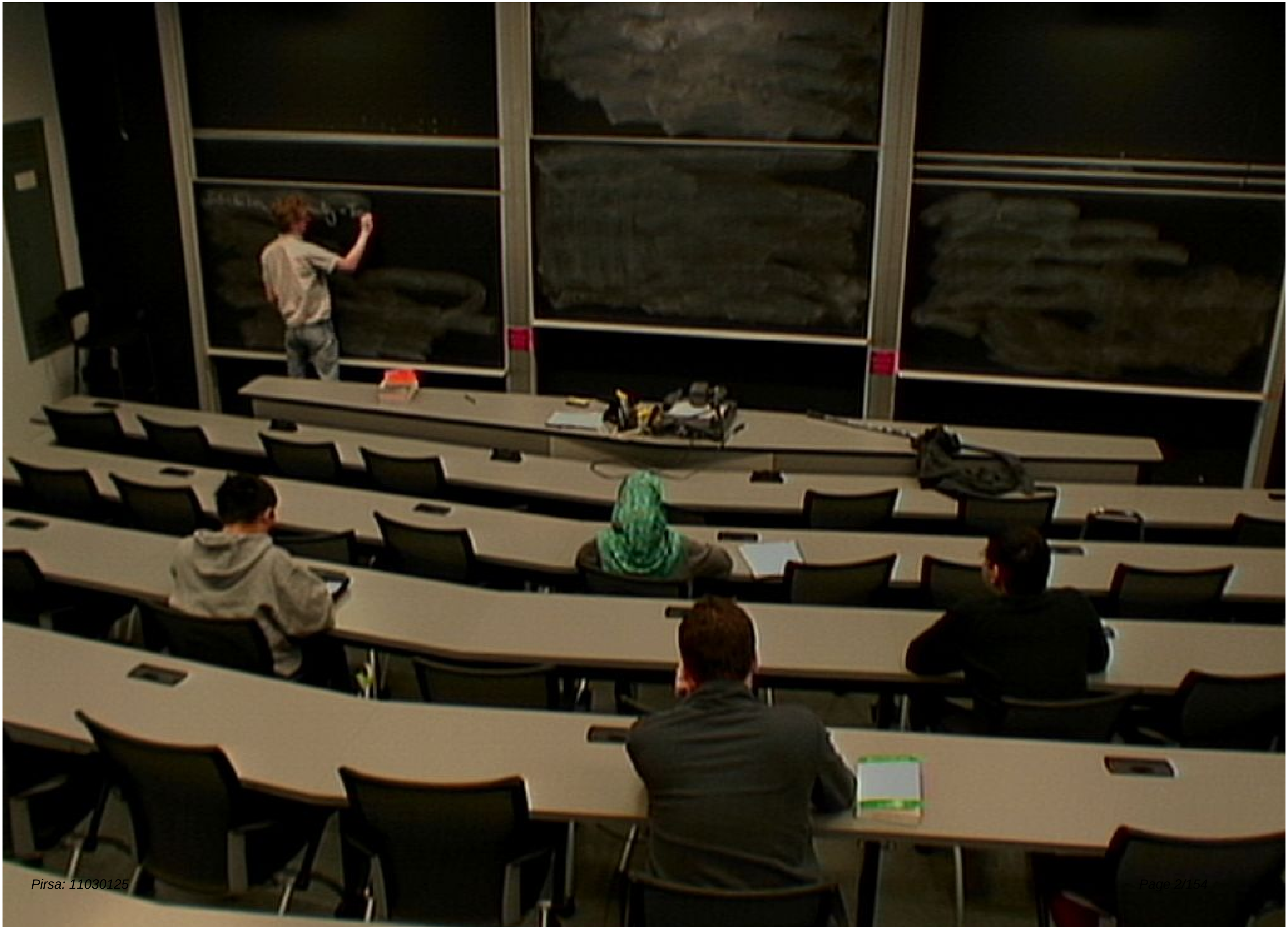
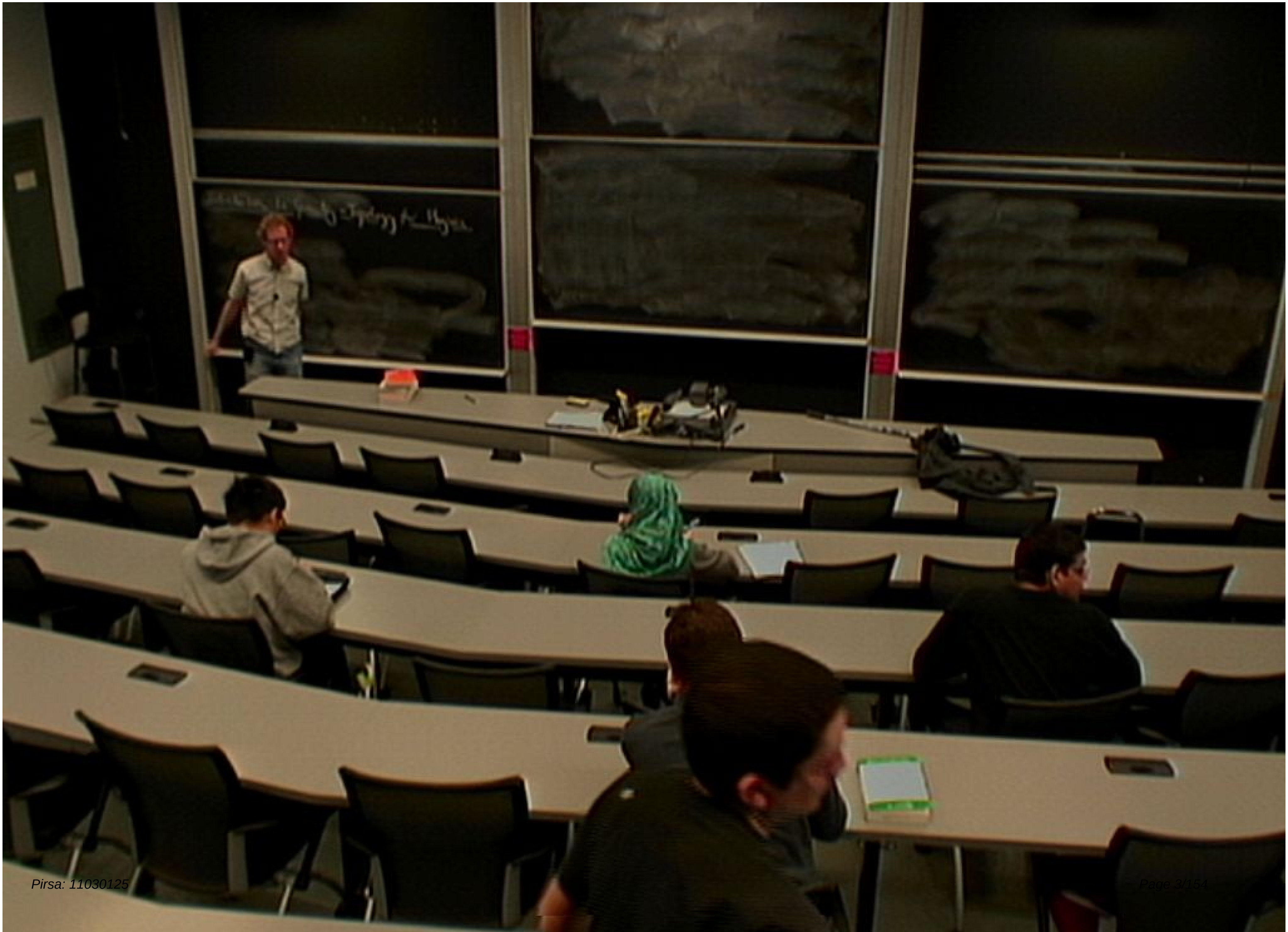
Title: Geometry & Topology for Physics - Lecture 1

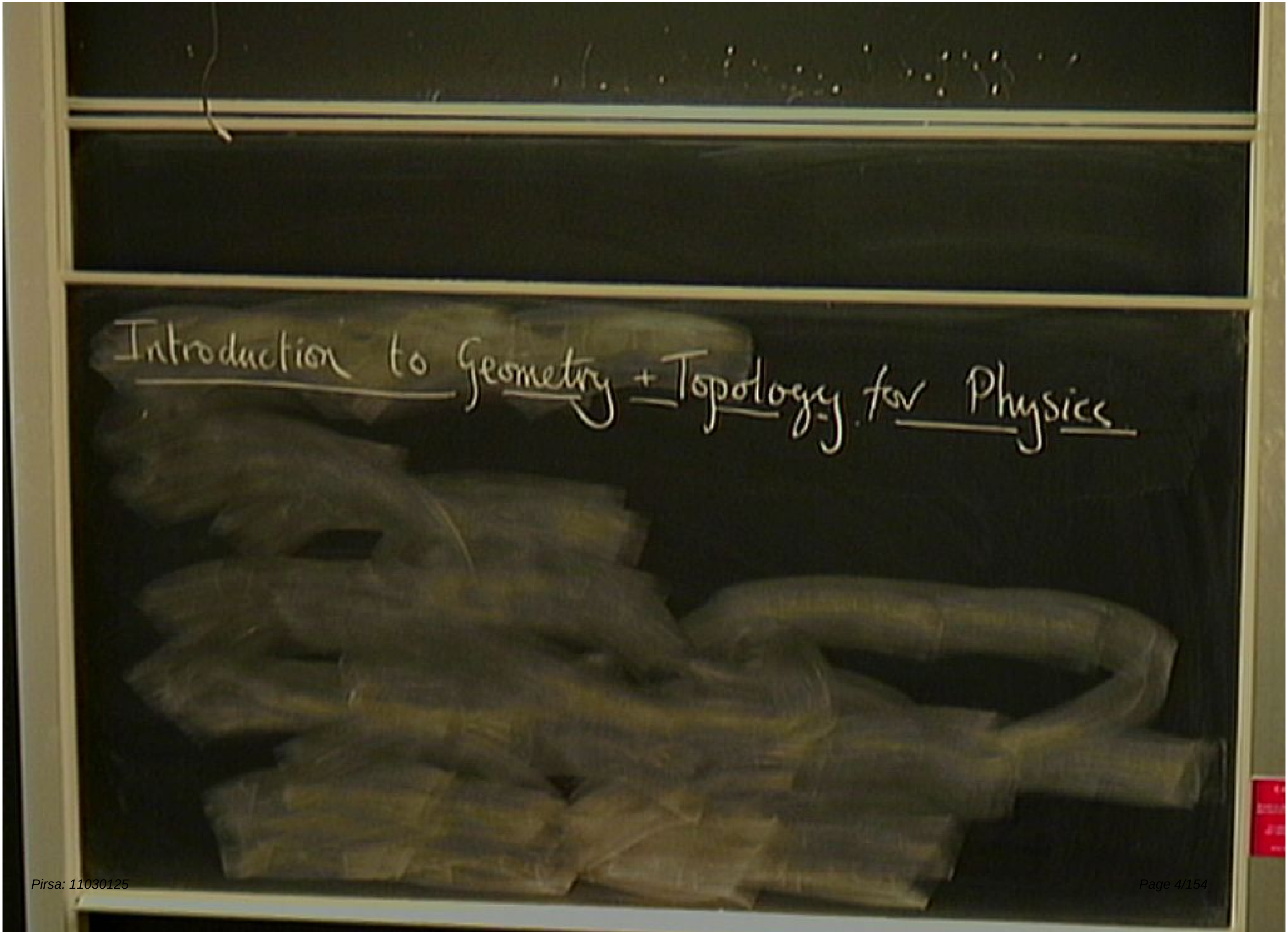
Date: Mar 28, 2011 02:00 PM

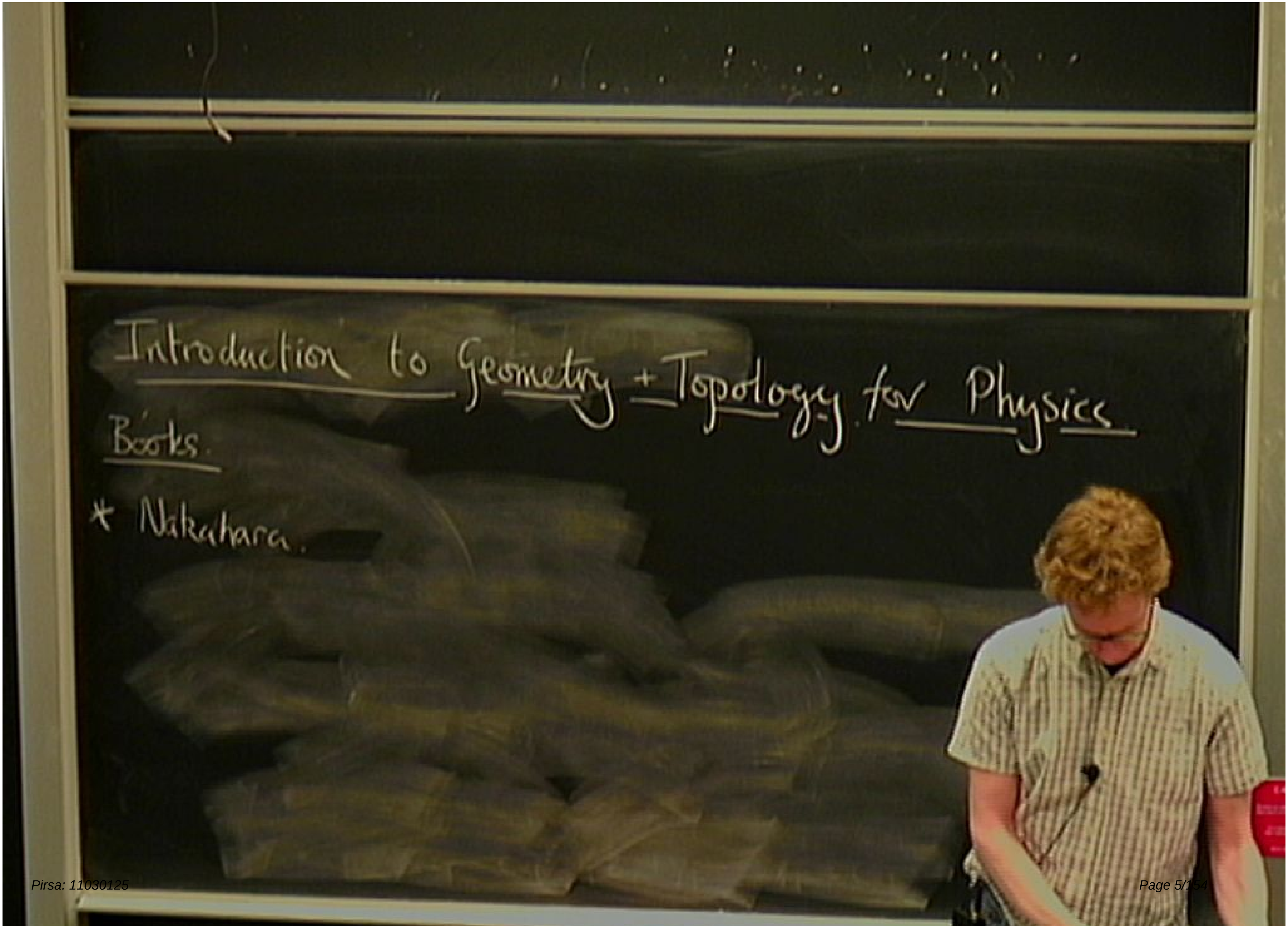
URL: http://pirsa.org/11030125

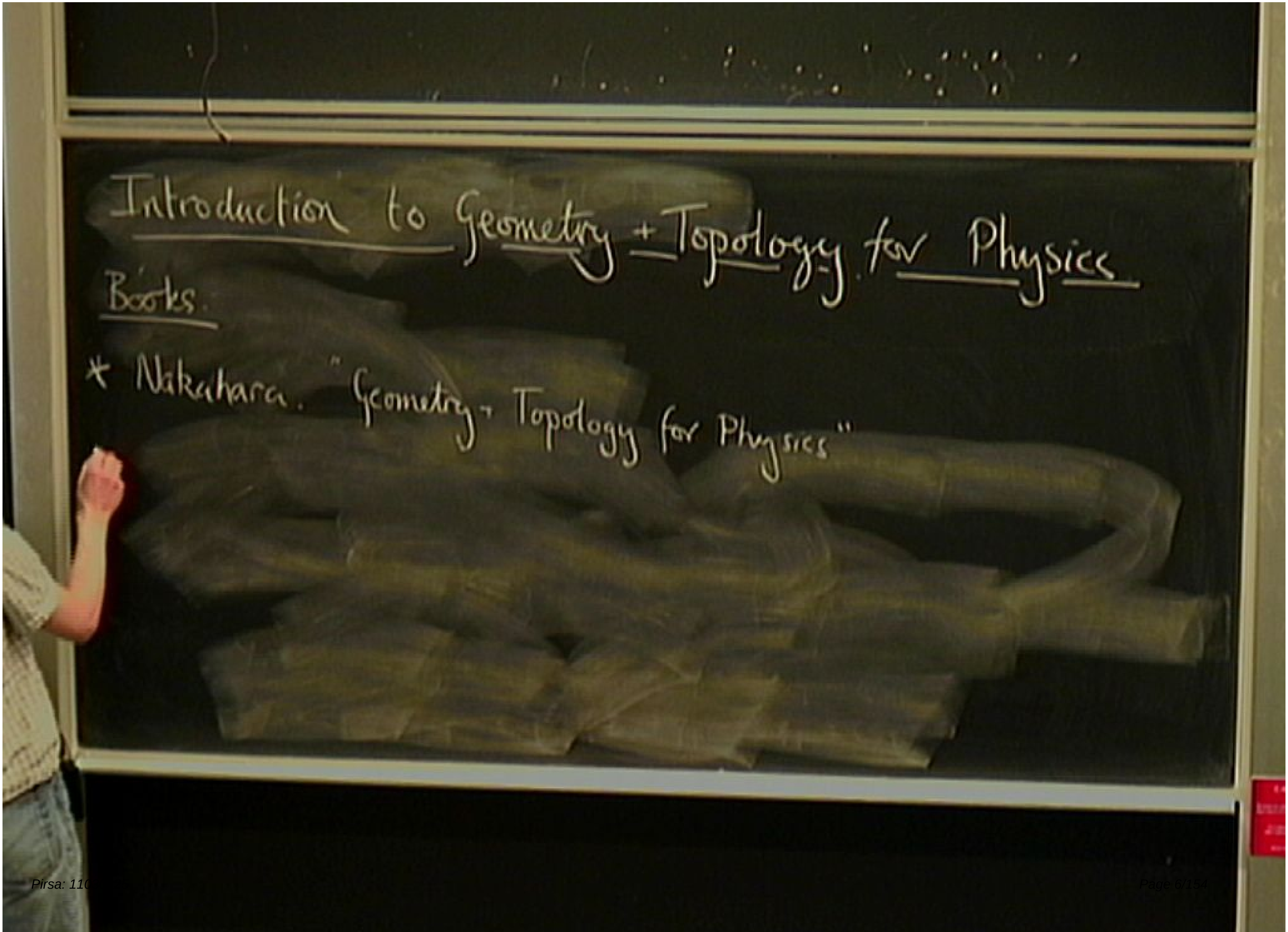
Abstract:






