Title: Explorations in Numerical Relativity - Lecture 3

Date: Apr 06, 2011 11:30 AM

URL: http://pirsa.org/11040042

Abstract:

perimeter scholars
international

Richardson Extrapolation

- When it works, Richardson extrapolation has an almost magical quality about it
- However, generally have to start with fairly accurate (on the order of a few %) solutions in order to see the dramatic improvement in accuracy suggested by (84).
- Still a struggle to achieve that sort of accuracy (i.e. a few %) for *any* computation in many areas of numerical relativity/astrophysics *and* keep the error smooth (which is necessary for Richardson extrapolation to be effective)
- Thus, techniques based on Richardson extrapolation have not had a major impact in this context, although higher-order $O(h^4)$, $O(h^6)$ etc. finite difference methods *are* increasingly common for the vacuum Einstein equations

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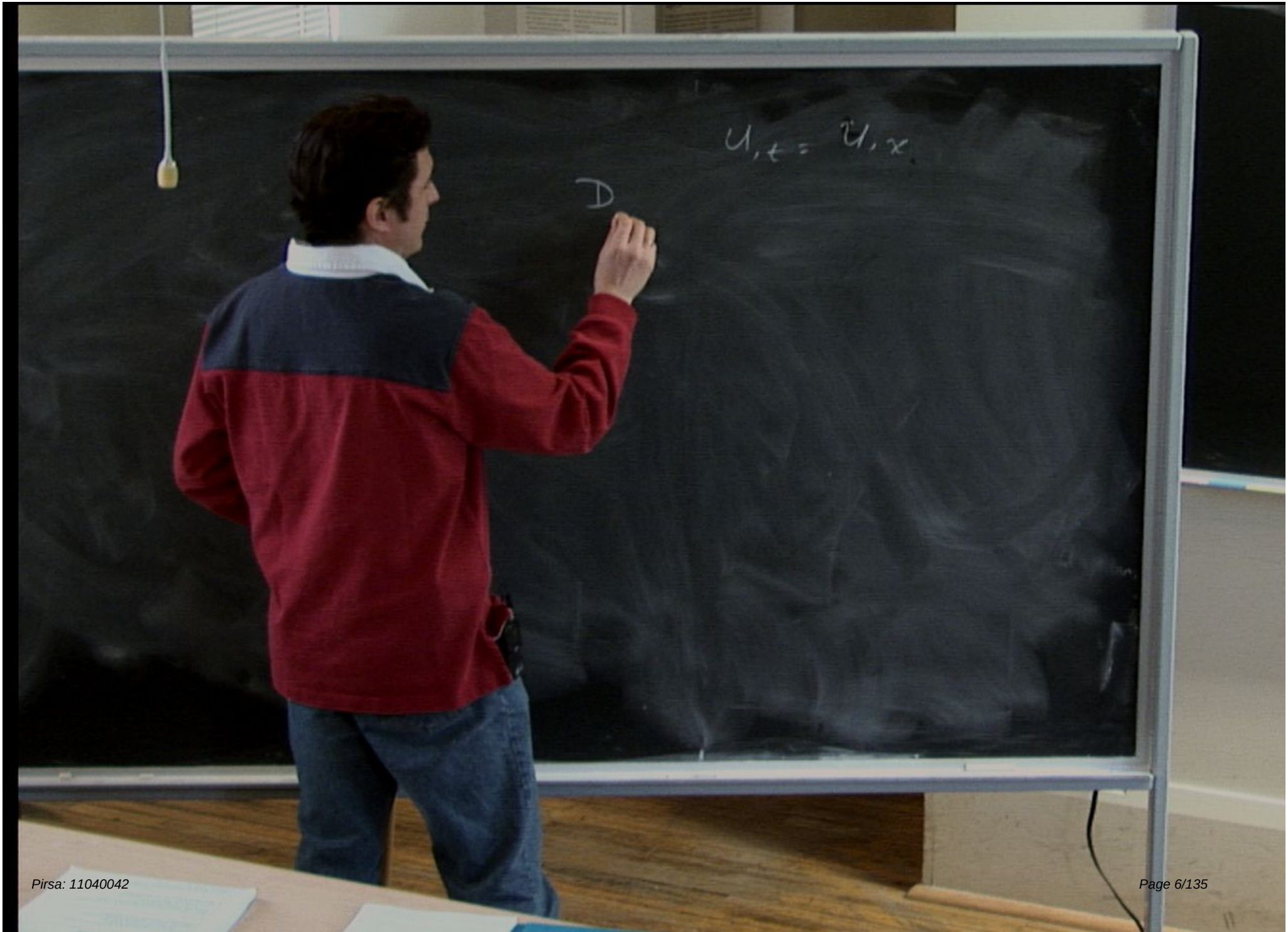
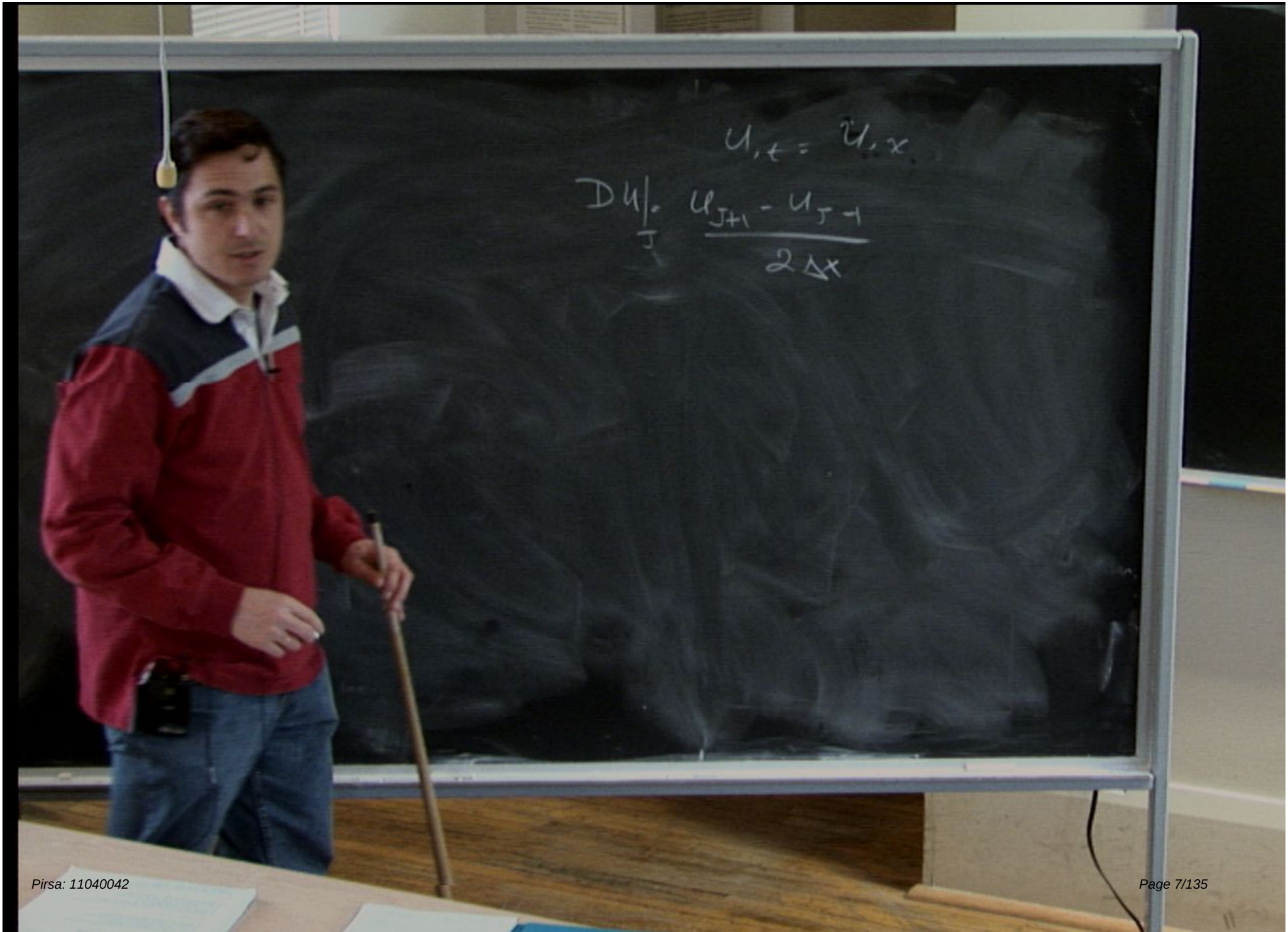
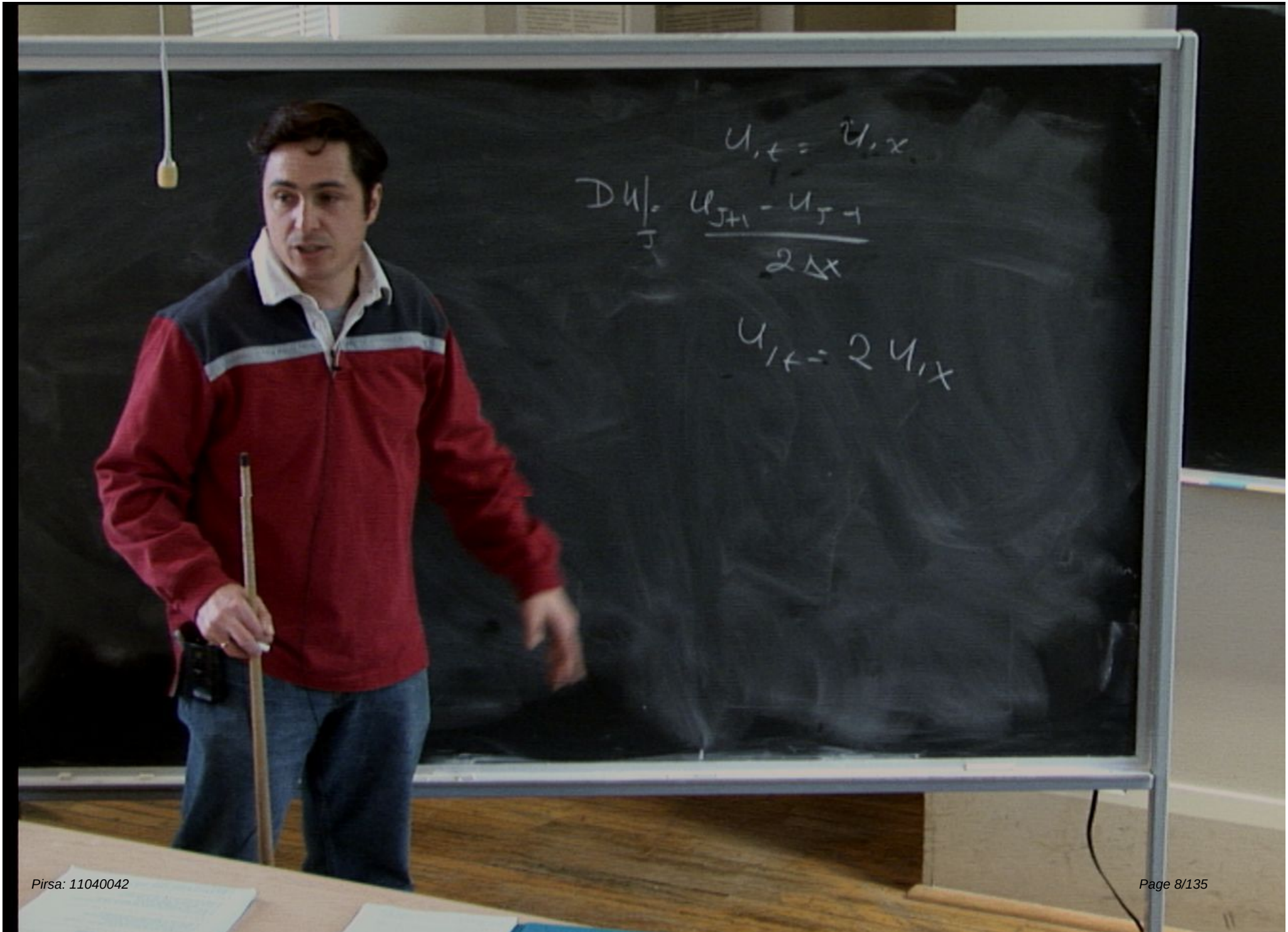

$u_{,t} = u_{,x}$
D


$u_{,t} = u_{,x}$
$D u|_j \; \frac{u_{j+1} - u_{j-1}}{2\Delta x}$


$u_{,t} = u_{,x}$
$D u|_j \; \frac{u_{j+1} - u_{j-1}}{2\Delta x}$
$u_{,t} = 2u_{,x}$

Independent Residual Evaluation

- Question that often arises in convergence testing: is the following:

  "OK, you've established that u^h is converging as $h \to 0$, but how do you know you're converging to u, the solution of the continuum problem?"

- Here, notion of an independent residual evaluation is very useful.

- Idea is as follows: have continuum PDE

$$Lu - f = 0 \tag{85}$$

  and FDA

$$L^h u^h - f^h = 0 \tag{86}$$

- Assume that u^h is apparently converging from, for example, computation of convergence factor (78) that looks like it tends to 4 as h tends to 0.

- However, do not know if we have derived and/or implemented our discrete operator L^h correctly.

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Independent Residual Evaluation

- Note that implicit in the "implementation" is the fact that, particularly for multi-dimensional and/or implicit and/or multi-component FDAs, considerable "work" (i.e. analysis and coding) may be involved in setting up and solving the algebraic equations for u^h.

- As a check that solution *is* converging to u, consider a *distinct* (i.e. independent) discretization of the PDE:

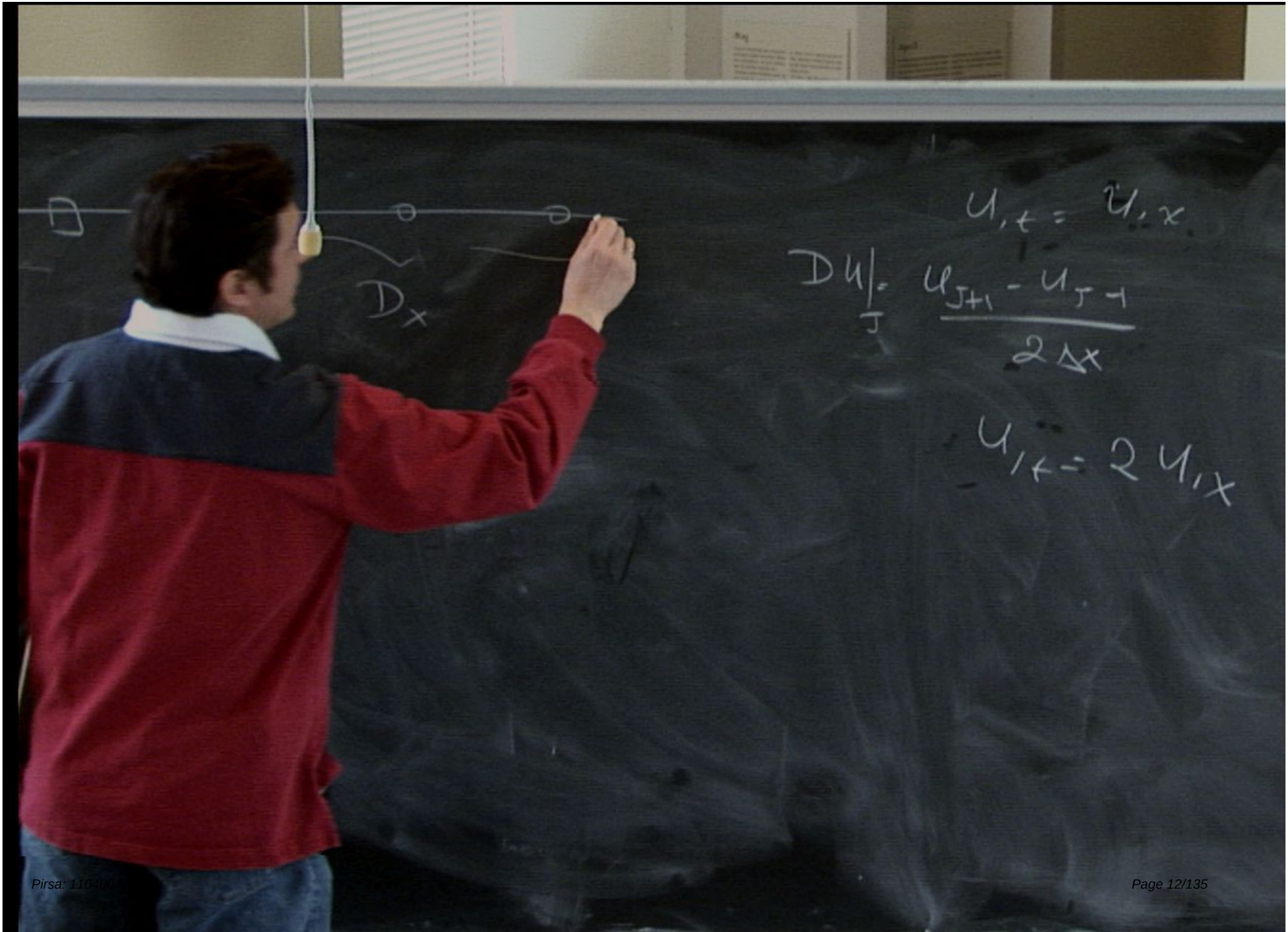
$$\tilde{L}^h \tilde{u}^h - f^h = 0 \tag{87}$$

- Only thing needed from this FDA for the purposes of the independent residual test is the new FD operator $\tilde{L}^h$.

- As with L^h, can expand $\tilde{L}^h$ in powers of the mesh spacing:

$$\tilde{L}^h = L + h^2 E_2 + h^4 E_4 + \cdots \tag{88}$$

where E_2, E_4, $\cdots$ are higher order (involve higher order derivatives than L) differential operators


$u_{,t} = u_{,x}$
$D u|_j \approx \frac{u_{j+1} - u_{j-1}}{2\Delta x}$
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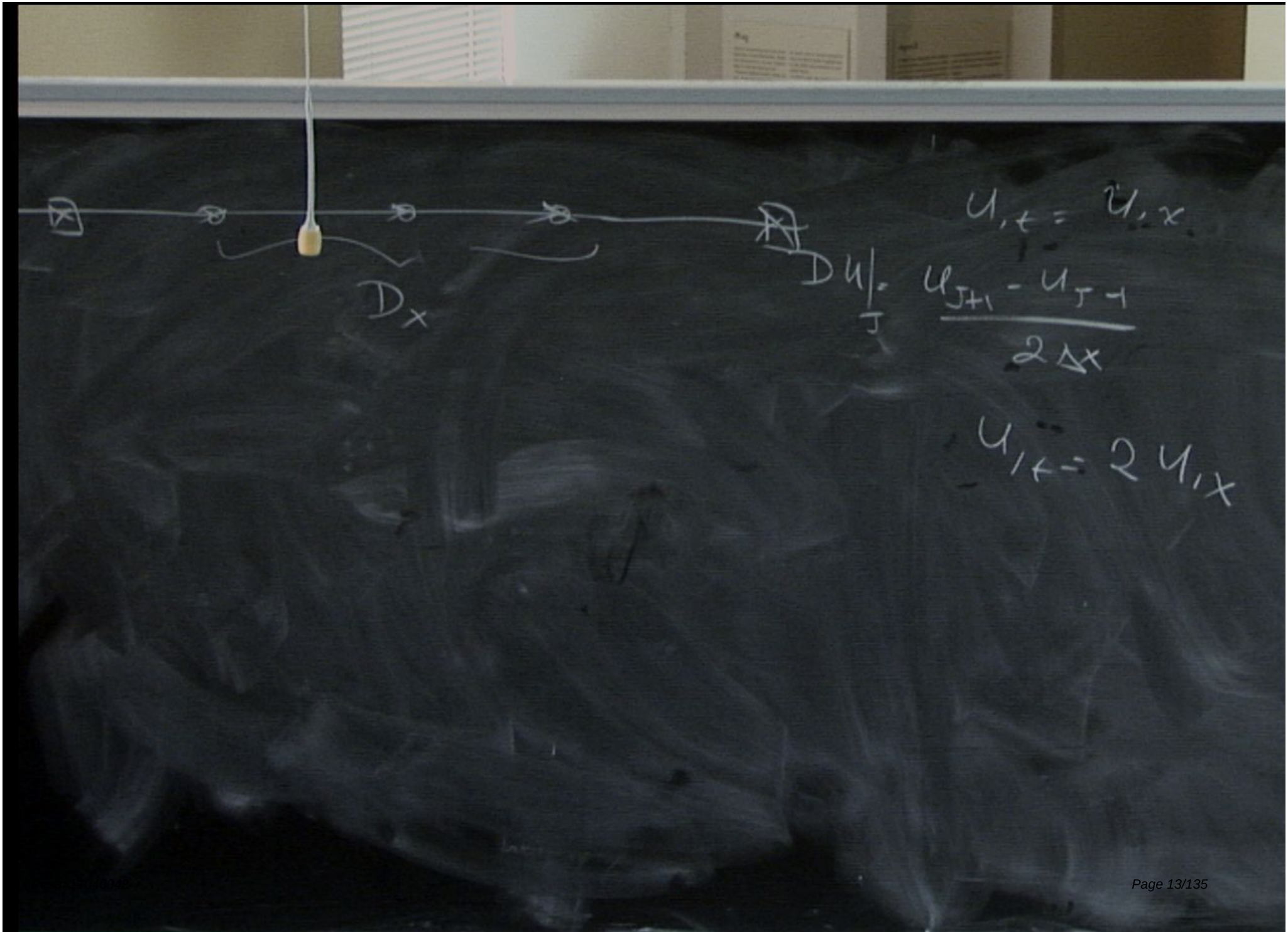
D_x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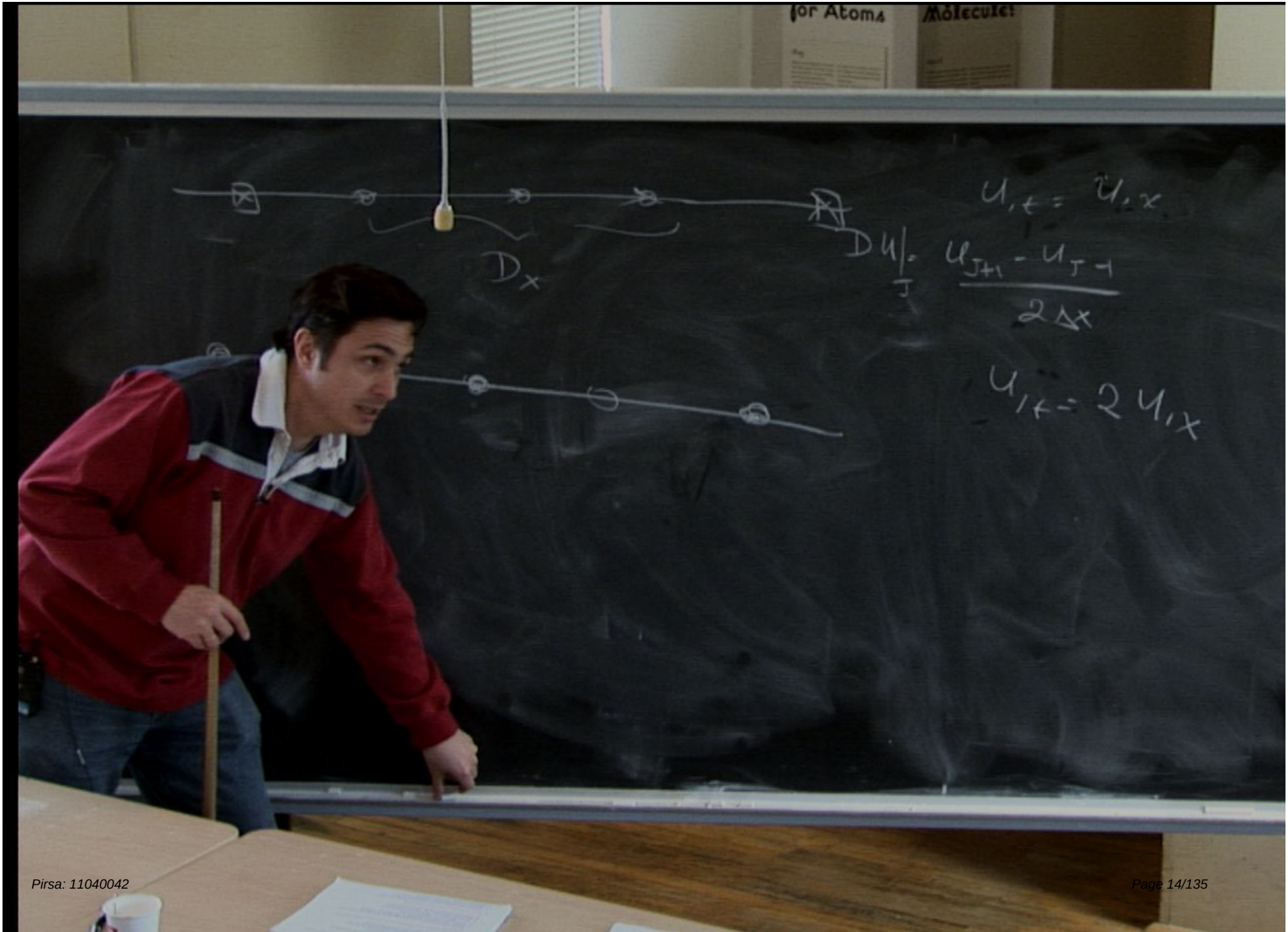
for Atoms
Molecule!
$u_{,t} = \ddot{u}_{,x}$
D_x
$D u|_j$
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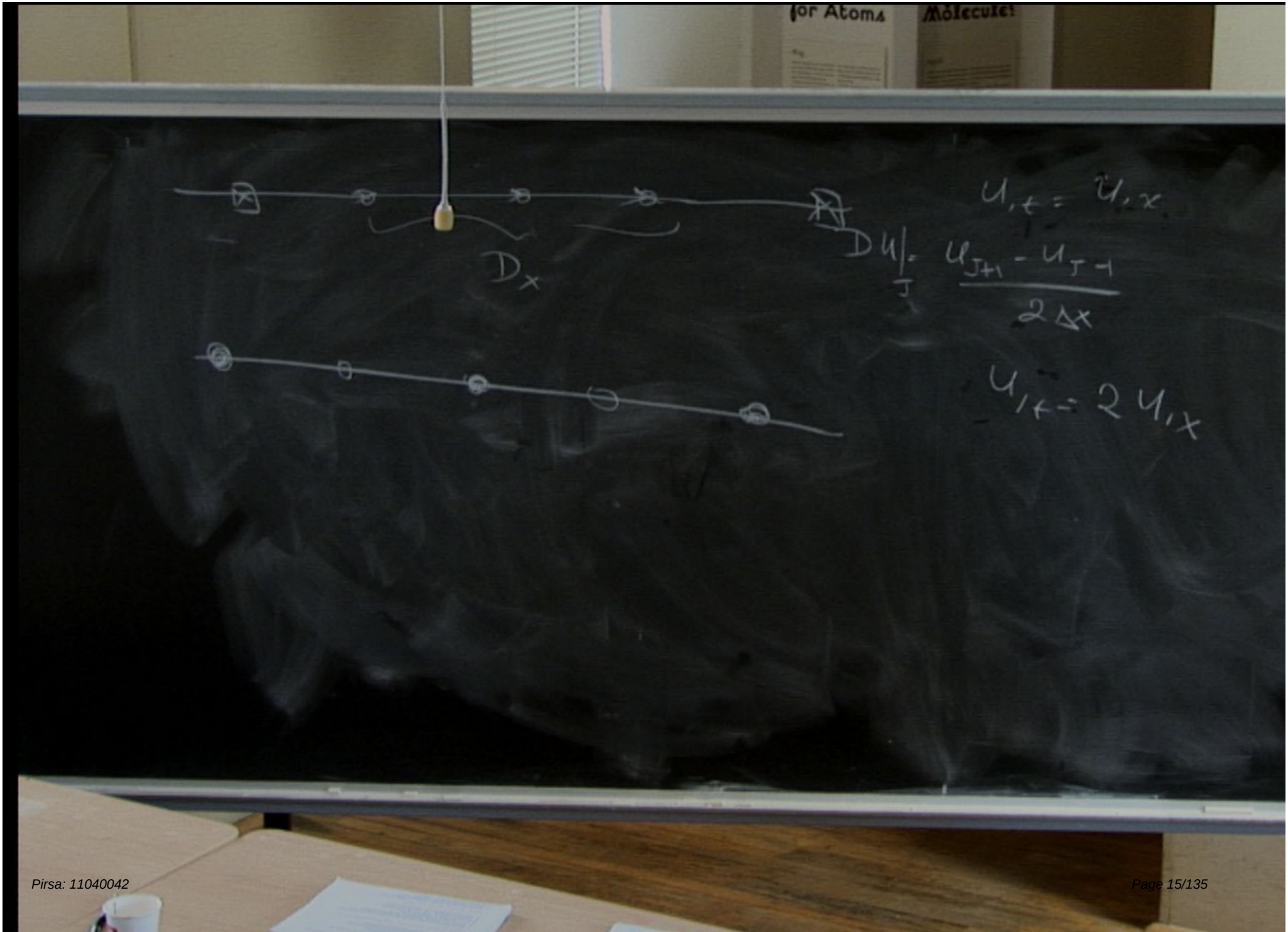
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Independent Residual Evaluation

- Conversely, assume there is a problem in the derivation and/or implementation of $L^h u^h = f^h = 0$, but there is still convergence; i.e. for example,

$$u^{2h} - u^h \to 0 \quad \text{as} \quad h \to 0 \tag{93}$$

- Then must have something like

$$u^h = u + e_0 + h e_1 + h^2 e_2 + \cdots \tag{94}$$

where crucial fact is that the error must have an $O(1)$ component, e_0.

