Title: Anyonic Statistics, Quantum Configuration Spaces, and Diffeomorphism Group Representations

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Abstract: We begin with a fundamental approach to quantum mechanics based on the unitary representations of the group of diffeomorphisms of physical space (and correspondingly, self-adjoint representations of a local current algebra). From these, various classes of quantum configuration spaces arise naturally. One obtains in addition the usual exchange statistics for spatial dimension d >2, induced by representations of the symmetric group, while for d = 2, the approach led to an early prediction of intermediate or Ã¢ÂÂ€ÂÂœanyonicÃ¢ÂÂ€ÂÂ• statistics induced by unitary representations of the braid group. After reviewing these ideas, which are based on joint work with R. Menikoff and D. H. Sharp at Los Alamos National Laboratory, we shall discuss briefly some analogous possibilities for infinite-dimensional configuration spaces, including anyonic statistics for extended objects in 3-dimensional space.

Anyon Statistics, Quantum Configuration Spaces, and Diffeomorphism Group Representations

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Perimeter Institute
Quantum Foundations Seminar
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Collaborators

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Ugo Moschella (Como)

Inspirations

T. Regge + M. Rasseti

A. Vershik, I. M. Gelfand, M. Graev

J. Leinaas + J. Myrheim

R. S. Ismagilov

S. Albeverio, Y. Kondratiev, M. Röckner

Some history of particle statistics

50 years!

- Bosons, fermions in QM (1920s)
- Topological ideas; parastatistics
  Aharanov-Bohm effect (1950s)
- Topology + quantum statistics
  Laidlaw-DeWitt (1971)
- Intermediate statistics in 1 and 2 dimensions
  Leinaas-Myrheim (1977)
- Current algebra + 2-dim. anyon statistics from induced reps. (1980-81),
  braid group (1983)
  higher-dim reps. (1985)
  Goldin-Menikoff-Sharp

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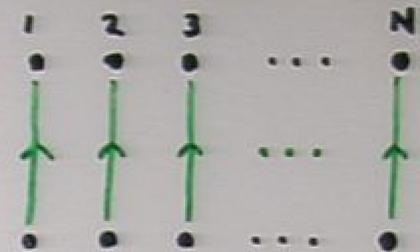
  higher-dim. reps. (1985)
  Goldin-Menikoff-Sharp

* Charged-particle / flux tubes composites (1982), Wilczek
* Q. Hall effect, surface phenomena, vorticity, etc.
  1980s
* High T_c superconductivity
  1988-1990s
* Quantum computing
  2000s

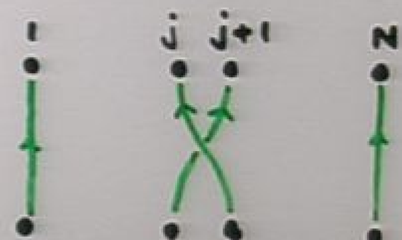
Braid group B_N (Artin)