Introduction to Geometry + Topology for Physics


Introduction to Geometry + Topology for Physics
Books.
* Nakahara.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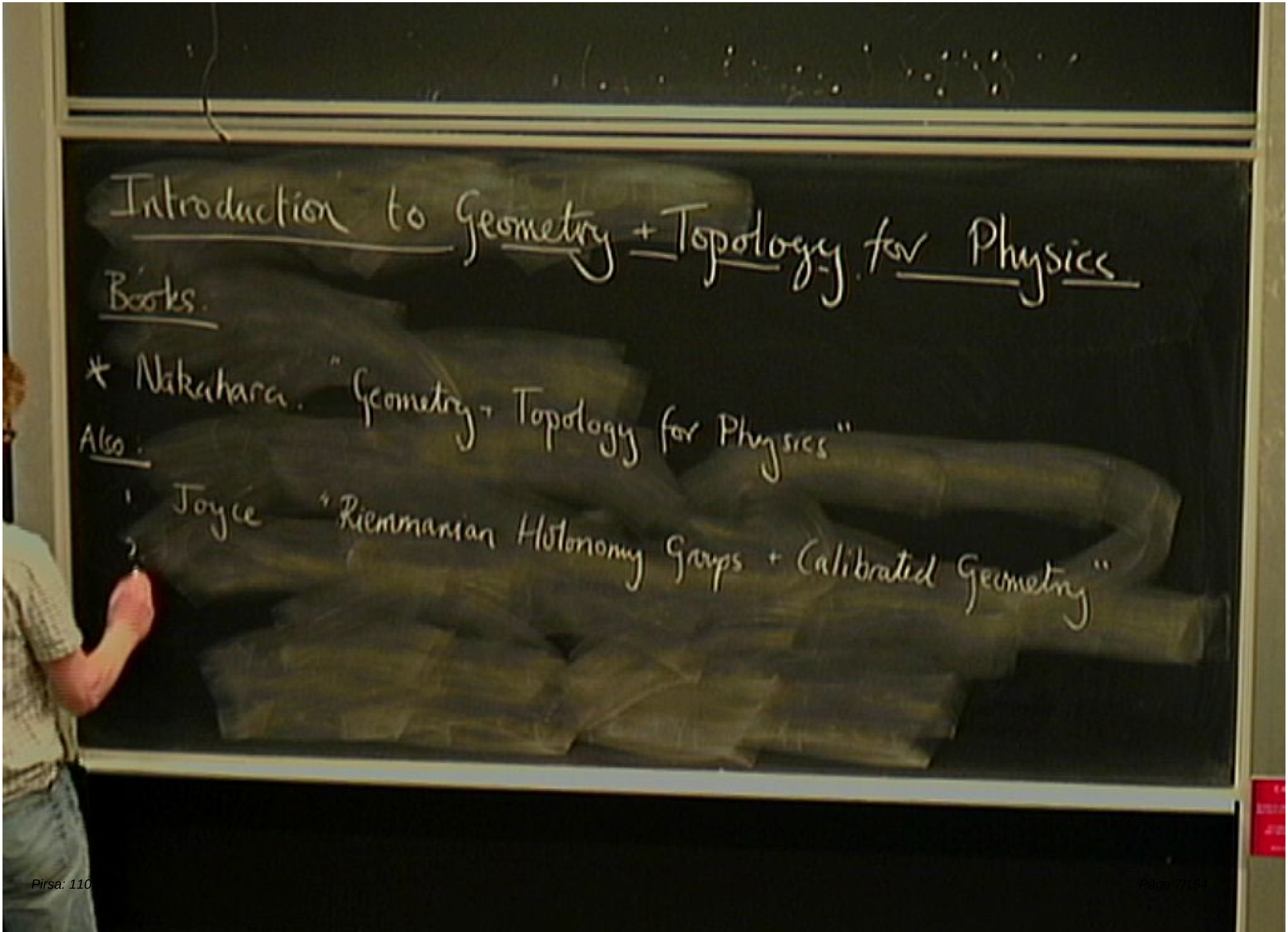
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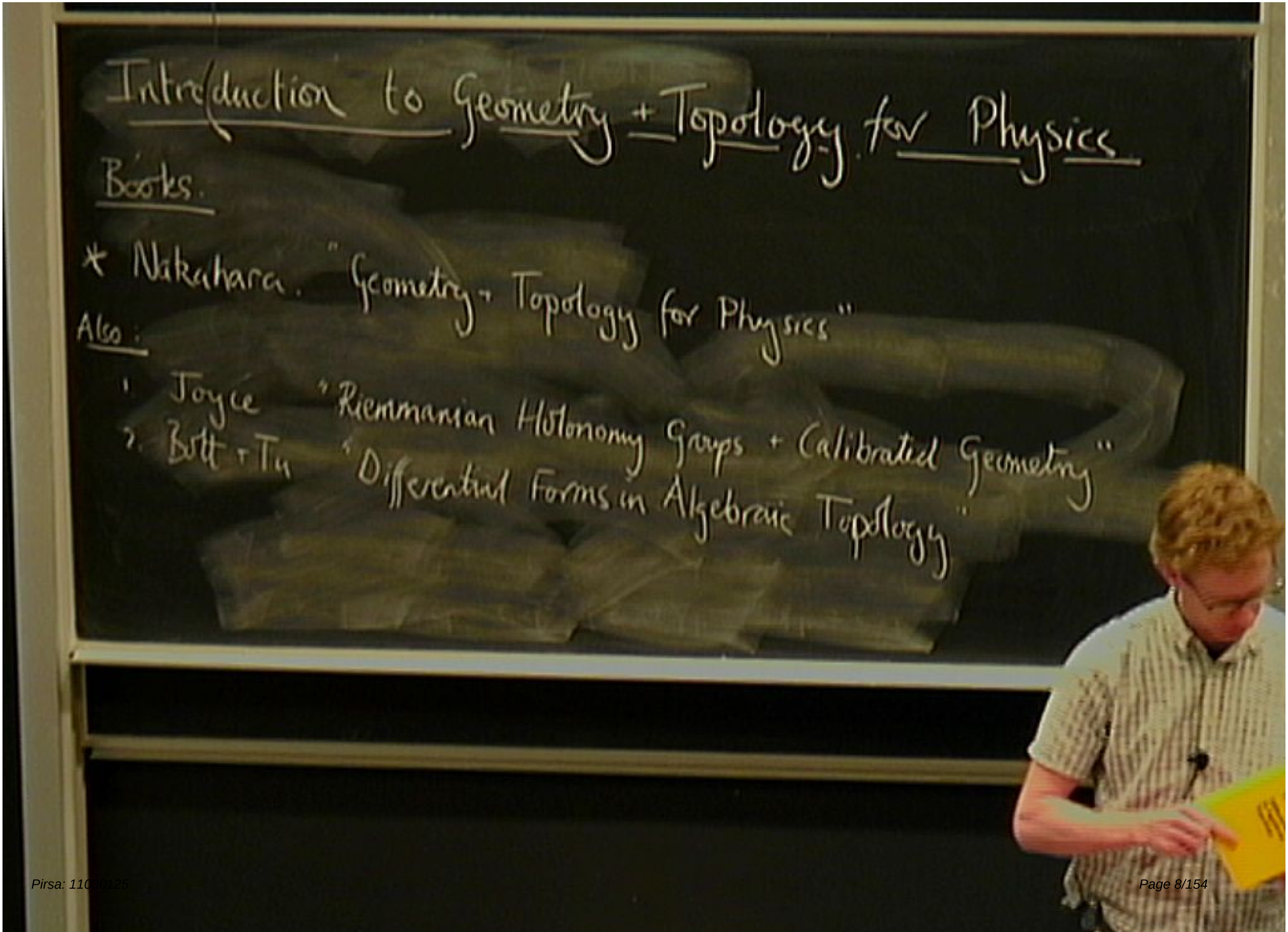
Also.
1 Joyce "Riemmanian Holonomy Groups + Calibrated Geometry"
2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Also:
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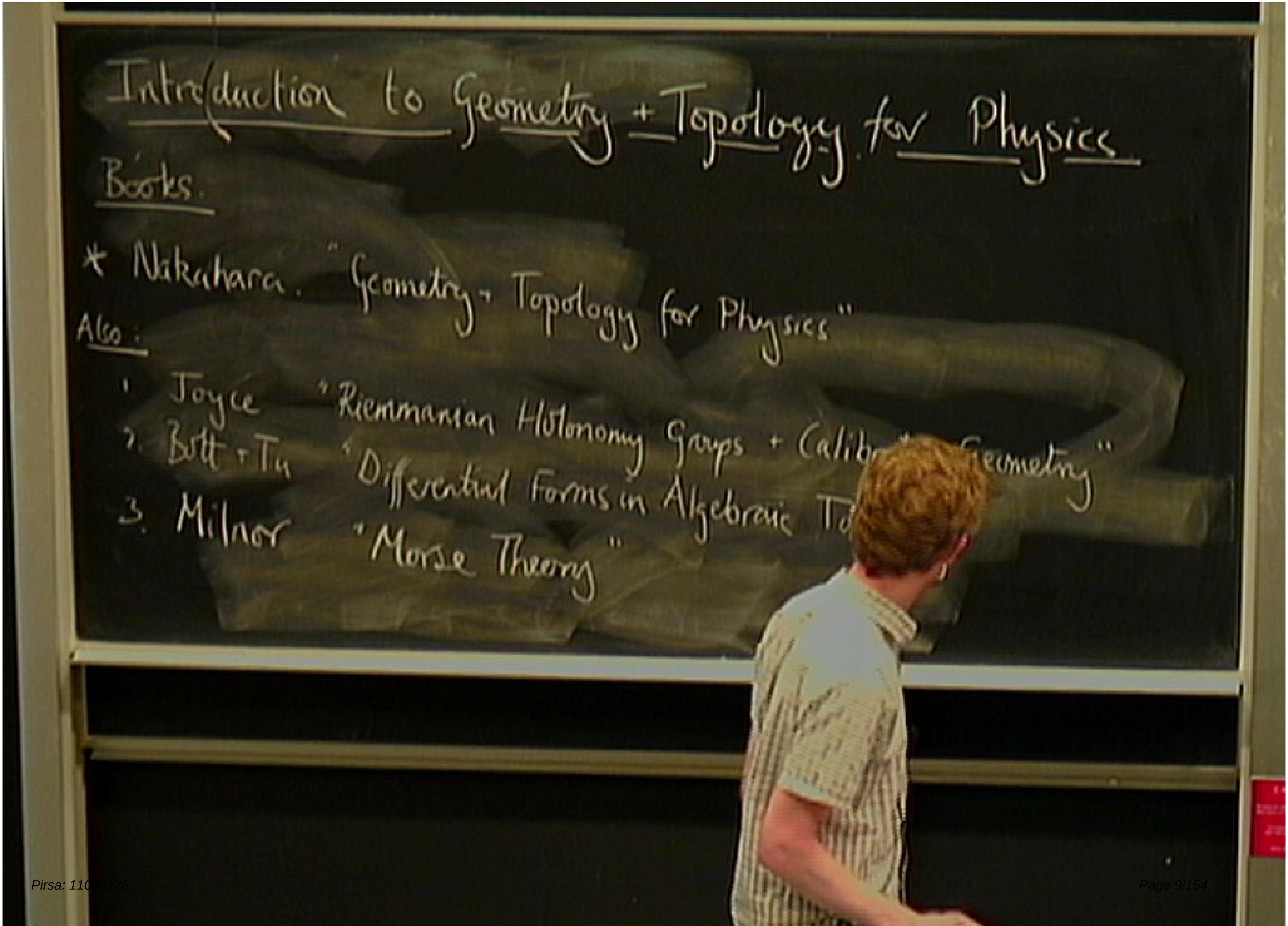
2. Bott + Tu "Differential Forms in Algebraic Topology"



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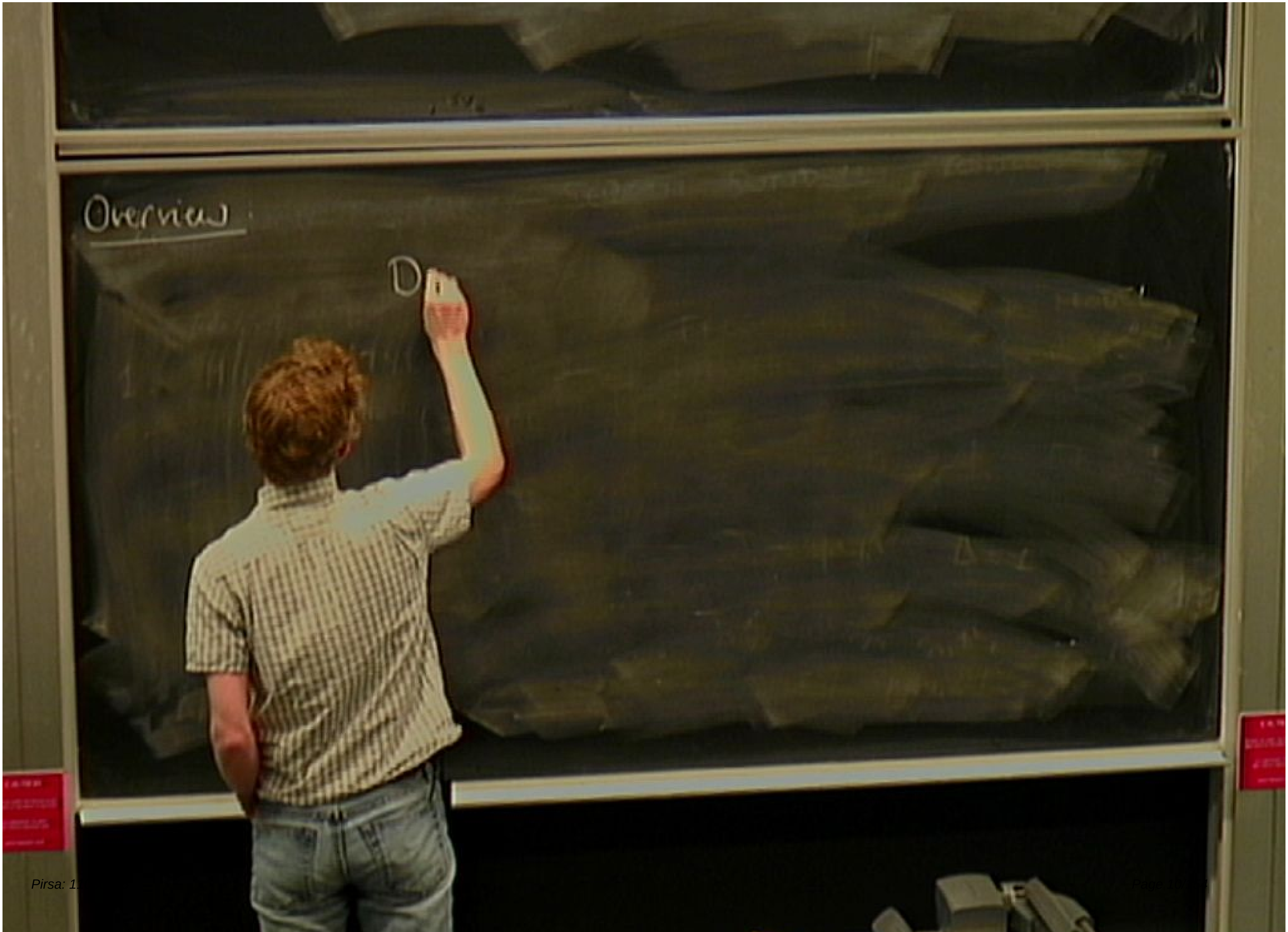
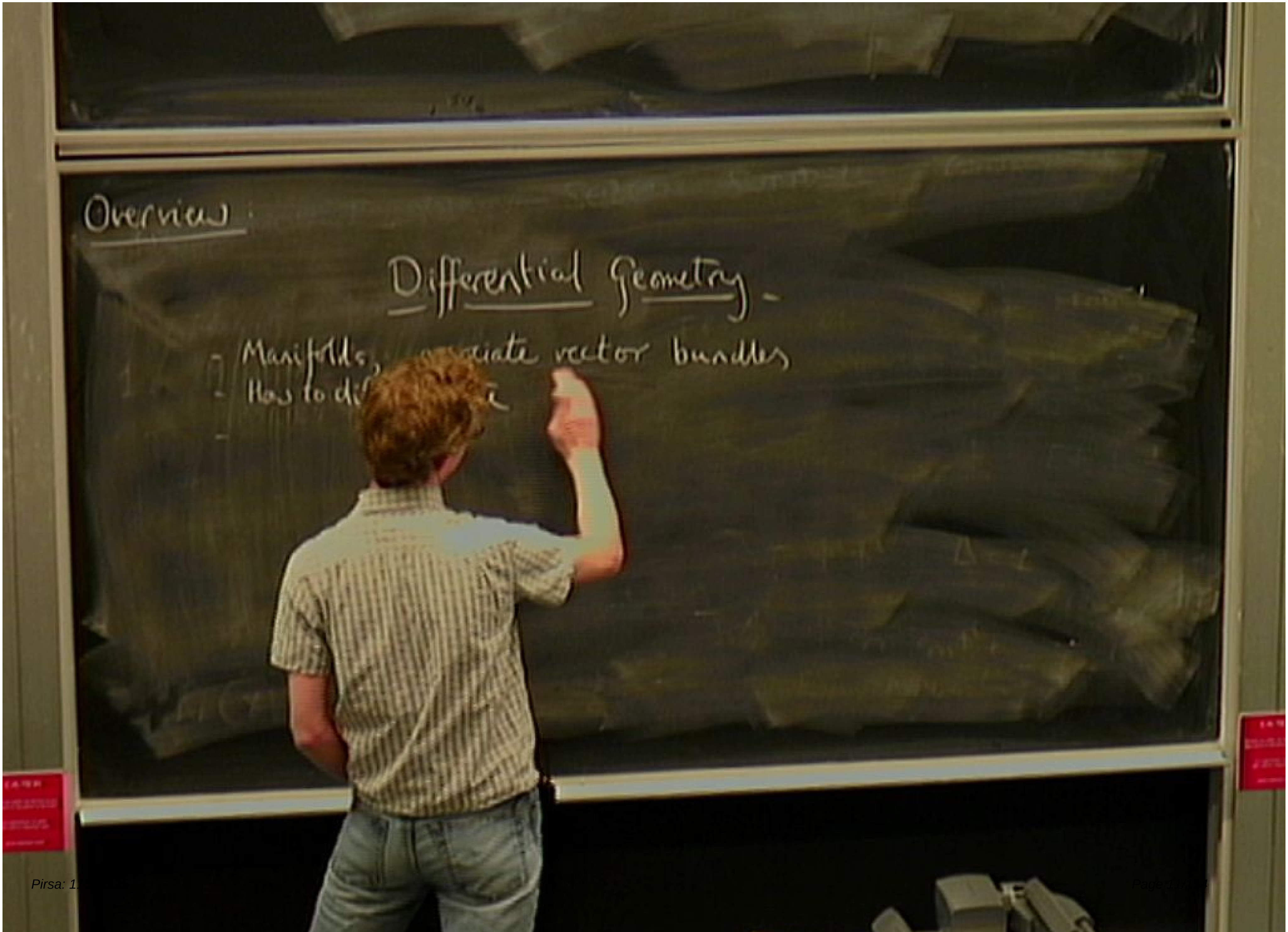
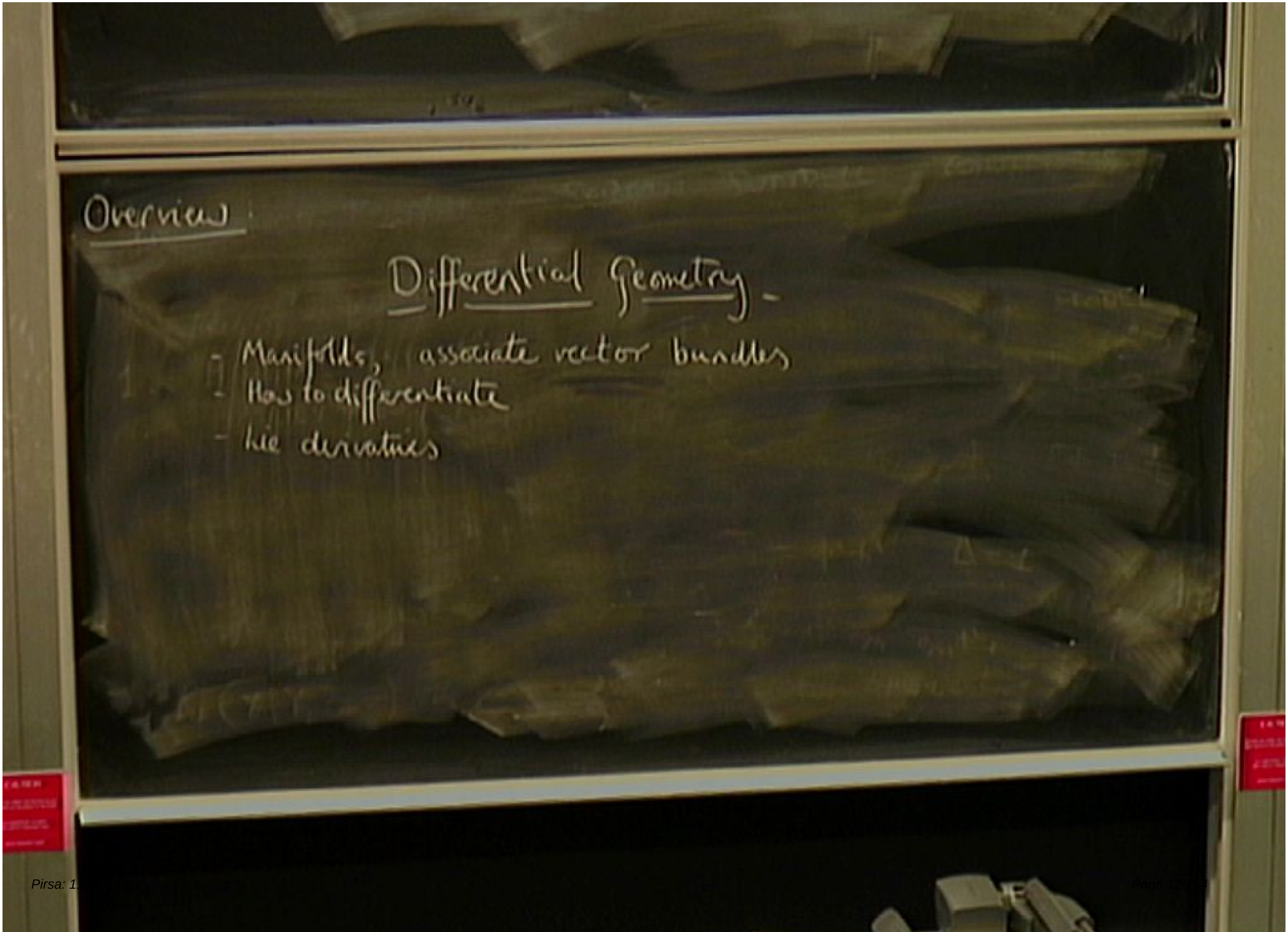
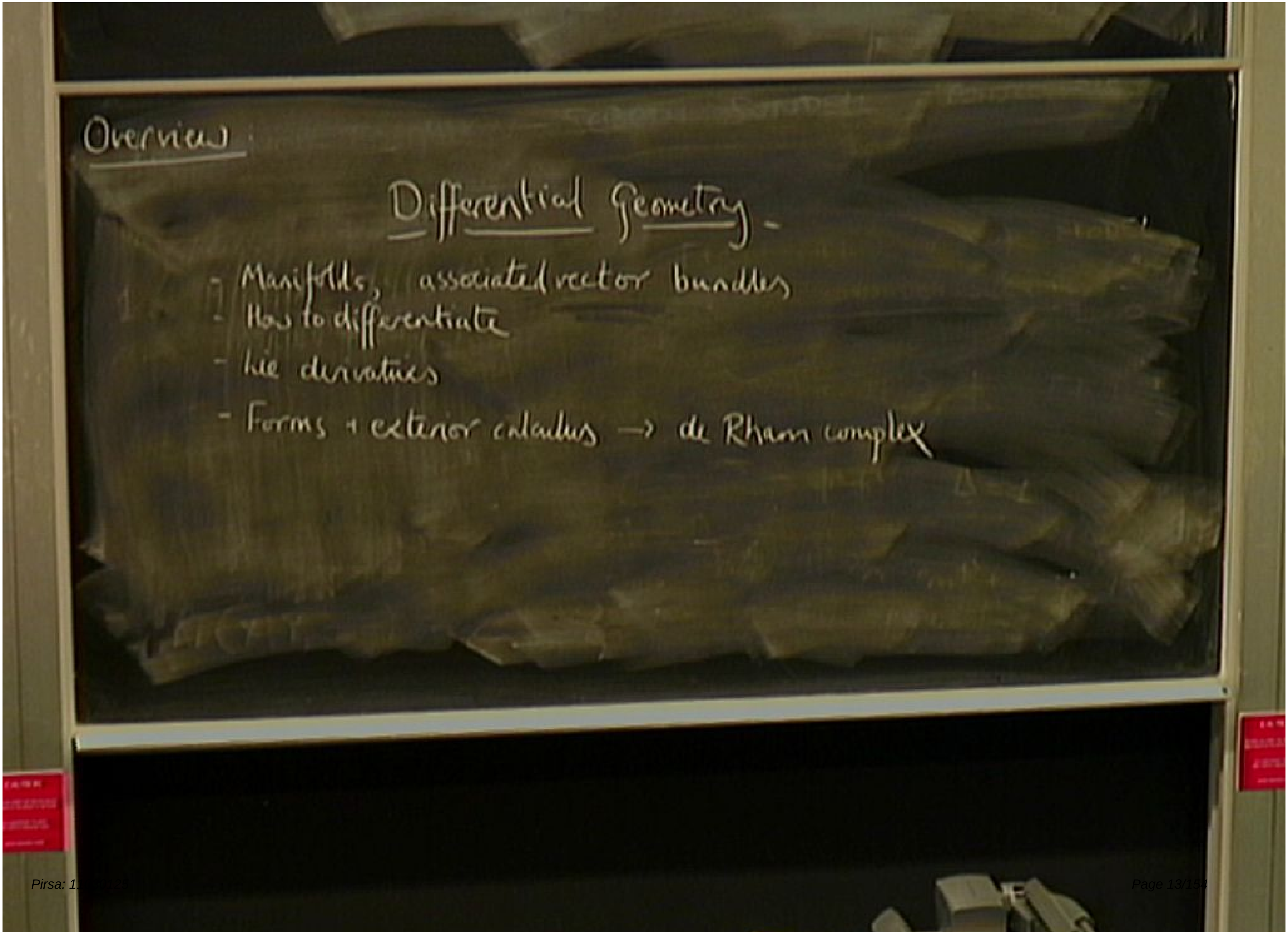
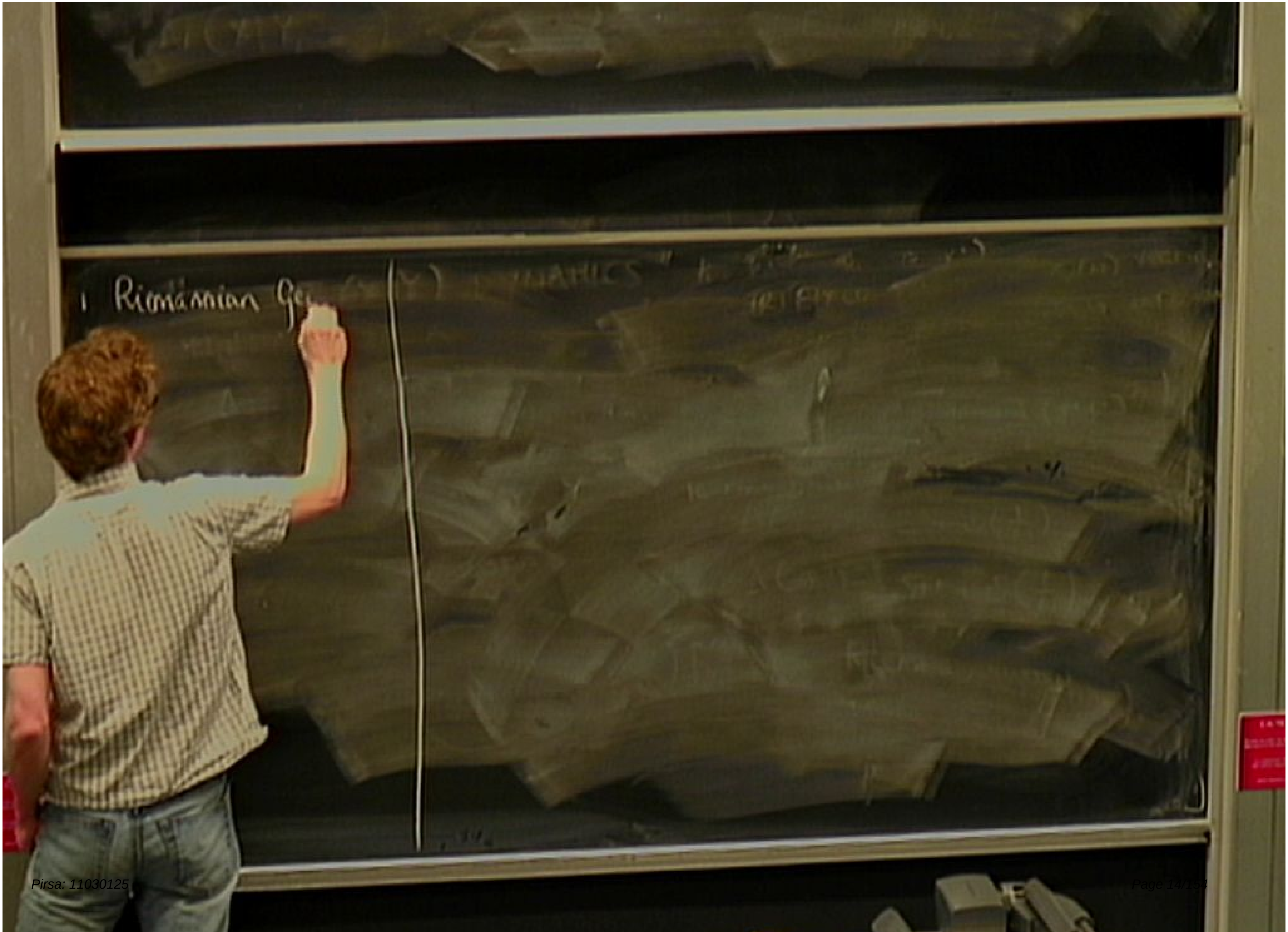
2. Bott + Tu "Differential Forms in Algebraic To
3. Milnor "Morse Theory"

