Title: Explorations in Numerical Relativity - Lecture 1

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Abstract:

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Explorations in Gravitational Physics–Numerical Relativity

Matt Choptuik, UBC
Luis Lehner, Perimeter/Guelph
Frans Pretorius, Princeton
Scott Noble, RIT

Perimeter Institute
Waterloo, ON
April 4–April 22, 2011

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Course Outline

- Solution of Classical Field Equations Using Finite Difference Techniques (Luis,Matt)

  1. Solving the wave equation using finite difference techniques
  2. $3+1$ approach to the Einstein equations
  3. Dynamical spherically symmetric spacetimes
  4. Spherically symmetric Einstein-Klein-Gordon Evolution
  5. Introduction to Black Hole Critical Phenomena

- General Relativistic Hydrodynamics Using Gudonov/HRSC Schemes (Scott,Luis)

  1. Mathematical structure; Linearly degenerate vs truly nonlinear eqns
  2. Burgers eqn; Godunov Methods & the Riemann problem
  3. $3+1$ Approach to GRHydrodynamics
  4. Stationary solutions, TOV stars & perturbations
  5. Magnetohydrodynamics & miscellaneous topics

- Topics in Numerical Relativity (Luis,Frans)

  1. Gravitational waves overview (nature in GR & sources)

4. Adaptive mesh refinement (AMR)/parallel computation
5. Miscellaneous topics: excision, apparent horizon finders, GW extraction

Week 1: References

- Mitchell, A. R., and D. F. Griffiths, **The Finite Difference Method in Partial Differential Equations**, New York: Wiley (1980)
- Richtmeyer, R. D., and Morton, K. W., **Difference Methods for Initial-Value Problems**, New York: Interscience (1967)
- H.-O. Kreiss and J. Oliger, **Methods for the Approximate Solution of Time Dependent Problems**, GARP Publications Series No. 10, (1973)
- Gustatsson, B., H. Kreiss and J. Oliger, **Time-Dependent Problems and Difference Methods**, New York: Wiley (1995)

5 (6 of 83) 76.6% Find

Solution of Classical Field Equations Using Finite Difference Techniques

1. Solving the wave equation using finite difference techniques

6 (7 of 83) 76.6% Find

Preliminaries

- Classical field equations ≡ time dependent partial differential equations (PDEs)
- Can divide time-dependent PDEs into two broad classes:

  1. **Initial-value Problems (Cauchy Problems)**, spatial domain has no boundaries (either infinite or "closed"—e.g. "periodic boundary conditions")
  2. **Initial-Boundary-Value Problems**, spatial domain *finite*, need to specify boundary conditions

- **Note:** Even if *physical* problem is really of type 1, finite computational resources ⟶ finite spatial domain ⟶ approximate as type 2; will hereafter loosely refer to either type as an IVP.
- *Working Definition:* **Initial Value Problem**
  - State of physical system arbitrarily (usually) specified at some initial time $t = t_0$.
  - Solution exists for $t \geq t_0$; uniquely determined by equations of motion (EOM) and boundary conditions (BCs).

Preliminaries

- Approximate solution of initial value problems using *any* numerical method, including finite differencing, will always involve three key steps
  1. Complete mathematical specification of system of PDEs, including boundary conditions and initial conditions
  2. Discretization of the system: replacement of continuous domain by discrete domain, and approximation of differential equations by algebraic equations for discrete unknowns
  3. Solution of discrete algebraic equations
- Will assume that the set of PDEs has a unique solution for given initial conditions and boundary conditions, and that the solution does not "blow up" in time, unless such blow up is expected from the physics
- Whenever this last condition holds for an initial value problem, we say that the problem is well posed
- Note that this is a non-trivial issue in general relativity, since there are in practice *many* distinct forms the PDEs can take for a given physical scenario (in principle infinitely many), and not all will be well-posed in general


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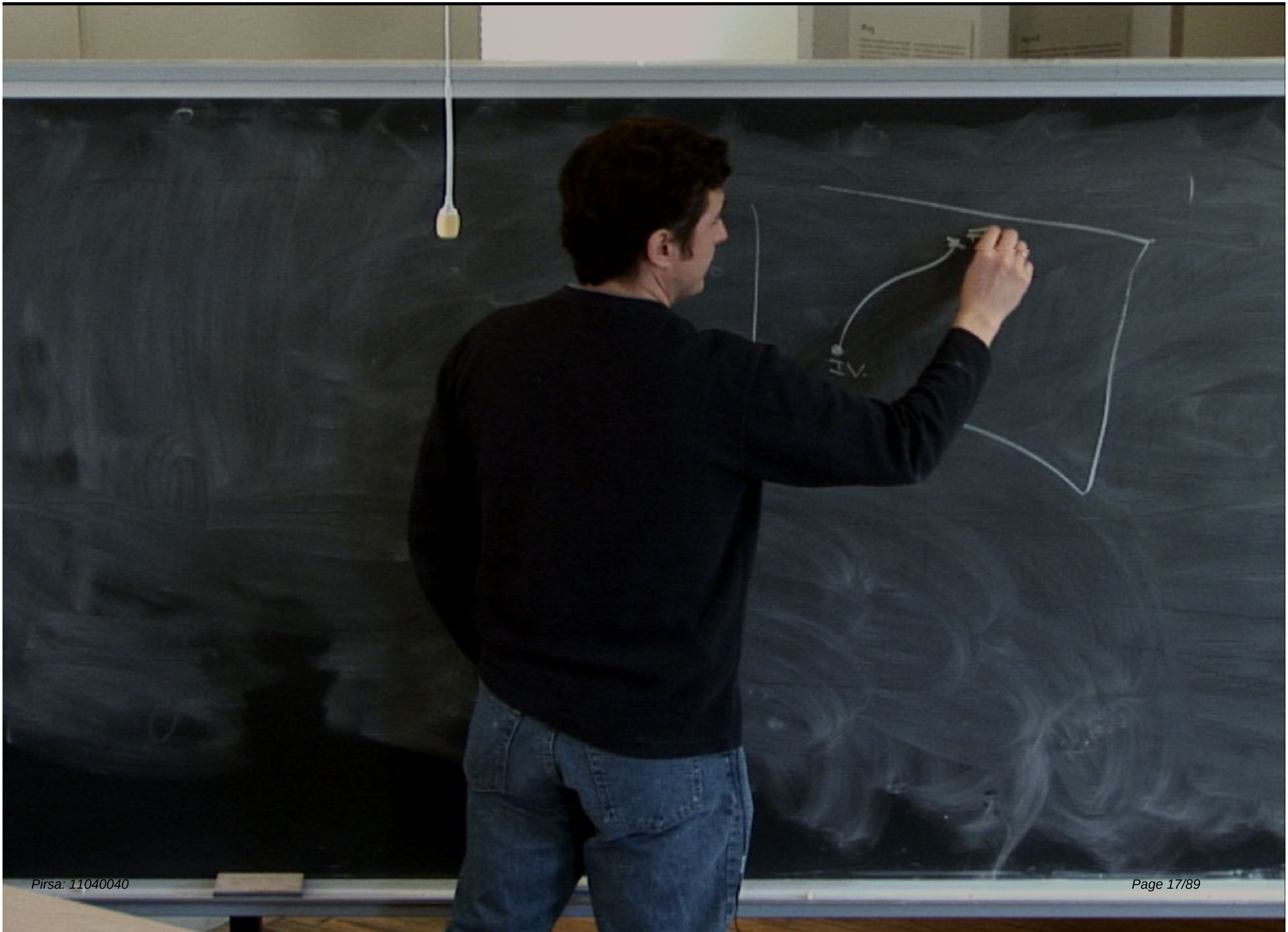
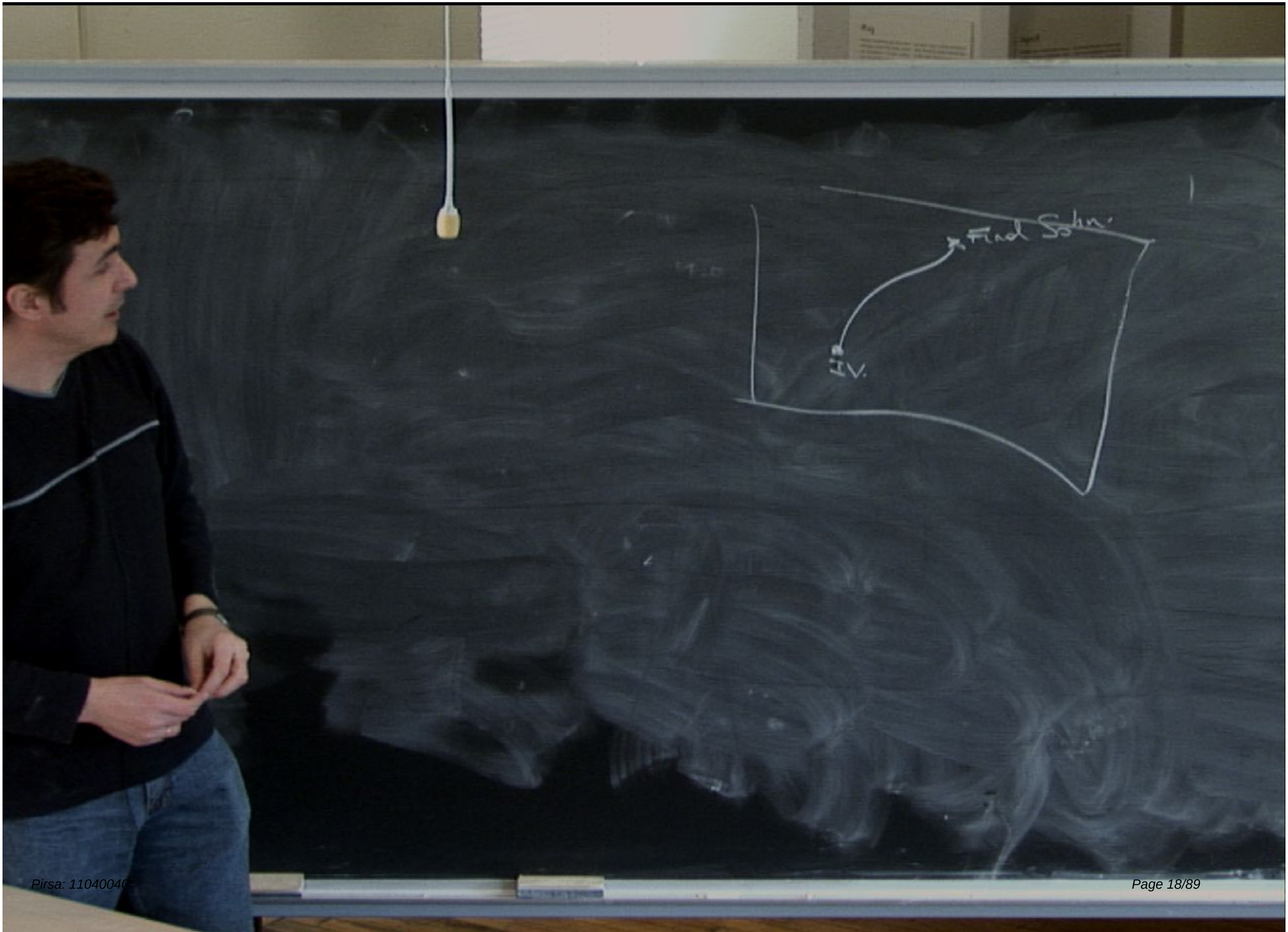

IV.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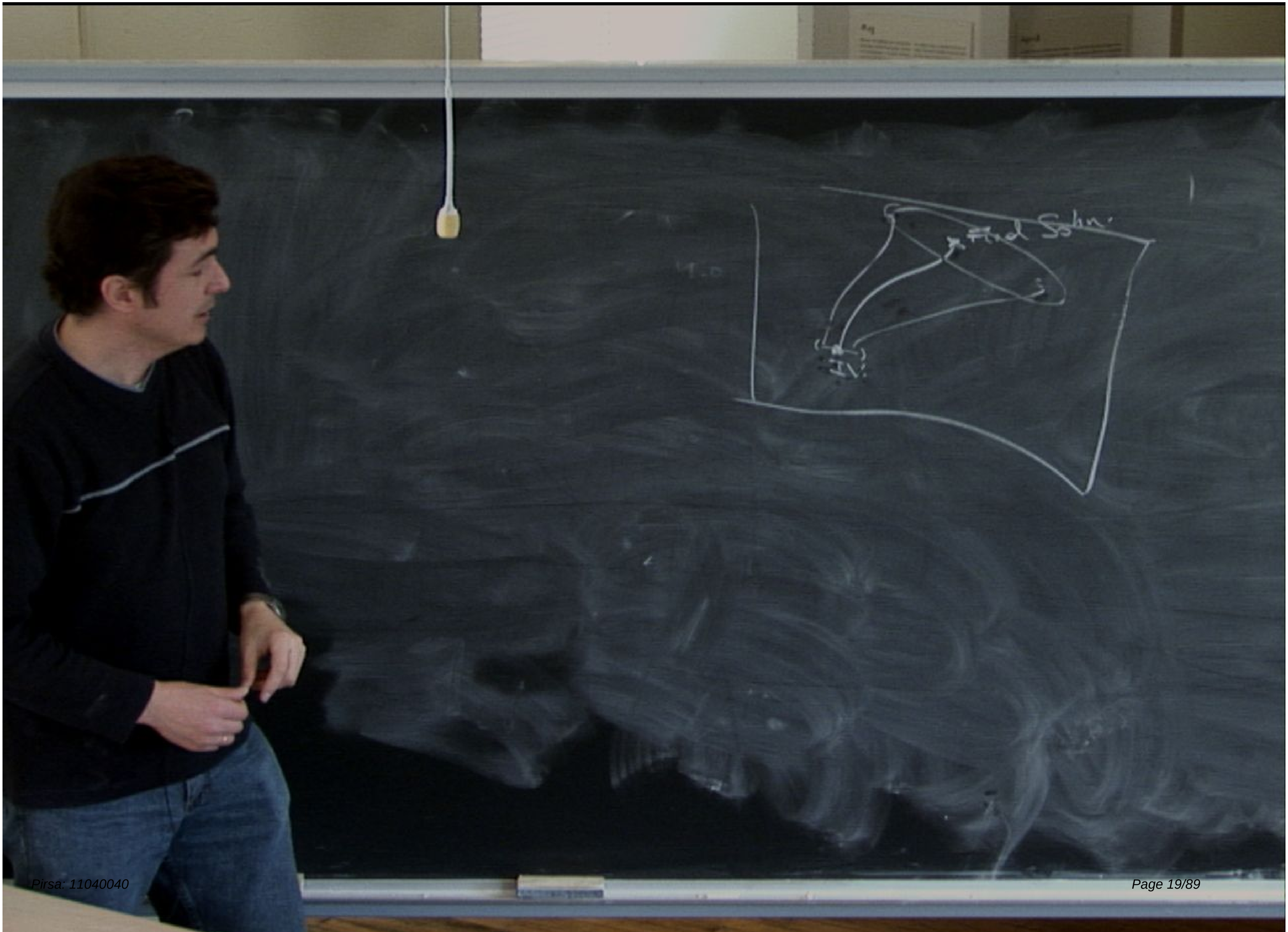
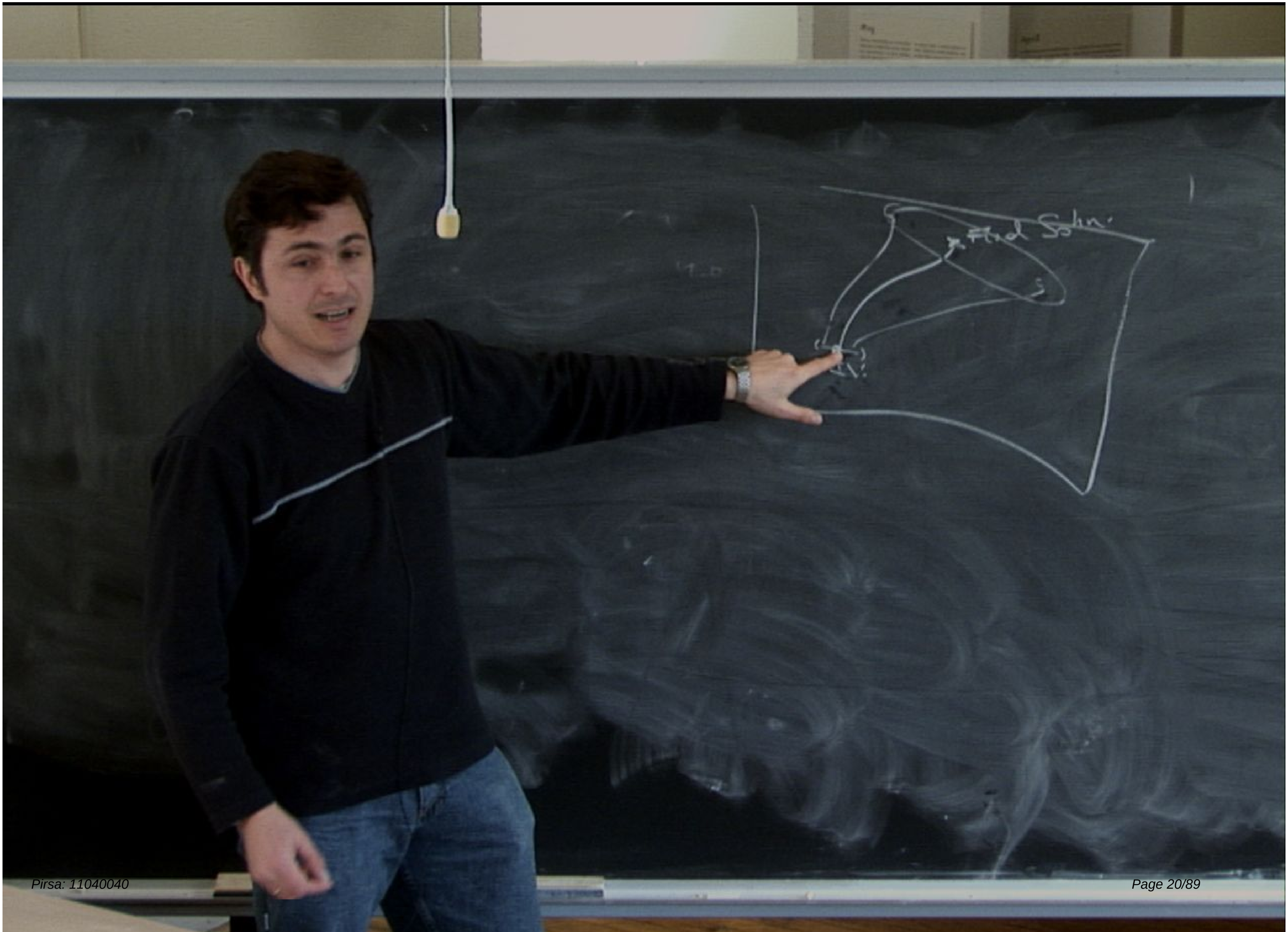
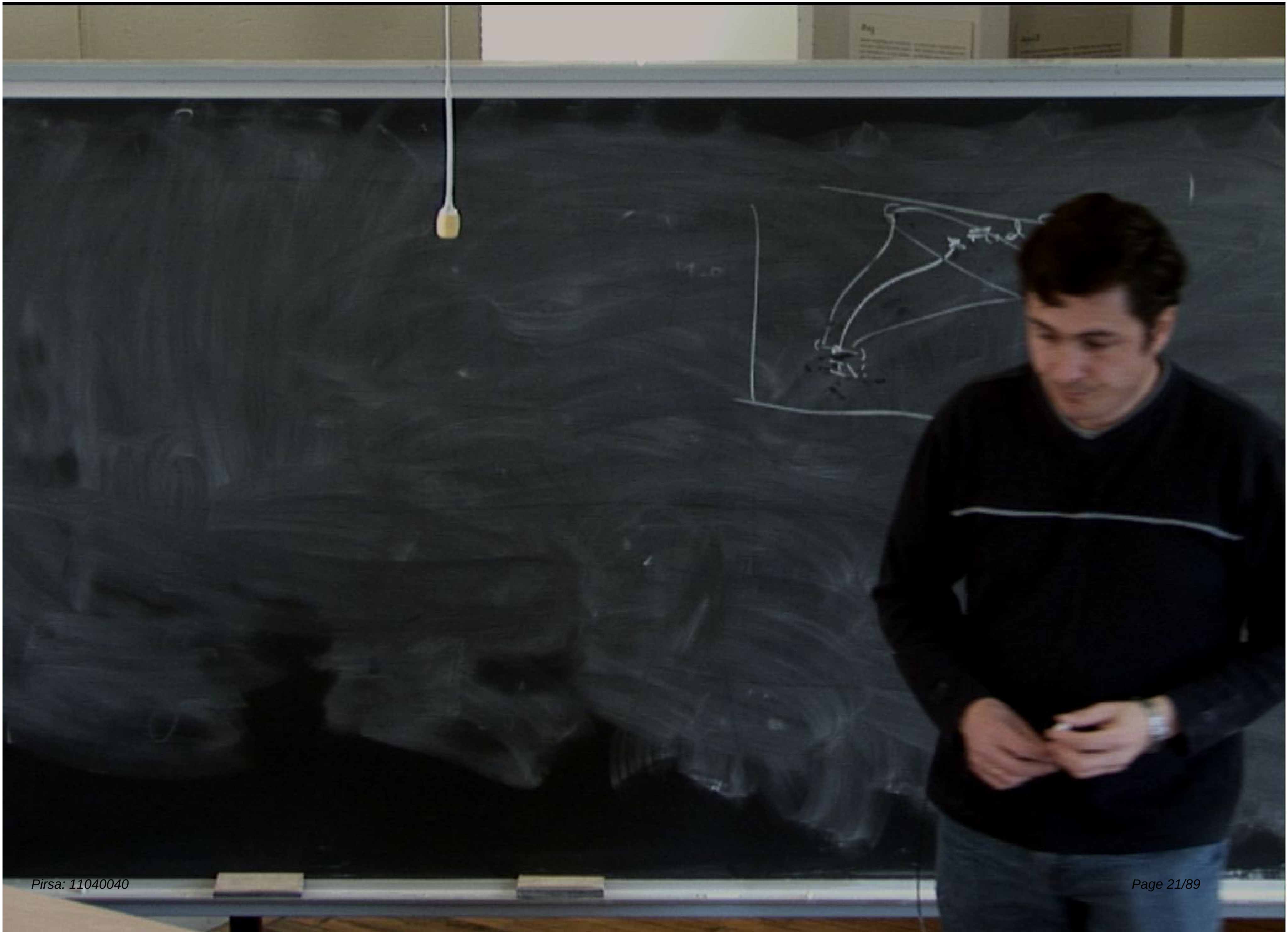
I.V.


Find Sol'n
IV


Fluid Sol'n



Why Finite Differencing?

