gives

- existence and uniqueness on $[\tau_i, \tau_c) \times \mathbb{R}$ for smooth initial data with compact support and
- $E_1(t) \leq e^{B_0 t} E_1(0)$, where

$$E_1(t) = E_1[\vec{\varphi}](t) = \int_{\mathbb{R}} \|\vec{\varphi}\|^2 d\rho.$$

Step Two

Introduce another energy integral

$$E_2(\tau) = E_2[\phi](\tau) = \int_{\mathbb{R}} -\alpha\phi_{,\tau}^2 + \gamma\phi_{,\rho}^2 + \ell(\ell+1)e^{\psi}\phi^2\, d\rho.$$

Key lemma: there exist $\tau_1 \in (0, \tau_c)$ and $m_0 > 0$, which depend only on G, ψ, ℓ such that

$$\frac{dE_2}{d\tau} \leq m_0 E_2, \quad x \in [\tau_1, \tau_c).$$

Then

$$E_2[\phi](\tau) \leq C_1 E_1[\vec{\varphi}](0), \quad x \in [\tau_i, \tau_c],$$

and so

$$\int_{\mathbb{R}} |\phi^2(\tau_c, \rho)| + |\phi_{,\rho}^2(\tau_c, \rho)|\, d\rho < C_2 E_1[\vec{\varphi}](0) < \infty. \qquad (1)$$

Step Three

For each $\tau \in [\tau_i, \tau_c)$, we can apply the Sobolev inequality

$$|\phi(\tau,\rho)|^2 \leq \frac{1}{2}\int |\phi(\tau,\bar{\rho})|^2 + |\phi_{,\bar{\rho}}(\tau,\bar{\rho})|^2 \, d\bar{\rho}.$$

Taking a limit yields

$$|\phi(\tau,\rho)|^2 \leq C_3 E_1[\vec{\varphi}](0), \quad \tau \in [\tau_i, \tau_c], \rho \in \mathbb{R}.$$

Note: Existence of the function $\phi|_{\mathrm{CH}}$ can be proven using Lipschitz property of $\phi_{,\tau}$ for $\tau < \tau_c$.

Notes

- $H^{1,2}$ and pointwise bounds: These hold for all smooth initial data with compact support. By applying the inequality to sequences of such data and taking a limit, it also holds for solutions with

$$\phi(\tau_i) \in H^{1,2}(\mathbb{R}), \quad \phi_{,\tau}(\tau_i) \in L^2(\mathbb{R}).$$

- Local energy bound extends to

$$\phi(\tau_i) \in H^{3,2}(\mathbb{R}), \quad \phi_{,\tau}(\tau_i) \in H^{2,2}(\mathbb{R}).$$

- The same results hold for $\phi_\kappa = r^\kappa \phi$, $\kappa \in [0, \kappa_*)$ with $\kappa_* > 0$ determined by G, ψ. Thus ϕ *need not* vanish at the origin. In fact blow up of ϕ at $\mathcal{O}$ may occur, but is not mediated to the Cauchy horizon.
- $H^{1,2}$ and pointwise bounds also hold for minimally coupled massive field

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Perturbations - Gerlach-Sengupta formalism

- Decompose non-spherical part of the perturbation into (scalar, vector, tensor) harmonics $\ell = 0, 1, \ldots$.

$$\delta g_{\mu\nu} = \left(\begin{array}{c|c} h_{AB} Y & h_A^{\text{even}} Y_a + h_A^{\text{odd}} S_a \\ \hline \text{symm} & r^2 K \gamma_{ab} Y + r^2 G Z_{ab} + h S_{ab} \end{array} \right).$$

- Identify gauge invariant parts of the metric and matter perturbations: tensor fields on the Lorentzian 2-space.
- Specify matter model: perturb within that model.
- Complete set of g.i. variables exists for $\ell \geq 2$; gauge fixing required for $\ell = 0, 1$.

Gauge invariant variables

- Even parity perturbations:

$$\text{metric:} \quad k_{AB}, k \qquad \text{matter:} \quad T_{AB}, T_A, T^2, T^3.$$

- Odd parity perturbations:

$$\text{metric:} \quad k_A \qquad \text{matter:} \quad L_A, L.$$

Perturbations - Gerlach-Sengupta formalism

- Decompose non-spherical part of the perturbation into (scalar, vector, tensor) harmonics $\ell = 0, 1, \ldots$.

$$\delta g_{\mu\nu} = \left(\begin{array}{c|c} h_{AB} Y & h_A^{\text{even}} Y_a + h_A^{\text{odd}} S_a \\ \hline \text{symm} & r^2 K \gamma_{ab} Y + r^2 G Z_{ab} + h S_{ab} \end{array} \right).$$

- Identify gauge invariant parts of the metric and matter perturbations: tensor fields on the Lorentzian 2-space.
- Specify matter model: perturb within that model.
- Complete set of g.i. variables exists for $\ell \geq 2$; gauge fixing required for $\ell = 0, 1$.

Gauge invariant variables

- Even parity perturbations:

$$\text{metric:} \quad k_{AB}, k \qquad \text{matter:} \quad T_{AB}, T_A, T^2, T^3.$$

- Odd parity perturbations:

$$\text{metric:} \quad k_A \qquad \text{matter:} \quad L_A, L.$$

Odd-parity (axial) perturbations

- Problem reduces to an inhomogeneous self-similar wave equation for a g.i. scalar Π.
- The inhomogeneity is an initial data function.
- Π is both (i) a potential for the g.i. metric perturbation and (ii) the gauge and tetrad invariant perturbation of the Weyl scalar Ψ_2.
- Results for Π as for ϕ above (Vaidya and LTB).