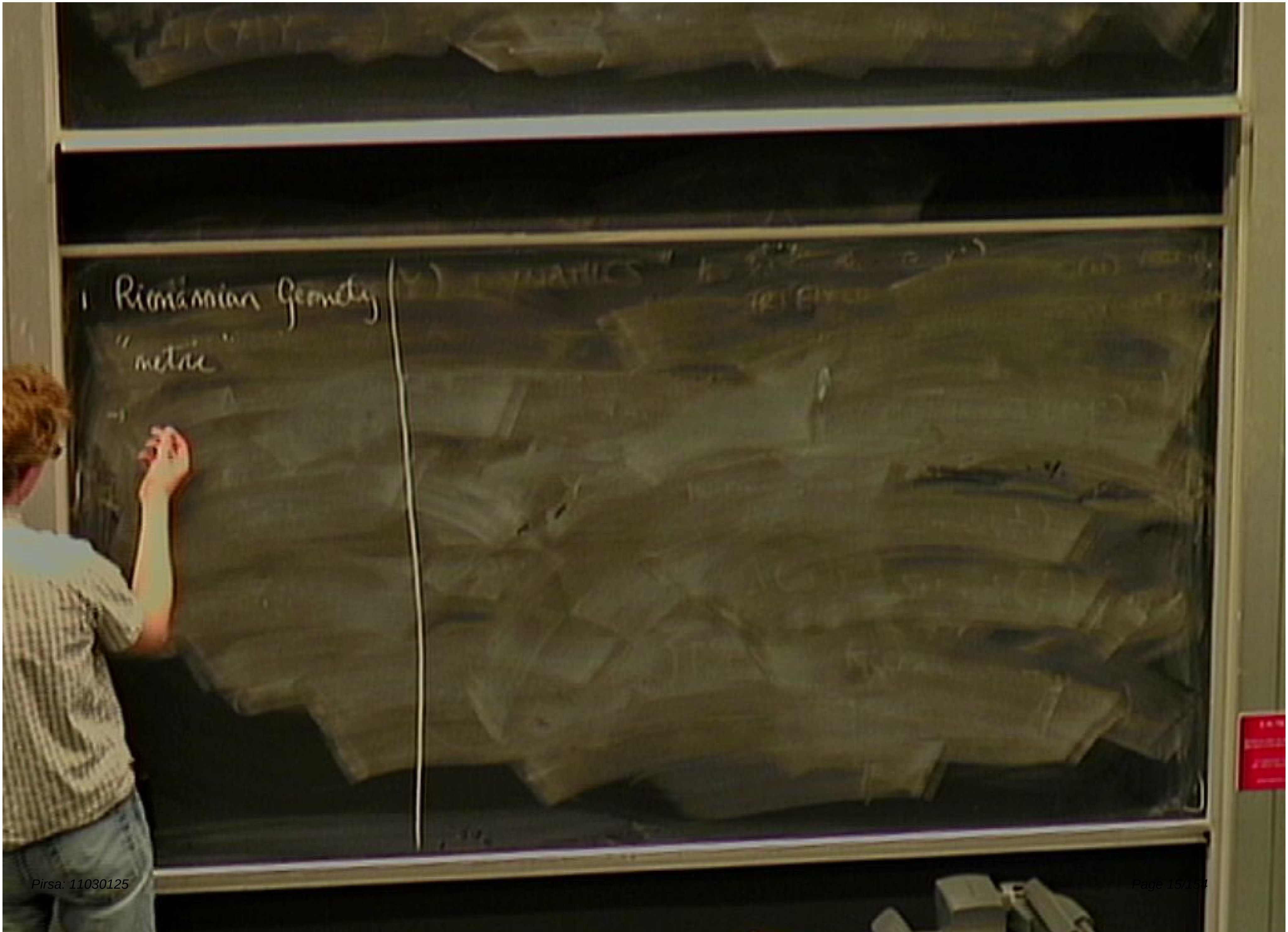

Overview

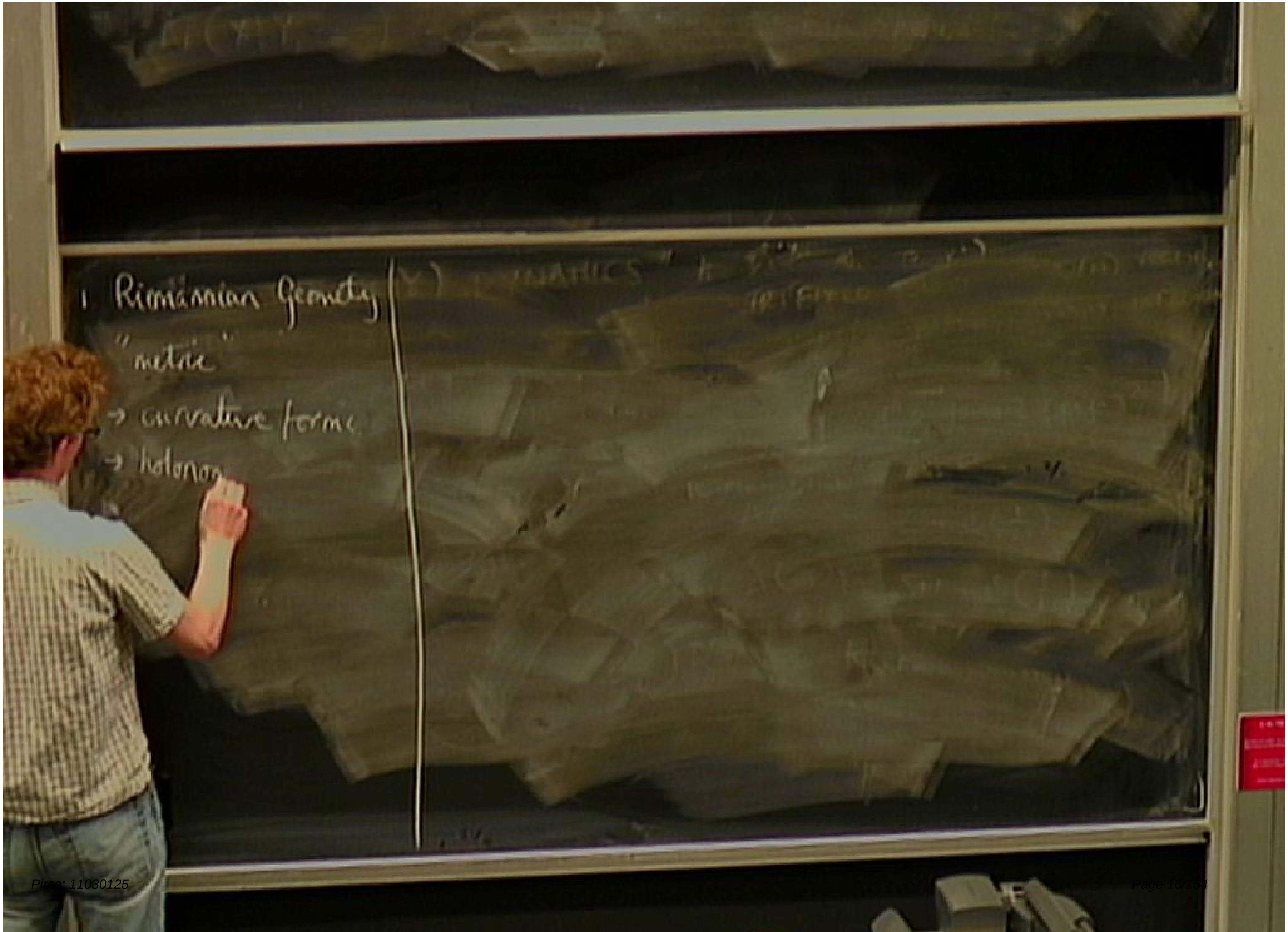

Overview
Differential Geometry
Manifolds,
vector bundles

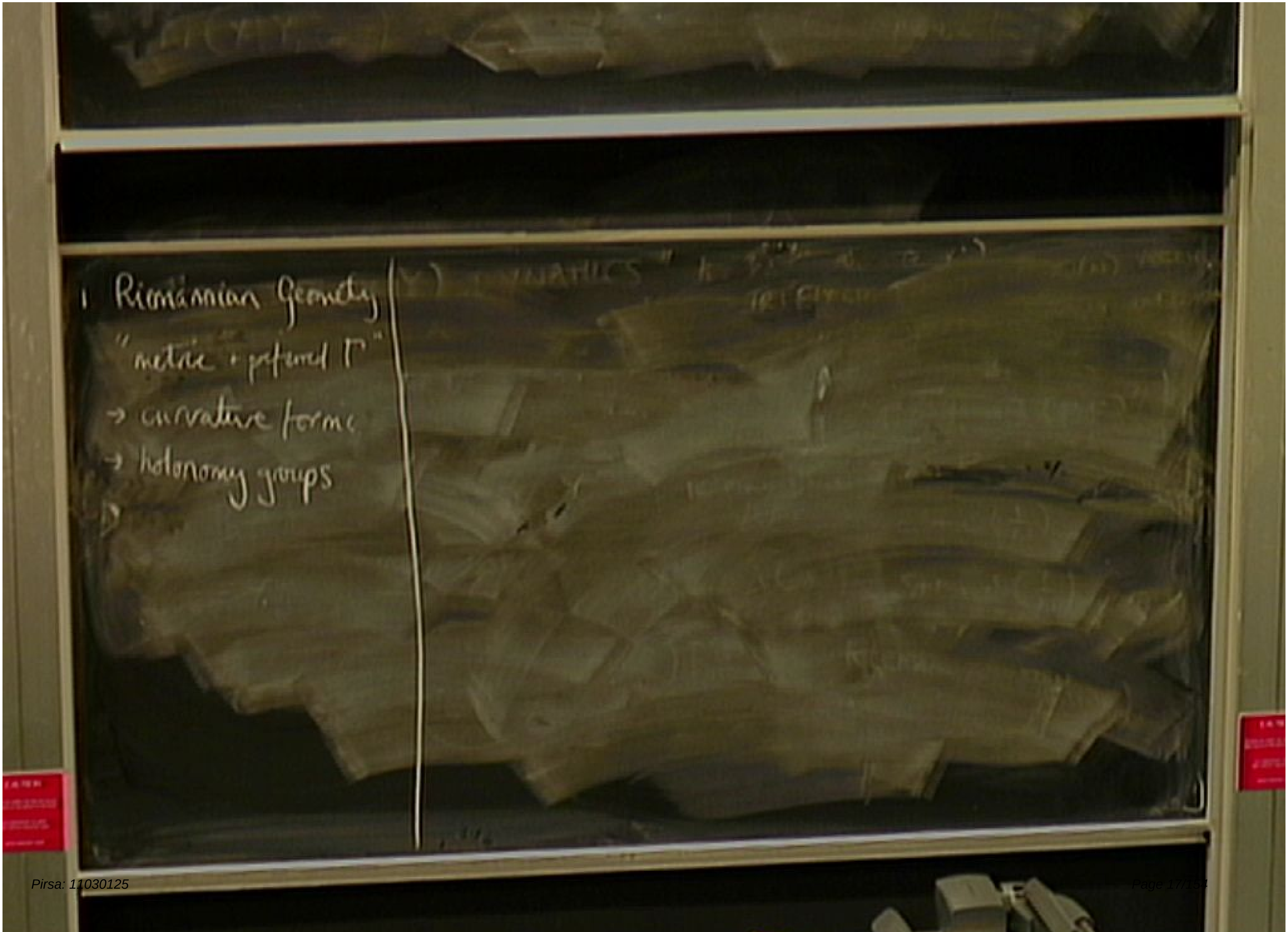

Overview
Differential Geometry
- Manifolds, associate vector bundles
- How to differentiate
- Lie derivatives

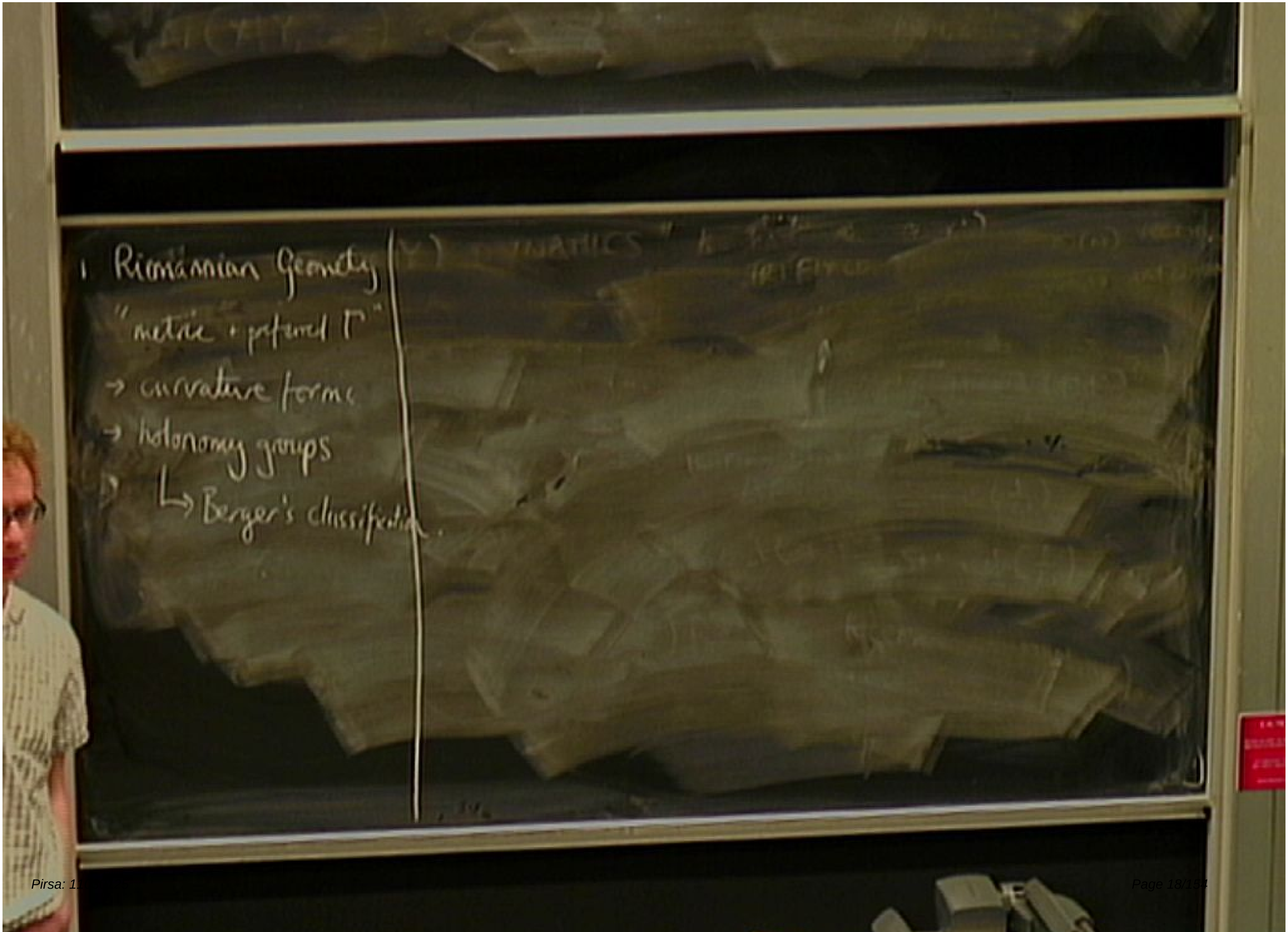

Overview
Differential Geometry
- Manifolds, associated vector bundles
- How to differentiate
- Lie derivatives
- Forms + exterior calculus → de Rham complex