identity e

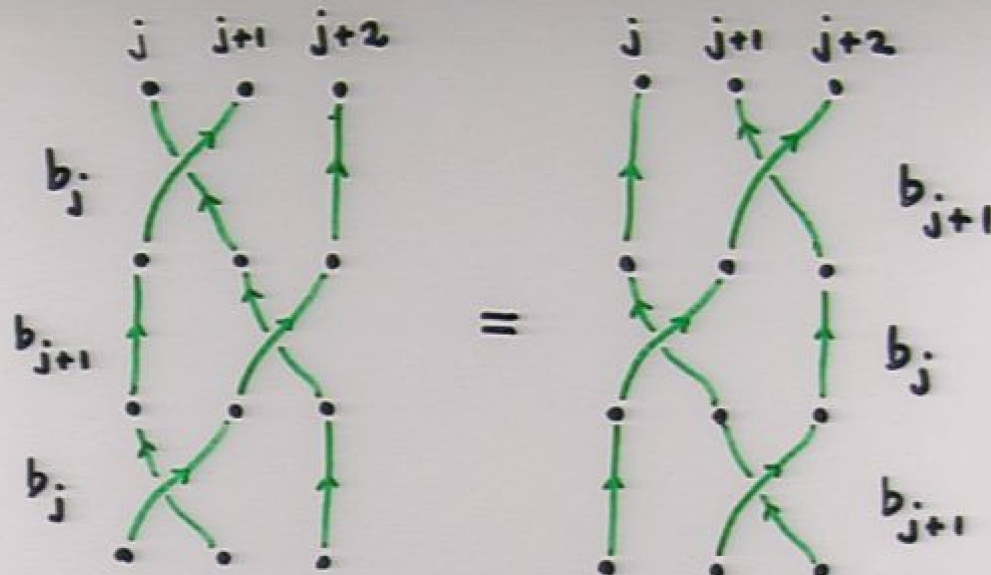
generator b_j

$j = 1, \cdots, N-1$



b_j^{-1}

Group multiplication by composition of braids.

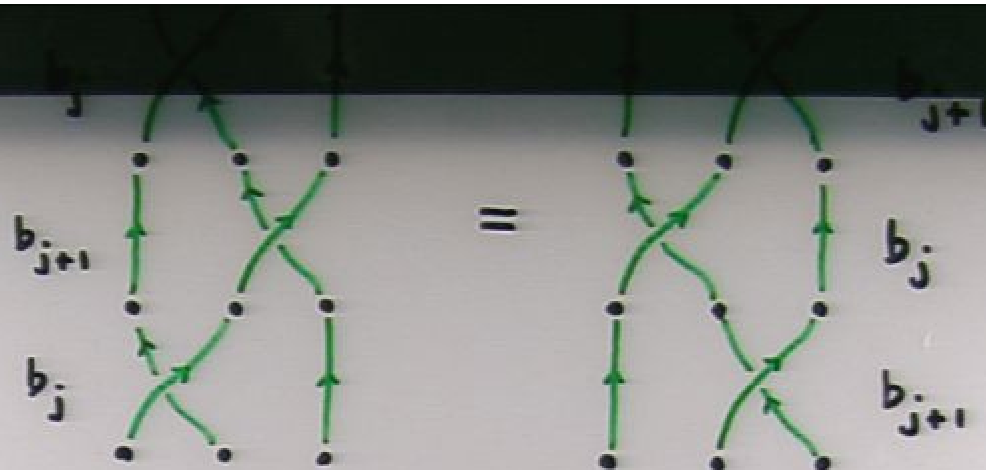


1-dim. unitary reps. labeled by Θ:

$$b_j \to e^{i\Theta} = q$$

$$b_j^{-1} \to e^{-i\Theta} = \bar{q}$$

$\Theta = 0$ bosons, $\Theta = \pi$ (fermions) give reps. of S_n.



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1. Introduction

Current algebra for quantum mechanics:

locality + symmetry

$\psi(x)$ a second-quantized field

$$\rho(x) = m\,\psi^*(x)\,\psi(x)$$

$$\vec{J}(x) = \frac{\hbar}{2i}\Big[\psi^*(x)\,\nabla\psi(x) - \big(\nabla\psi^*(x)\big)\,\psi(x)\Big]$$

Average ρ and $\vec{J}$ at fixed time with test functions

$$f \in C_0^\infty(M),\quad \vec{g} \in \mathrm{vect}_0(M),$$

so that we have ∞-dim. Lie alg.

(M = manifold of physical space).

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$$[\rho(f_1), \rho(f_2)] = 0$$

$$[\rho(f), J(\vec{g})] = i\hbar \rho(\vec{g} \cdot \nabla f)$$

$$[J(\vec{g}_1), J(\vec{g}_2)] = -i\hbar J([\vec{g}_1, \vec{g}_2])$$

$$[\vec{g}_1, \vec{g}_2] = \vec{g}_1 \cdot \nabla \vec{g}_2 - \vec{g}_2 \cdot \nabla \vec{g}_1$$

$$\varphi_s^{\vec{g}} \in \mathrm{Diff}(M)$$

$$\frac{\partial \varphi_s^{\vec{g}}}{\partial s} = \vec{g} \circ \varphi_s^{\vec{g}}$$

$$\varphi_{s=0}(x) \equiv x \; .$$

$$V(\varphi_s^{\vec{g}}) = e^{i \frac{s}{\hbar} J(\vec{g})}$$

$$U(sf) = e^{i \frac{s}{\hbar} \rho(f)}$$

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Exponentiating this "current algebra" of quantum mechanics leads to study of continuous unitary reps. (CURs) of

$$\boxed{C_0^\infty(M) \rtimes \mathrm{Diff}_0(M)}$$

in Hilbert space $\mathcal{H}$.