- There are several general approaches to the numerical solution of time dependent PDEs, including

  1. Finite differences
  2. Finite volume
  3. Finite elements
  4. Spectral

- Finite difference (FD) methods are particularly appropriate when the solution is expected to be smooth (infinitely differentiable) given that the initial data is smooth

- This is the case for many classical field theories including those for a scalar (linear/nonlinear Klein Gordon), vector (electromagnetism [Maxwell]), rank-2 symmetric tensor (general relativity [Einstein])

- In cases where solutions do *not* remain smooth, even if the initial data is—as happens in compressible hydrodynamics, for example, where shocks can form—the finite volume approach is the method of choice (next week)

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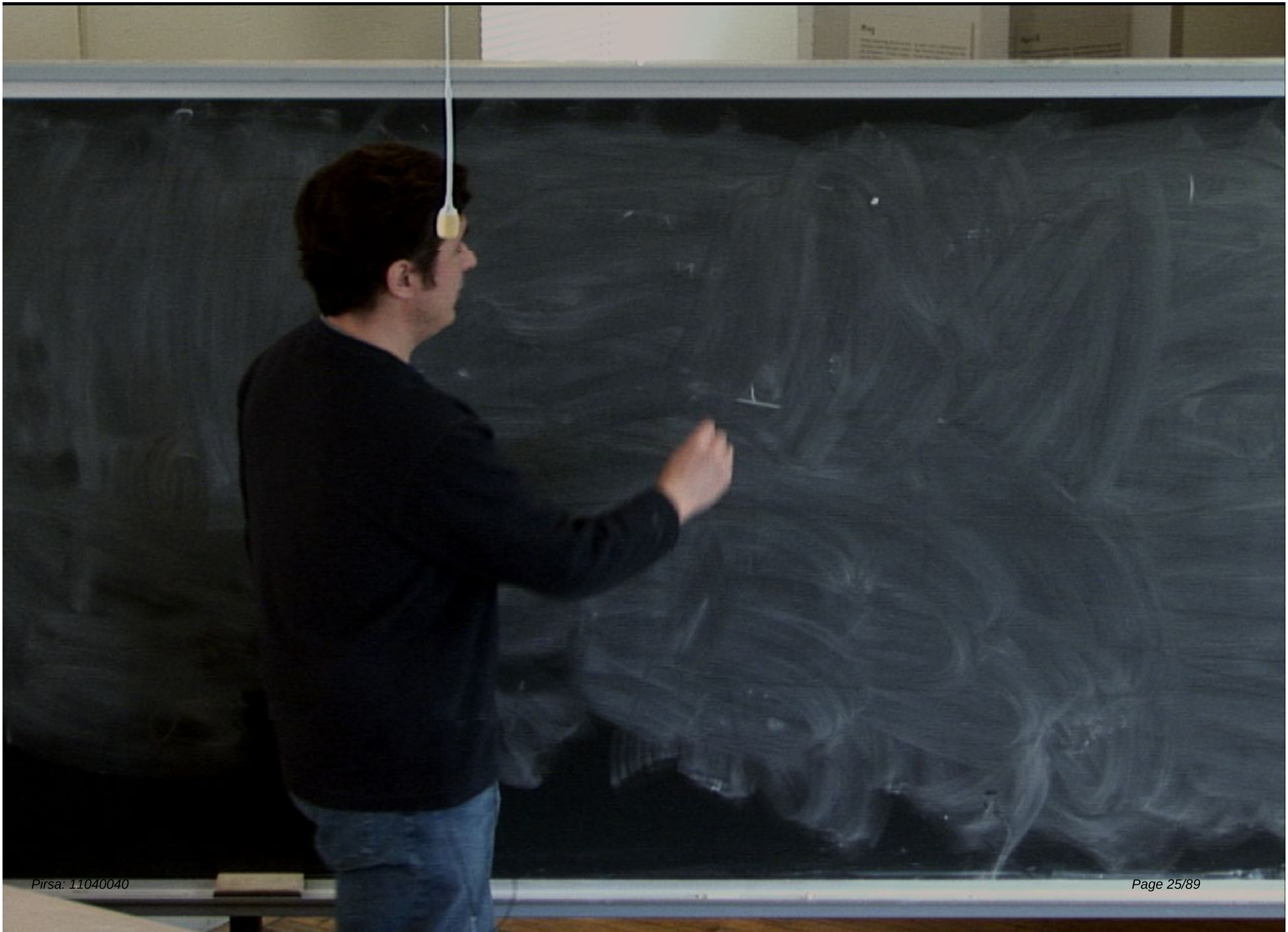
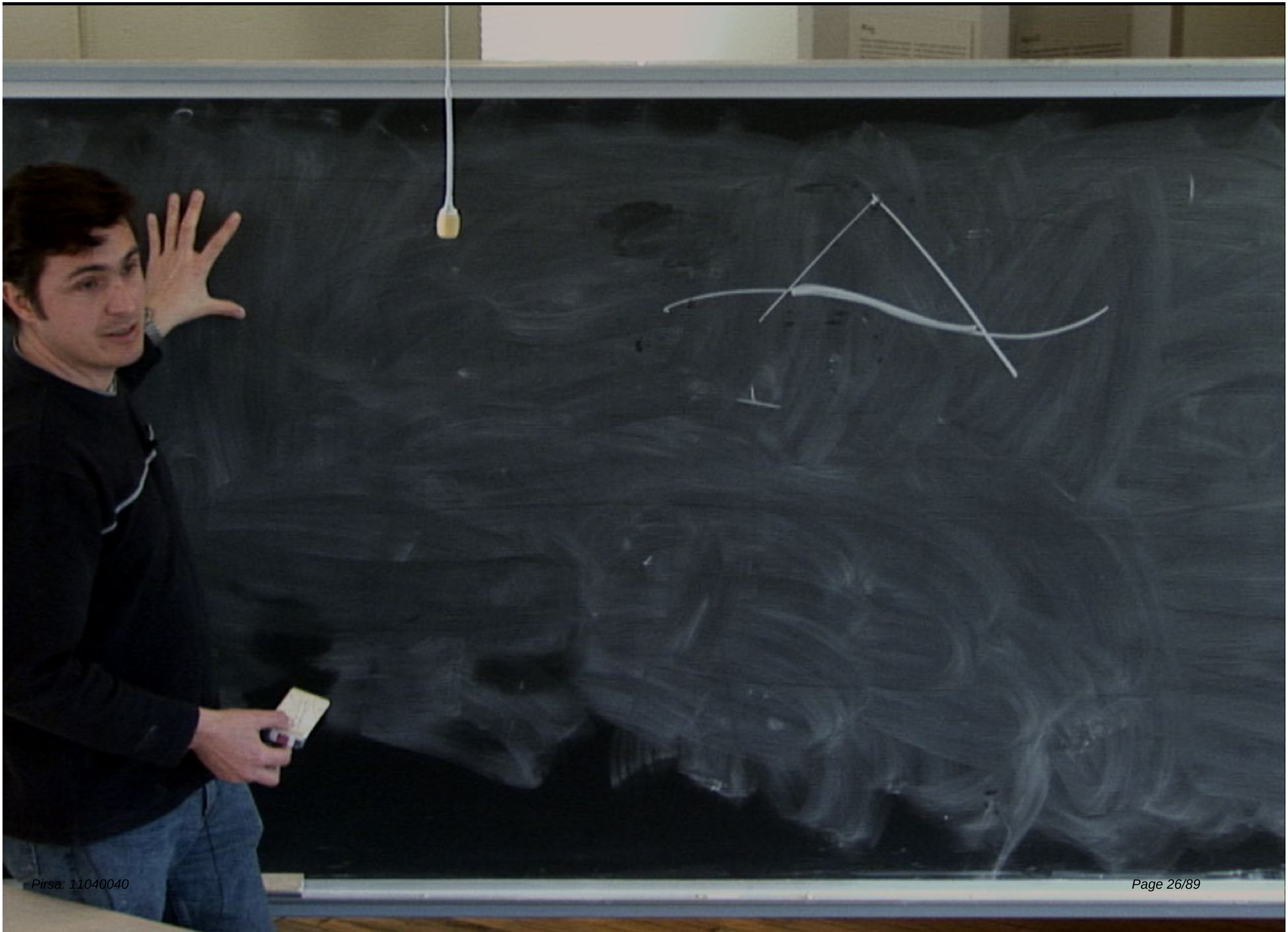
Why Finite Differencing?

- Accessibility: Requires a minimum of mathematical background: if you're mathematically mature enough to understand the nature of the PDEs you need to solve, you're mathematically mature enough to understand finite differencing
- Flexibility: Technique can be used for essentially any system of PDEs that has smooth solutions, irrespective of
  - Number of dependent variables (unknown functions)
  - Number of independent variables (a.k.a. "dimensionality" of the system: nomenclature "1-D" means dependence on one spatial dimension plus time, "2-D", "3-D" similarly mean dependence on two/three dimensions, plus time, respectively)
  - Nonlinearity
  - Form of equations: technique does not require that the system of equations has any particular/special form (contrast with finite volume methods where one generally wants to cast the equations in so-called conservation-law form)





1. Mathematical Formulation

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The 1-D Wave Equation

- Consider the following initial value (Cauchy) problem for the scalar function $\phi(t,x)$

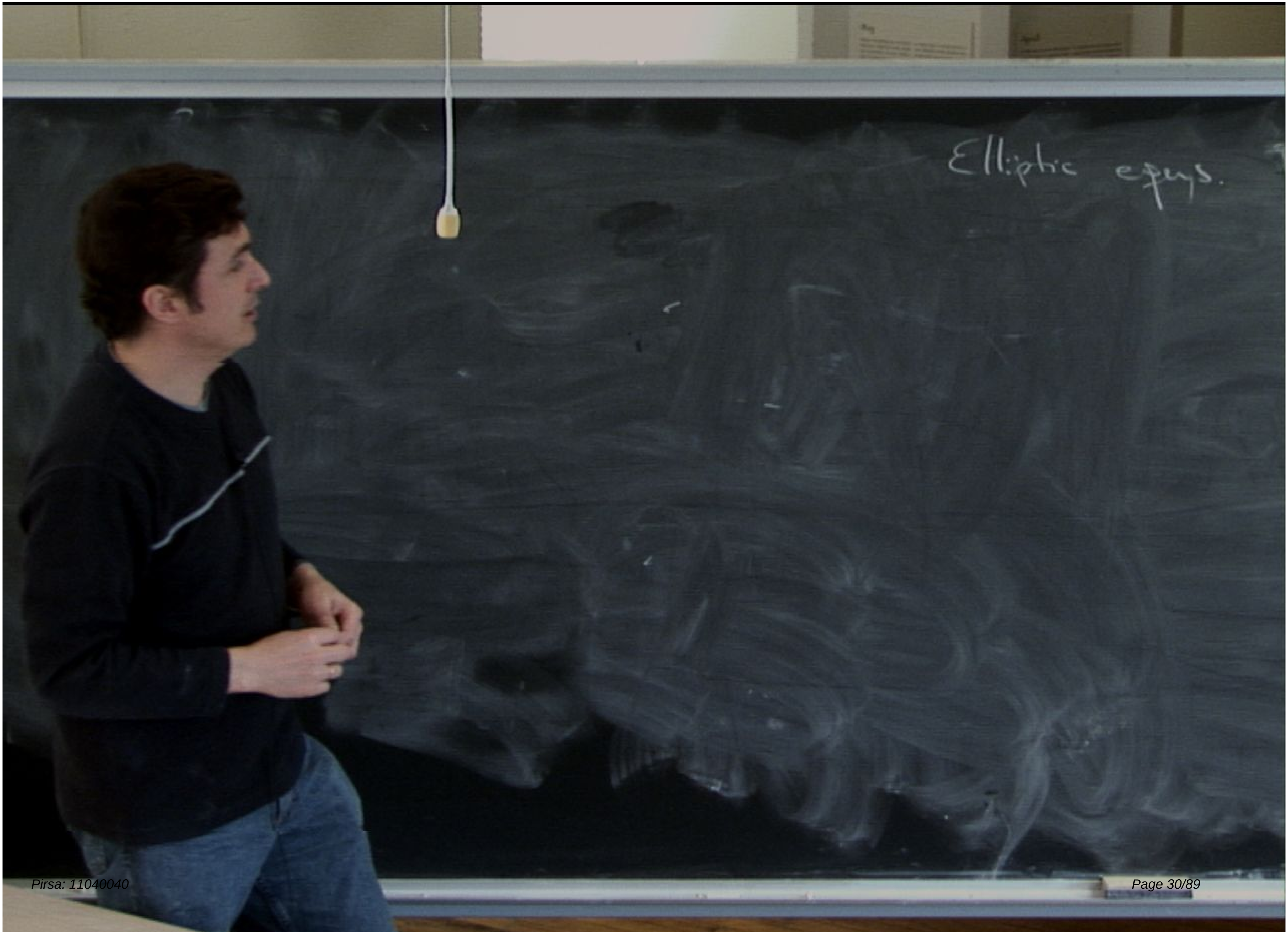
$$\phi_{tt} = c^2 \phi_{xx}\,, \quad -\infty \le x \le \infty\,, \quad t \ge 0 \qquad (1)$$

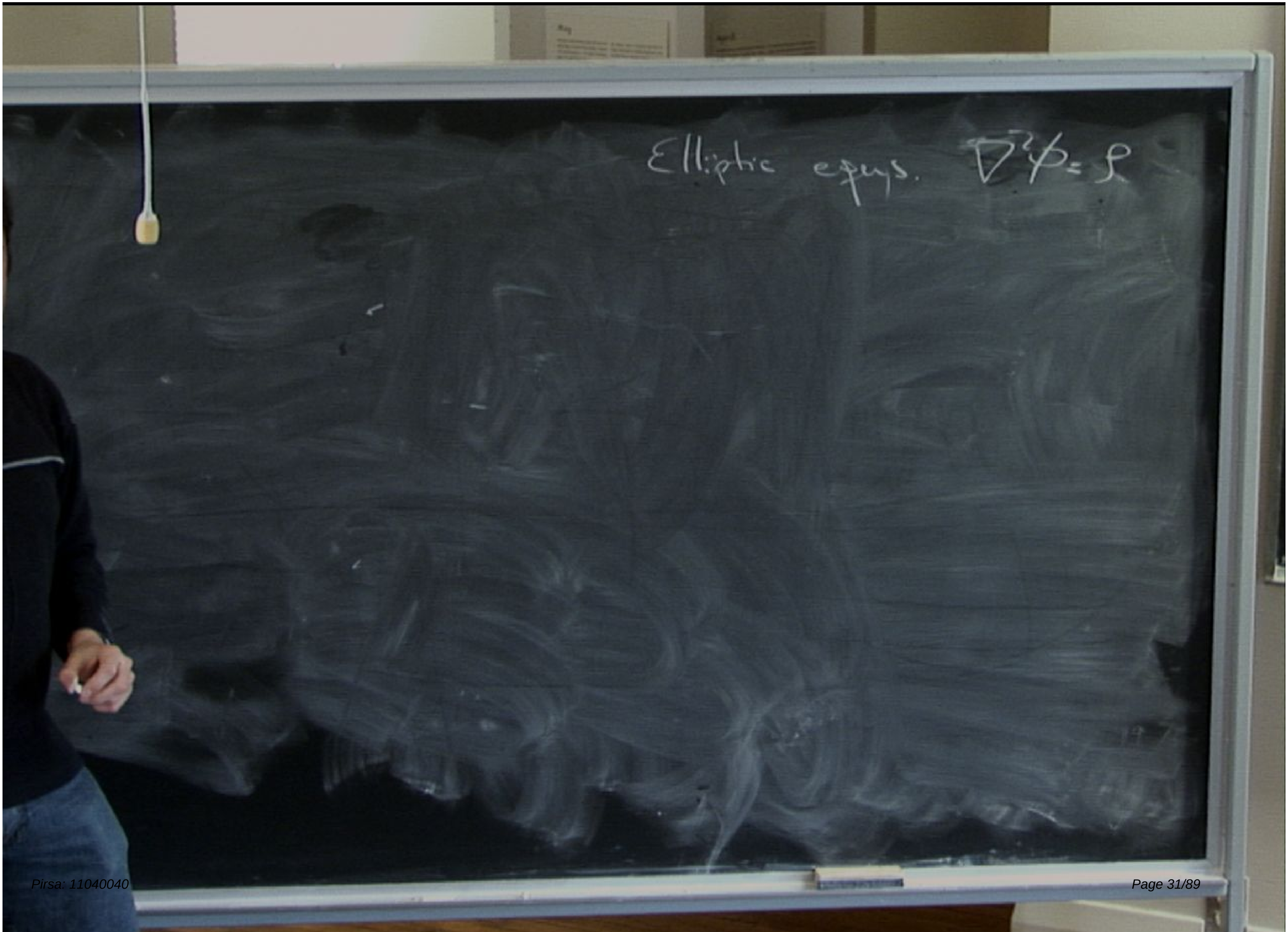
$$\phi(0,x) = \phi_0(x) \qquad (2)$$

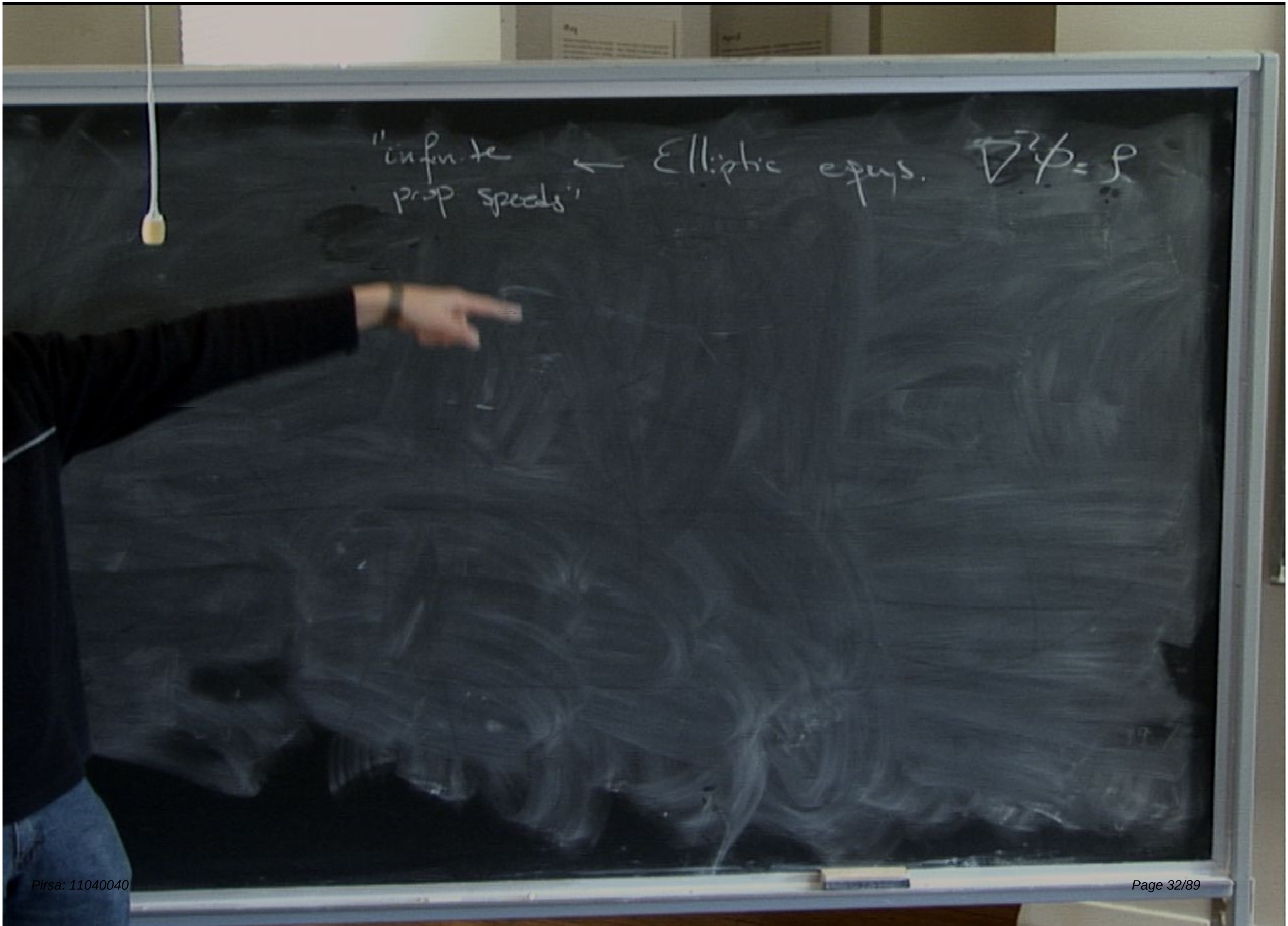
$$\phi_t(0,x) = \Pi_0(x) \qquad (3)$$

  where c is a positive constant, we have adopted the subscript notation for partial differentiation, e.g. $\phi_{tt} \equiv \partial^2\phi/\partial t^2$, and we wish to determine $\phi(t,x)$ in the solution domain from the initial conditions (2–3) and the governing equation (1)

- Note the following:
  - Since the spatial domain is unbounded, there are *no* boundary conditions
  - Since the equation is second order in time, two functions-worth of initial data must be specified: the initial scalar field profile, $\phi_0(x)$, and the initial time derivative, $\Pi_0(x)$
  - This system is well posed, and if the initial conditions $\phi_0(x)$ and $\Pi_0(x)$ are smooth—which we will hereafter assume—so is the complete solution $\phi(t,x)$


Elliptic eqns.


Elliptic eqns. ∇²φ = ρ


"infinite
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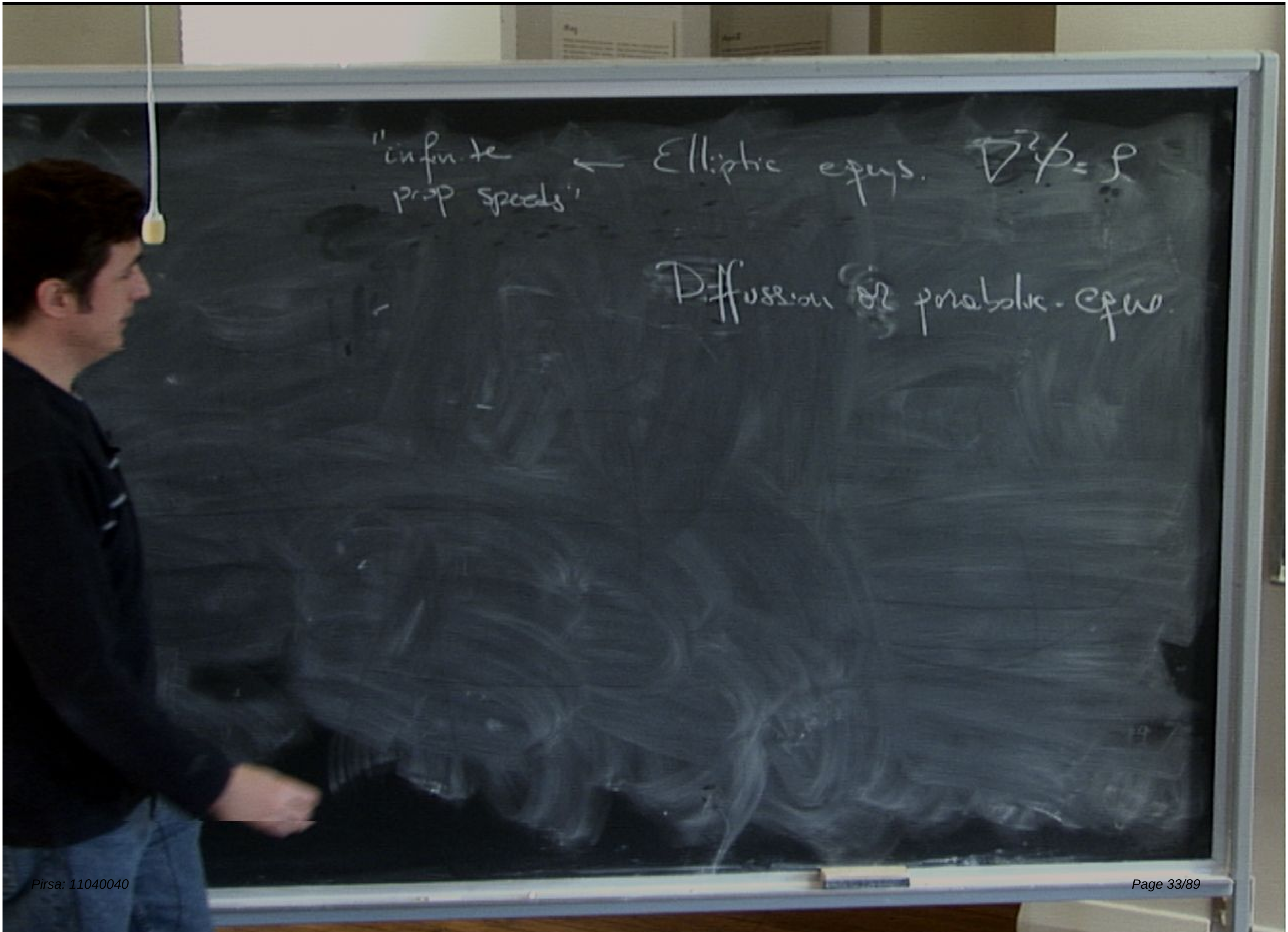
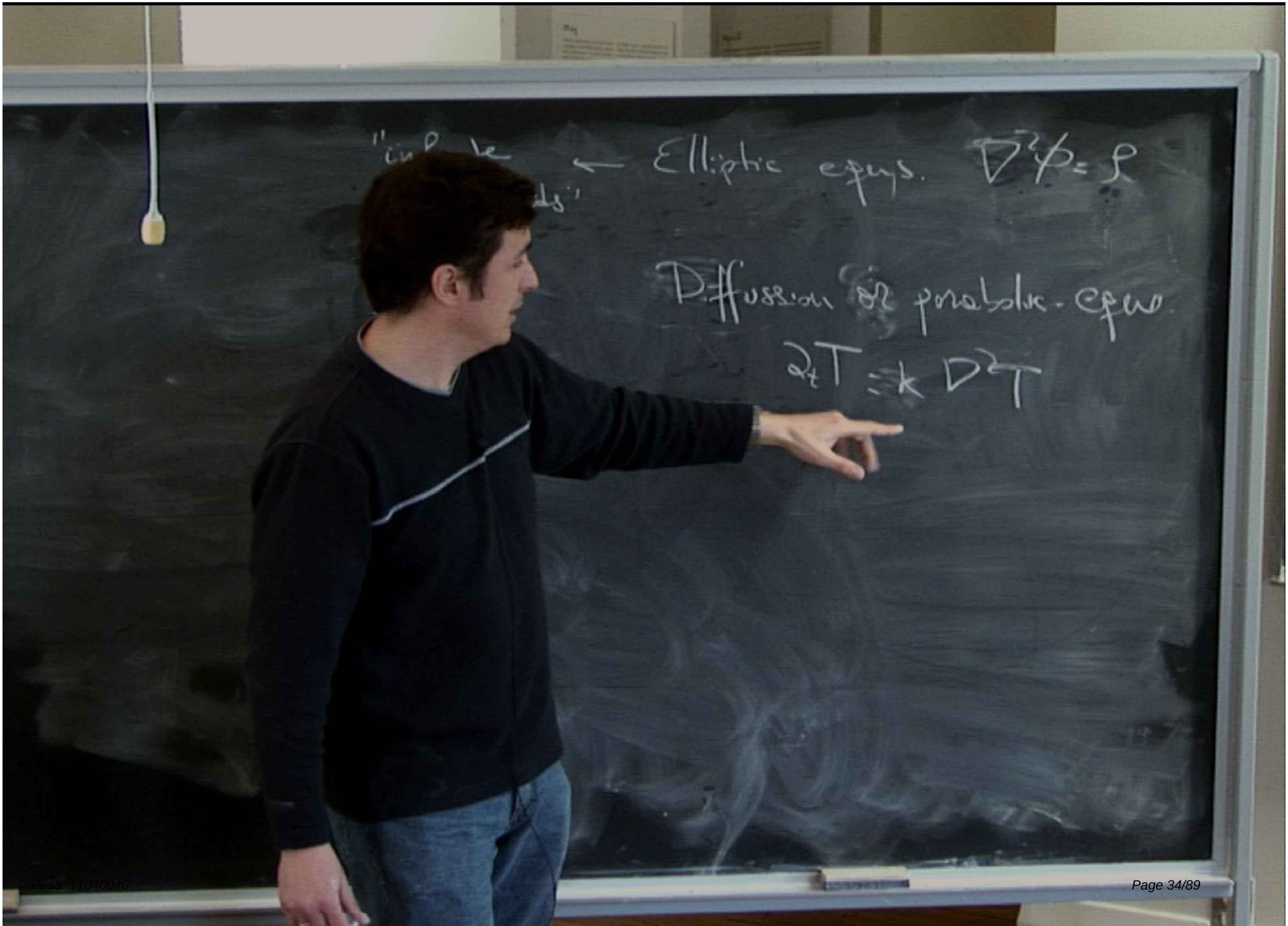
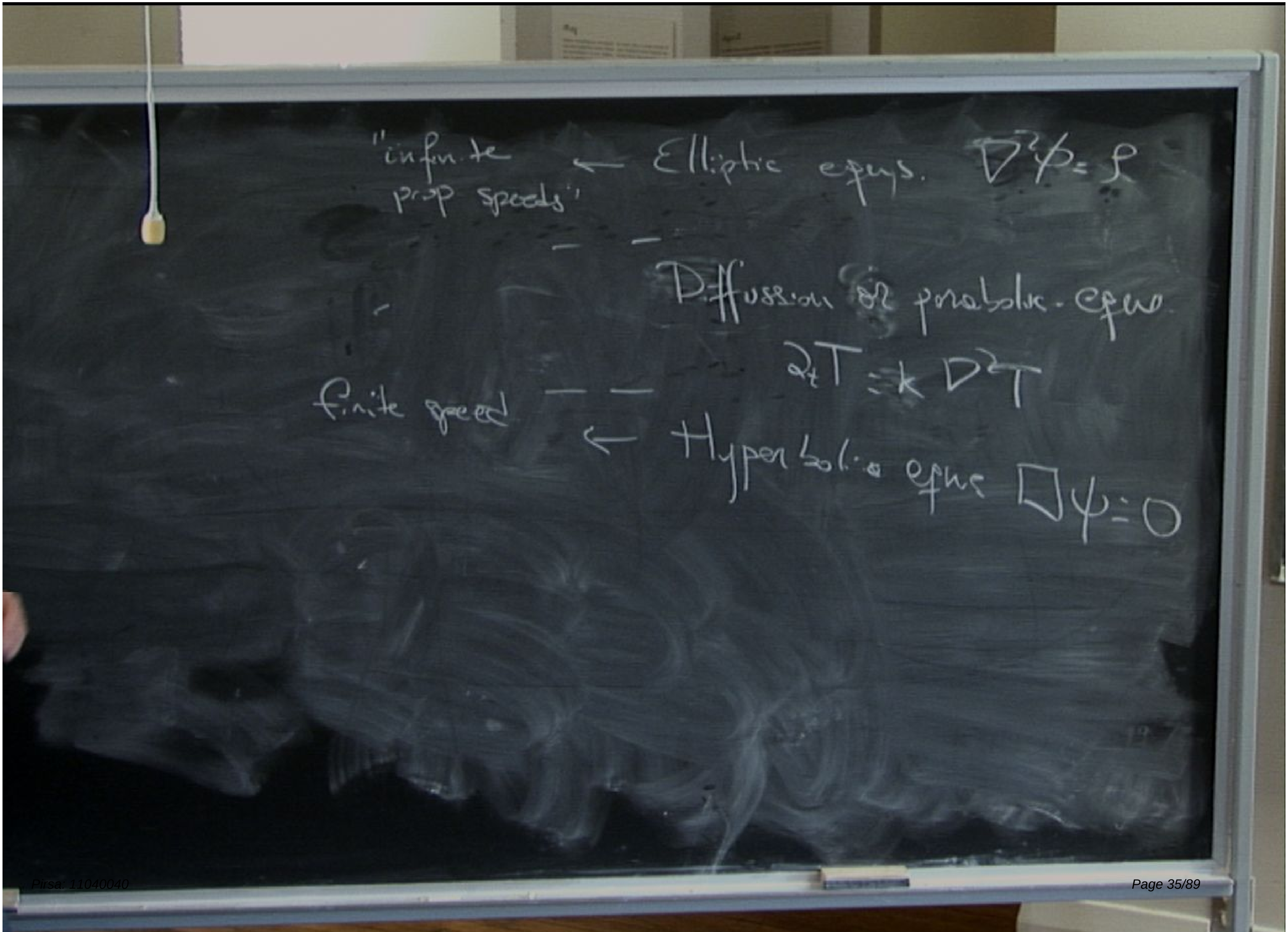
← Elliptic eqns. ∇²φ = ρ


