Title: Explorations in Numerical Relativity - Lecture 5

Date: Apr 08, 2011 11:30 AM

URL: http://pirsa.org/11040044

Abstract:

Applications Places System Fri Apr 8,

obe Reader

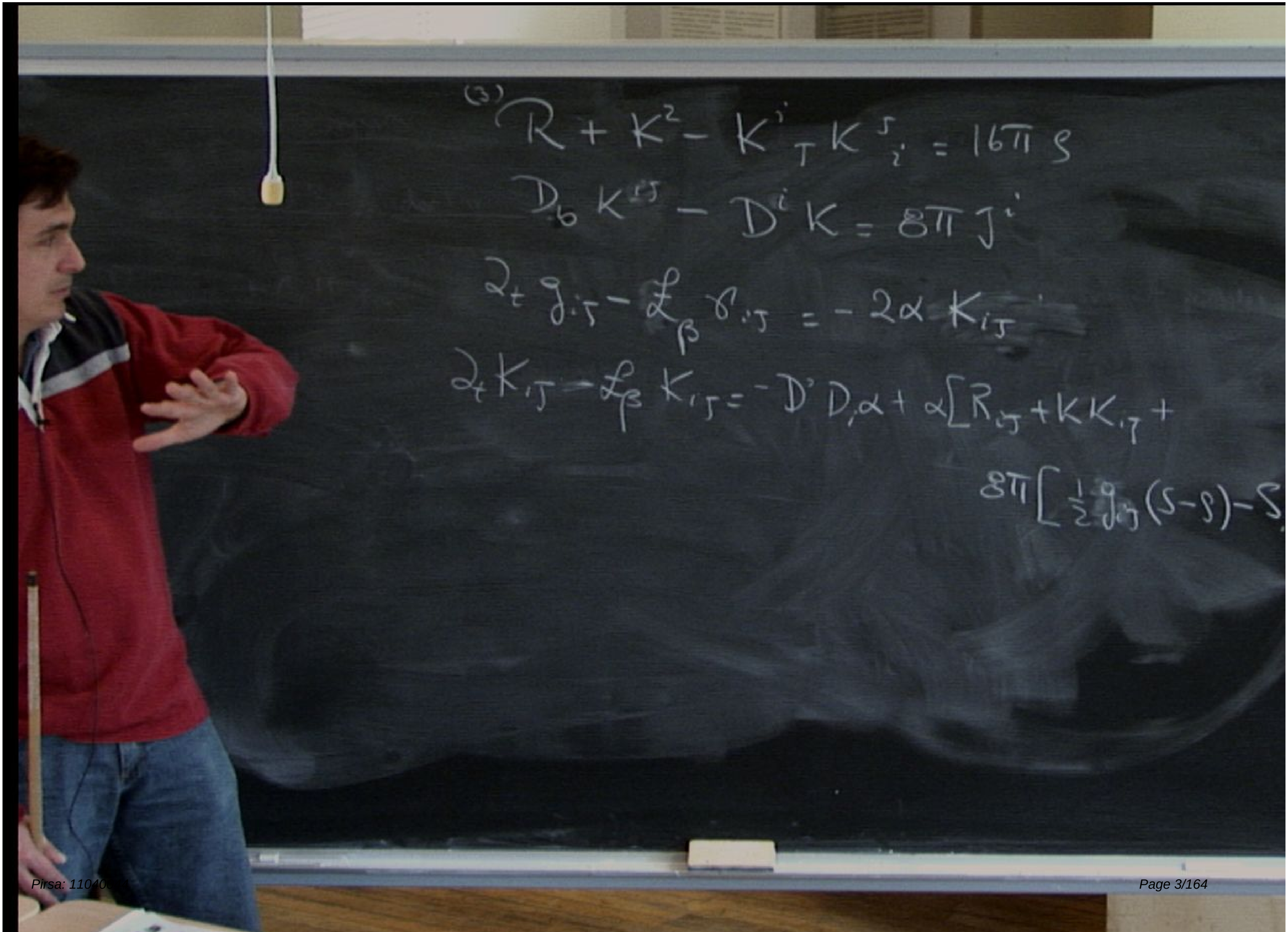
Tools Window Help

1 / 57

PHY 387N SPHERICAL SYMMETRY (SS) 1

- IN 3D SPACETIME NEED MATTER FOR DYNAMICS (BIRKHOFF'S THM, UNIQUENESS of SCHWARZSCHILD SOL^N AS SOL^N of $G_{ab} = 0$)

- WILL RESTRICT MATTER CONTENT TO SINGLE, MASSLESS, MINIMALLY-COUPLED SCALAR FIELD, ϕ
  - GOOD MODEL PROBLEM FOR STUDYING STRONG-FIELD, RADIATIVE S.T'S - INCLUDING BLACK HOLE FORMATION
  - EXHIBITS INTERESTING PHYSICAL BEHAVIOUR → CRITICAL PHENOMENA aka BLACK HOLE THRESHOLD PHENOMENA


$^{(3)}R + K^2 - K^i{}_j K^j{}_i = 16\pi\rho$
$D_j K^{ij} - D^i K = 8\pi j^i$
$\partial_t g_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}$
$\partial_t K_{ij} - \mathcal{L}_\beta K_{ij} = -D_i D_j \alpha + \alpha[R_{ij} + K K_{ij} +$
$8\pi[\frac{1}{2} g_{ij}(S-\rho) - S$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 1 / 57 Find

PHY 387N SPHERICAL SYMMETRY (SS)

- IN 3D SPACETIME NEED MATTER FOR DYNAMICS (BIRKHOFF'S THM, UNIQUENESS of SCHWARZSCHILD SOL^N AS SOL^N of $G_{ab} = 0$)

- WILL RESTRICT MATTER CONTENT TO SINGLE, MASSLESS, MINIMALLY-COUPLED SCALAR FIELD, ϕ
  - GOOD MODEL PROBLEM FOR STUDYING STRONG-FIELD, RADIATIVE S.T.'S — INCLUDING BLACK HOLE FORMATION
  - EXHIBITS INTERESTING PHYSICAL BEHAVIOUR → CRITICAL PHENOMENA aka BLACK HOLE THRESHOLD PHENOMENA

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 1 / 57 Find

AS EMKG & EKG (EINSTEIN-MASSLESS KLEIN-GORDON)

LAGRANGIAN DENSITY FOR EMKG SYSTEM

$$\mathcal{L} = \mathcal{L}_{GRAV} + \mathcal{L}_{\Phi}$$

$$= \sqrt{-g}\left(R - \frac{1}{2} \nabla_a \Phi \nabla^a \Phi \right)$$

"COVARIANT" E.O.M

$$G_{ab} = 8\pi T_{ab} = 8\pi \left(\nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab} \nabla_c \Phi \nabla^c \Phi \right)$$

$$\Box \Phi \equiv \nabla^a \nabla_a \Phi = 0$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 1 / 57 Find

AS EMKG & EKG (EINSTEIN-MASSLESS KLEIN-GORDON)

LAGRANGIAN DENSITY FOR EMKG SYSTEM

$$\mathcal{L} = \mathcal{L}_{GRAV} + \mathcal{L}_{\phi}$$

$$= \sqrt{-g}\left(R - \frac{1}{2} \nabla_a \phi \nabla^a \phi \right)$$

"COVARIANT" E.O.M

$$G_{ab} = 8\pi T_{ab} = 8\pi \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi \right)$$

$$\Box \phi = \nabla^a \nabla_a \phi = 0$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

$$G_{ab} = 8\pi T_{ab} = 8\pi \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi \right)$$

$$\Box \phi \equiv \nabla^a \nabla_a \phi = 0$$

3.1 FORM OF SPACETIME METRIC IN SS

- COORDINATES (t, r, θ, φ) ADAPTED TO S.S.

METRIC ON UNIT 2-SPHERE $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

- THEN MOST GENERAL 3-METRIC IS

$$\gamma_{ij} = \mathrm{diag}\left(a^2(r,t),\ r^2 b^2(r,t),\ r^2 b^2 \sin^2\theta \right) \qquad (1)$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 2 / 57 Find

3.1 FORM OF SPACETIME METRIC IN SS

- COORDINATES (t, r, θ, φ) ADAPTED TO S.S.

METRIC ON UNIT 2-SPHERE $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

- THEN MOST GENERAL 3-METRIC IS

$$\gamma_{ij} = \mathrm{diag}\left(a^2(r,t),\ r^2 b^2(r,t),\ r^2 b^2 \sin^2\theta\right) \quad (1)$$

THE LAPSE FUNCTION IS $\alpha(r,t)$, AND THE SHIFT VECTOR $\beta^i(r,t)$ HAS ONLY A RADIAL COMPONENT, $\beta(r,t)$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

3.1 FORM OF SPACETIME METRIC IN SS

- COORDINATES (t, r, θ, φ) ADAPTED TO S.S.

METRIC ON UNIT 2-SPHERE $d\underline{\Omega}^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

- THEN MOST GENERAL 3-METRIC IS

$$\gamma_{ij} = \mathrm{diag}\left(a^2(r,t),\ r^2 b^2(r,t),\ r^2 b^2 \sin^2\theta\right) \qquad (1)$$

THE LAPSE FUNCTION IS $\alpha(r,t)$, AND THE SHIFT VECTOR $\beta^i(r,t)$ HAS ONLY A RADIAL COMPONENT, $\beta(r,t)$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

- THEN MOST GENERAL 3-METRIC IS

$$\gamma_{ij} = \mathrm{diag}\left(a^2(r,t),\ r^2 b^2(r,t),\ r^2 b^2 \sin^2\theta\right) \quad (1)$$

THE LAPSE FUNCTION IS $\alpha(r,t)$, AND THE SHIFT VECTOR $\beta^i(r,t)$ HAS ONLY A RADIAL COMPONENT, $\beta(r,t)$

$$\beta^i = (\beta, 0, 0) \quad (2)$$

$$\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0) \quad (3)$$

- THE MOST GENERAL 4-METRIC IS THEN

- THEN MOST GENERAL 3-METRIC IS

$$\gamma_{ij} = \mathrm{diag}\left(a^2(r,t),\ r^2 b^2(r,t),\ r^2 b^2 \sin^2\theta\right) \qquad (1)$$

THE LAPSE FUNCTION IS $\alpha(r,t)$, AND THE SHIFT VECTOR $\beta^i(r,t)$ HAS ONLY A RADIAL COMPONENT, $\beta(r,t)$

$$\beta^i = (\beta, 0, 0) \qquad (2)$$

$$\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0) \qquad (3)$$

- THE MOST GENERAL 4-METRIC IS THEN

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

VECTOR $\beta^i(r,t)$ ~~HAS~~ ONLY ~~A~~ RADIAL COMPONENT, $\beta(r,t)$

$$\beta^i = (\beta, 0, 0) \qquad (2)$$

$$\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0) \qquad (3)$$

THE MOST GENERAL 4-METRIC IS THEN

$$ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_i\,dt\,dx^i + \gamma_{ij}\,dx^i dx^j$$

$$= (-\alpha^2 + a^2\beta^2)\,dt^2 + 2a^2\beta\,dt\,dr + a^2dr^2 + r^2b^2d\Omega^2 \quad (4)$$

CORRESPONDING EXTRINSIC CURVATURE TENSOR, $K^i{}_j$,

LIKE γ_{ij}, HAS ONLY TWO INDEPENDENT COMPONENTS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

$$\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0) \qquad (3)$$

- THE MOST GENERAL 4-METRIC IS THEN

$$ds^2 = (-\alpha^2 + \beta^i\beta_i)dt^2 + 2\beta_i\,dt\,dx^i + \gamma_{ij}dx^i dx^j$$

$$= (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta\,dt\,dr + a^2dr^2 + r^2b^2d\Omega^2 \qquad (4)$$

- CORRESPONDING EXTRINSIC CURVATURE TENSOR, $K^i{}_j$, LIKE γ_{ij}, HAS ONLY TWO INDEPENDENT COMPONENTS

$$K^i{}_j = \mathrm{diag}(K^r{}_r(r,t),\ \underbrace{K^\theta{}_\theta(r,t),\ K^\theta{}_\theta}_{\text{EASY TO SHOW EQUALITY}}) \qquad (5)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

$$\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0) \qquad (3)$$

- THE MOST GENERAL 4-METRIC IS THEN

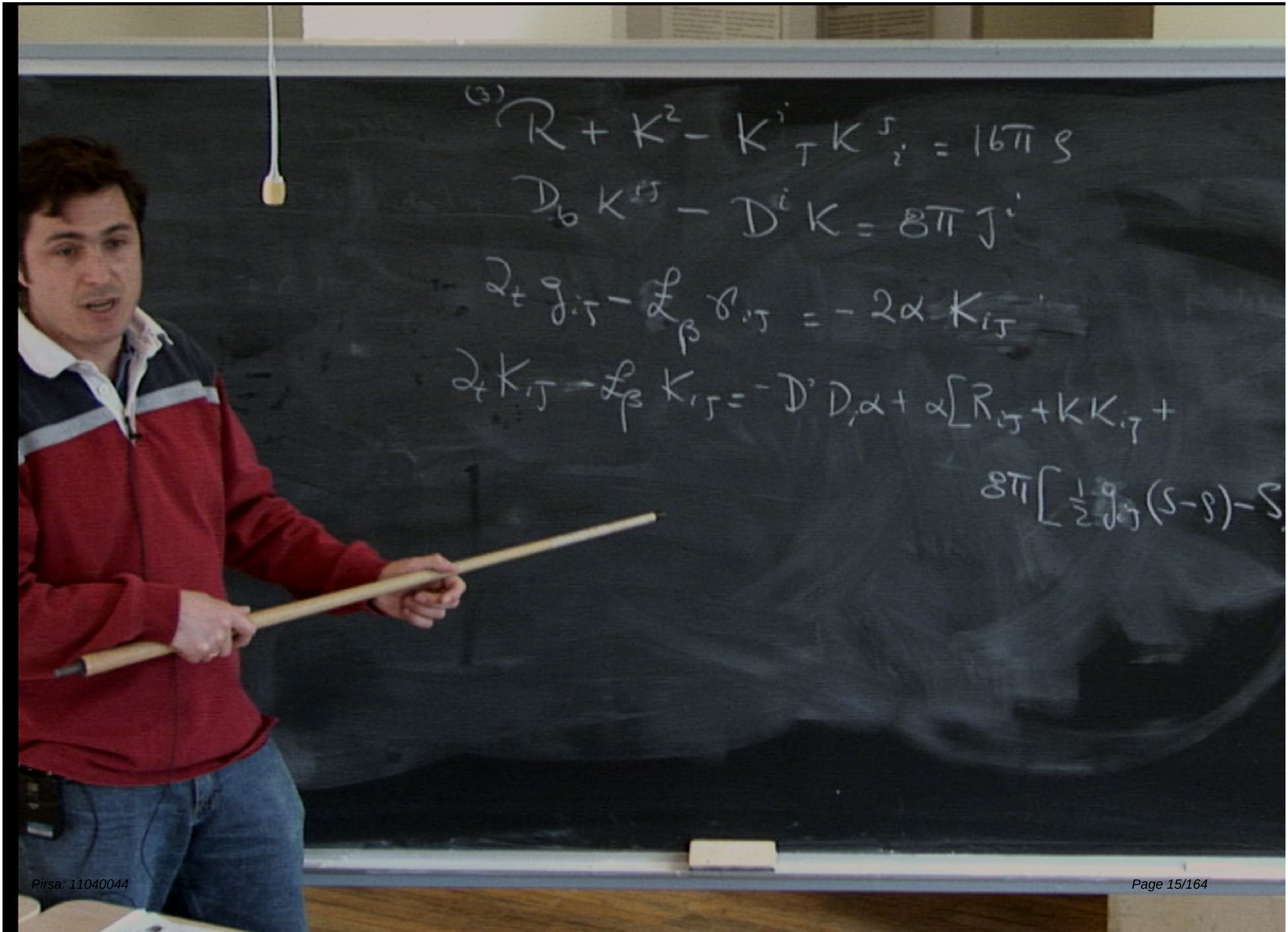
$$ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_i\,dt\,dx^i + \gamma_{ij}\,dx^i dx^j$$

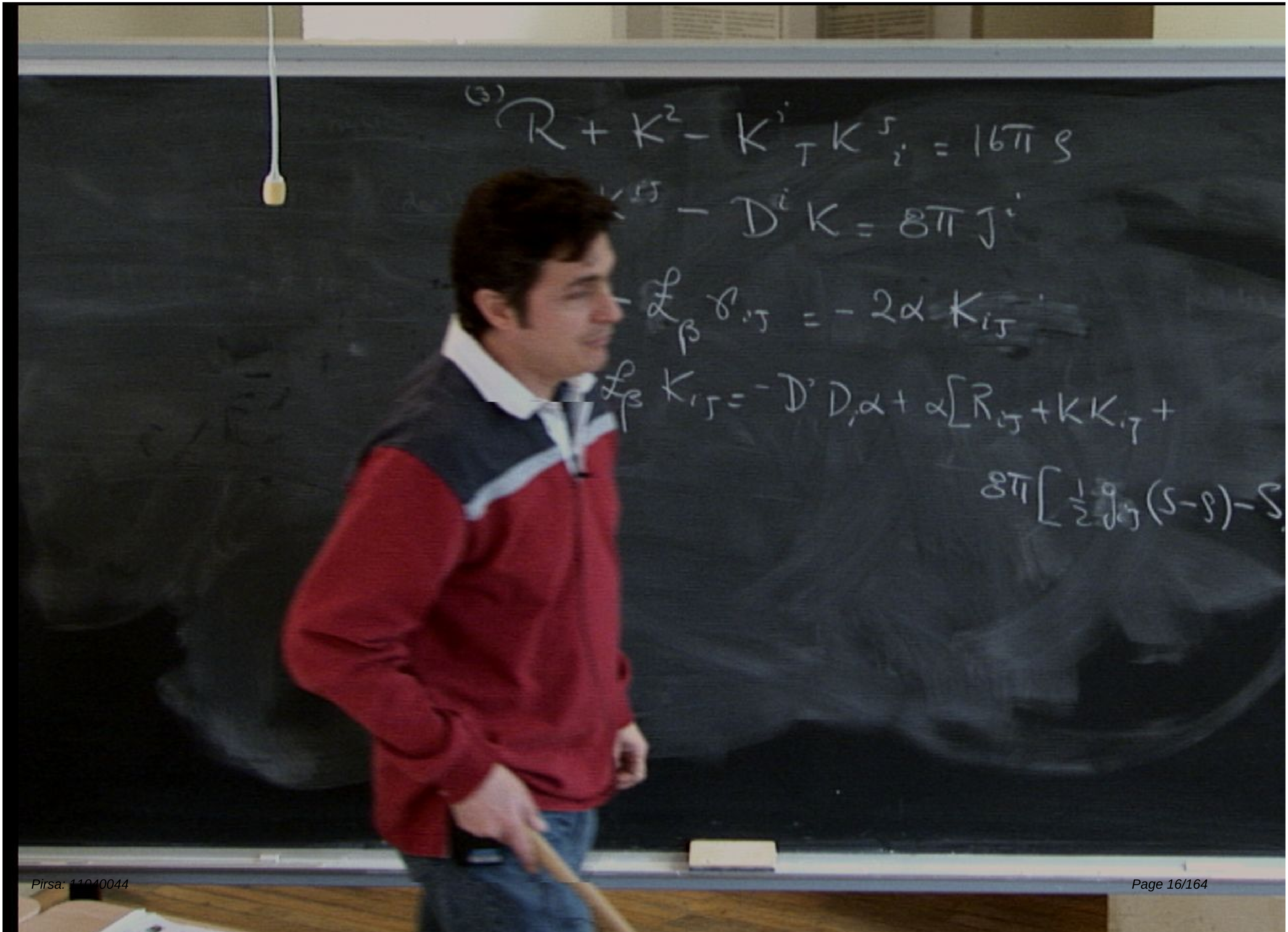
$$= (-\alpha^2 + a^2\beta^2)\,dt^2 + 2a^2\beta\,dt\,dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \qquad (4)$$

- CORRESPONDING EXTRINSIC CURVATURE TENSOR, $K^i{}_j$, LIKE γ_{ij}, HAS ONLY TWO INDEPENDENT COMPONENTS

$$K^i{}_j = \mathrm{diag}(K^r{}_r(r,t),\ K^\theta{}_\theta(r,t),\ K^\theta{}_\theta) \qquad (5)$$

EASY TO SHOW EQUALITY FROM


${}^{(3)}R + K^2 - K^i{}_j K^j{}_i = 16\pi\rho$
$D_b K^{ij} - D^i K = 8\pi j^i$
$\partial_t g_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}$
$\partial_t K_{ij} - \mathcal{L}_\beta K_{ij} = -D_i D_j \alpha + \alpha[R_{ij} + K K_{ij} +$
$8\pi[\frac{1}{2} g_{ij}(S-\rho) - S$


$^{(3)}R + K^2 - K^i{}_j K^j{}_i = 16\pi \rho$
$K^{ij} - D^i K = 8\pi j^i$
$\mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}$
$\mathcal{L}_\beta K_{ij} = -D_i D_j \alpha + \alpha [R_{ij} + K K_{ij} +$
$8\pi [\frac{1}{2} g_{ij} (S - \rho) - S$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

$$\beta_i = \gamma_{ij}\beta^j = (a^2\beta, 0, 0) \qquad (3)$$

• THE MOST GENERAL 4-METRIC IS THEN

$$ds^2 = (-\alpha^2 + \beta^i\beta_i)dt^2 + 2\beta_i\, dt\, dx^i + \gamma_{ij}\, dx^i dx^j$$

$$= (-\alpha^2 + a^2\beta^2)dt^2 + 2a^2\beta\, dt\, dr + a^2 dr^2 + r^2 b^2 d\Omega^2 \qquad (4)$$

• CORRESPONDING EXTRINSIC CURVATURE TENSOR, $K^i{}_j$, LIKE γ_{ij}, HAS ONLY TWO INDEPENDENT COMPONENTS

$$K^i{}_j = \mathrm{diag}(K^r{}_r(r,t),\ K^\theta{}_\theta(r,t),\ K^\theta{}_\theta) \qquad (5)$$

EASY TO SHOW EQUALITY FROM

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 2 / 57 Find

• CORRESPONDING EXTRINSIC CURVATURE TENSOR, $K^i{}_j$, LIKE γ_{ij}, HAS ONLY TWO INDEPENDENT COMPONENTS

$$K^i{}_j = \mathrm{diag}\left(K^r{}_r(r,t),\ \underbrace{K^\theta{}_\theta(r,t),\ K^\theta{}_\theta}\right) \qquad (5)$$

EASY TO SHOW EQUALITY FROM

$$K_{ij} = (2\alpha)^{-1}\left(-\partial_t \gamma_{ij} + D_i\beta_j + D_j\beta_i\right)$$

PALM BEAM SPHERICAL SYMMETRY ②

• SO, HAVE REDUCED TOTAL # OF GRAVITATIONAL KIN. DYN VBLS FROM 16 TO 6, AND, OF COURSE, THESE VBLS ARE FUNCTIONS ONLY of (r,t) RATHER THAN