Generators of 1-parameter subgroups are self-adjoint mass density + momentum density operators.

What diffeomorphisms do:

$\varphi : M \longrightarrow M$

1-1, invertible, inverse 1-1

smooth (C^∞)

compactly supported

group under composition

identity element:

$\varphi(x) \equiv x$.

Diffeomorphisms act on configurations.

Diffeomorphisms associated with regions (locality)

Locality:

associate operators with
points or regions in space-time

Symmetry

describe features invariant
under some (Lie) group action

Local symmetry

associate group(s) with
points or regions in space-time

→ local current groups
(or algebras)
diffeomorphism groups
(or algebras of vector fields)

Very general way to describe unitary reps. with

$f \in C_0^\infty(M)$, $\varphi \in \mathrm{Diff}_0(M)$

Γ a configuration space over M; write $\gamma \in \Gamma$,

(i.e., natural action of $\mathrm{Diff}_0(M)$ on Γ) $(\varphi, \gamma) \to \varphi\gamma$

μ a measure on Γ quasi-invariant under $\mathrm{Diff}_0(M)$; then if we can identify γ with element of $C_0^\infty(M)'$, write:

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$$\mathcal{H} = L^2_\mu(\Gamma, \mathcal{M})$$

$$\Psi \in \mathcal{H}, \quad \Psi(\gamma) \in \mathcal{M}$$

$\mathcal{M}$ an inner product space

$$(\Phi, \Psi) = \int_\Gamma \langle \Phi(\gamma), \Psi(\gamma) \rangle_{\mathcal{M}} \, d\mu(\gamma)$$

$$U(f)\Psi(\gamma) = e^{i\langle \gamma, f \rangle}\Psi(\gamma)$$

$$V(\varphi)\Psi(\gamma) =$$

$$\underbrace{\chi_\varphi(\gamma)}_{\substack{\text{unitary 1-cocycle} \\ \text{acting in } \mathcal{M}}} \Psi(\varphi\gamma) \underbrace{\sqrt{\frac{d\mu_\varphi}{d\mu}(\gamma)}}_{\substack{\text{Radon-Nikodym} \\ \text{derivative} \\ \text{(real cocycle).}}}$$

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Can obtain cocycles in some cases from an <u>inducing construction</u>, beginning with a CUR of the <u>stability subgroup</u>

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$$G_\gamma = \left\{ \varphi \in \mathrm{Diff}_0(M) \,\middle|\, \varphi\gamma = \gamma \right\}.$$

2. Classification, prediction, unification of wide variety of quantum systems from CURs of $\mathrm{Diff}_0(M)$:

multiparticle systems
particle statistics
    bosons + fermions
    anyons + plektons
    parastatistics
    general topological effects
Aharanov-Bohm, etc.
∞ quantum systems
spin systems
quantum dipoles, etc.
nonlinear QM

of quantum systems from

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nonlinear QM

12

N-point configuration spaces

(QM of N particles):

$$\gamma = \{x_1, \cdots, x_N\} \subset M.$$

$$\varphi\gamma = \{\varphi(x_1), \cdots, \varphi(x_N)\}.$$

If $\varphi\gamma = \gamma$, then

φ implements a permutation

of the N points.

So for $M = \mathbb{R}^d$, $d \geq 2$,

$G_\gamma \longrightarrow S_N$ (symmetric group)

1-dim. unitary reps. of $S_N \longrightarrow$

bosons, fermions. $(M = \mathbb{C})$.