- In this case, will compute

$$\begin{aligned} \tilde{L}^h u^h &= \left(L + h^2 E_2 + O(h^4)\right)\left(u + e_0 + h e_1 + h^2 e_2 + O(h^4)\right) \\ &= Lu + Le_0 + hLe_1 + O(h^2) \\ &= Le_0 + O(h) \end{aligned}$$

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- Unless one is *extraordinarily* (un) lucky, and $L\,e_0$ vanishes, will *not* observe the

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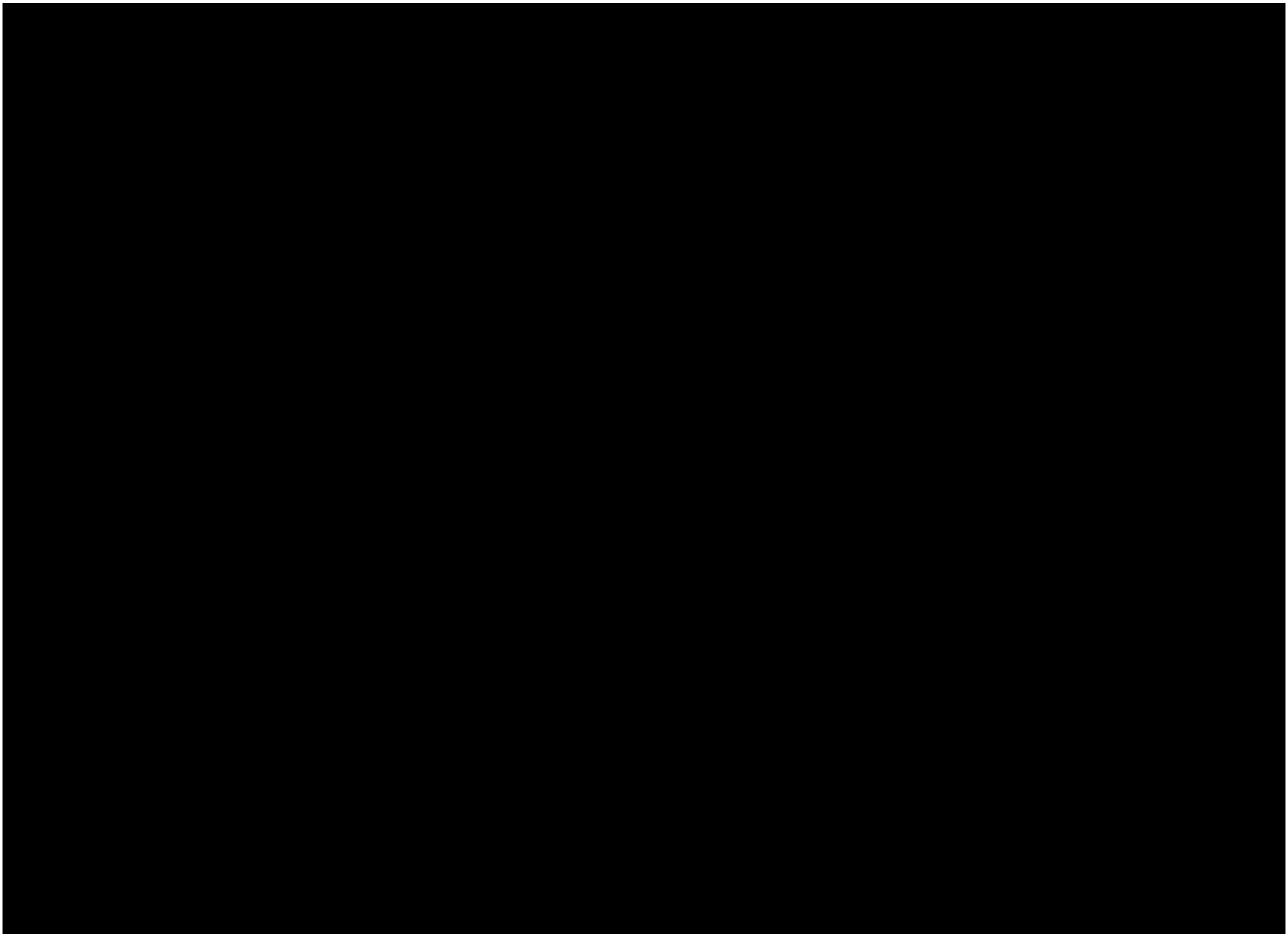
(3) WALD, APP. E.2 ; CHAPTER 10

(4) ARNOWITT, DESER & MISNER (1962) "THE DYN of GR" IN "GRAVITATION: AN INTRODUCTION TO CURRENT RESEARCH" (L. WITTEN ed)



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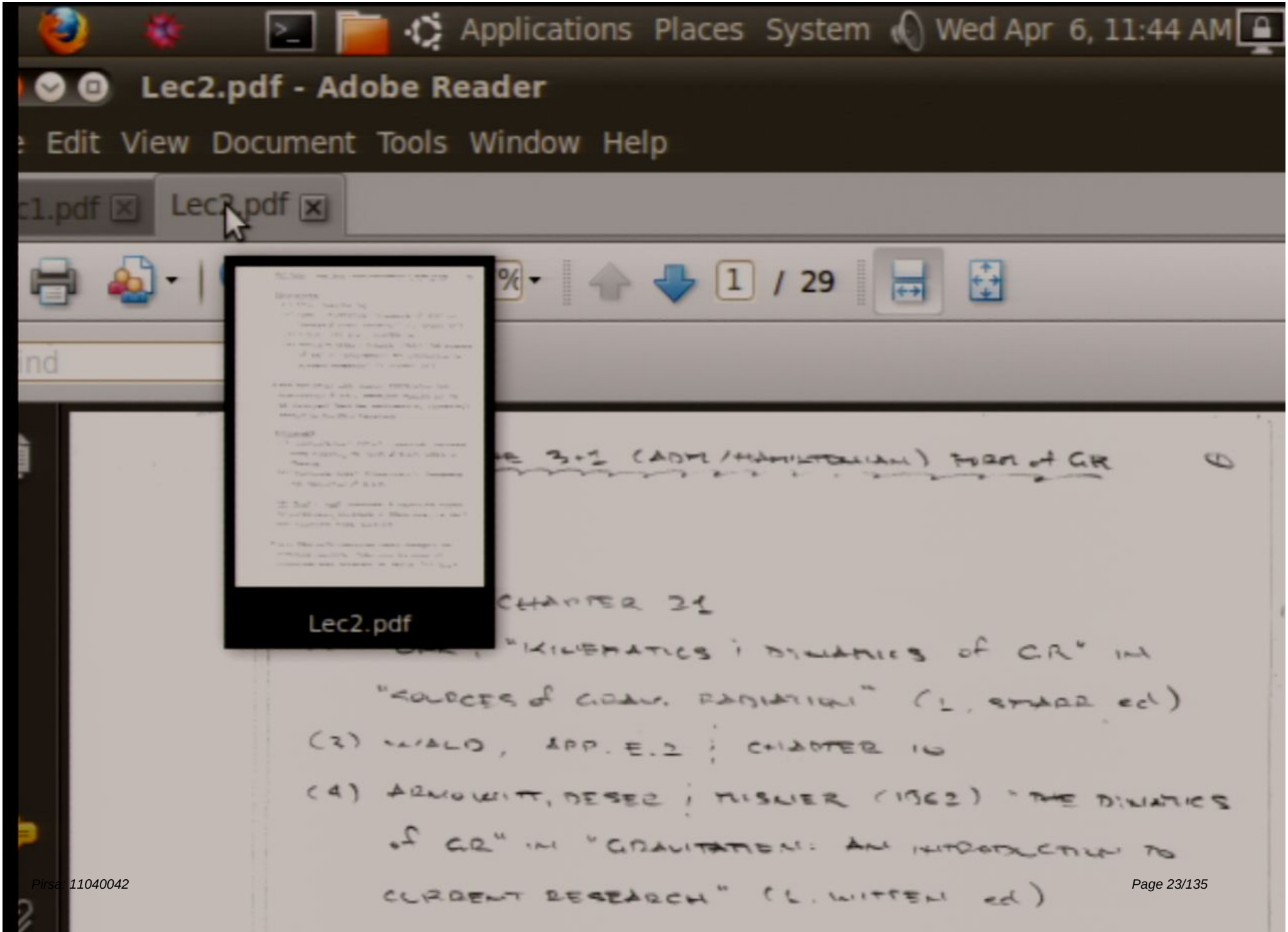
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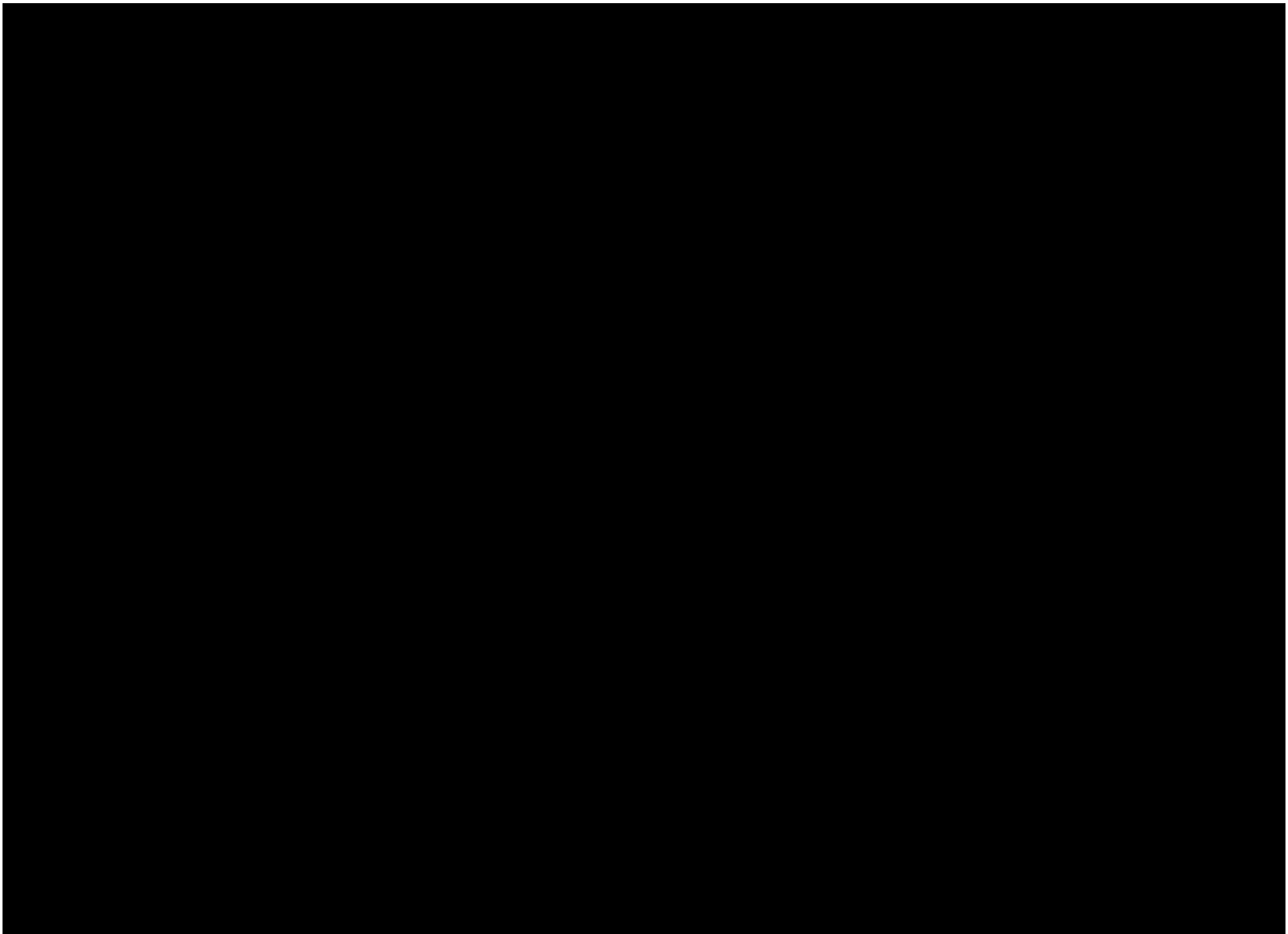

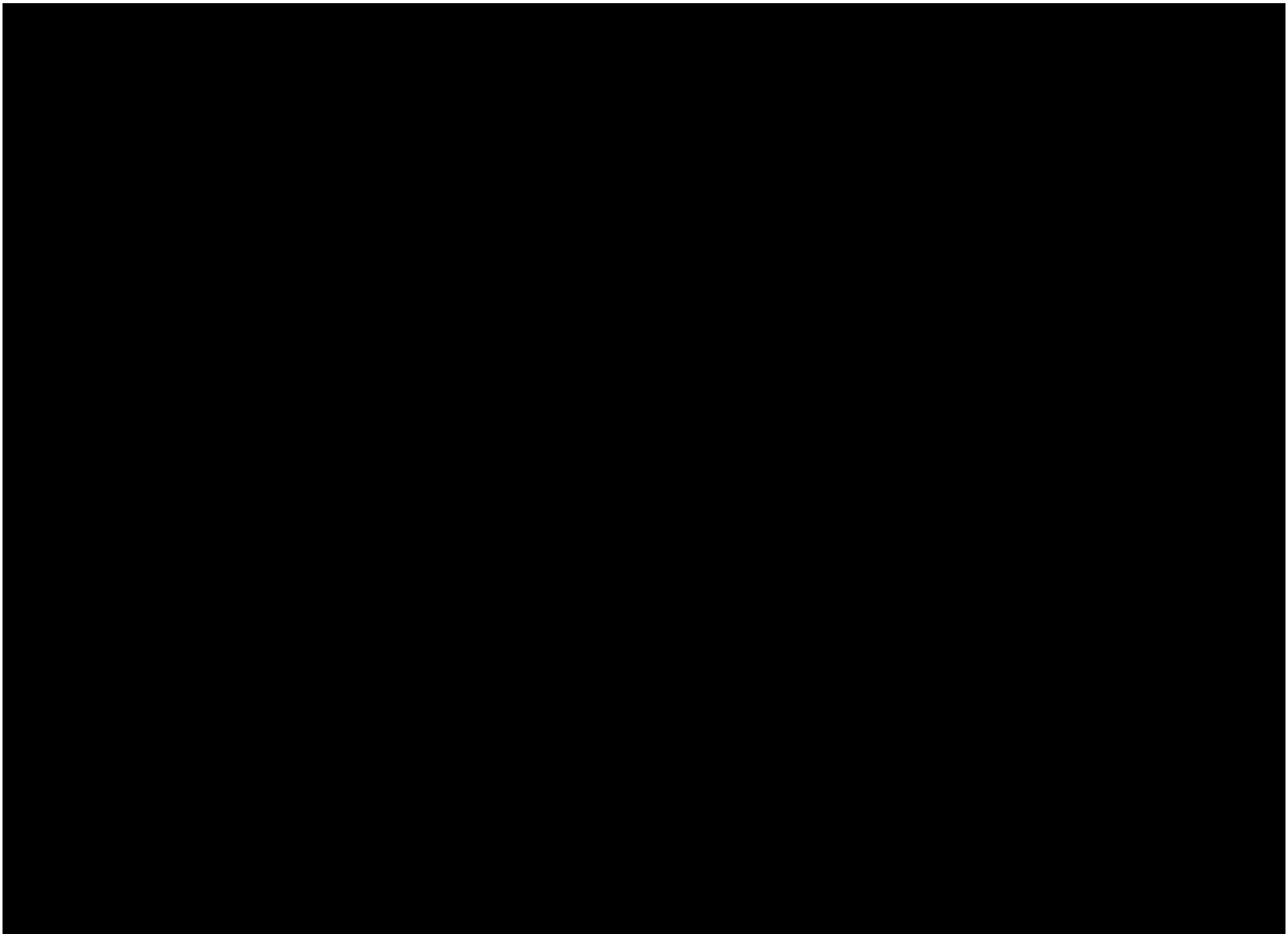

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• ADM MOTIVATION WAS, AGAIN, PREPARATION FOR QUANTIZATION of GR; FORMALISM TURNED OUT TO BE EXCELLENT BASIS FOR COMPUTATIONAL (NUMERICAL) ASSAULT ON EINSTEIN EQUATIONS

APPROACHES

(1) "COORDINATE-FUL" (MTW) : INTUITIVE, CONNECTS MORE DIRECTLY TO FORM of E.O.M USED IN PRACTICE

(2) "COORDINATE-FREE" (YORK/WALD) : PREFERABLE

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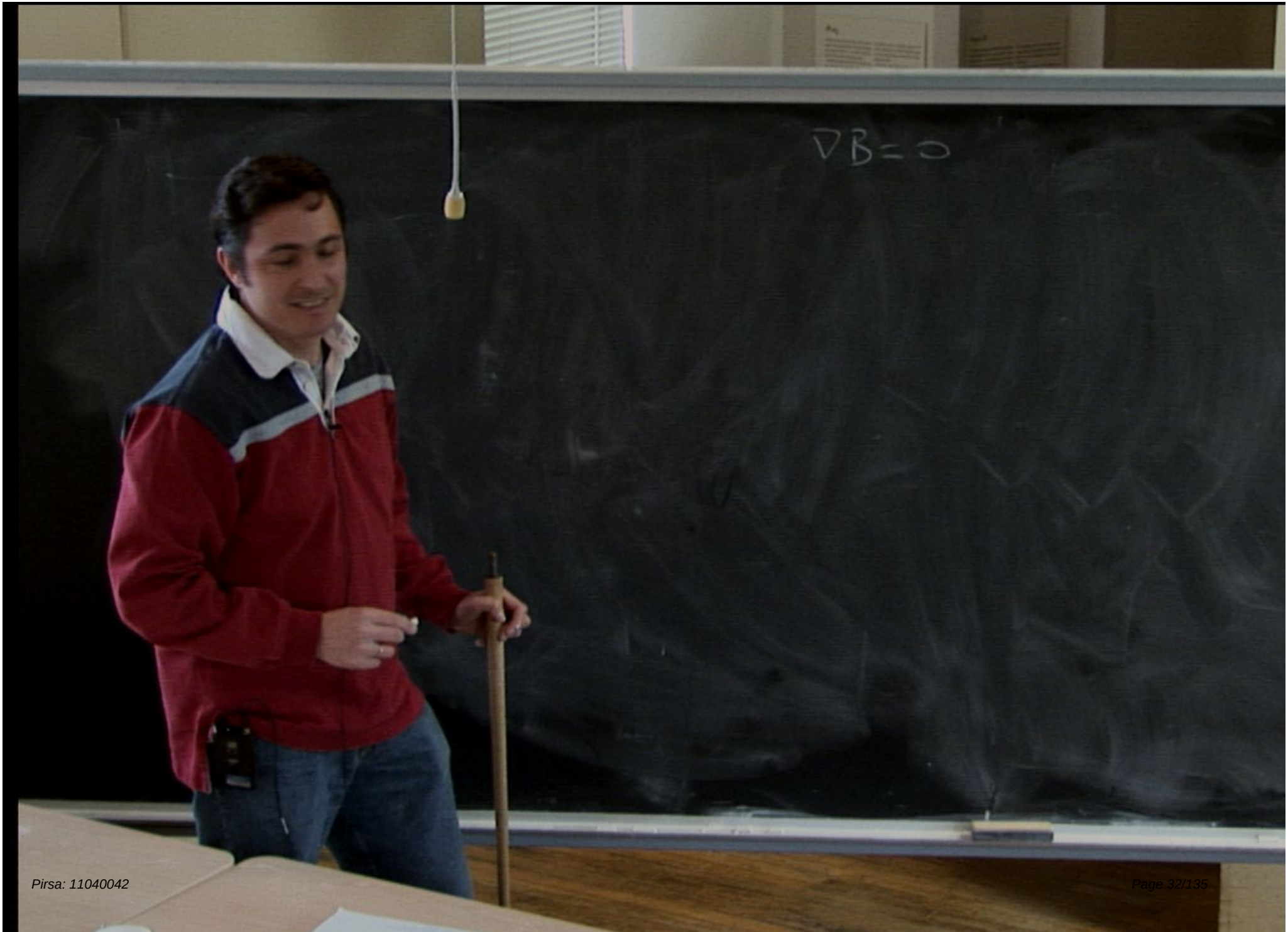
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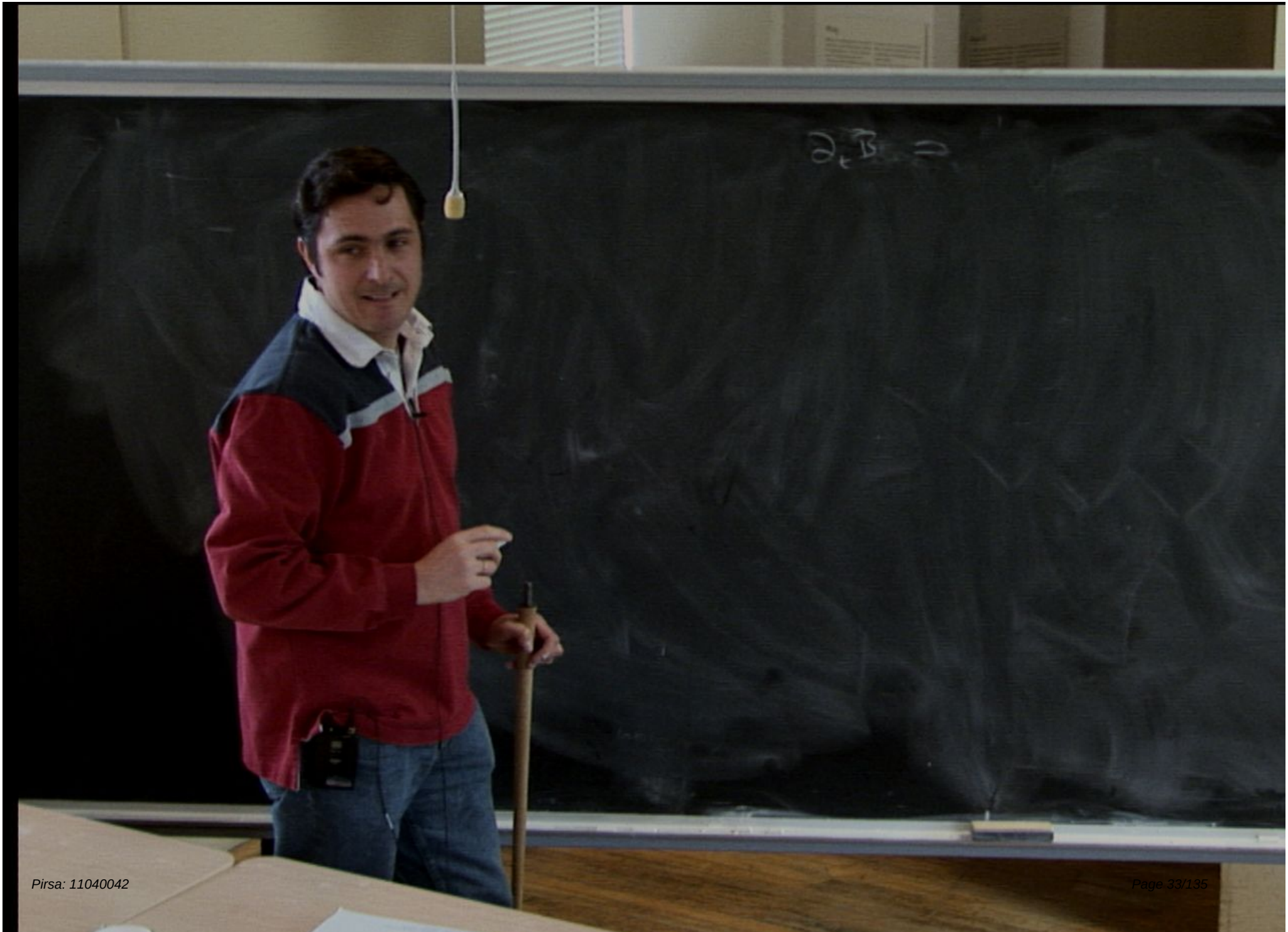

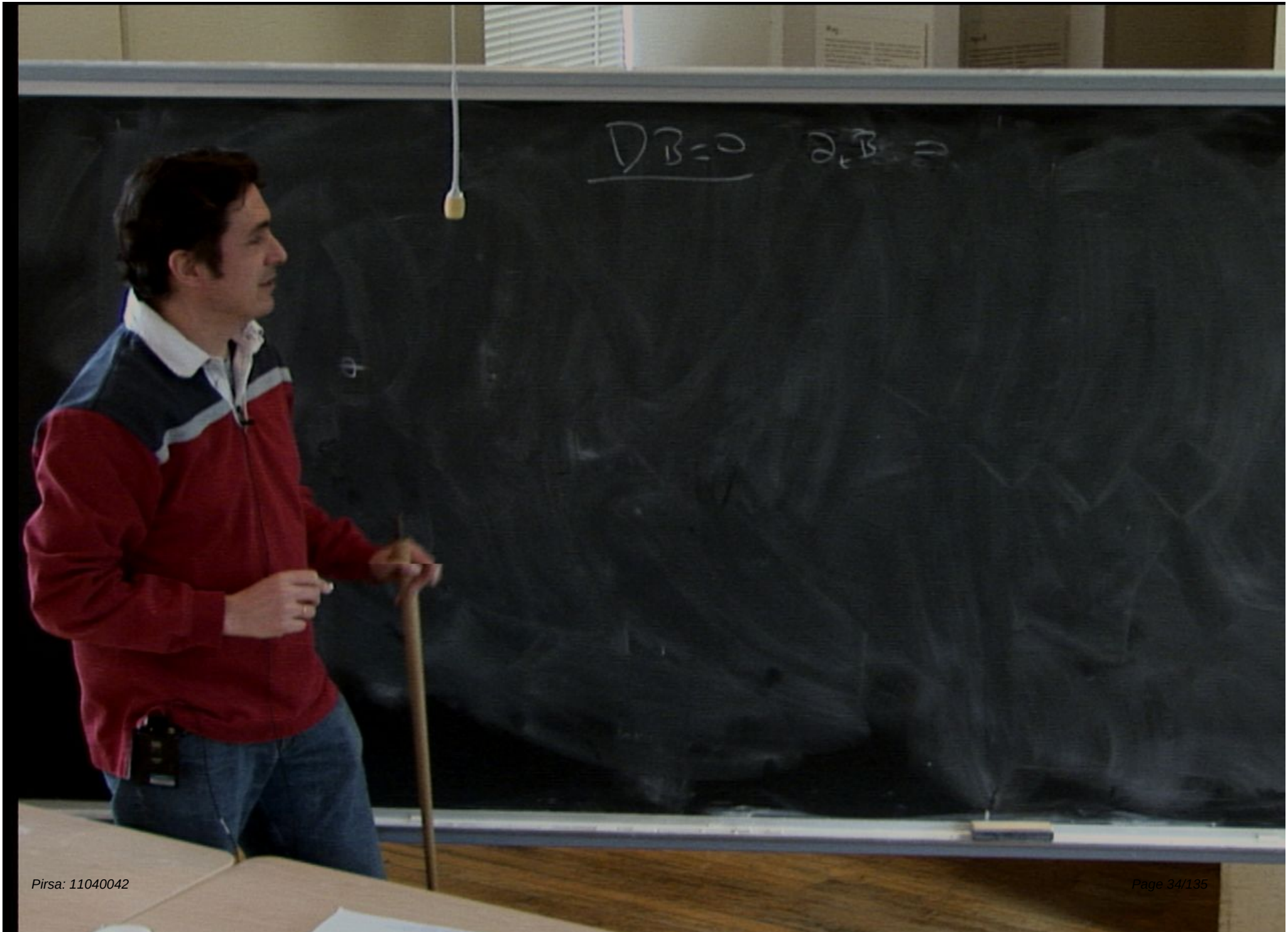
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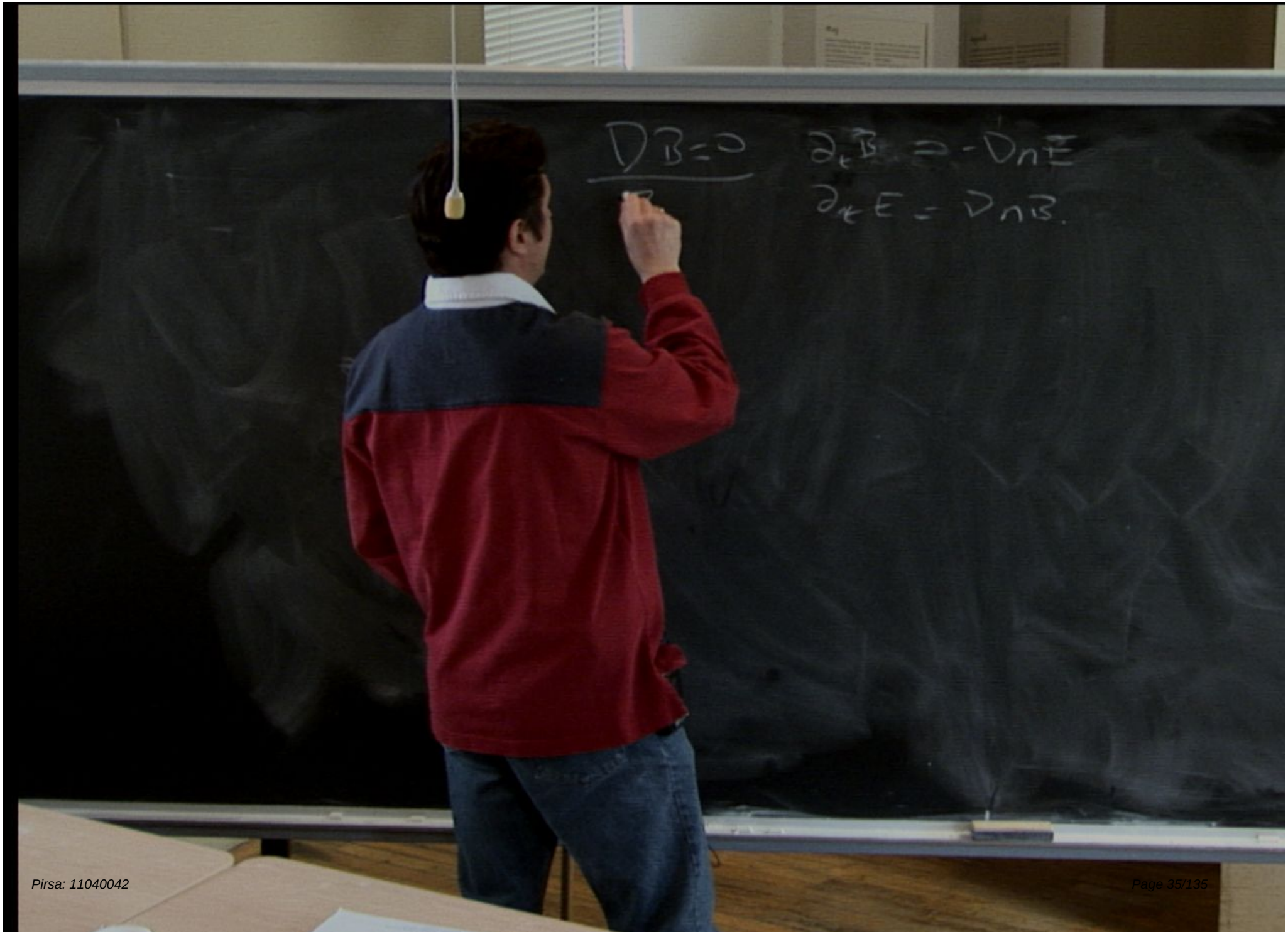
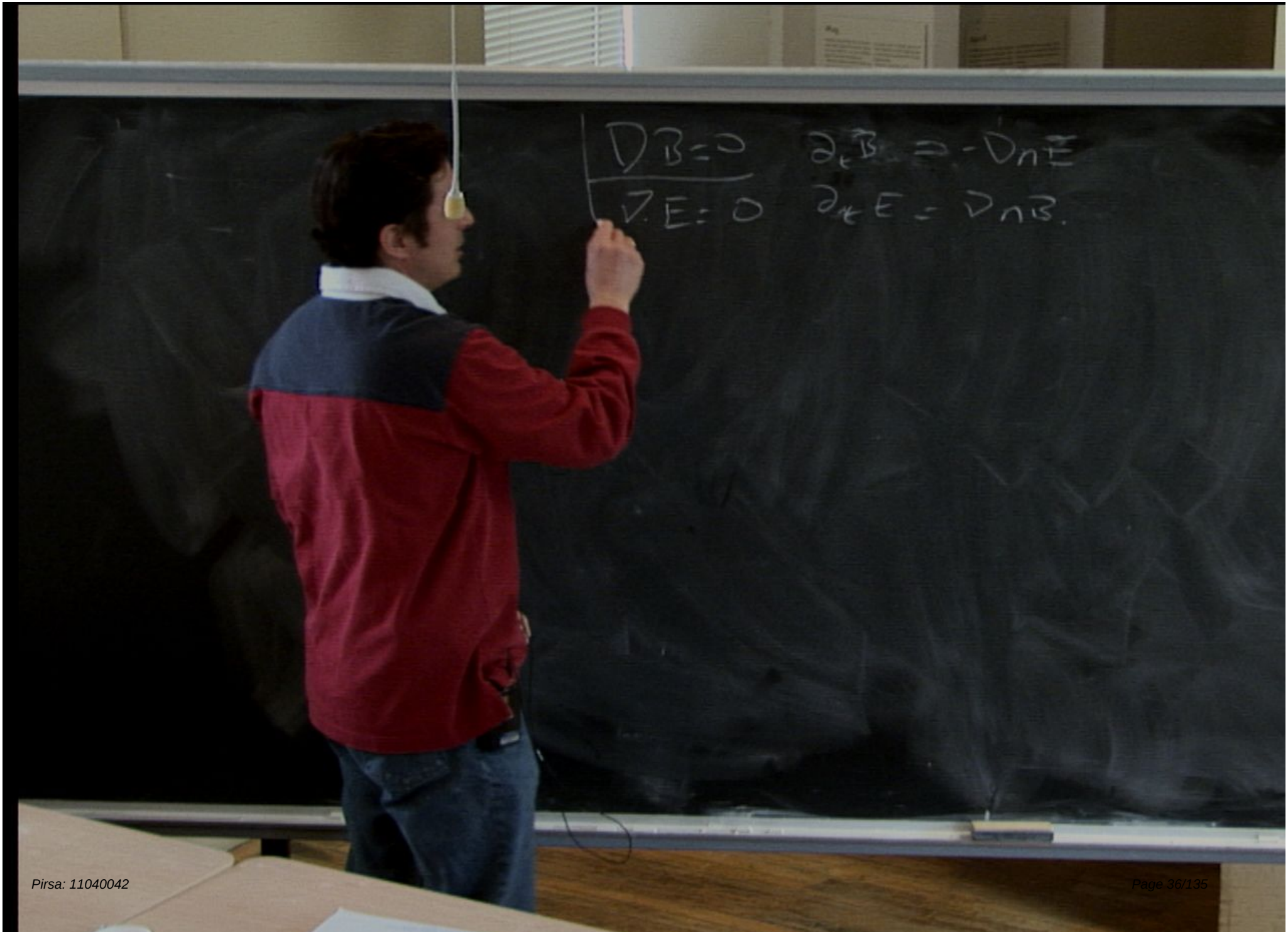