"infinite
prop speeds" ← Elliptic eqns. $\nabla^2 \phi = \rho$
Diffusion or parabolic eqns.


Elliptic eqns.
$\nabla^2 \phi = \rho$
Diffusion
$\partial_t T = k \nabla^2 T$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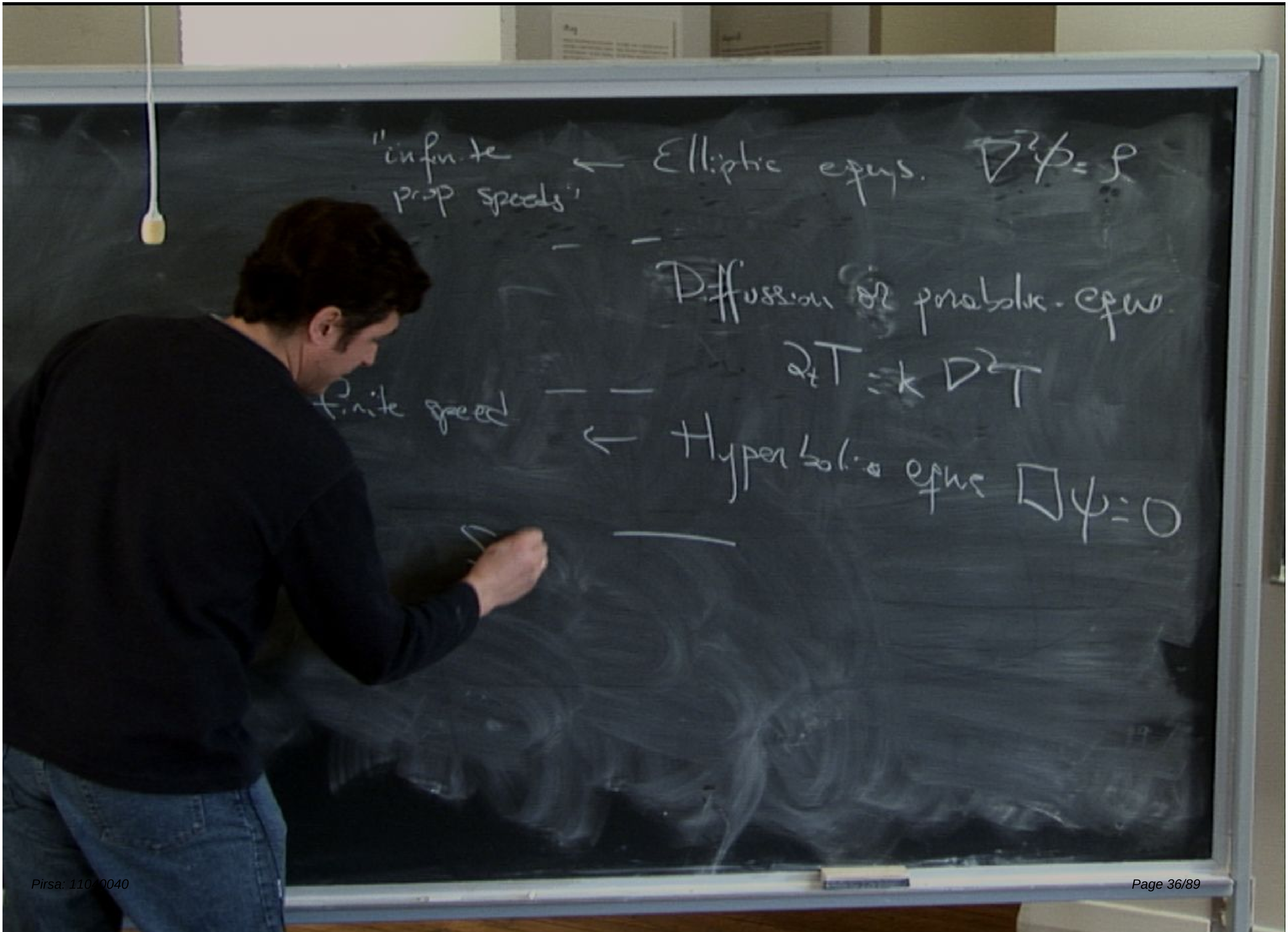
Diffusion or parabolic eqns.
$\partial_t T = k \nabla^2 T$
finite speed ← Hyperbolic eqns $\Box\psi = 0$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13 (14 of 83) 76.6% Find

The 1-D Wave Equation

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The 1-D Wave Equation

- Eqn. (1) is a *hyperbolic* PDE, and as such, its solutions generically describe the propagation of disturbances at some finite speed(s), which in this case is c

- Without loss of generality, we can assume that we have adopted units in which this speed satisfies $c = 1$. Our problem then becomes

$$\phi_{tt} = \phi_{xx}\,, \quad -\infty \le x \le \infty\,, \quad t \ge 0 \tag{4}$$

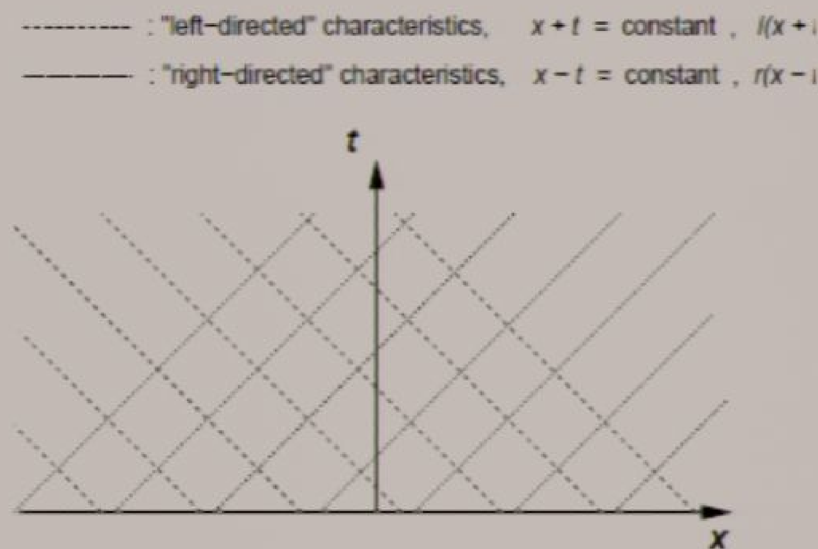
$$\phi(0,x) = \phi_0(x) \tag{5}$$

$$\phi_t(0,x) = \Pi_0(x) \tag{6}$$

- In the study of the solutions of hyperbolic PDEs, using either closed form (preferred to "analytic") or numerical approaches, the concept of characteristic is crucial

- Loosely, in a spacetime diagram, characteristics are the lines/surfaces along which information/signals propagate(s).

The 1-D Wave Equation



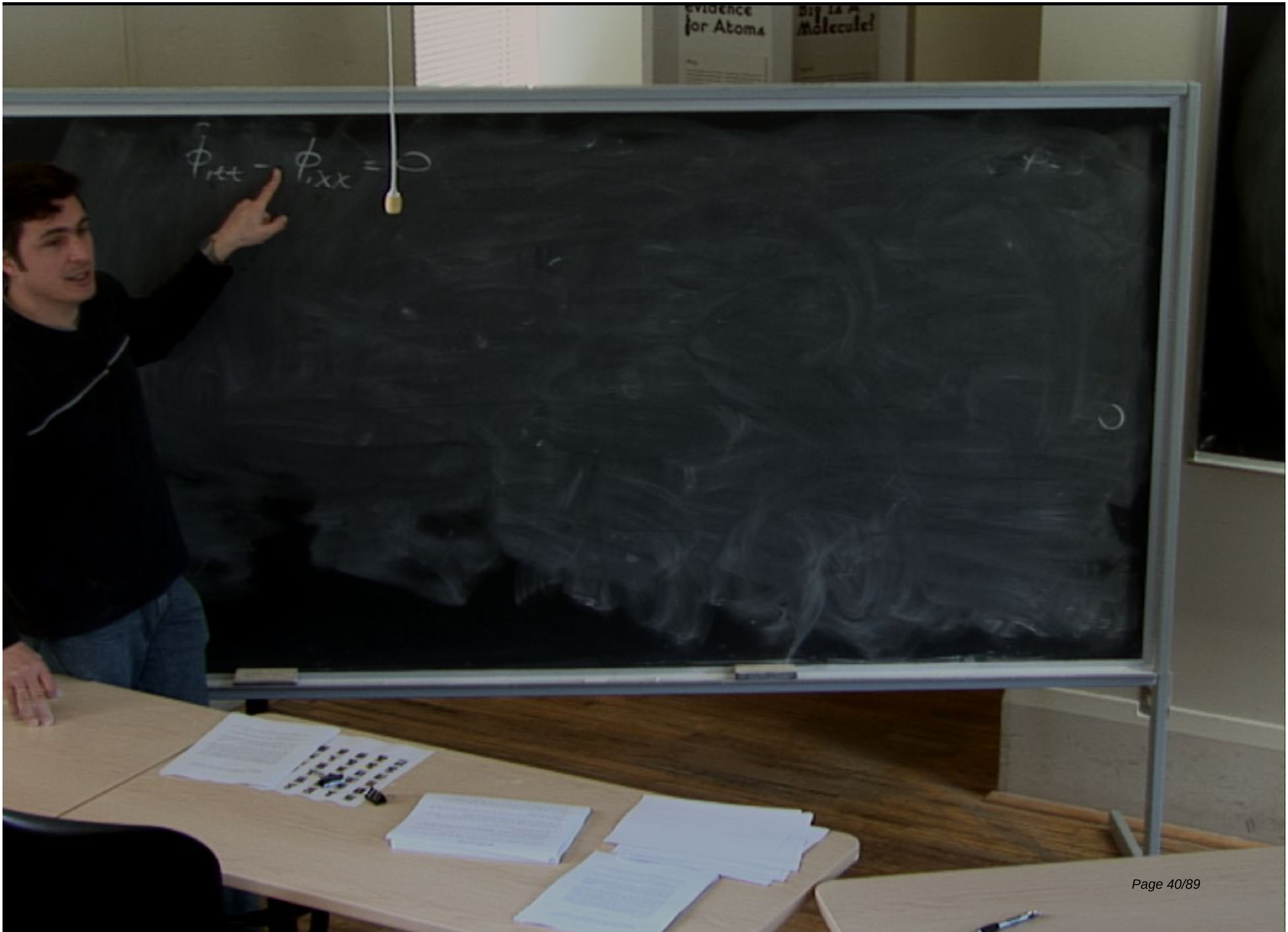
- General solution of (4) is a superposition of an arbitrary *left-moving* profile ($v = -c = -1$), and an arbitrary *right-moving* profile ($v = +c = +1$); i.e.

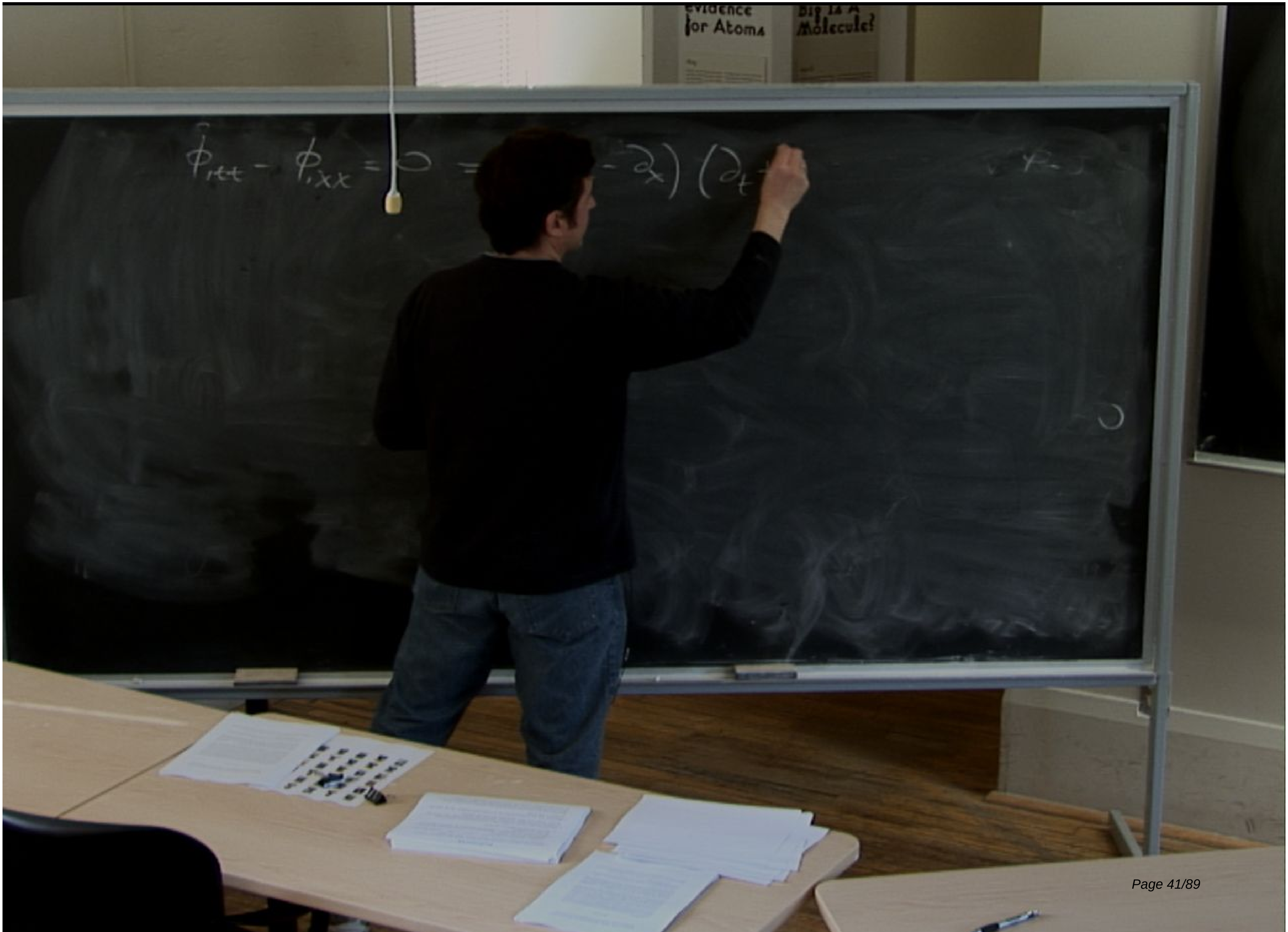
$$\phi(t, x) = \ell(x + t) + r(x - t) \qquad (7)$$

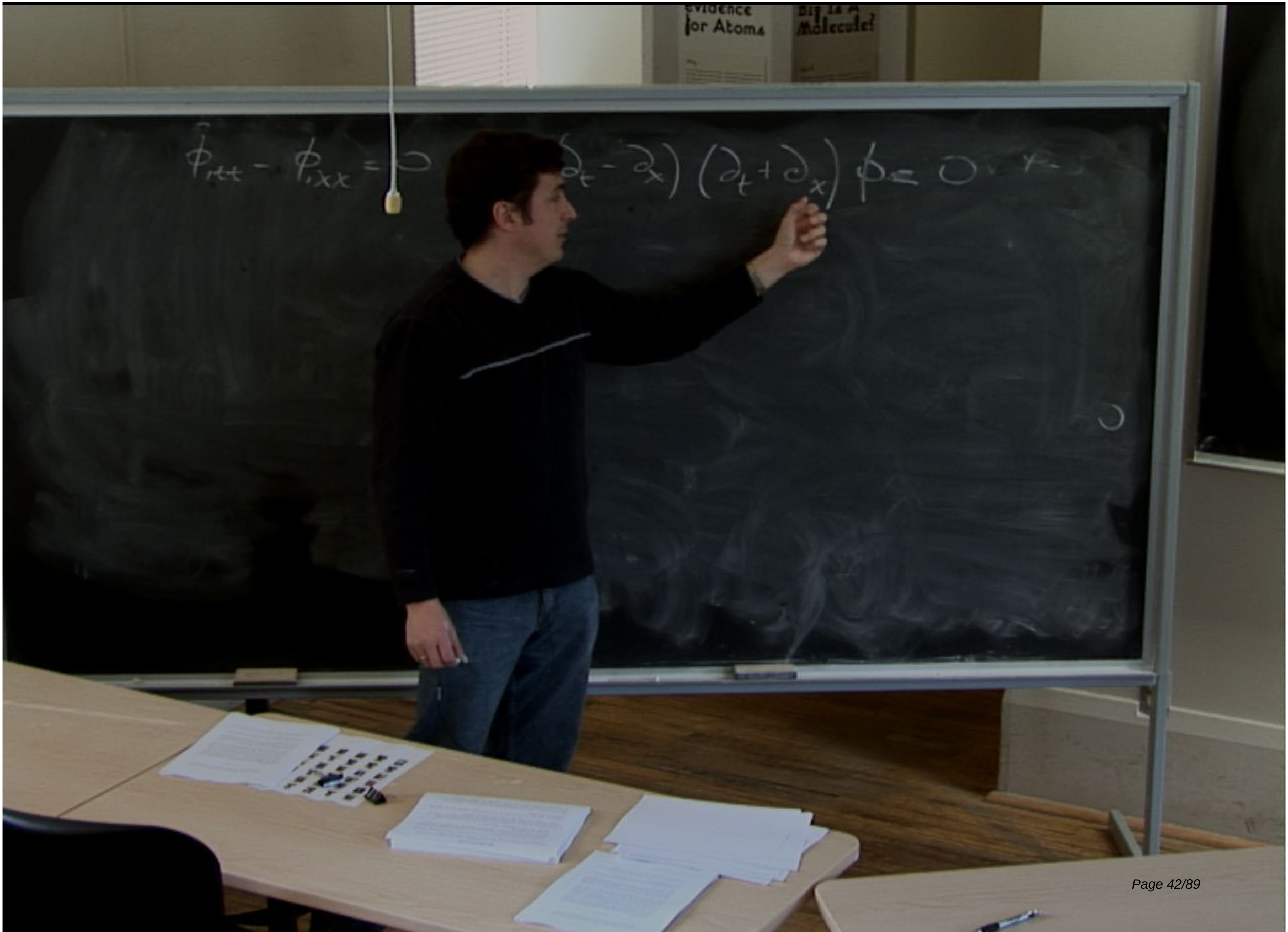
where

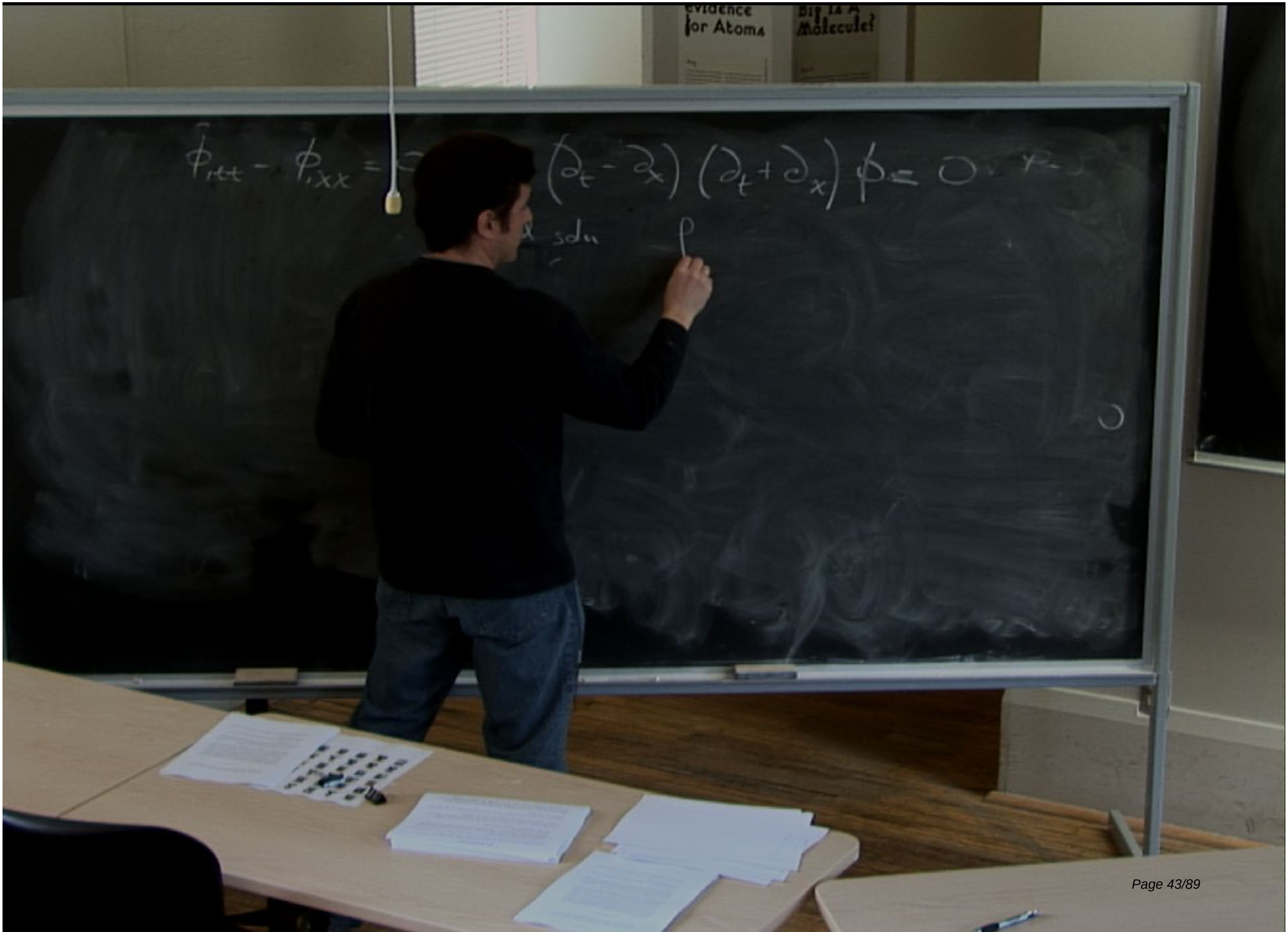
ℓ : constant along "left-directed" characteristics

r : constant along "right-directed" characteristics


for Atoms
$\phi_{,tt} - \phi_{,xx} = 0$


for Atoms
Molecule?