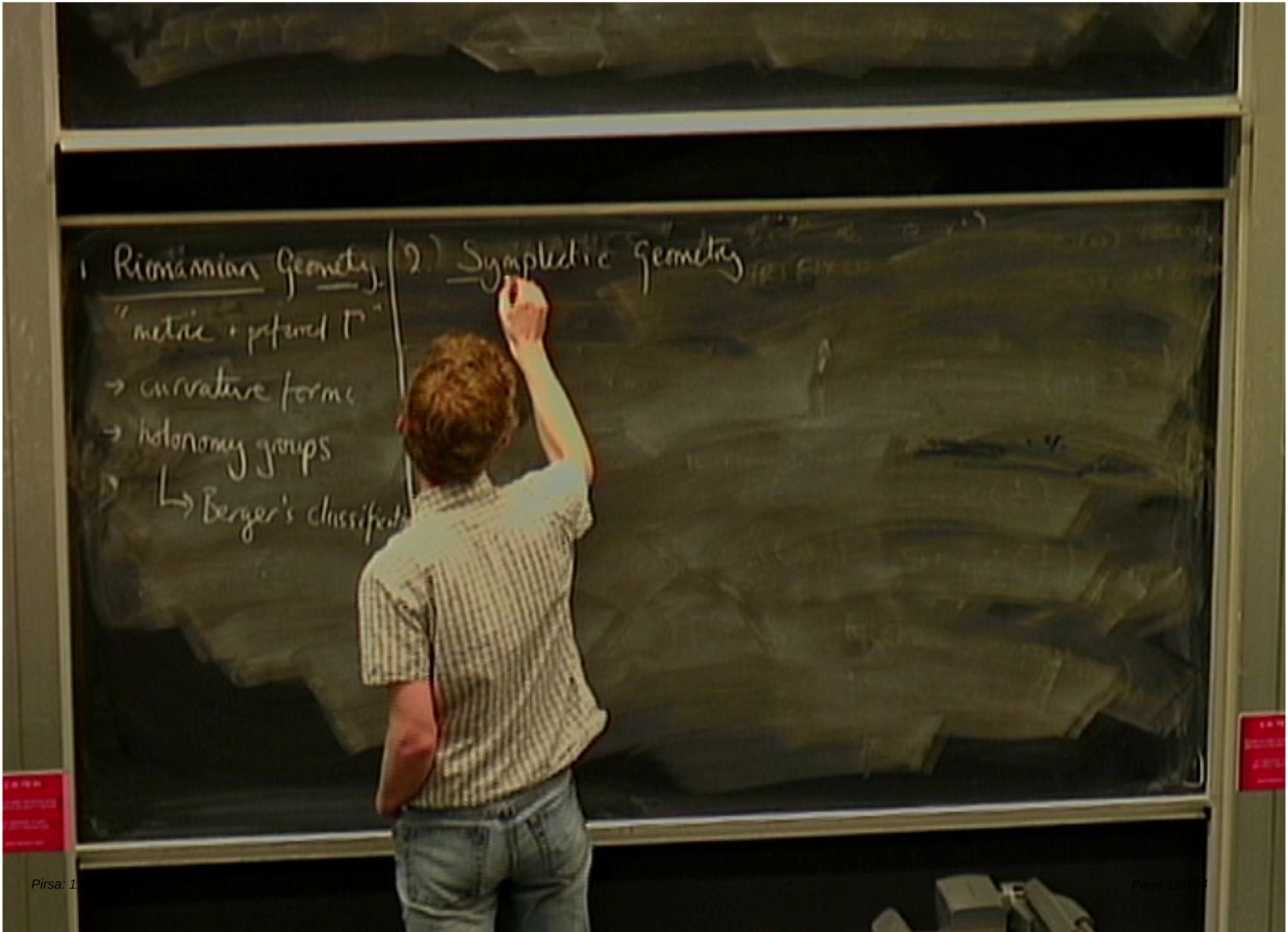

Riemannian

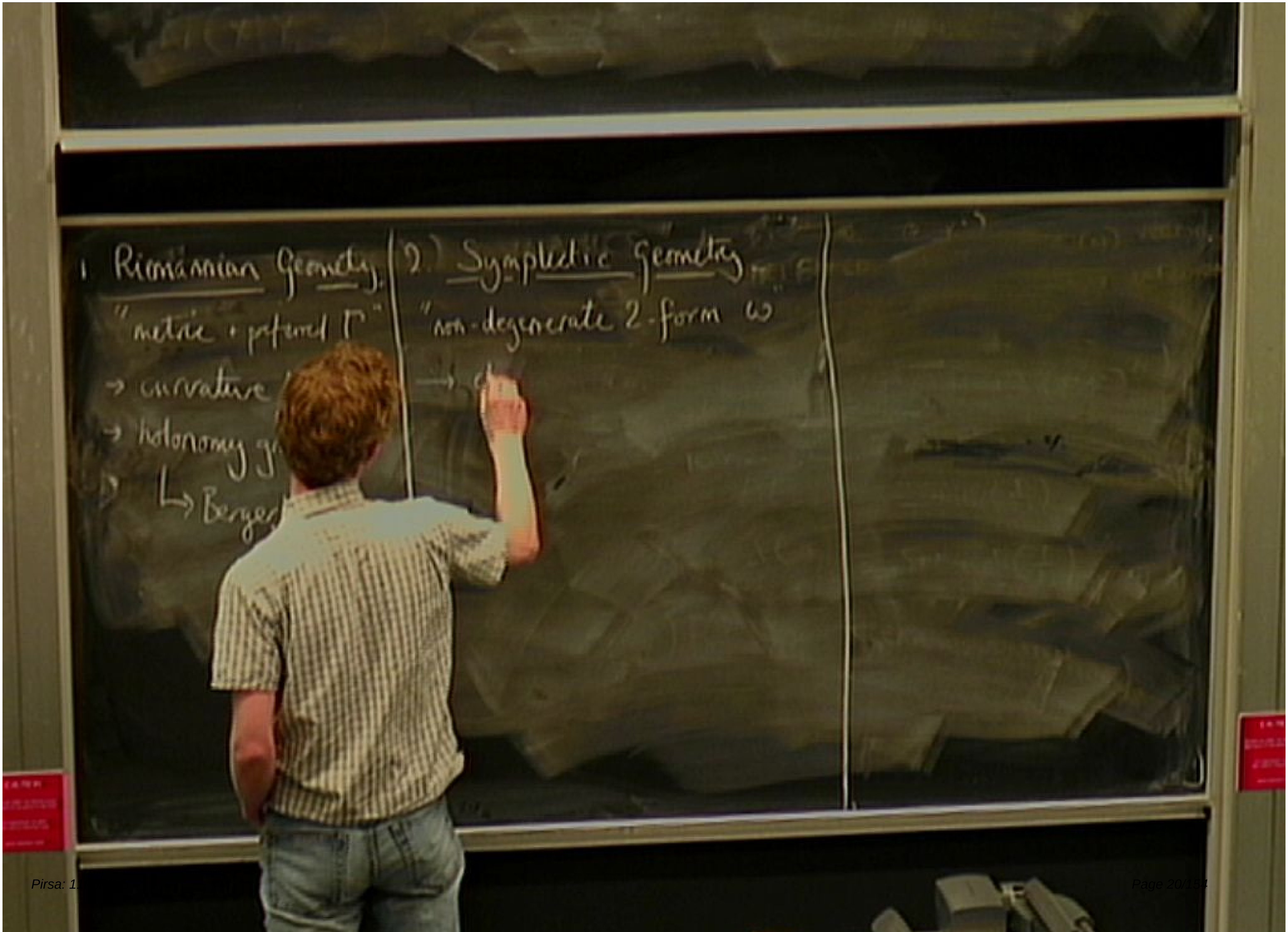

1 Riemannian Geometry
"metric"

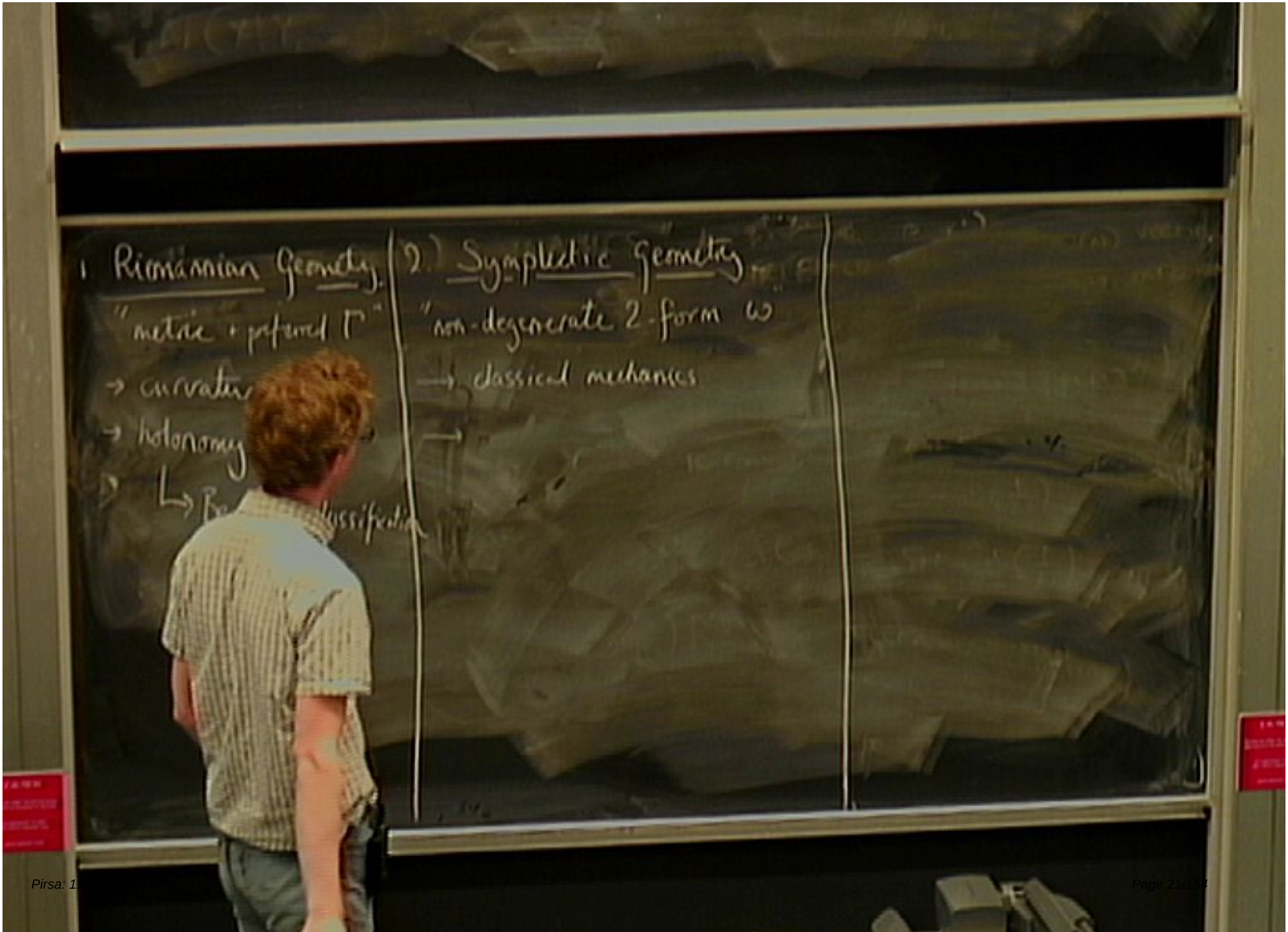

1 Riemannian Geometry
"metric"
→ curvature forms
→ holonom

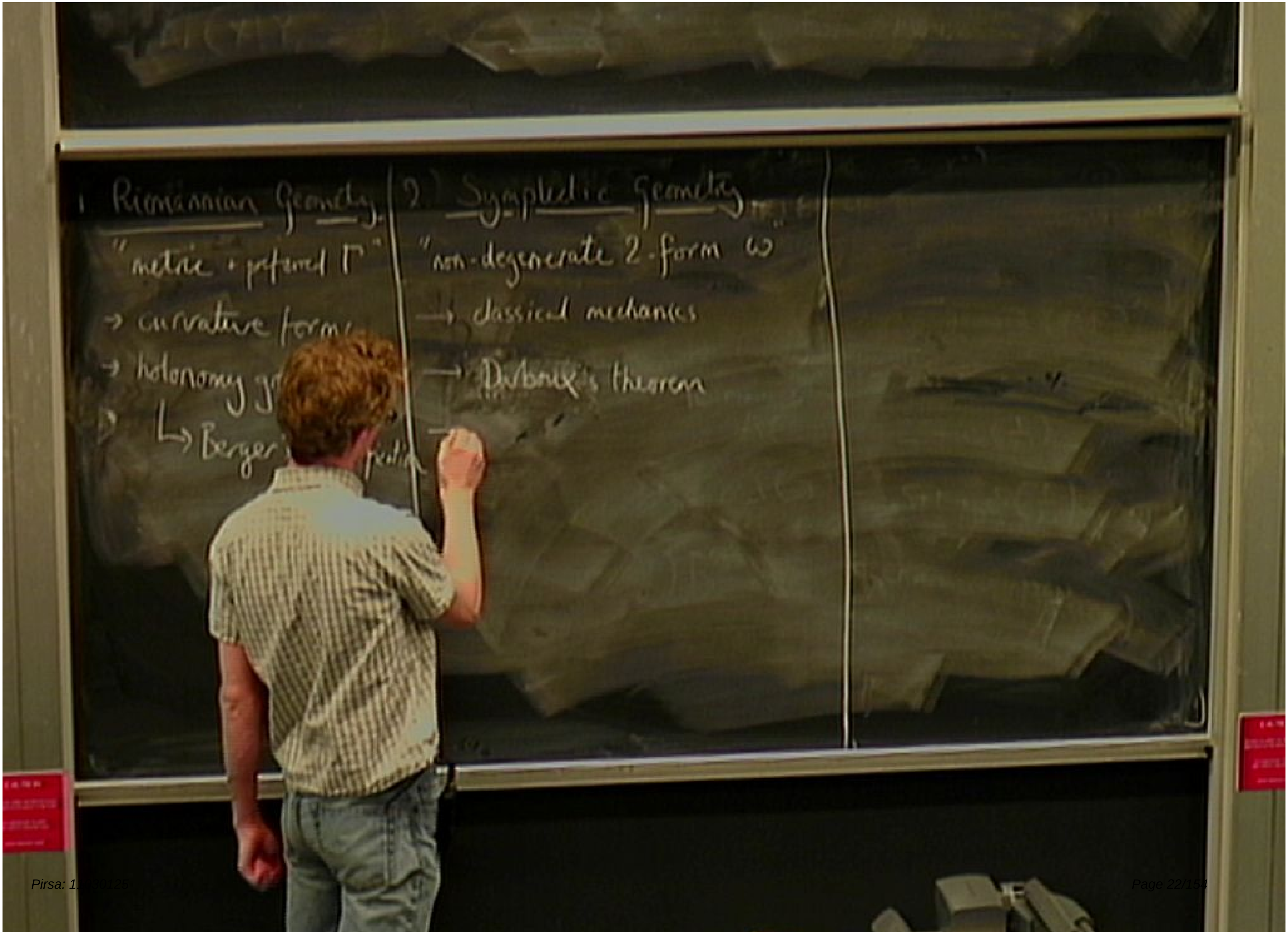

1 Riemannian Geometry
"metric + preferred Γ"
→ curvature forms
→ holonomy groups

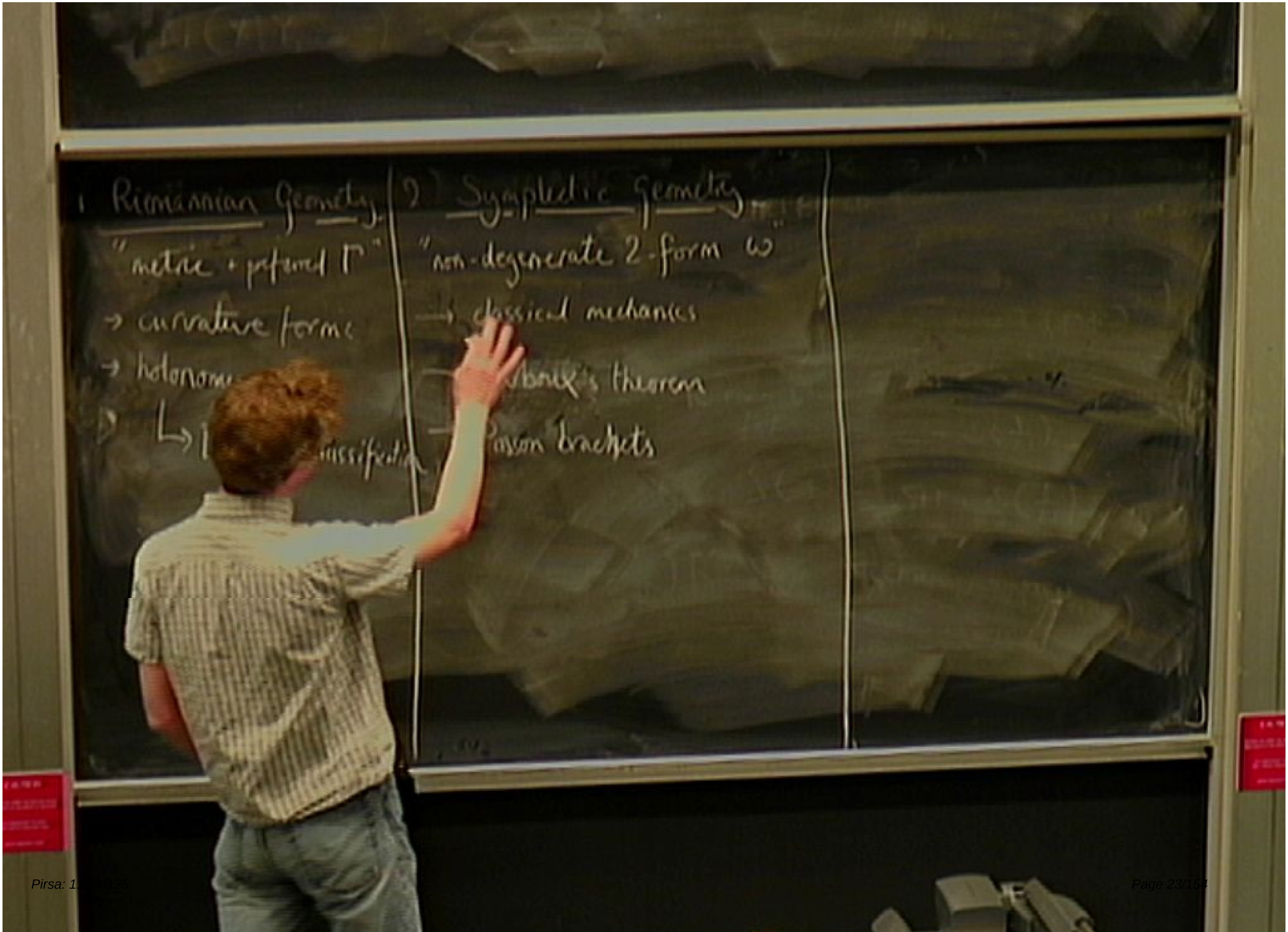

1. Riemannian Geometry
"metric + preferred Γ"
→ curvature forms
→ holonomy groups
↳ Berger's classification.


1) Riemannian Geometry
"metric + preferred Γ"
→ curvature forms
→ holonomy groups
↳ Berger's classificat
2) Symplectic Geometry