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 3 / 57 Find

EASY TO SHOW EQUALITY FROM

$$K_{ij} = (2\alpha)^{-1}\left(-\partial_t \gamma_{ij} + D_i \beta_j + D_j \beta_i\right)$$

PHY 382N SPHERICAL SYMMETRY ②

SO, HAVE REDUCED TOTAL # OF GRAVITATIONAL KIN. DYN VBLS FROM 16 TO 6, AND, OF COURSE, THESE VBLS ARE FUNCTIONS ONLY of (r, t) RATHER THAN (x, y, z, t)

EINSTEIN EQUATIONS

(1) CONSTRAINTS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 3 / 57 Find

(x, y, z, t)

EINSTEIN EQUATIONS

(1) CONSTRAINTS

$$\boxed{\begin{aligned} &R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho &(6)\\ &D_j K_i{}^j - D_i K = 8\pi j_i &(7)\end{aligned}}$$

NOTE INDEX SHIFT RELATIVE TO PREVIOUS FORM

WHERE $\rho = n_\mu n_\nu T^{\mu\nu}$ (8)

$j_i = \gamma_{ik} j^k = -n_\mu T^\mu{}_i$ (9)

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 3 / 57 Find

$$R - K^{i}{}_{j} K^{j}{}_{i} + K^{2} = 16\pi \rho \qquad (6)$$

$$D_j K_i{}^{j} - D_i K = 8\pi j_i \qquad (7)$$

↳ NOTE INDEX SHIFT RELATIVE TO PREVIOUS TEXT

WHERE $\rho = n_\mu n_\nu T^{\mu\nu}$ (8)

$$j_i = \gamma_{ik} j^k = - n_\mu T^{\mu}{}_{i} \qquad (9)$$

RECALL: $n_\mu = (-\alpha, 0, 0, 0)$ (10)

(2) EVOLUTION EQUATIONS $\left(\dot{} \equiv \frac{\partial}{\partial t} \equiv \partial_t \right)$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 3 / 57 Find

$$\boxed{\begin{aligned} &R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho && (6) \\ &D_j K_i{}^j - D_i K = 8\pi j_i && (7) \end{aligned}}$$

NOTE INDEX SHIFT RELATIVE TO PREVIOUS TEXT

WHERE $\rho = n_\mu n_\nu T^{\mu\nu}$ (8)

$$j_i = \gamma_{ik} j^k = -n_\mu T^\mu{}_i \qquad (9)$$

RECALL: $n_\mu = (-\alpha, 0, 0, 0)$ (10)

(ii) EVOLUTION EQUATIONS $\left(\dot{} \equiv \frac{\partial}{\partial t} \equiv \partial_t \right)$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 3 / 57 Find

$$\boxed{\begin{aligned} &R - K^{i}{}_{j} K^{j}{}_{i} + K^2 = 16\pi \rho &(6)\\ &D_j K_i{}^{j} - D_i K = 8\pi j_i &(7)\end{aligned}}$$

NOTE INDEX SHIFT RELATIVE TO PREVIOUS TEXT

WHERE $\rho = n_\mu n_\nu T^{\mu\nu}$ (8)

$$j_i = \gamma_{ik} j^k = -n_\mu T^\mu{}_i \quad (9)$$

RECALL: $n_\mu = (-\alpha, 0, 0, 0)$ (10)

(2) EVOLUTION EQUATIONS $\left(\dot{} \equiv \frac{\partial}{\partial t} \equiv \partial_t \right)$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 3 / 57 Find

WHERE $\rho = n_\mu n_\nu T^{\mu\nu}$ (8)

$$j_i = \gamma_{ik} j^k = -n_\mu T^\mu{}_i \quad (9)$$

RECALL: $n_\mu = (-\alpha, 0, 0, 0)$ (10)

(2) EVOLUTION EQUATIONS $\left(\cdot \equiv \frac{\partial}{\partial t} \equiv \partial_t \right)$

$$\dot{\gamma}_{ij} = -2\alpha \gamma_{ik} K^k{}_j + \beta^k \partial_k \gamma_{ij} + \gamma_{ik} \partial_j \beta^k + \gamma_{kj} \partial_i \beta^k \quad (11)$$

$$\dot{K}^i{}_j = \beta^k \partial_k K^i{}_j - \partial_k \beta^i K^k{}_j + \partial_j \beta^k K^i{}_k - D^i D_j \alpha$$

$$(12) \quad + \alpha \left(R^i{}_j + K K^i{}_j + 4\pi (S - \rho) \delta^i{}_j - 8\pi S^i{}_j \right)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 3 / 57 Find

WHERE $\rho = n_\mu n_\nu T^{\mu\nu}$ (8)

$$j_i = \gamma_{ik} j^k = -n_\mu T^\mu{}_i \qquad (9)$$

RECALL: $n_\mu = (-\alpha, 0, 0, 0)$ (10)

(3) EVOLUTION EQUATIONS $\left(\dot{} \equiv \frac{\partial}{\partial t} \equiv \partial_t \right)$

$$\dot{\gamma}_{ij} = -2\alpha \gamma_{ik} K^k{}_j + \beta^k \partial_k \gamma_{ij} + \gamma_{ik} \partial_j \beta^k + \gamma_{kj} \partial_i \beta^k \qquad (11)$$

$$\dot{K}^i{}_j = \beta^k \partial_k K^i{}_j - \partial_k \beta^i K^k{}_j + \partial_j \beta^k K^i{}_k - D^i D_j \alpha$$

(12) $$+ \alpha \left(R^i{}_j + K K^i{}_j + 4\pi (S - \rho) \delta^i{}_j - 8\pi S^i{}_j \right)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 3 / 57 Find

$$\dot{\gamma}_{ij} = -2\alpha \gamma_{ik} K^k{}_j + \beta^k \partial_k \gamma_{ij} + \gamma_{ik} \partial_j \beta^k + \gamma_{kj} \partial_i \beta^k \quad (11)$$

$$\dot{K}^i{}_j = \beta^k \partial_k K^i{}_j - \partial_k \beta^i K^k{}_j + \partial_j \beta^k K^i{}_k - D^i D_j \alpha$$

$$(12) \qquad + \alpha \left(R^i{}_j + K K^i{}_j + 4\pi (S-\rho)\delta^i{}_j - 8\pi S^i{}_j \right)$$

WHERE $\underset{(13)}{S_{ij} = T_{ij}}$, $\underset{(14)}{S^i{}_j = \gamma^{ik} S_{kj}}$, $\underset{(15)}{S = S^i{}_i}$

- NEED CHRISTOFFEL SYMBOLS $\Gamma^i{}_{jk}$, RICCI COMPONENTS $R^i{}_j$ AND RICCI SCALAR R ASSOCIATED WITH γ_{ij} (1). USING STANDARD FORMULAE FIND FOLLOWING NON-VANISHING $\Gamma^i{}_{jk}$ ($' \equiv \frac{\partial}{\partial r} \equiv \partial_r$)

$\Gamma^r \ldots = a' \ldots \quad \Gamma^r \ldots (r^2 b^2)' \quad \Gamma^r \ldots \Gamma^r$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 4 / 57 Find

(13) (14) (15)

- NEED CHRISTOFFEL SYMBOLS $\Gamma^i{}_{jk}$, RICCI COMPONENTS $R^i{}_j$ AND RICCI SCALAR R ASSOCIATED WITH γ_{ij} (1). USING STANDARD FORMULAE FIND FOLLOWING NON-VANISHING $\Gamma^i{}_{jk}$

$\left(' \equiv \frac{\partial}{\partial r} \equiv \partial_r \right)$

$$\Gamma^r{}_{rr} = \frac{a'}{a} \qquad \Gamma^r{}_{\theta\theta} = -\frac{(r^2b^2)'}{2a^2} \qquad \Gamma^r{}_{\phi\phi} = \sin^2\theta \, \Gamma^r{}_{\theta\theta}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{(r^2b^2)'}{2(r^2b^2)} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \qquad (16a\text{-}g)$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \Gamma^\theta{}_{\theta r} \qquad \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 4 / 57 Find

$$2(r^2 b^2)$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \Gamma^{\theta}{}_{\theta r} \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

FROM THESE WE COMPUTE NON-VANISHING $R^i{}_j$

$$R^r{}_r = \frac{-2}{arb}\left(\frac{(rb)'}{a}\right)' \qquad (17a)$$

$$R^\theta{}_\theta = R^\phi{}_\phi = \frac{1}{a(rb)^2}\left(a - \left(\frac{rb}{a}(rb)'\right)'\right) \qquad (17b)$$

AND FINALLY, THE SCALAR CURVATURE, R, IS

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 4 / 57 Find

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \Gamma^{\theta}{}_{\theta r} \qquad \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta$$

FROM THESE WE COMPUTE NON-VANISHING $R^{i}{}_{j}$

$$R^{r}{}_{r} = \frac{-2}{arb}\left(\frac{(rb)'}{a}\right)' \qquad (17a)$$

$$R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = \frac{1}{a(rb)^2}\left(a - \left(\frac{rb}{a}(rb)'\right)'\right) \qquad (17b)$$

AND FINALLY, THE SCALAR CURVATURE, R, IS

$$R = R^{r}{}_{r} + R^{\theta}{}_{\theta} + R^{\phi}{}_{\phi} = R^{r}{}_{r} + 2R^{\theta}{}_{\theta}$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 4 / 57 Find

AND FINALLY, THE SCALAR CURVATURE, $\mathcal{R}$; IS

$$\mathcal{R} = \mathcal{R}^r{}_r + \mathcal{R}^\theta{}_\theta + \mathcal{R}^\phi{}_\phi = \mathcal{R}^r{}_r + 2\mathcal{R}^\theta{}_\theta$$

$$= -\frac{2}{arb}\left(\left(\frac{(rb)'}{a}\right)' + \frac{1}{rb}\left(\left(\frac{rb}{a}(rb)'\right)' - a\right)\right) \quad C(E)$$

IN THE EVOLUTION EQUATION FOR $K^i{}_j$ WE NEED TO EVALUATE $D^i D_j \alpha$:

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 5 / 57 Find

DAN BEAM SPHERICAL SYMMETRY (5)

$$D^i D_j \alpha = \gamma^{ik} D_k D_j \alpha = \gamma^{ik} D_k \partial_j \alpha$$

$$= \gamma^{ik} \left(\partial_k \partial_j \alpha - \Gamma^m{}_{jk} \partial_m \alpha \right)$$

USING RESULTS FROM ABOVE, WE FIND

$$D^r D_r \alpha = \frac{1}{a} \left(\frac{\alpha'}{a} \right)' \qquad (19a)$$

$$D^\theta D_\theta \alpha = D^\phi D_\phi \alpha = \frac{\alpha' (rb)'}{\ldots} \qquad (19b)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 5 / 57 Find

DHM 3E7M SPHERICAL SYMMETRY ⑤

$$D^i D_j \alpha = \gamma^{ik} D_k D_j \alpha = \gamma^{ik} D_k \partial_j \alpha$$

$$= \gamma^{ik} \left(\partial_k \partial_j \alpha - \Gamma^m{}_{jk} \partial_m \alpha \right)$$

USING RESULTS FROM ABOVE, WE FIND

$$D^r D_r \alpha = \frac{1}{a} \left(\frac{\alpha'}{a} \right)' \qquad (19a)$$

$$D^\theta D_\theta \alpha = D^\phi D_\phi \alpha = \frac{\alpha' (rb)'}{a^2 rb} \qquad (19b)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 5 / 57 Find

$$D^b D_b \alpha = D^d D_d \alpha = \frac{\alpha'(rb)'}{a^2 rb} \qquad (19b)$$

- ALSO NEED STRESS-TENSOR "COMPONENTS" ρ, j_i, $S^i{}_j$

- IN SPIRIT of HAMILTONIAN APPROACH, IT IS CONVENIENT TO INTRODUCE AUXILIARY FUNCTIONS

$$\Phi(r,t) \equiv \phi'(r,t) = \partial_r \phi(r,t) \qquad (20)$$

$$\Pi(r,t) \equiv \frac{a}{\alpha}\left(\dot\phi - \beta\phi'\right) \qquad (21)$$

VIEW Φ, Π AS "REAL" DYNAMICAL VBLS FOR

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 5 / 57 Find

* IN SPIRIT OF HAMILTONIAN APPROACH, IT IS CONVENIENT TO INTRODUCE AUXILIARY FUNCTIONS

$$\Phi(r,t) \equiv \phi'(r,t) = \partial_r \phi(r,t) \quad (20)$$

$$\pi(r,t) \equiv \frac{a}{\alpha}\left(\dot{\phi} - \beta \phi'\right) \quad (21)$$

VIEW Φ, π AS "REAL" DYNAMICAL VBLS FOR SCALAR FIELD; NOTE: FOR MASSLESS FIELD, VALUE OF ϕ IS MEANINGLESS ($\phi \to \phi + \text{cnst}$ STILL SATISFIES $\Box\phi = 0$) ALL "ACTION" IS IN GRADIENTS OF ϕ (I.E. IN Φ AND π)

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 5 / 57 Find

$$\pi(r,t) \equiv \frac{a}{\alpha}\left(\dot\phi - \beta\phi'\right) \qquad (21)$$

VIEW Φ, π AS "REAL" DYNAMICAL VARS FOR SCALAR FIELD; NOTE: FOR MASSLESS FIELD, VALUE of ϕ IS MEANINGLESS ($\phi \to \phi +$ const STILL SATISFIES $\Box\phi = 0$) ALL "ACTION" IS IN GRADIENTS OF ϕ (I.E. IN Φ AND π)

ALSO NOTE THAT WE HAVE (c/f (4))

$$g_{tt} = -\alpha^2 + a^2\beta^2 \qquad g_{tr} = g_{rt} = a^2\beta$$

(22a-e)

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 6 / 57 Find

ALSO NOTE THAT WE HAVE (c/f (a))

$$g_{tt} = -\alpha^2 + a^2\beta^2 \qquad g_{tr} = g_{rt} = a^2\beta$$

(22a-e)

$$g_{rr} = a^2 \qquad g_{\theta\theta} = r^2 b^2 \qquad g_{\phi\phi} = r^2 b^2 \sin^2\theta$$

AND

$$g^{tt} = -\alpha^{-2} \qquad g^{tr} = g^{rt} = \beta\alpha^{-2}$$

(23a-e)

$$g^{rr} = a^{-2} - \beta^2\alpha^{-2} \qquad g^{\theta\theta} = (rb)^{-2} \qquad g^{\phi\phi} = (rb\sin\theta)^{-2}$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 6 / 57 Find

AND

$$g^{tt} = -\alpha^{-2} \qquad g^{tr} = g^{rt} = \beta\alpha^{-2} \qquad (23a\text{-}e)$$

$$g^{rr} = a^{-2} - \beta^2\alpha^{-2} \qquad g^{\theta\theta} = (rb)^{-2} \qquad g^{\phi\phi} = (rb\sin\theta)^{-2}$$

THEN WE FIND

$$\nabla\phi \equiv \phi = g\,\partial_t\phi + g\,\partial_r\phi = -\frac{\Pi}{\alpha a} \qquad (24)$$

$$(\nabla^\mu\phi)(\nabla_\mu\phi) = \phi^{,\mu}\phi_{,\mu} = \frac{\Phi^2 - \Pi^2}{a^2} \qquad (25)$$

AND, AGAIN RECALLING THAT $n_\mu = (-\alpha, 0, 0, 0)$, WE FIND

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 6 / 57 Find

THEN WE FIND

$$\nabla^t \phi \equiv \phi^{;t} = g^{tt}\partial_t\phi + g^{tr}\partial_r\phi = -\frac{\Pi}{\alpha a} \qquad (24)$$

$$(\nabla^\mu\phi)(\nabla_\mu\phi) = \phi^{;\mu}\phi_{;\mu} = \frac{\Phi^2-\Pi^2}{a^2} \qquad (25)$$

AND, AGAIN RECALLING ~~THAT~~ $n_\mu = (-\alpha, 0, 0, 0)$, WE FIND

$$\rho = n_\mu n_\nu T^{\mu\nu} = \alpha^2 T^{tt} = \alpha^2\left(\phi^{;t}\phi^{;t} - \frac{1}{2}g^{tt}\phi^{;\mu}\phi_{;\mu}\right)$$

$$= \frac{\Phi^2+\Pi^2}{2a^2} \qquad (26)$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 6 / 57 Find

$$(\nabla^\mu\phi)(\nabla_\mu\phi) = \phi^{,\mu}\phi_{,\mu} = \frac{\Phi^2-\Pi^2}{a^2} \qquad (25)$$

AND, AGAIN RECALLING ~~THAT~~ $n_\mu = (-\alpha, 0, 0, 0)$, WE FIND

$$\rho = n_\mu n_\nu T^{\mu\nu} = \alpha^2 T^{tt} = \alpha^2\left(\phi^{,t}\phi^{,t} - \tfrac{1}{2} g^{tt}\phi^{,\mu}\phi_{,\mu}\right)$$

$$= \frac{\Phi^2+\Pi^2}{2a^2} \qquad (26)$$

$$j_i = (j_r, 0, 0)$$

$$j_r = -n_\mu T^\mu{}_i = \alpha T^0{}_i = \alpha\, \phi^{,t}\phi_{,r} = -\frac{\Phi\Pi}{a} \qquad (27)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 6 / 57 Find

AND, AGAIN RECALLING ~~THAT~~ $n_\mu = (-\alpha, 0, 0, 0)$, WE FIND

$$\rho = n_\mu n_\nu T^{\mu\nu} = \alpha^2 T^{tt} = \alpha^2 \left(\phi^{,t} \phi^{,t} - \tfrac{1}{2} g^{tt} \phi^{,\mu} \phi_{,\mu} \right)$$

$$= \frac{\Phi^2 + \Pi^2}{2a^2} \qquad (26)$$

$$j_i = (j_r, 0, 0)$$

$$j_r = -n_\mu T^\mu{}_i = \alpha T^0{}_i = \alpha \, \phi^{,t} \phi_{,r} = -\frac{\Phi \Pi}{a} \qquad (27)$$

PHY 387M SPHERICAL SYMMETRY ⑦

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

$$= \frac{\Phi^2 + \Pi^2}{2a^2} \qquad (26)$$

$$j_i = (j_r, 0, 0)$$

$$j_r = -n_\mu T^\mu{}_i = \alpha T^0{}_i = \alpha\, \phi^{,t} \phi_{,r} = -\frac{\Phi \Pi}{a} \qquad (27)$$

PHY 387M SPHERICAL SYMMETRY ④

FINALLY, WE HAVE THE SPATIAL STRESS COMPONENTS

$$S^i{}_j = \gamma^{ik} S_{kj} = \gamma^{ik} T_{kj}$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

$2a$

$$j_i = (j_r, 0, 0)$$

$$j_r = -n_\mu T^\mu{}_i = \alpha T^0{}_i = \alpha \phi'^+ \phi_{,r} = -\frac{\dot{\phi}\pi}{a} \quad (27)$$

PHY 387M SPHERICAL SYMMETRY ④

FINALLY, WE HAVE THE SPATIAL STRESS COMPONENTS

$$S^i{}_j = \gamma^{ik} S_{kj} = \gamma^{ik} T_{kj}$$

FROM WHICH WE FIND

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

$$S^i{}_j = \gamma^{ik} S_{kj} = \gamma^{ik} T_{kj}$$

FROM WHICH WE FIND

$$S^r{}_r = \rho = \frac{\Phi^2 + \Pi^2}{2a^2} \qquad (28)$$

$$S^\theta{}_\theta = S^\phi{}_\phi = \frac{\Pi^2 - \Phi^2}{2a^2} \qquad (29)$$

$$S - \rho = 2S^\theta{}_\theta = \frac{\Pi^2 - \Phi^2}{a^2} \qquad (30)$$

• WE CAN NOW ASSEMBLE THE ABOVE RESULTS TO PRODUCE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

$$a^2$$

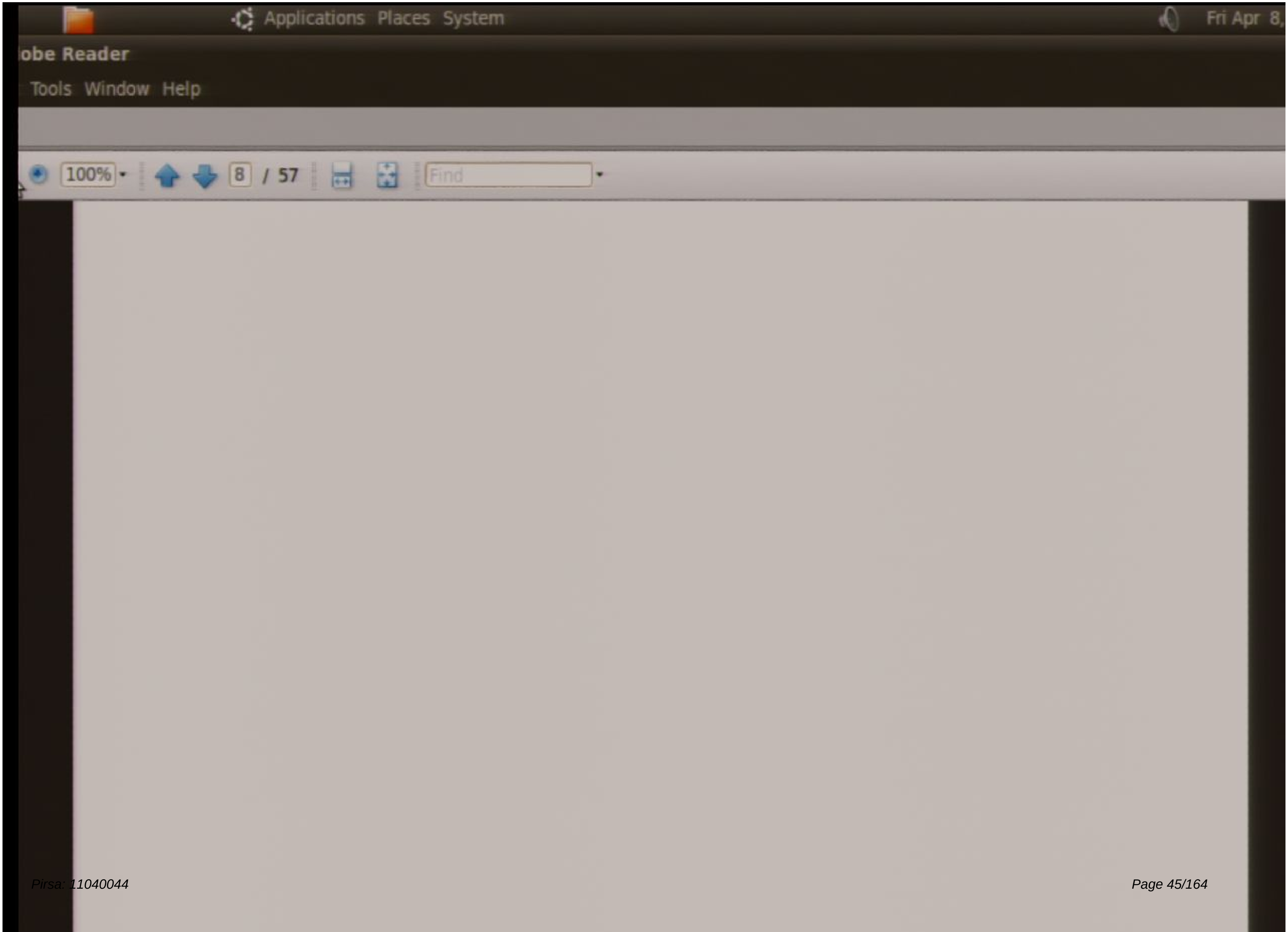
- WE CAN NOW ASSEMBLE THE ABOVE RESULTS TO PROJECT THE SPHERICALLY-SYMMETRIC SPECIALIZATION of THE GENERAL 3+1 EQUATIONS (6), (7), (11) ; (12)