∇B=0
∂tB = -∇∩E
∂tE = ∇∩B.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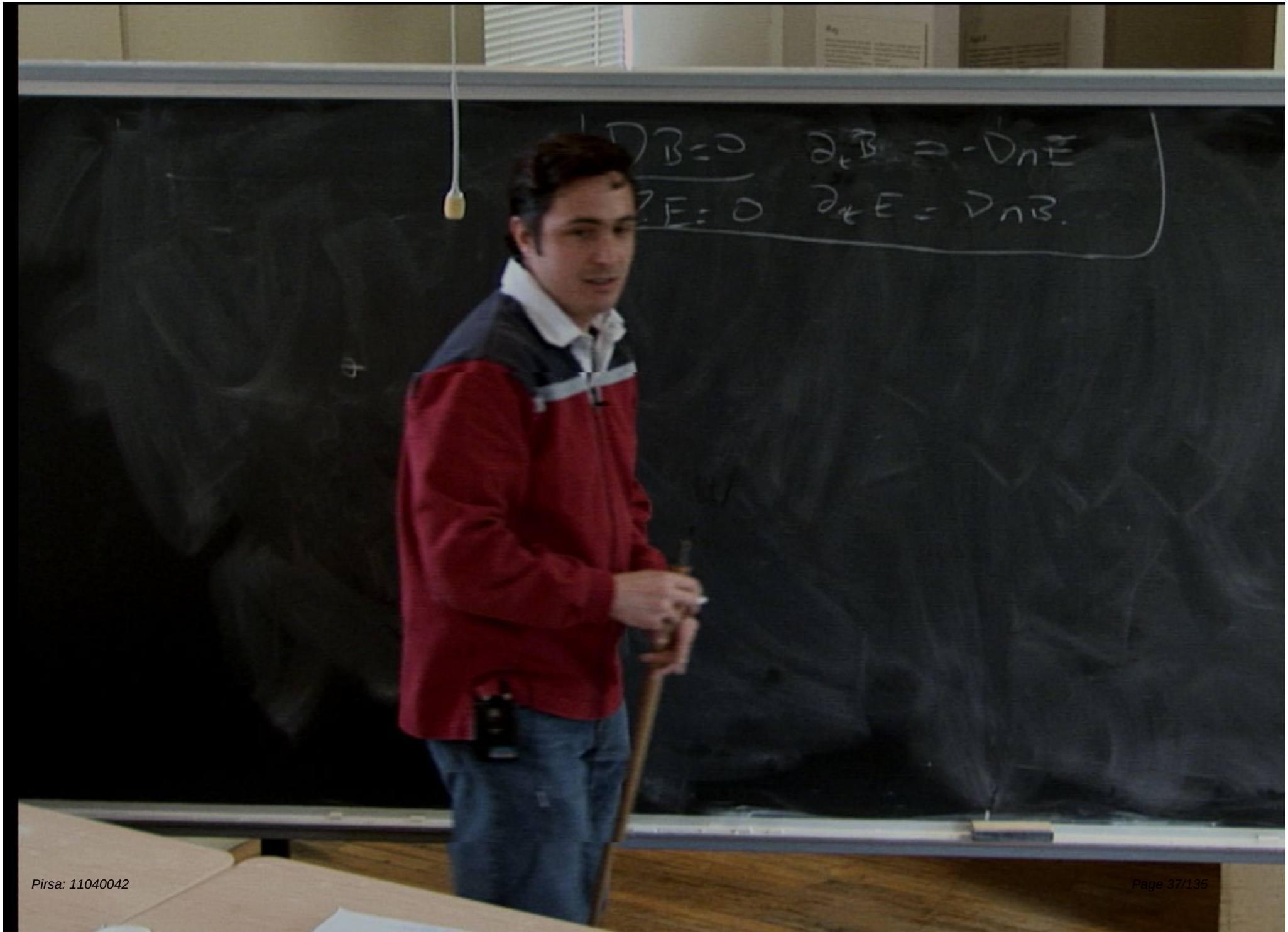
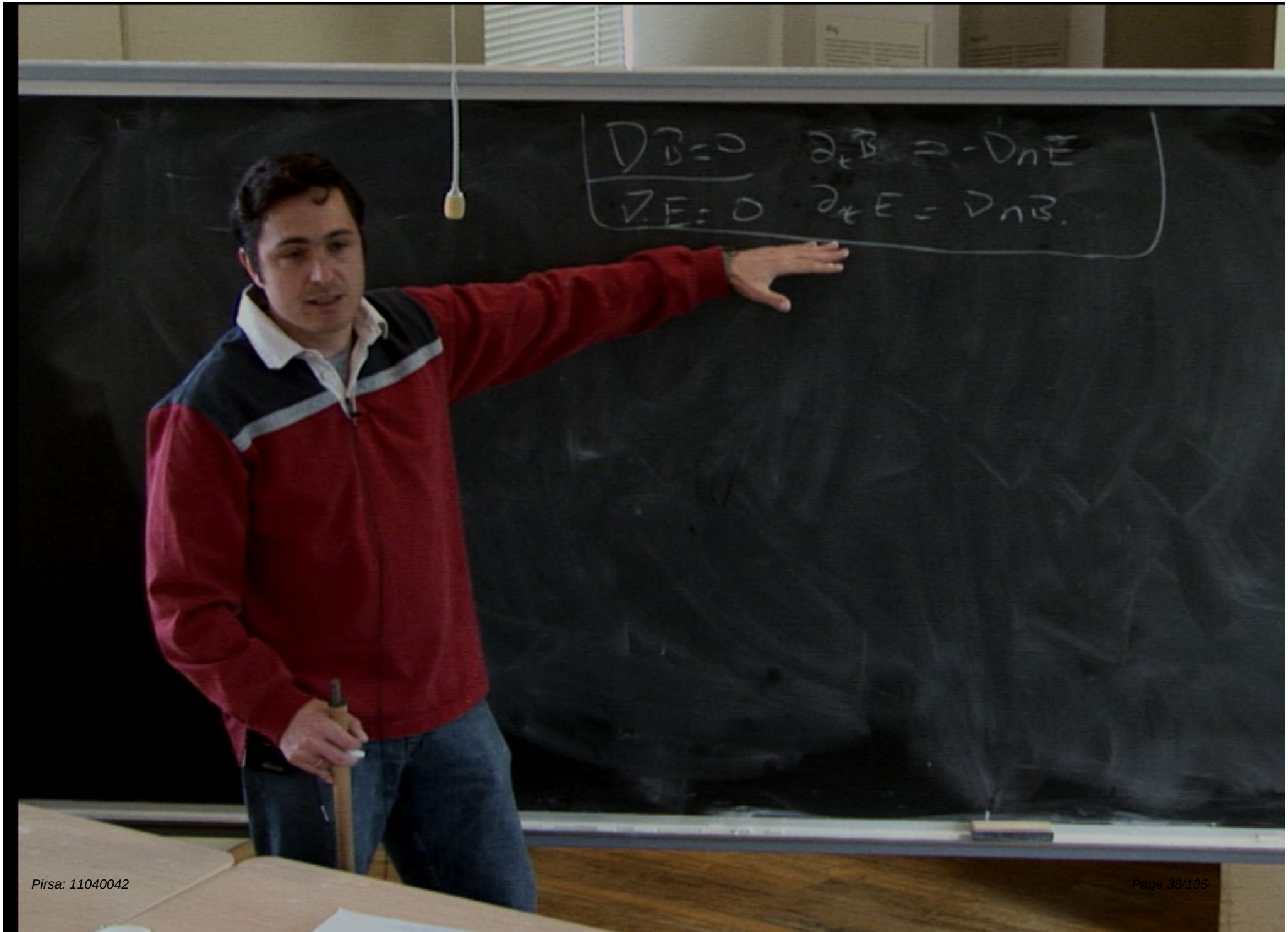

∇B=0 ∂tB = -∇∩E
∇E=0 ∂tE = ∇∩B.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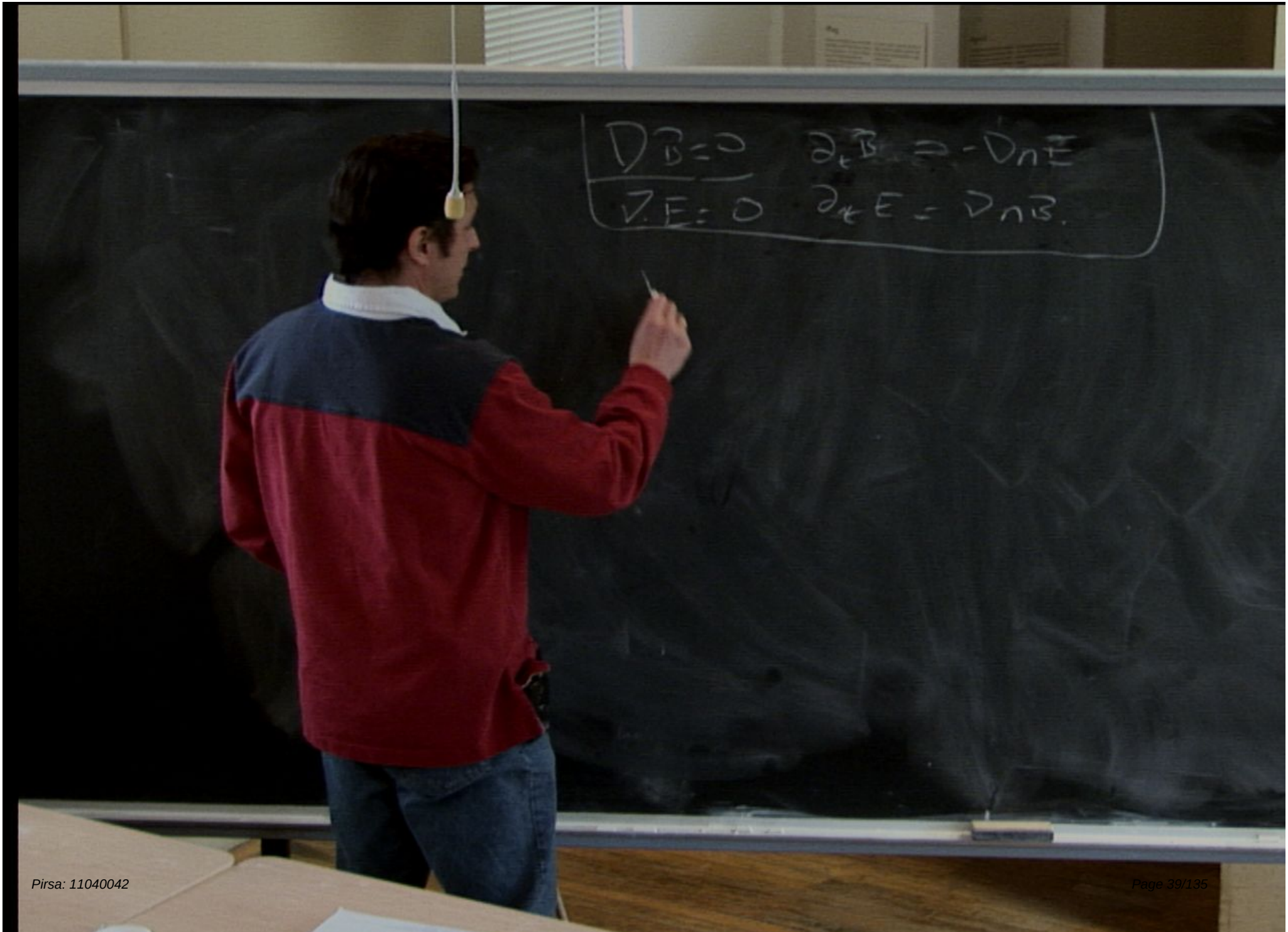
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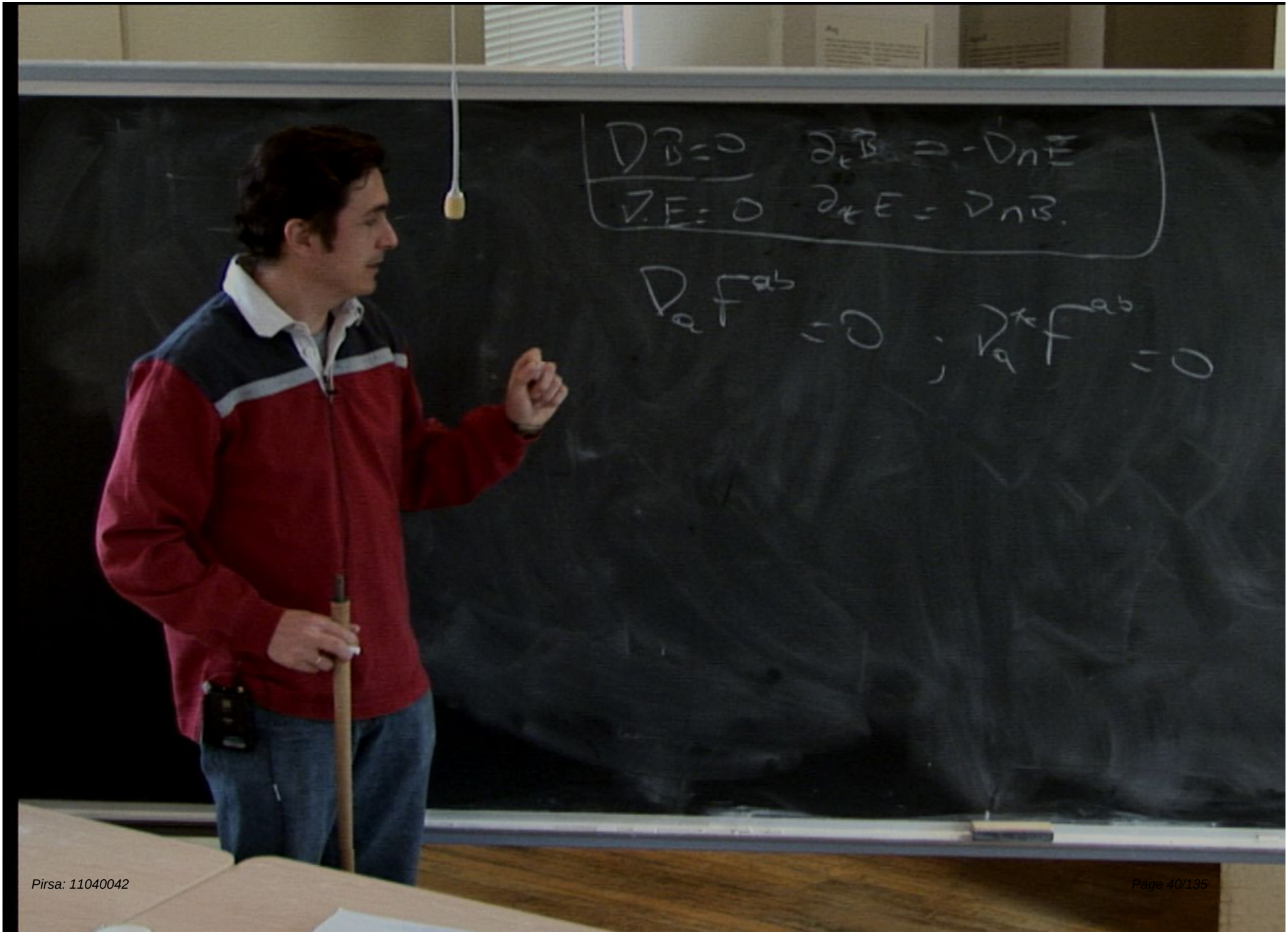
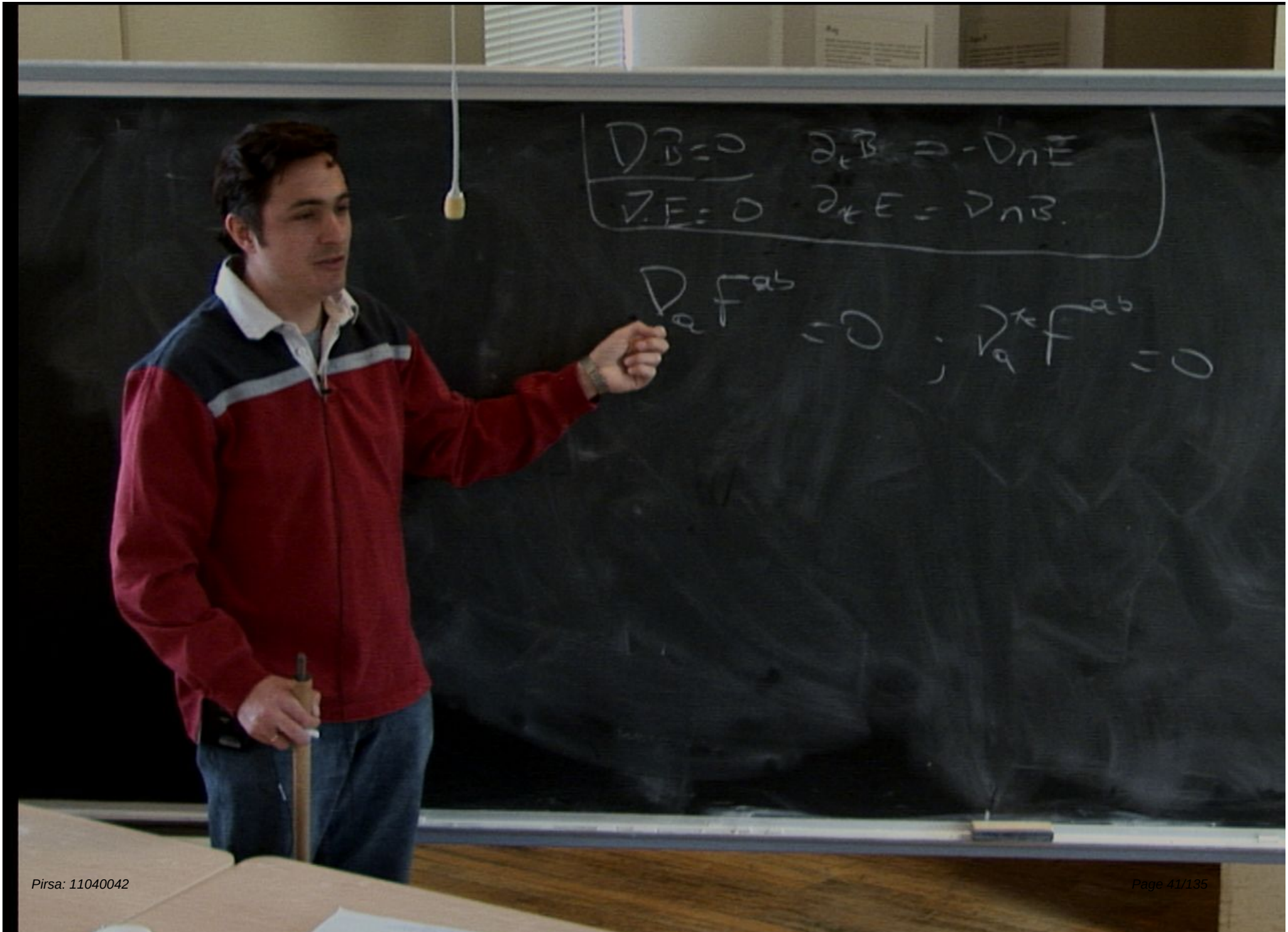