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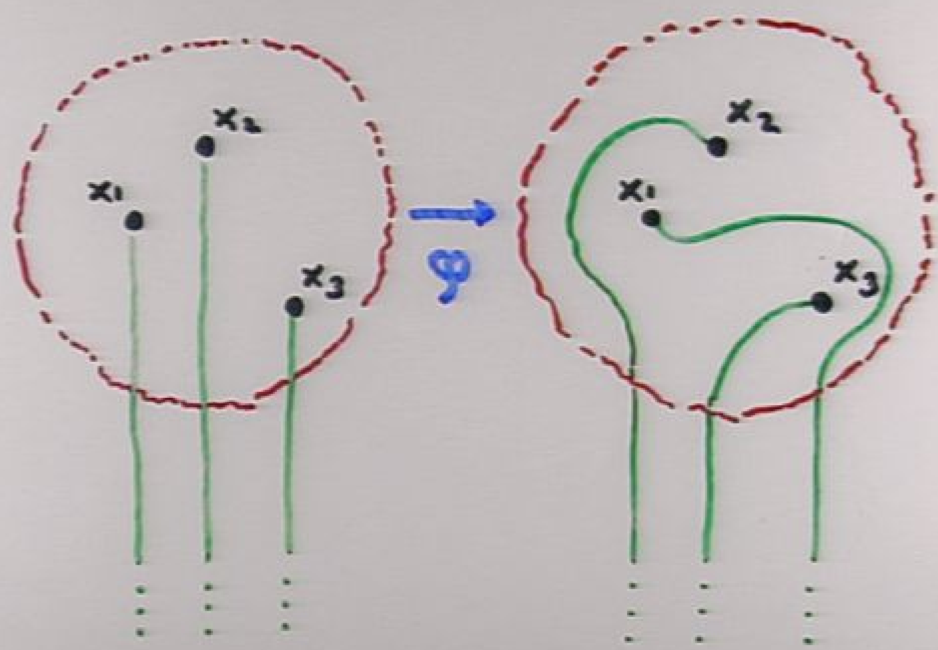
$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

diffeomorphism with compact support



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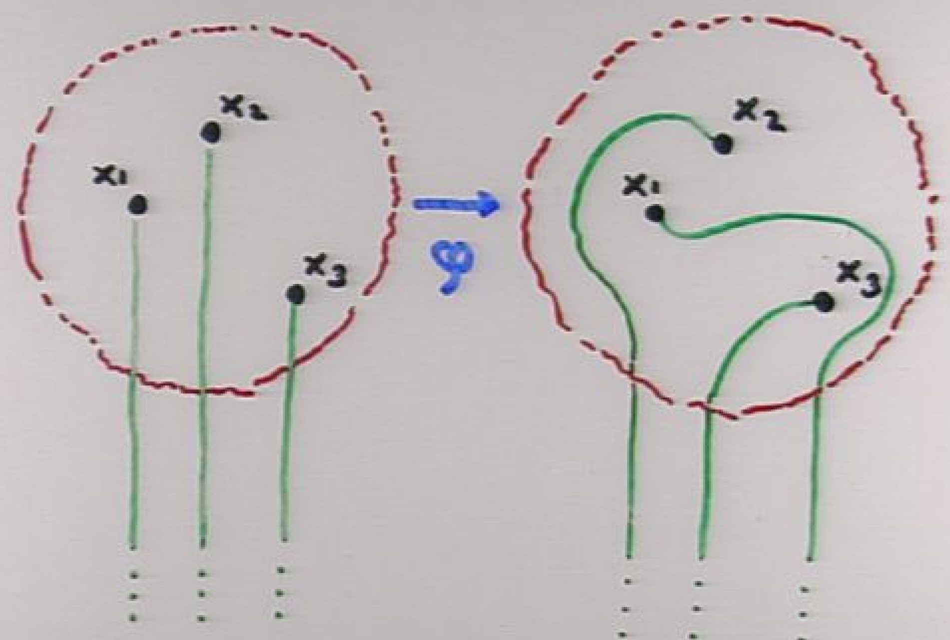