A) HAMILTONIAN CONSTRAINT

$$R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho$$

$$-K^i{}_j K^j{}_i + K^2 = -\left(K^r{}_r{}^2 + 2K^\theta{}_\theta{}^2\right) + \left(K^r{}_r + 2K^\theta{}_\theta\right)^2$$

$$= 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2$$


Applications Places System
Fri Apr 8,
obe Reader
Tools Window Help
100%
8 / 57
Find

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 8 / 57 Find

$$= K^r{}_r + 2\Gamma^r{}_{r\theta}(K^r{}_r - K^\theta{}_\theta)$$

$$D_r K = (K^r{}_r + 2K^\theta{}_\theta)'$$

THEN WE HAVE FROM (7) AND (27)

$$\boxed{K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = 4\pi \frac{\Phi \Pi}{a} \qquad (32)}$$

c) EVOLUTION EQUATIONS FOR γ_{ij} (a, b)

- FOLLOW DIRECTLY FROM (11), RECALL, CAN BE VIEWED AS DEF^{N} OF $K^i{}_j$

$$\dot{a} = -\alpha a K^r{}_r + (a\beta)' \qquad (33)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

$$R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho$$

$$-K^i{}_j K^j{}_i + K^2 = -(K^r{}_r{}^2 + 2K^\theta{}_\theta{}^2) + (K^r{}_r + 2K^\theta{}_\theta)^2$$

$$= 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2$$

$$\boxed{R + 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2 = 8\pi\,\frac{\Phi^2 + \Pi^2}{a^2} \quad (31)}$$

SEE (18)

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NON-TRIVIAL)

• FIRST NOTE THAT

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

A) HAMILTONIAN CONSTRAINT

$$R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho$$

$$-K^i{}_j K^j{}_i + K^2 = -\left(K^r{}_r{}^2 + 2K^\theta{}_\theta{}^2\right) + \left(K^r{}_r + 2K^\theta{}_\theta\right)^2$$

$$= 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2$$

$$\boxed{R + 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2 = 8\pi \frac{\Phi^2 + \Pi^2}{a^2} \quad (31)}$$

SEE (18)

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NONTRIVIAL)

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

GENERAL 3+1 EQUATIONS (6), (7), (11) ; (12)

A) HAMILTONIAN CONSTRAINT

$$R - K^i{}_j K^j{}_i + K^2 = 16\pi\rho$$

$$-K^i{}_j K^j{}_i + K^2 = -\left(K^r{}_r{}^2 + 2K^\theta{}_\theta{}^2\right) + \left(K^r{}_r + 2K^\theta{}_\theta\right)^2$$

$$= 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2$$

$$\boxed{R + 4K^r{}_r K^\theta{}_\theta + 2K^\theta{}_\theta{}^2 = 8\pi\,\frac{\Phi^2 + \Pi^2}{a^2} \quad (31)}$$

SEE (18)

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NONTRIVIAL)

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 8 / 57 Find

$$R + 4K^r{}_r K^\theta{}_\theta + 2 K^\theta{}_\theta{}^2 = 8\pi \frac{\Phi^2 + \Pi^2}{a^2} \quad (31)$$

SEE (1ɛ)

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NONTRIVIAL)

• FIRST NOTE THAT

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

$$= K^r{}_r{}' + 2\Gamma^\theta{}_{r\theta}\left(K^r{}_r - K^\theta{}_\theta\right)$$

$$D_r K = \left(K^r{}_r + 2K^\theta{}_\theta\right)'$$

THEN WE HAVE FROM (3) AND (27)

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 8 / 57 Find

$$\ldots = 8\pi \frac{\rho}{a^2} + \ldots \quad (31)$$

SEE (18)

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NONTRIVIAL)

• FIRST NOTE THAT

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

$$= K^r{}_r{}' + 2\Gamma^\theta{}_{r\theta}\left(K^r{}_r - K^\theta{}_\theta\right)$$

$$D_r K = \left(K^r{}_r + 2K^\theta{}_\theta\right)'$$

THEN WE HAVE FROM (3) AND (27)

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 8 / 57 Find

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

$$= K^r{}_r{}' + 2\Gamma^\theta{}_{r\theta}\left(K^r{}_r - K^\theta{}_\theta\right)$$

$$D_r K = \left(K^r{}_r + 2K^\theta{}_\theta\right)'$$

THEN WE HAVE FROM (3) AND (27)

$$\boxed{K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = 4\pi \frac{\Phi \Pi}{a} \qquad (32)}$$

c) EVOLUTION EQUATIONS FOR γ_{ij} (a, b)

FOLLOW DIRECTLY FROM (11). RECALL, CAN BE VIEWED AS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 8 / 57 Find

$$rb \qquad\qquad a$$

c) EVOLUTION EQUATIONS FOR γ_{ij} (a, b)

- FOLLOW DIRECTLY FROM (11). RECALL, CAN BE VIEWED AS DEFN OF $K^i{}_j$

$$\boxed{\begin{aligned} \dot{a} &= -\alpha a K^r{}_r + (a\beta)' &(33)\\ \dot{b} &= -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' &(34)\end{aligned}}$$

D) EVOLUTION EQUATIONS FOR $K^i{}_j$ $(K^r{}_r, K^\theta{}_\theta)$

- FOLLOW DIRECTLY FROM (12) AND ...

c) EVOLUTION EQUATIONS FOR γ_{ij} (a, b)

• FOLLOW DIRECTLY FROM (11). RECALL, CAN BE VIEWED AS DEF^{N} OF $K^i{}_j$

$$\boxed{\begin{aligned} \dot{a} &= -\alpha a K^r{}_r + (a\beta)' && (33) \\ \dot{b} &= -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' && (34) \end{aligned}}$$

D) EVOLUTION EQUATIONS FOR $K^i{}_j$ $(K^r{}_r, K^\theta{}_\theta)$

• FOLLOW DIRECTLY FROM (12) AND OTHER RESULTS ABOVE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 8 / 57 Find

- FOLLOW DIRECTLY FROM (11), RECALL, CAN BE VIEWED AS DEF^{N} OF $K^i{}_j$

$$\boxed{\begin{aligned} \dot{a} &= -\alpha a K^r{}_r + (a\beta)' \qquad (33) \\ \dot{b} &= -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' \qquad (34) \end{aligned}}$$

D) EVOLUTION EQUATIONS FOR $K^i{}_j$ $(K^r{}_r, K^\theta{}_\theta)$

- FOLLOW DIRECTLY FROM (12) AND OTHER RESULTS ABOVE

$$\dot{K}^r{}_r = \beta K^r{}_r{}' - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left(-\frac{2}{rab}\left(\frac{(rb)'}{a}\right)' + K K^r{}_r - 8\pi \frac{\Phi^2}{a^2}\right)$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 8 / 57 Find

• FOLLOW DIRECTLY FROM (11), RECALL, CAN BE VIEWED AS DEF^{N} OF $K^i{}_j$

$$\boxed{\begin{aligned} \dot{a} &= -\alpha a K^r{}_r + (a\beta)' \qquad (33) \\ \dot{b} &= -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' \qquad (34) \end{aligned}}$$

D) EVOLUTION EQUATIONS FOR $K^i{}_j$ $(K^r{}_r, K^\theta{}_\theta)$

• FOLLOW DIRECTLY FROM (12) AND OTHER RESULTS ABOVE

$$\dot{K}^r{}_r = \beta K^r{}_r{}' - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left(-\frac{2}{rab}\left(\frac{(rb)'}{a}\right)' + K K^r{}_r - 8\pi \frac{\Phi^2}{a^2}\right) \qquad (35)$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 9 / 57 Find

D) EVOLUTION EQUATIONS FOR $K^i{}_j$ $(K^r{}_r, K^\theta{}_\theta)$

- FOLLOW DIRECTLY FROM (12) AND OTHER RESULTS ABOVE

$$\boxed{\dot K^r{}_r = \beta K^r{}_r{}' - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left(-\frac{2}{rab}\left(\frac{(rb)'}{a}\right)' + K K^r{}_r - 8\pi\frac{\Phi^2}{a^2}\right) \qquad (35)}$$

$$\boxed{\dot K^\theta{}_\theta = \beta K^\theta{}_\theta{}' + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left(\frac{\alpha rb}{a}(rb)'\right)' + \alpha K K^\theta{}_\theta \qquad (36)}$$

- (31)-(36) ARE THE COMPLETE 3+1 EQUATIONS FOR THE GEOMETRIC VARIABLES (NOTE: WE HAVE ...

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 7 / 57 Find

PHY 387M SPHERICAL SYMMETRY

4

FINALLY, WE HAVE THE SPATIAL STRESS COMPONENTS

$$S^i{}_j = \gamma^{ik} S_{kj} = \gamma^{ik} T_{kj}$$

FROM WHICH WE FIND

$$S^r{}_r = \rho = \frac{\Phi^2 + \Pi^2}{2a^2} \qquad (28)$$

$$S^\theta{}_\theta = S^\phi{}_\phi = \frac{\Pi^2 - \Phi^2}{2a^2} \qquad (29)$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 9 / 57 Find

$$\dot{K}^{\theta}{}_{\theta} = \beta K^{\theta}{}_{\theta}{}' + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left(\frac{\alpha rb}{a}(rb)'\right)' + \alpha K K^{\theta}{}_{\theta} \qquad (36)$$

← (31)-(36) ARE THE COMPLETE 3+1 EQUATIONS FOR THE GEOMETRIC VARIABLES (NOTE: WE HAVE SAID NOTHING YET RE COORDINATE CHOICES, I.E. RE SPECIFICATIONS OF α AND β)

MASSLESS KLEIN GORDON EQUATION

← NEED E.O.M. FOR Φ AND Π

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

• NEED E.O.M. FOR Φ AND Π

• RECALL DEF^{N} OF Π, (21)

$$\Pi = \frac{a}{\alpha}\left(\dot{\phi} - \beta\phi'\right)$$

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\Pi + \beta\phi' = \frac{\alpha}{a}\Pi + \beta\Phi$$

• BUT $\dot{\phi}' = \dot{\Phi}$, SO

$$\boxed{\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{a}\Pi\right)' \qquad (32)}$$

• TO FIND $\dot{\Pi}$ EQN, RECALL THAT

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

PHY 387M SPHERICAL SYMMETRY (5)

CHARACTERISTIC ANALYSIS of THE SCALAR FIELD

REF: COURANT & HILBERT "METHODS of MATH. PHYS",
VOL II, ch 5

• EQN^S (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u = (\Phi, \Pi)^T$$

WE CAN WRITE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 10 / 57 Find

$$\rightarrow \partial_t\left(\sqrt{-g}\, g^{\nu t} \partial_\nu \phi\right) + \partial_r\left(\sqrt{-g}\, g^{\nu r} \partial_\nu \phi\right)$$

$$= \partial_t\left(\alpha a r^2 b^2 \left(-\alpha^{-2} \dot{\phi} + \beta \alpha^{-2} \phi'\right)\right)$$

$$+ \partial_r\left(\alpha a r^2 b^2 \left(\left(a^{-2} - \beta^2 \alpha^{-2}\right) \phi' + \beta \alpha^{-2} \dot{\phi}\right)\right)$$

$$= 0$$

$$\rightarrow \dot{\left(r^2 b^2 \Pi\right)} = \left(r^2 b^2 \left(\beta \Pi + \frac{\alpha}{a} \Phi\right)\right)'$$

$\hookrightarrow$ WRITE AS $(r^2 b^2) \dot{\Pi} + \dot{(r^2 b^2)} \Pi$ AND USE

EVOLUTION EQUATION (39) FOR $\dot{b}$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 8 / 57 Find

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NONTRIVIAL)

• FIRST NOTE THAT

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

$$= K^r{}_r{}' + 2\Gamma^\theta{}_{r\theta}\left(K^r{}_r - K^\theta{}_\theta\right)$$

$$D_r K = \left(K^r{}_r + 2K^\theta{}_\theta\right)'$$

THEN WE HAVE FROM (7) AND (27)

$$\boxed{K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = 4\pi \frac{\Phi \Pi}{a}} \qquad (32)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 8 / 57 Find

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NONTRIVIAL)

• FIRST NOTE THAT

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

$$= K^r{}_r{}' + 2\Gamma^\theta{}_{r\theta}\left(K^r{}_r - K^\theta{}_\theta\right)$$

$$D_r K = \left(K^r{}_r + 2K^\theta{}_\theta\right)'$$

THEN WE HAVE FROM (7) AND (27)

$$\boxed{K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = 4\pi \frac{\Phi \Pi}{a} \qquad (32)}$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 8 / 57 Find

B) MOMENTUM CONSTRAINT (ONLY r-COMPONENT IS NON-TRIVIAL)

• FIRST NOTE THAT

$$D_i K^i{}_r = \partial_i K^i{}_r + \Gamma^i{}_{mi} K^m{}_r - \Gamma^m{}_{ri} K^i{}_m$$

$$= K^r{}_r{}' + 2\Gamma^\theta{}_{r\theta}\left(K^r{}_r - K^\theta{}_\theta\right)$$

$$D_r K = \left(K^r{}_r + 2K^\theta{}_\theta\right)'$$

THEN WE HAVE FROM (7) AND (27)

$$\boxed{K^\theta{}_\theta{}' + \frac{(rb)'}{rb}\left(K^\theta{}_\theta - K^r{}_r\right) = 4\pi \frac{\Phi\Pi}{a} \qquad (32)}$$

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 8 / 57 Find

- FOLLOW DIRECTLY FROM (11), RECALL, CAN BE VIEWED AS DEF^{N} OF $K^i{}_j$

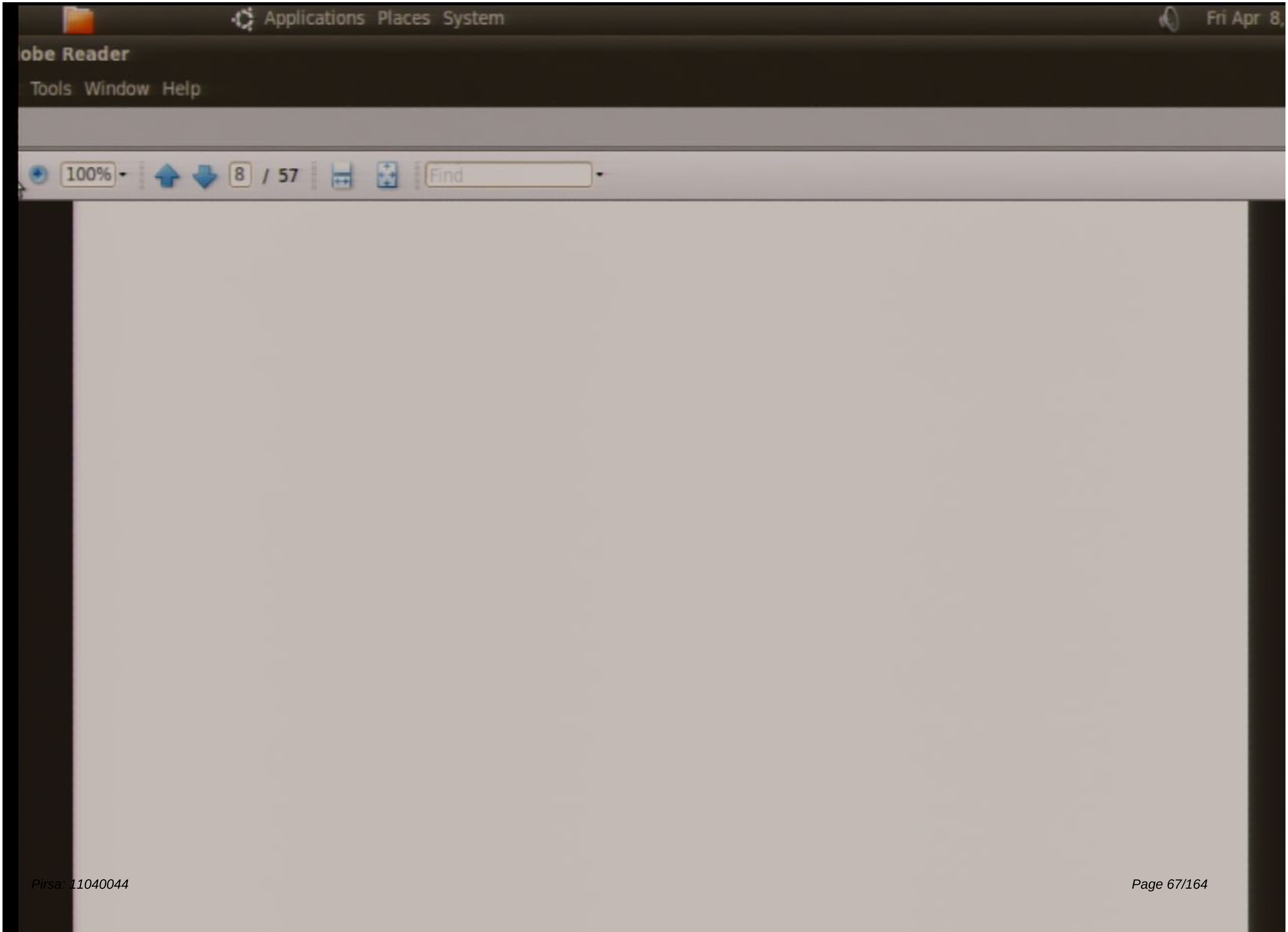
$$\dot a = -\alpha a K^r{}_r + (a\beta)' \qquad (33)$$

$$\dot b = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)' \qquad (34)$$

D) EVOLUTION EQUATIONS FOR $K^i{}_j$ $(K^r{}_r, K^\theta{}_\theta)$

- FOLLOW DIRECTLY FROM (12) AND OTHER RESULTS ABOVE

$$\dot K^r{}_r = \beta K^r{}_r{}' - \frac{1}{a}\left(\frac{\alpha'}{a}\right)' + \alpha\left(-\frac{2}{ra}\left(\frac{(rb)'}{b}\right)' + K K^r{}_r - 8\pi \frac{\Phi^2}{a^2}\right)$$


Applications Places System
Fri Apr 8,
obe Reader
Tools Window Help
100%
8 / 57
Find

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

• BUT $\dot{\phi}' = \dot{\Phi}$, SO

$$\boxed{\dot{\Phi} = \left(\lambda \Phi + \frac{\alpha}{a} \Pi \right)' \qquad (32)}$$

• TO FIND $\dot{\Pi}$ EQN, RECALL THAT

$$\Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right)$$

$$\Box \phi = 0 \rightarrow \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

GEOMETRIC VARIABLES (NOTE: WE HAVE SAID NOTHING YET RE COORDINATE CHOICES, I.E. RE SPECIFICATIONS OF α AND β)

MASSLESS KLEIN GORDON EQUATION

* NEED E.O.M. FOR Φ AND Π

* RECALL DEF^{N} OF Π, (21)

$$\Pi = \frac{a}{\alpha}\left(\dot{\phi} - \beta \phi'\right)$$

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\Pi + \beta\phi' = \frac{\alpha}{a}\Pi + \beta\,\Phi$$

* BUT $\dot{\phi}' = \dot{\Phi}$, SO

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

PHY 387N SPHERICAL SYMMETRY ⑤

$$\boxed{\dot{K}^{\theta}_{\theta} = \beta K^{\theta\,\prime}_{\theta} + \frac{\alpha}{(rb)^2} - \frac{1}{a(rb)^2}\left(\frac{\alpha rb}{a}(rb)'\right)' + \alpha K K^{\theta}_{\theta}} \quad (36)$$

- (31)-(36) ARE THE COMPLETE 3+1 EQUATIONS FOR THE GEOMETRIC VARIABLES (NOTE: WE HAVE SAID NOTHING YET RE COORDINATE CHOICES, I.E. RE SPECIFICATIONS OF α AND β)

MASSLESS KLEIN GORDON EQUATION

- NEED E.O.M. FOR Φ AND Π

- RECALL DEF^{N} OF Π (24)

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

YET THE COORDINATE CHOICES, I.E. THE SPECIFICATIONS OF α AND β)

MASSLESS KLEIN GORDON EQUATION

* NEED E.O.M. FOR Φ AND π

* RECALL DEF^{N} OF π, (2.1)

$$\pi = \frac{a}{\alpha}(\dot{\phi} - \beta\phi')$$

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\pi + \beta\phi' = \frac{\alpha}{a}\pi + \beta\Phi$$

* BUT $\dot{\phi}' = \dot{\Phi}$, SO

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

OF α AND β)

MASSLESS KLEIN GORDON EQUATION

* NEED E.O.M. FOR Φ AND Π

* RECALL DEF^{N} OF Π, (2.1)

$$\Pi = \frac{a}{\alpha}\left(\dot{\phi} - \beta \phi'\right)$$

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\Pi + \beta \phi' = \frac{\alpha}{a}\Pi + \beta \Phi$$

* BUT $\dot{\phi}' = \dot{\Phi}$, SO

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

• NEED E.O.M. FOR Φ AND Π

• RECALL DEF^{N} OF Π, (21)

$$\Pi = \frac{a}{\alpha}(\dot{\phi} - \beta\phi')$$

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\Pi + \beta\phi' = \frac{\alpha}{a}\Pi + \beta\Phi$$

• BUT $\dot{\phi}' = \dot{\Phi}$, SO

$$\boxed{\dot{\Phi} = \left(\beta\Phi + \frac{\alpha}{a}\Pi\right)' \qquad (32)}$$

• TO FIND $\dot{\Pi}$ EQN, RECALL THAT

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

- RECALL DEF^{N} OF Π, (21)

$$\Pi = \frac{a}{\alpha}\left(\dot{\phi} - \beta \phi'\right)$$

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\Pi + \beta\phi' = \frac{\alpha}{a}\Pi + \beta\,\Phi$$

- BUT $\dot{\phi}' = \dot{\Phi}$, SO

$$\boxed{\dot{\Phi} = \left(\beta\,\Phi + \frac{\alpha}{a}\Pi\right)' \qquad (32)}$$

- TO FIND $\dot{\Pi}$ EQN, RECALL THAT

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 9 / 57 Find

$$\rightarrow \quad \dot{\phi} = \frac{\alpha}{a}\Pi + \beta\phi' = \frac{\alpha}{a}\Pi + \beta\,\Phi$$

- BUT $\dot{\phi}' = \dot{\Phi}$, SO

$$\boxed{\dot{\Phi} = \left(\beta\,\Phi + \frac{\alpha}{a}\Pi\right)' \qquad (32)}$$