$\nabla \cdot B = 0$
$\partial_t B = -\nabla \times E$
$\nabla \cdot E = 0$
$\partial_t E = \nabla \times B$
$\nabla_a F^{ab} = 0$; $\nabla_a {}^*F^{ab} = 0$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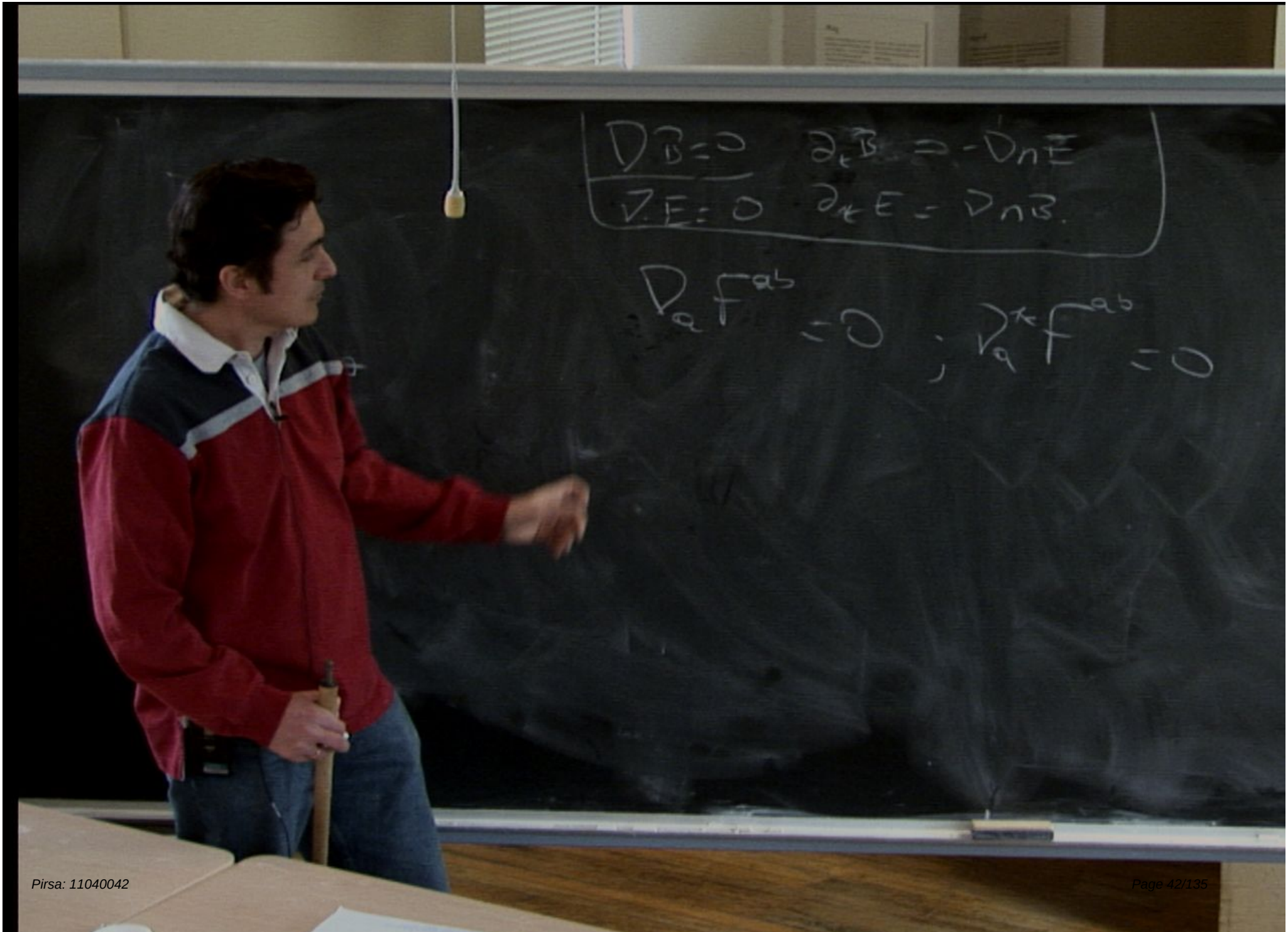
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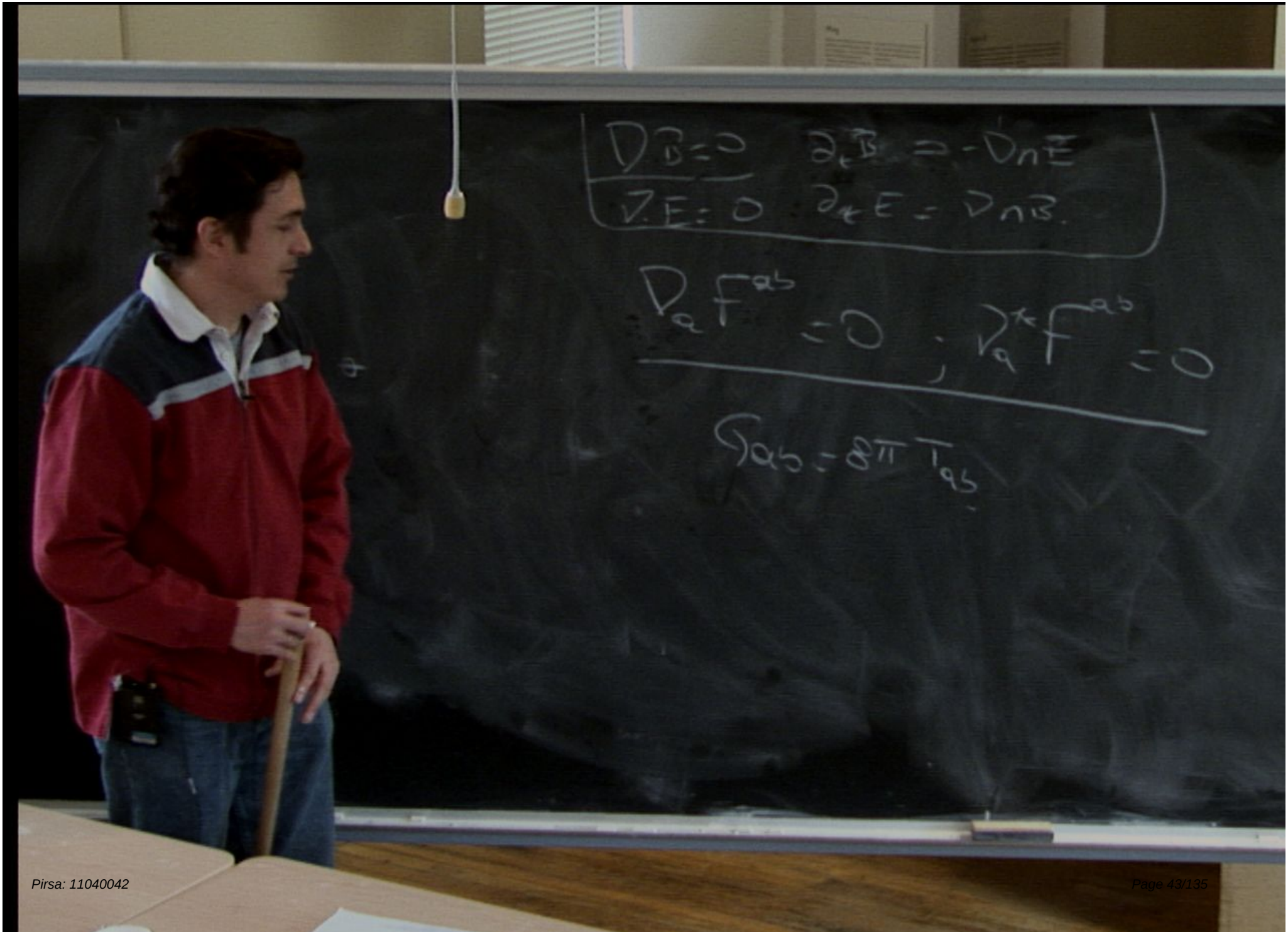
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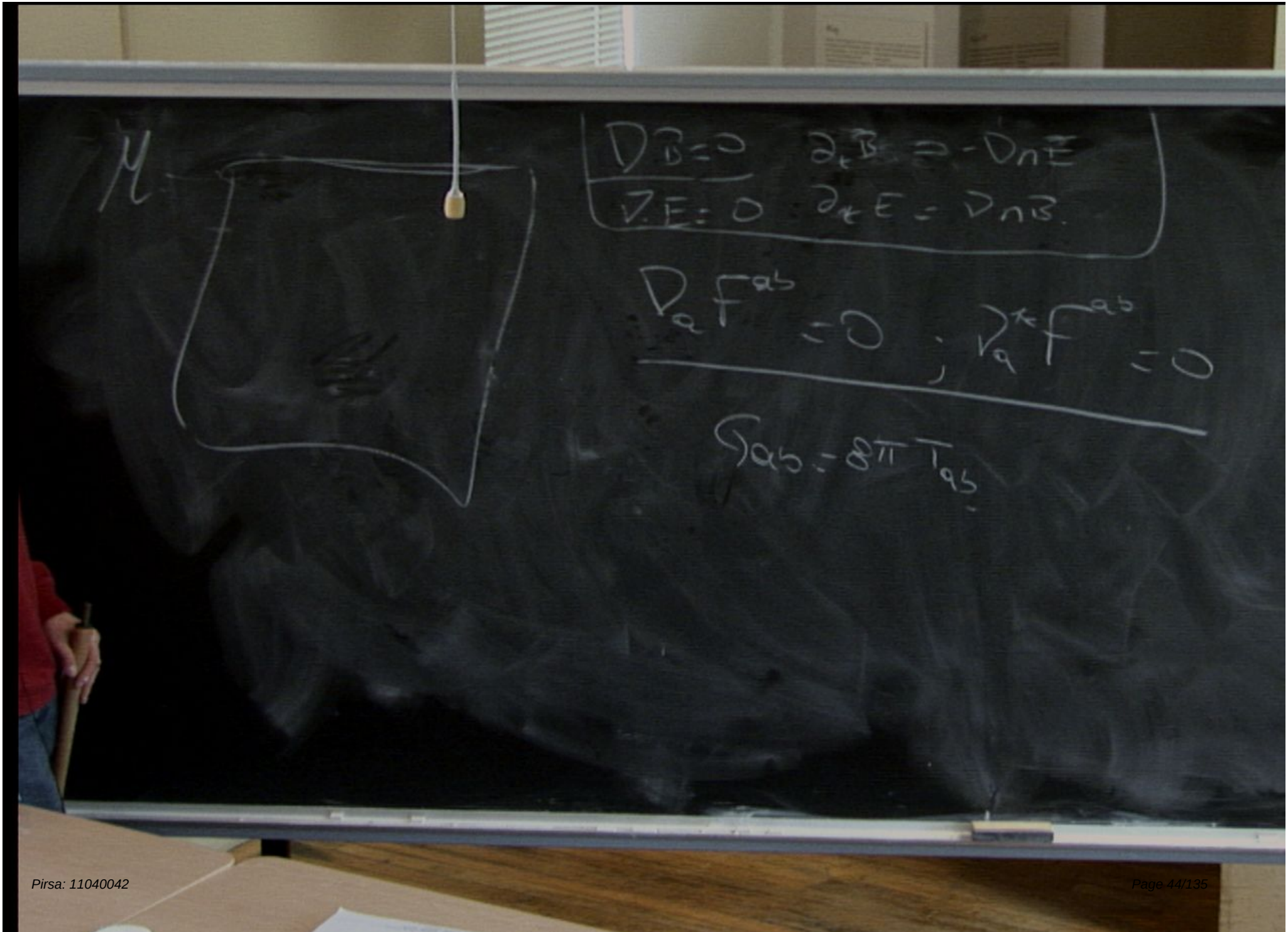
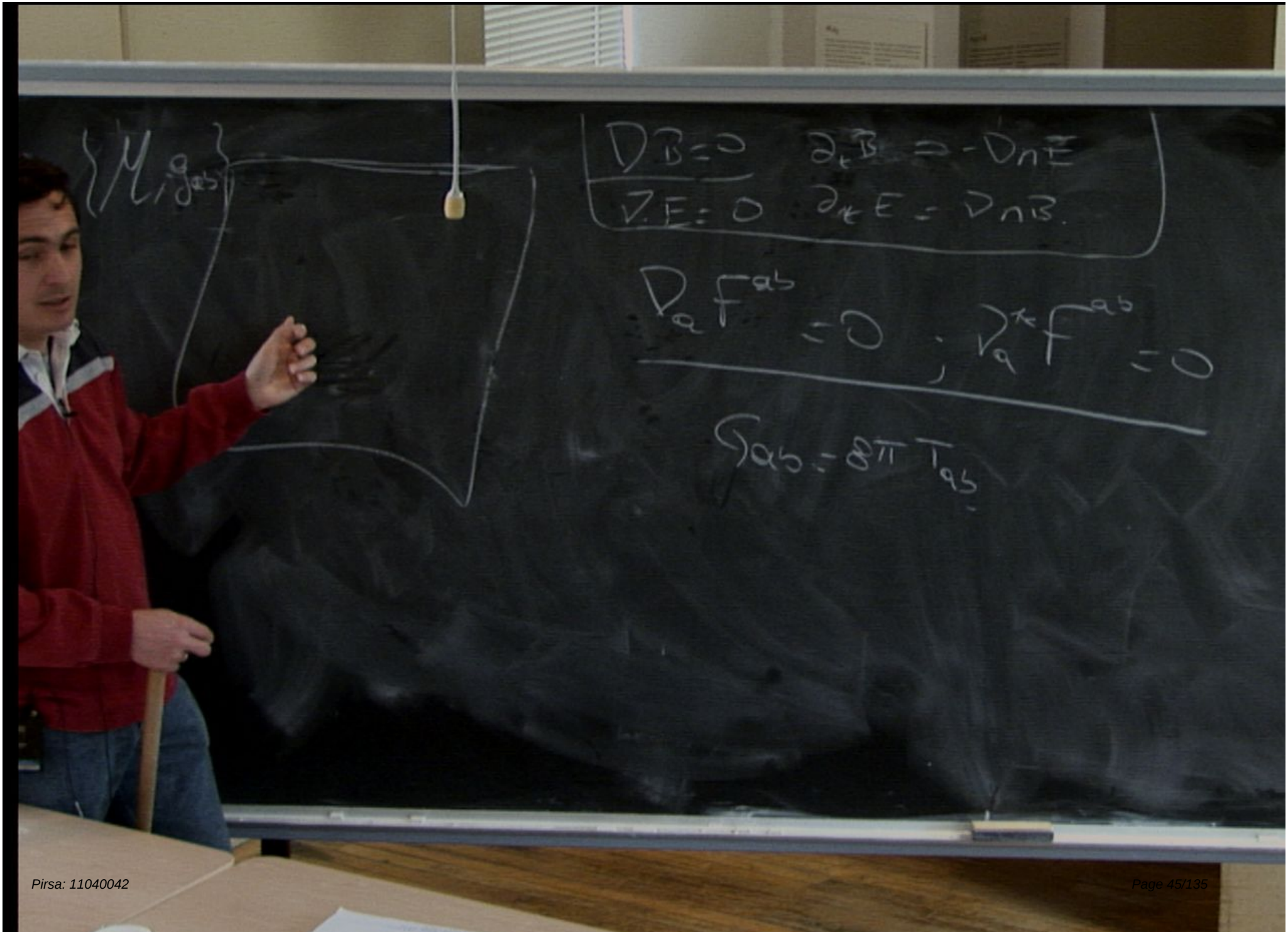
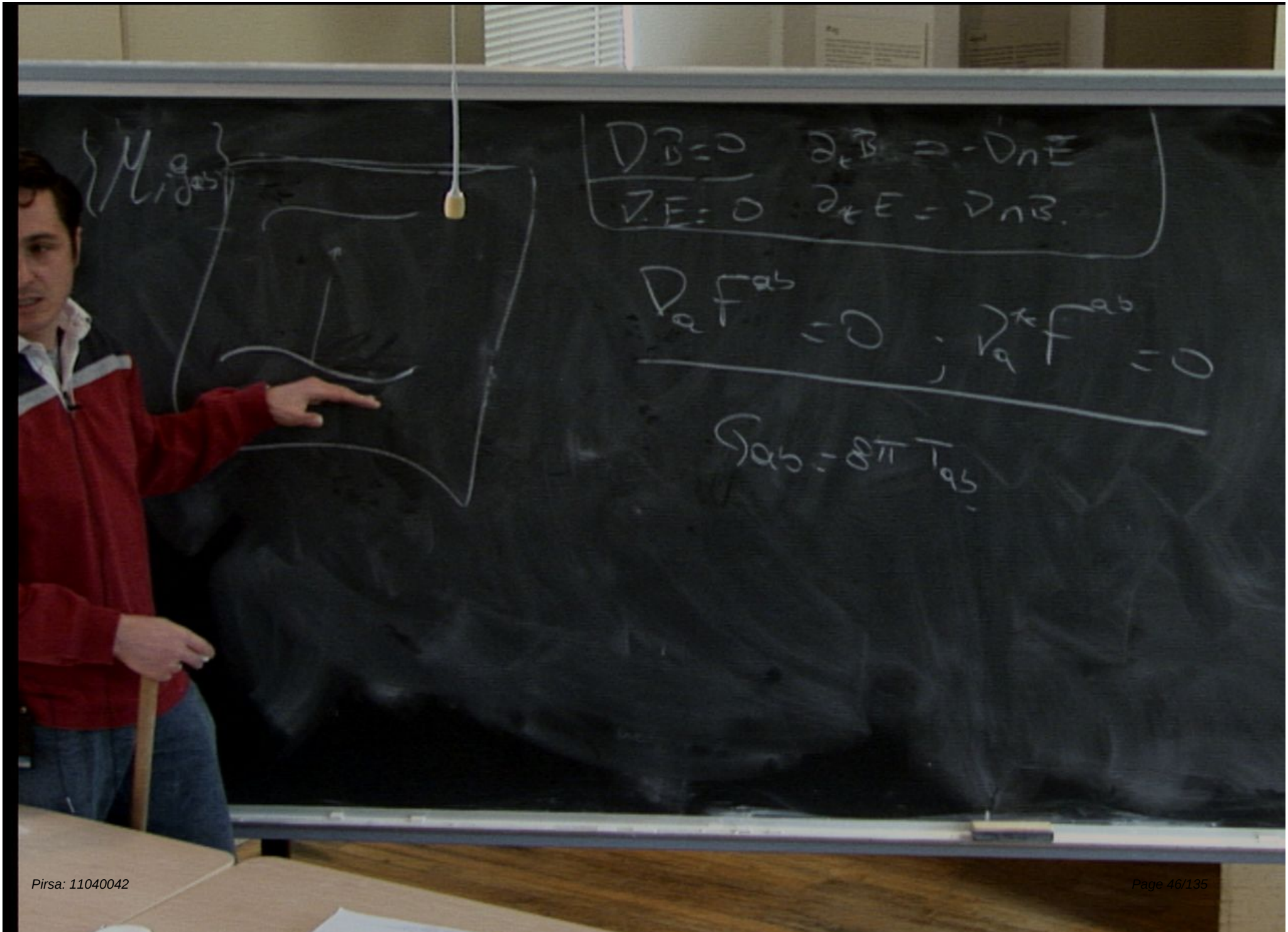
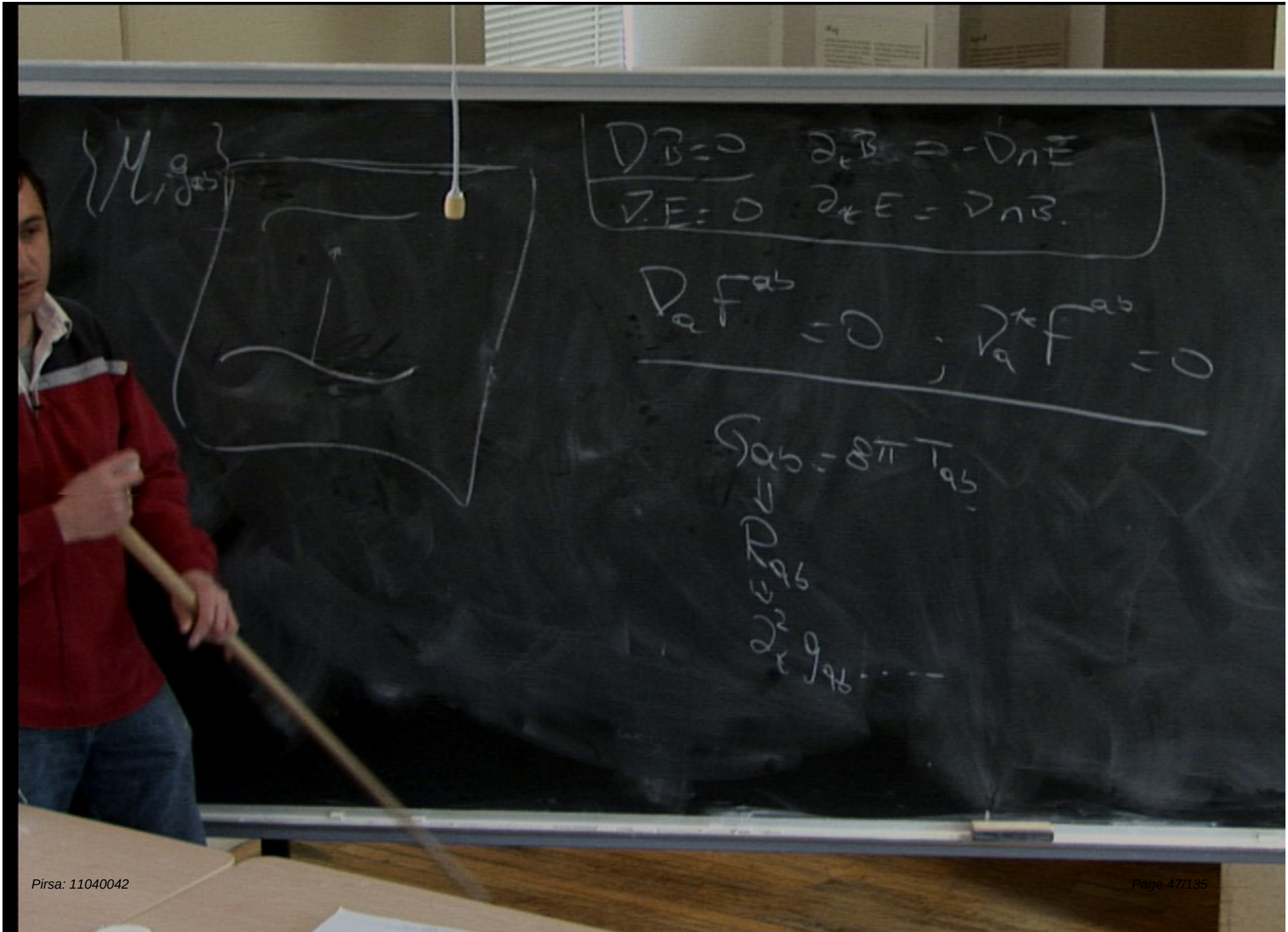
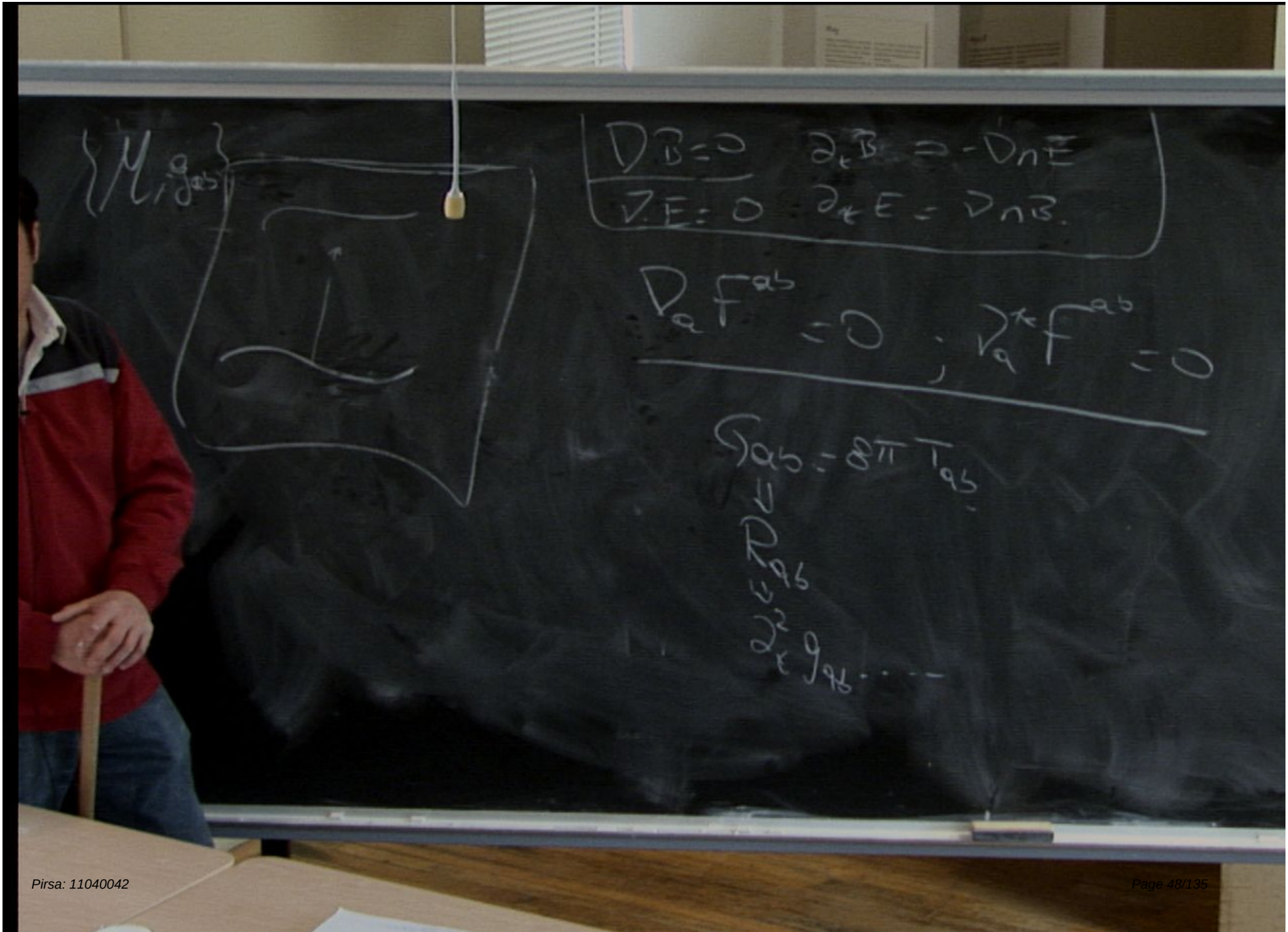

M
∇·B = 0 ∂_t B = −∇∧E
∇·E = 0 ∂_t E = ∇∧B
∇_a F^ab = 0 ; ∇_a *F^ab = 0
G_ab = 8π T_ab


$\{M, g_{ab}\}$
$D \cdot B = 0$
$\partial_t B = -D \wedge E$
$D \cdot E = 0$
$\partial_t E = D \wedge B$
$D_a F^{ab} = 0 \; ; \; D_a {}^*F^{ab} = 0$
$G_{ab} = 8\pi T_{ab}$


$\{M_{ij}^{ab}\}$
$D\vec{B}=0$ $\partial_t \vec{B} = -D\wedge E$
$D\vec{E}=0$ $\partial_t \vec{E} = D\wedge B$
$D_a F^{ab} = 0$; $D_a {}^{*}F^{ab} = 0$
$G_{ab} = 8\pi T_{ab}$


$\{M_{ij}^{ab}\}$
$\nabla \cdot B = 0$
$\partial_t B = -\nabla \times E$
$\nabla \cdot E = 0$
$\partial_t E = \nabla \times B$
$\nabla_a F^{ab} = 0 \; ; \; \nabla_a {}^{*}F^{ab} = 0$
$G_{ab} = 8\pi T_{ab}$
$R = R_{ab} = \sum_i g_{ab} \cdots$


$\{M, g_{ab}\}$
$\nabla \cdot B = 0$
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$\nabla \cdot E = 0$
$\partial_t E = \nabla \times B$
$\nabla_a F^{ab} = 0 \; ; \; \nabla_a {}^{*}F^{ab} = 0$
$G_{ab} = 8\pi T_{ab}$
R_{ab}
$\partial_t^2 g_{ab} \cdots$

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PHY 3E7M THE 3+1 (ADM / HAMILTONIAN) FORM of GR

REFERENCES

(1) MTW CHAPTER 21

(2) YORK, "KINEMATICS & DYNAMICS of GR" IN "SOURCES of GRAV. RADIATION" (L. SMARR ed)

(3) WALD, APP. E.2 ; CHAPTER 10

(4) ARNOWITT, DESER & MISNER (1962) "THE DYNAMICS of GR" IN "GRAVITATION: AN INTRODUCTION TO CURRENT RESEARCH" (L. WITTEN ed)

• ADM MOTIVATION WAS, AGAIN, PREPARATION FOR QUANTIZATION of GR; FORMALISM TURNED OUT TO BE EXCELLENT BASIS FOR COMPUTATIONAL (NUMERICAL)

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(4) Ap... of GR" IN "GRAVITATION: AN INTRODUCTION TO CURRENT RESEARCH" (L. WITTEN ed)

- ADM MOTIVATION WAS, AGAIN, PREPARATION FOR QUANTIZATION of GR; FORMALISM TURNED OUT TO BE EXCELLENT BASIS FOR COMPUTATIONAL (NUMERICAL) ASSAULT ON EINSTEIN EQUATIONS

APPROACHES

(1) "COORDINATE-FUL" (MTW): INTUITIVE, CONNECTS MORE DIRECTLY TO FORM of E.O.M USED IN PRACTICE

(2) "COORDINATE FREE" (YORK/WALD): PREFERABLE FOR DERIVATION of E.O.M.

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STAY COORDINATE FREE FOREVER

- WILL ~~START~~ WITH COORDINATE BASED APPROACH TO INTRODUCE CONCEPTS, THEN WILL GO OVER TO COORDINATE-FREE APPROACH TO DERIVE 3+1 EQNS

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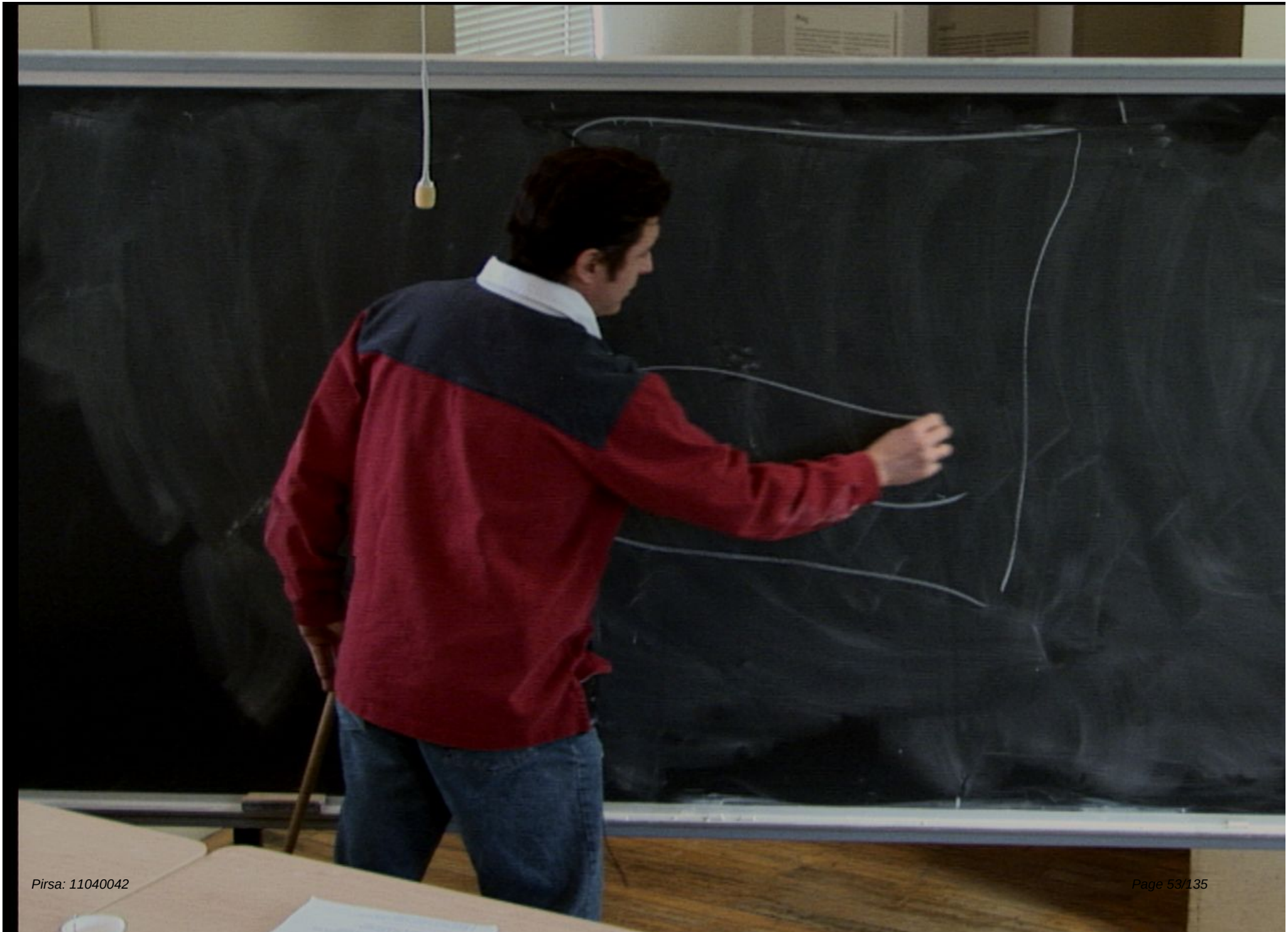
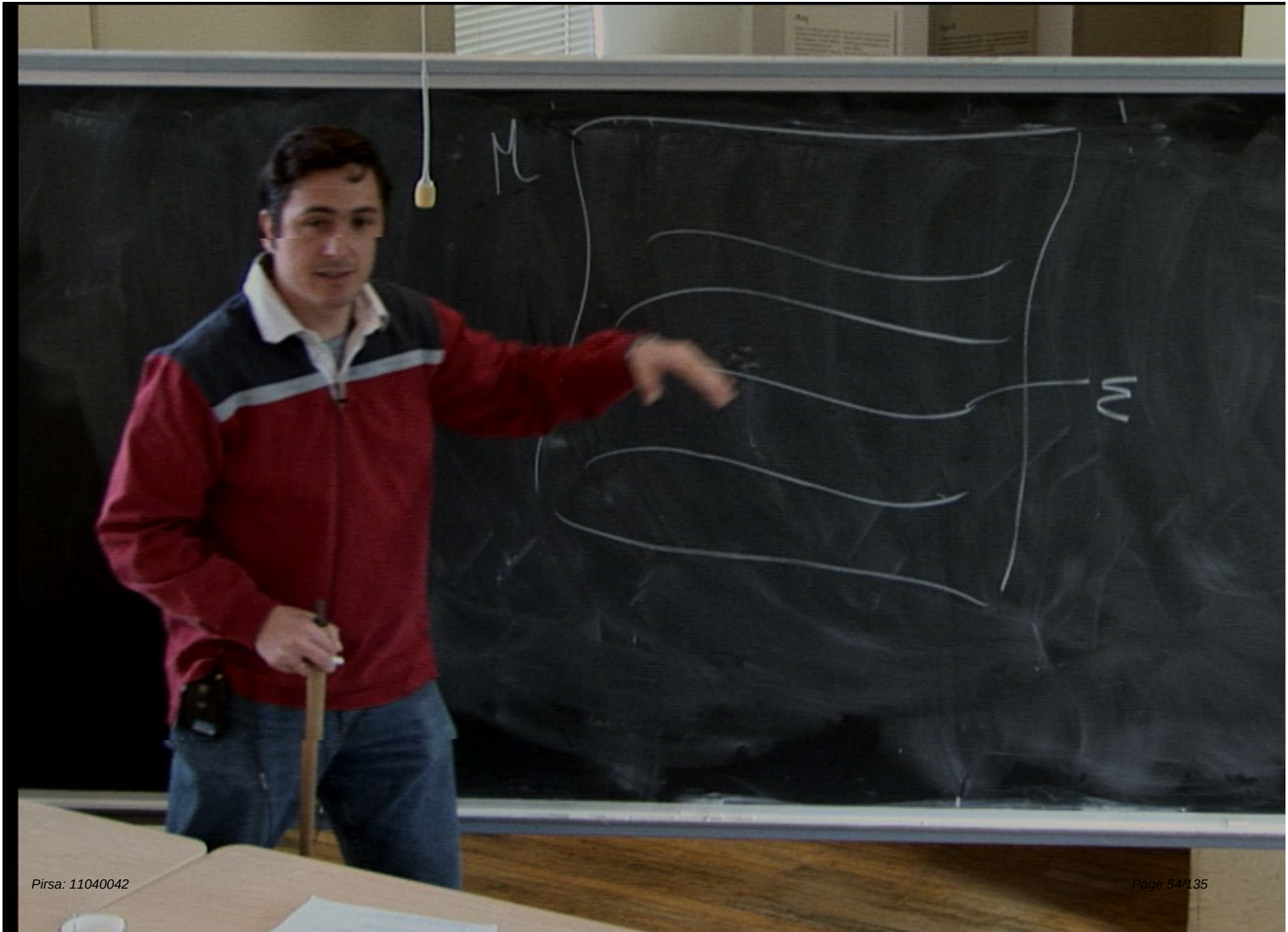
$$G_{ab} = 8\pi T_{ab}$$

AS SYSTEM of FIRST-ORDER (IN TIME) PDE'S FOR THE GRAV. FIELD VBL'S, WHICH CAN THEN BE SOLVED AS AN "INITIAL-VALUE" or "CAUCHY" PROBLEM

SHIFT IN PERSPECTIVE: UP TO NOW $G_{ab} = 8\pi T_{ab}$ DESCRIBED THE LINKAGE OF THE GEOMETRY of SPACETIME (4-D) TO THE DISTRIBUTION of MATTER-STRESS-ENERGY IN S.T.