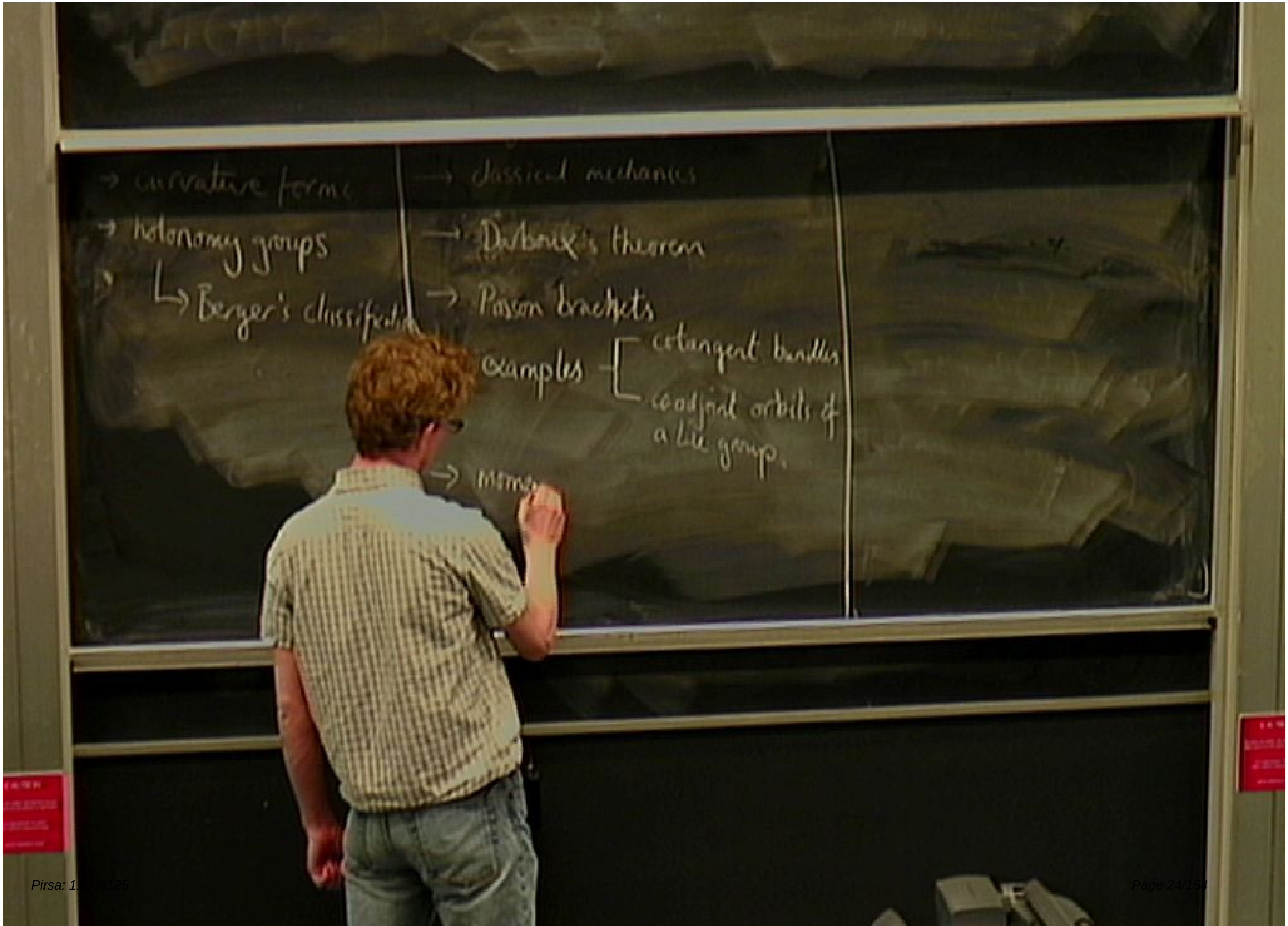

1 Riemannian Geometry
"metric + preferred Γ"
→ curvature
→ holonomy
↳ Berger
2) Symplectic Geometry
"non-degenerate 2-form ω"

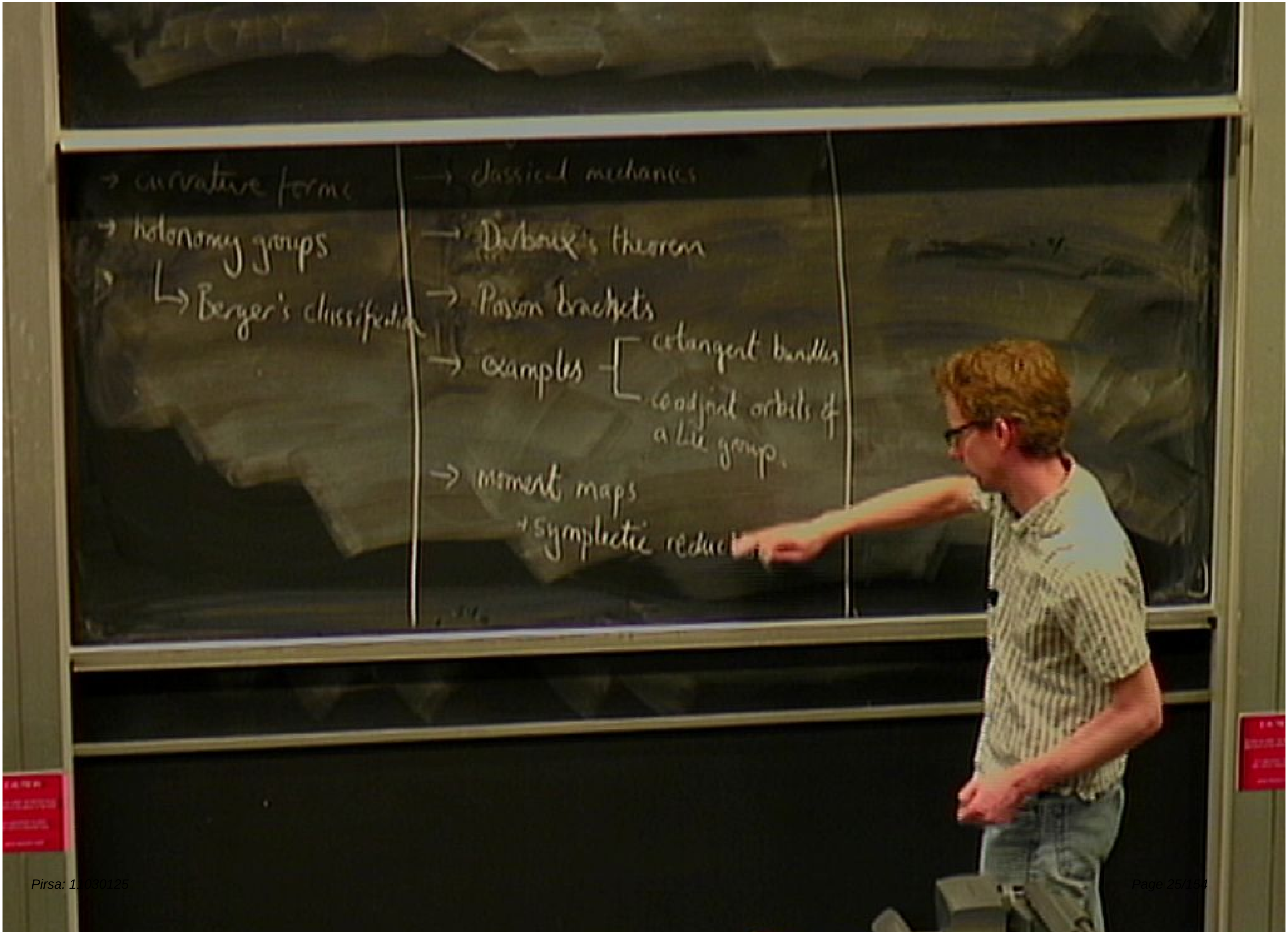

1) Riemannian Geometry
"metric + preferred Γ"
→ curvature
→ holonomy
2) Symplectic Geometry
"non-degenerate 2-form ω"
→ classical mechanics

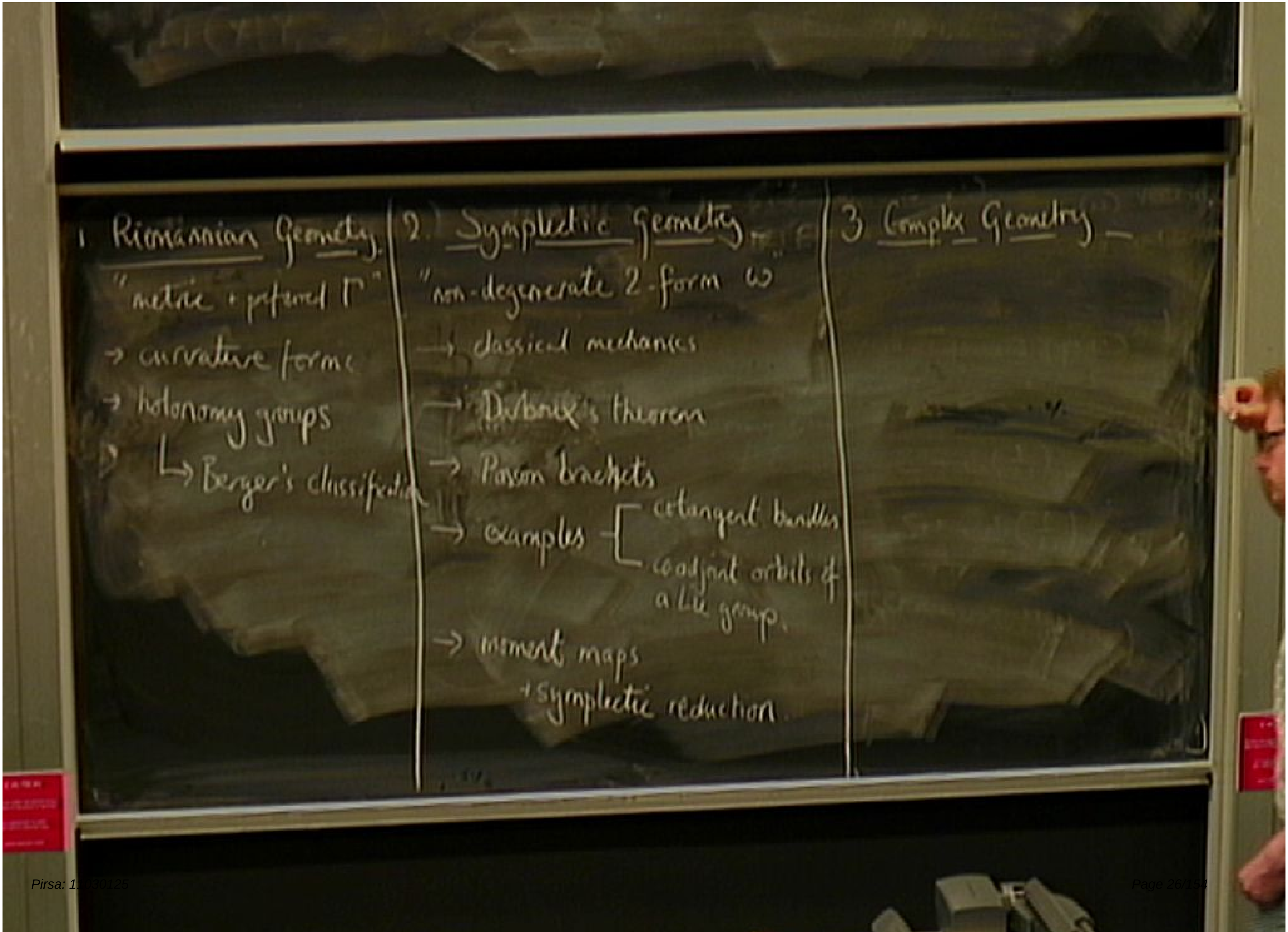

1 Riemannian Geometry
"metric + preferred Γ"
→ curvature form
→ holonomy
↳ Berger's
2 Symplectic Geometry
"non-degenerate 2-form ω
→ classical mechanics
→ Darboux's theorem



1 Riemannian Geometry
"metric + preferred Γ"
→ curvature forms
→ holonomy
2) Symplectic Geometry
"non-degenerate 2-form ω"
→ classical mechanics
's theorem
brackets


→ curvature forms
→ holonomy groups
↳ Berger's classificati
→ classical mechanics
→ Darboux's theorem
→ Poisson brackets
examples
cotangent bundles
coadjoint orbits of a Lie group.


→ curvature forms
→ holonomy groups
↳ Berger's classification
→ classical mechanics
→ Darboux's theorem
→ Poisson brackets
→ examples
cotangent bundles
coadjoint orbits of a Lie group.
→ moment maps
+ symplectic


1 Riemannian Geometry
"metric + preferred Γ"
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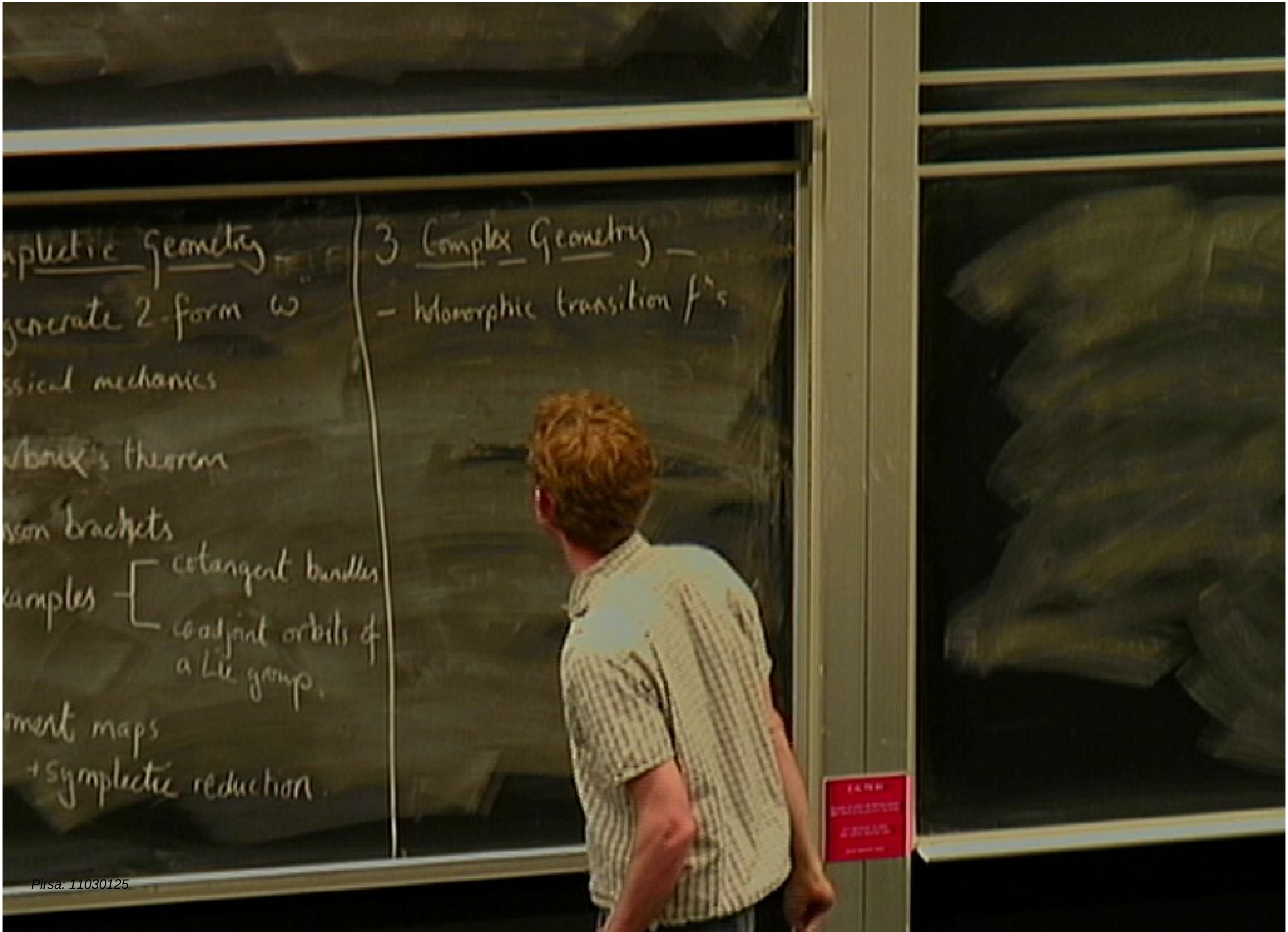
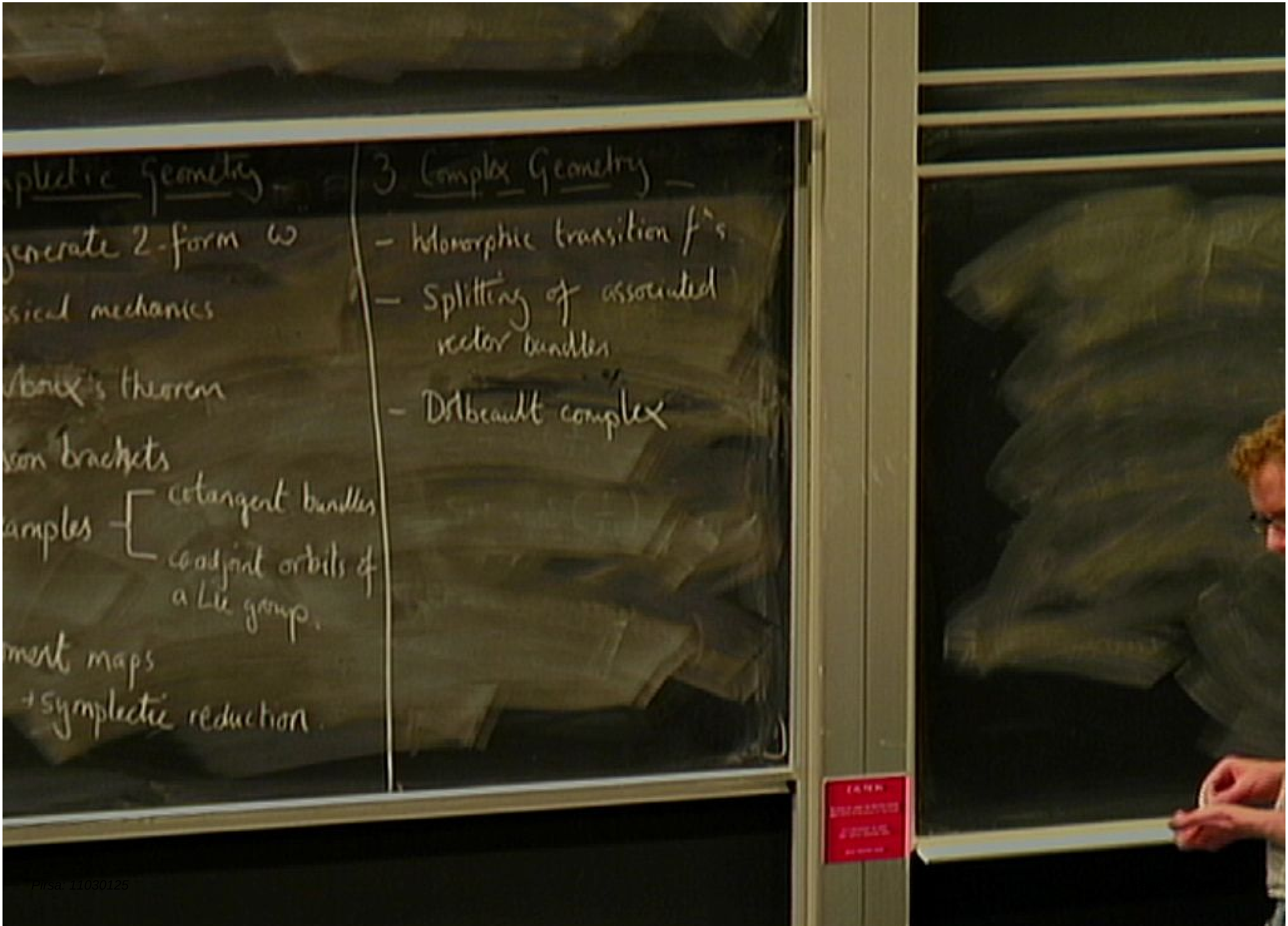
↳ Berger's classification
2. Symplectic Geometry
"non-degenerate 2-form ω
→ classical mechanics
→ Darboux's theorem
→ Poisson brackets
→ examples
cotangent bundles
coadjoint orbits of a Lie group.
→ moment maps
+ symplectic reduction
3 Complex Geometry