- TO FIND $\dot{\Pi}$ EQN, RECALL THAT

$$\Box\phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$$

$$\Box\phi = 0 \rightarrow \partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) = 0$$

$\pi + \beta\,\bar{\Phi}$

$)$

$$^{(3)}R + K^2 - K^i{}_j K^j{}_i = 16\pi\rho$$

$$D_j K^{ij} - D^i K = 8\pi j^i$$

$$\partial_t g_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}$$

$$\partial_t K_{ij} - \mathcal{L}_\beta K_{ij} = -D_i D_j \alpha + \alpha\big[R_{ij} + K K_{ij} +$$

$$8\pi\big[\tfrac{1}{2} g_{ij}(S-\rho) - S$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 10 / 57 Find

• TO FIND π EQN, RECALL THAT

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$$

$$\Box\phi = 0 \rightarrow \partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) = 0$$

$$\rightarrow \partial_t\left(\sqrt{-g}\,g^{\nu t}\partial_\nu\phi\right) + \partial_r\left(\sqrt{-g}\,g^{\nu r}\partial_\nu\phi\right)$$

$$= \partial_t\left(\alpha a r^2 b^2\left(-\alpha^{-2}\dot\phi + \beta\alpha^{-2}\phi'\right)\right)$$

$$+ \partial_r\left(\alpha a r^2 b^2\left(\left(a^{-2} - \beta^2\alpha^{-2}\right)\phi' + \beta\alpha^{-2}\dot\phi\right)\right)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 10 / 57 Find

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right)$$

$$\Box\phi = 0 \to \partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) = 0$$

$$\to \partial_t\left(\sqrt{-g}\,g^{\nu t}\partial_\nu\phi\right) + \partial_r\left(\sqrt{-g}\,g^{\nu r}\partial_\nu\phi\right)$$

$$= \partial_t\left(\alpha a r^2 b^2\left(-\alpha^{-2}\dot\phi + \beta\alpha^{-2}\phi'\right)\right)$$

$$+\partial_r\left(\alpha a r^2 b^2\left(\left(a^{-2}-\beta^2\alpha^{-2}\right)\phi' + \beta\alpha^{-2}\dot\phi\right)\right)$$

$$= 0$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 10 / 57 Find

$$\frac{1}{\sqrt{-g}} \partial \left(\quad \right)$$

$$\Box\phi = 0 \to \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi \right) = 0$$

$$\to \partial_t \left(\sqrt{-g}\, g^{\nu t} \partial_\nu \phi \right) + \partial_r \left(\sqrt{-g}\, g^{\nu r} \partial_\nu \phi \right)$$

$$= \partial_t \left(\alpha a r^2 b^2 \left(-\alpha^{-2} \dot\phi + \beta \alpha^{-2} \phi' \right) \right)$$

$$+ \partial_r \left(\alpha a r^2 b^2 \left(\left(a^{-2} - \beta^2 \alpha^{-2} \right) \phi' + \beta \alpha^{-2} \dot\phi \right) \right)$$

$$= 0$$

$$\to \dot{\left(r^2 b^2 \Pi \right)} = \left(r^2 b^2 \left(\beta \Pi + \frac{\alpha}{a} \Phi \right) \right)'$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 10 / 57 Find

$$\Box\phi = 0 \rightarrow \partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) = 0$$

$$\rightarrow \partial_t\left(\sqrt{-g}\, g^{\nu t}\partial_\nu\phi\right) + \partial_r\left(\sqrt{-g}\, g^{\nu r}\partial_\nu\phi\right)$$

$$= \partial_t\left(\alpha a r^2 b^2\left(-\alpha^{-2}\dot{\phi} + \beta\alpha^{-2}\phi'\right)\right)$$

$$+ \partial_r\left(\alpha a r^2 b^2\left(\left(a^{-2} - \beta^2\alpha^{-2}\right)\phi' + \beta\alpha^{-2}\dot{\phi}\right)\right)$$

$$= 0$$

$$\rightarrow \dot{\left(r^2 b^2 \Pi\right)} = \left(r^2 b^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right)'$$

$\hookrightarrow$ WRITE AS $\left(r^2b^2\right)\dot{\Pi} + \dot{\left(r^2b^2\right)}\Pi$ AND USE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 10 / 57 Find

$= 0$

$$\rightarrow \left(r^2 b^2 \dot{\Pi}\right) = \left(r^2 b^2 \left(\beta \Pi + \frac{\alpha}{a} \Phi\right)\right)'$$

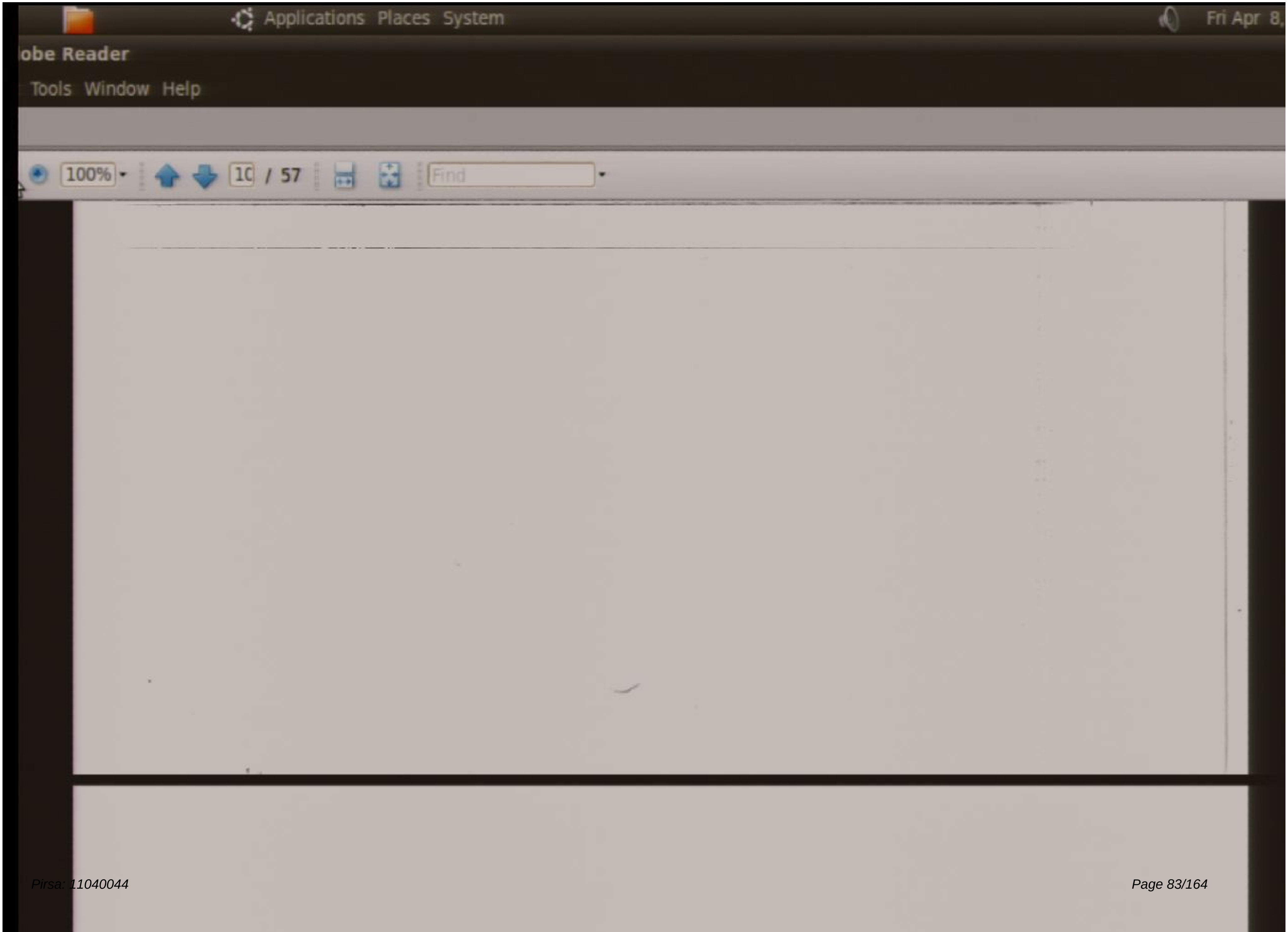
WRITE AS $(r^2 b^2)\dot{\Pi} + (r^2 \dot{b}^2)\Pi$ AND USE

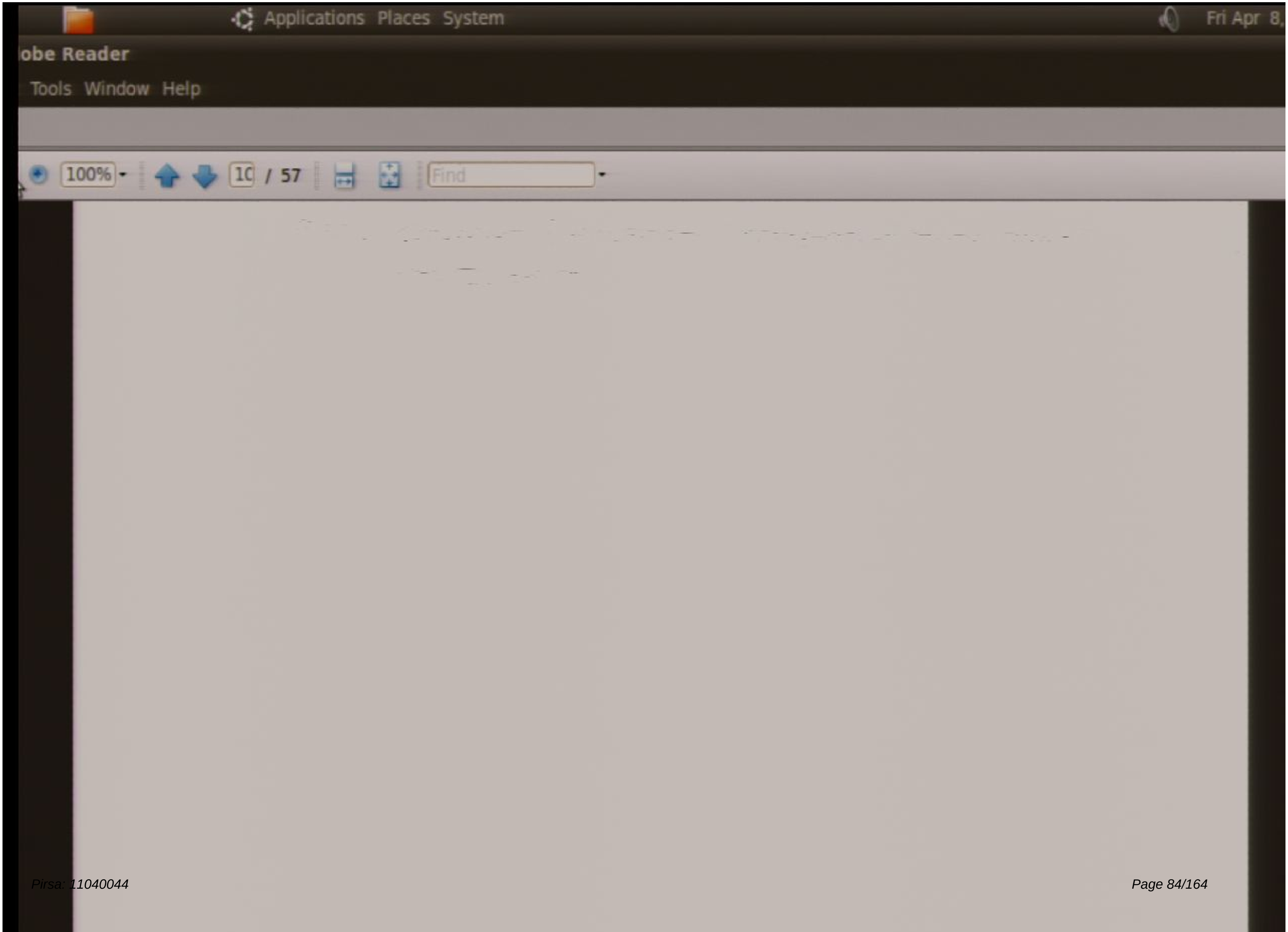
EVOLUTION EQUATION (3a) FOR $\dot{b}$

$$\boxed{\dot{\Pi} = \frac{1}{r^2 b^2}\left(r^2 b^2 \left(\beta \Pi + \frac{\alpha}{a} \Phi\right)\right)' + 2\left(\alpha K^\theta{}_\theta - \beta \frac{(rb)'}{rb}\right)\Pi} \qquad (3b)$$

↳ WRITE AS $(r^2b^2)\dot{\Pi} + (\dot{r^2b^2})\Pi$ AND USE EVOLUTION EQUATION (39) FOR $\dot{b}$

$$\boxed{\dot{\Pi} = \frac{1}{r^2b^2}\left(r^2b^2\left(\beta\Pi + \frac{\alpha}{a}\Phi\right)\right)' + 2\left(\alpha K^{\theta}{}_{\theta} - \beta\frac{(rb)'}{rb}\right)\Pi} \qquad (38)$$


Applications Places System
Fri Apr 8,
obe Reader
Tools Window Help
100%
10 / 57
Find


Applications Places System
Fri Apr 8,
obe Reader
Tools Window Help
100%
10 / 57
Find

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

REF: COURANT : HILBERT "METHODS of MATH. PHYS",

VOL II, ch 5

• EQN^5 (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u \equiv (\Phi, \Pi)^T$$

WE CAN WRITE

WE CAN WRITE

$$u_t + A u_x = B \qquad (39)$$

$$A = -\begin{pmatrix} \beta & x/a \\ \end{pmatrix} \qquad (40)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

PHY 387M SPHERICAL SYMMETRY ⑥

CHARACTERISTIC ANALYSIS of THE SCALAR FIELD

REF: COURANT : HILBERT "METHODS of MATH. PHYS",
VOL II, ch 5

• EQN^S (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u \equiv (\Phi, \Pi)^T$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

PHY 387M SPHERICAL SYMMETRY 6

CHARACTERISTIC ANALYSIS of THE SCALAR FIELD

REF: COURANT & HILBERT "METHODS of MATH. PHYS", VOL II, ch 5

- EQN^S (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

PHY 387M SPHERICAL SYMMETRY (5)

CHARACTERISTIC ANALYSIS of THE SCALAR FIELD

REF: COURANT : HILBERT "METHODS of MATH. PHYS", VOL II, ch 5

- EQN[S] (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u \equiv (\Phi, \Pi)^T$$

WE CAN WRITE

obe Reader

Tools Window Help

11 / 57

PHY 387M SPHERICAL SYMMETRY (5)

CHARACTERISTIC ANALYSIS of THE SCALAR FIELD

REF: COURANT & HILBERT "METHODS of MATH. PHYS", VOL II, CH 5

- EQNS (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u \equiv (\Phi, \Pi)^T$$

WE CAN WRITE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

REF: COURANT & HILBERT "METHODS of MATH. PHYS",
VOL II, ch 5

EQN^S (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS
FOR OUR RADIATION FIELD. DEFINING

$$u \equiv (\Phi, \Pi)^T$$

WE CAN WRITE

$$u_t + A u_x = B \qquad (39)$$

$$A = -\begin{pmatrix} \beta & \alpha' a \\ \ldots & \ldots \end{pmatrix} \qquad (40)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

VOL II, ccl 5

- EQN^S (37)-(38) ARE A 1ST-ORDER, QUASI-LINEAR SYS FOR OUR RADIATION FIELD. DEFINING

$$u = (\Phi, \Pi)^T$$

WE CAN WRITE

$$u_t + A u_x = B \qquad (39)$$

$$A = -\begin{pmatrix} \beta & \alpha/a \\ \alpha/a & \beta \end{pmatrix} \qquad (40)$$

AND B IS ANOTHER MATRIX WHICH DOES NOT INVOLVE DERIVATIVES of u.

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

$$A = -\begin{pmatrix} \beta & \alpha/a \\ \alpha/a & \beta \end{pmatrix} \qquad (40)$$

AND B IS ANOTHER MATRIX WHICH DOES <u>NOT</u> INVOLVE DERIVATIVES of u.

- THE CHARACTERISTIC DIRECTIONS $\tau \equiv dr/dt$ OF (39)-(40) ARE GIVEN BY

$$|A - \tau \mathbb{1}| = 0$$

$$\rightarrow \qquad \boxed{\tau = -\beta \pm \frac{\alpha}{a}} \qquad (41)$$

THESE ARE THE "LOCAL SIGNAL SPEEDS" FOR THE SCALAR

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 11 / 57 Find

DERIVATIVES OF u.

- THE CHARACTERISTIC DIRECTIONS $\tau = dr/dt$ OF (39)-(40) ARE GIVEN BY

$$|A - \tau \mathbb{1}| = 0$$

$$\rightarrow \qquad \boxed{\tau = -\beta \pm \frac{\alpha}{a}} \qquad (41)$$

THESE ARE THE "LOCAL SIGNAL SPEEDS" FOR THE SCALAR FIELD

MASSLESS SCALAR FIELD – WEAK (NON-SELF GRAVITATING)

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 12 / 57 Find

$$\rightarrow \quad \boxed{\tau = -\beta \pm \frac{\alpha}{a}} \qquad (41)$$

THESE ARE THE "LOCAL SIGNAL SPEEDS" FOR THE SCALAR FIELD

• MASSLESS SCALAR FIELD – WEAK (NON-SELF GRAVITATING) SIGNALS TRAVEL ALONG NULL GEODESICS → ALTERNATE DERIVATION of (41)

$$ds^2 = -\alpha^2 dt^2 + a^2(dr + \beta dt)^2 = 0$$

REGULARITY / LOCAL FLATNESS AT $r=0$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 12 / 57 Find

• MASSLESS SCALAR FIELD – WEAK (NON-SELF GRAVITATING)

SIGNALS TRAVEL ALONG NULL GEODESICS → ALTERNATE DERIVATION OF (A1)

$$ds^2 = -\alpha^2 dt^2 + a^2(dr + \beta dt)^2 = 0$$

REGULARITY / LOCAL FLATNESS AT $r=0$

• OUR CAUCHY PROBLEM FOR THE EMKG MODEL IS TO BE SOLVED ON

$$t \geq 0, \quad r \geq 0$$

• BOUNDARY CONDITIONS AS $r \to \infty$ WILL FOLLOW FROM ASYMPTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 12 / 57 Find

- OUR CAUCHY PROBLEM FOR THE EMKG MODEL IS TO BE SOLVED ON

$$t \geq 0 \ , \quad r \geq 0$$

- BOUNDARY CONDITIONS AS $r \to \infty$ WILL FOLLOW FROM ASYMPTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE DIFFERENCING) IS EFFECTIVELY ONE

- GET CONDITIONS AT $r=0$ BY DEMANDING THAT SCALAR + GRAV. FIELDS BE REGULAR, AND THAT THE S.T. BE LOCALLY FLAT THERE

TRICKY ISSUE IN GENERAL UNLESS WE MAKE ASSUMPTIONS

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 12 / 57 Find

DERIVATION of (A.1)

$$ds^2 = -\alpha^2 dt^2 + a^2(dr + \beta dt)^2 = 0$$

REGULARITY / LOCAL FLATNESS AT $r=0$

- OUR CAUCHY PROBLEM FOR THE EMKG MODEL IS TO BE SOLVED ON

$$t \geq 0 \, , \quad r \geq 0$$

- BOUNDARY CONDITIONS AS $r \to \infty$ WILL FOLLOW FROM ASYMPTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE DIFFERENCING) IS EFFECTIVELY ONE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 12 / 57 Find

DERIVATION of (A.1)

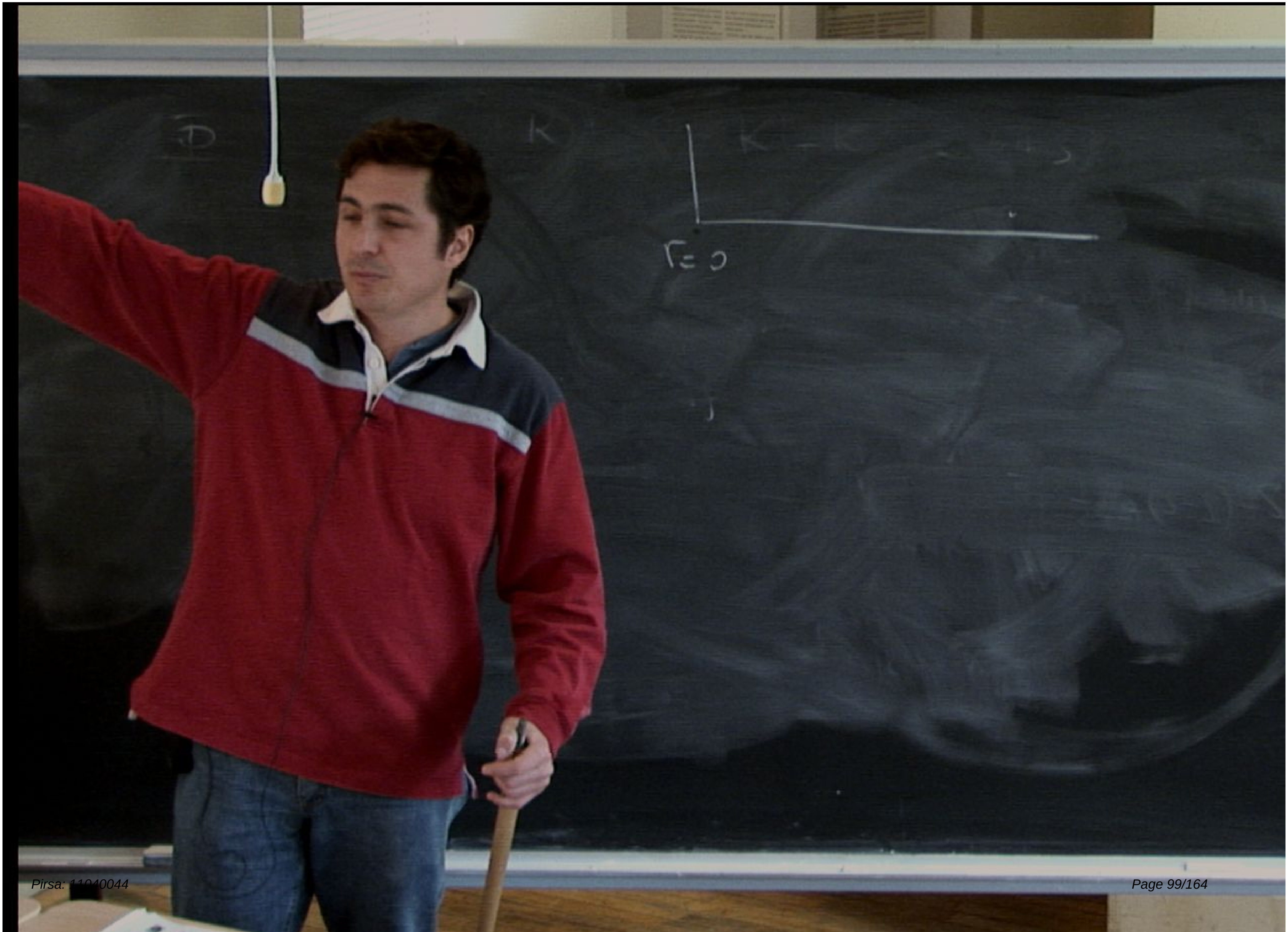
$$ds^2 = -\alpha^2 dt^2 + a^2(dr + \beta dt)^2 = 0$$