for Atoms
Molecule!
φ,tt − φ,xx = 0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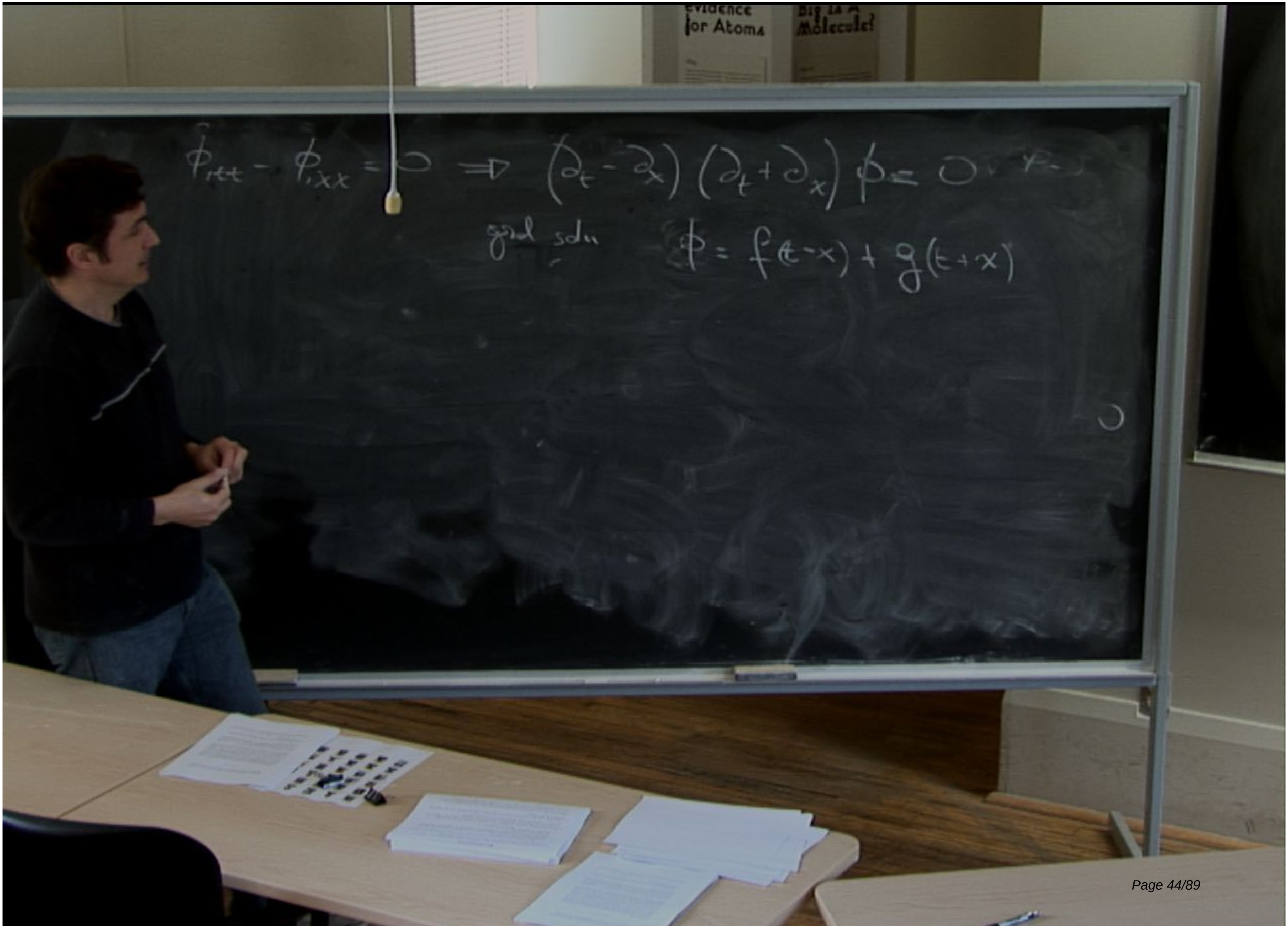
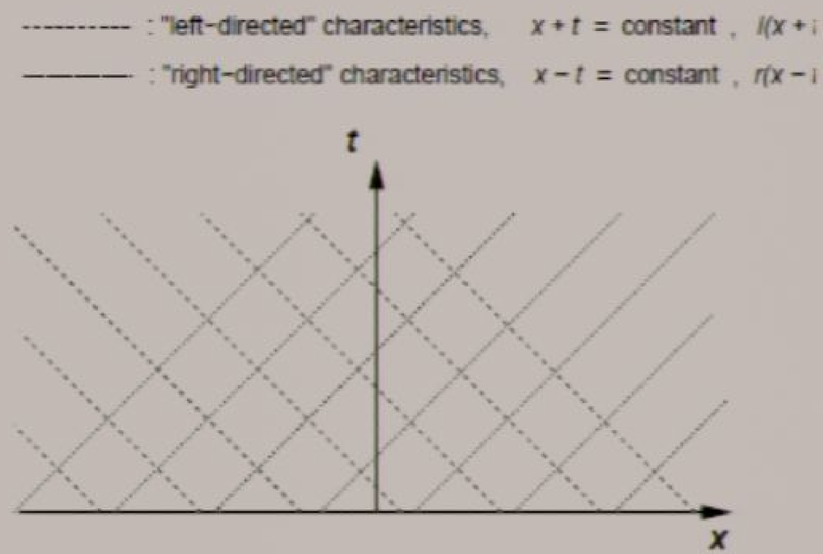


$\phi_{,tt} - \phi_{,xx} = 0 \Rightarrow (\partial_t - \partial_x)(\partial_t + \partial_x)\phi = 0$
gen'l soln $\phi = f(t-x) + g(t+x)$

The 1-D Wave Equation



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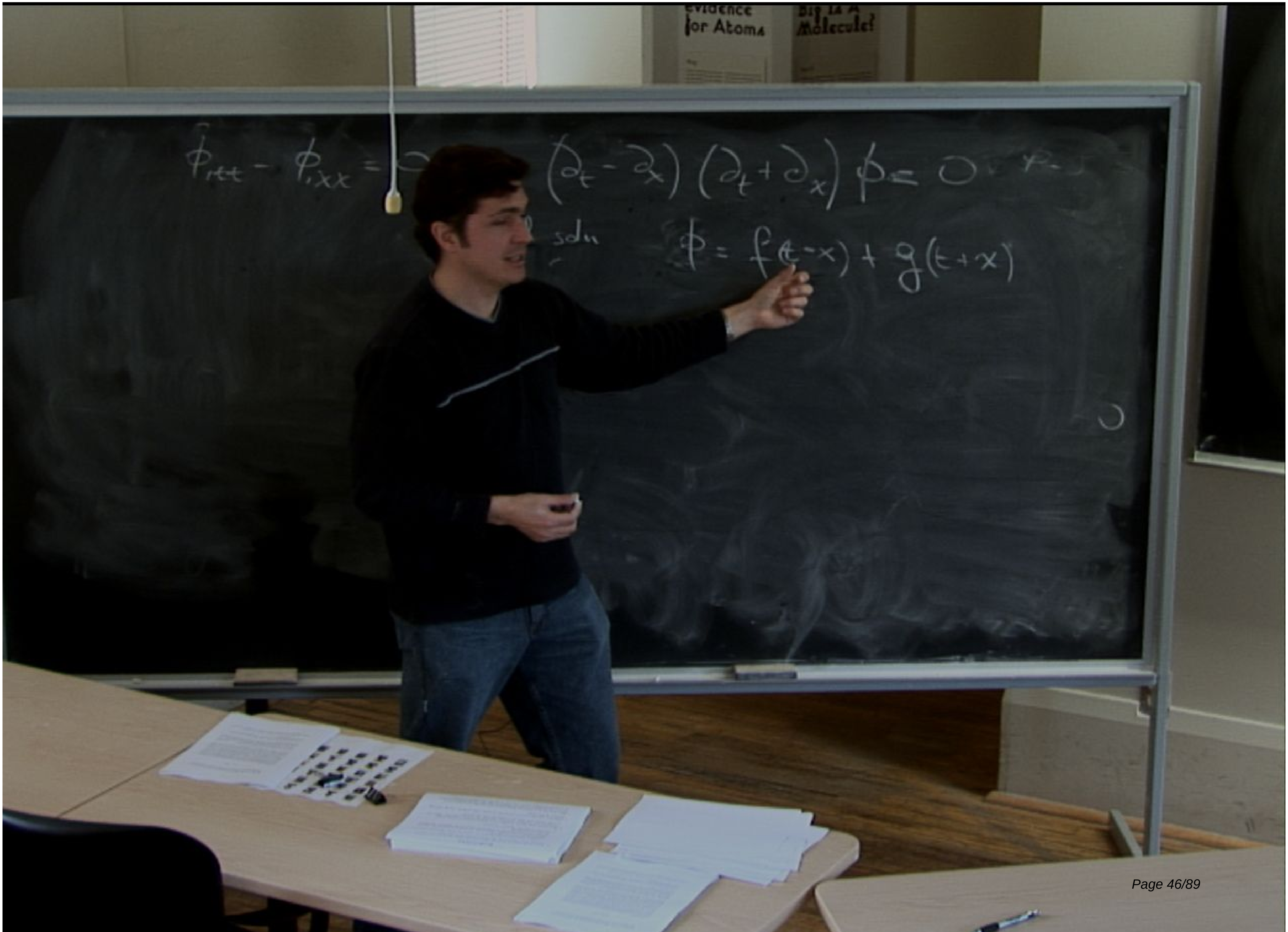
evidence
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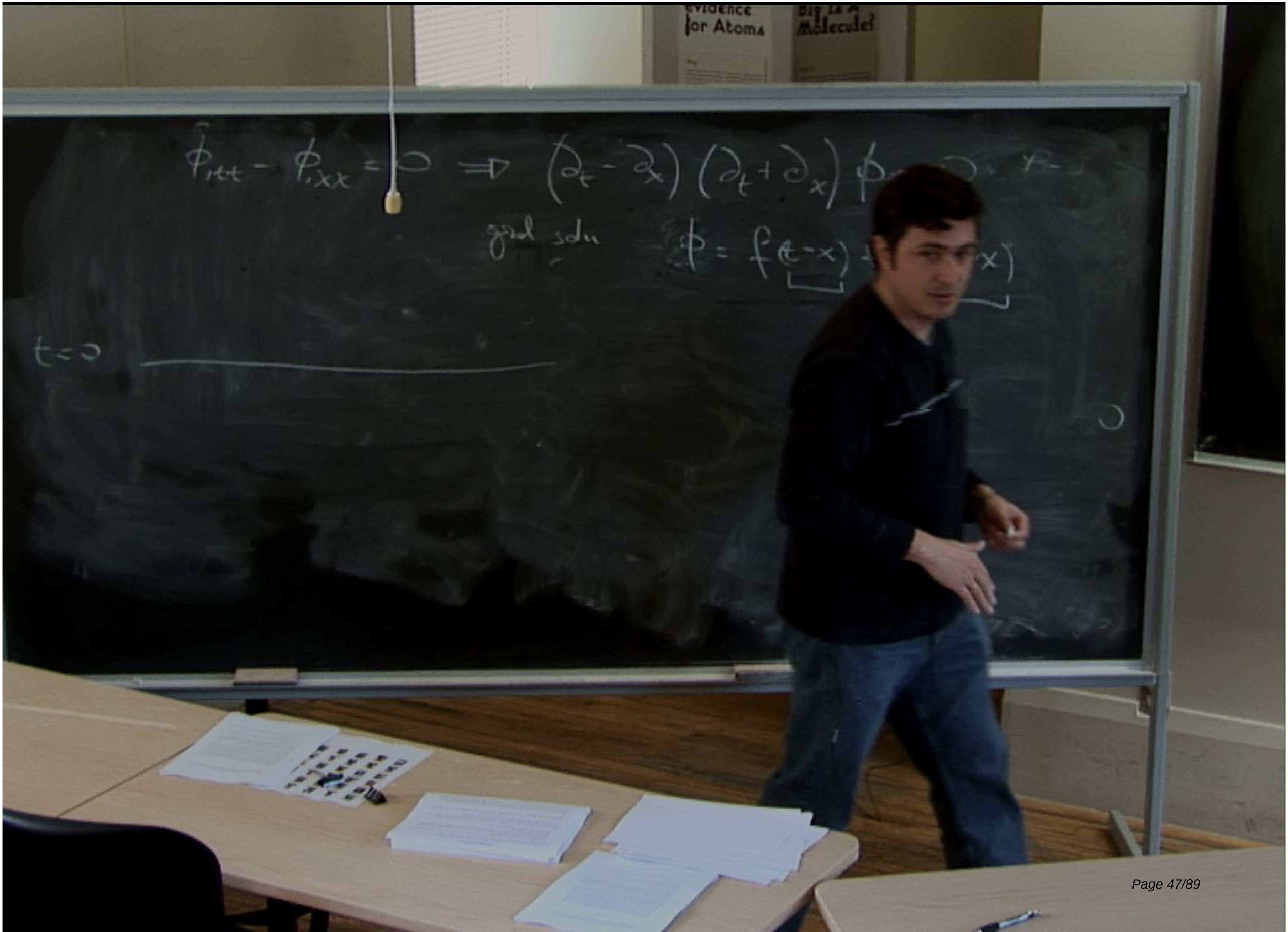
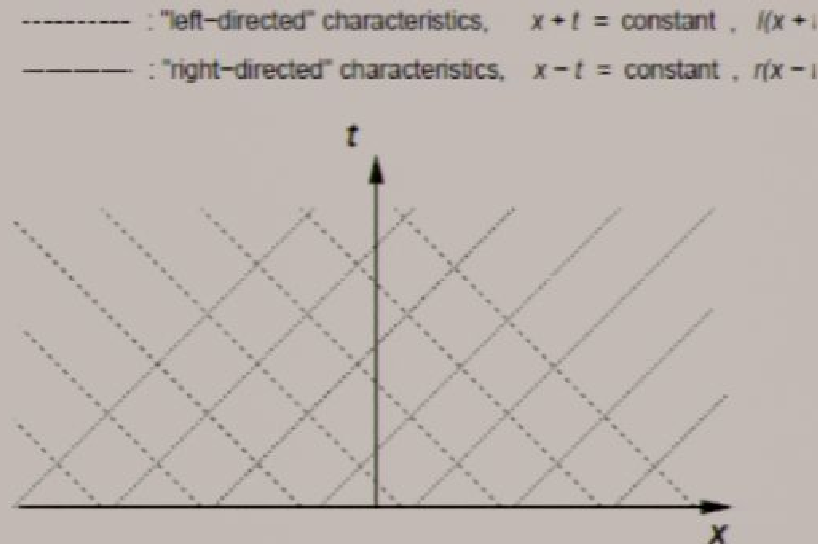


evidence
for Atoms
Molecule?
good soln
t=0

The 1-D Wave Equation



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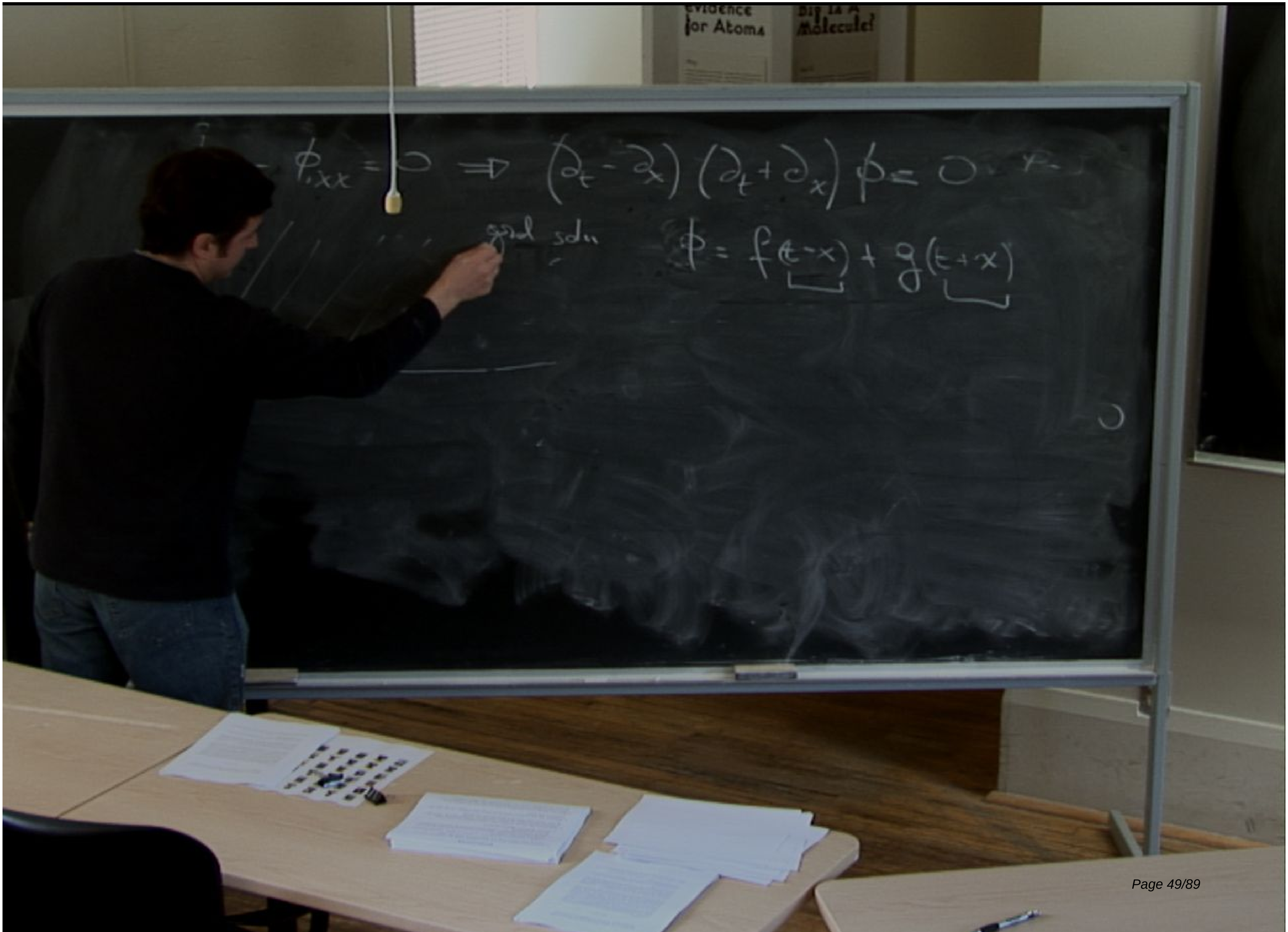
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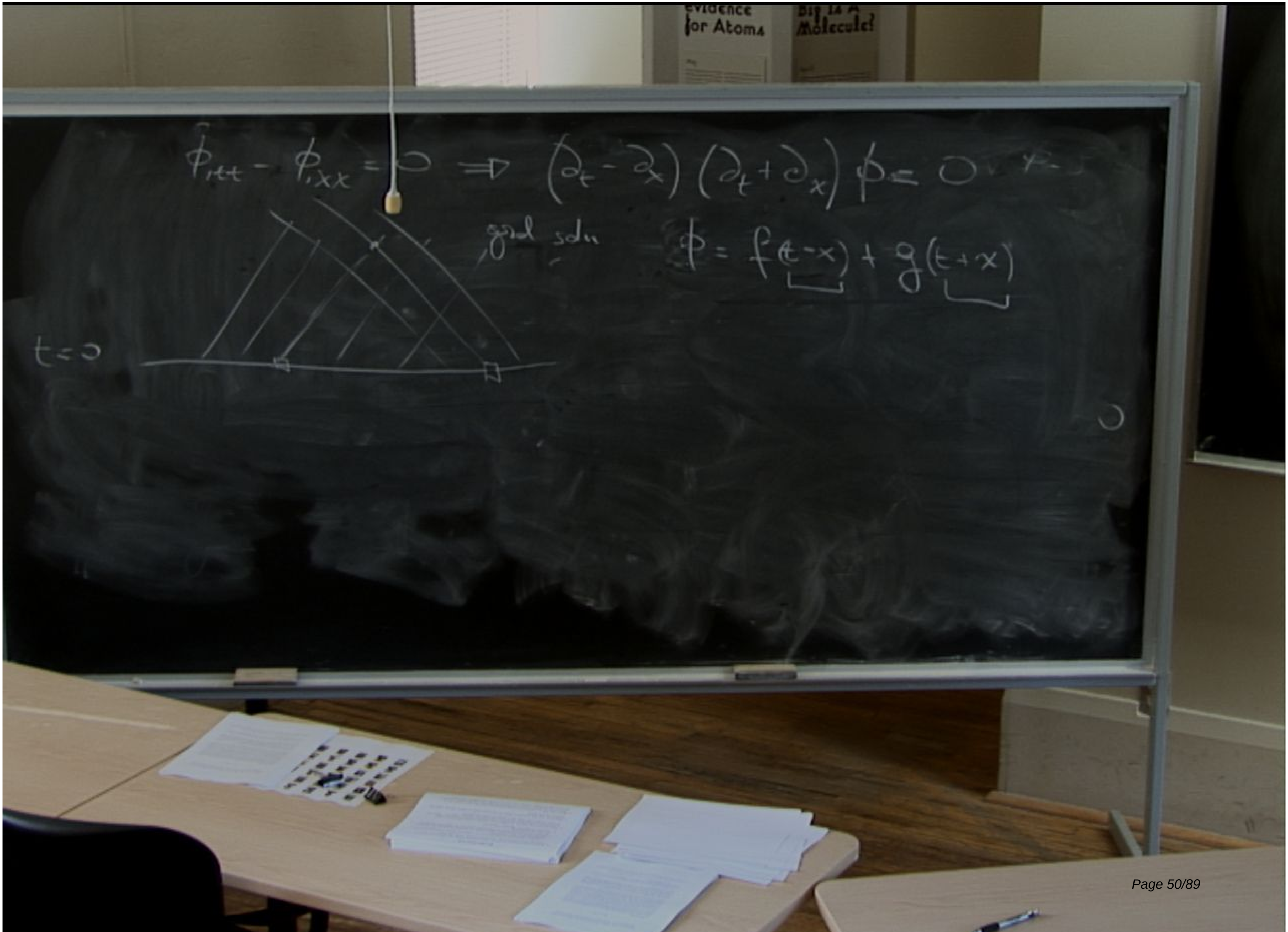
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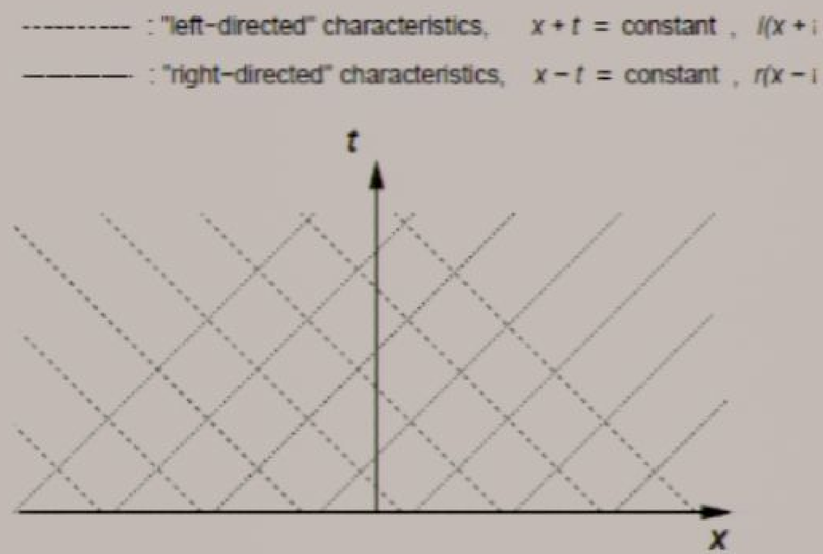
The 1-D Wave Equation



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The 1-D Wave Equation

- Observation provides alternative way of specifying initial values—often convenient in practice
- Rather than specifying $u(x,0)$ and $u_t(x,0)$ directly, specify *initial* left-moving and right-moving parts of the solution, $\ell(x)$ and $r(x)$
- Specifically, set

$$\phi(x,0) = \ell(x) + r(x) \quad (8)$$

$$\phi_t(x,0) = \ell'(x) - r'(x) \equiv \frac{d\ell}{dx}(x) - \frac{dr}{dx}(x) \quad (9)$$

- For illustrative purposes will frequently take profile functions $\phi_0(x)$, $\ell(x)$, $r(x)$ to be "gaussians", e.g.