...mplectic Geometry
...generate 2-form ω
...ssical mechanics
...boux's theorem
...sson brackets
...amples
cotangent bundles
coadjoint orbits of a Lie group.
...ment maps
+ symplectic reduction
3 Complex Geometry
- holomorphic transition f^ns


plectic Geometry
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ssical mechanics
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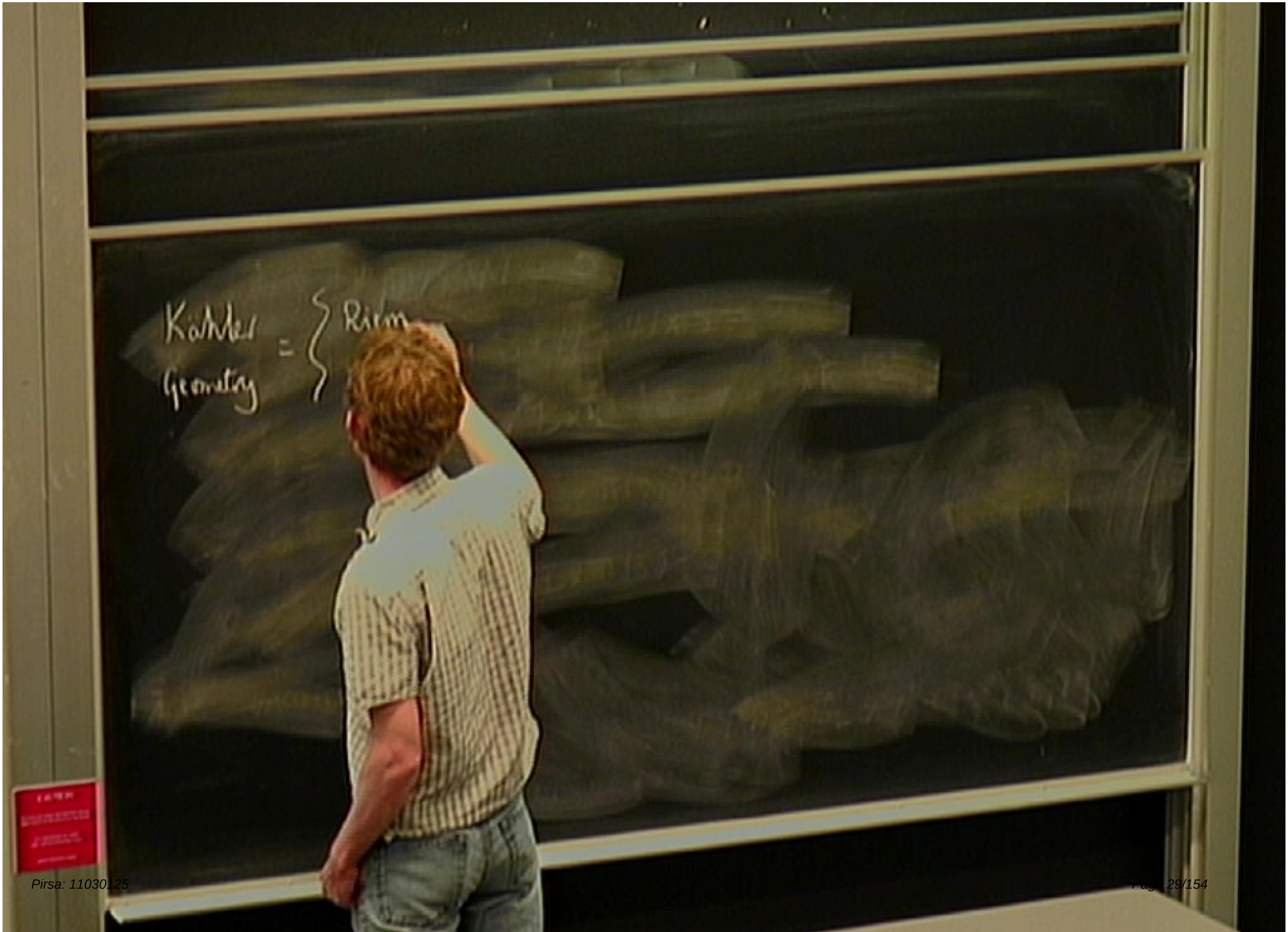
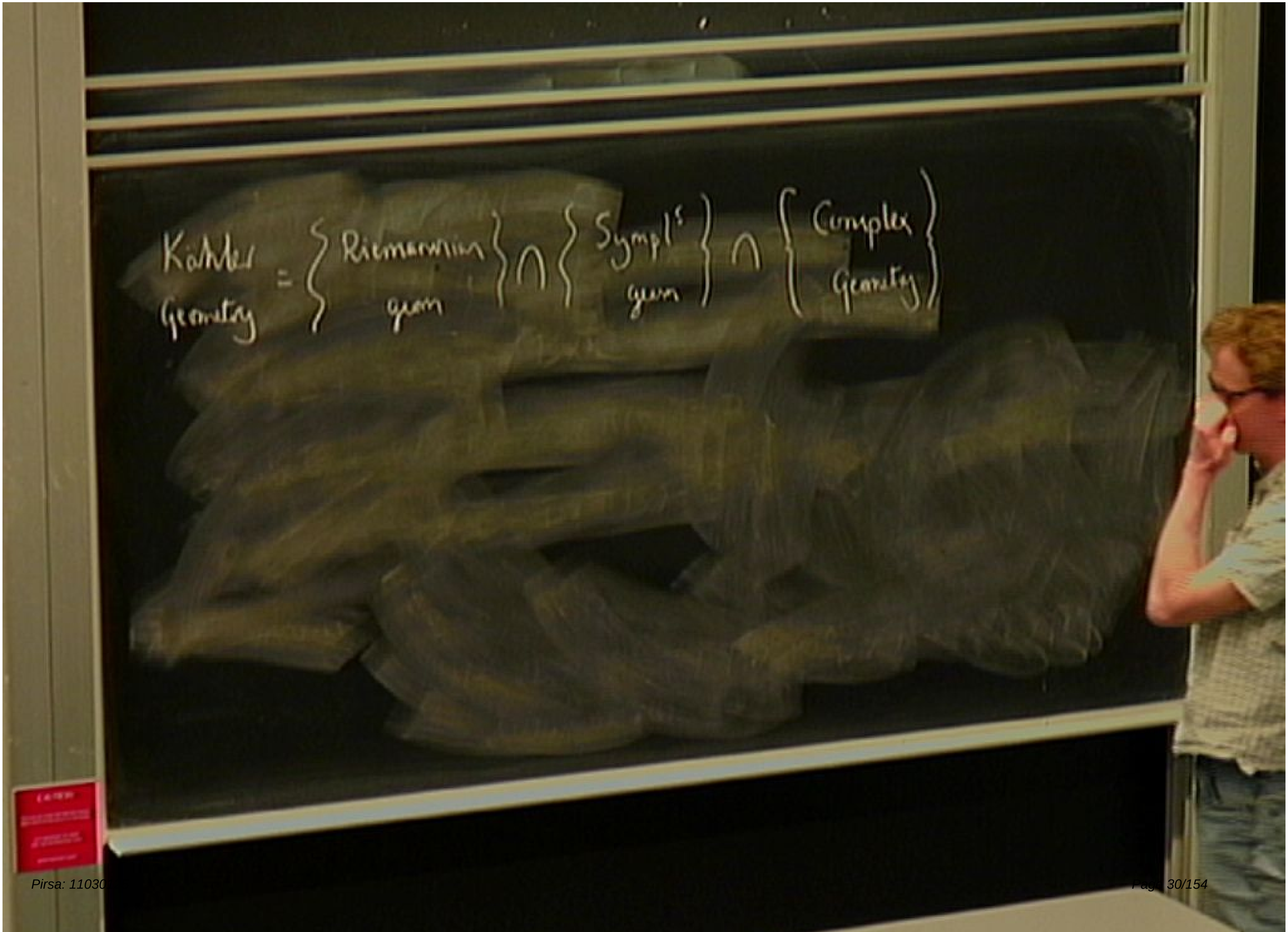
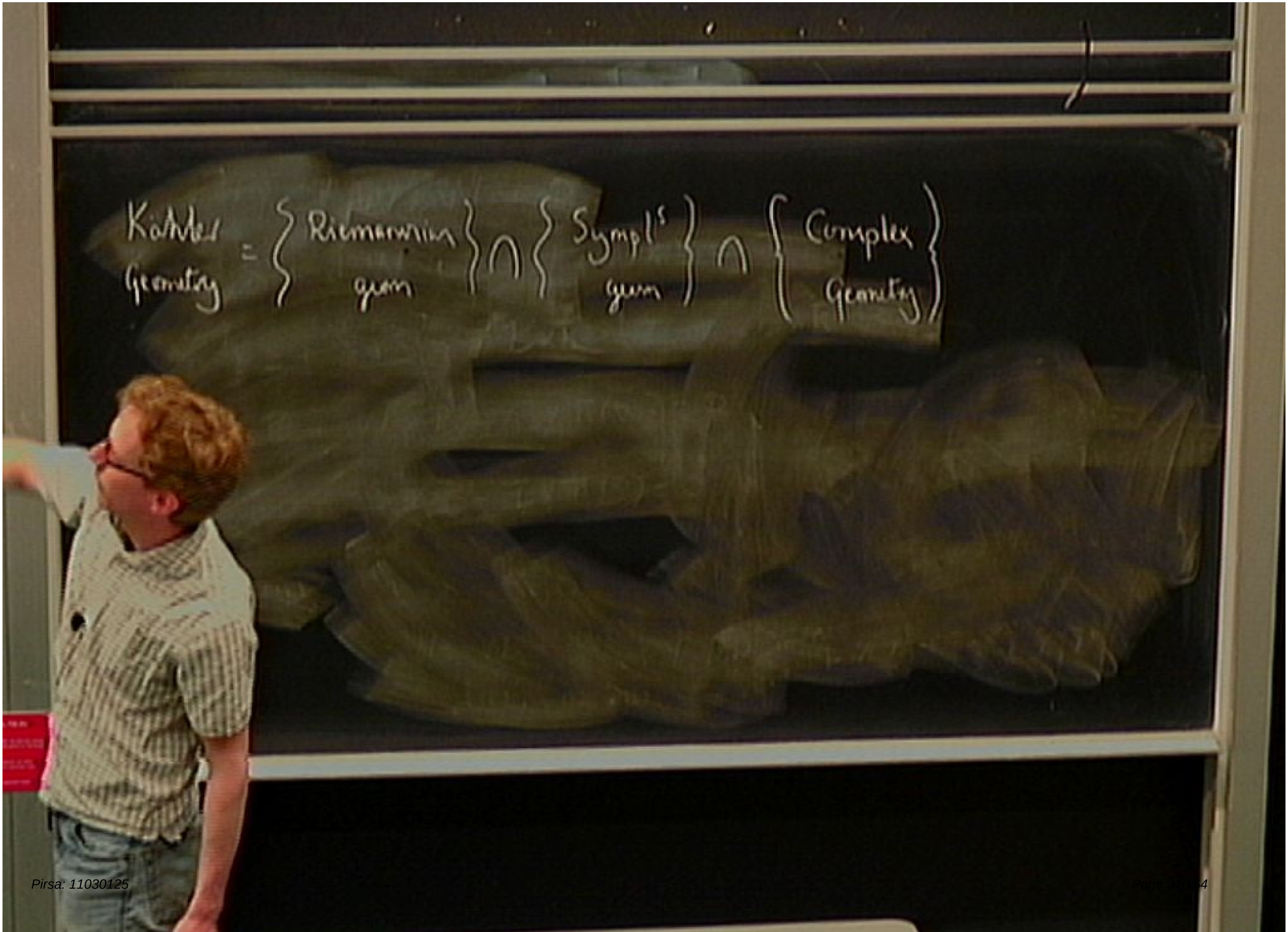
ment maps
+ symplectic reduction.
3 Complex Geometry
- holomorphic transition f^ns
- Splitting of associated vector bundles
- Dolbeault complex


Kähler
Geometry
=
Riem


Kähler Geometry = { Riemannian geom } ∩ { Sympl^c geom } ∩ { Complex Geometry }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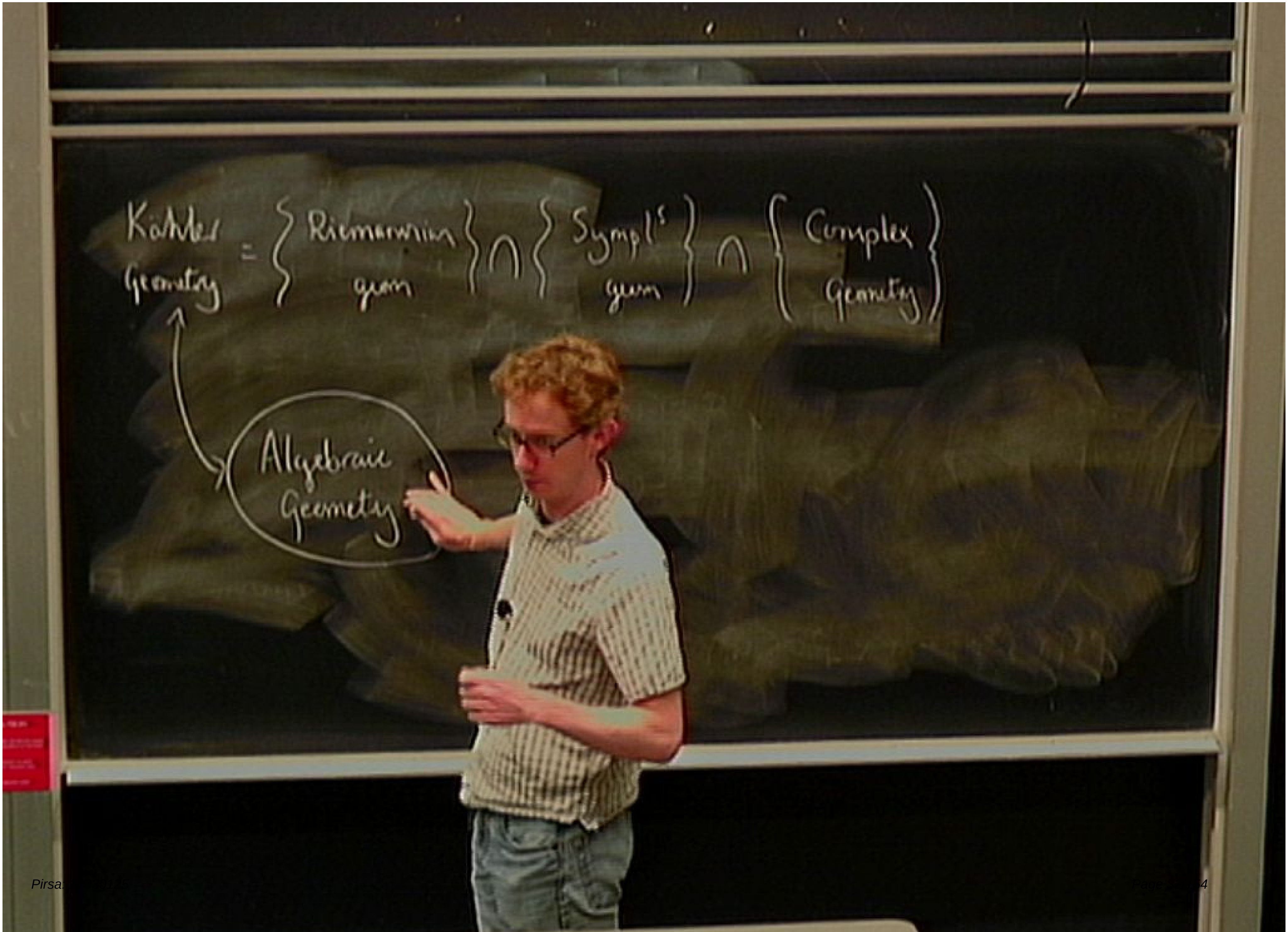
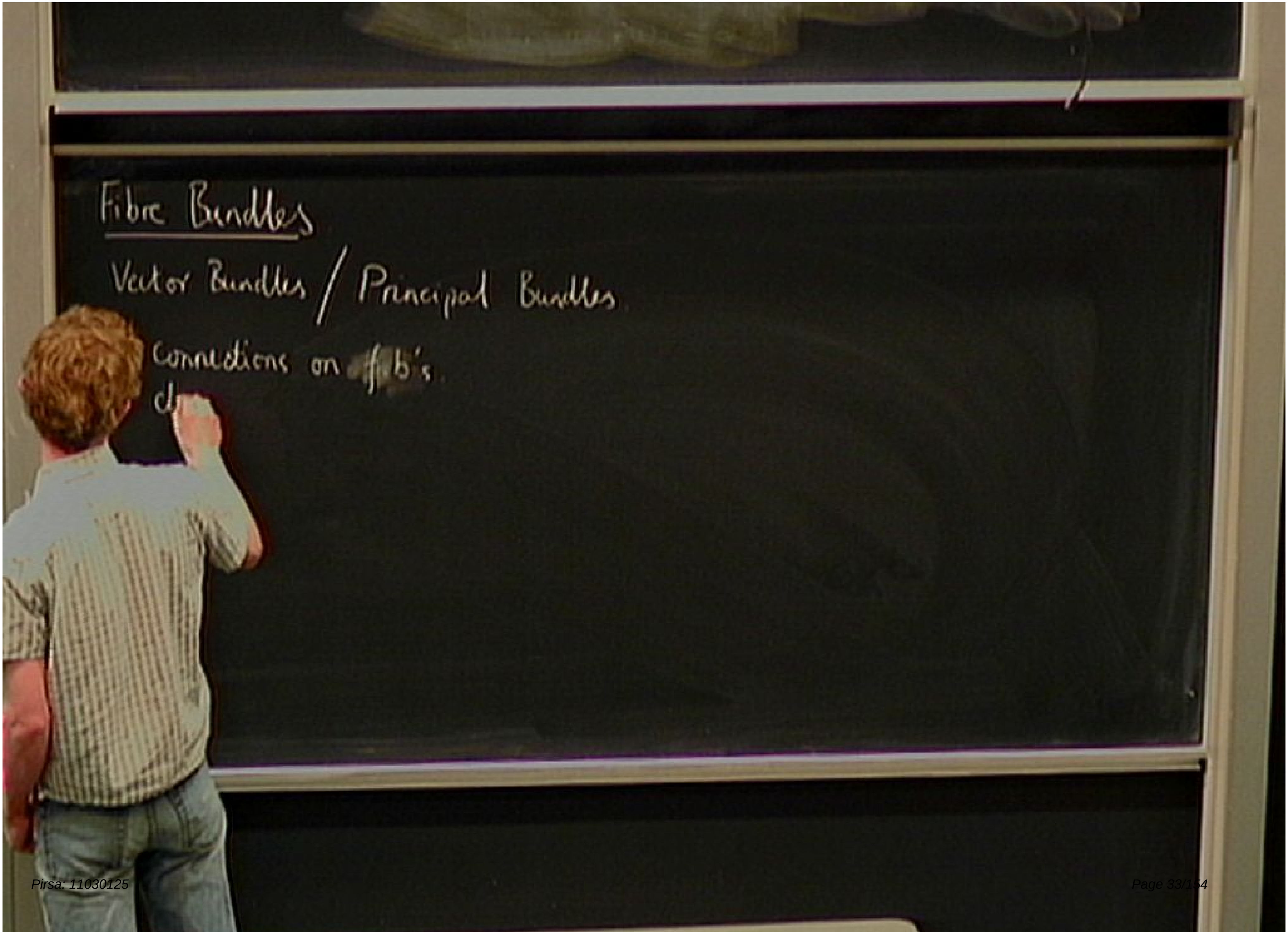
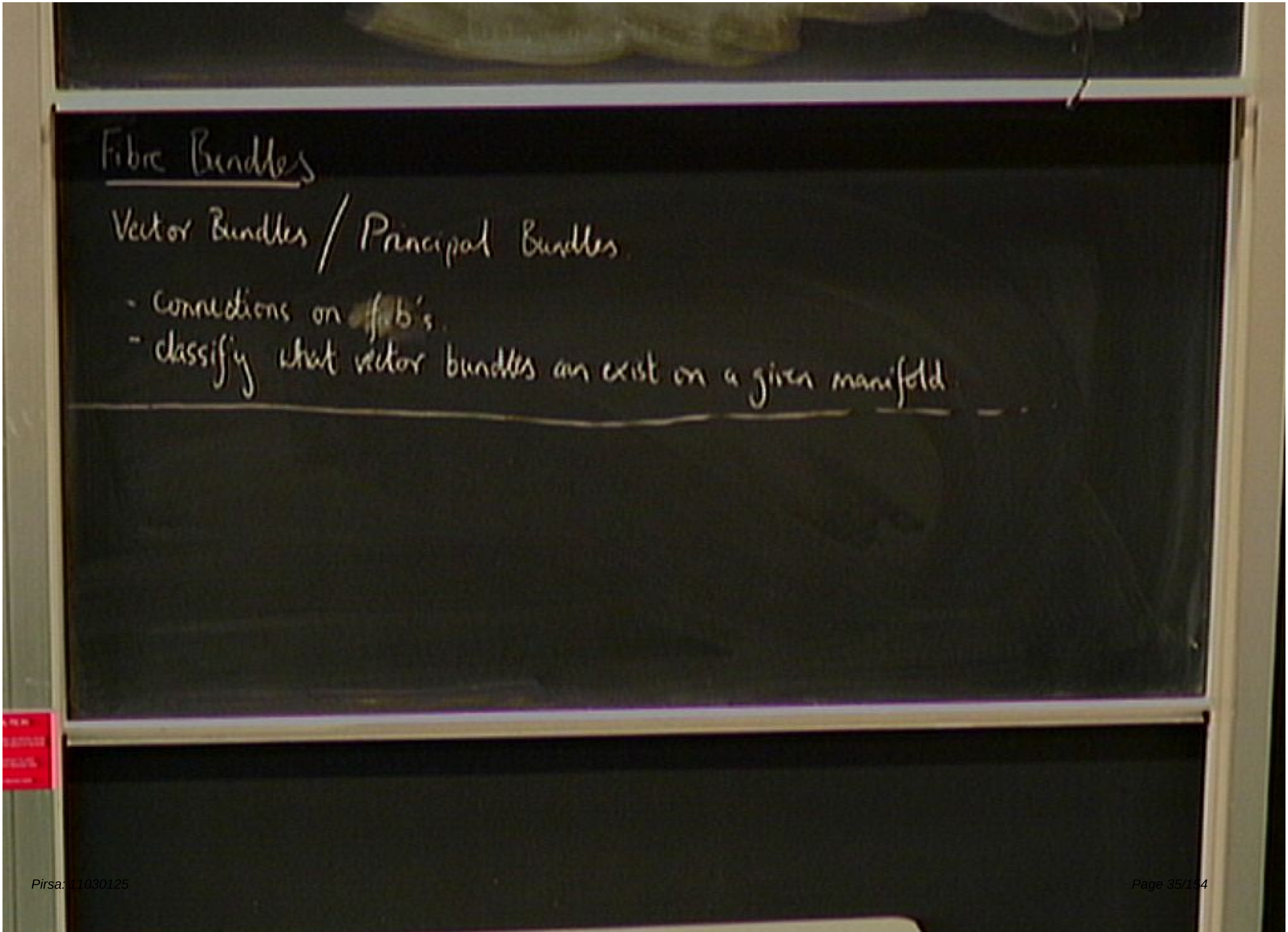

Kähler Geometry = { Riemannian geom } ∩ { Sympl'c geom } ∩ { Complex Geometry }
Algebraic Geometry


Fibre Bundles
Vector Bundles / Principal Bundles
connections on f.b's

Fibre Bundles

Vector Bundles / Principal Bundles

- connections on f.b's.
- classify what vector bundles can exist on a given manifold.