REGULARITY / LOCAL FLATNESS AT $r=0$

- OUR CAUCHY PROBLEM FOR THE EMKG MODEL IS TO BE SOLVED ON

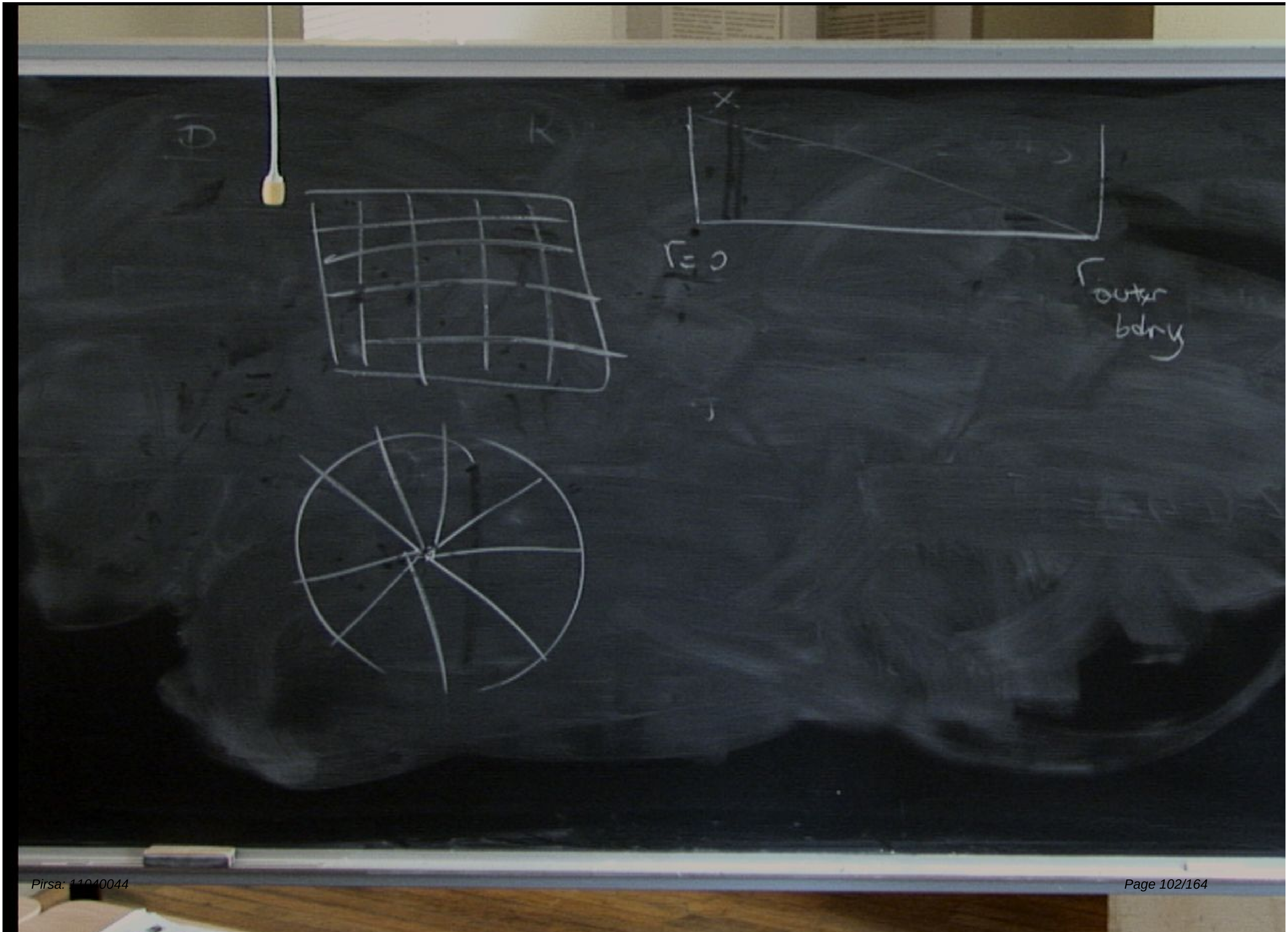
$$t \geq 0 , \quad r \geq 0$$

- BOUNDARY CONDITIONS AS $r \to \infty$ WILL FOLLOW FROM ASYMPTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE DIFFERENCING) IS EFFECTIVELY ONE


r = 0


r=0
r outer bdry


x
r = 0
r outer bdry


x
r = 0
r
outer
bdry

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 12 / 57 Find

SOLVED ON

$$t \geq 0 \ , \quad r \geq 0$$

- BOUNDARY CONDITIONS AS $r \to \infty$ WILL FOLLOW FROM ASYMPTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE DIFFERENCING) IS EFFECTIVELY ONE

- GET CONDITIONS AT $r=0$ BY DEMANDING THAT SCALAR + GRAV. FIELDS BE REGULAR, AND THAT THE S.T. BE LOCALLY FLAT THERE

- TRICKY ISSUE IN GENERAL UNLESS WE MAKE ASSUMPTIONS ABOUT SLICING, SPATIAL COORDINATES (SEE BARDEEN; PIRAN, PHYSICS REPORTS 96 (1983) 205-250); WE WILL

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 12 / 57 Find

...PTOTIC FLATNESS, NO INCOMING RADIATION; $r=0$ NOT A REAL BOUNDARY, BUT COMPUTATIONALLY (I.E. WHEN FINITE DIFFERENCING) IS EFFECTIVELY ONE

- GET CONDITIONS AT $r=0$ BY DEMANDING THAT SCALAR GRAV. FIELDS BE REGULAR, AND THAT THE S.T. BE LOCALLY FLAT THERE

- TRICKY ISSUE IN GENERAL UNLESS WE MAKE ASSUMPTIONS ABOUT SLICING, SPATIAL COORDINATES (SEE BARDEEN; PIRAN, PHYSICS REPORTS 96 (1983) 205-250); WE WILL ASSUME EVERYWHERE SMOOTH SLICINGS AND SPATIAL COORDS

(a) SCALAR FIELD

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 12 / 57 Find

(a) SCALAR FIELD

$$\nabla \phi = \phi' \hat{r}$$

NOT DEFINED AT $r=0$ UNLESS

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 13 / 57 Find

COMPONENTS CAN BE EXPANDED IN NON-NEG. POWERS OF x, y, z:

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

- APPLYING TO OUR CASE (WITH ASSUMPTION OF SMOOTH COORDS) FIND

$$g(r,t) = g_0(t) + r^2 g_2(t) + \dots$$

WHERE g IS ANY OF $a, b, K^r{}_r$ OR $K^\theta{}_\theta$

$$\rightarrow \quad a'(0,t) = b'(0,t) = K^r{}_r{}'(0,t) = K^\theta{}_\theta{}'(0,t) = 0 \qquad (43a\text{-}d)$$

- THESE CONDITIONS WILL BE CONSISTENT WITH EVOL.

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 13 / 57 Find

$x = r\sin\theta\cos\varphi \quad y = r\sin\theta\sin\varphi \quad z = r\cos\theta$

- APPLYING TO OUR CASE (WITH ASSUMPTION OF SMOOTH COORDS) FIND

$$g(r,t) = g_0(t) + r^2 g_2(t) + \dots$$

WHERE g IS ANY OF $a, b, K^r{}_r$ OR $K^\theta{}_\theta$

$$\rightarrow \quad a'(0,t) = b'(0,t) = K^r{}_r{}'(0,t) = K^\theta{}_\theta{}'(0,t) = 0$$

(43a-d)

- THESE CONDITIONS WILL BE CONSISTENT WITH EVOL. EQNS ONLY IF

WHERE g IS ANY OF $a, b, K^r{}_r$ OR $K^t{}_t$

$$a'(0,t) = b'(0,t) = K^r{}_r{}'(0,t) = K^t{}_t{}'(0,t) = 0 \qquad (43a\text{-}d)$$

- THESE CONDITIONS WILL BE CONSISTENT WITH EVOL. EQNS ONLY IF

$$\alpha'(0,t) = \beta(0,t) = 0 \qquad (44a)$$

$$\beta(0,t) = r\beta_1(t) + O(r^3)$$

(SMOOTHNESS of COORDINATES)

- LOCAL FLATNESS: CONSIDER A TRANSPORT OF AN ARBITRARY VECTOR ABOUT ...

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

$$\Gamma, \quad \mathcal{J}(a,t) = r\mathcal{J}_1(t) + O(r^3)$$

(SMOOTHNESS of COORDINATES)

LOCAL FLATNESS: CONSIDER // TRANSPORT OF AN ARBITRARY VECTOR ABOUT CLOSED LOOP ENCLOSING

$r=0$ ON $\theta = \frac{\pi}{2}$ (EQUATORIAL PLANE); DEMAND THAT THERE BE NO NET ROTATION IN LIMIT LOOP SHRUNK TO POINT $\Rightarrow \lim_{r\to 0}$ PROPER CIRCUM / PROPER RADIUS $= 2\pi$ AND

$$\nabla^\mu(rb)\nabla_\mu(rb) = 1$$

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

LOCAL FLATNESS: CONSIDER ∥ TRANSPORT OF AN ARBITRARY VECTOR ABOUT CLOSED LOOP ENCLOSING

$r=0$ ON $\theta = \frac{\pi}{2}$ (EQUATORIAL PLANE). DEMAND THAT THERE BE NO NET ROTATION IN LIMIT LOOP SHRUNK TO POINT $\Rightarrow$ $\lim_{r\to 0}$ PROPER CIRCUM / PROPER RADIUS $= 2\pi$

FIND

$$\nabla^\mu (rb) \nabla_\mu (rb) = 1$$

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS

FIND

$$a'(0,t) - b(0,t) \qquad (45)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

$r=0$ ON $\theta = \frac{\pi}{2}$ (EQUATORIAL PLANE); DEMAND THAT THERE BE NO NET ROTATION IN LIMIT LOOP SHRUNK TO POINT $\Rightarrow$ $\lim_{r\to 0}$ PROPER CIRCUM / PROPER RADIUS $= 2\pi$

FIND

$$\nabla^\mu (rb) \nabla_\mu (rb) = 1$$

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS FIND

$$a'(0,t) = b(0,t) \qquad (45)$$

AS + $\dot{a}, \dot{b}$ EQUAL + REG. THEN IMPLY

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 14 / 57 Find

THIS + $\dot{a}, \dot{b}$ EQNS + REG. THEN IMPLY

$$K^r{}_r(0,t) = K^\theta{}_\theta(0,t) \qquad (A6)$$

• NOTE: REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

TRAPPED SURFACES / APPARENT HORIZONS

• WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

- WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

- USEFUL TO BE ABLE TO COMPUTE "INSTANTANEOUS" APPROX. (I.E. ON ANY HYPERSURFACE $\Sigma(t)$) TO EH. — PROVIDED BY APPARENT HORIZON = OUTERMOST MARGINALLY TRAPPED SURFACE

PHYSICS SPHERICAL SYMMETRY

- TRAPPED SURFACE: 2-SURFACE WITH TOPOLOGY S^2 SUCH THAT DIVERGENCE OF OUTGOING NULL

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS

FIND

$$a'(0,t) = b(0,t) \qquad (45)$$

THIS + $\dot{a}, \dot{b}$ BEING + DEG. THEN IMPLY

$$K^r{}_r(0,t) = K^\theta{}_\theta(0,t) \qquad (46)$$

* NOTE : REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY ; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

(SMOOTHNESS of COORDINATES)

§ LOCAL FLATNESS: CONSIDER // TRANSPORT OF AN ARBITRARY VECTOR ABOUT CLOSED LOOP ENCLOSING

$r \to 0$ ON $\theta = \frac{\pi}{2}$ (EQUATORIAL PLANE); DEMAND THAT THERE BE NO NET ROTATION IN LIMIT LOOP SHRUNK TO POINT $\Rightarrow$ $\lim_{r \to 0}$ PROPER CIRCUM / PROPER RADIUS $= 2\pi$

AND

$$\nabla^{\mu}(rb)\nabla_{\mu}(rb) = 1$$

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS

FIND

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 14 / 57 Find

LOCAL FLATNESS: CONSIDER // TRANSPORT OF AN ARBITRARY VECTOR ABOUT CLOSED LOOP ENCLOSING

$r=0$ ON $\theta = \frac{\pi}{2}$ (EQUATORIAL PLANE); DEMAND THAT THERE BE NO NET ROTATION IN LIMIT LOOP SHRUNK TO POINT $\Rightarrow$ $\lim_{r\to 0}$ PROPER CIRCUM / PROPER RADIUS $= 2\pi$

FIND

$$\nabla^\mu(rb)\nabla_\mu(rb) = 1$$

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS FIND

$$a(0,t) = b(0,t) \quad (45)$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

USING PREVIOUS RESULTS INCLUDING ABOVE REG. CONDITIONS

FIND

$$a(0,t) = b(0,t) \qquad (45)$$

THIS + $\dot{a}, \dot{b}$ EQNS + REG. THEN IMPLY

$$K^r{}_r(0,t) = K^\theta{}_\theta(0,t) \qquad (46)$$

NOTE: REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

- NOTE: REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

TRAPPED SURFACES / APPARENT HORIZONS

- WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

- USEFUL TO BE ABLE TO COMPUTE "INSTANTANEOUS" APPROX. (I.E. ON ANY HYPERSURFACE $\Sigma(t)$) TO EH. — PROVIDED BY APPARENT HORIZON = OUTERMOST MARGINALLY TRAPPED SURFACE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 14 / 57 Find

NOTE: REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

TRAPPED SURFACES / APPARENT HORIZONS

- WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

- USEFUL TO BE ABLE TO COMPUTE "INSTANTANEOUS" APPROX. (I.E. ON ANY HYPERSURFACE $\Sigma(t)$) TO EH. — PROVIDED BY APPARENT HORIZON = OUTERMOST MARGINALLY TRAPPED SURFACE

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

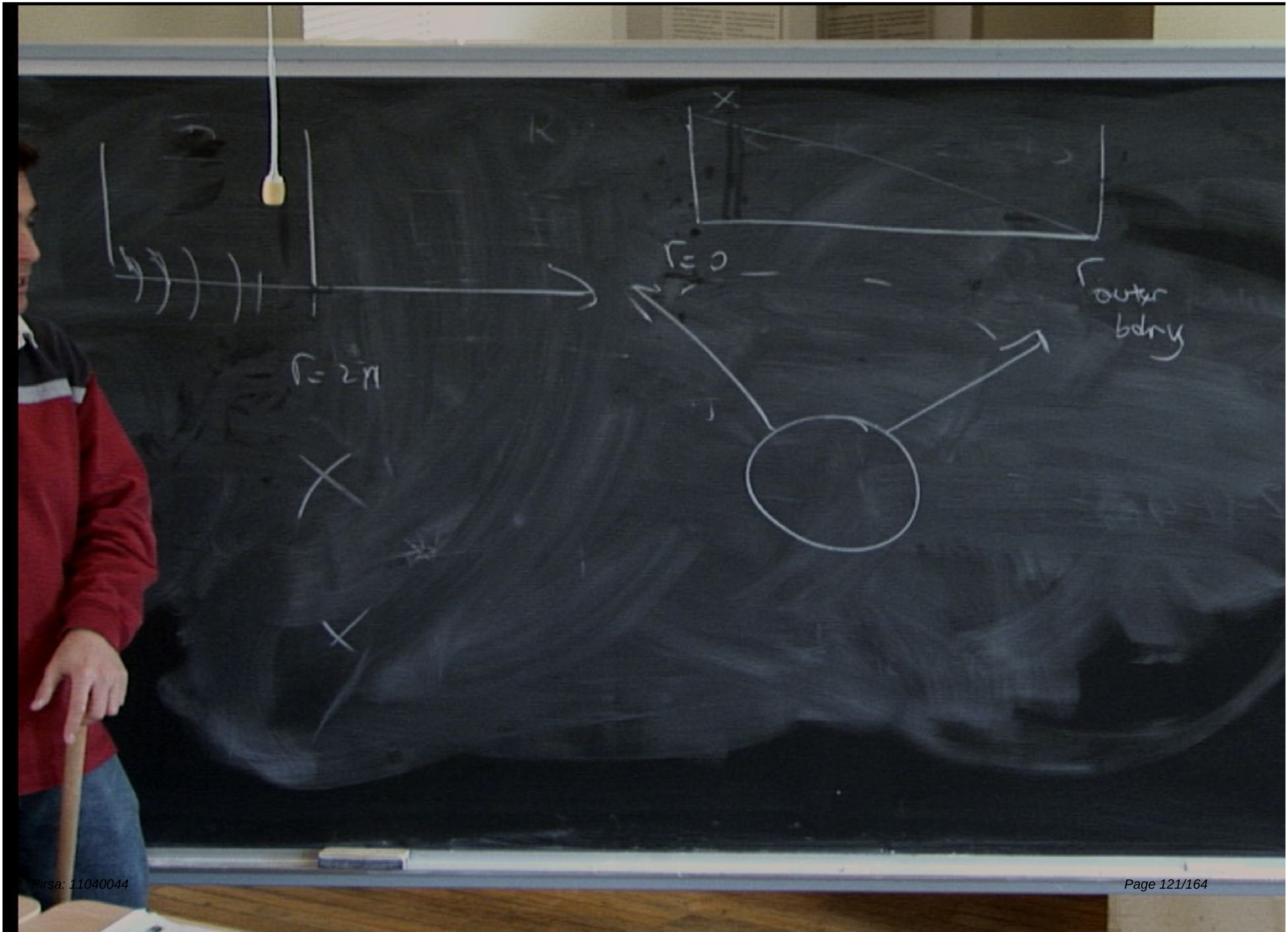
100% 14 / 57 Find

NOTE: REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

TRAPPED SURFACES / APPARENT HORIZONS

- WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

- USEFUL TO BE ABLE TO COMPUTE "INSTANTANEOUS" APPROX. (I.E. ON ANY HYPERSURFACE $\Sigma(t)$) TO EH. — PROVIDED BY APPARENT HORIZON $\equiv$ OUTERMOST MARGINALLY TRAPPED SURFACE


r = 2M
r = 0
r outer bdry

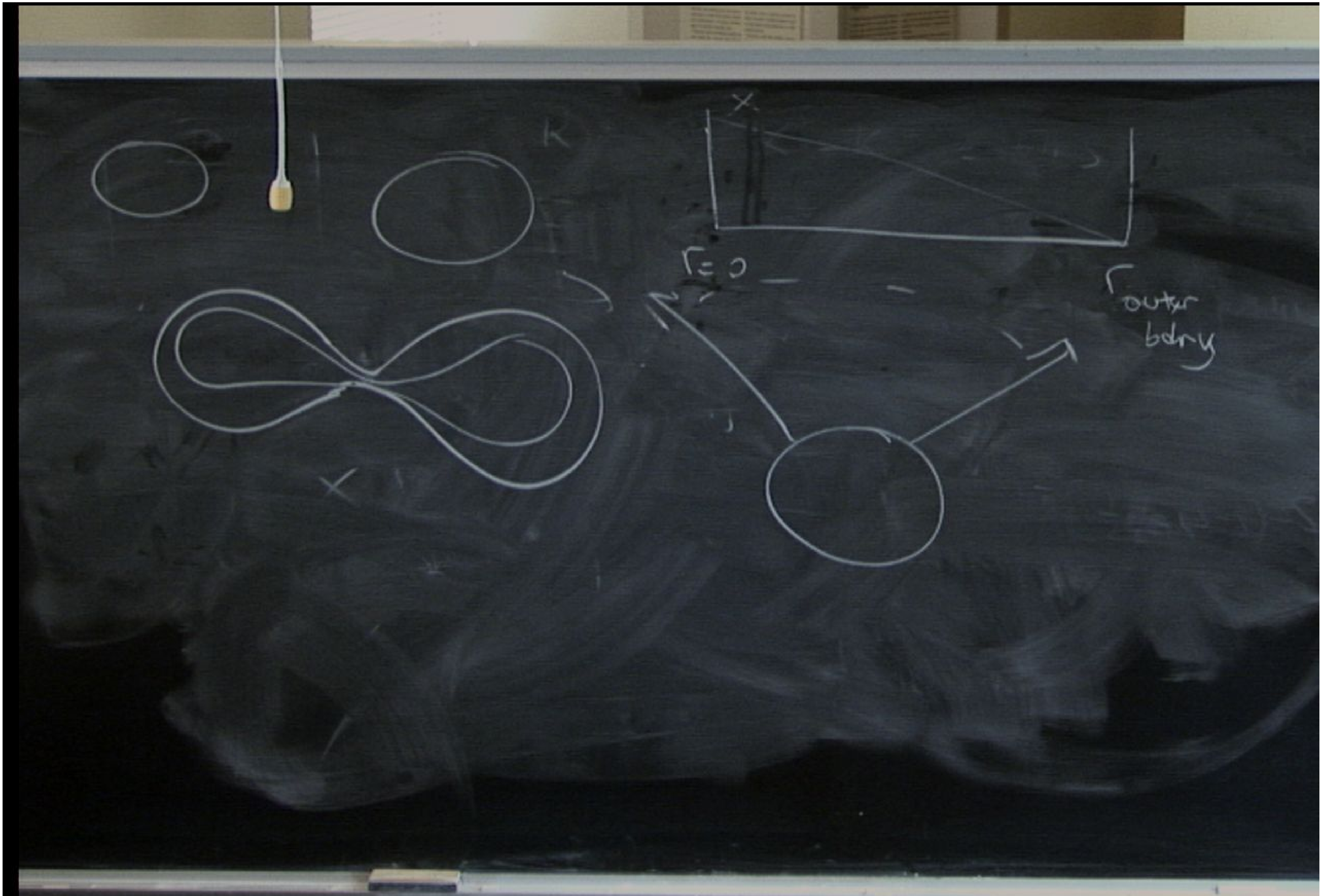
Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 14 / 57 Find

• NOTE: REGULARITY CONDITIONS MUCH MORE INVOLVED FOR AXISYMMETRY; MAINTAINING REGULARITY IN AXI. SIMULATIONS ALSO MUCH MORE PROBLEMATIC THAN S.S.

TRAPPED SURFACES / APPARENT HORIZONS

• WANT TO STUDY BLACK HOLE FORMATION; BH CHARACTERIZED BY AN EVENT HORIZON WHICH CAN ONLY BE DETERMINED ONCE COMPLETE S.T. HAS BEEN CONSTRUCTED

• USEFUL TO BE ABLE TO COMPUTE "INSTANTANEOUS" APPROX. (I.E. ON ANY HYPERSURFACE $\Sigma(t)$) TO EH. — PROVIDED BY APPARENT HORIZON = OUTERMOST MARGINALLY TRAPPED SURFACE


r = 0
r outer
bdry

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 14 / 57 Find

- WILL TEND TO USE TERMS AH, TB & MTS INTER-CHANGEABLY, BUT SHOULD BE AWARE OF DISTINCTIONS

(MARGINALLY) TRAPPED SURFACE EQN (AH EQN)

- CONSIDER A 2-SURFACE WITH OUTGOING NULL TANGENT u^a WHICH IS MARGINALLY TRAPPED, THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 16 / 57 Find

NOW, CAN WRITE u^a AS

$$u^a = s^a + n^a$$

UNIT FUTURE-DIRECTED TIMELIKE NORMAL TO Σ

UNIT OUTWARDS-POINTING SPACELIKE NORMAL TO 2-SURF.