$$\phi_0(x) = A \exp\left[-\left(\left(x - x_0\right)/\delta\right)^2\right] \quad (10)$$

where A, x_0 and δ are viewed as adjustable parameters that control the overall

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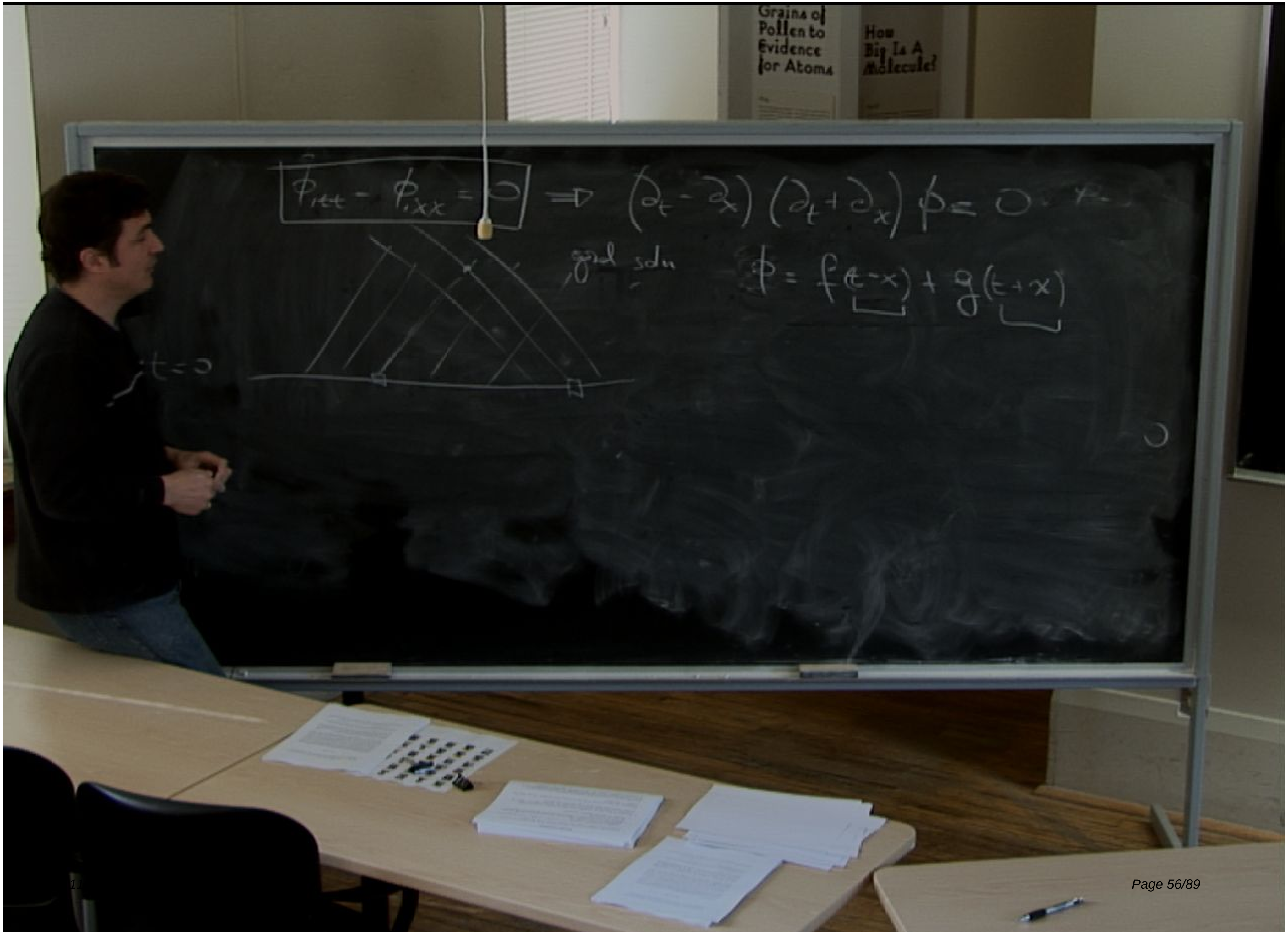
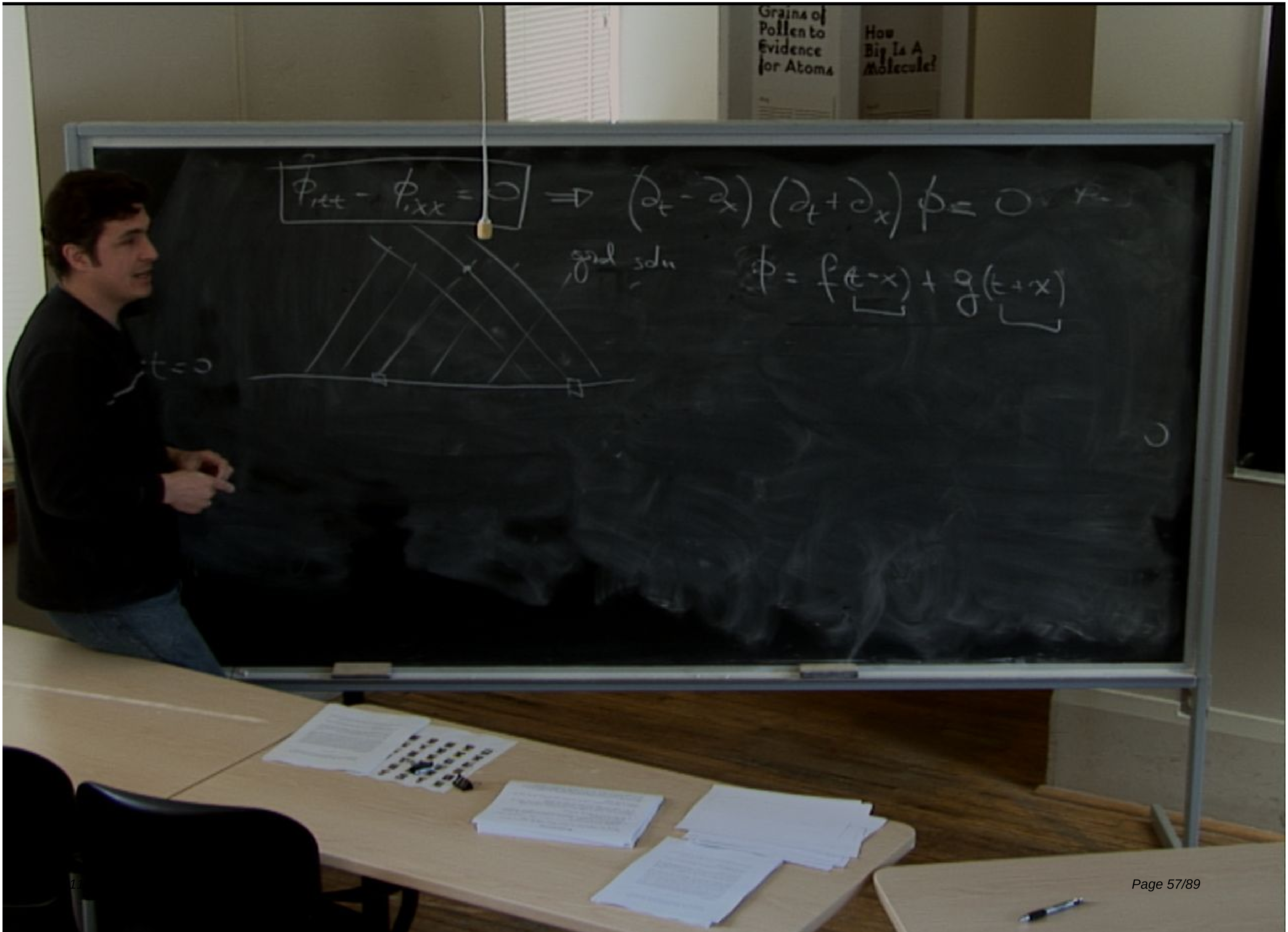
2. Discretization

Deriving Finite Difference Formulae

- Essence of finite-difference approximation of a PDE:
  - Replacement of the continuum by a discrete lattice of grid points
  - Replacement of derivatives/differential operators by finite-difference expressions
- Finite-difference expressions (finite-difference quotients) approximate the derivatives of functions at grid points, using the grid values themselves. All operators and expressions needed here can easily be worked out using Taylor series techniques.
- Example: Consider task of approximating the first derivative $u_x(x)$ of a function $u(x)$, given a discrete set of values $u_j \equiv u(jh)$


Grains of Pollen to Evidence for Atoms
How Big Is A Molecule?
$\phi_{,tt} - \phi_{,xx} = 0 \Rightarrow (\partial_t - \partial_x)(\partial_t + \partial_x)\phi = 0$
good soln
$\phi = f(t-x) + g(t+x)$
t=0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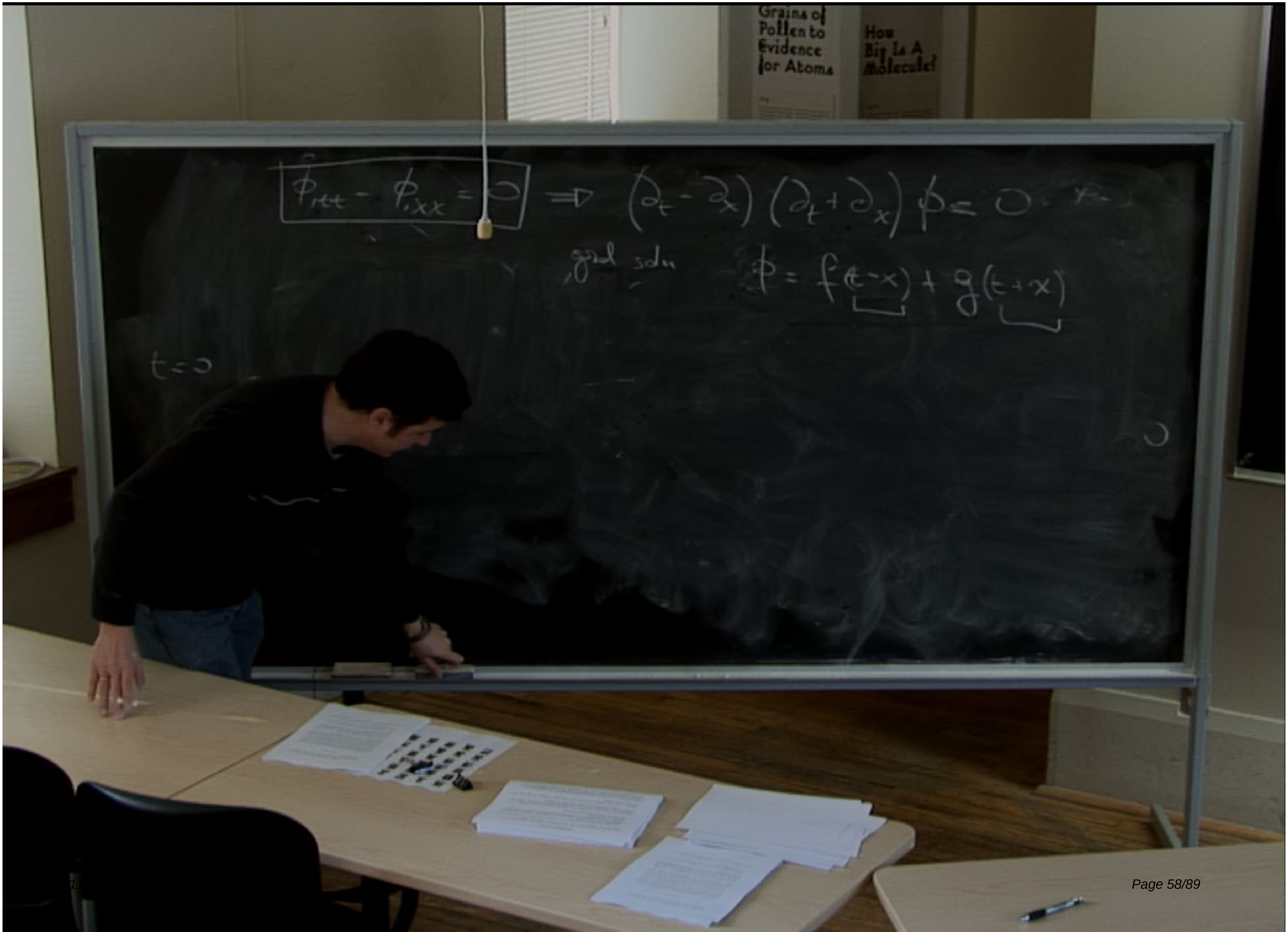
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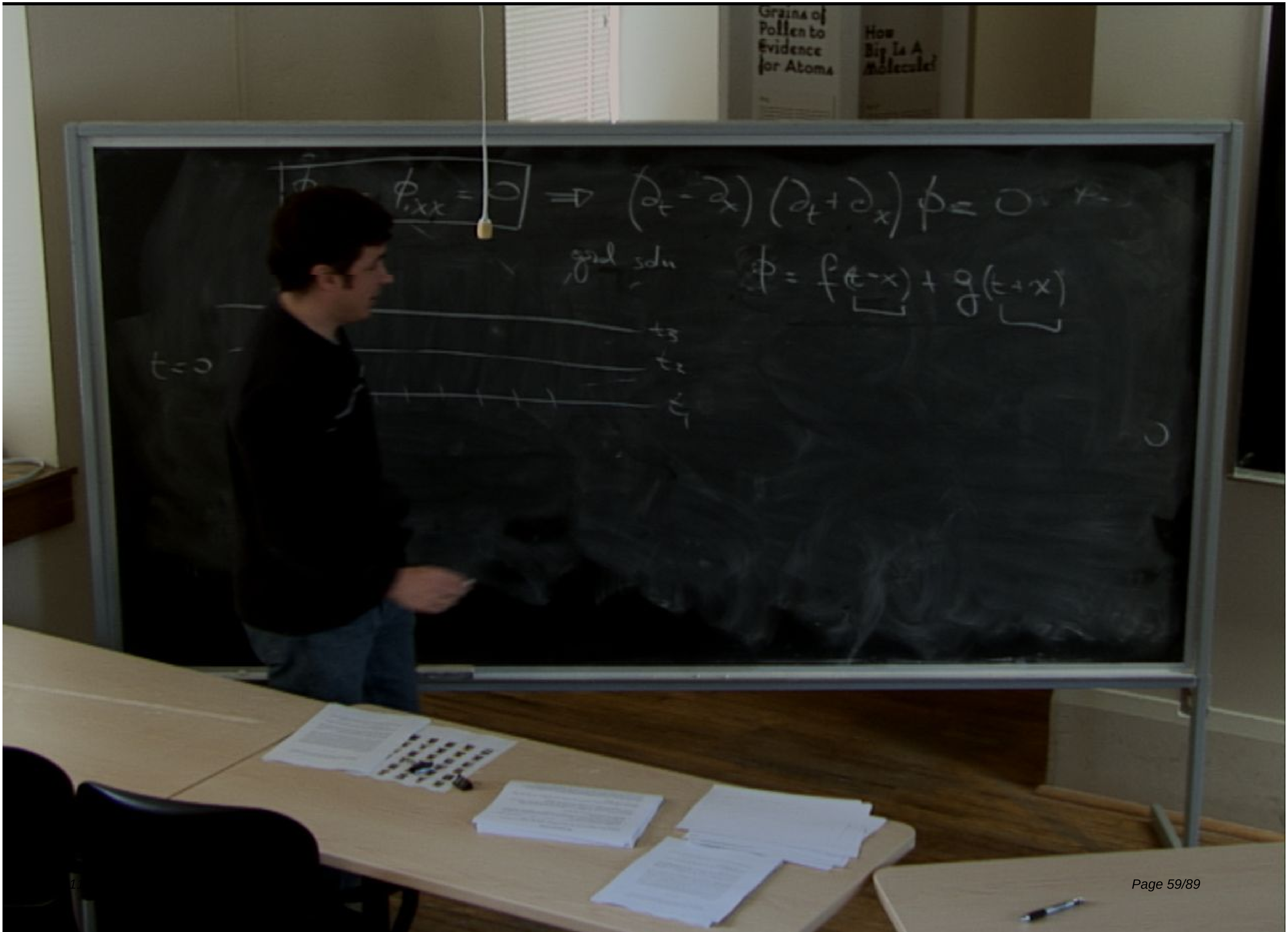
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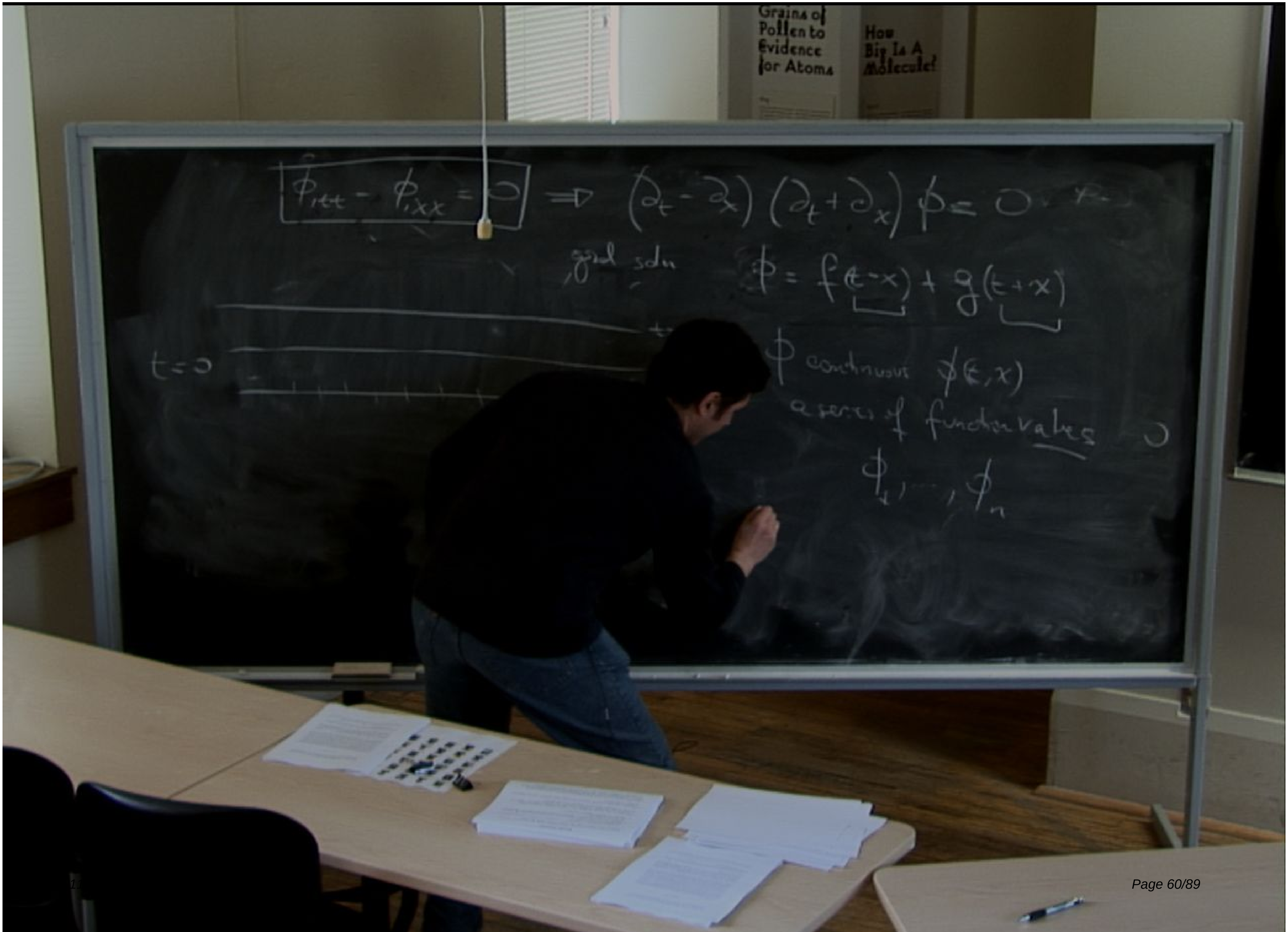
$t=0$
t_3
t_2
t_1


Grains of Pollen to Evidence for Atoms
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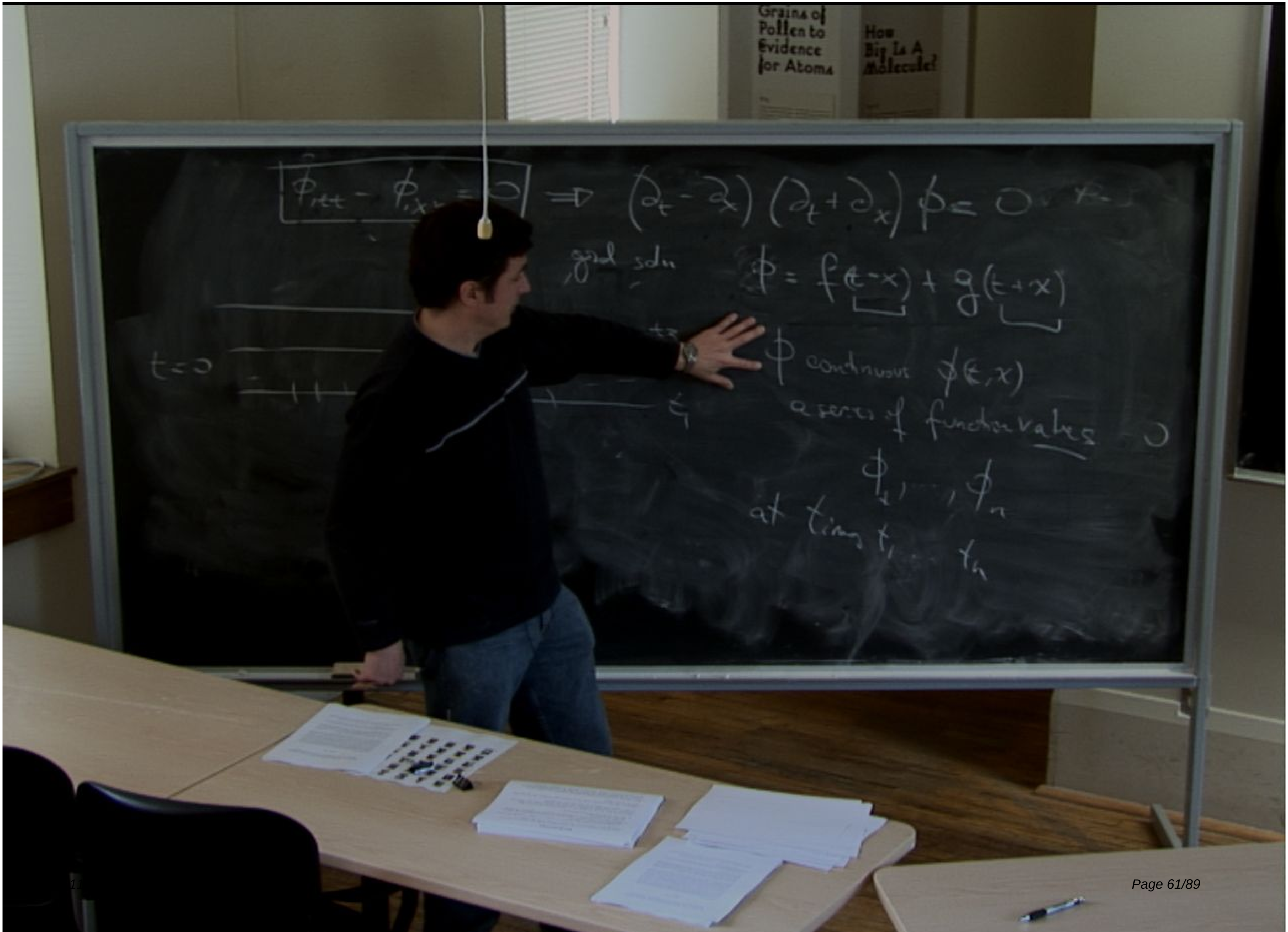
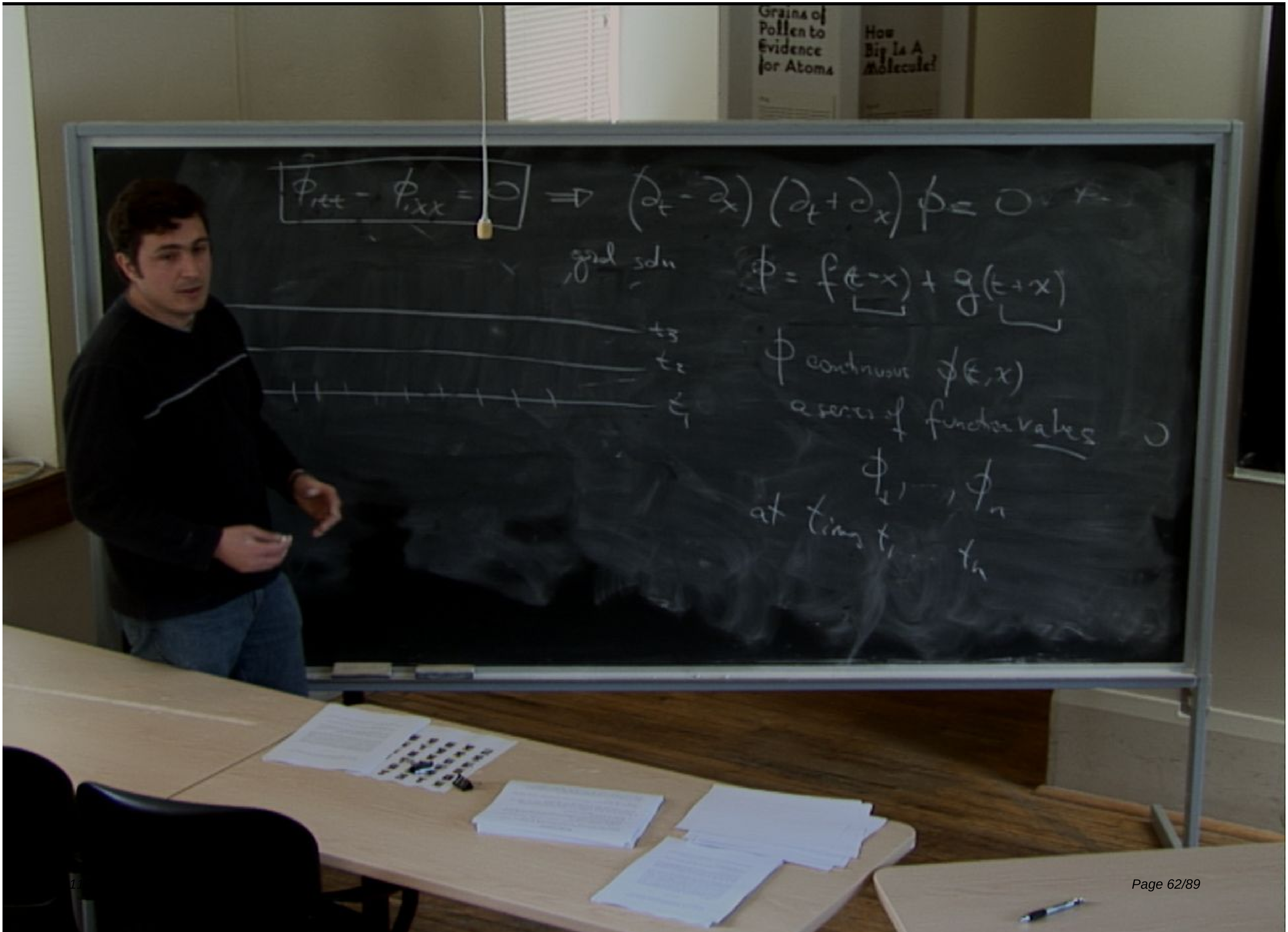
φ = f(t−x) + g(t+x)
t=0
φ continuous φ(t,x)
a series of function values
φ1, ..., φn