Fibre Bundles
Vector Bundles / Principal Bundles
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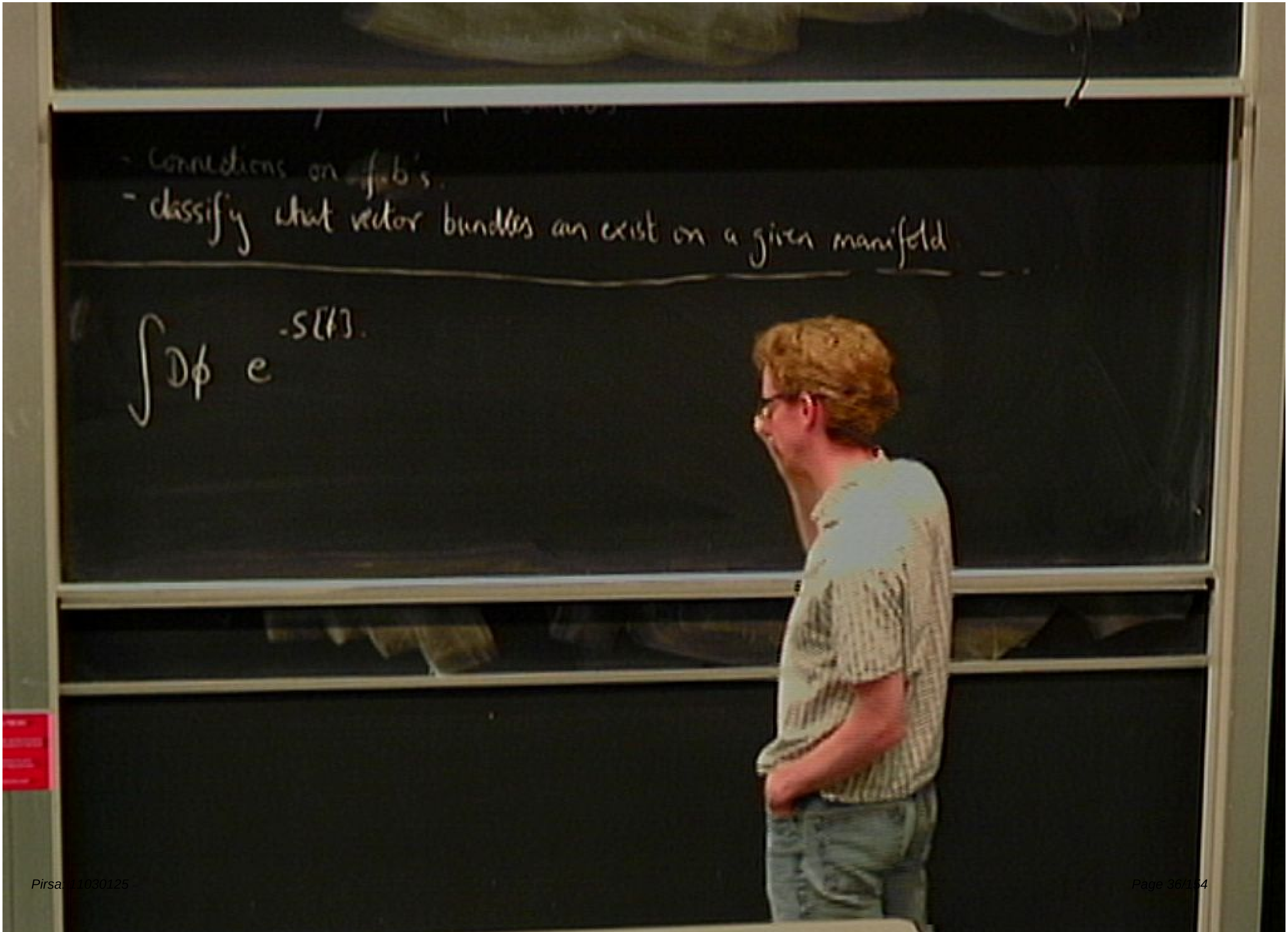
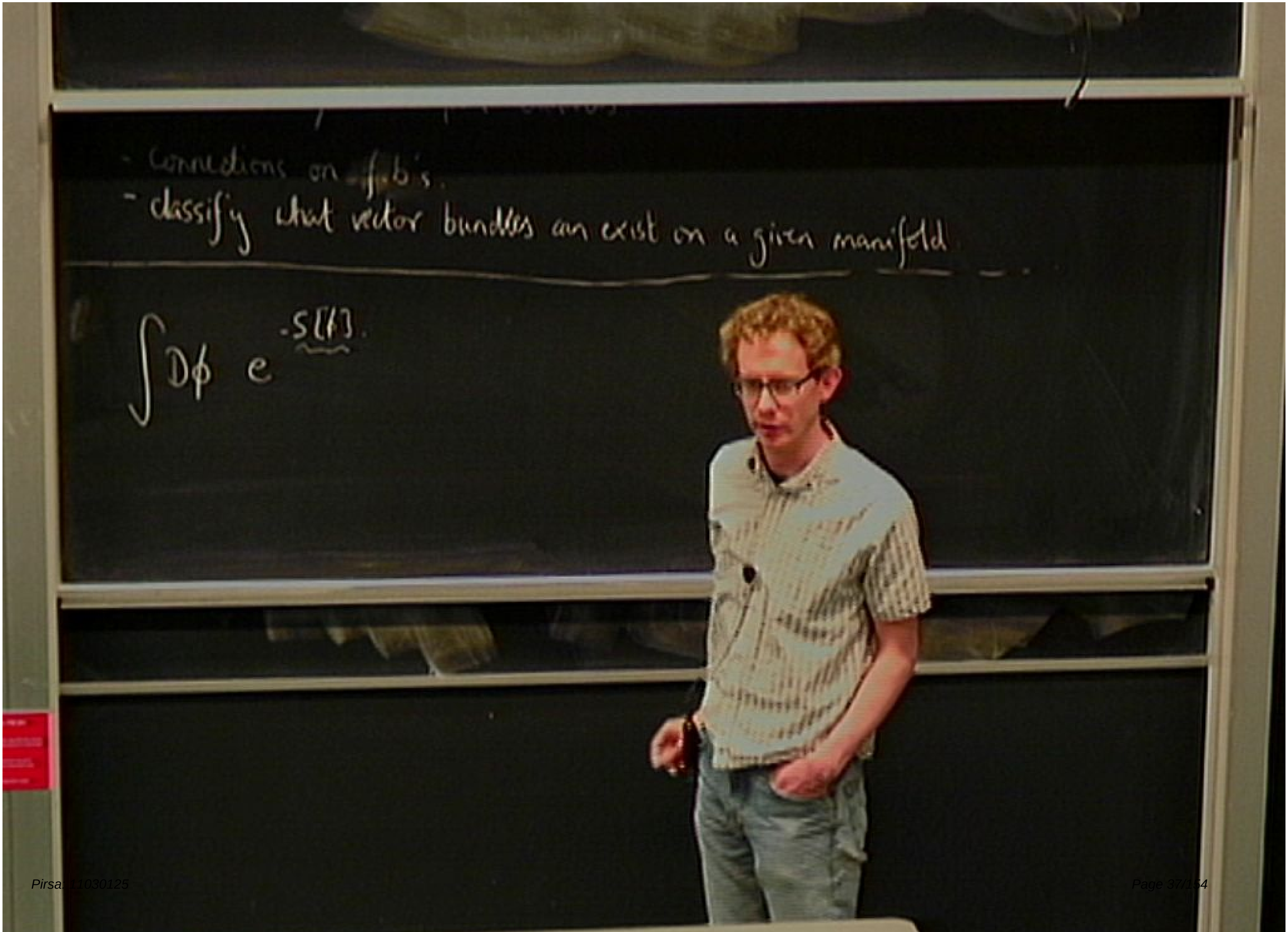

- Connections on f.b's.
- classify what vector bundles can exist on a given manifold
$\int D\phi \, e^{-S[\phi]}$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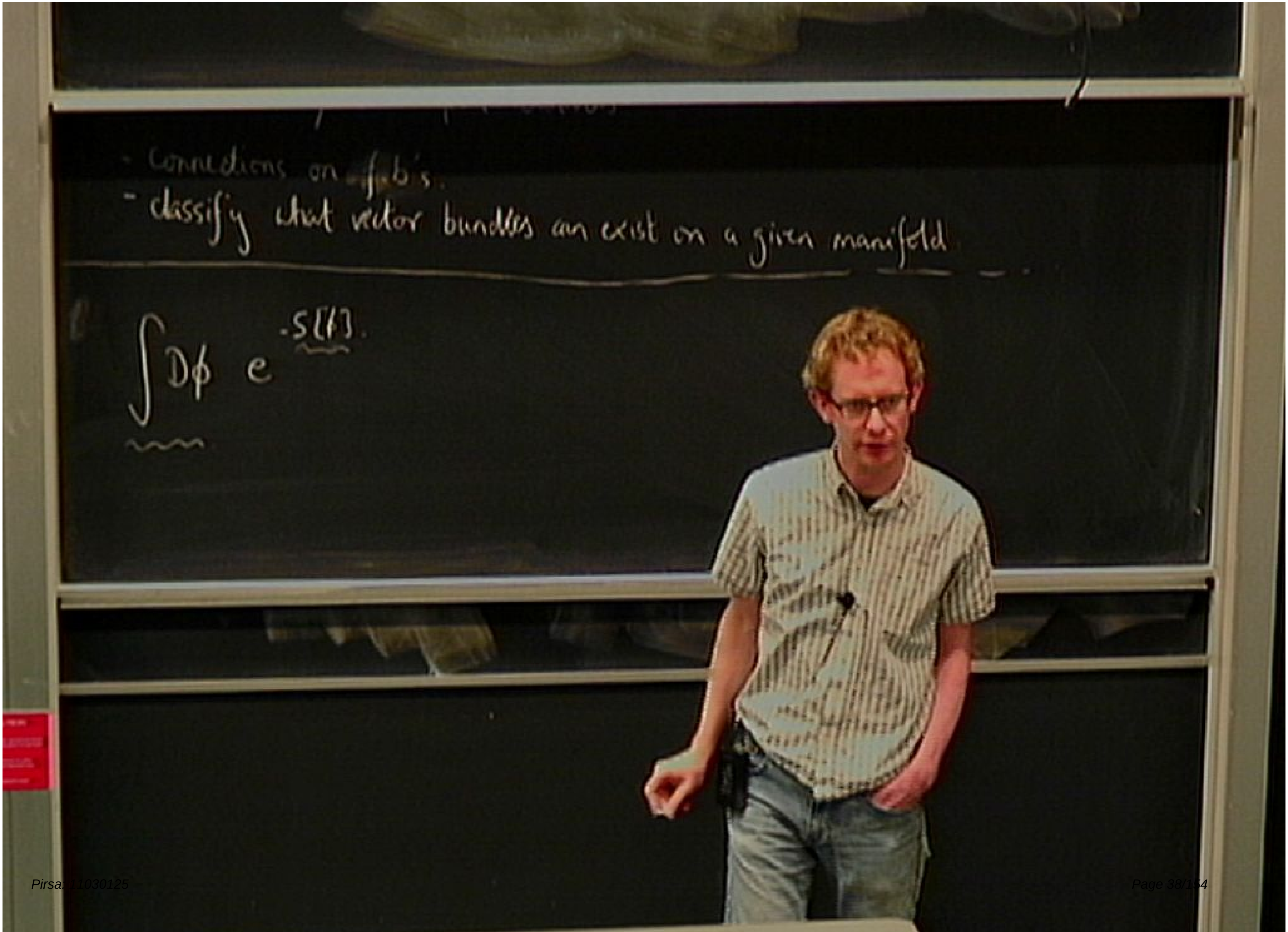
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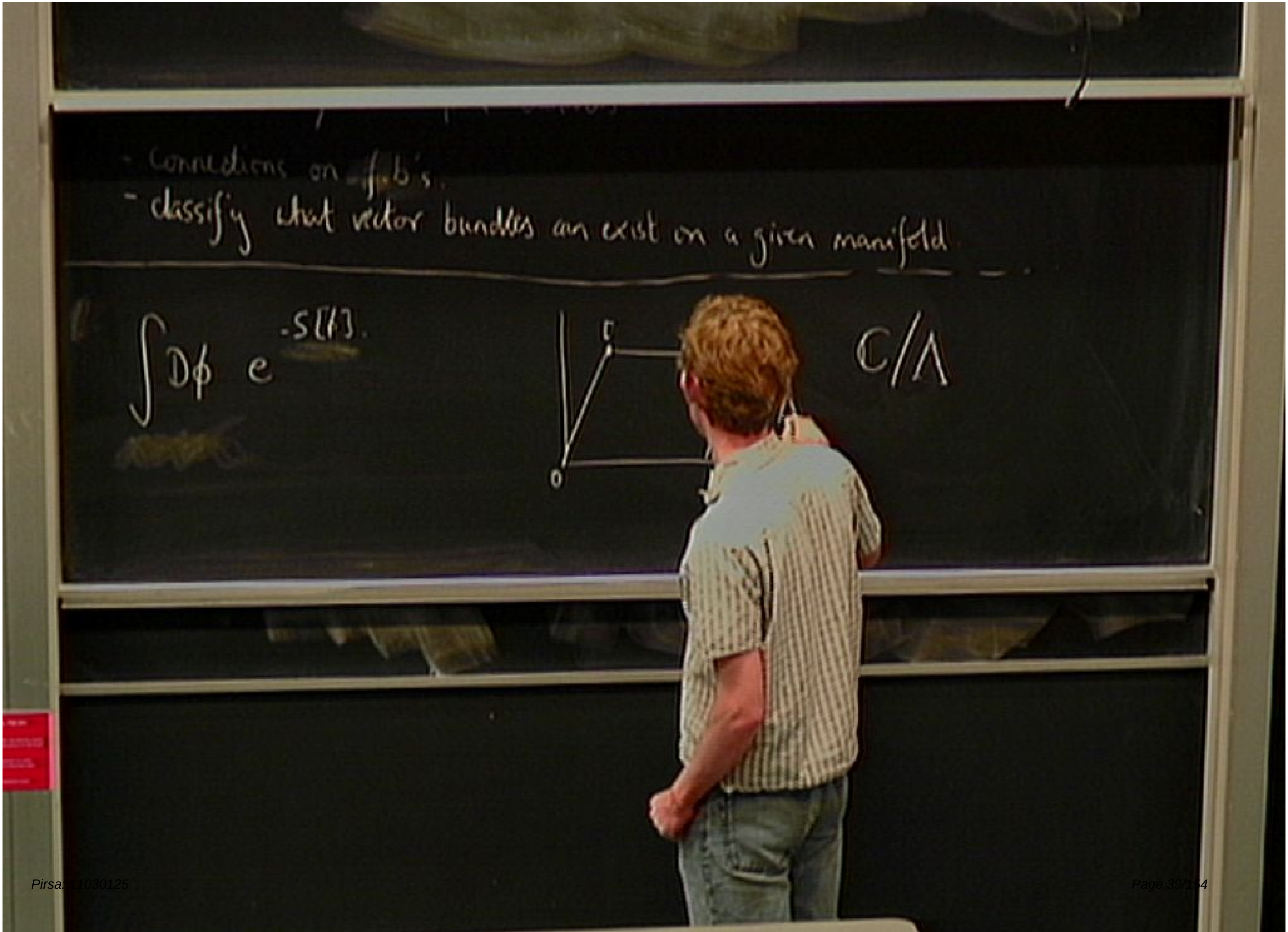
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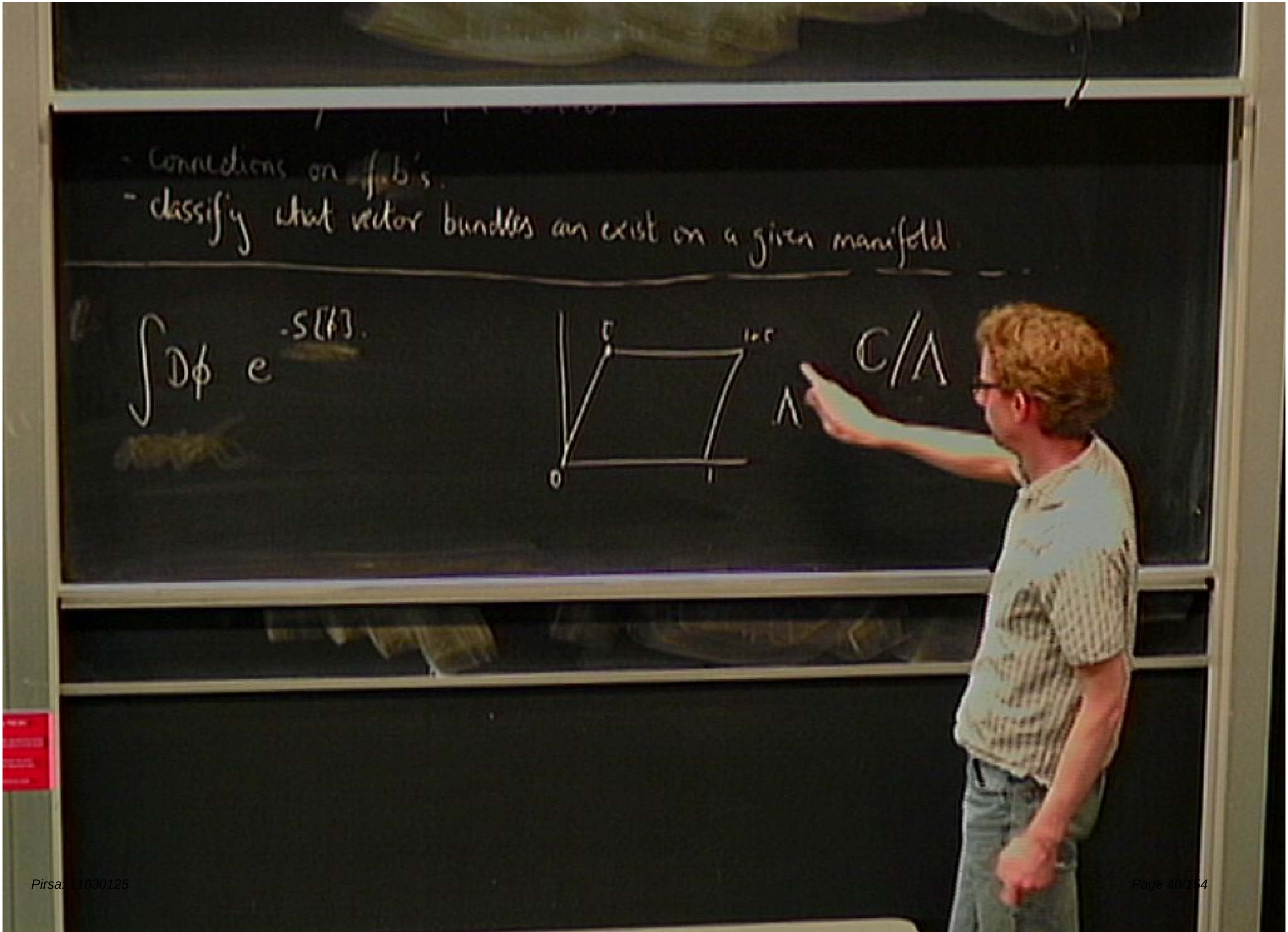
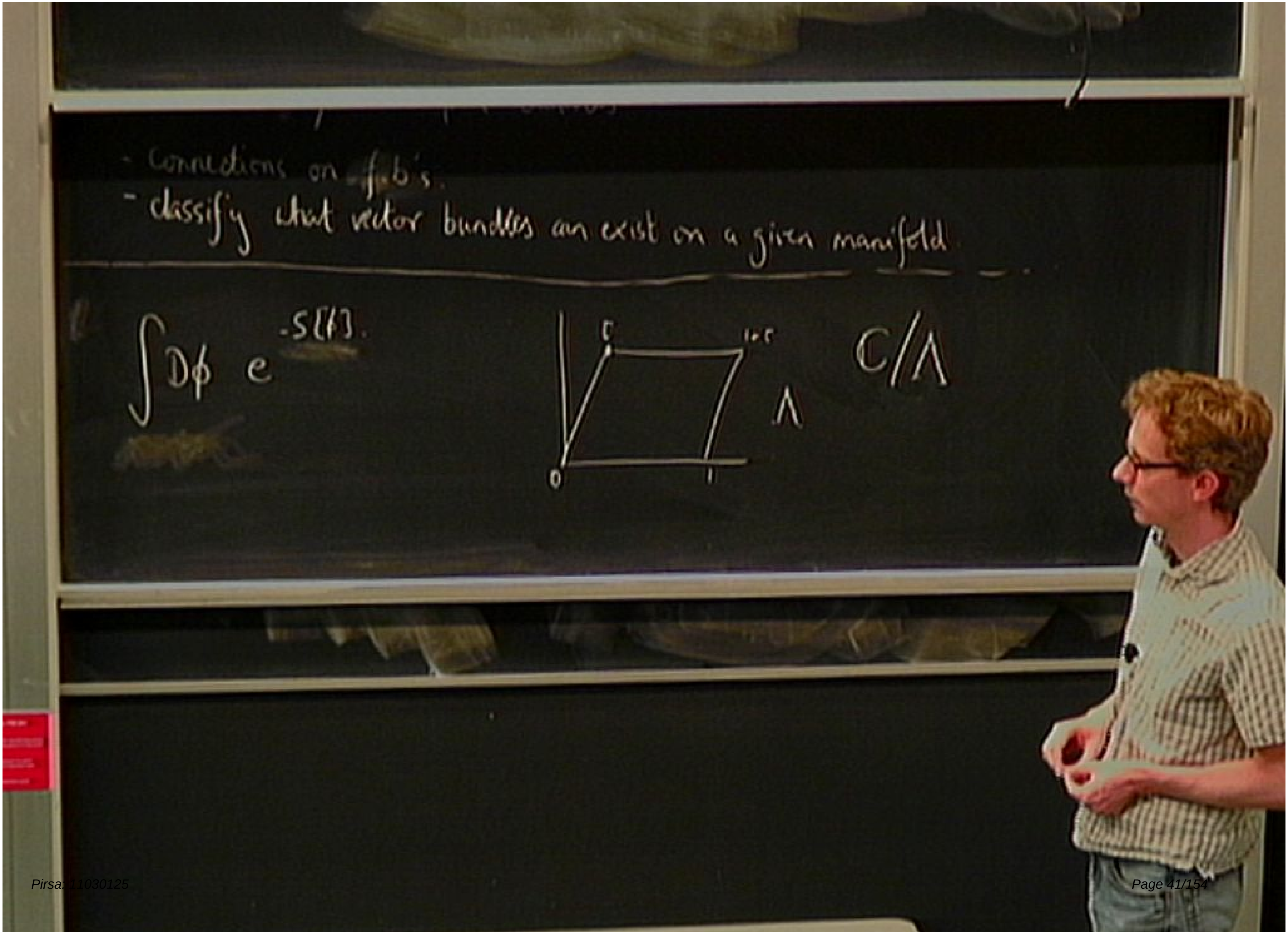
- classify what vector bundles can exist on a given manifold.
∫Dφ e^{-S[φ]}
C/Λ