NEW VIEW: GEOMETRY of S.T IS "TIME-HISTORY" (EVOLUTION) of GEOMETRY of A SPACELIKE HYPERSURFACE (3-D) =) GEOMETRODYNAMICS; "VIEW" NOT UNIQUE SINCE THERE ARE MANY MANY WAYS of "SLICING UP" GIVEN S.T. INTO A

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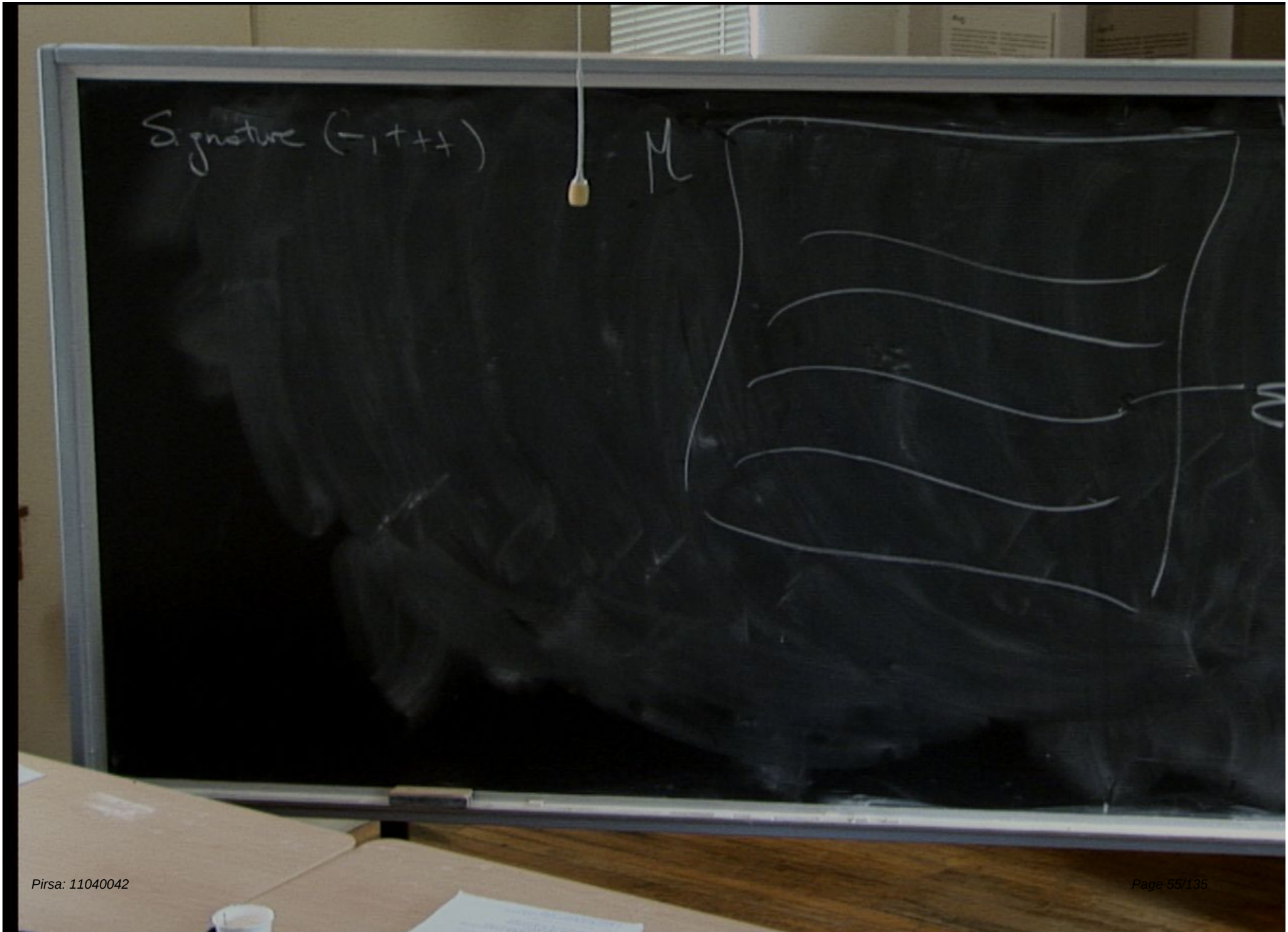
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$$G_{ab} = \varepsilon\pi T_{ab}$$

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THE LINKAGE OF THE GEOMETRY of SPACETIME (4-D)
TO THE DISTRIBUTION of MATTER-STRESS-ENERGY IN S.T

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of GEOMETRY of A SPACELIKE HYPERSURFACE (3-D) =)
GEOMETRODYNAMICS; "VIEW" NOT UNIQUE SINCE THERE ARE
CO'NLY MANY WAYS of "SLICING UP" GIVEN S.T. INTO A
FAMILY OF S.L. HYPERSURFACES; SUCH PREFERRED (i.e.
DYNAMICALLY MORE RELEVANT) IN GENERAL

SPLITTING SPACETIME INTO SPACE-PLUS-TIME (THE 3+1 SPLIT)

- SPACETIME IS 4-DIMENSIONAL MANIFOLD M, WITH LORENTZIAN SIGNATURE METRIC g_{ab} (−+++)
- INTRODUCE COORDINATES $\{x^\mu\} = \{t, x^i\}$ (MAY NOT COVER ENTIRE S.T., BUT WILL GENERALLY PRETEND THEY DO)

NOTATION/CONVENTIONS

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SPLITTING SPACETIME INTO SPACE-PLUS-TIME (THE 3+1 SPLIT)

- SPACETIME IS 4-DIMENSIONAL MANIFOLD M, WITH LORENTZIAN SIGNATURE METRIC g_{ab} $(-+++)$
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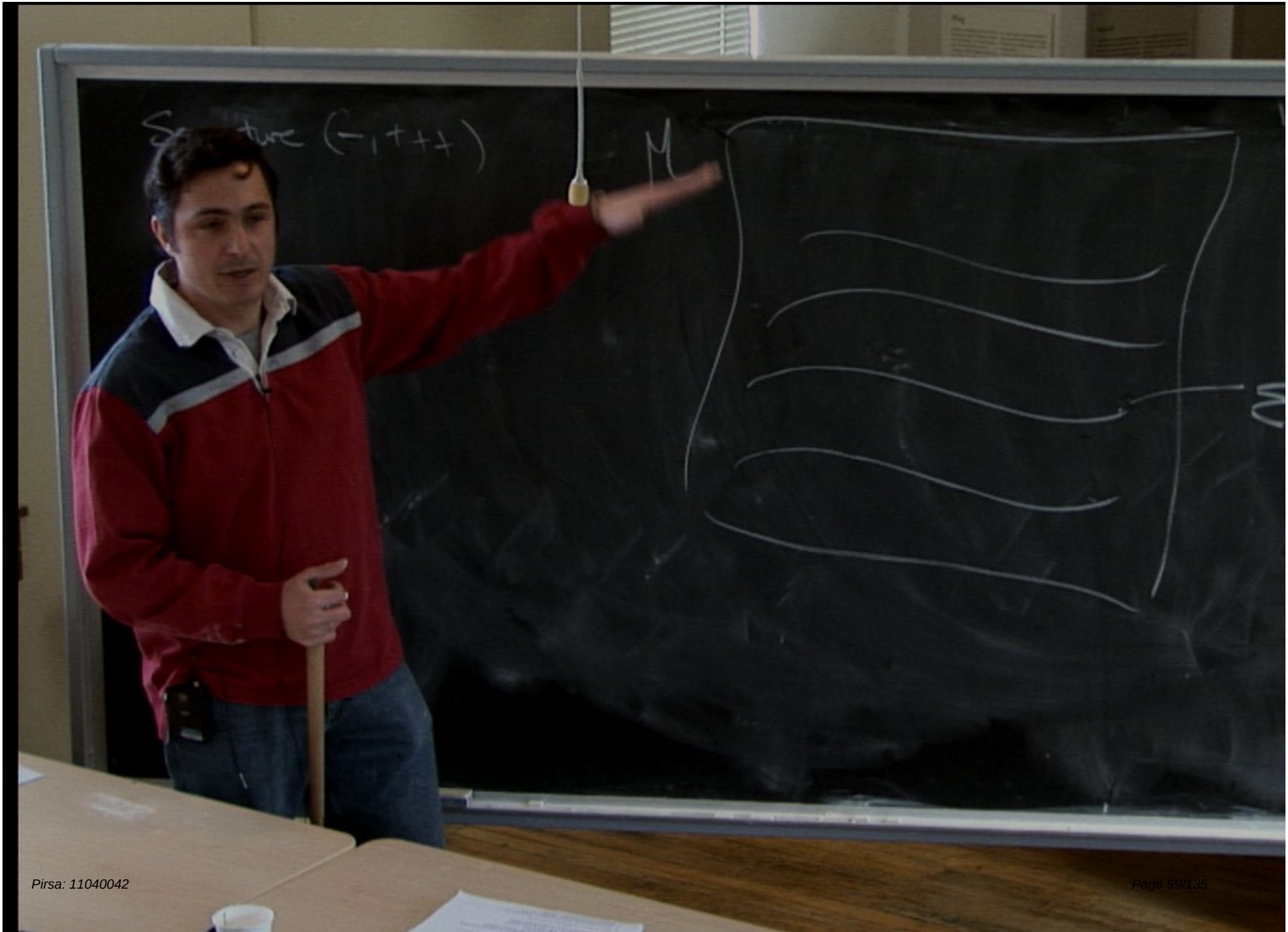
NOTATION / CONVENTIONS:

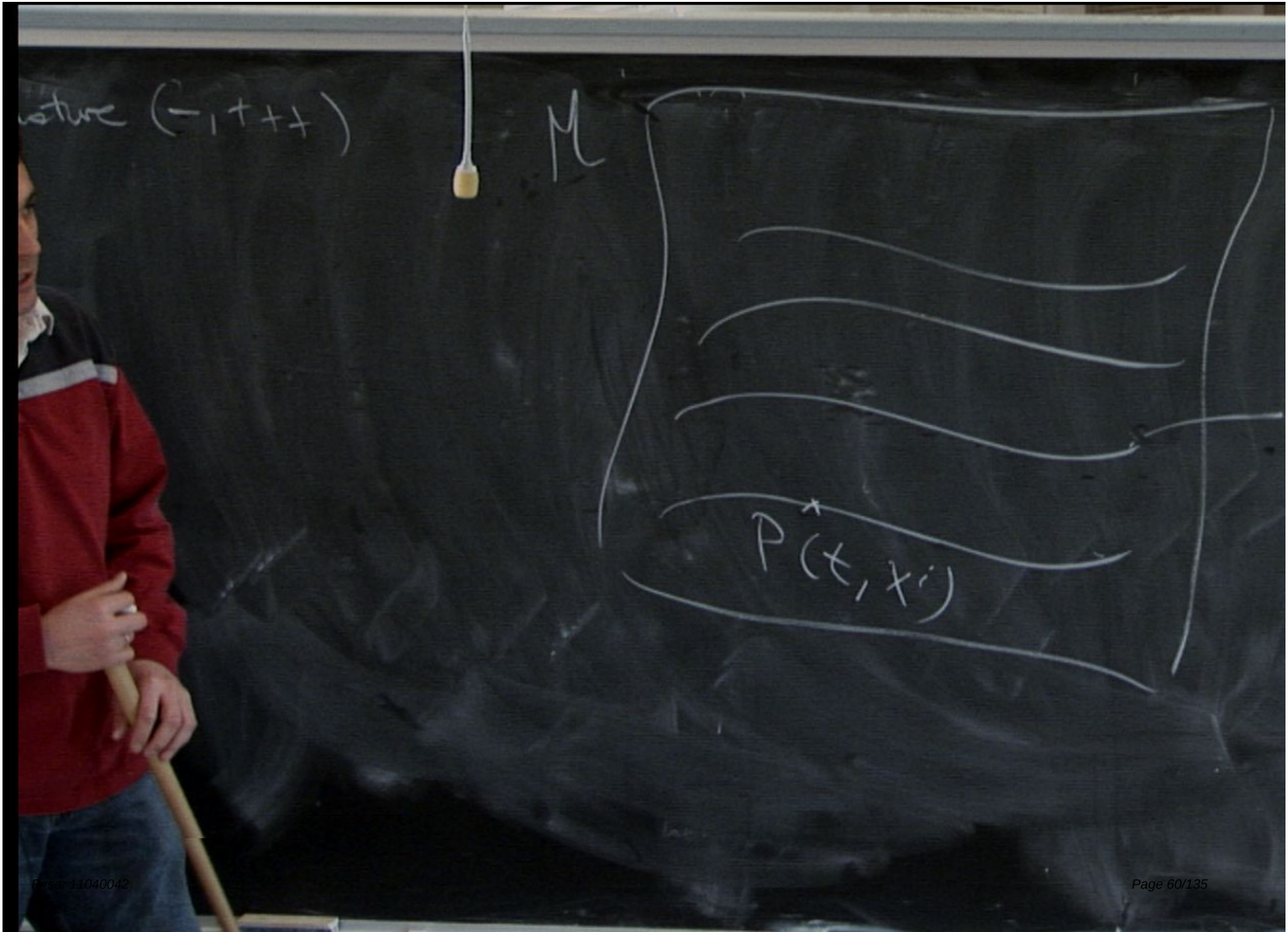
GREEK INDICES: $(\mu, \nu$ etc.) $0, 1, 2, 3$

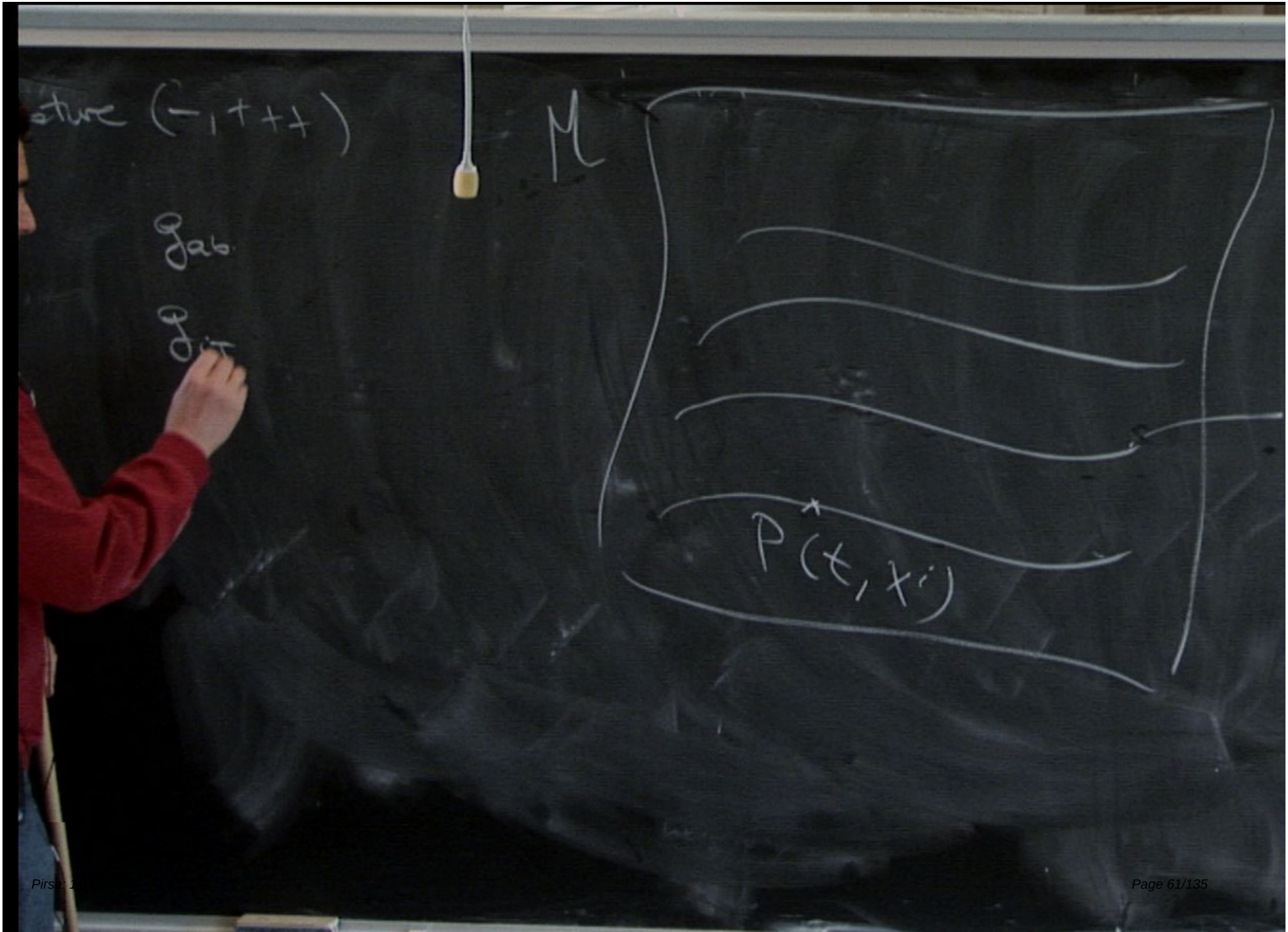
"INTEGER" LATIN " (i, j, k, l, m, n) $1, 2, 3$ (SPATIAL)

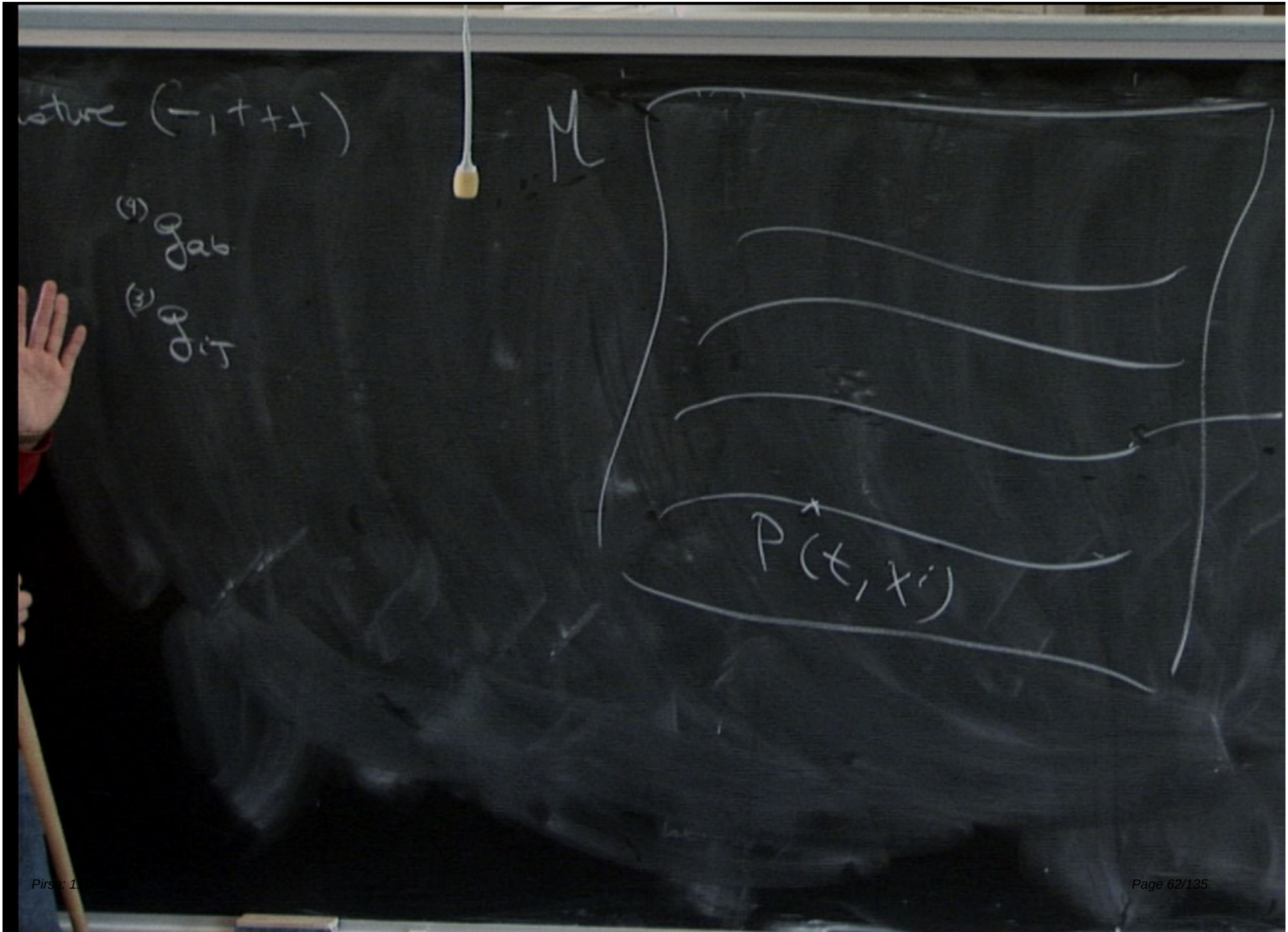
WILL ADOPT USUAL EINSTEIN SUMMATION CONVENTION FOR BOTH TYPES!

- DEMAND THAT $t =$ CONST SURFACES (HYPERSURFACES), $\Sigma(t)$, ARE SPACELIKE; I.E. IF DISTINCT EVENTS P, P'


M


M
P(t, x)


ature (−, +++)
M
g_{ab}
g_{ij}
$P(t, x^i)$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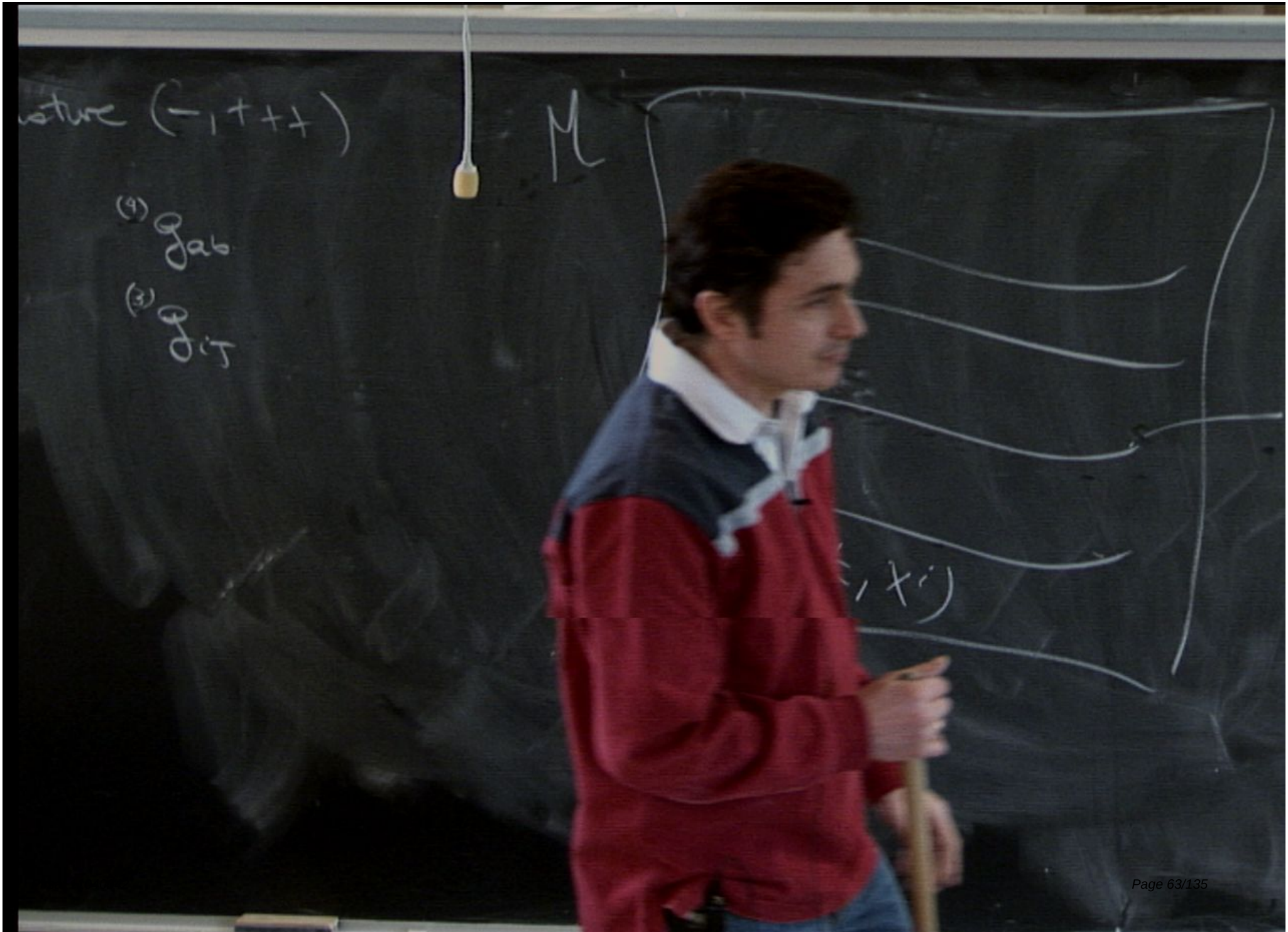
(3) g_ij
P(t, x^i)


(9) g_{ab}
(3) g_{ij}
M

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- EACH $\Sigma(t)$ IS A DIFFERENTIABLE MANIFOLD IN ITS OWN RIGHT, WITH A 3-METRIC ${}^{(3)}g_{ij} = {}^{(3)}g_{(ij)}$ WHICH IS INDUCED ON $\Sigma(t)$ BY THE 4-METRIC ${}^{(4)}g_{\mu\nu}$ OF THE ENVELOPING SPACETIME

- ANY GIVEN FOLIATION (= CHOICE OF TIME COORD = CHOICE OF SLICING) ALSO DEFINES A NATURAL, UNIT-NORM VECTOR FIELD, n^μ, WHICH IS NORMAL (ORTHOGONAL) TO THE SLICES, AND POINTS "TO THE FUTURE"

$$n^\mu n_\mu = {}^{(4)}g_{\mu\nu} n^\mu n^\nu = -1$$

SPACETIME DISPLACEMENT IN THE 3+1 SPLIT

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WHICH IS INDUCED ON $\Sigma(t)$ BY THE 4-METRIC
(4) $g_{\mu\nu}$ OF THE ENVELOPING SPACETIME

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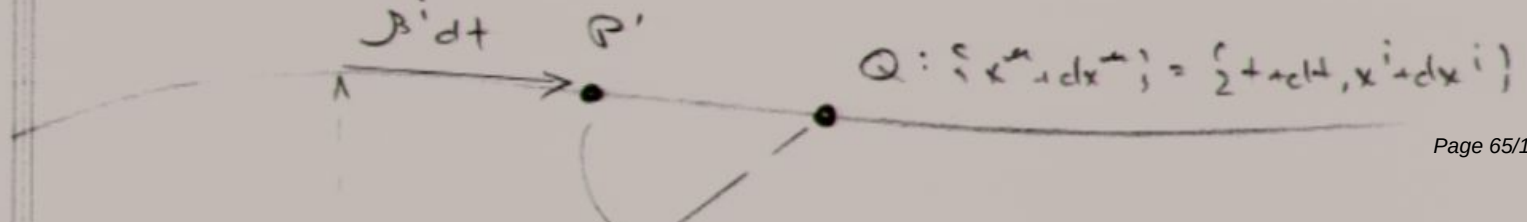
$$n^\mu n_\mu \overset{(4)}{=} g_{\mu\nu} n^\mu n^\nu = -1$$

SPACETIME DISPLACEMENT IN THE 3+1 SPLIT



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$^{(4)}g_{\mu\nu}$ OF THE ENVELOPING SPACETIME

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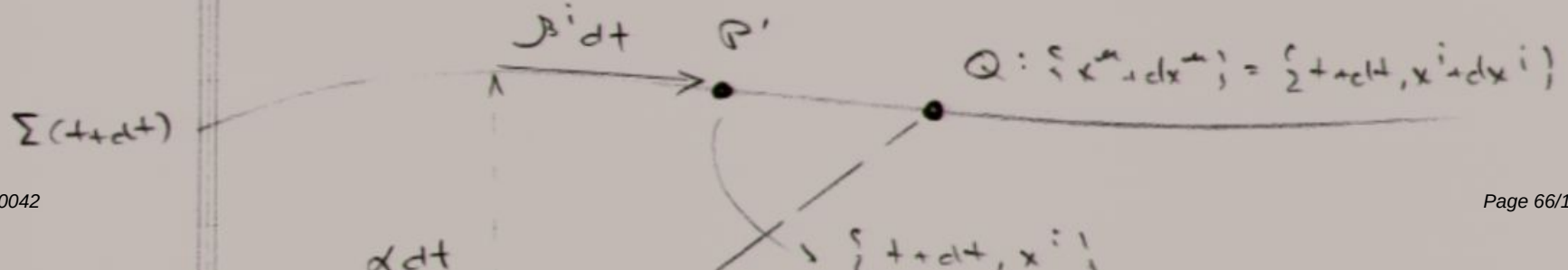