Grains of Pollen to Evidence for Atoms
How Big Is A Molecule?
continuous
a series of function values
at times


Grains of Pollen to Evidence for Atoms
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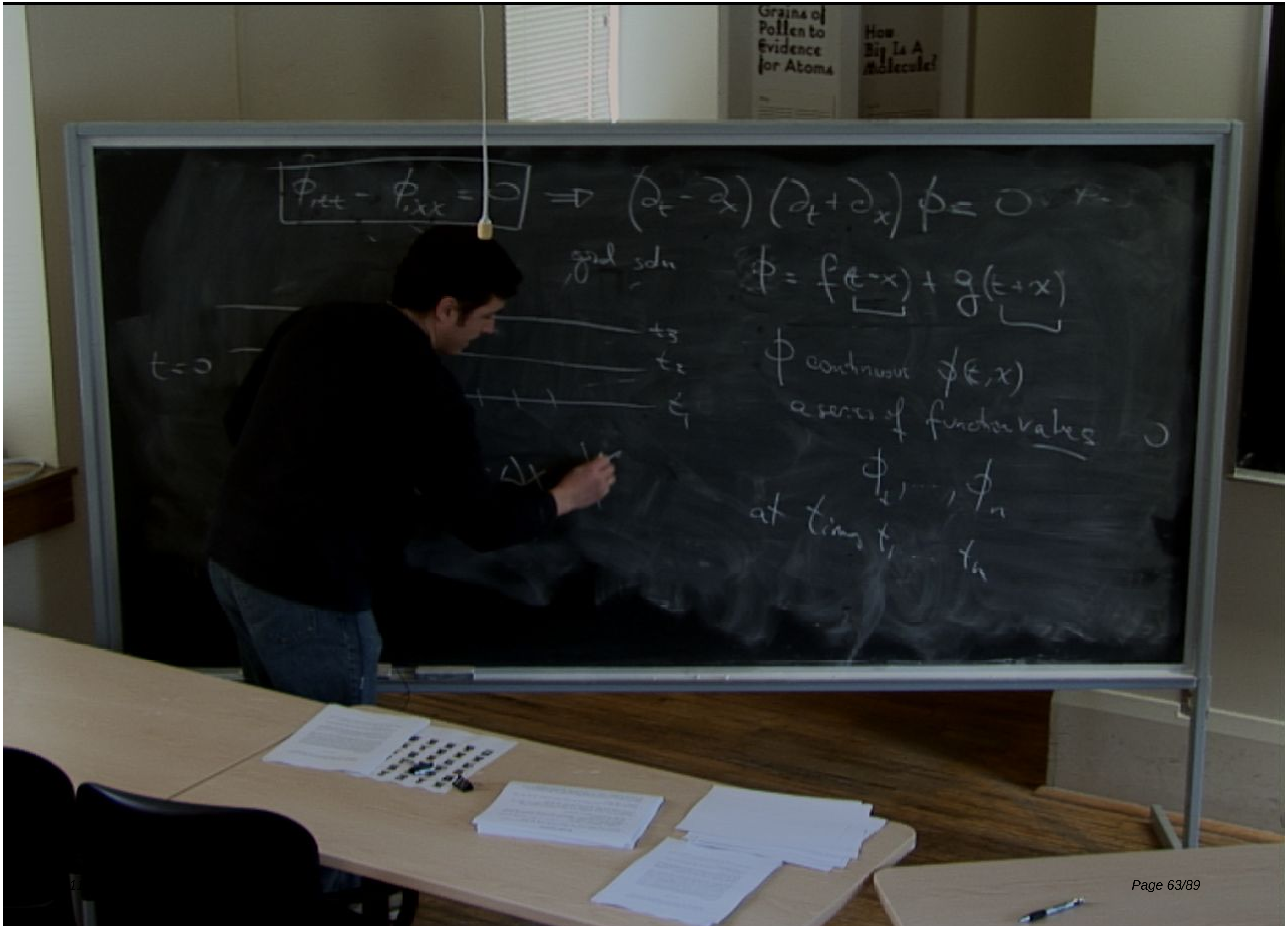
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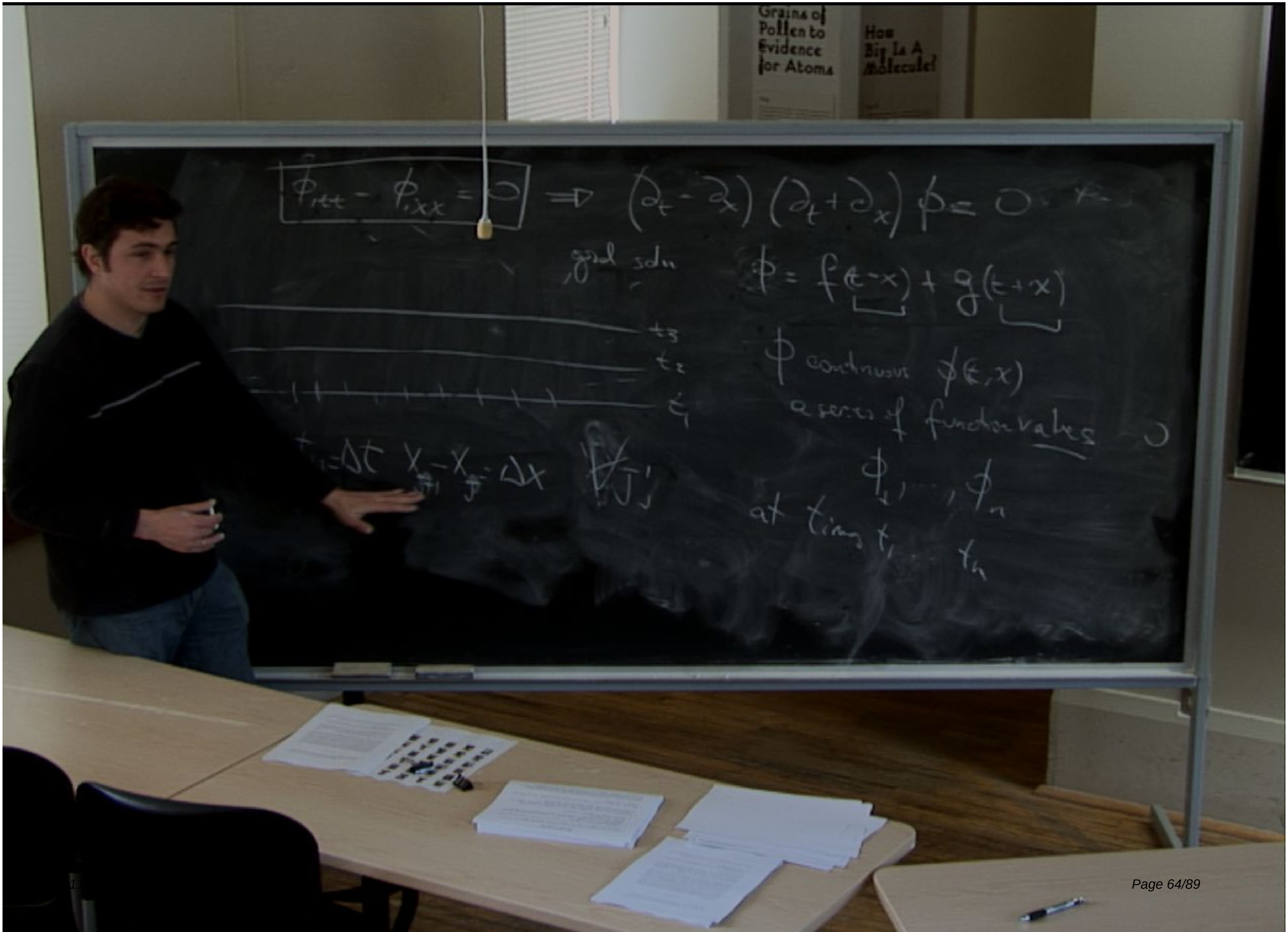
$\phi = f(t-x) + g(t+x)$
t_3
$t=0$
t_2
t_1
ϕ continuous $\phi(t,x)$
a series of function values
$\phi_1, \dots, \phi_n$
at times $t_1 \dots t_n$


Grains of Pollen to Evidence for Atoms
How Big Is A Molecule?
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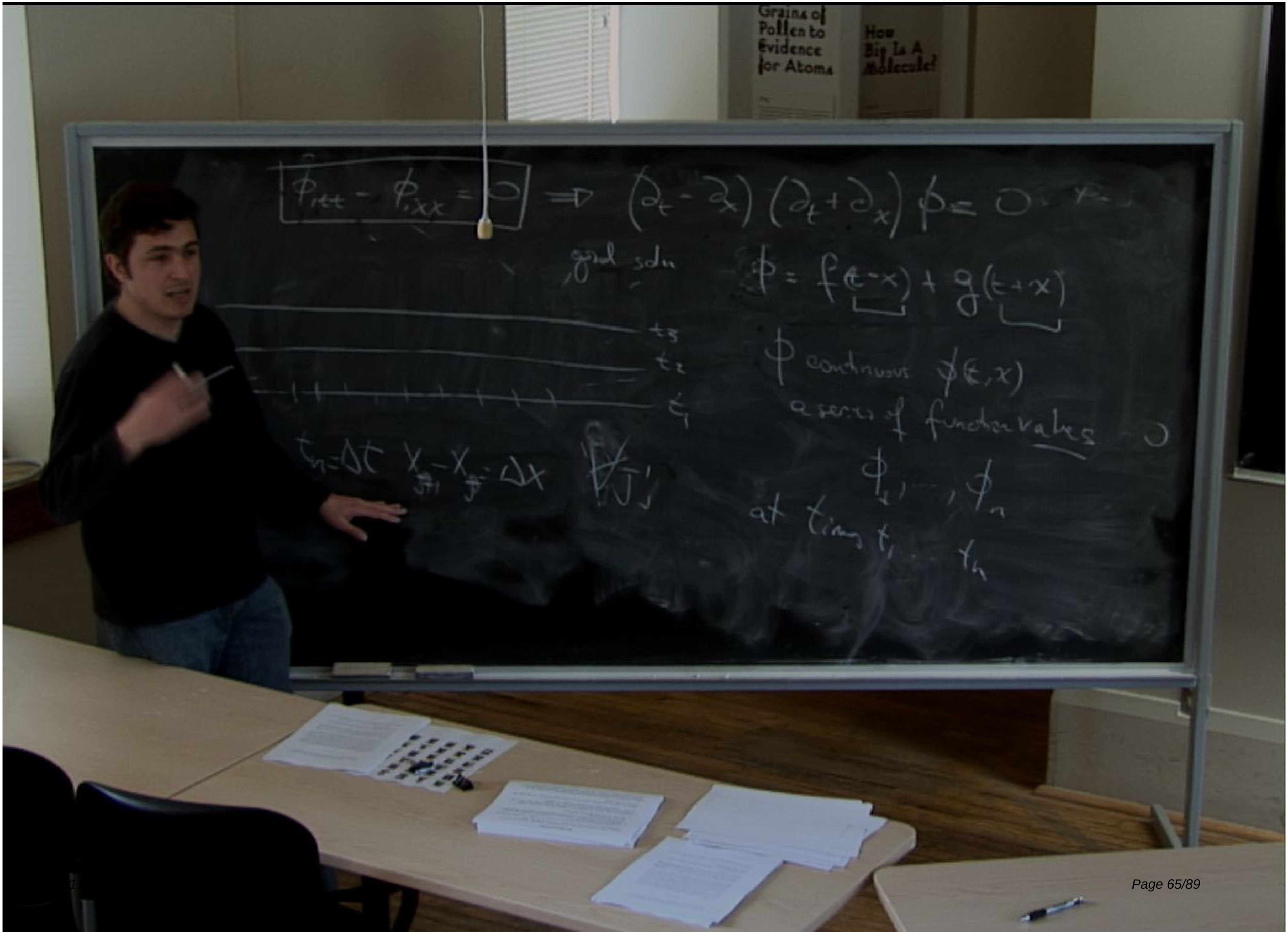
at times


Grains of Pollen to Evidence for Atoms
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φ = f(t−x) + g(t+x)
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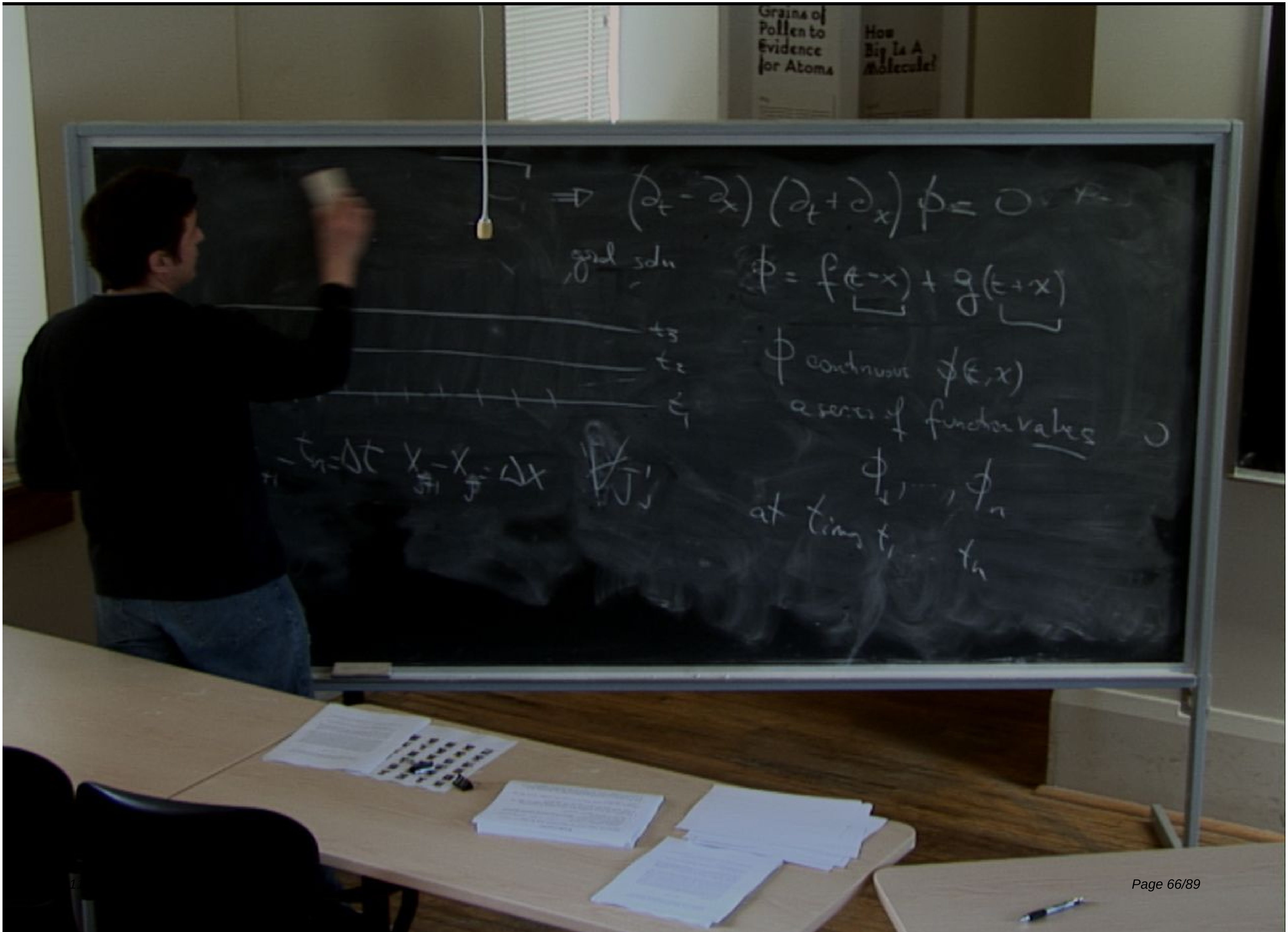
φ1, ..., φn
at times t1 ... tn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t_2
t_1
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a series of function values
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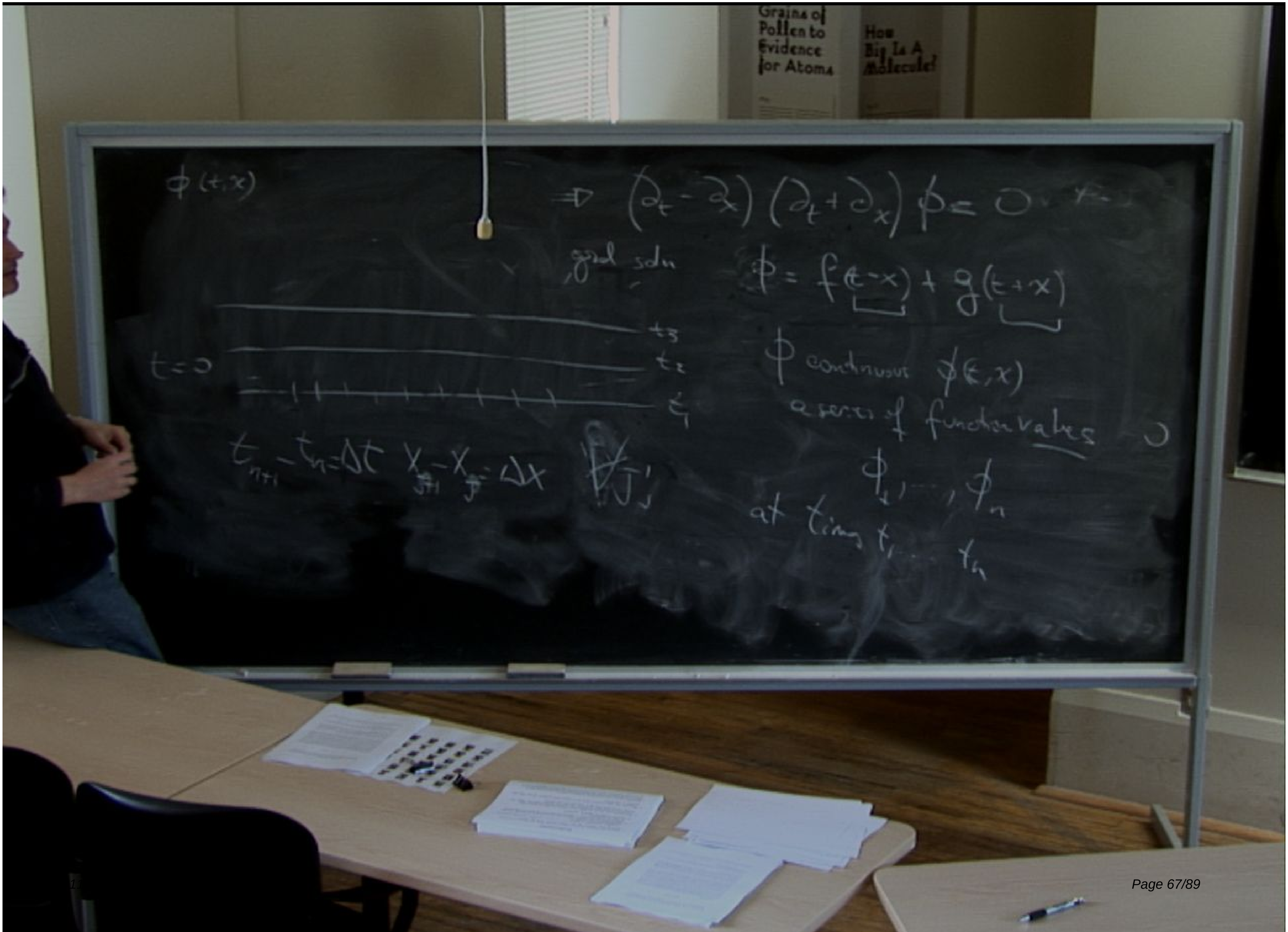
at times t_1 ... t_n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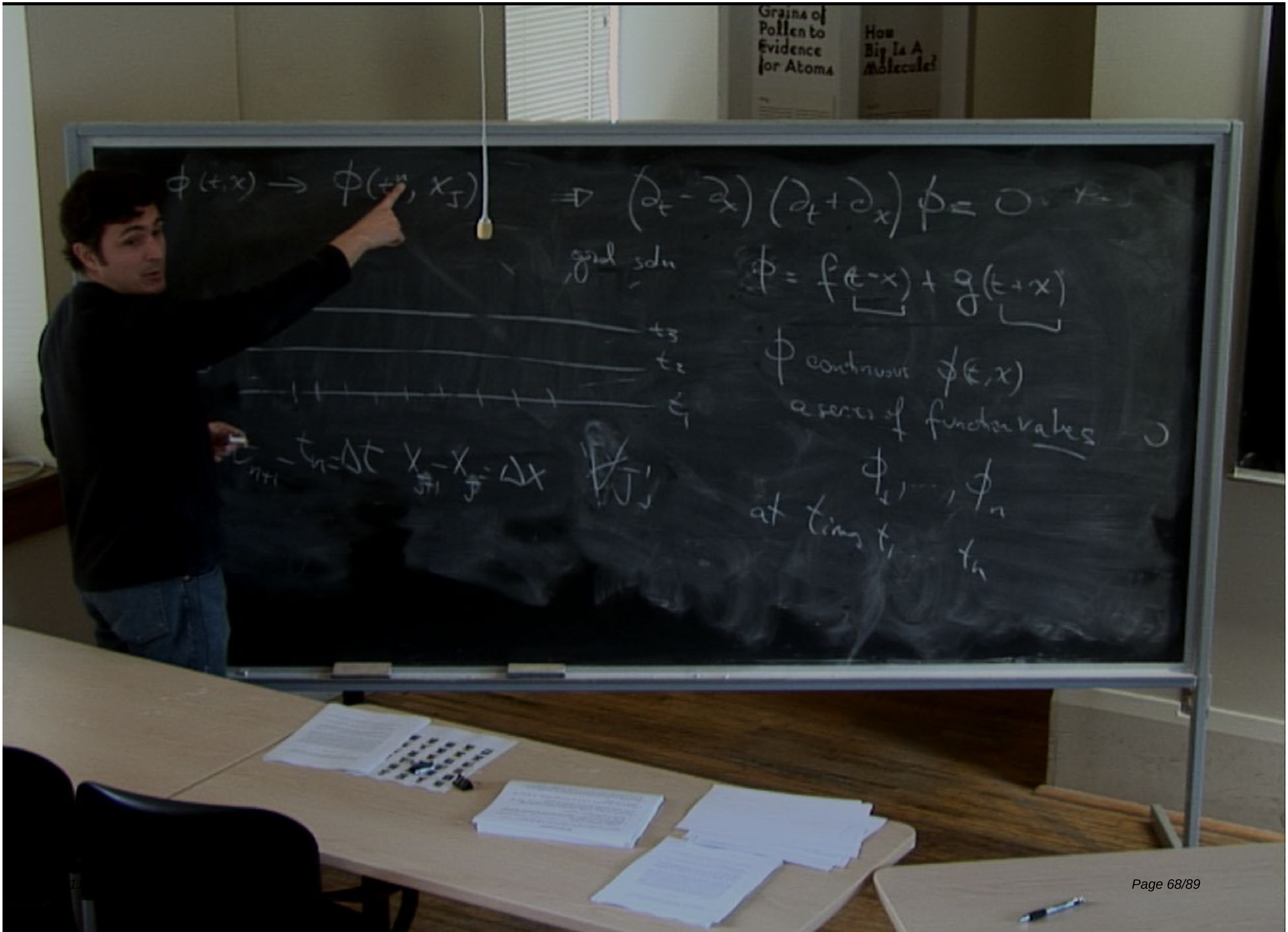
t3
t=0
t2
t1
φ continuous φ(t,x)
a series of function values
φ1, ..., φn
at times t1 ... tn
tn+1 − tn = Δt
xj+1 − xj = Δx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a series of function values
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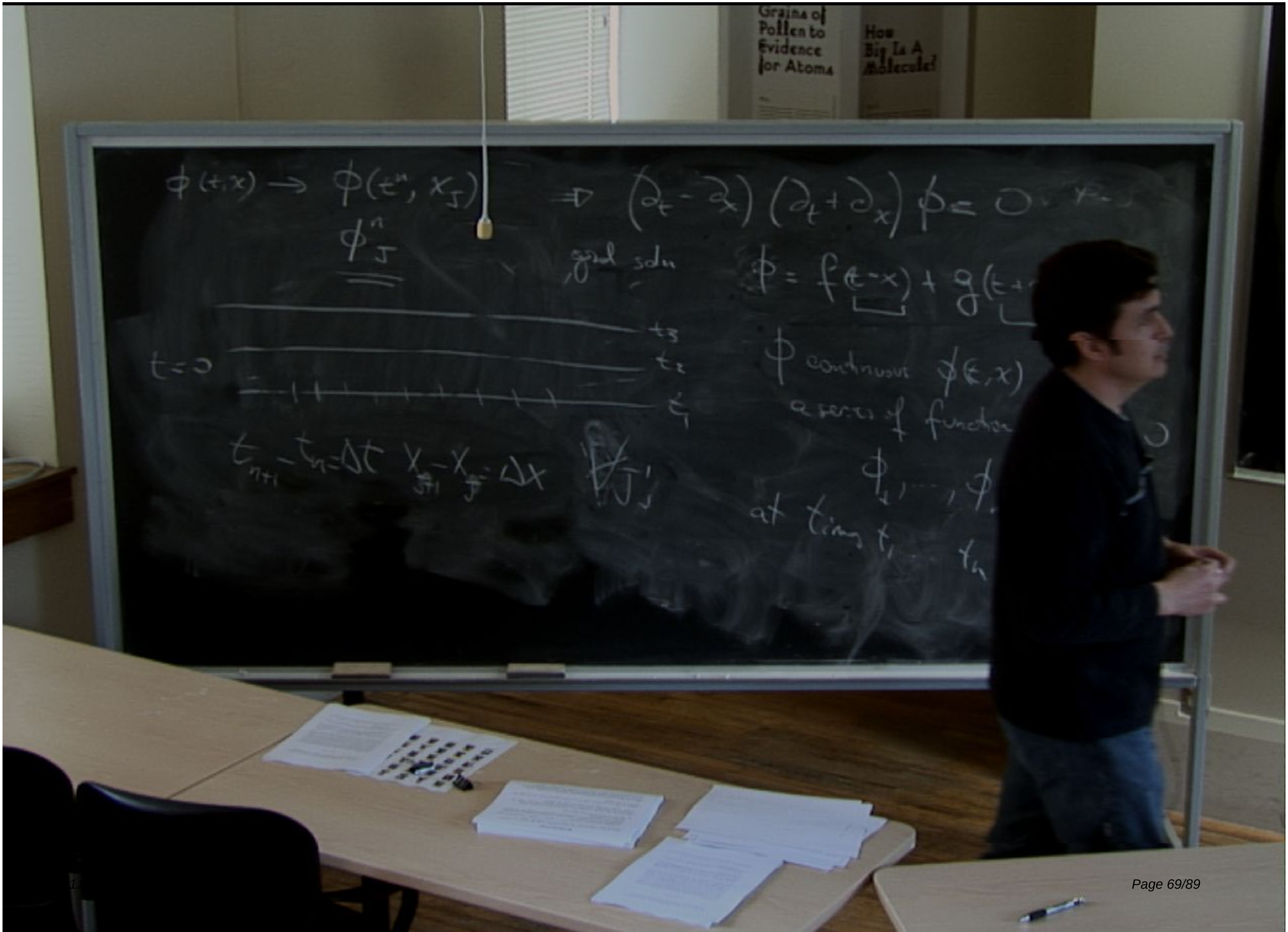
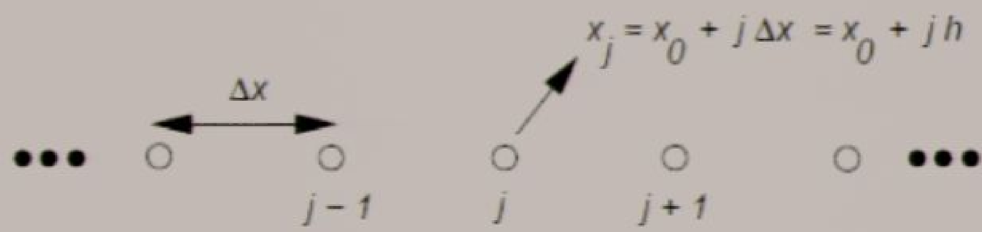
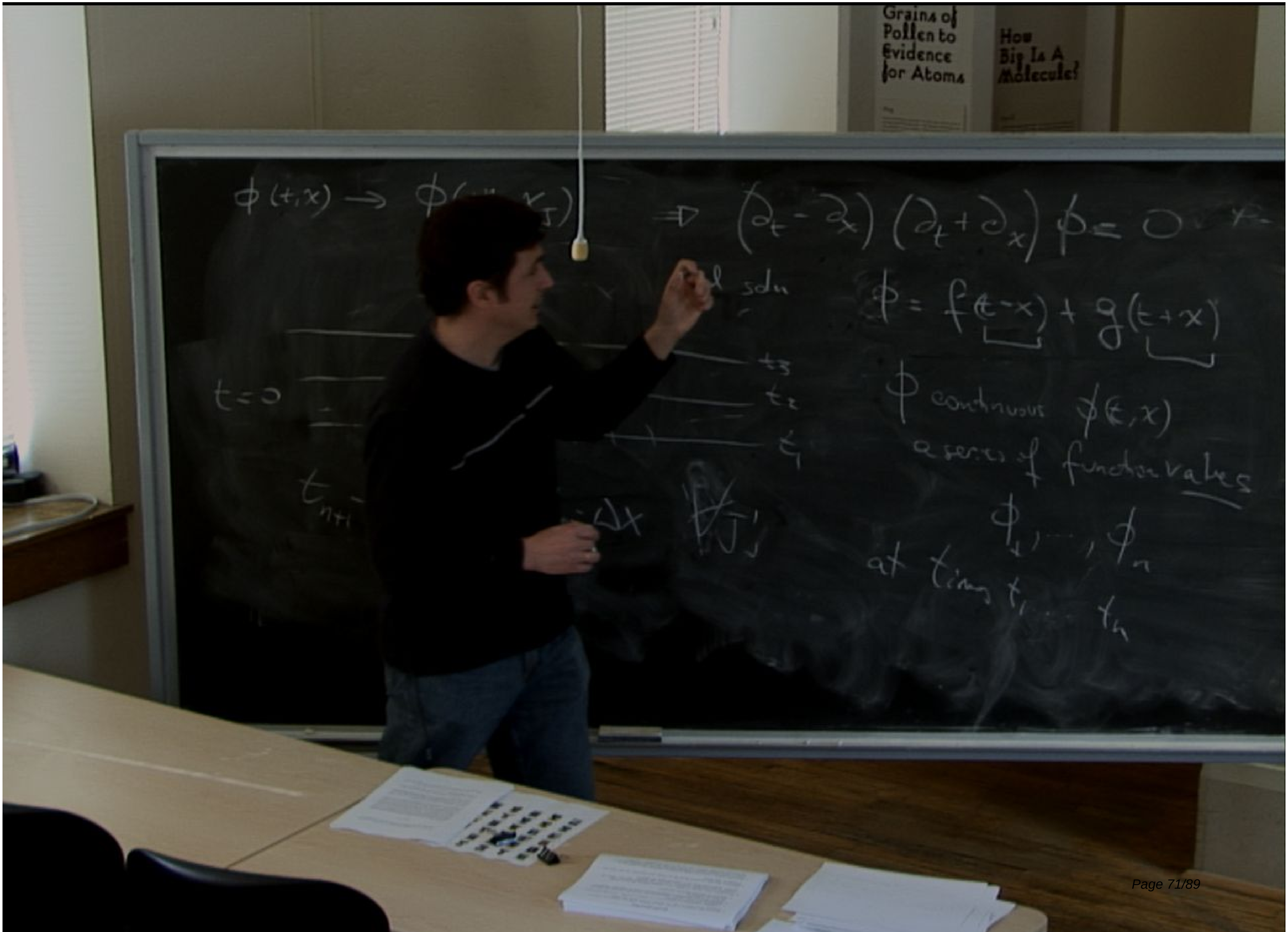
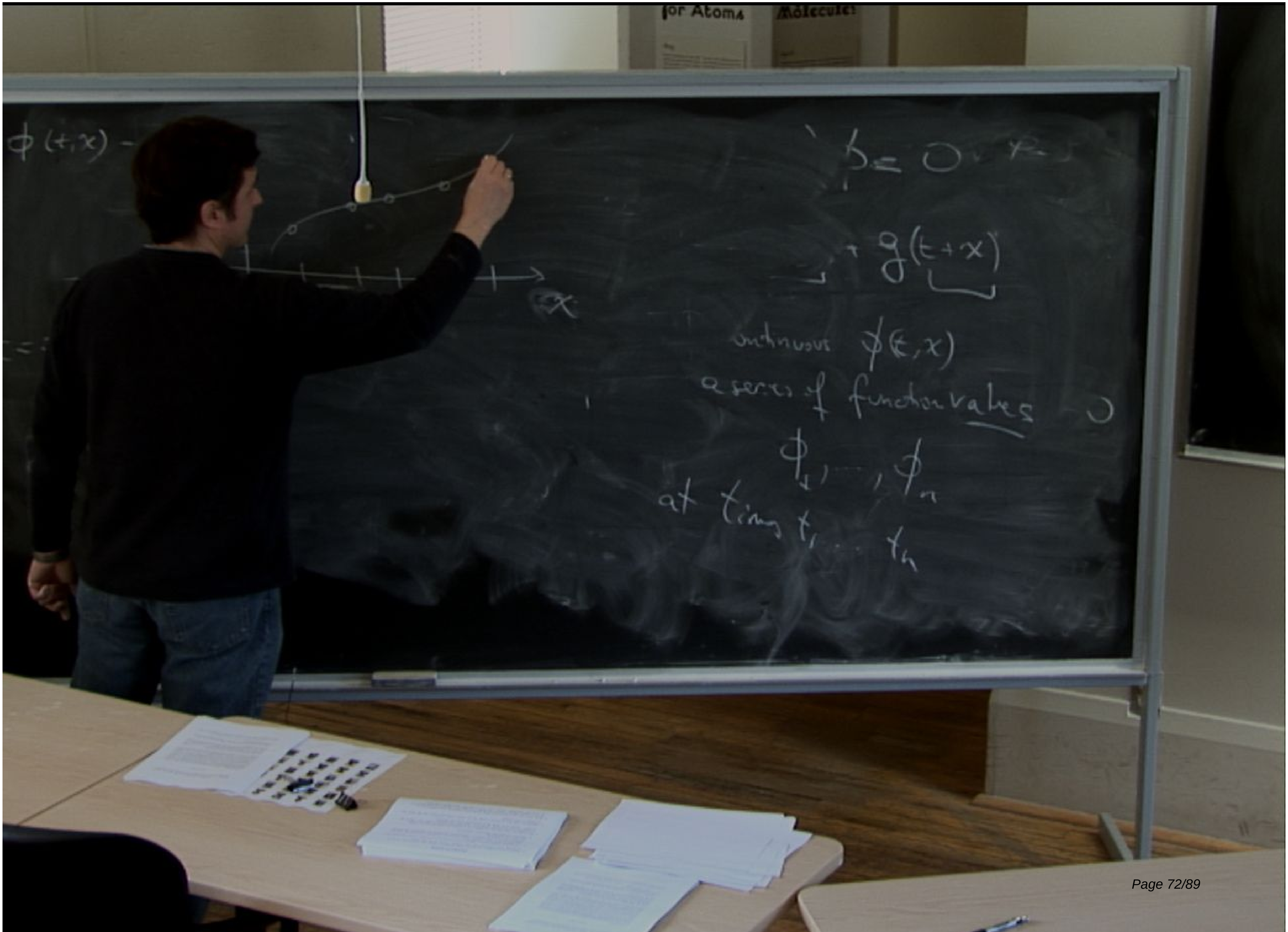