- For odd-parity, there is a full set of tetrad and gauge invariant perturbed Weyl scalars, $\delta\Psi_A, A = 0-4$.
- These involve terms in $\alpha(\tau)\Pi_{,\tau\tau}$.
- Introduce null (characteristic) coordinates (U, V), with $V_{\rm CH} = 0$. Wave equation reads

$$\Pi_{UV} + F(U, V)\Pi = J,$$

and F is *analytic* on sufficiently small characteristic diamonds with the Cauchy horizon on the N-W boundary.
- By work above, characteristic data on S-W and S-E boundaries may be assumed continuous. Calculating $\delta\Psi_A(U, V)$ shows that these are all finite at the Cauchy horizon.
- Conclusion: odd-parity linear stability of the naked singularity in self-similar Vaidya and LTB spacetime.

Even parity (polar) perturbations of LTB.

- The g.i. treatment yields a first order symmetric hyperbolic system in 5 dimensions with a (non-trivial) propagating constraint.
- Variable is $\vec{u}$ - metric, matter perturbations plus their time derivatives.
- System has the same general properties as the homogeneous scalar case - coefficients are smooth, uniformly bounded but with time derivatives of the form $\alpha(\tau)\partial_\tau$.
- Existence, uniqueness on $[\tau_i, \tau_c)$ straightforward, along with (exponentially growing) bound on the energy.

Evolution equations

- These have the form

$$(\tau - \tau_c)\partial_\tau w + D(\tau)\partial_\rho w + A(\tau)w = 0,$$

with $w \in \mathbb{R}^5$.

- $D = D(\tau)$, $A = A(\tau)$ analytic on $[\tau_i, \tau_c]$, but A is problematic (eigenvalues $0^4, -k^2$) at $\tau = \tau_c$.

Results 1

- Take data $w(\tau_i) \in C_0^\infty(\mathbb{R}, \mathbb{R}^5) = X$.
- Define the blow-up condition

$$B : \|w(\tau)\|_{L^p} \to \infty \quad \text{as} \quad \tau \to \tau_c \quad \text{for all} \quad 1 \leq p < \infty.$$

- Theorem 1: There is a subset $X_0 \subseteq X$ such that (i) solutions with data in X_0 satisfy the blow-up condition B and (ii) $X_1 = X \setminus X_0$ has codimension 1 in X.
- Theorem 2: There is an open dense subset of $(X, \|\cdot\|_{L^1})$ such that all solutions with initial data in this set satisfy the blow-up condition B.

Proof:

- Study the behaviour of

$$\bar{w}(\tau) = \int_{\mathbb{R}} w(\tau, \rho) d\rho,$$

which satisfies an ODE (singular at τ_c). The general solution satisfies

$$\bar{w} \sim (\tau_c - \tau)^{-} k^2, \quad \tau \to \tau_c.$$

- Generic blow-up of this quantity yields generic power-law blow-up in L^1 of w.
- Support spreads as $\ln|\tau - \tau_c|$.
- L^p embedding:

$$\|\vec{w}(\tau)\|_{L^p} \leq \text{Vol}[\text{supp}(\vec{w}(\tau))])^{1/p-1/q} \|\vec{w}\|_{L^q}$$

then yields B.

Results 2

- Define $u = \tau^{k^2} w$.
- Apply previous methods to show that u remains *finite* at the Cauchy horizon.
- However, this finite value might be zero...

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- Define $u = \tau^{k^2} w$.
- Apply previous methods to show that u remains *finite* at the Cauchy horizon.
- However, this finite value might be zero...
- Theorem 3: It's not zero.
- Theorem 3: There is an open dense subset $Y_0 \subseteq Y$ such that solutions with initial data in Y_0 satisfy

$$\|w(\tau)\|_{L^2} \to \infty \qquad \text{as} \qquad \tau \to \tau_c$$

Furthermore, there is an interval I such that for all $\rho \in I$,

$$\lim_{\tau \to \tau_c} |w(\tau, \rho)| = +\infty.$$

Note

Consider function

$$f(t,x) = \frac{\sin(tx)}{\sqrt{x}},$$

and take its even extension to the real line. Then

(i) $\lim_{t\to 0} f(t,x) = 0$ for all $x \in \mathbb{R}$;

(ii) $\int_{\mathbb{R}} f(t,x)dx = \sqrt{\frac{2\pi}{t}} \to \infty$ as $t \to 0$.

Proof requires the construction of a Cauchy sequence of functions $f_n(\rho) = x(\tau_n, \rho)$ with $\tau_n \to \tau_c$ and use of L^p completeness and the dominated convergence theorem to show that $x(\tau_c, \rho)$ cannot vanish everywhere.

Conclusions and comments.

- Overall, the Cauchy horizon of self-similar LTB space-time is unstable to linear perturbations.
- Instability is found in the even parity sector and corresponds to divergence of gauge invariant perturbed curvature scalars.
- Next: study perfect fluid space-times.
  - Try to address analytically the stability of the Cauchy horizon in critical collapse space-times.
  - Likewise for the general relativistic Larson-Penston space-times - numerically, attractors with a naked singularity.