AN // TO 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED ON THE 2-SURFACE BY PROJECTION

$$h^{ab} = \gamma^{ab} - s^a s^b = g^{ab} + n^a n^b - s^a s^b$$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 15 / 57 Find

(MARGINALLY) TRAPPED SURFACE EQN (AH EQN)

CONSIDER A 2-SURFACE WITH OUTGOING NULL TANGENT u^a WHICH IS MARGINALLY TRAPPED, THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 15 / 57 Find

- WILL TEND TO USE TERMS AH, B ; MTS INTER-CHANGABLY, BUT SHOULD BE AWARE OF DISTINCTIONS

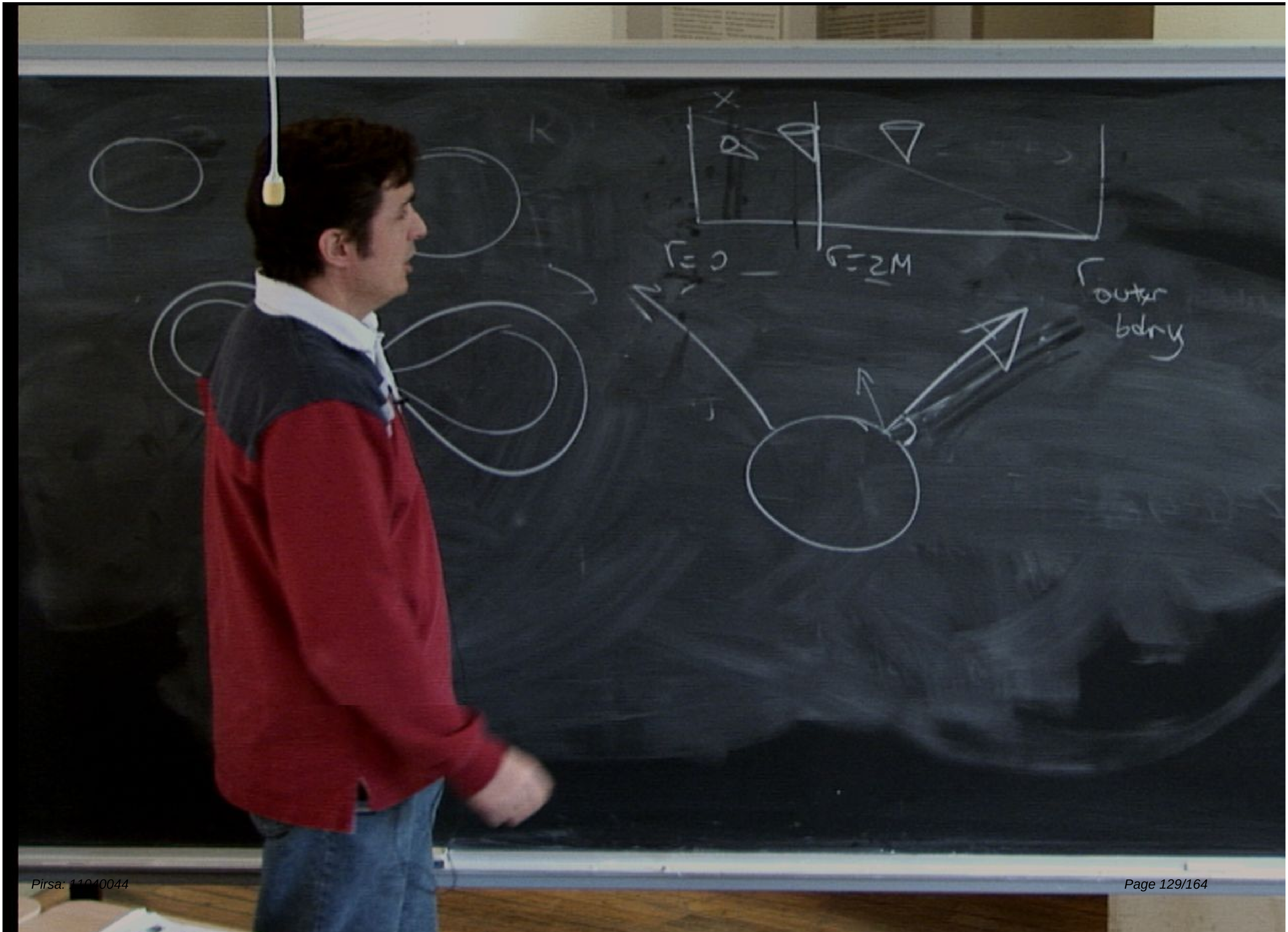
(MARGINALLY) TRAPPED SURFACE EQN (AH EQN)

- CONSIDER A 2-SURFACE WITH OUTGOING NULL TANGENT u^a WHICH IS MARGINALLY TRAPPED, THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS


r = 0
r outer
bdry


r=0
r=2M
r outer bdry

r=2M
r outer
bdry

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 15 / 57 Find

- WILL TEND TO USE TERMS AH, B & MTS INTER-CHANGABLY, BUT SHOULD BE AWARE OF DISTINCTIONS

(MARGINALLY) TRAPPED SURFACE EQN (AH EQN)

- CONSIDER A 2-SURFACE WITH OUTGOING NULL TANGENT u^a WHICH IS MARGINALLY TRAPPED, THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 16 / 57 Find

THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

$$u^a = s^a + n^a$$

UNIT FUTURE-DIRECTED TIMELIKE NORMAL TO Σ

UNIT OUTWARDS-POINTING SPACELIKE NORMAL TO 2-SURF.

IN // TO 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED

ON THE 2-SURFACE BY PROJECTION

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 16 / 57 Find

THEN

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

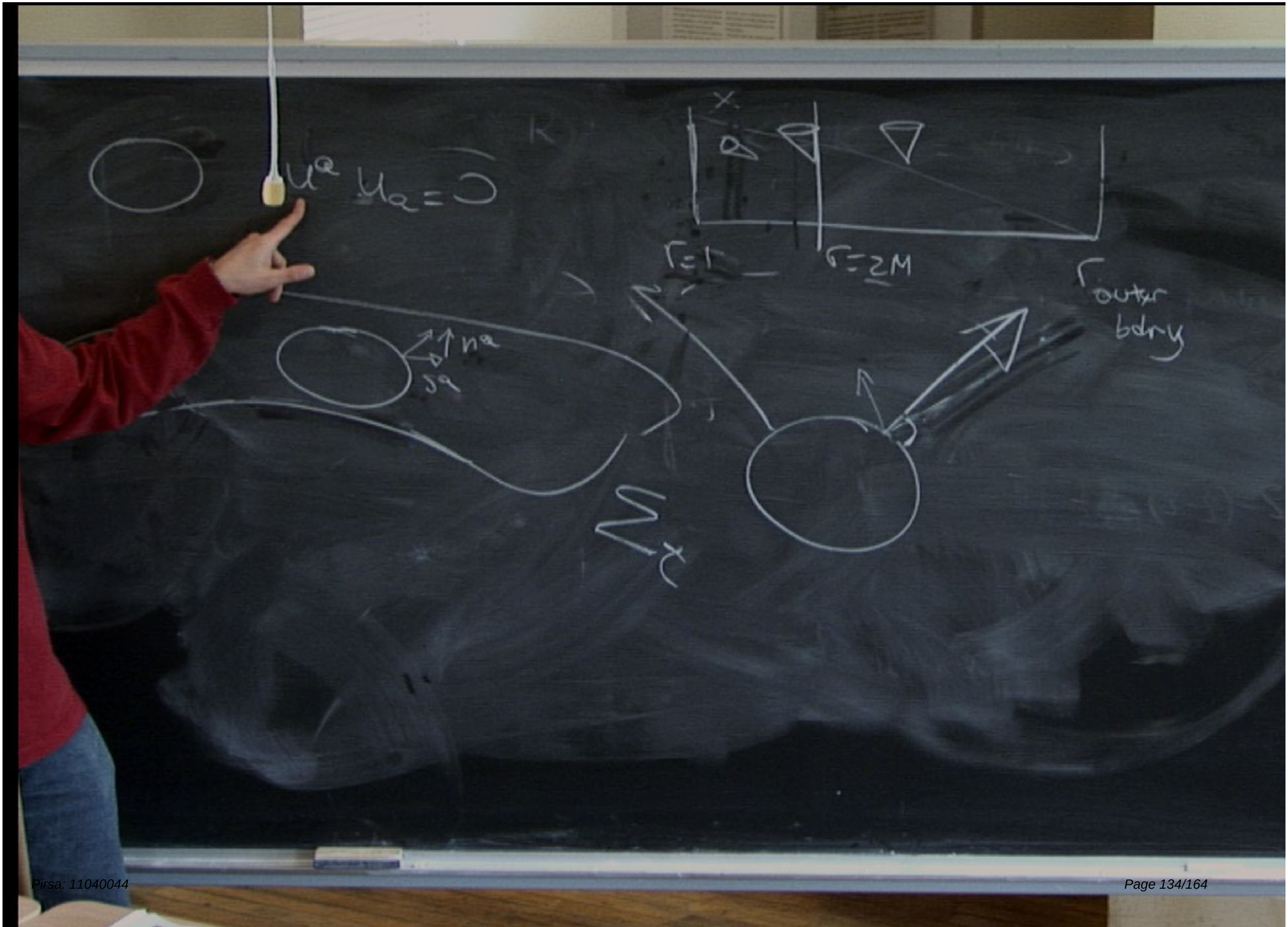
$$u^a = s^a + n^a$$

UNIT FUTURE-DIRECTED TIMELIKE NORMAL TO Σ

UNIT OUTWARDS-POINTING SPACELIKE NORMAL TO 2-SURF.

IN / TO 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED

ON THE 2-SURFACE BY PROJECTION



$u^a u_a = 0$
n^a
ξ^a
$r=r_1$
$r=2M$
r_{outer} bdry
Σ_t

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 16 / 57 Find

$$\nabla_a u^a = 0$$

NOW, CAN WRITE u^a AS

$$u^a = s^a + n^a$$

UNIT FUTURE-DIRECTED TIMELIKE NORMAL TO Σ

UNIT OUTWARDS-POINTING SPACELIKE NORMAL TO 2-SURF.

IN // TO 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED

ON THE 2-SURFACE BY PROJECTION

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 16 / 57 Find

UNIT FUTURE-DIRECTED TIMELIKE NORMAL TO Σ

UNIT OUTWARDS-POINTING SPACELIKE NORMAL TO 2-SURF.

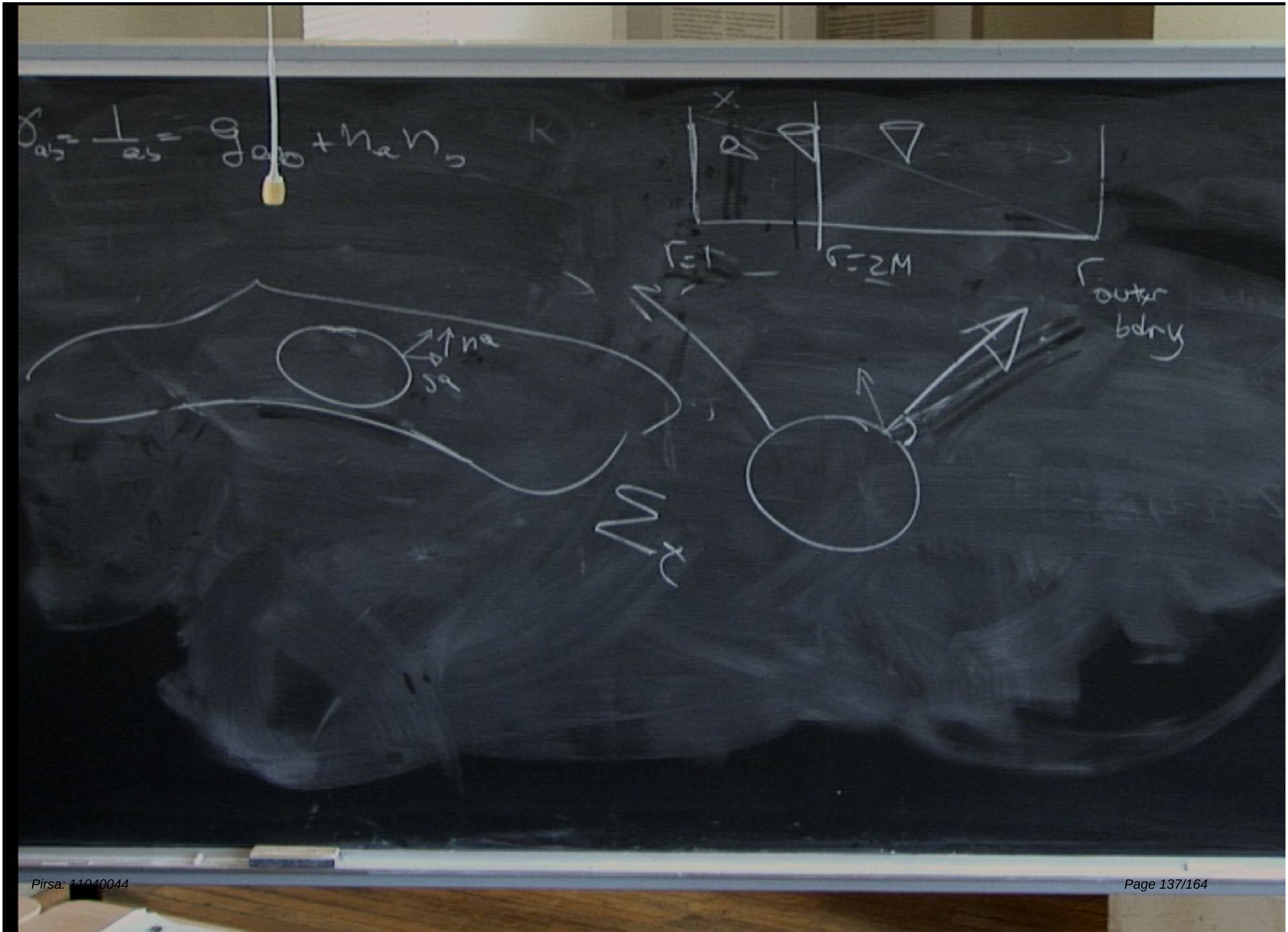
AS IN 3+1 DECOMPOSITION, METRIC h_{ab}, h^{ab} IS INDUCED ON THE 2-SURFACE BY PROJECTION

$$h^{ab} = \gamma^{ab} - s^a s^b = g^{ab} + n^a n^b - s^a s^b$$

CAN SHOW (EXERCISE) THAT $\nabla_a u^a$ IS A "2-TENSOR"; I.E IS INTRINSIC TO 2-SURFACE; I.E. DOES NOT DEPEND ON HOW 2-SURFACE IS EMBEDDED IN Σ, THEN

$$\nabla_a u^a = g^{ab} \nabla_a u_b = h^{ab} \nabla_a u_b$$

$$= h^{ab} \nabla_a (s_b + n_b)$$



$\gamma_{ab} = \perp_{ab} = g_{ab} + n_a n_b$
R
X
$r=r$
$r=2M$
r outer bdry
n^a
ξ^a
Σ_t


4D → Σ
Σ → S
$\gamma_{ab} = \frac{1}{ab} = g_{ab} + n_a n_b$
$h_{ab} = (\gamma_{ab} - S_a S_b)$
n^a
S^a
S
Σ_t
$r = 2M$

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 17 / 57 Find

PHY 3E?M SPHERICAL SYMMETRY ⑤

THUS, OUR TRAPPED SURFACE (AH) EQUATION IS

$$\boxed{D^a s_a - K + s^a s^b K_{ab} = 0} \tag{47}$$

AND ARGUING AS WE DID FOR THE 3+1 EQNS WE GET A VALID COMPONENT FORM OF THIS EQUATION BY TAKING $a \to i$, $b \to j$

$$\boxed{D^i s_i - K + s^i s^j K_{ij} = 0} \tag{48}$$

• SPECIALIZING NOW TO SPHERICAL SYMMETRY

Applications Places System Fri Apr 8,
obe Reader
Tools Window Help
100% 18 / 57 Find

NOW, RECALL EVOL. EQN (34) FOR b

$$\dot{b} = -\alpha b K^\theta{}_\theta + \frac{\beta}{r}(rb)'$$

$$\Rightarrow K^\theta{}_\theta = -(\alpha b)^{-1}\left(\dot{b} - \frac{\beta}{r}(rb)'\right)$$

$$rK^\theta{}_\theta = -(\alpha b)^{-1}\left((r\dot{b}) - \beta(rb)'\right)$$

SO (49) CAN BE REWRITTEN AS

$$\boxed{(r\dot{b}) + \left(\frac{\alpha}{a} - \beta\right)(rb)' = 0} \qquad (50)$$

WHICH SAYS THAT THE SURFACE OF CONSTANT AREAL RADIUS

Applications Places System Fri Apr 8,

obe Reader

Tools Window Help

100% 18 / 57 Find

NOW, RECALL EVOL. EQN (34) FOR b

$$\dot{b} = -\alpha b K^{\theta}{}_{\theta} + \frac{\beta}{r}(rb)'$$

$$\Rightarrow K^{\theta}{}_{\theta} = -(\alpha b)^{-1}\left(\dot{b} - \frac{\beta}{r}(rb)'\right)$$

$$rK^{\theta}{}_{\theta} = -(\alpha b)^{-1}\left((r\dot{b}) - \beta(rb)'\right)$$

SO (49) CAN BE REWRITTEN AS

$$\boxed{(r\dot{b}) + \left(\frac{\alpha}{a} - \beta\right)(rb)' = 0} \qquad (50)$$

WHICH SAYS THAT THE SURFACE OF CONSTANT AREAL RADIUS

[redacted]

Stability Analysis

- One of the most frustrating/fascinating features of FD solutions of time dependent problems: discrete solutions often "blow up"—e.g. floating-point overflows are generated at some point in the evolution

- 'Blow-ups" can sometimes be caused by legitimate (!) "bugs"—i.e. an incorrect implementation—at other times it is simply the *nature of the FD scheme* which causes problems.

- Are thus lead to consider the *stability* of solutions of difference equations

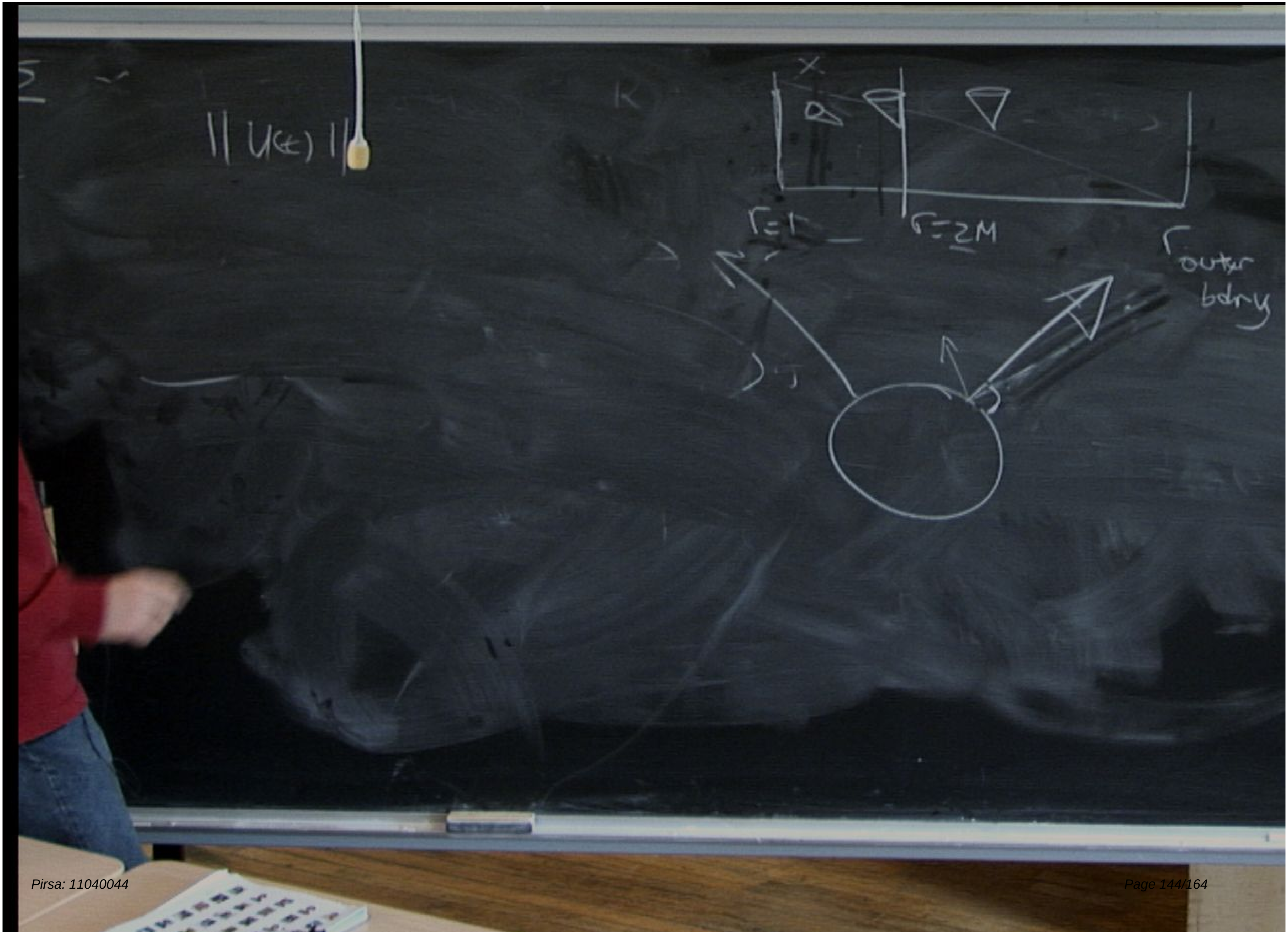
- Again consider the 1-d wave equation, $u_{tt} = u_{xx}$