Grains of Pollen to Evidence for Atoms
How Big Is A Molecule?
good soln
continuous
a series of functions
at times

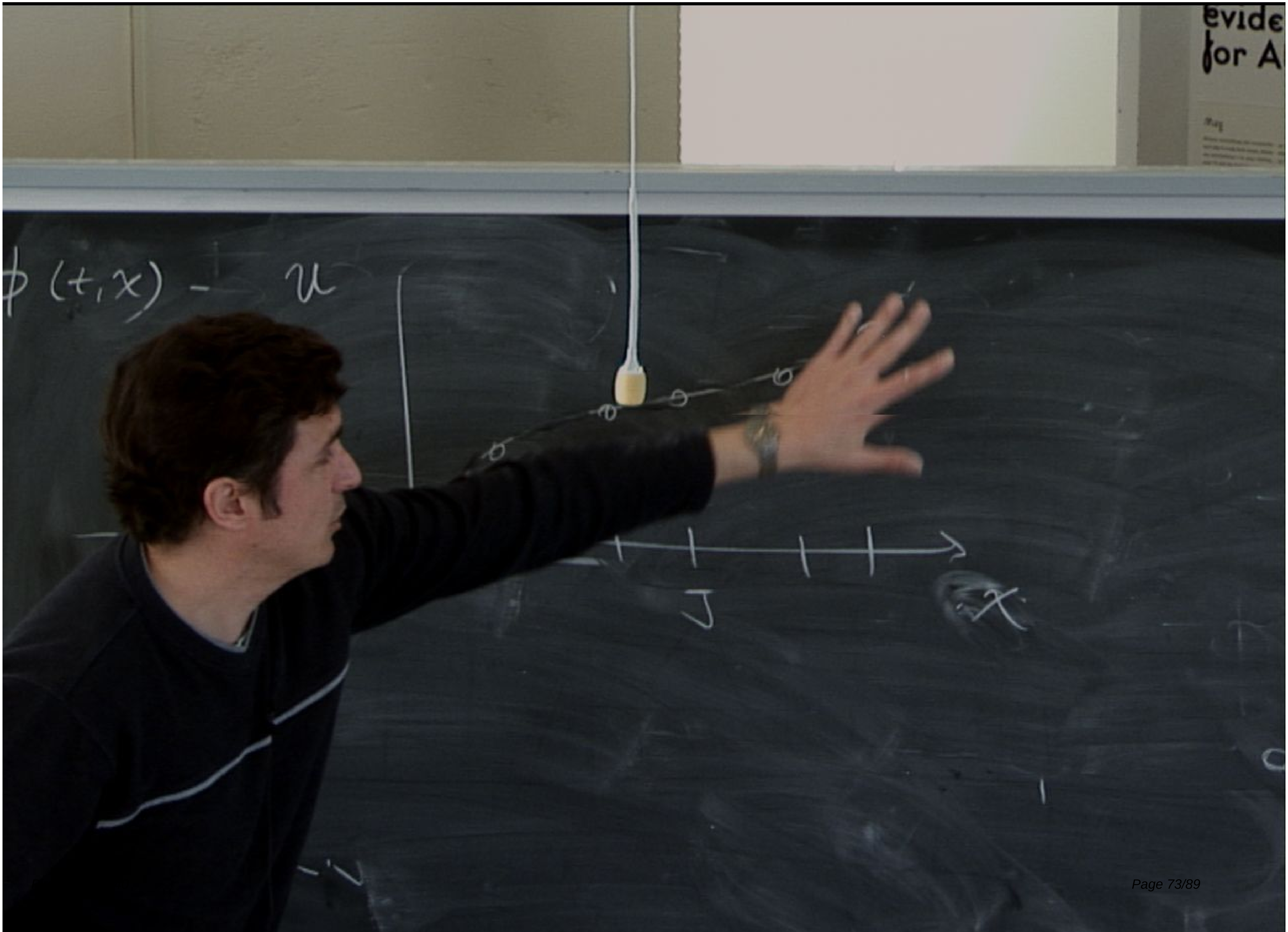
Deriving Finite Difference Formulae

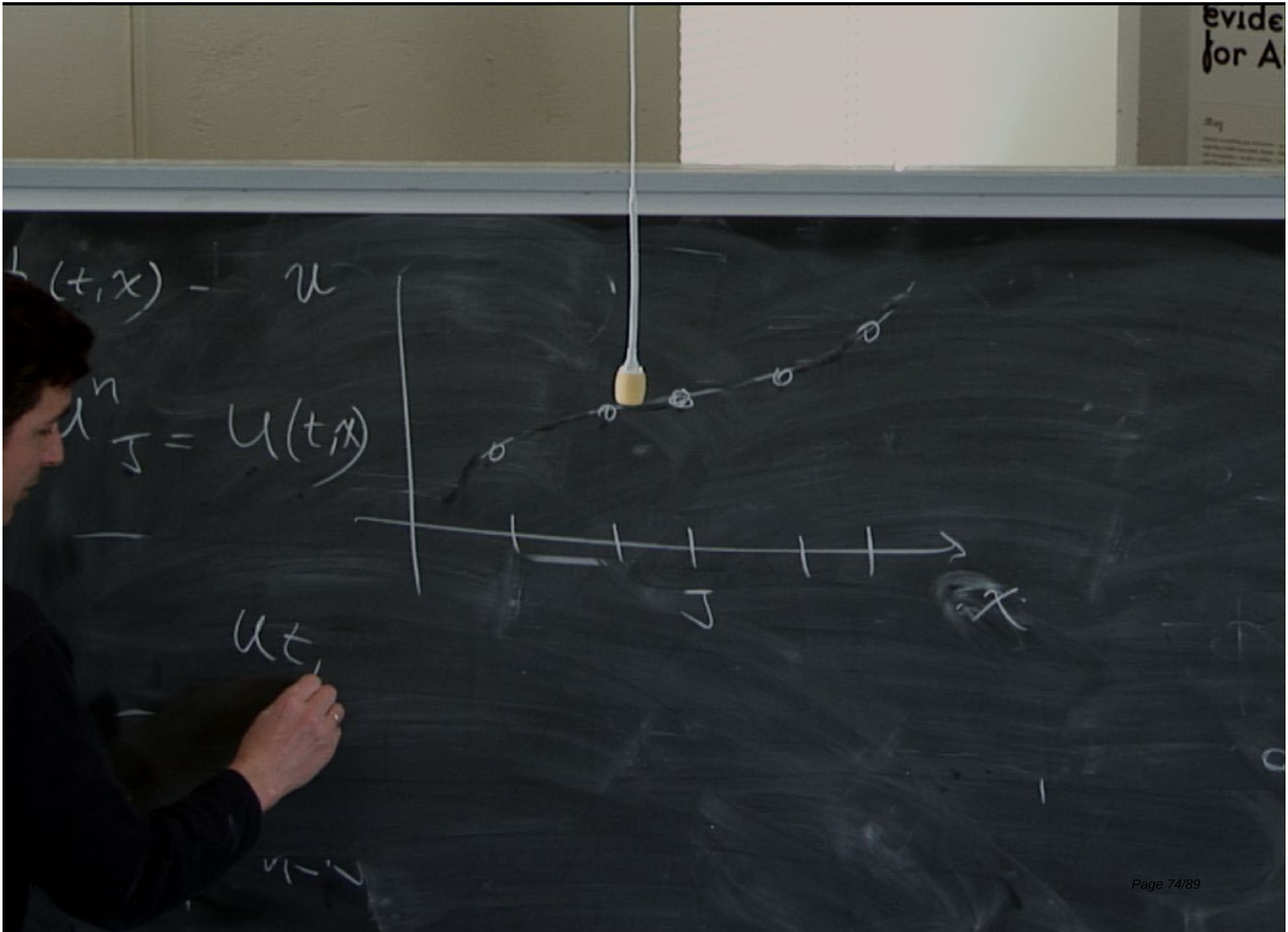


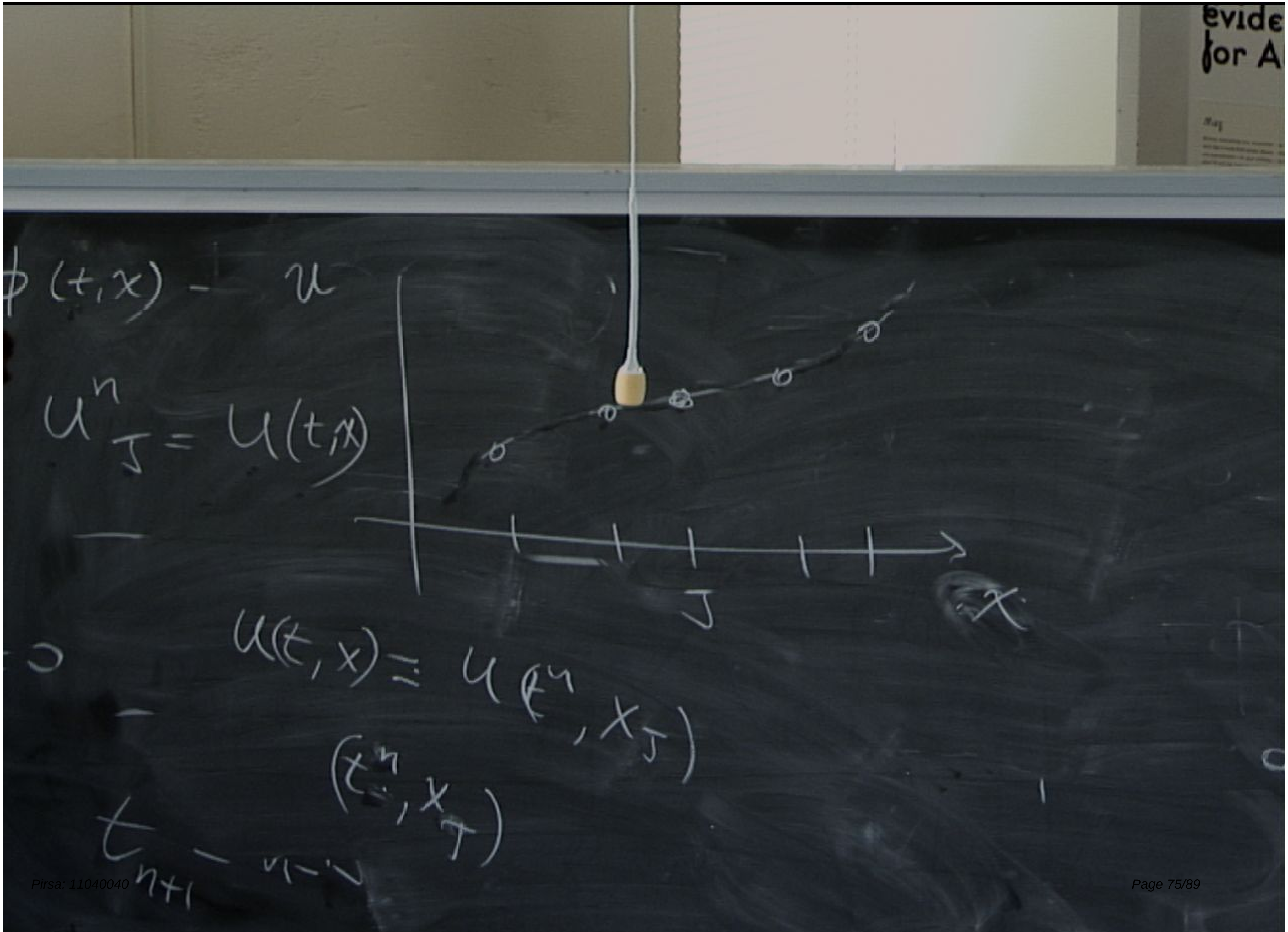
- One-dimensional, uniform finite difference mesh.
- Note that the spacing, $\Delta x = h$, between adjacent mesh points is *constant*.
- Will tacitly assume that the origin, x_0, of coordinate system is $x_0 = 0$.