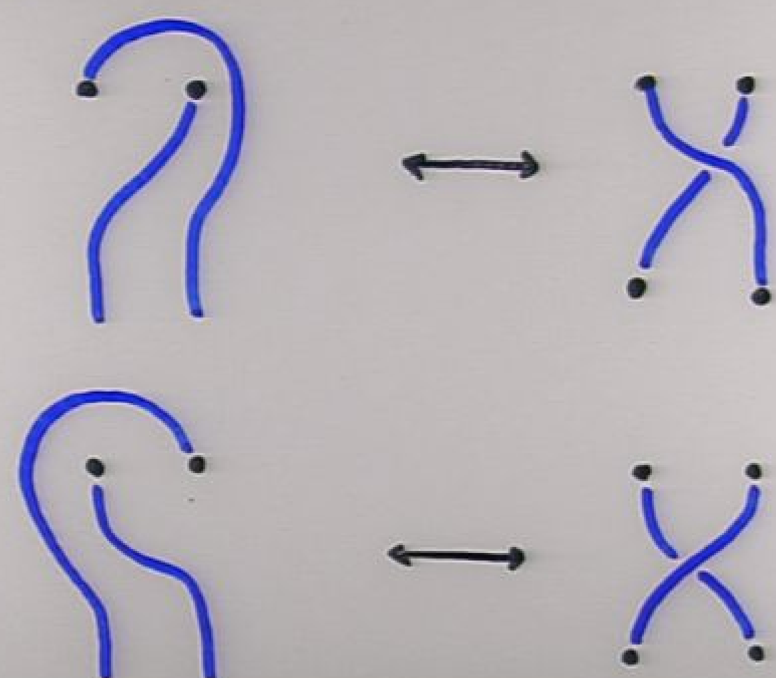
$\varphi : \{x_1, x_2, x_3\} \rightarrow \{x_1, x_2, x_3\}$

permutes the points

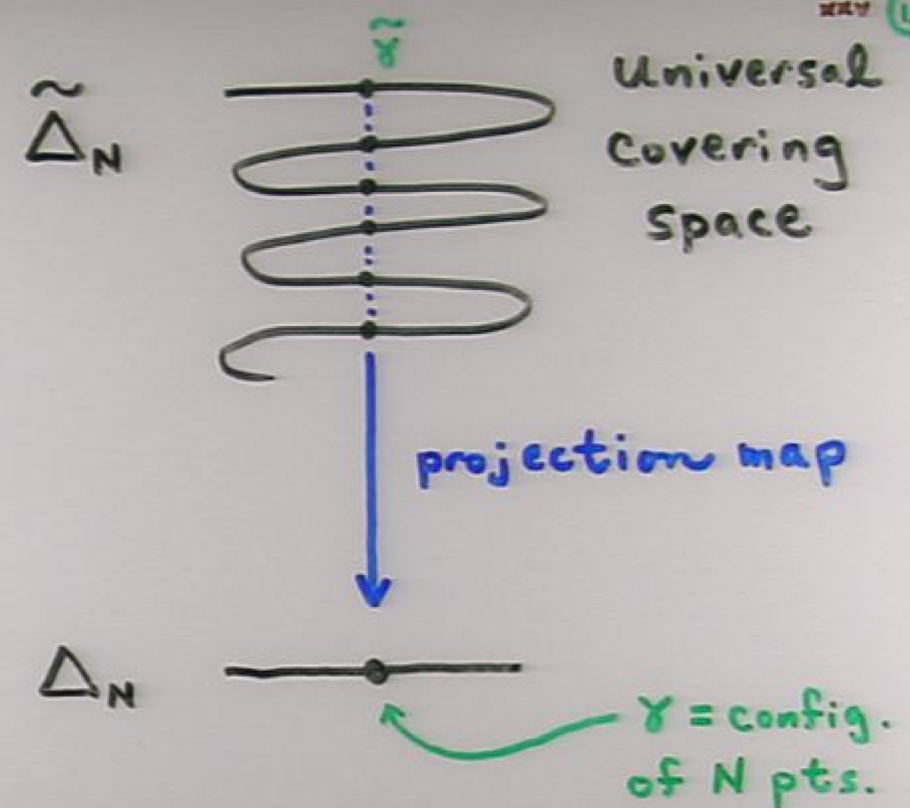
φ establishes a homotopy class

of paths from infinity

Correspondence of planar system of strands to braid:



Unitary rep. of braid group B_N



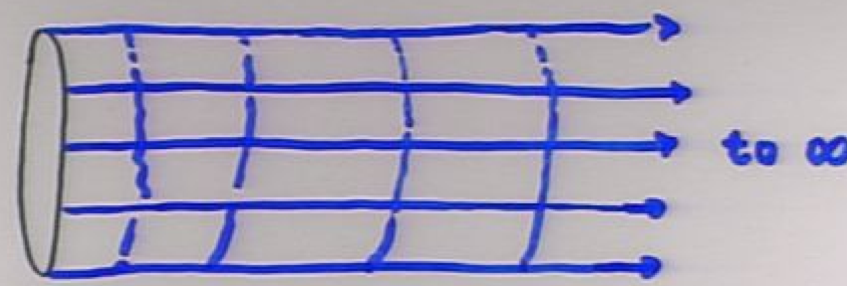
For $s \geq 3$, $\tilde{\Delta}_N$ has $N!$ sheets

$s = 2$, $\tilde{\Delta}_N$ has ∞ sheets

Unusual statistics for $s = 2$

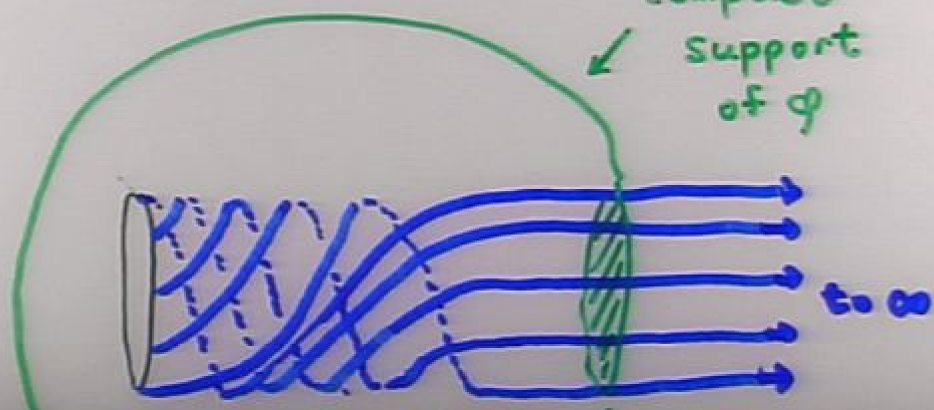
does not require identical

Single loop in $\mathbb{R}^3$:

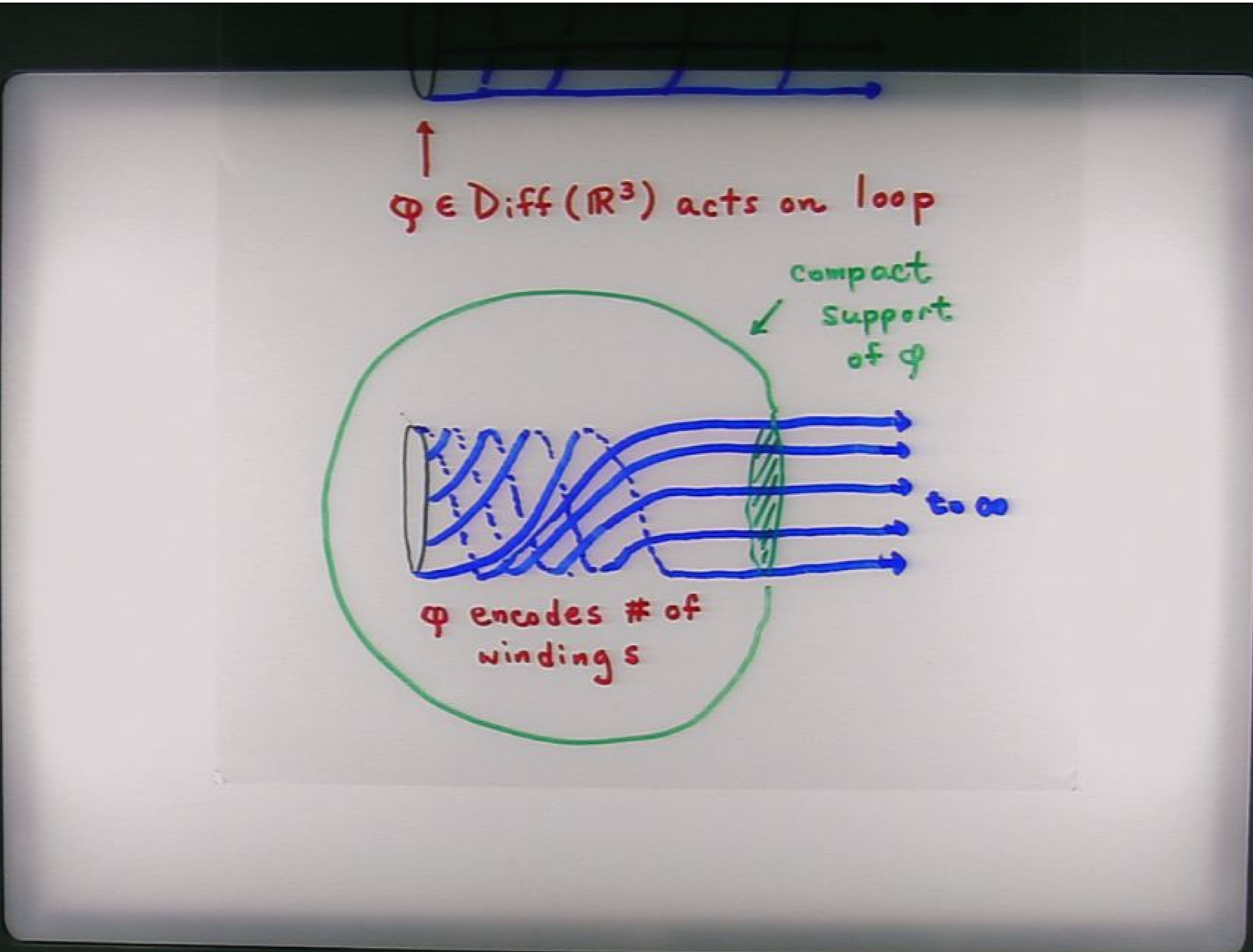


$\varphi \in \mathrm{Diff}(\mathbb{R}^3)$ acts on loop

compact support of φ



φ encodes # of windings


φ ∈ Diff (ℝ³) acts on loop
compact support of φ
to ∞
φ encodes # of windings

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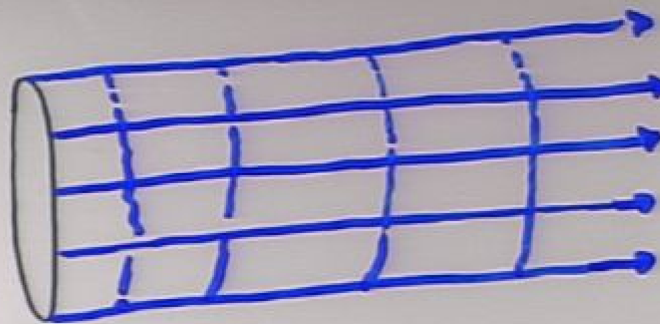

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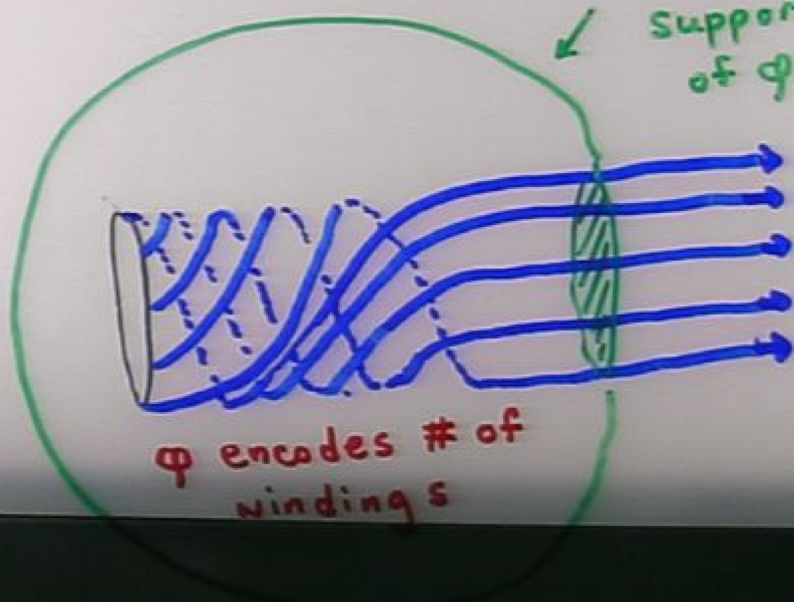

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Two loops