- Note that it is a *linear, non-dispersive* wave equation

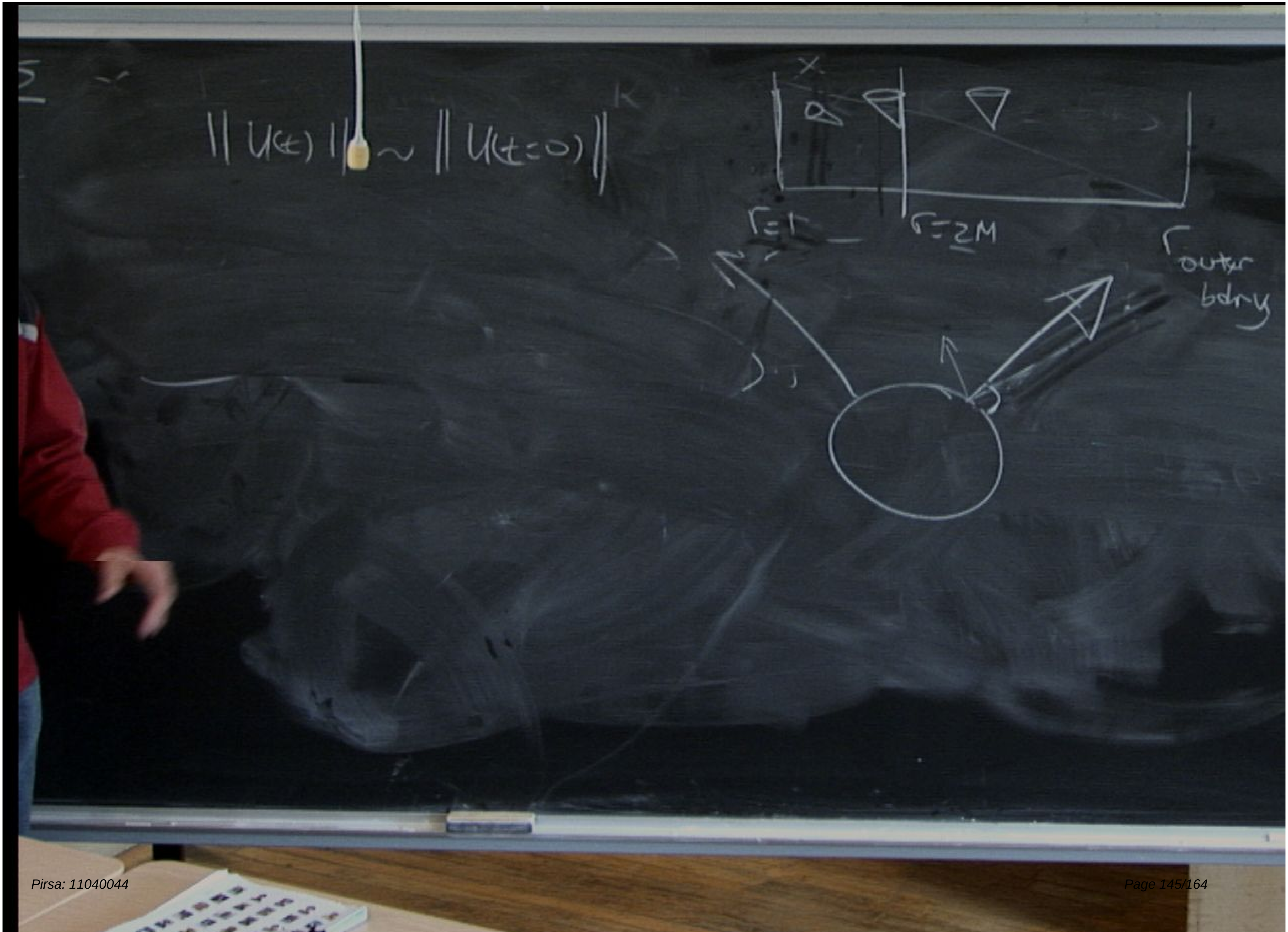
- Thus the "size" of the solution does *not* change with time:

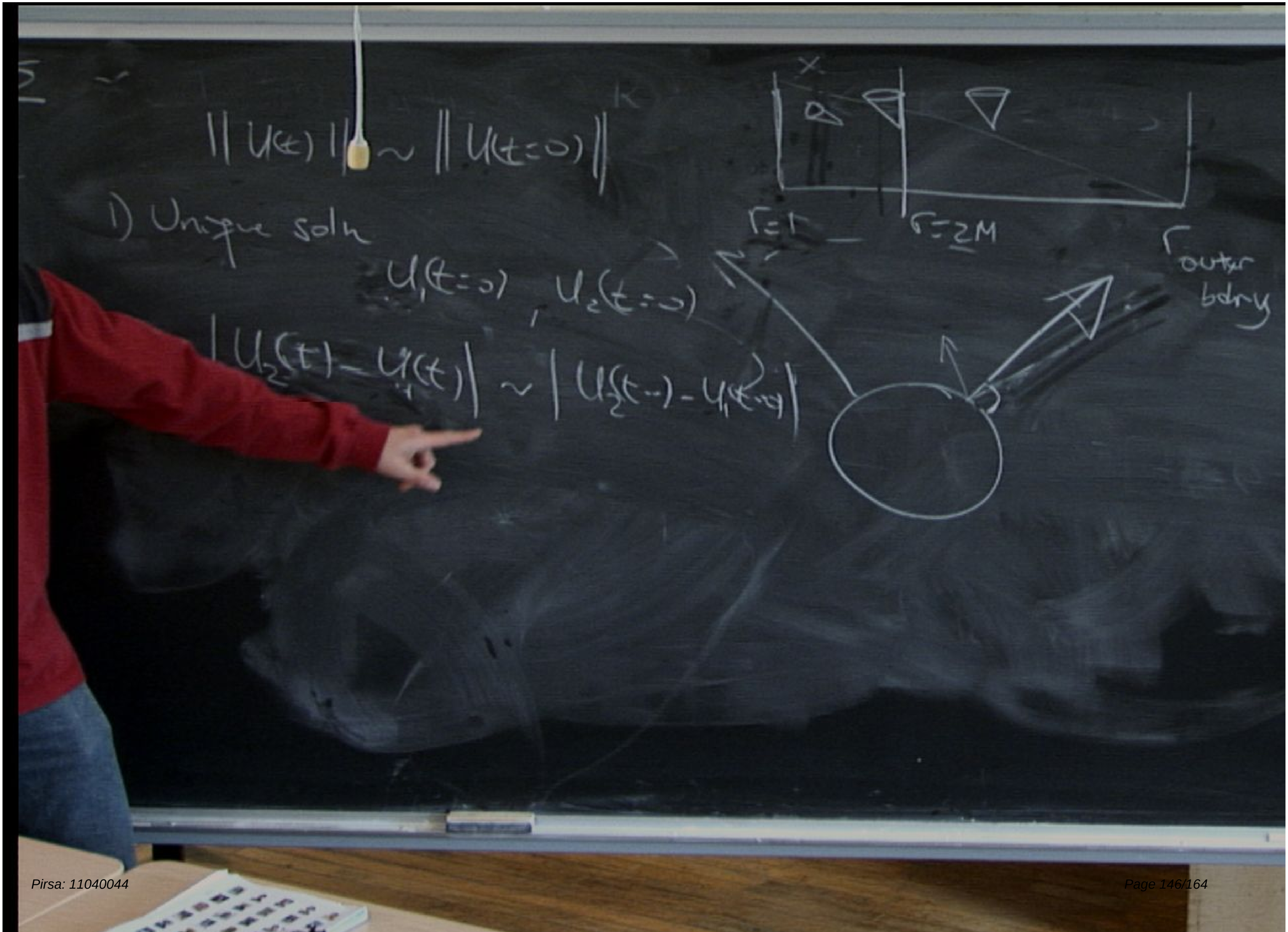
$$\|u(x,t)\| \sim \|u(x,0)\|\,, \tag{95}$$

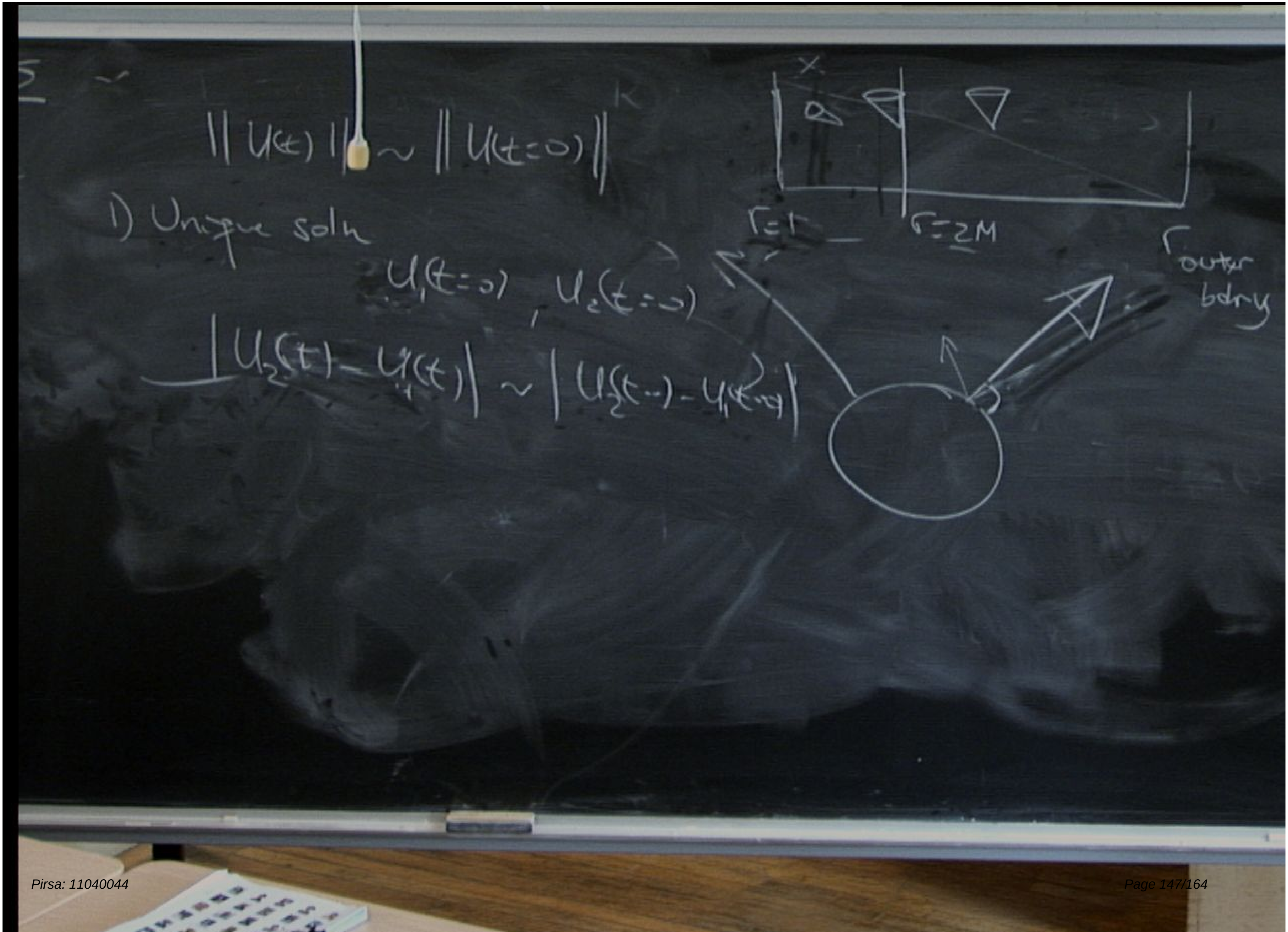
where $\|\cdot\|$ is an suitable norm, such as the L_2 norm:



|| U(t) ||
X
r=r
r=2M
r outer bdry


||U(t)|| ~ ||U(t=0)||
r=2M
r outer bdry


||U(t)|| ~ ||U(t=0)||
1) Unique soln
U1(t=0) , U2(t=0)
|U2(t) - U1(t)| ~ |U2(t..) - U1(t..)|
r=r
r=2M
r outer bdry



||U(t)|| ~ ||U(t=0)||
1) Unique soln
U1(t=0) , U2(t=0)
|U2(t) - U1(t)| ~ |U2(t=0) - U1(t=0)|
r=2M
r outer bdry

Stability Analysis

- Will use the property captured by ($\boxed{95}$) as working definition of stability.
- In particular, if you believe ($\boxed{95}$) is true for the wave equation, then you believe the wave equation is stable.
- Fundamentally, if FDA approximation *converges*, then expect the same behaviour for the difference solution:

$$\|u_j^n\| \sim \|u_j^0\| . \tag{97}$$

- FD solution constructed by *iterating in time*, generating

$$u_j^0,\ u_j^1,\ u_j^2,\ u_j^3,\ u_j^4,\ \cdots$$

in succession, using the FD equation

$$u_j^{n+1} = 2u_j^n - u_j^{n-1} + \lambda^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) .$$

Stability Analysis

- *Not* guaranteed that (97) holds for all values of $\lambda \equiv \triangle t \,/\, \triangle x$.
- For certain λ, have
$$\|u_j^n\| \gg \|u_j^0\|\,,$$
and for those λ, $\|u^n\|$ *diverges* from u, even (especially!) as $h \to 0$—that is, the difference scheme is *unstable*.
- For many wave problems (including all linear problems), given that a FD scheme is *consistent* (i.e. so that $\hat{\tau} \to 0$ as $h \to 0$), *stability is the necessary and sufficient condition for convergence* (Lax's theorem).

Heuristic Stability Analysis

- Write general time-dependent FDA in the form

$$\mathbf{u}^{n+1} = \mathbf{G}[\mathbf{u}^n]\,, \tag{98}$$

- $\mathbf{G}$ is some *update operator* (linear in our example problem)

- $\mathbf{u}$ is a column vector containing sufficient unknowns to write the problem in first-order-in-time form.

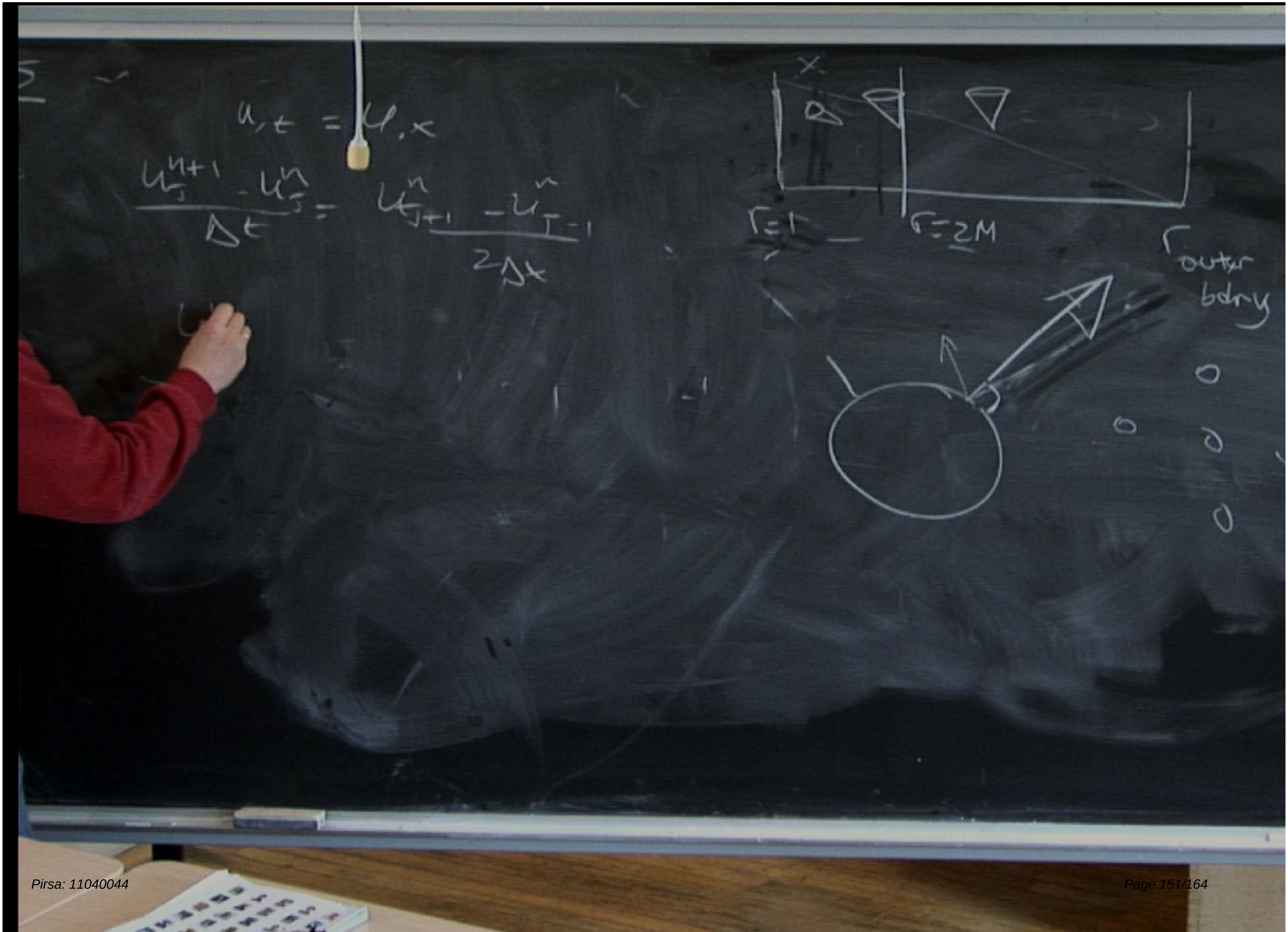
- Example: introduce new, auxiliary set of unknowns, v_j^n, defined by

$$v_j^n = u_j^{n-1}\,,$$

then can rewrite differenced-wave-equation (16) as

$$u_j^{n+1} = 2u_j^n - v_j^n + \lambda^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\,, \tag{99}$$

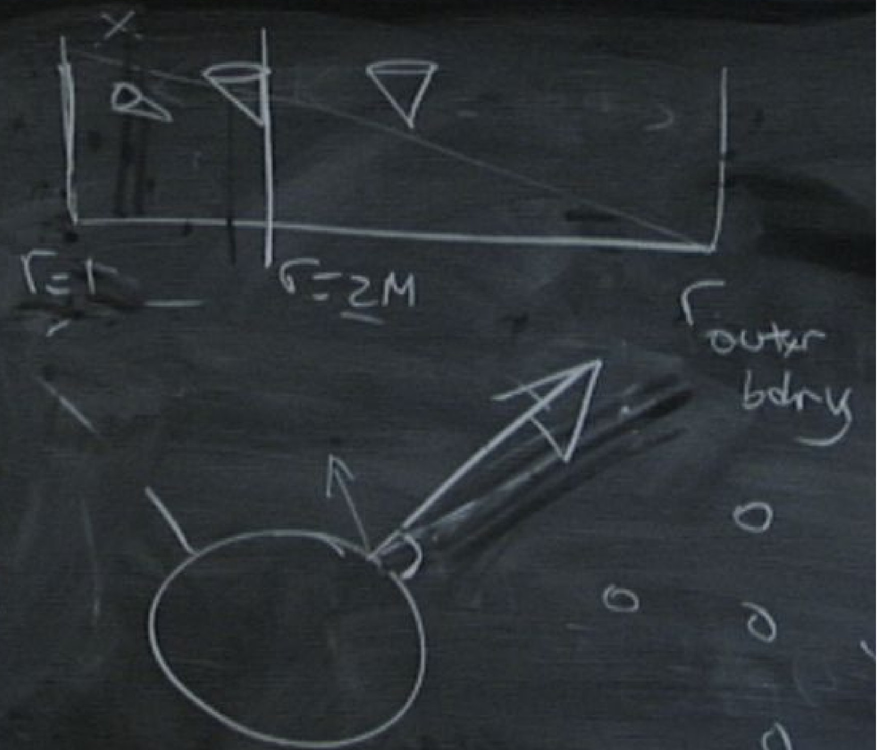
$$v_j^{n+1} = u_j^n\,, \tag{100}$$


$u_{,t} = u_{,x}$
$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$
x
r=1
r=2M
r outer bdry

$$u_{,t} = u_{,x}$$

$$\frac{u^{n+1}_{j} - u^{n}_{j}}{\Delta t} = \frac{u^{n}_{j+1} - u^{n}_{j-1}}{2\Delta x}$$

$$u^{n+1}_{j} = \left\{ u^{n}_{j},\ u^{n}_{j+1},\ u^{n}_{j-1} \right\}$$



Heuristic Stability Analysis

- Write general time-dependent FDA in the form

$$\mathbf{u}^{n+1} = \mathbf{G}[\mathbf{u}^n]\,, \tag{98}$$

- $\mathbf{G}$ is some *update operator* (linear in our example problem)
- $\mathbf{u}$ is a column vector containing sufficient unknowns to write the problem in first-order-in-time form.
- Example: introduce new, auxiliary set of unknowns, v_j^n, defined by

$$v_j^n = u_j^{n-1}\,,$$

then can rewrite differenced-wave-equation (16) as

$$u_j^{n+1} \;=\; 2u_j^n - v_j^n + \lambda^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\,, \tag{99}$$

$$v_j^{n+1} \;=\; u_j^n\,, \tag{100}$$

Heuristic Stability Analysis

- If difference scheme is to be stable, must have

$$|\mu_k| \le 1 \qquad k = 1, 2, \cdots J \tag{104}$$

(Note: μ_k will be complex in general, so $|\mu|$ denotes the complex modulus, $|\mu| \equiv \sqrt{\mu\mu^\star}$).