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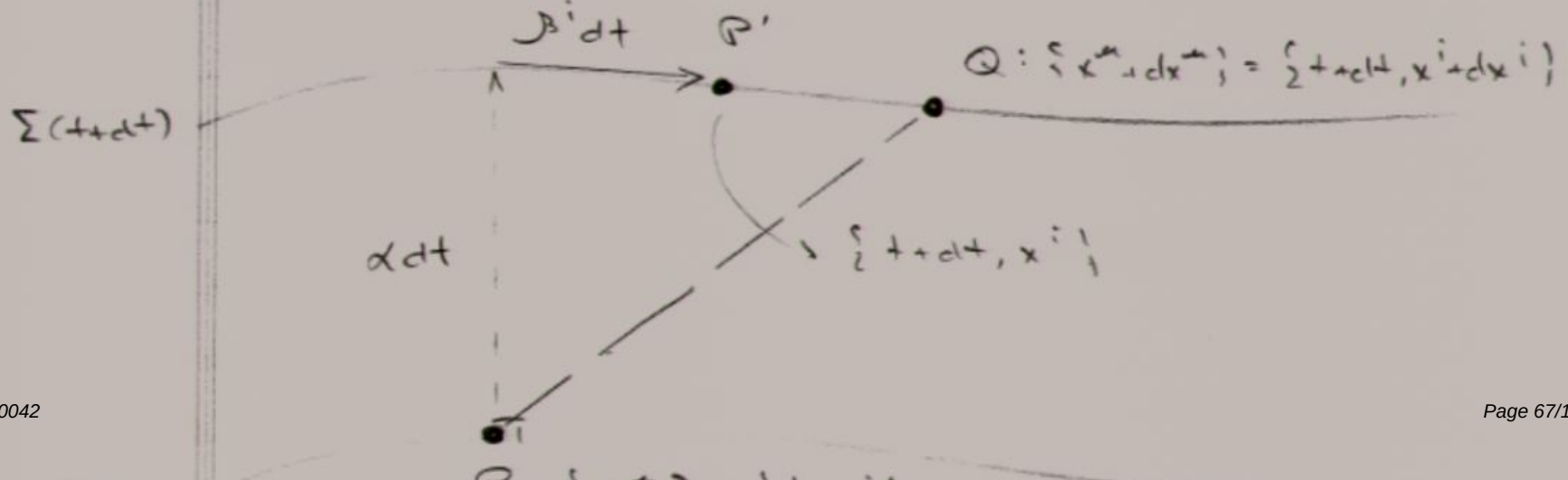
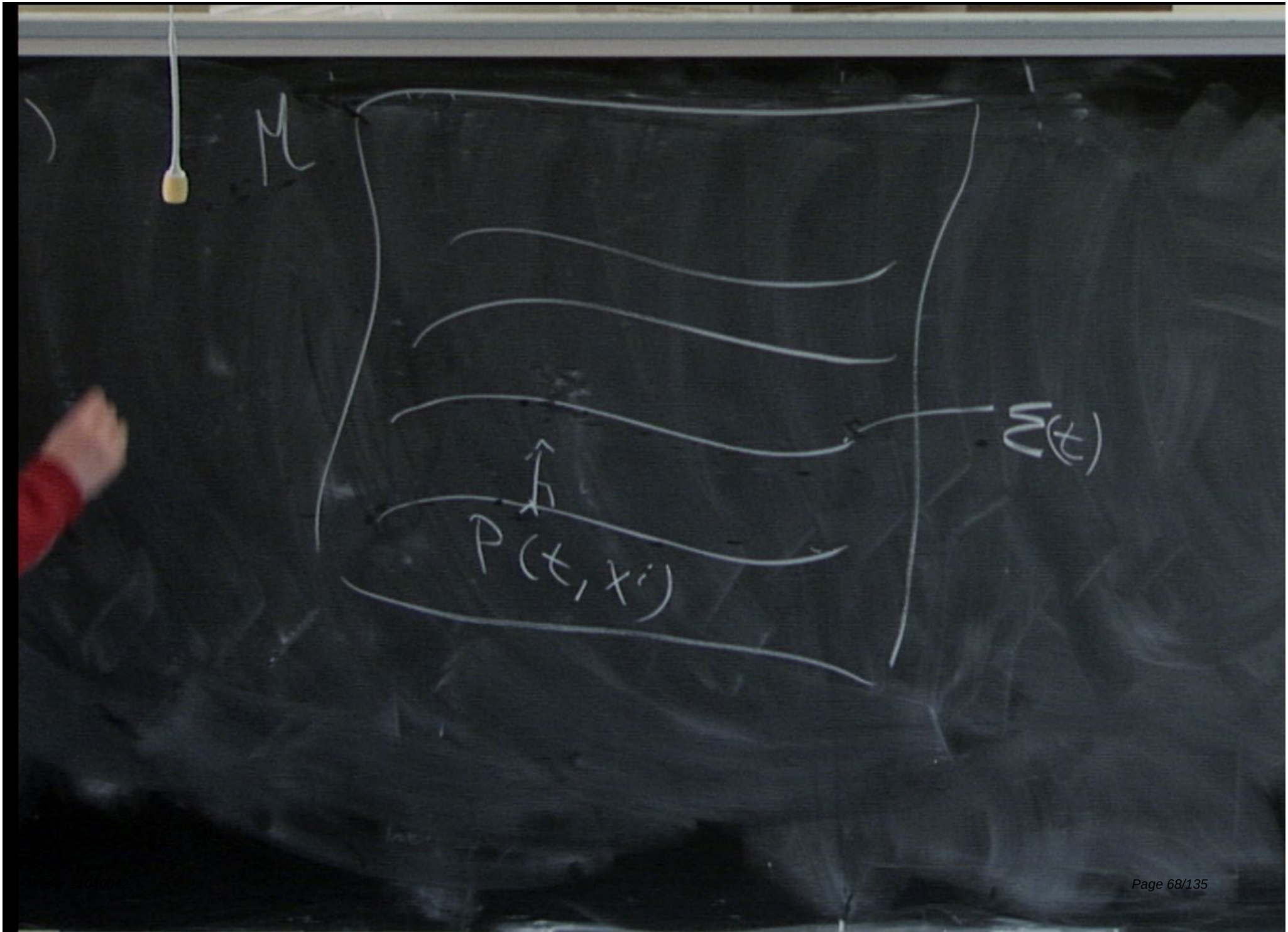
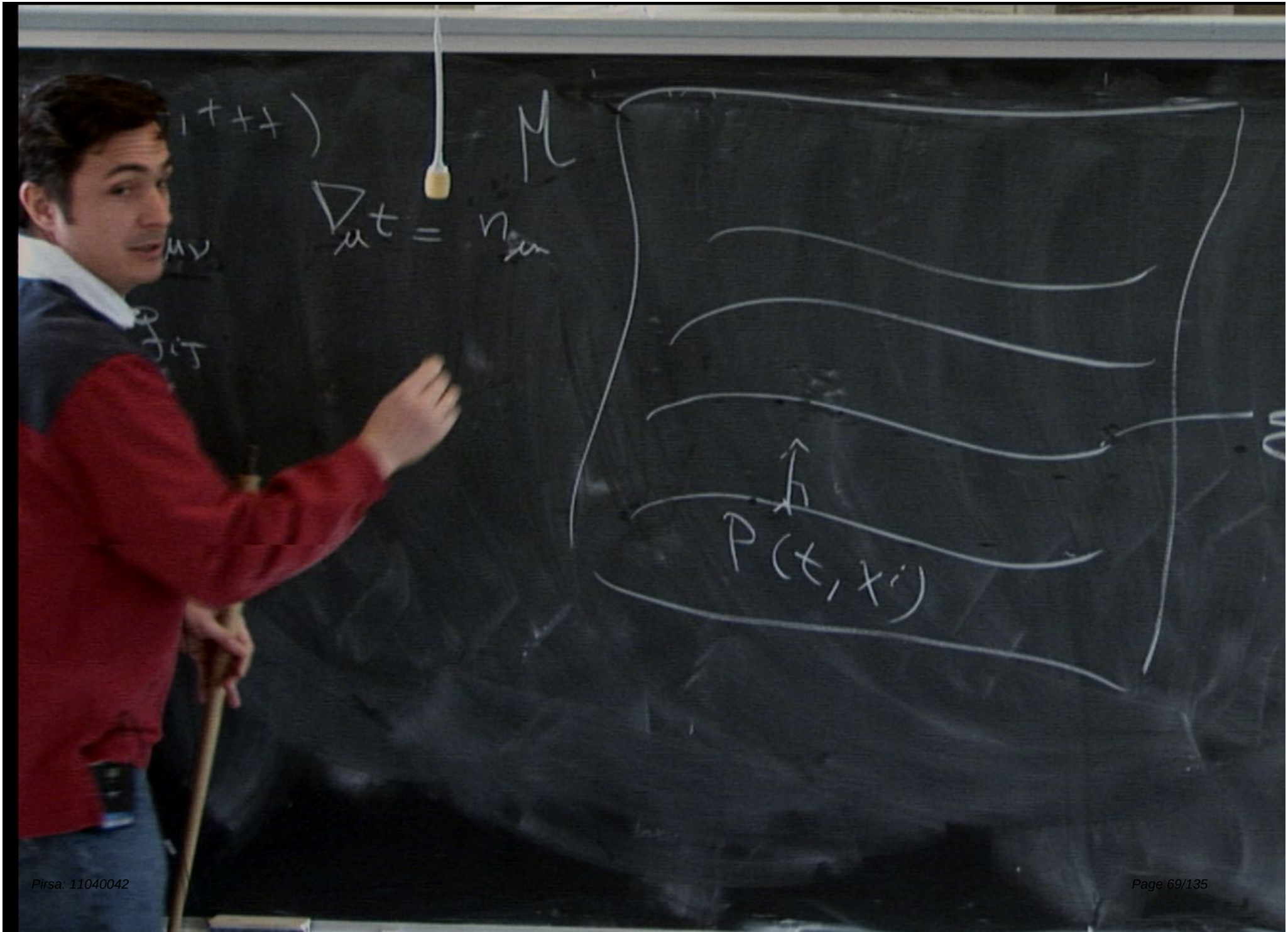
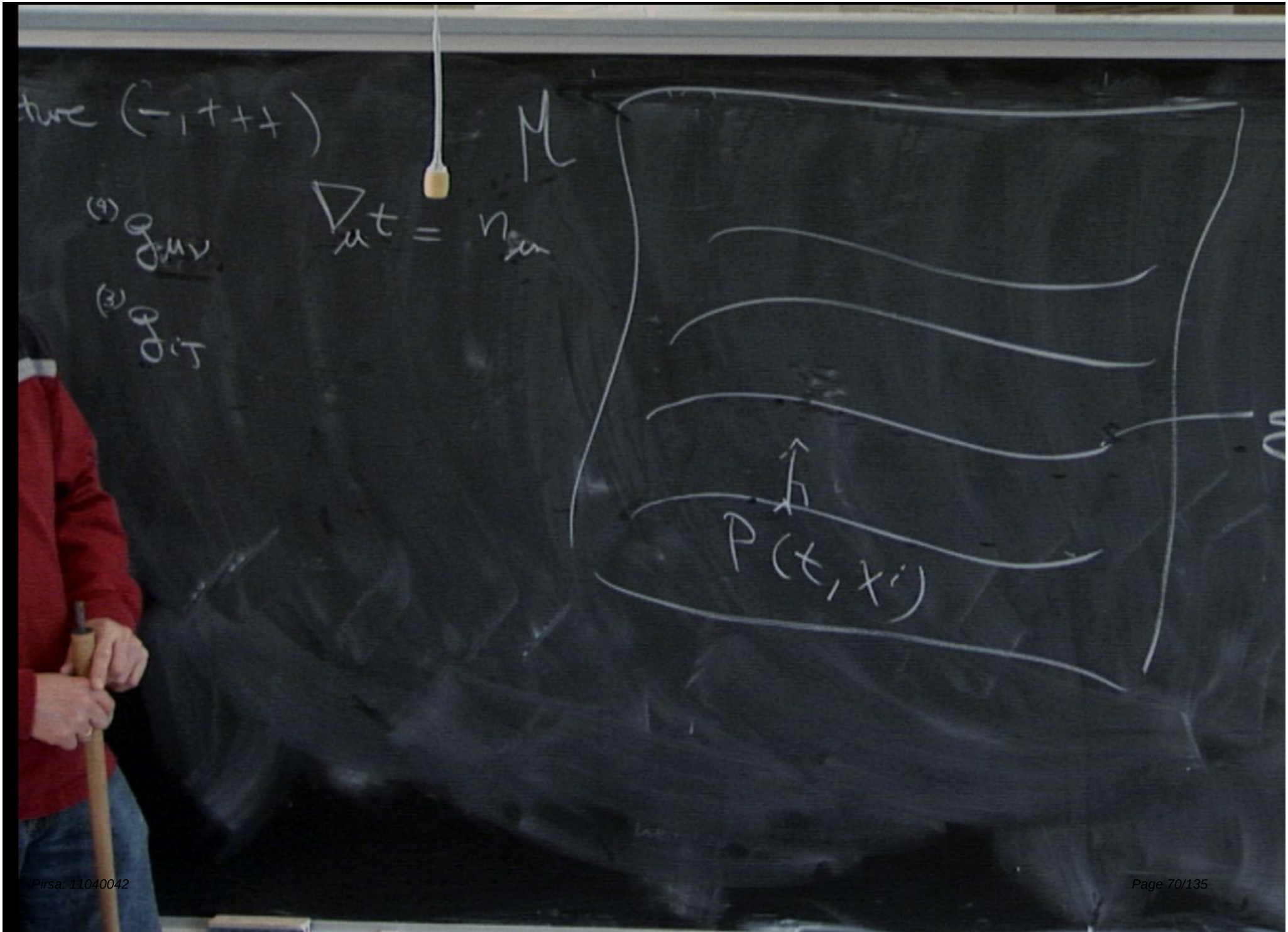
SPACETIME DISPLACEMENT IN THE 3+1 SPLIT




M
Σ(t)
P(t, x)


M
P(t, x)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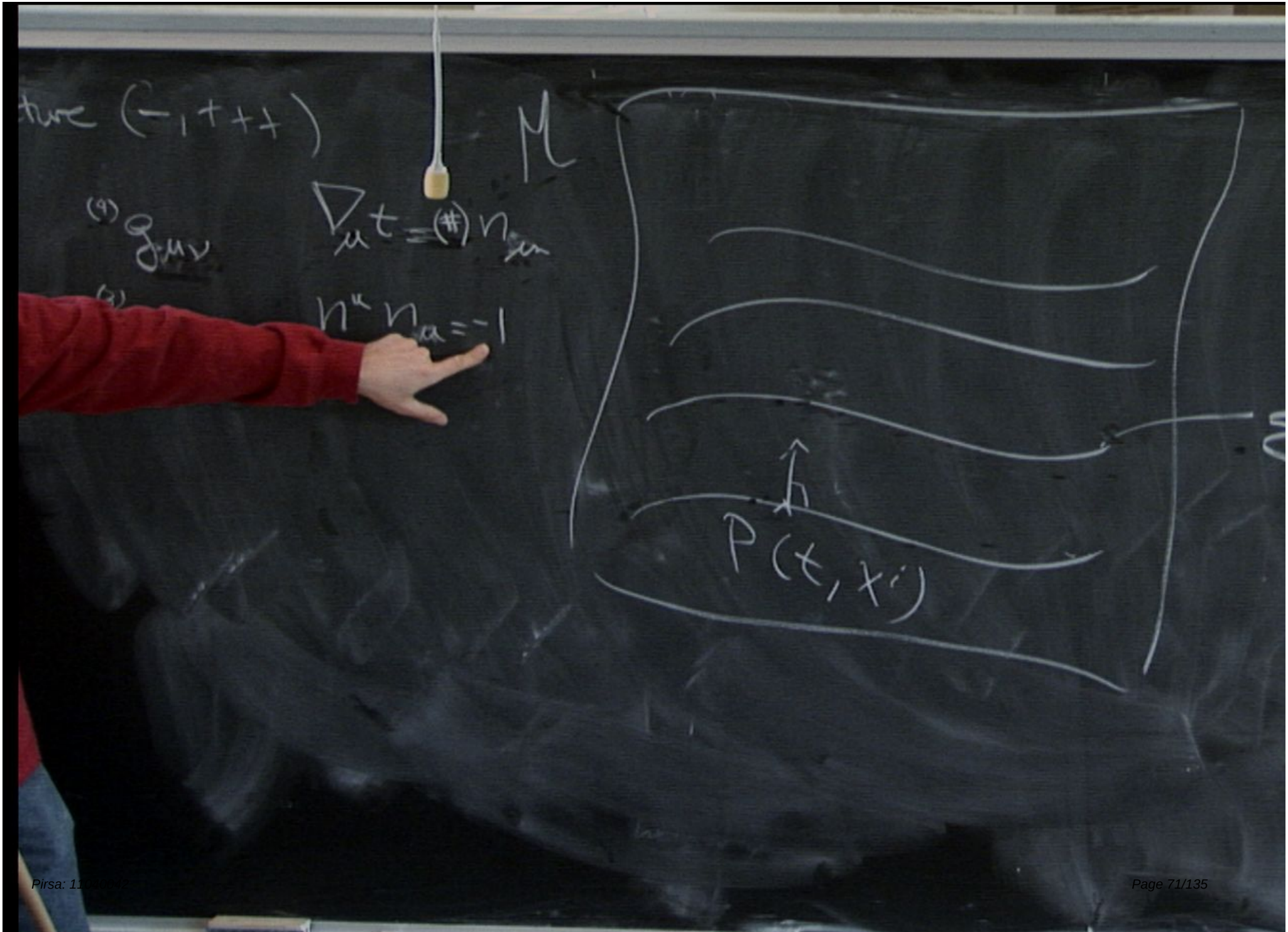
M
$\nabla_\mu t = n_\mu$
(1) $g_{\mu\nu}$
(3) g_{iT}
P(t, x^i)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$\nabla_\mu t = (\#) n_\mu$
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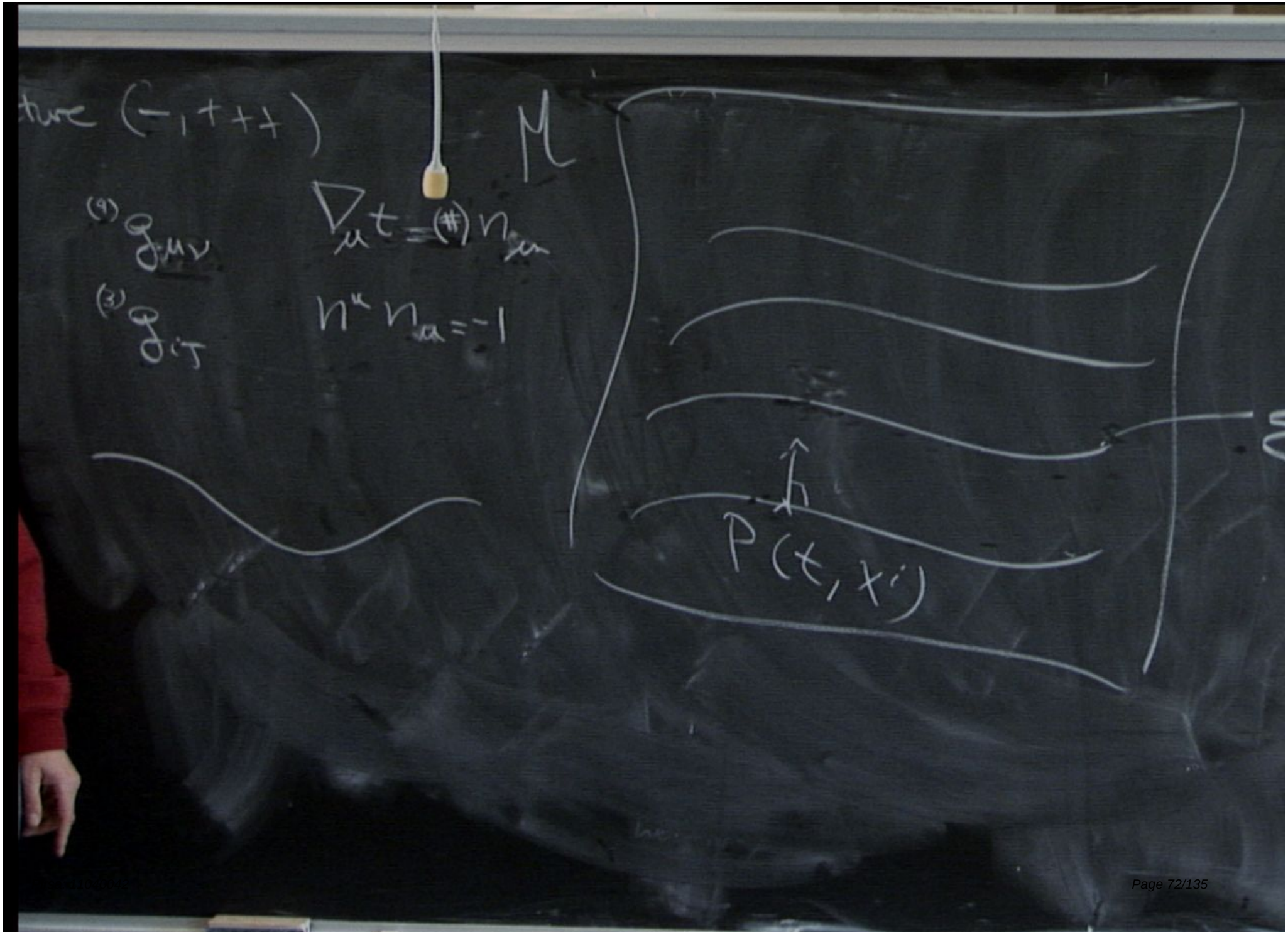
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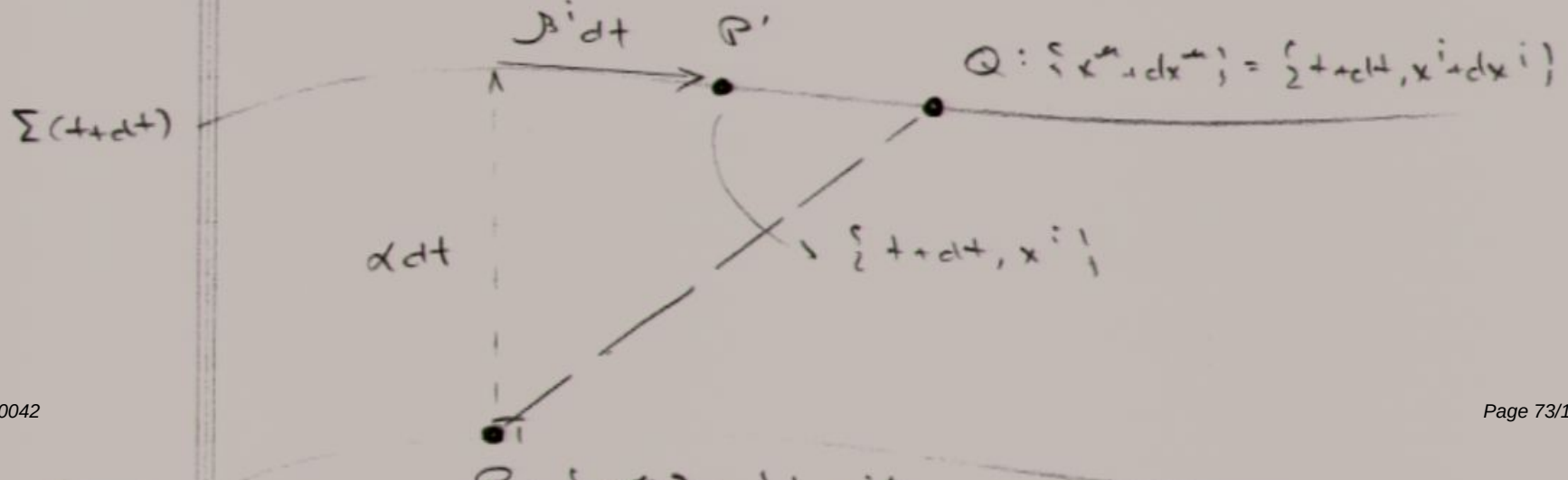

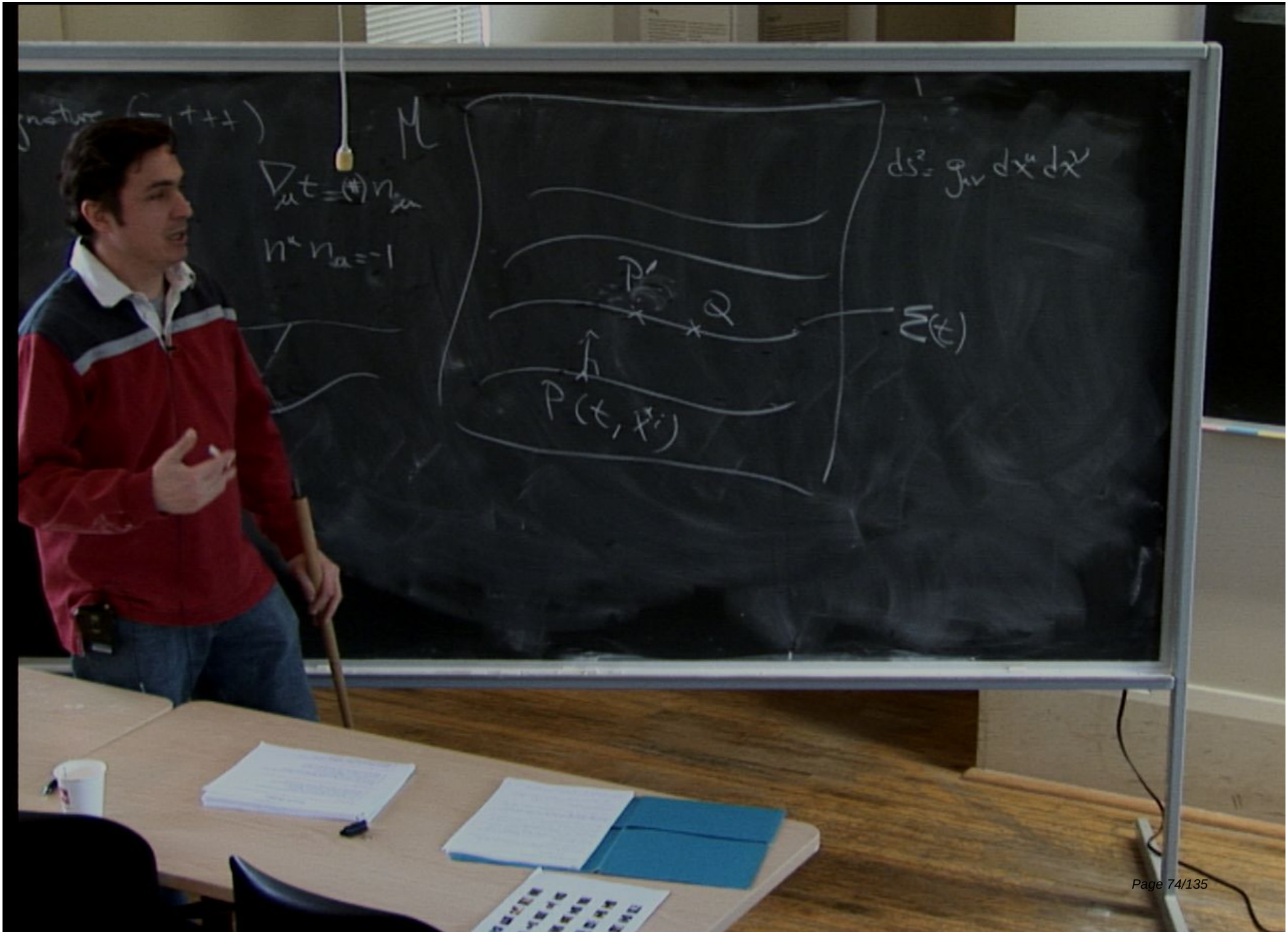
M
$\nabla_\mu t = (*) n_\mu$
$n^a n_a = -1$
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
P'
Q
$\Sigma(t)$
P(t, x^i)

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$$= (-\alpha^2 + {}^{(3)}g_{ij}\beta^i\beta^j)\,dt^2 + 2\,{}^{(3)}g_{ij}\beta^i dx^j dt + {}^{(3)}g_{ij}dx^i dx^j \qquad (1)$$

NOMENCLATURE:

$\alpha \equiv \alpha(t, x^i)$: LAPSE FUNCTION ("LAPSE") : GIVES LAPSE OF PROPER TIME PER UNIT COORDINATE TIME FOR AN OBSERVER MOVING NORMAL TO THE SLICES

$\beta^i \equiv \beta^i(t, x^i)$: SHIFT VECTOR ("SHIFT") : 3-VECTOR DESCRIBING

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$$\text{distance}(P,Q)^2 = {}^{(4)}ds^2$$

$$= -\alpha^2 dt^2 + {}^{(3)}g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$$= (-\alpha^2 + {}^{(3)}g_{ij}\beta^i\beta^j)dt^2 + 2\,{}^{(3)}g_{ij}\beta^i dx^j dt$$

$$+ {}^{(3)}g_{ij}dx^i dx^j \qquad (1)$$

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"SPACETIME PYTHAGOREAN THM" ($\lim dt \to 0$) $\Rightarrow$

$$\text{distance}(P,Q)^2 \overset{(a)}{=} ds^2$$

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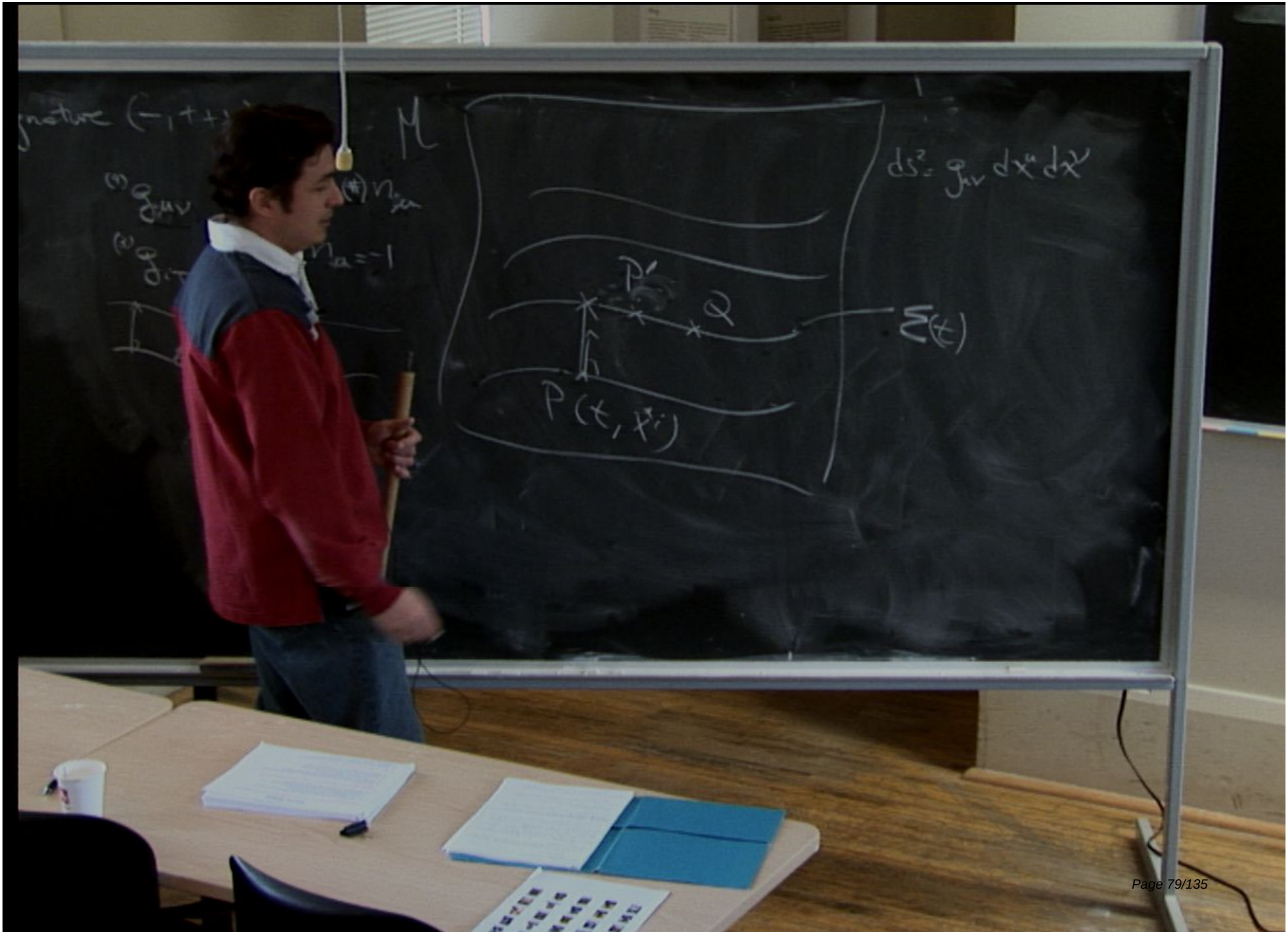
M
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
P'
Q
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$$\cdots + g_{ij}\beta^i\beta^j)\,dt^2 + 2\,g_{ij}\beta^i dx^j dt$$

$$+ {}^{(3)}g_{ij}\,dx^i dx^j \qquad (1)$$

NOMENCLATURE:

$\alpha \equiv \alpha(t, x^i)$: LAPSE FUNCTION ("LAPSE"): GIVES LAPSE OF PROPER TIME PER UNIT COORDINATE TIME FOR AN OBSERVER MOVING NORMAL TO THE SLICES

$\beta^i \equiv \beta^i(t, x^i)$: SHIFT VECTOR ("SHIFT"): 3-VECTOR DESCRIBING STRETCH OF SPATIAL COORDINATES RELATIVE TO "NORMAL PROPAGATION"

TOGETHER $\{\alpha, \beta^i\}$ CONSTITUTE 4-FOLD COORD. FREEDOM

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* TOGETHER $\{\alpha, \beta^i\}$ CONSTITUTE 4-FOLD COORD. FREEDOM of GR

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NOMENCLATURE:

$\alpha \equiv \alpha(t, x^i)$: LAPSE FUNCTION ("LAPSE"): GIVES LAPSE OF PROPER TIME PER UNIT COORDINATE TIME FOR AN OBSERVER MOVING NORMAL TO THE SLICES

$\beta^i \equiv \beta^i(t, x^i)$: SHIFT VECTOR ("SHIFT"): 3-VECTOR DESCRIBING SHIFTING OF SPATIAL COORDINATES RELATIVE TO "NORMAL PROPAGATION"

- TOGETHER $\{\alpha, \beta^i\}$ CONSTITUTE 4-FOLD COORD. FREEDOM of GR

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(COORDINATE FREEDOM)

(2) SOME PRESCRIPTION FOR α, β^i MUST BE GIVEN "FROM OUTSIDE" - I.E. E.O.M. (EINSTEIN EQ^{N}) ALONE WILL NOT, IN GENERAL, DETERMINE THEM ("GAUGE FIXING")

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NOTE: TENSORS SUCH AS β^i ARE DEFINED ON $\Sigma(t)$, AND ARE CALLED SPATIAL TENSORS

CLEARLY, ${}^{(3)}g_{ij}$ IS A SPATIAL TENSOR; IT HAS AN ASSOCIATED INVERSE ${}^{(3)}g^{ij}$ SATISFYING

$$ {}^{(3)}g^{ij}\,{}^{(3)}g_{jk} = \delta^i{}_k $$

INDICES OF 3-TENSORS ARE RAISED/LOWERED WITH ${}^{(3)}g^{ij}$, ${}^{(3)}g_{ij}$; THUS, WE CAN REWRITE (1) AS

$$ {}^{(4)}ds^2 = (-\alpha^2 + \beta^i\beta_i)\,dt^2 + 2\beta_j\,dx^j\,dt + {}^{(3)}g_{ij}\,dx^i\,dx^j $$

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SO, WE HAVE

$$^{(4)}g_{\mu\nu} = \begin{bmatrix} ^{(4)}g_{00} & ^{(4)}g_{0j} \\ ^{(4)}g_{i0} & ^{(4)}g_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha^2 + \beta^k\beta_k & \beta_j \\ \beta_i & ^{(3)}g_{ij} \end{bmatrix} \qquad (2)$$

IN PARTICULAR, NOTE THAT

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IN PARTICULAR, NOTE THAT

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^j}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & {}^{(3)}g^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{bmatrix} $$

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SO, WE HAVE

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$$ {}^{(4)}g_{ij} = {}^{(3)}g_{ij} $$

I.E. SPATIAL COVARIANT COMPONENTS OF 4- AND 3-METRICS ARE IDENTICAL

THIS IS A GENERAL RESULT; GIVEN ANY 4-TENSOR OF TYPE $(0,k)$ (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE COMPONENTS OF A TYPE $(0,k)$ 3-TENSOR

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$$\begin{bmatrix} {}^{(4)}g_{i0} & {}^{(4)}g_{ij} \end{bmatrix} \quad \begin{bmatrix} \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad (2)$$

IN PARTICULAR, NOTE THAT

$${}^{(4)}g_{ij} = {}^{(3)}g_{ij}$$

I.E. SPATIAL COVARIANT COMPONENTS OF 4- AND 3-METRICS ARE IDENTICAL

- THIS IS A GENERAL RESULT: GIVEN ANY 4-TENSOR OF TYPE $(0,k)$ (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE COMPONENTS OF A TYPE $(0,k)$ 3-TENSOR

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• THIS IS A GENERAL RESULT: GIVEN ANY 1-TENSOR OF TYPE $(0,k)$ (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE COMPONENTS OF A TYPE $(0,k)$ 3-TENSOR

WHY? RECALL THAT COVARIANT TENSOR COMPONENTS CAN BE DEFINED IN TERMS OF THE ACTION OF THE TENSOR ON THE COORDINATE BASIS VECTORS $\vec{e}_\mu = {}^{(4)}\vec{e}_\mu$, $\mu = 0, 1, 2, 3$

E.G. $T_{\mu\nu} = T(\vec{e}_\mu, \vec{e}_\nu)$

AND $T_{ij} = T(\vec{e}_i, \vec{e}_j)$

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OF A TYPE $(0,k)$ 3-TENSOR

WHY? RECALL THAT COVARIANT TENSOR COMPONENTS CAN BE DEFINED IN TERMS OF THE ACTION OF THE TENSOR ON THE COORDINATE BASIS VECTORS $^{(a)}\vec{e}_\mu$, $\mu = 0, 1, 2, 3$

E.G. $^{(a)}t_{\mu\nu} = {}^{(a)}t(\,{}^{(a)}\vec{e}_\mu, {}^{(a)}\vec{e}_\nu)$

AND $^{(a)}t_{ij} = {}^{(a)}t(\,{}^{(a)}\vec{e}_i, {}^{(a)}\vec{e}_j)$

BUT CLEARLY THE $\{{}^{(a)}\vec{e}_i\}$ ARE DIRECTLY THE COORDINATE BASIS VECTORS $\{{}^{(3)}\vec{e}_i\}$ FOR $\Sigma(t)$ WITH COORDINATES $\{x^i\}$; THUS INTERPRETING $^{(a)}t(\ldots)$

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$$ {}^{(2)}t_{ij} = {}^{(4)}t_{ij} $$

ON THE OTHER HAND, THE SPATIAL METRICS $\{ {}^{(4)}\underline{\omega}^i \}$ OF THE DUAL BASIS $\{ {}^{(4)}\underline{\omega}^m \}$, WILL NOT, IN GENERAL COINCIDE WITH $\{ {}^{(3)}\underline{\omega}^i \}$ - THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

• THUS, IN GENERAL

$$ {}^{(3)}t_{ij} \neq {}^{(4)}t_{ij} $$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$ \begin{vmatrix} -1/N^2 & N^j/N^2 \\ \cdots & \cdots \end{vmatrix} $$

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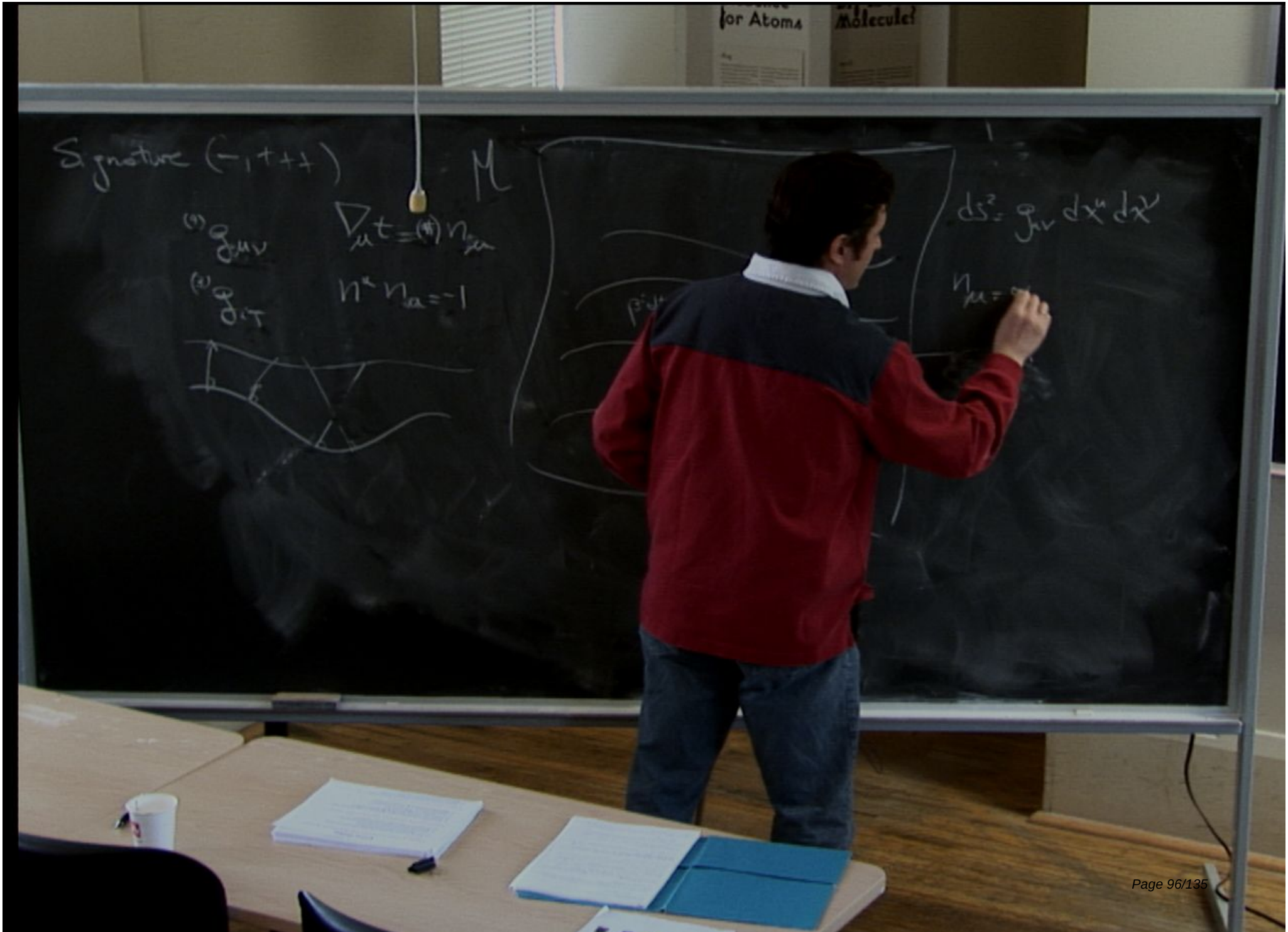
• THUS, IN GENERAL

$$ {}^{(3)}\Gamma_{ij} \neq {}^{(a)}\Gamma_{ij} $$

EXAMPLE : CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$ {}^{(a)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3) $$

(EXERCISE: VERIFY)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$^{(4)}g_{\mu\nu}$
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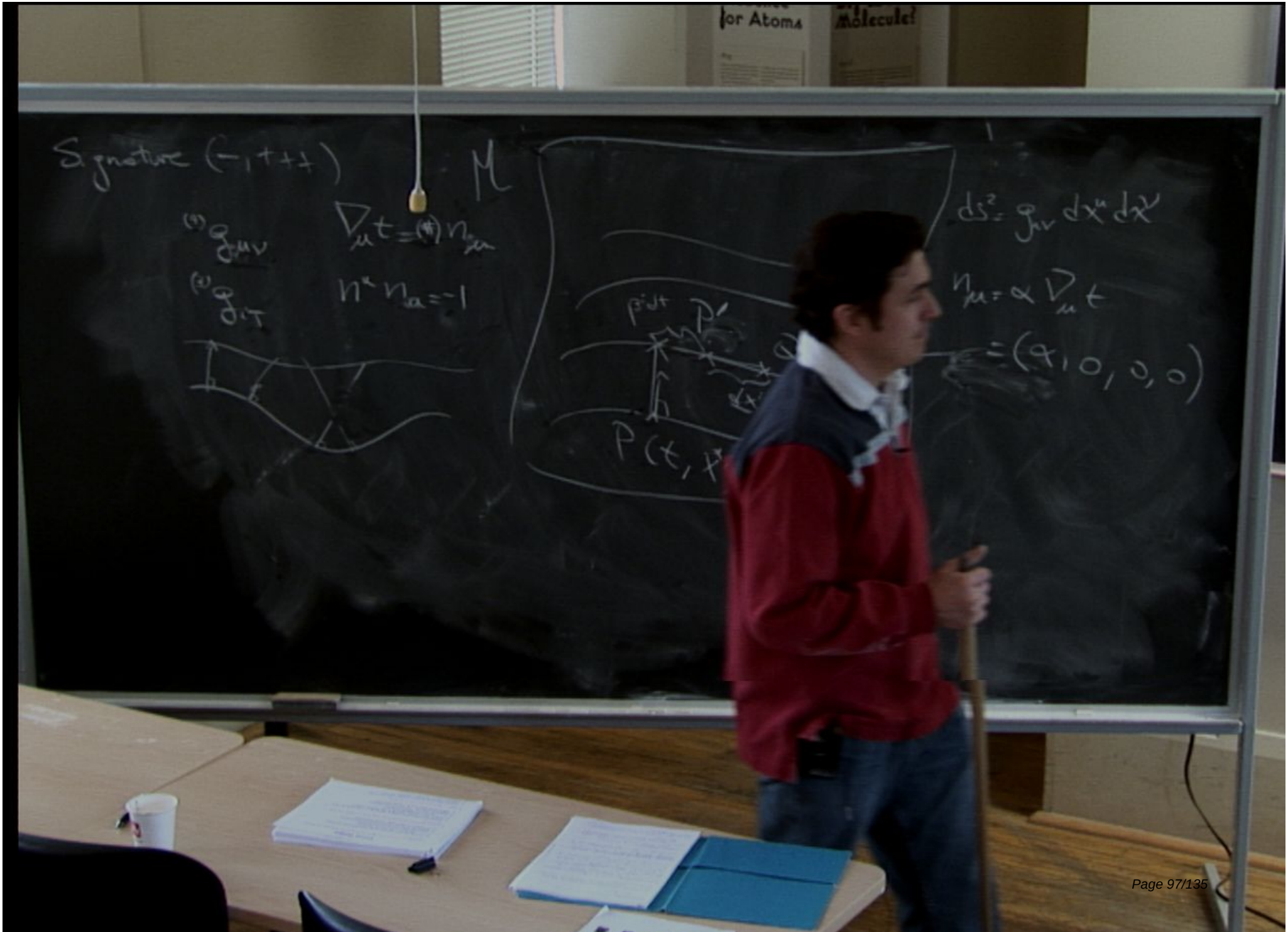
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$n^a n_a = -1$
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
$n_\mu = \alpha \nabla_\mu t$
$= (\alpha, 0, 0, 0)$
$P(t, x^*$

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• THUS, IN GENERAL

$$ {}^{(3)}\Gamma_{ij} \neq {}^{(4)}\Gamma_{ij} $$

EXAMPLE : CONSIDER THE INVERSE 4-METRIC COMPONENTS FROM (2)

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3) $$

(EXERCISE: VERIFY)

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FROM (2)

$$ {}^{(4)}g^{\mu\nu} = \begin{bmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^i/\alpha^2 & {}^{(3)}g^{ij} - \beta^i\beta^j/\alpha^2 \end{bmatrix} \qquad (3) $$

(EXERCISE: VERIFY)

DAY 387M THE 3+1 FORMULATION OF GR ④

- FROM (2) WE CAN ALSO COMPUTE THE USEFUL RESULT

$$ \sqrt{-{}^{(4)}g} = \alpha\sqrt{{}^{(3)}g} \qquad (4) $$

THE NORMAL VECTOR FIELD n^μ

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PHY 357H THE 3+1 FORMULATION of GR ④

- FROM (2) WE CAN ALSO COMPUTE THE USEFUL RESULT

$$\sqrt{-{}^{(4)}g} = \alpha \sqrt{{}^{(3)}g} \qquad (4)$$

THE NORMAL VECTOR FIELD n^μ

- EASIEST TO START WITH ASSOC. DUAL-VECTOR (ONE-FORM) FIELD, n_μ

GEOM. INTERP of DUAL VECTOR FIELD: LEVEL SURFACES of SCALAR FUNCTION → DUAL MOTION TO "INFINITESIMAL DISPLACEMENT" (VECTOR)

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THE NORMAL VECTOR FIELD n^μ

EASIEST TO START WITH ASSOC. DUAL-VECTOR (ONE-FORM) FIELD, n_μ

CLEAR, INSTEAD OF DUAL VECTOR FIELD: LEVEL SURFACES OF SCALAR FUNCTION → DUAL MOTION TO "INFINITESIMAL DISPLACEMENT" (VECTOR)

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GEOM. INTERP. OF DUAL-VECTOR FIELD: LEVEL SURFACES of SCALAR FUNCTION ⇒ DUAL MOTION TO "INFINITESIMAL DISPLACEMENT" (VECTOR)

$\Rightarrow \langle \vec{v}, \underline{df} \rangle = v^\mu (df)_\mu$: # of level surfaces "pierced" by $\vec{v}$

• HERE OUR SCALAR FUNCTION IS THE TIME COORDINATE t, WITH ASSOCIATED DUAL-VECTOR FIELD $\underline{dt}$, THEN

$$\underline{\sim} \propto \underline{dt}$$

OR IN COMPONENT FORM

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$$n_\mu = (-\alpha, 0, 0, 0) \qquad (5)$$

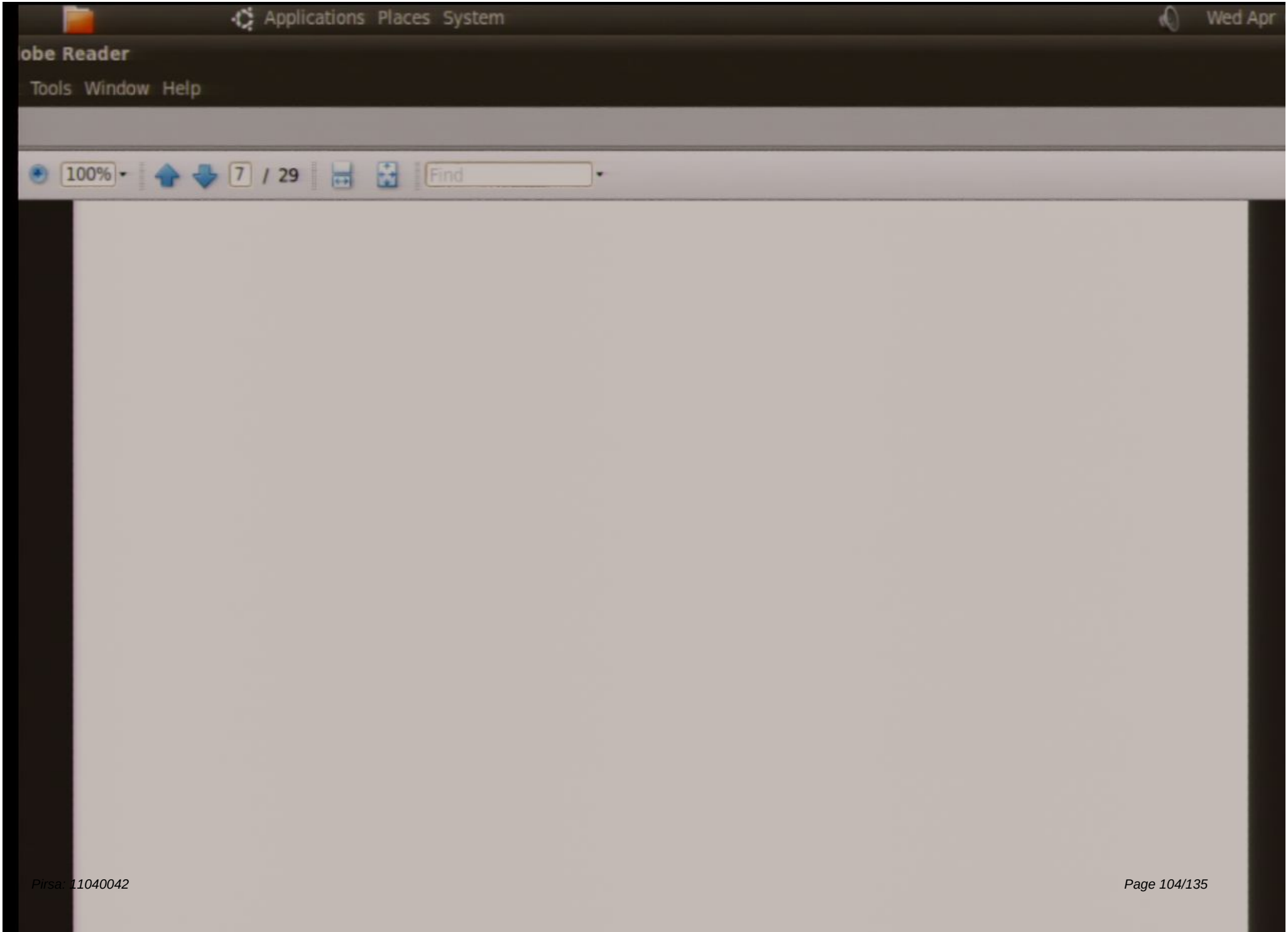
AND THEN

$$n^\mu = {}^{(4)}g^{\mu\nu} n_\nu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right) \qquad (6)$$

EXTRINSIC CURVATURE

- THE INTRINSIC GEOMETRY of $\Sigma(t)$ IS DESCRIBED BY
  $^{(3)}g_{ij}$ WHICH ENCODES ALL GEOM. INFO. WHICH MAY BE DEDUCED BY MAKING MEASUREMENTS ON $\Sigma(t)$ ALONE

- HOWEVER, A GIVEN 3-GEOMETRY (SLICE, HYPERSURFACE) MAY BE EMBEDDED IN SPACETIME IN INFINITELY MANY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• HOWEVER, A GIVEN 3-GEOMETRY (SLICE, HYPERSURFACE) MAY BE EMBEDDED IN SPACETIME IN INFINITELY MANY DISTINGUISHABLE (BY 4-D MEASUREMENTS) WAYS

EXAMPLE: 2D EMBEDDED IN 3D; FLAT SURFACE EMBEDDED WITH / WITHOUT EXTRINSIC CURVATURE

• THE MANNER IN WHICH $\Sigma(t)$ IS EMBEDDED CAN BE CHARACTERIZED BY INVESTIGATING THE CHANGE IN THE DIRECTION OF THE NORMAL FIELD AS A FCN OF POS ON $\Sigma(t)$ — THIS DEFINES THE EXTRINSIC CURVATURE TENSOR (AKA THE SECOND FUNDAMENTAL FORM)

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WE HAVE

SIGN CHOSEN SO THAT n^μ IS "FUTURE-DIRECTED"

$$n_\mu = (-\alpha, 0, 0, 0) \qquad (5)$$

AND THEN

$$n^\mu = {}^{(4)}g^{\mu\nu} n_\nu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha} \right) \qquad (6)$$

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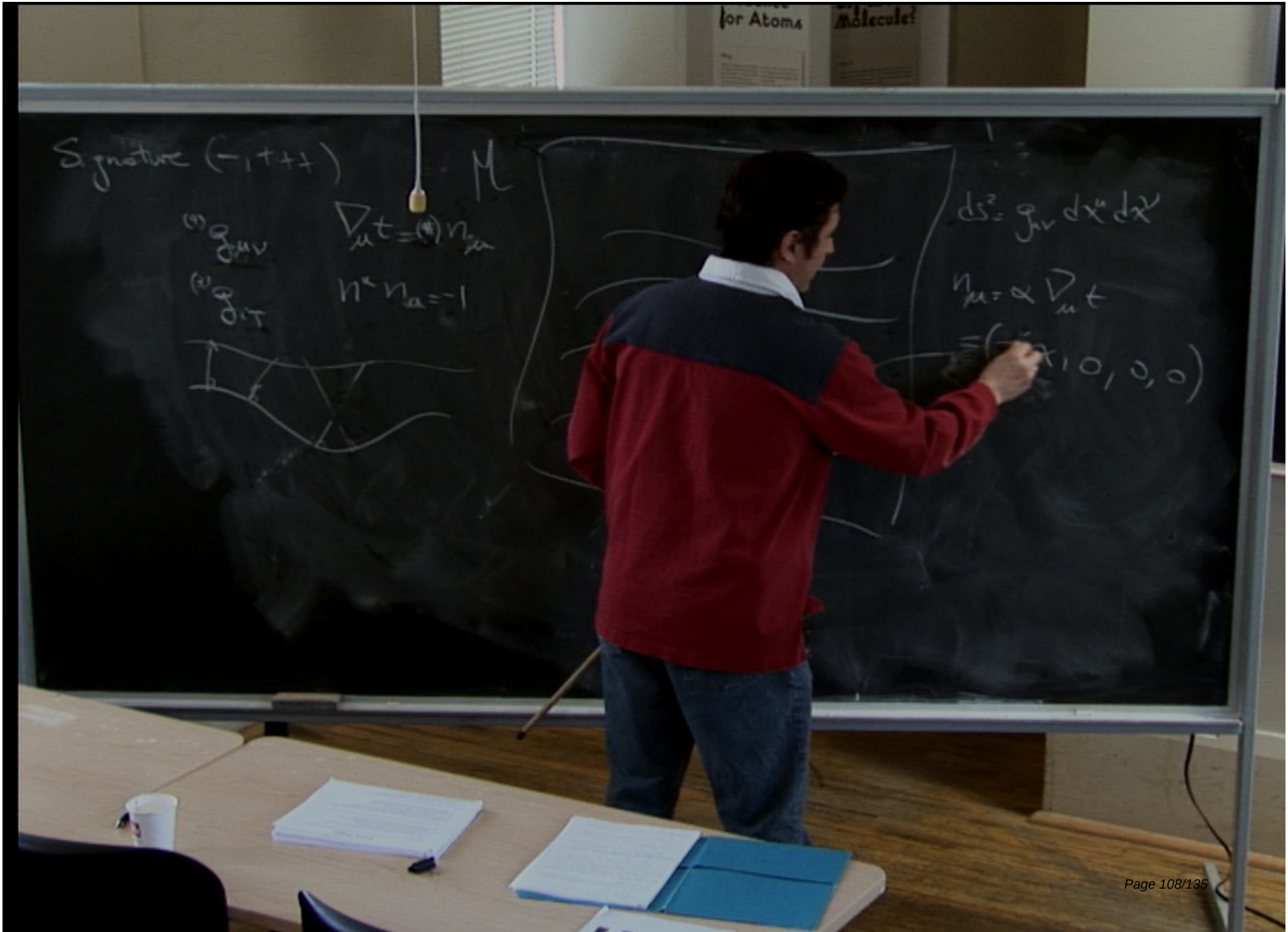
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$$K_{ij} = -\nabla_i n_j = -\nabla(_i n_{j)} \qquad (7)$$

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$$K_{ij} = -\nabla_i n_j = -\nabla(_i n_{j)} \qquad (7)$$

$$\nabla_i n_j = \partial_i n_j - \Gamma^m{}_{ij} n_m$$

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PHY 3[illegible] THE 3+1 FORMULATION of GR

5

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_\mu$$

BUT $n_\mu = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha\, {}^{(4)}\Gamma^{0}{}_{ij}$$

$$= \alpha \left({}^{(4)}g^{00}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0l}\, {}^{(4)}\Gamma_{lij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2} \left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - {}^{(4)}g_{ij,0} \right)$$

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$$K_{ij} = -\nabla_i n_j = -\nabla(_i n_{j)} \qquad (7)$$

ADM 3+1 THE 3+1 FORMULATION of GR (5)

$$\nabla_i n_j = \partial_i n_j - \Gamma^{\mu}{}_{ij} n_\mu$$

BUT $n_\mu = (-\alpha, 0, 0, 0)$ SO

$$\nabla_i n_j = \alpha^{(4)}\Gamma^0{}_{ij}$$

$$= \alpha\left({}^{(4)}g^{0\alpha}\,{}^{(4)}\Gamma_{\alpha ij} + {}^{(4)}g^{0k}\,{}^{(4)}\Gamma_{kij}\right)$$

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PHY 3[illegible] THE 3+1 FORMULATION of GR

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$$= \alpha \left({}^{(4)}g^{00}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0l}\, {}^{(4)}\Gamma_{lij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2} \left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - \underline{{}^{(4)}g_{ij,0}} \right)$$

THUS

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PHY 3[illegible] THE 3+1 FORMULATION of GR