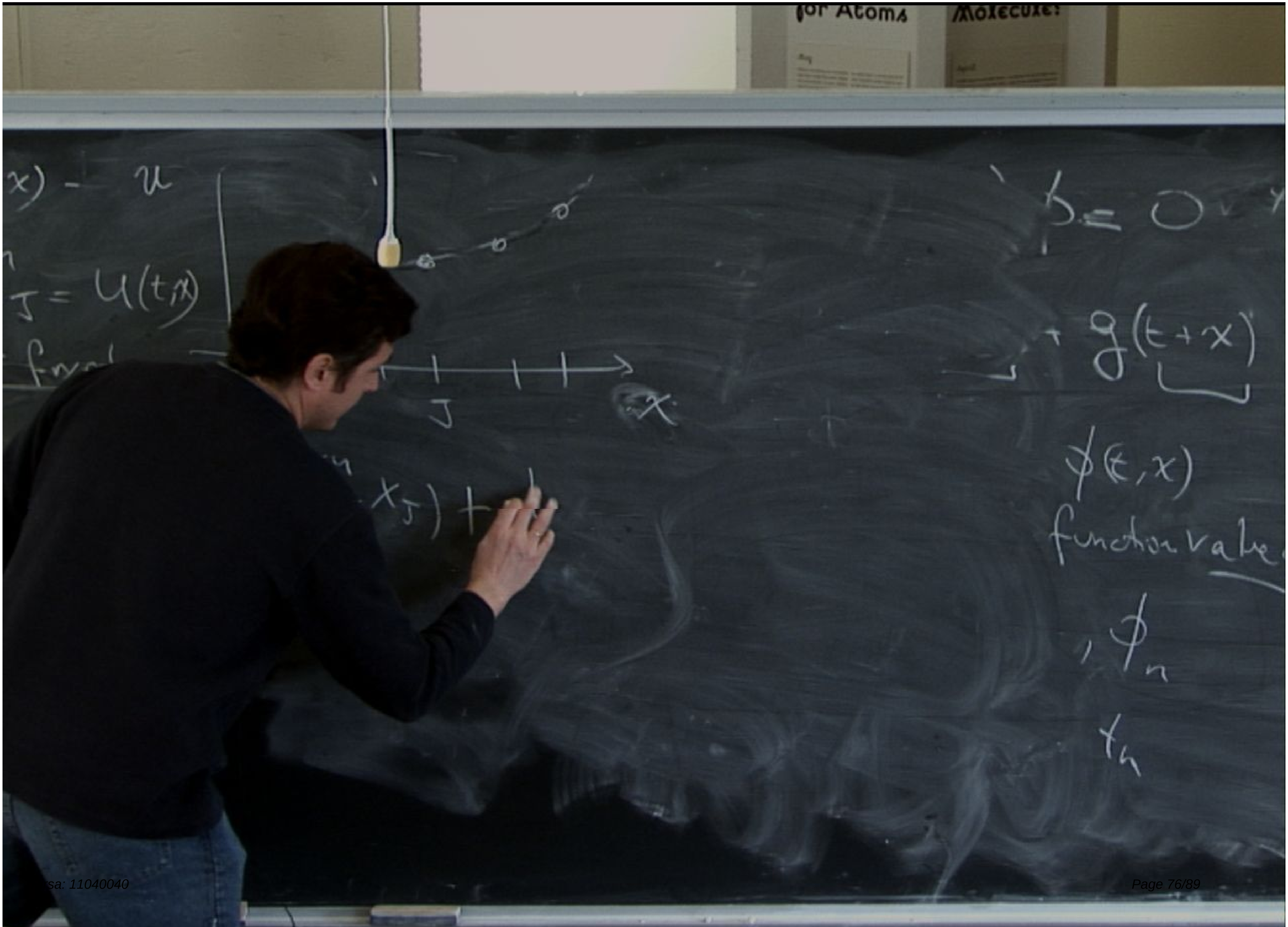

Grains of Pollen to Evidence for Atoms
How Big Is A Molecule?
ϕ continuous ϕ(t,x)
a series of function values
at times

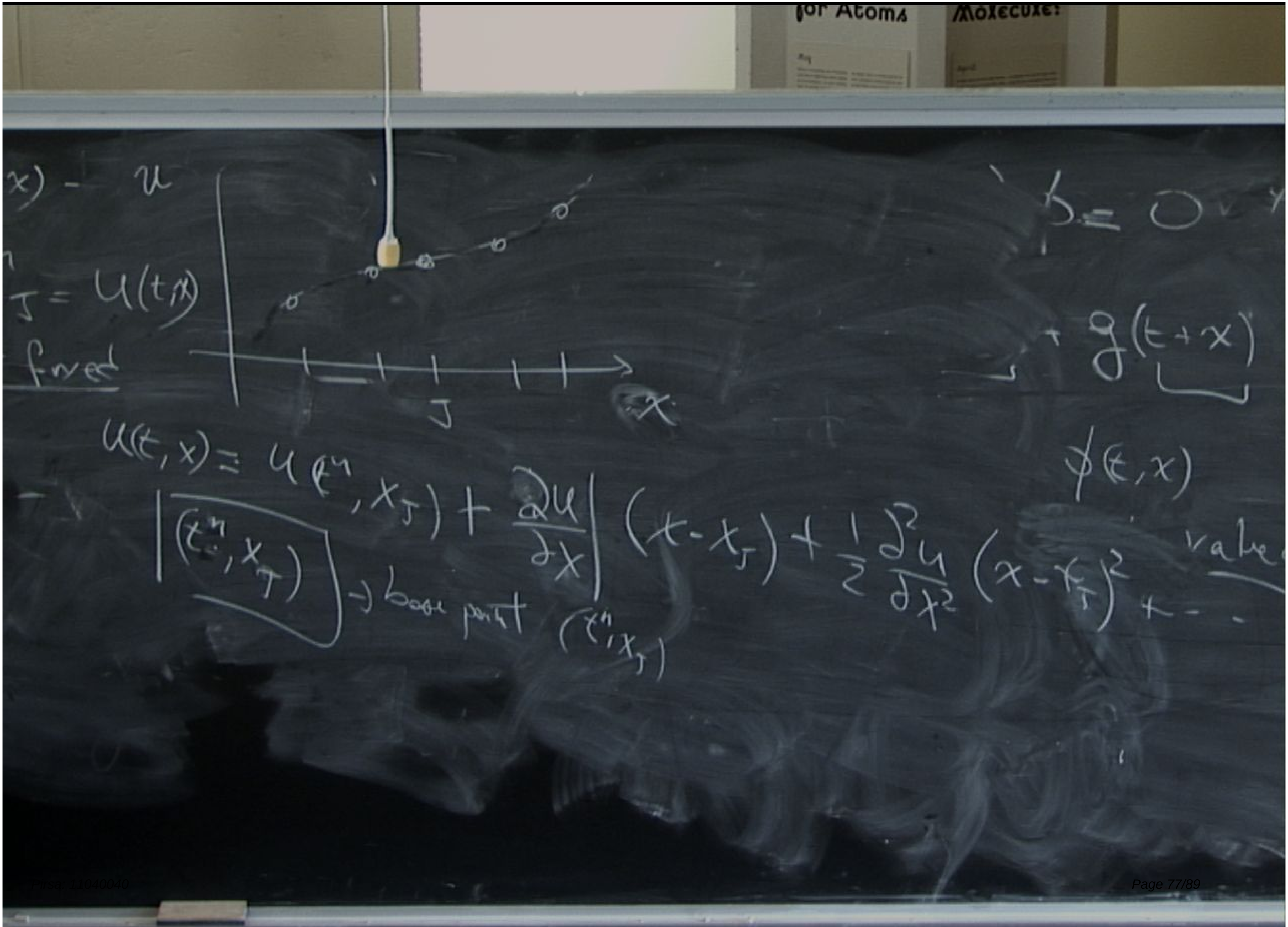

ϕ(t,x)
x
g(t+x)
ϕ(t,x)
a series of function values
ϕ1, ..., ϕn
at times t1 ... tn

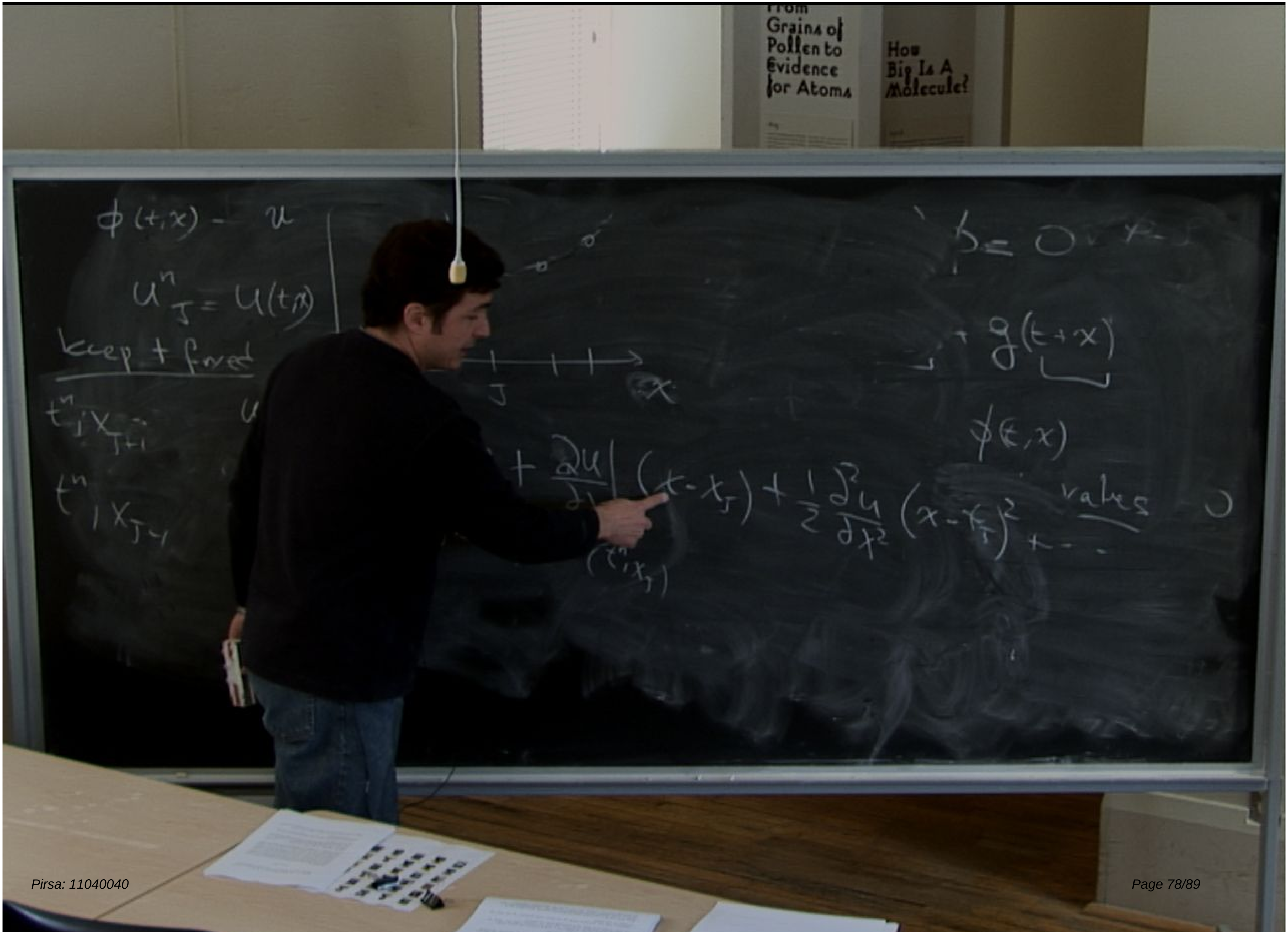

$\phi(t,x)$ - u
J
x
evide
for A


(t,x) – u
u^n_J = U(t,x)
J
x
ut


ϕ (t, x) – u
u^n_J = u(t,x)
J
x
u(t, x) = u(t^n, x_J)
(t^n, x_J)
t_{n+1}


for Atoms
U(t,x)
g(t+x)
φ(t,x)


for Atoms
Molecules
u(t, x) = u(t^n, x_J) + ∂u/∂x (x - x_J) + 1/2 ∂²u/∂x² (x - x_J)² + ...
(t^n, x_J) → base point (t^n, x_J)
J = u(t, x)
fixed
g(t + x)
φ(t, x)
value


Grains of
Pollen to
Evidence
for Atoms
How
Big Is A
Molecule?
keep t fixed
values

Deriving Finite Difference Formulae

- Given the three values $u(x_j - h), u(x_j)$ and $u(x_j + h)$, denoted u_{j-1}, u_j, and u_{j+1} respectively, can compute an $O(h^2)$ approximation to $u_x(x_j) \equiv (u_x)_j$ as follows

- Taylor expanding, have

$$
\begin{aligned}
u_{j-1} &= u_j - h(u_x)_j + \frac{1}{2}h^2(u_{xx})_j - \frac{1}{6}h^3(u_{xxx})_j + \frac{1}{24}h^4(u_{xxxx})_j + O(h^5) \\
u_j &= u_j \\
u_{j+1} &= u_j + h(u_x)_j + \frac{1}{2}h^2(u_{xx})_j + \frac{1}{6}h^3(u_{xxx})_j + \frac{1}{24}h^4(u_{xxxx})_j + O(h^5)
\end{aligned}
$$

- Now seek a linear combination of u_{j-1}, u_j, and u_{j+1} which yields $(u_x)_j$ to $O(h^2)$ accuracy, i.e. we seek c_-, c_0 and c_+ such that

$$
c_- \, u_{j-1} + c_0 \, u_j + c_+ \, u_{j+1} = (u_x)_j + O(h^2)
$$

Deriving Finite Difference Formulae

- Results in a system of three linear equations for u_{j-1}, u_j, and u_{j+1}:

$$
\begin{aligned}
c_- + c_0 + c_+ &= 0 \\
-hc_- + hc_+ &= 1 \\
\frac{1}{2}h^2 c_- + \frac{1}{2}h^2 c_+ &= 0
\end{aligned}
$$

which has the solution

$$
\begin{aligned}
c_- &= -\frac{1}{2h} \\
c_0 &= 0 \\
c_+ &= +\frac{1}{2h}
\end{aligned}
$$

- Thus, $O(h^2)$ FDA (finite difference approximation) for the first derivative is

$$\frac{u(x+h) - u(x-h)}{2h} = u_x(x) + O(h^2) \qquad (11)$$

Deriving Finite Difference Formulae

- Given the three values $u(x_j - h), u(x_j)$ and $u(x_j + h)$, denoted u_{j-1}, u_j, and u_{j+1} respectively, can compute an $O(h^2)$ approximation to $u_x(x_j) \equiv (u_x)_j$ as follows

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$$c_-\, u_{j-1} + c_0\, u_j + c_+\, u_{j+1} = (u_x)_j + O(h^2)$$

21 (22 of 83) 76.6% Find

Deriving Finite Difference Formulae

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\frac{1}{2}h^2 c_- + \frac{1}{2}h^2 c_+ &= 0
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$$

which has the solution

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c_- &= -\frac{1}{2h} \\
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$$

- Thus, $O(h^2)$ FDA (finite difference approximation) for the first derivative is

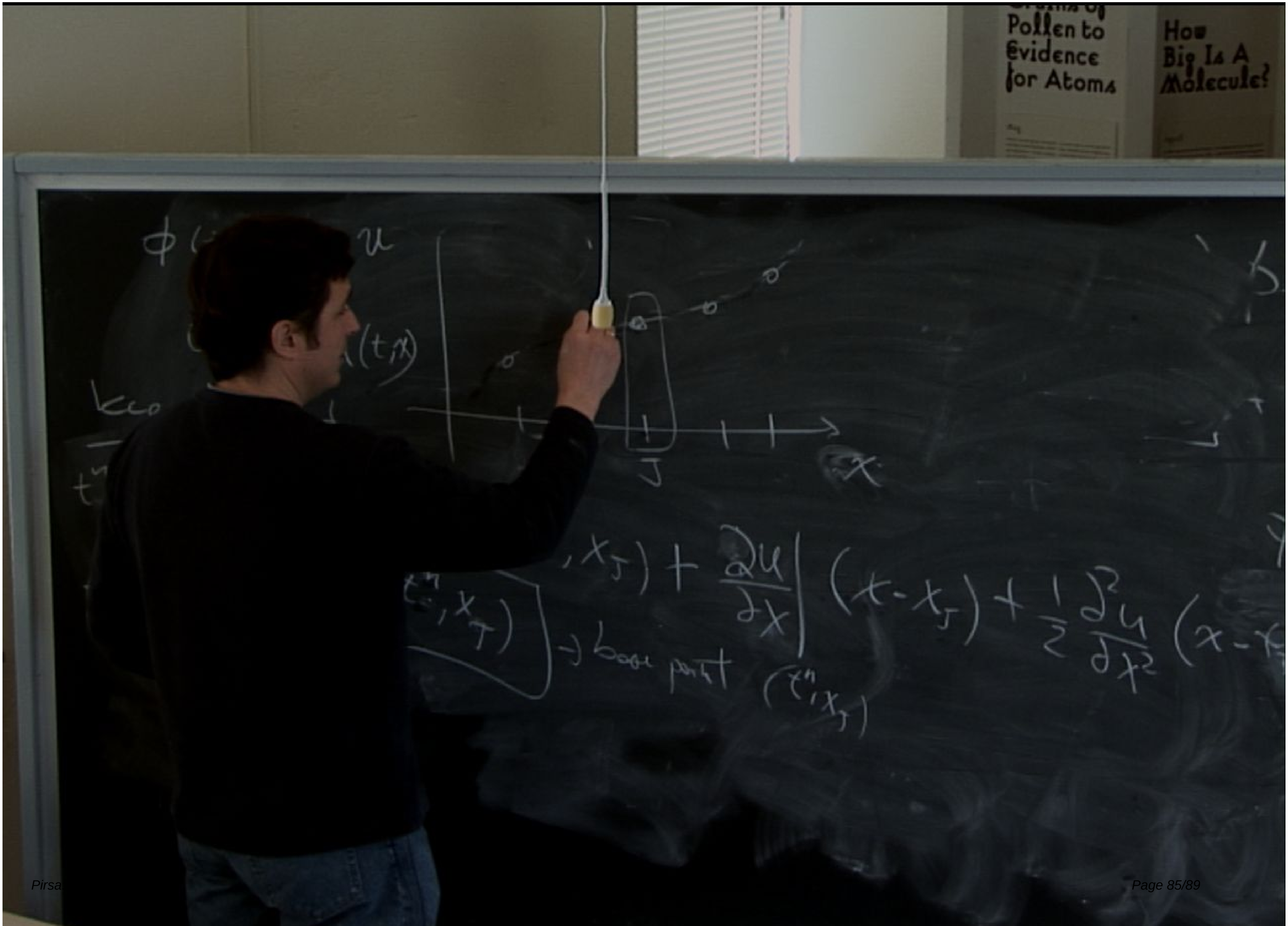
$$\frac{u(x+h) - u(x-h)}{2h} = u_x(x) + O(h^2) \qquad (11)$$

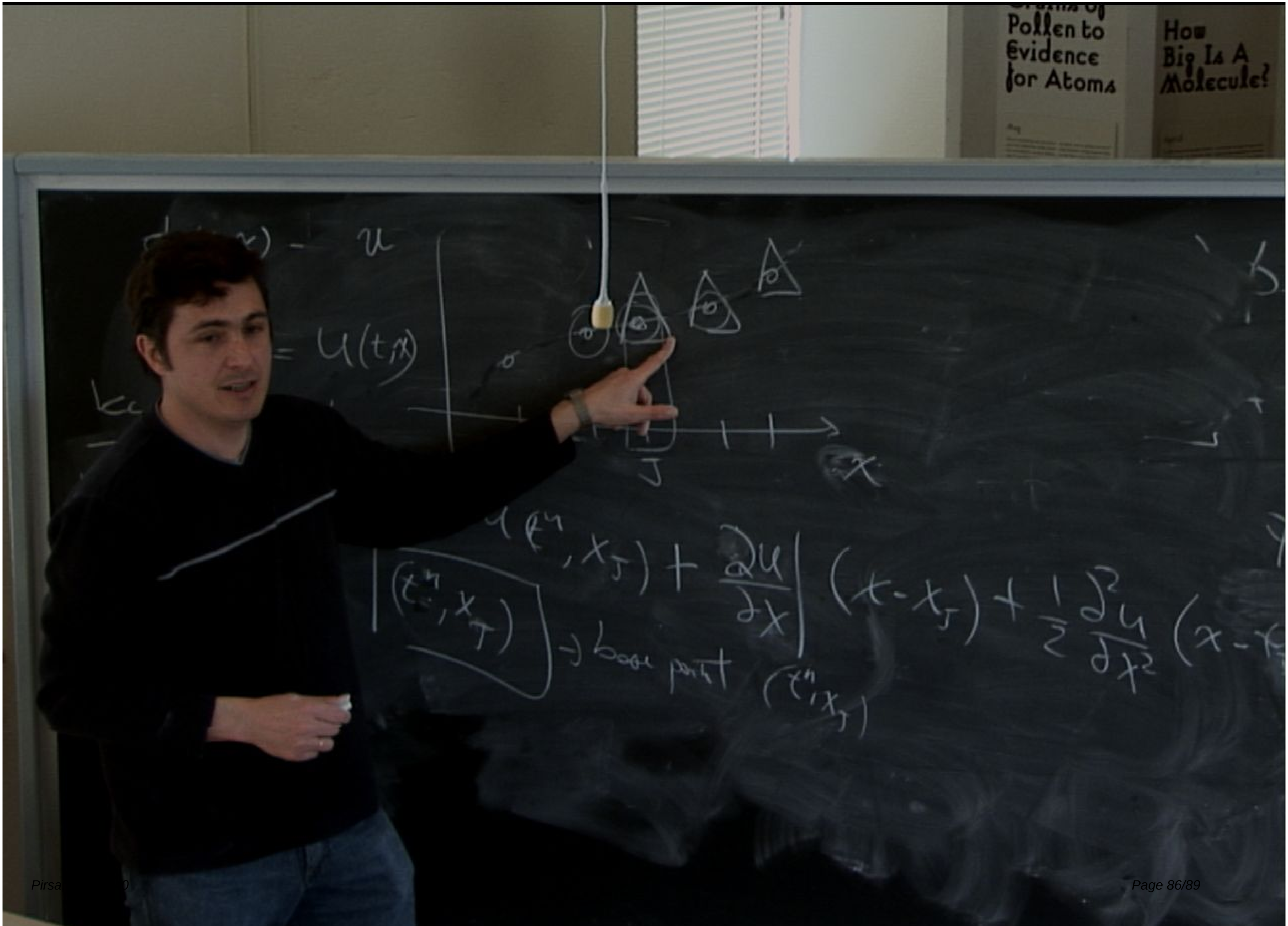
Deriving Finite Difference Formulae

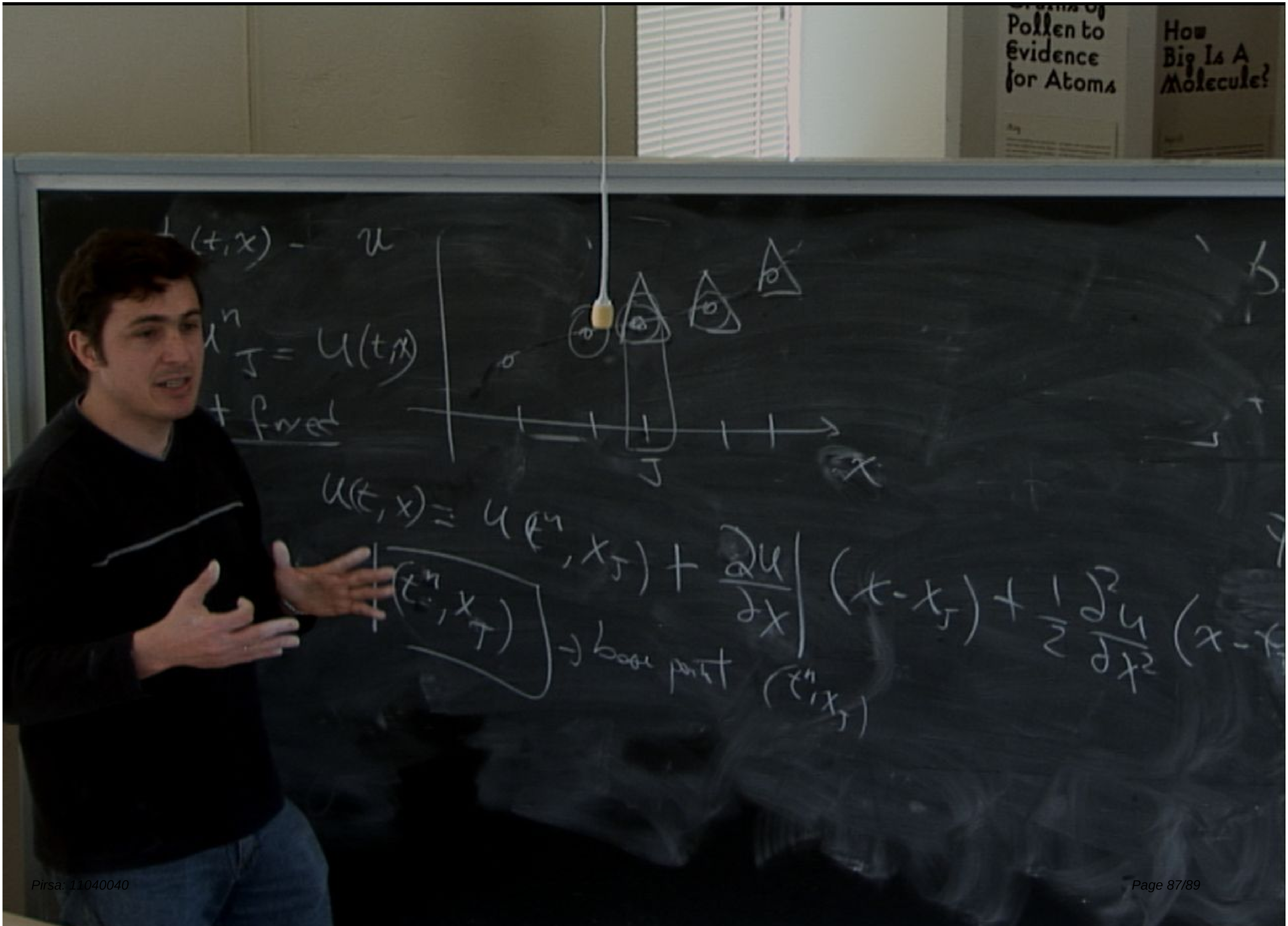
- May not be obvious *a priori*, that the truncation error of approximation *is* $O(h^2)$
- Naive consideration of the number of terms in the Taylor series expansion which can be eliminated using 2 values (namely $u(x+h)$ and $u(x-h)$) suggests that the error might be $O(h)$.
- Fact that the $O(h)$ term "drops out" a consequence of the *symmetry*, or *centering* of the stencil: common theme in such FDA, called *centred* difference approximations
- Using same technique, can easily generate $O(h^2)$ expression for the *second* derivative, which uses the same difference stencil as the above approximation for the first derivative.

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u_{xx}(x) + O(h^2) \qquad (12)$$

- *Exercise:* Compute the precise form of the $O(h^2)$ terms in expressions (11) and (12).


Pollen to
Evidence
for Atoms
How
Big Is A
Molecule?
boost point


Pollen to Evidence for Atoms
How Big Is A Molecule?
base point


Pollen to
Evidence
for Atoms
How
Big Is A
Molecule?
$u_J^n = u(t,x)$
t fixed
base point (t^n, x_J)

Sample FDA for the 1-D Wave Equation

- Let us consider the 1-D wave equation again, but this time on the finite spatial domain, $0 \le x \le 1$, where we will prescribe fixed (Dirichlet) boundary conditions

- Then we wish to solve

$$\begin{aligned}
\phi_{tt} &= \phi_{xx} \qquad (c = 1) \qquad 0 \le x \le 1, \quad t \ge 0 && (13)\\
\phi(0, x) &= \phi_0(x) \\
\phi_t(0, x) &= \Pi_0(x) \\
\phi(t, 0) &= \phi(t, 1) = 0 && (14)
\end{aligned}$$

- We will again require that the initial data functions, $\phi_0(x)$ and $\Pi_0(x)$ be smooth

- Moreover, in order to ensure a smooth solution everywhere, the initial values must be compatible with the boundary conditions, i.e.

$$\phi_0(0) = \phi_0(1) = \Pi_0(0) = \Pi_0(1) = 0 \qquad (15)$$

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