- Geometric interpretation: eigenvalues of the update matrix must lie on or within the unit circle

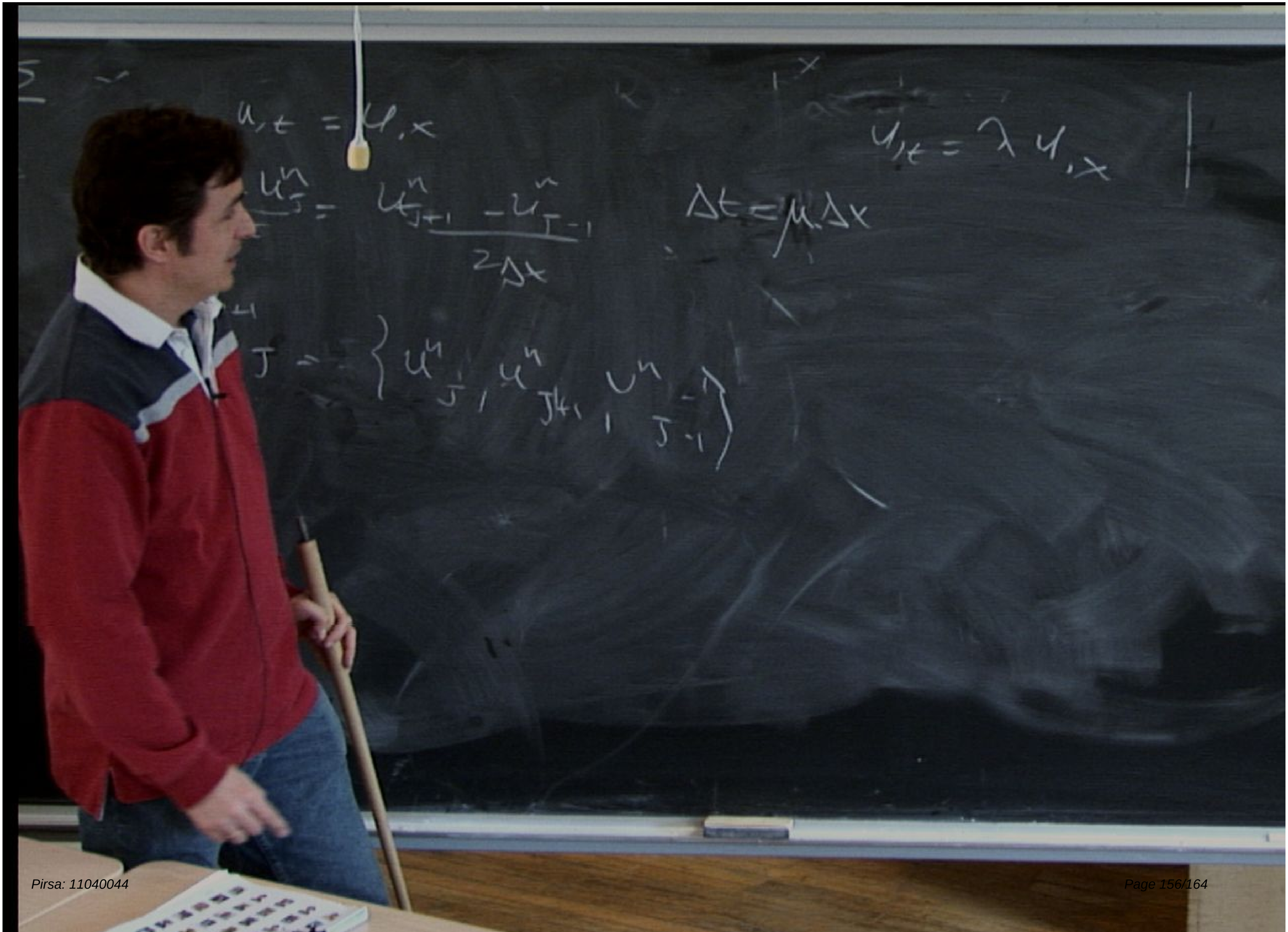
Heuristic Stability Analysis

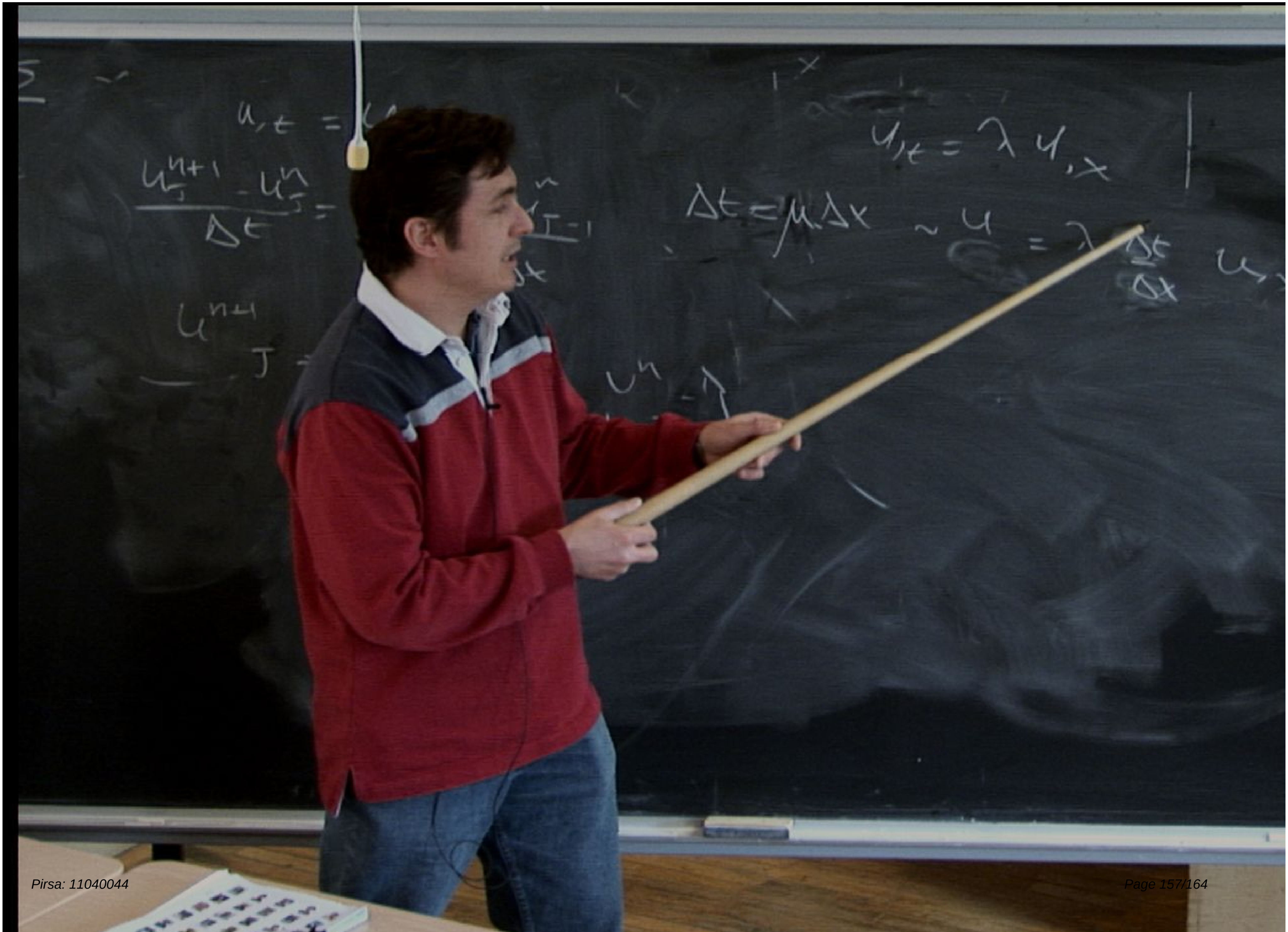
- If difference scheme is to be stable, must have

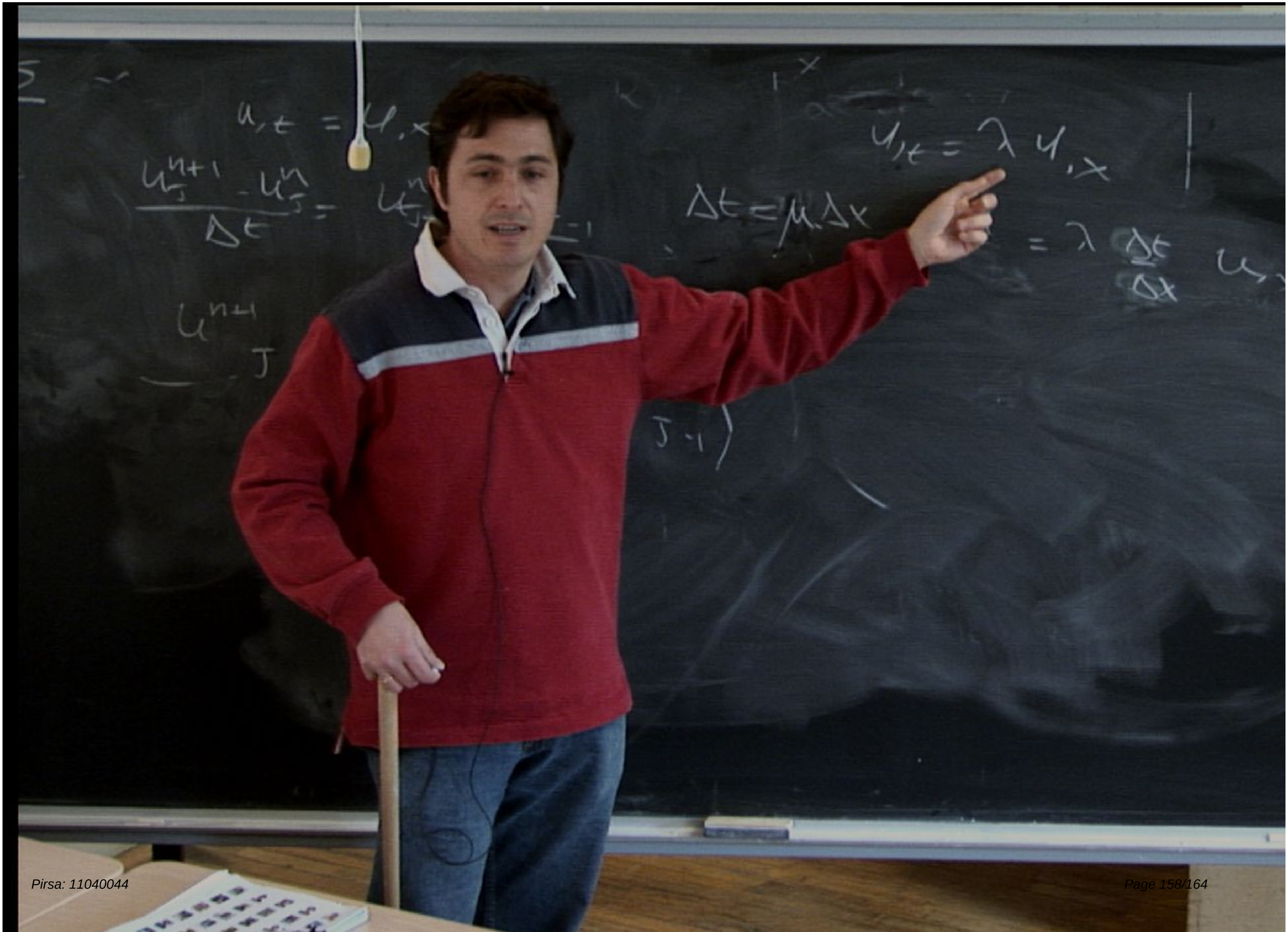
$$|\mu_k| \le 1 \qquad k = 1, 2, \cdots J \tag{104}$$

(Note: μ_k will be complex in general, so $|\mu|$ denotes the complex modulus, $|\mu| \equiv \sqrt{\mu\mu^\star}$).

- Geometric interpretation: eigenvalues of the update matrix must lie on or within the unit circle


u,t = u,x
Δt = μ Δx
u,t = λ u,x
2Δx


u,t = λ u,x
Δt = μ Δx


$u_{,t} = \lambda u_{,x}$
$\Delta t = \mu \Delta x$

$$u_{,t} = u_{,x}$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \qquad \Delta t = \mu \Delta x$$

$$u_{,t} = \lambda u_{,x}$$

$$\sim u = \lambda \mu \; u_{,x}$$

$$u_j^{n+1} = \{u_j^n, u_{j+1}^n, u_{j-1}^n\}$$

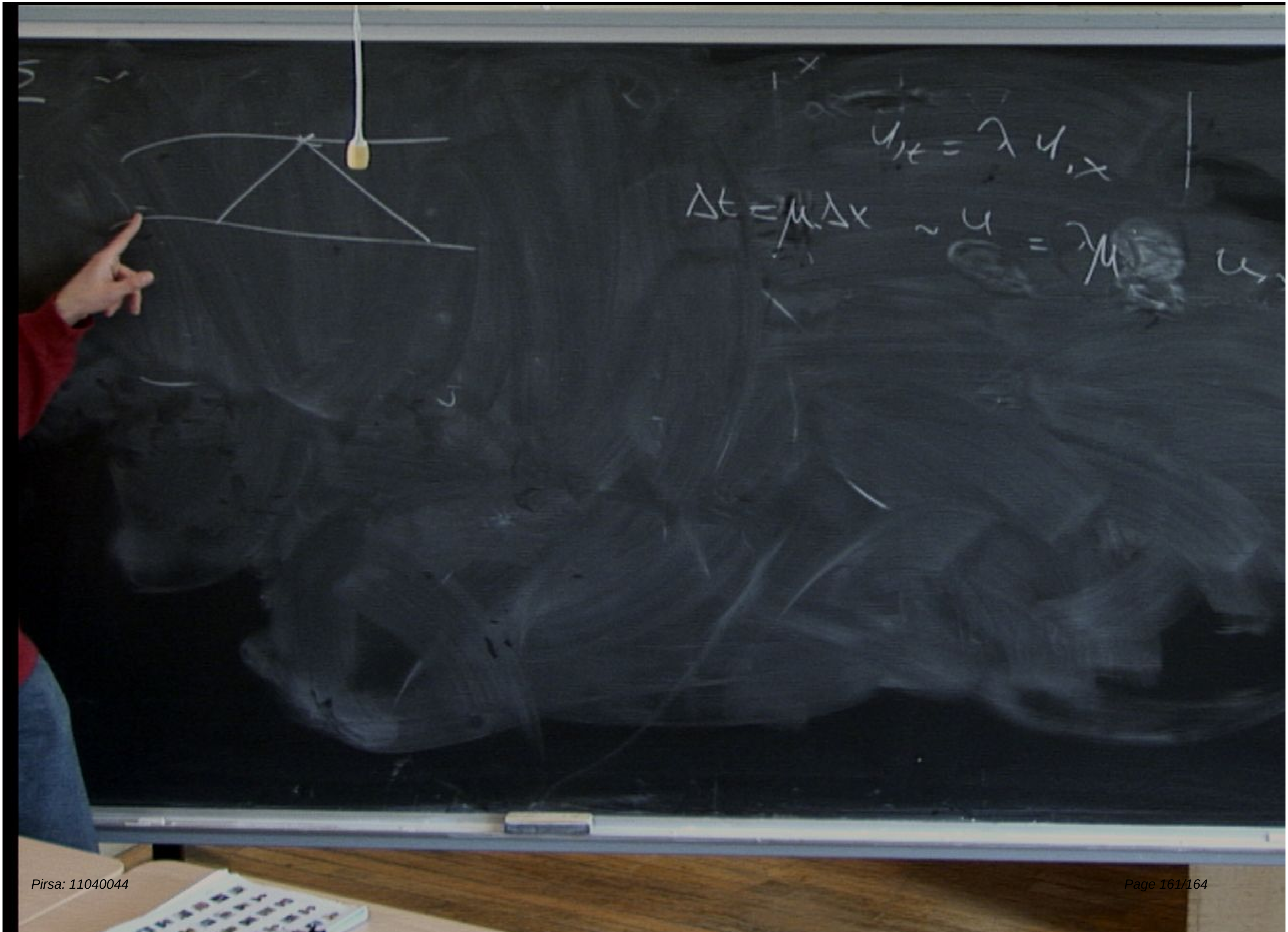
Von-Neumann (Fourier) Stability Analysis (Summary)

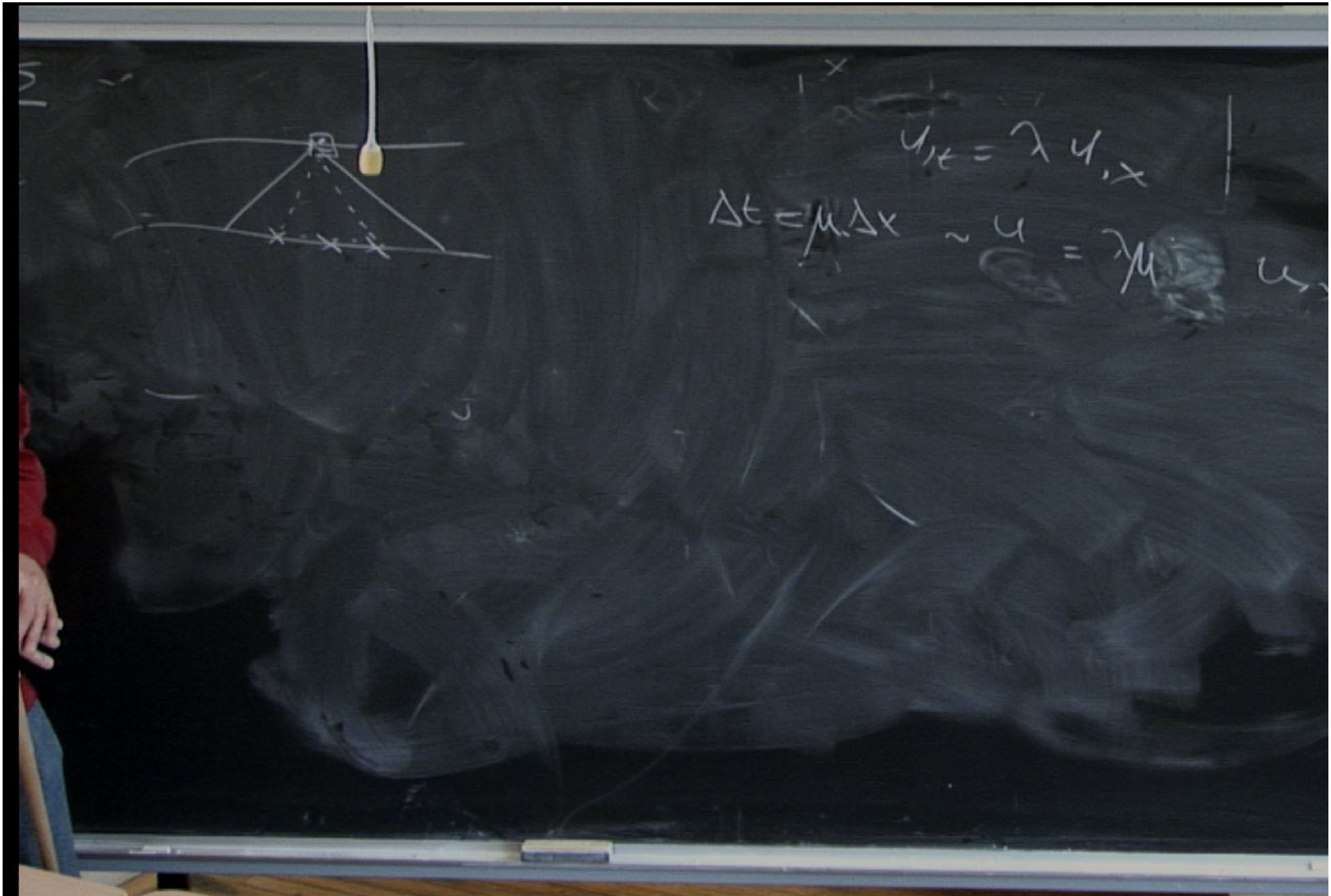
- Von-Neumann (VN) stability analysis based on the ideas sketched above
- Assumes that difference equation is linear with constant coefficients, periodic boundary conditions boundary conditions are periodic
- Can then use Fourier analysis: difference operators in real-space variable $x \longrightarrow$ algebraic operations in Fourier-space variable k
- VN applied to wave-equation example shows that must have

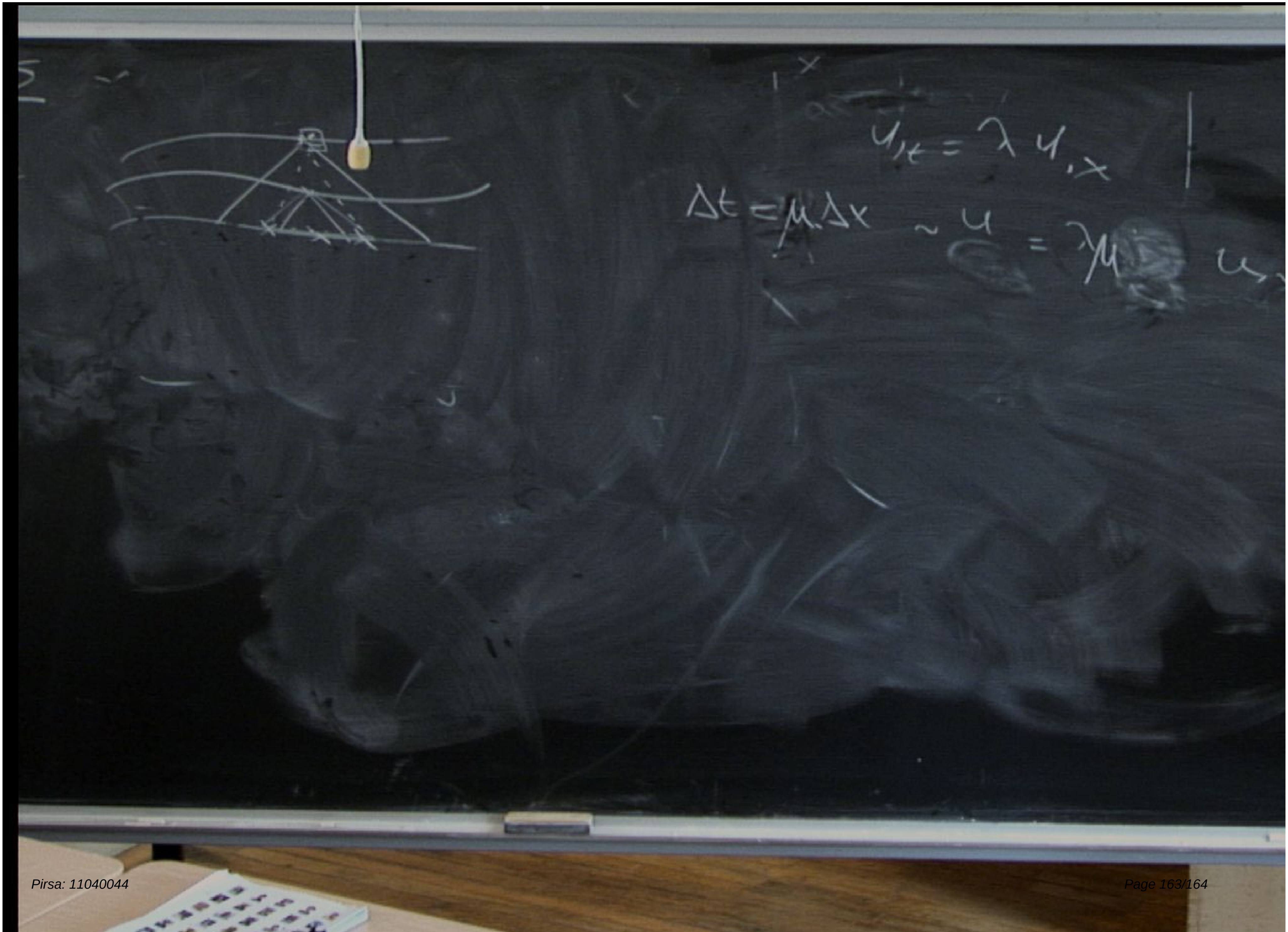
$$\lambda \equiv \frac{\triangle t}{\triangle x} \leq 1 \,,$$

for stability of scheme ([16]).

- Condition is often called the CFL condition—after Courant, Friedrichs and Lewy who derived it in 1928
- This type of instability has "physical" interpretation, often summarized by the


u,t = λ u,x
Δt = μ Δx
~ u = λμ u,x


$u_{,t} = \lambda u_{,x}$
$\Delta t = \mu \Delta x$
$\sim u = \lambda\mu \, u_{,x}$



$u_{,t} = \lambda u_{,x}$
$\Delta t = \mu \Delta x$
$\sim u = \lambda\mu \; u_{,x}$

Heuristic Stability Analysis

- If difference scheme is to be stable, must have

$$|\mu_k| \le 1 \qquad k = 1, 2, \cdots J \tag{104}$$

(Note: μ_k will be complex in general, so $|\mu|$ denotes the complex modulus, $|\mu| \equiv \sqrt{\mu\mu^\star}$).

- Geometric interpretation: eigenvalues of the update matrix must lie on or within the unit circle