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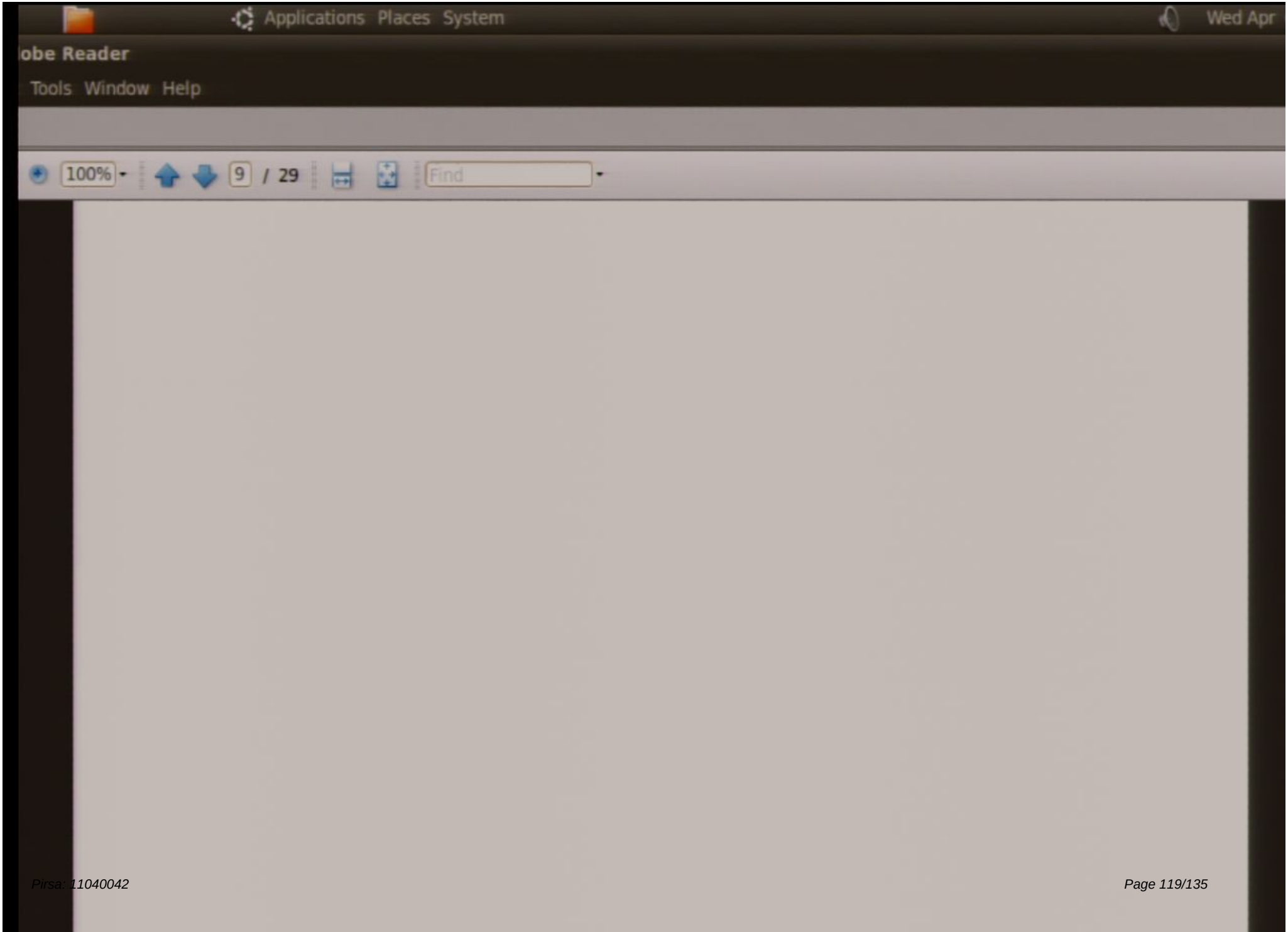
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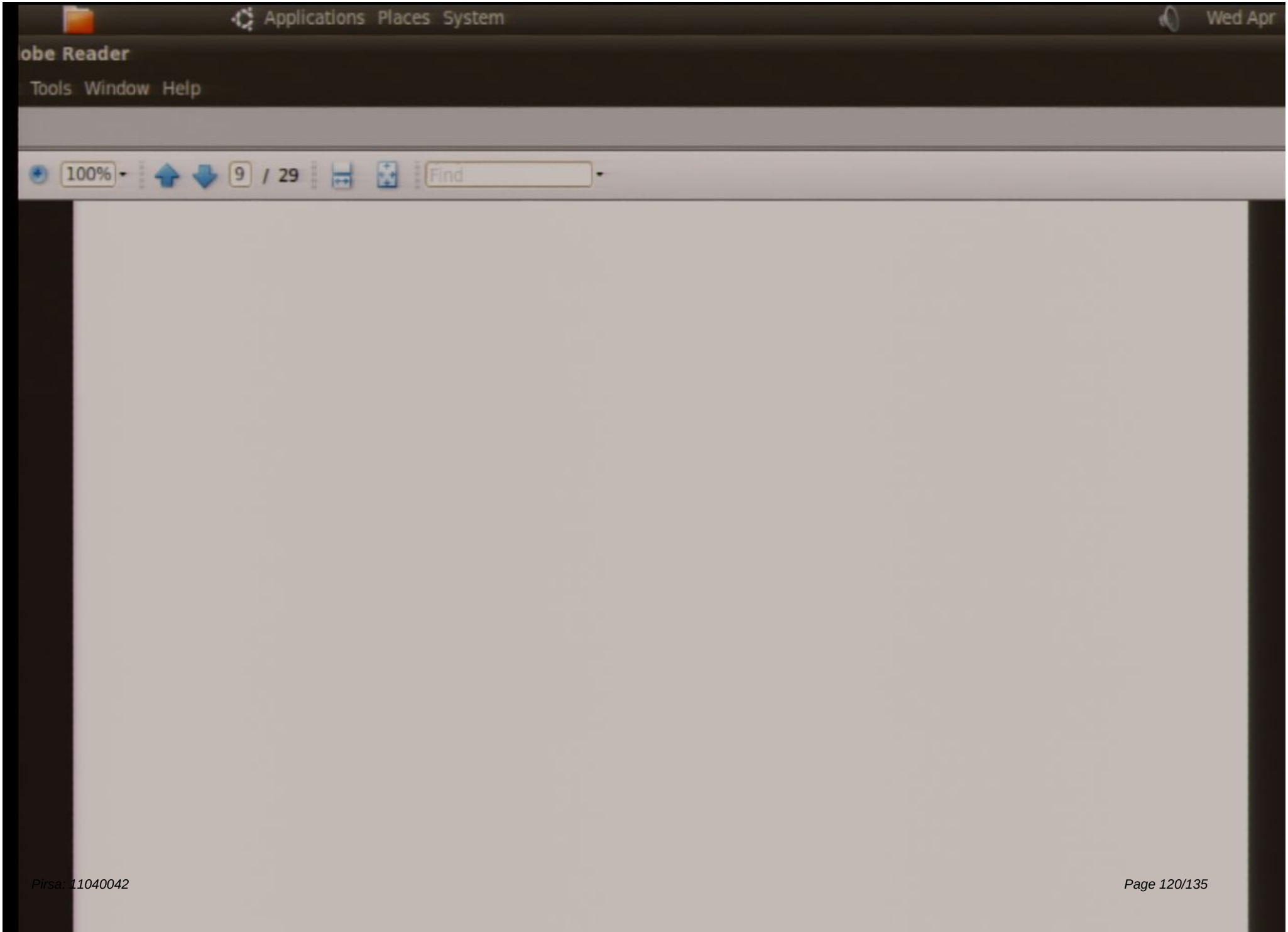
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COORDINATES $\{x^i\}$, THUS INTERPRETING $t(\ldots)$ AS A 3-TENSOR $^{(2)}t(\ldots)$, WE NECESSARILY HAVE

$$^{(3)}t_{ij} = {}^{(A)}t_{ij}$$

ON THE OTHER HAND, THE SPATIAL VECTORS $\{{}^{(A)}\underline{\omega}^i\}$ OF THE DUAL BASIS $\{{}^{(A)}\underline{\omega}^m\}$, WILL NOT, IN GENERAL COINCIDE WITH $\{{}^{(3)}\underline{\omega}^i\}$ - THE DUAL BASIS ON $\Sigma(t)$ DEPENDS ON HOW $\Sigma(t)$ IS EMBEDDED IN THE S.T.

THUS, IN GENERAL

$$^{(3)}t_{ij} \neq {}^{(A)}t_{ij}$$

EXAMPLE: CONSIDER THE INVERSE 4-METRIC COMPONENTS

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I.E. SPATIAL COVARIANT COMPONENTS OF 4- AND 3-METRICS ARE IDENTICAL

- THIS IS A GENERAL RESULT: GIVEN ANY 4-TENSOR OF
- THIS IS A GENERAL RESULT: GIVEN ANY 4-TENSOR OF TYPE (0, k) (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE COMPONENTS OF A TYPE (0,k) 3-TENSOR

WHY? RECALL THAT COVARIANT TENSOR COMPONENTS CAN BE DEFINED IN TERMS OF THE ACTION OF THE TENSOR ON THE COORDINATE BASIS VECTORS $e_{(\mu)} = e_\mu$, $\mu = 0, 1, 2, 3$

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$$ds^2 = (-\alpha^2 + \beta^i \beta_i)dt^2 + 2\beta_j dx^j dt + {}^{(3)}g_{ij} dx^i dx^j$$

SO, WE HAVE

$$ {}^{(4)}g_{\mu\nu} = \begin{bmatrix} {}^{(4)}g_{00} & {}^{(4)}g_{0j} \\ {}^{(4)}g_{i0} & {}^{(4)}g_{ij} \end{bmatrix} = \begin{bmatrix} -\alpha^2 + \beta^k \beta_k & \beta_j \\ \beta_i & {}^{(3)}g_{ij} \end{bmatrix} \qquad (2)$$

IN PARTICULAR, NOTE THAT

$$ {}^{(4)}g_{ij} = {}^{(3)}g_{ij}$$

I.E. SPATIAL COVARIANT COMPONENTS OF 4- AND 3-METRICS ARE IDENTICAL

THIS IS A GENERAL RESULT; GIVEN ANY 4-TENSOR OF TYPE $(0,k)$ (COVARIANT TENSOR)

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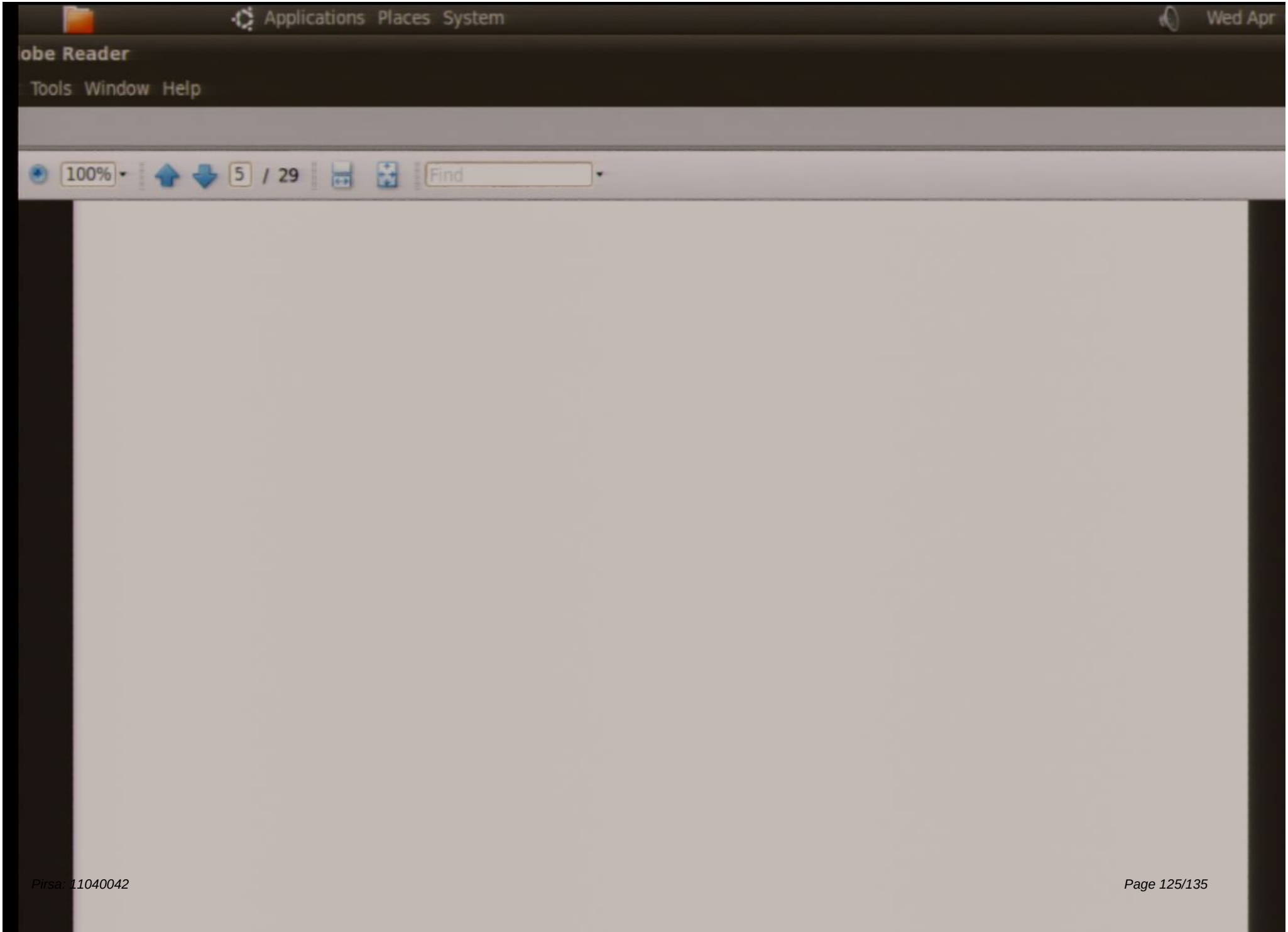
THIS IS A GENERAL RESULT: GIVEN ANY 1-TENSOR OF TYPE (0, k) (COVARIANT TENSOR), THE SPATIAL COMPONENTS OF THAT TENSOR CAN BE IDENTIFIED AS THE COMPONENTS OF A TYPE (0,k) 3-TENSOR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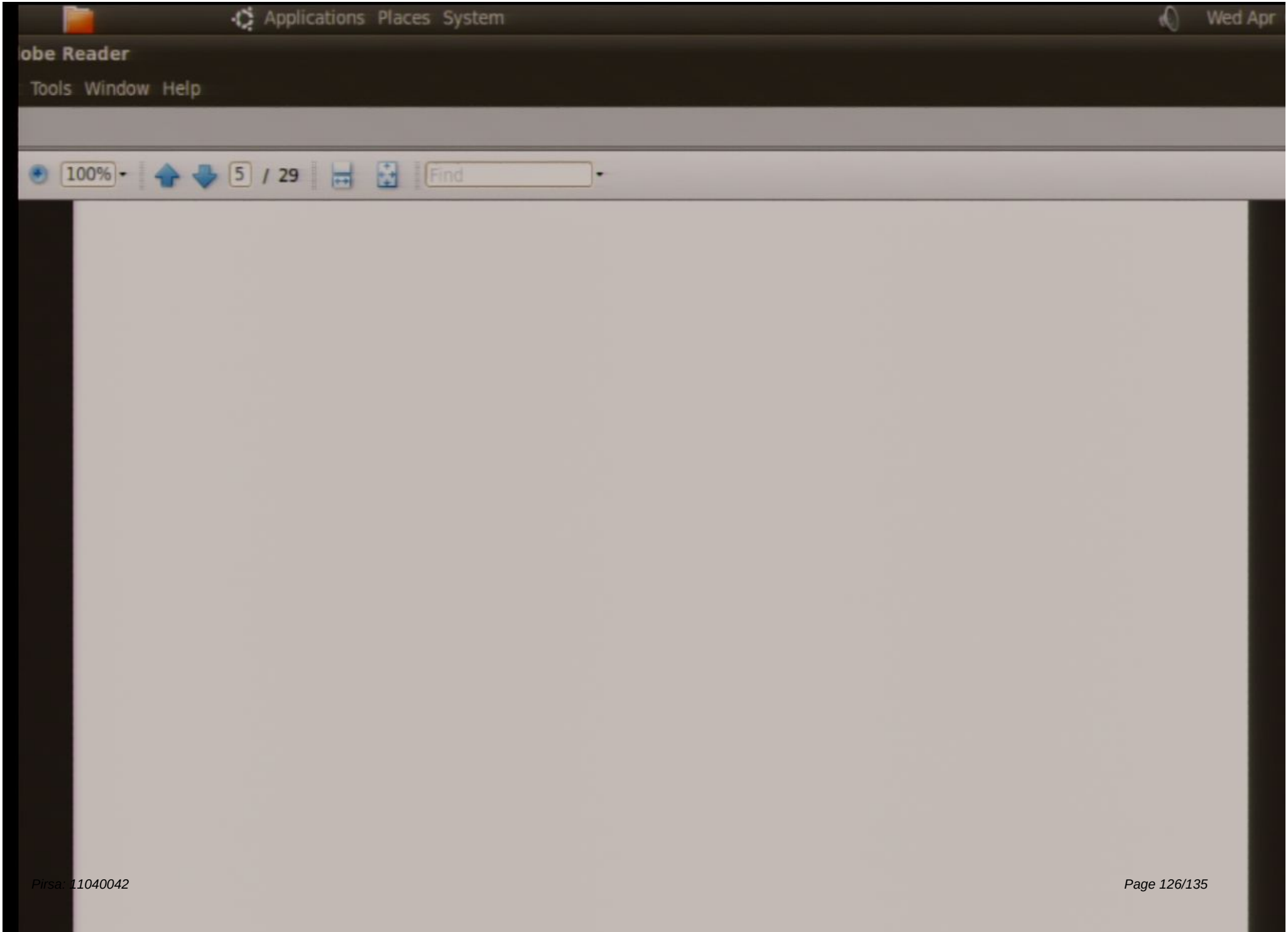
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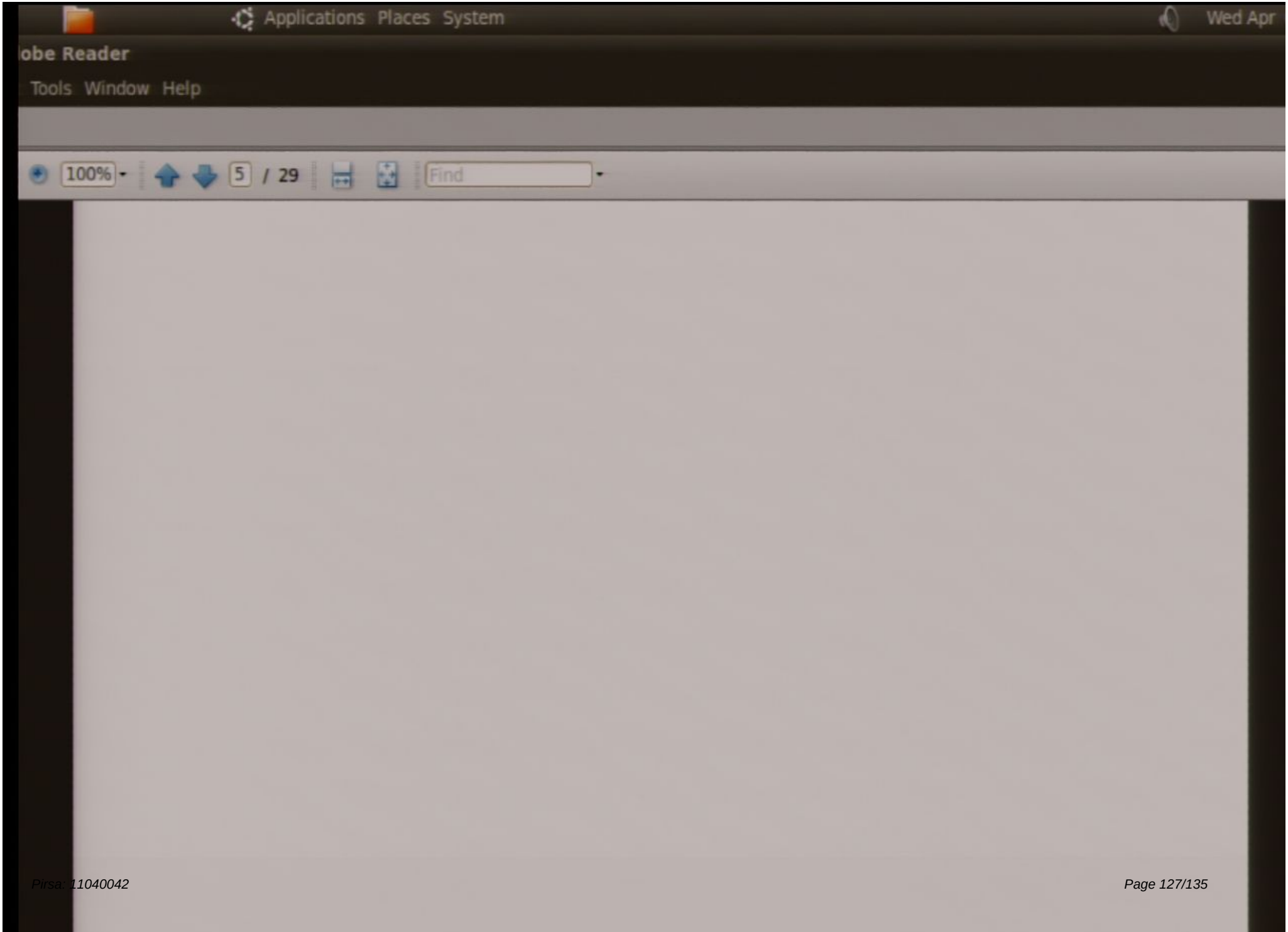
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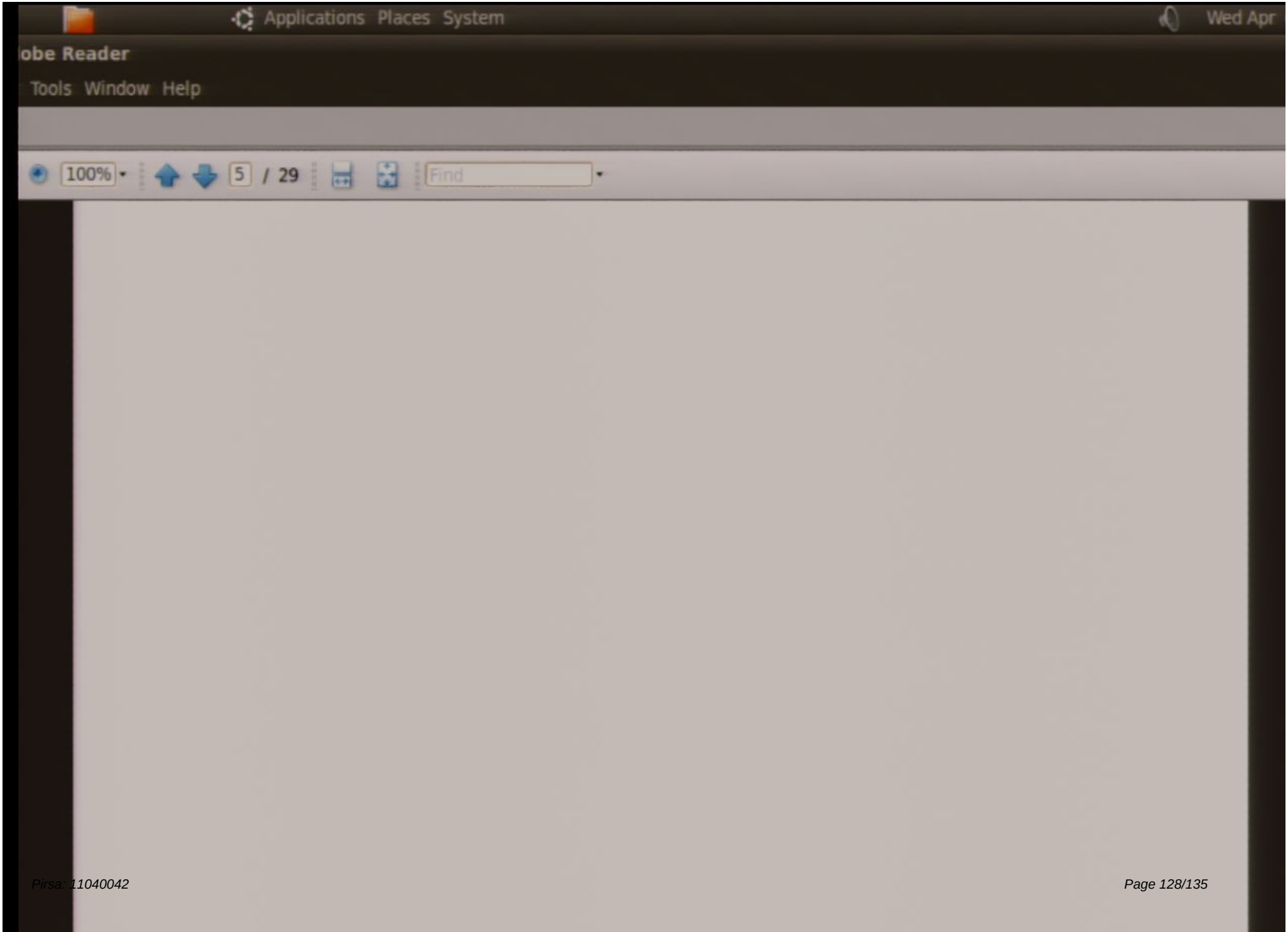
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$$g^{ab}\Omega_a\Omega_b = -\alpha^{-2}\,, \tag{3}$$

where α is the lapse function, as previously. Thus we can construct the unit-norm dual-vector field, n_a, via

$$n_a = -\alpha\Omega_a = -\alpha\nabla_a\tau\,, \tag{4}$$

where the sign is chosen so that the associated unit-norm, hypersurface-orthogonal vector field, n^a,

$$n^a = g^{ab}n_b\,, \tag{5}$$

is future-directed. Note that we can view n^a as the 4-velocity field of a congruence of observers moving orthogonally to the slices (not necessarily coordinate-stationary). Such observers will have a 4-acceleration, a^b given by

$$a^b = n^a\nabla_a n^b\,. \tag{6}$$

2) The Projection Tensor and the Spatial Metric

In the derivation of the 3+1 form of the Einstein equations, we will necessarily be interested in decomposing various spacetime tensors into hypersurface-tangential ("spatial") and hypersurface-orthogonal ("temporal") pieces. Determining the "temporal" part of a tensor is straightforward, we simply contract with n^a. York throws in a slight twist by introducing a relative minus depending on whether a vector or covector index is being projected. Thus, for a vector field, W^a, we define

$$W^{\hat{n}} = -W^a n_a\,, \tag{7}$$

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THE 3+1 EINSTEIN EQUATIONS

These notes rework the calculation of the 3+1 equations as presented in *Kinematics and Dynamics of General Relativity*, by J. W. York, Jr., which itself is contained in the volume *Sources of Gravitational Radiation*, edited by L. Smarr. Many calculational details omitted from that source are included here.

(**Note:** A colleague of mine, who will remain anonymous this week, has used these notes to demonstrate the power of his symbolic manipulation software and has found at least one error

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$$\nabla_i n_j = \alpha \, {}^{(4)}\Gamma^{0}{}_{ij}$$

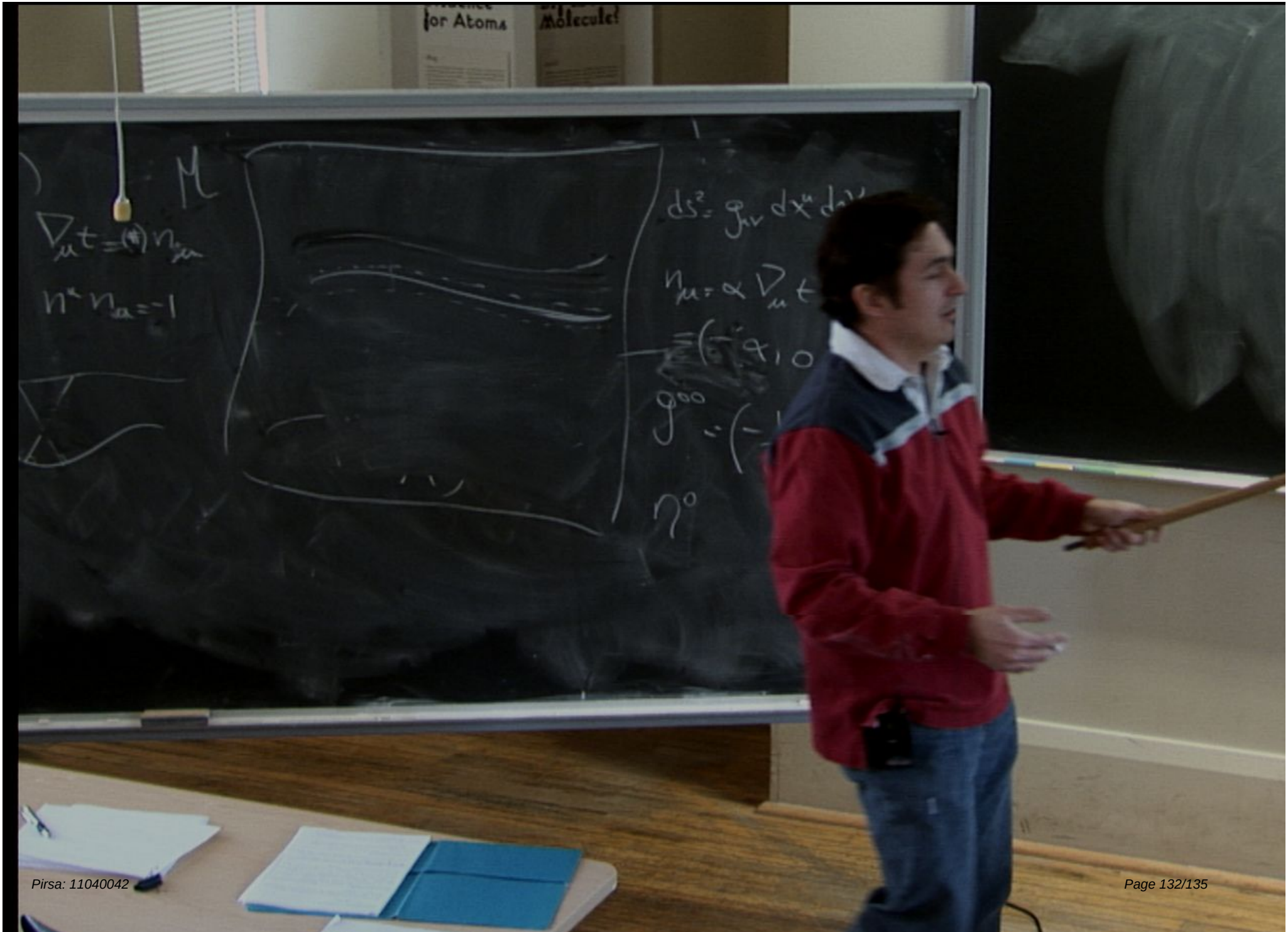
$$= \alpha \left({}^{(4)}g^{00}\, {}^{(4)}\Gamma_{0ij} + {}^{(4)}g^{0l}\, {}^{(4)}\Gamma_{lij} \right)$$

$$\Gamma_{0ij} = \frac{1}{2}\left({}^{(4)}g_{0i,j} + {}^{(4)}g_{0j,i} - \underbrace{{}^{(4)}g_{ij,0}} \right)$$

THUS

$$K_{ij} = -\frac{1}{2\alpha}\frac{\partial\, {}^{(3)}g_{ij}}{\partial t} + \ldots$$

SO WE CAN VIEW THE EXTRINSIC CURVATURE AS THE "VELOCITY" OF THE 3-METRIC ${}^{(3)}g_{ij}$ ("CONJUGATE MOMENTA")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$$\nabla_i n_j = \alpha \, {}^{(4)}\Gamma_{ij}$$

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1) Foliations and Normals

As before, we consider a spacetime M with metric g_{ab} which is sliced into a foliation $\{\Sigma\}$ defined by the isosurfaces of a scalar field τ (the time parameter). Then the spacelike hypersurfaces are, at least locally, described by a *closed* one-form (dual vector field), Ω_a:

$$\Omega_a = \nabla_a \tau \,. \tag{1}$$

Note that since Ω_a is the gradient of a scalar function, and ∇_a is torsion-free, we have

$$\nabla_{[a}\Omega_{b]} = \nabla_{[a}\nabla_{b]}\tau = 0 \,. \tag{2}$$

The norm of Ω_a is given by

$$g^{ab}\Omega_a\Omega_b = -\alpha^{-2} \,, \tag{3}$$

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