- Connections on f.b.'s.
- classify what vector bundles can exist on a given manifold
∫Dφ e^{-S[φ]}
τ
1+τ
Λ
C/Λ
0
1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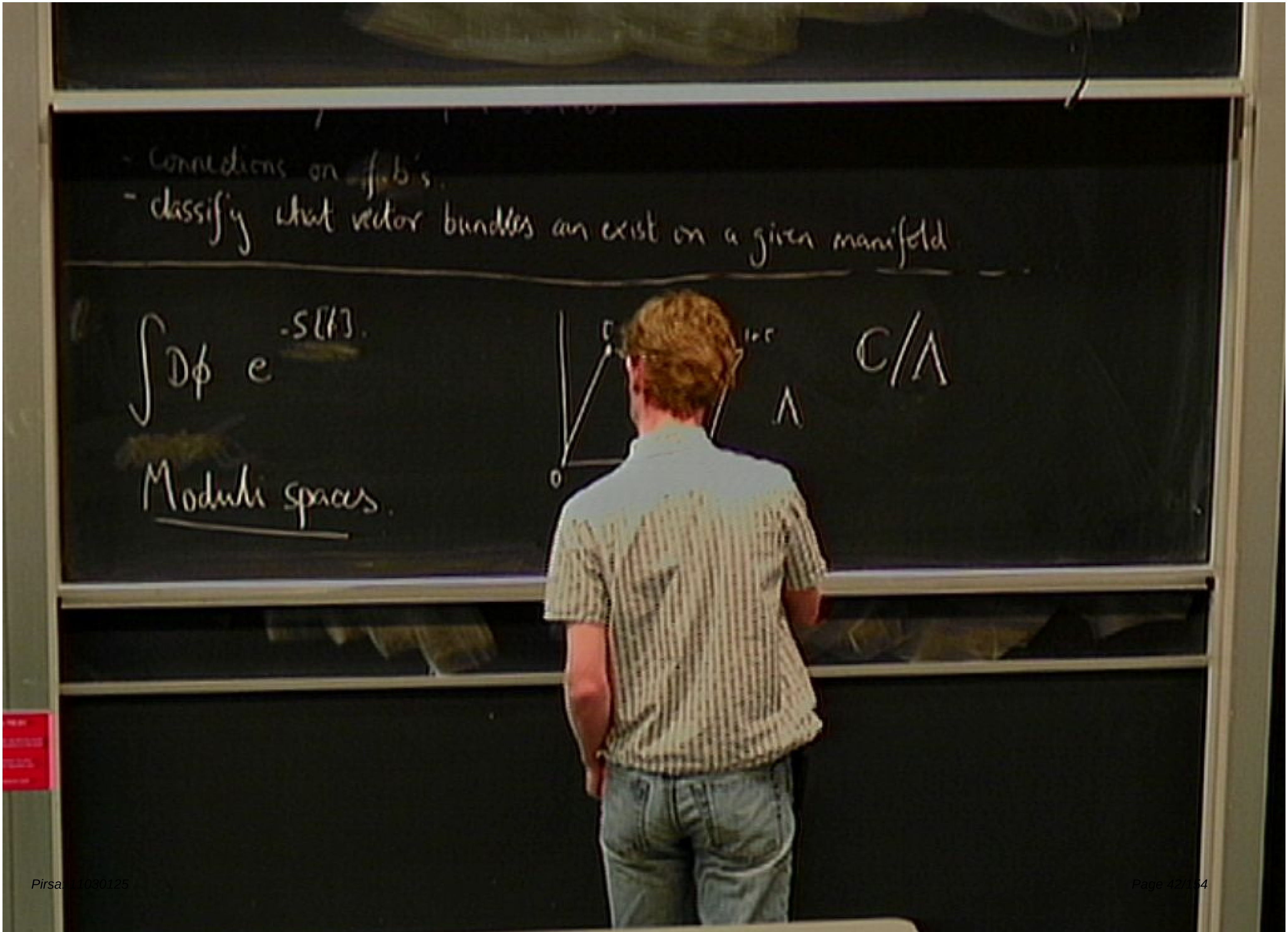
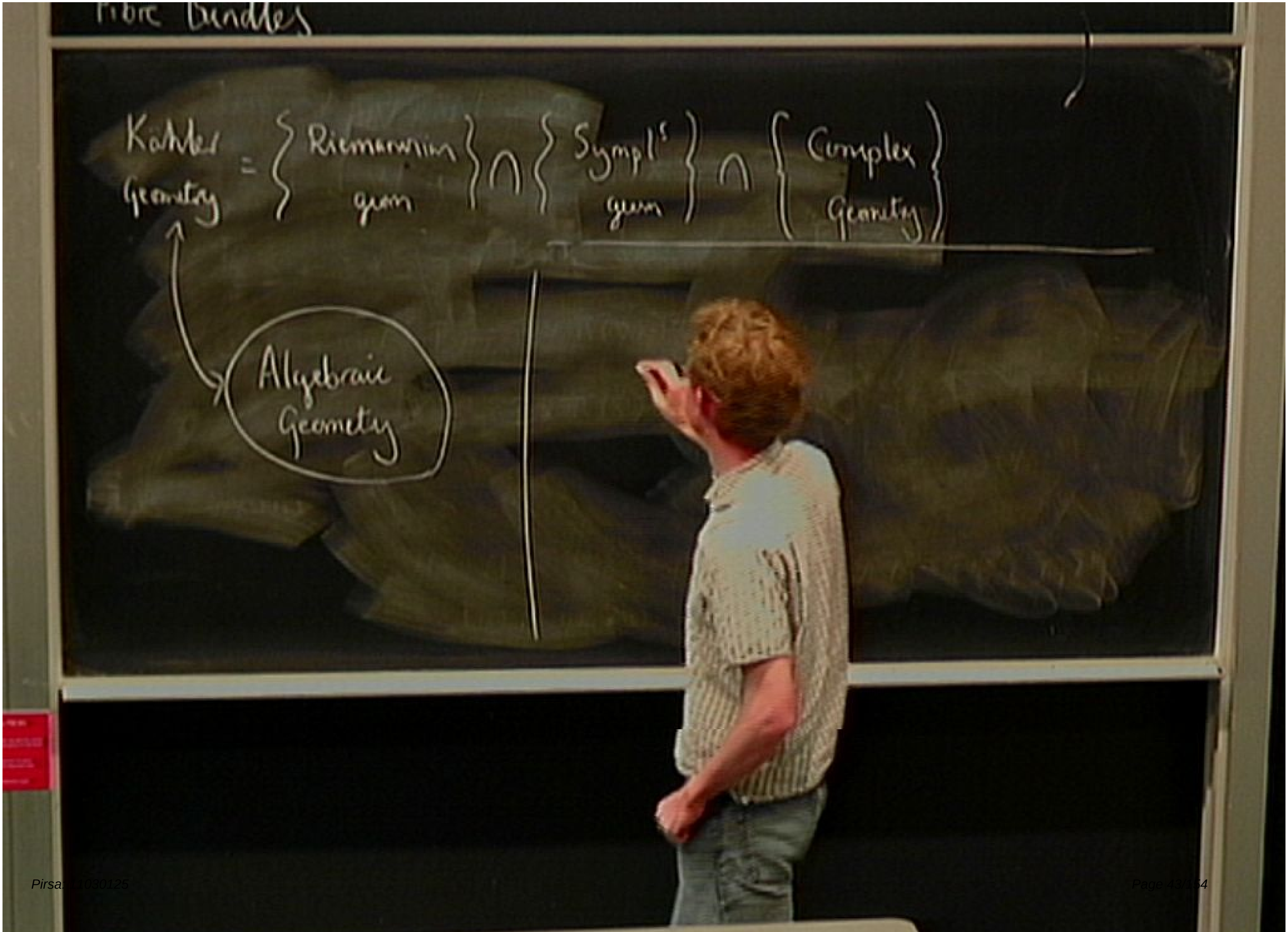
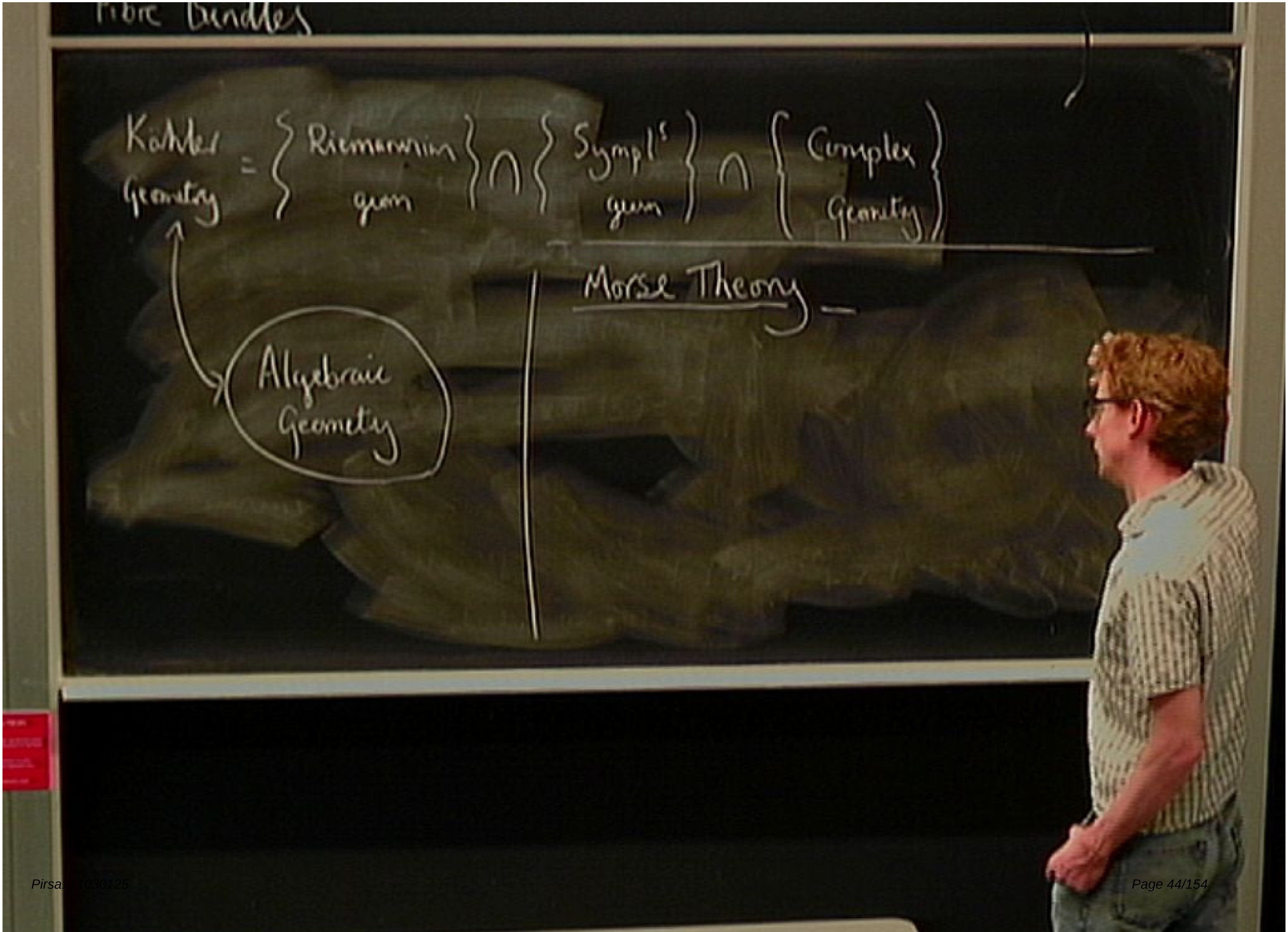
∫Dφ e^{-S[φ]}
τ
1+τ
0
1
Λ
C/Λ


- Connections on f.b's.
- classify what vector bundles can exist on a given manifold
∫Dφ e^{-S[φ]}
C/Λ
Λ
0
Moduli spaces.


Fibre Bundles
Kähler Geometry = {Riemannian geom} ∩ {Sympl'c geom} ∩ {Complex Geometry}
Algebraic Geometry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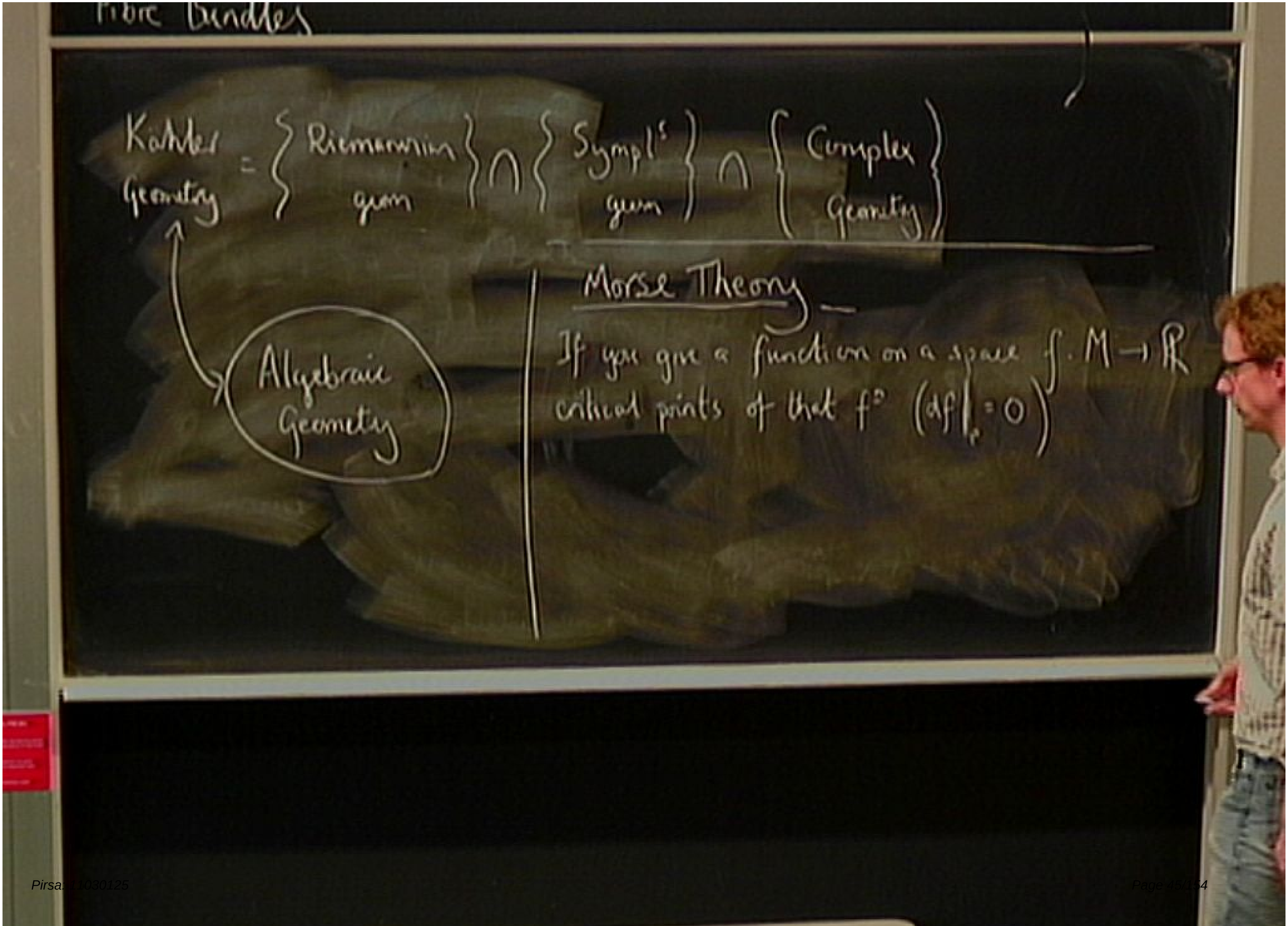
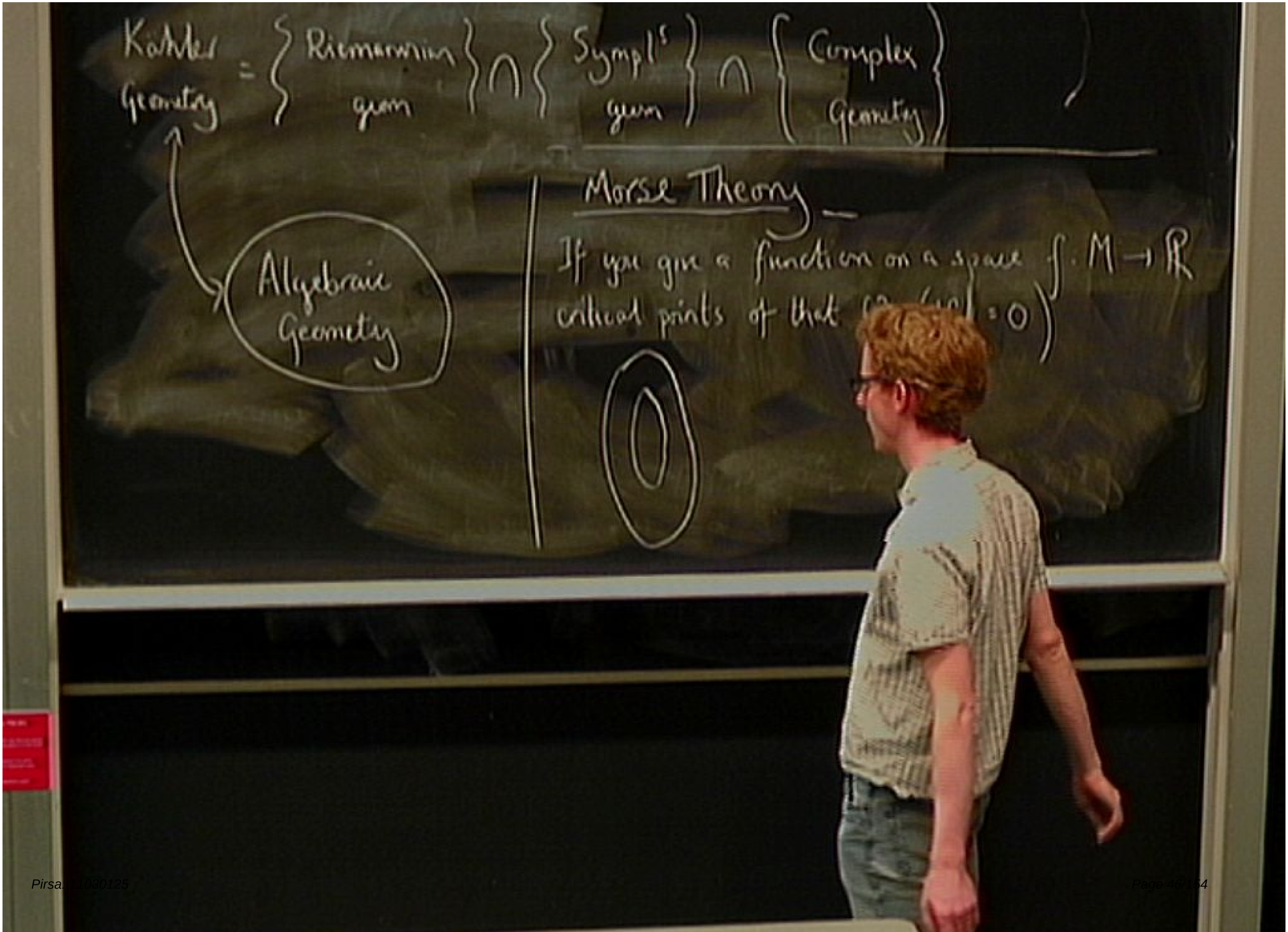
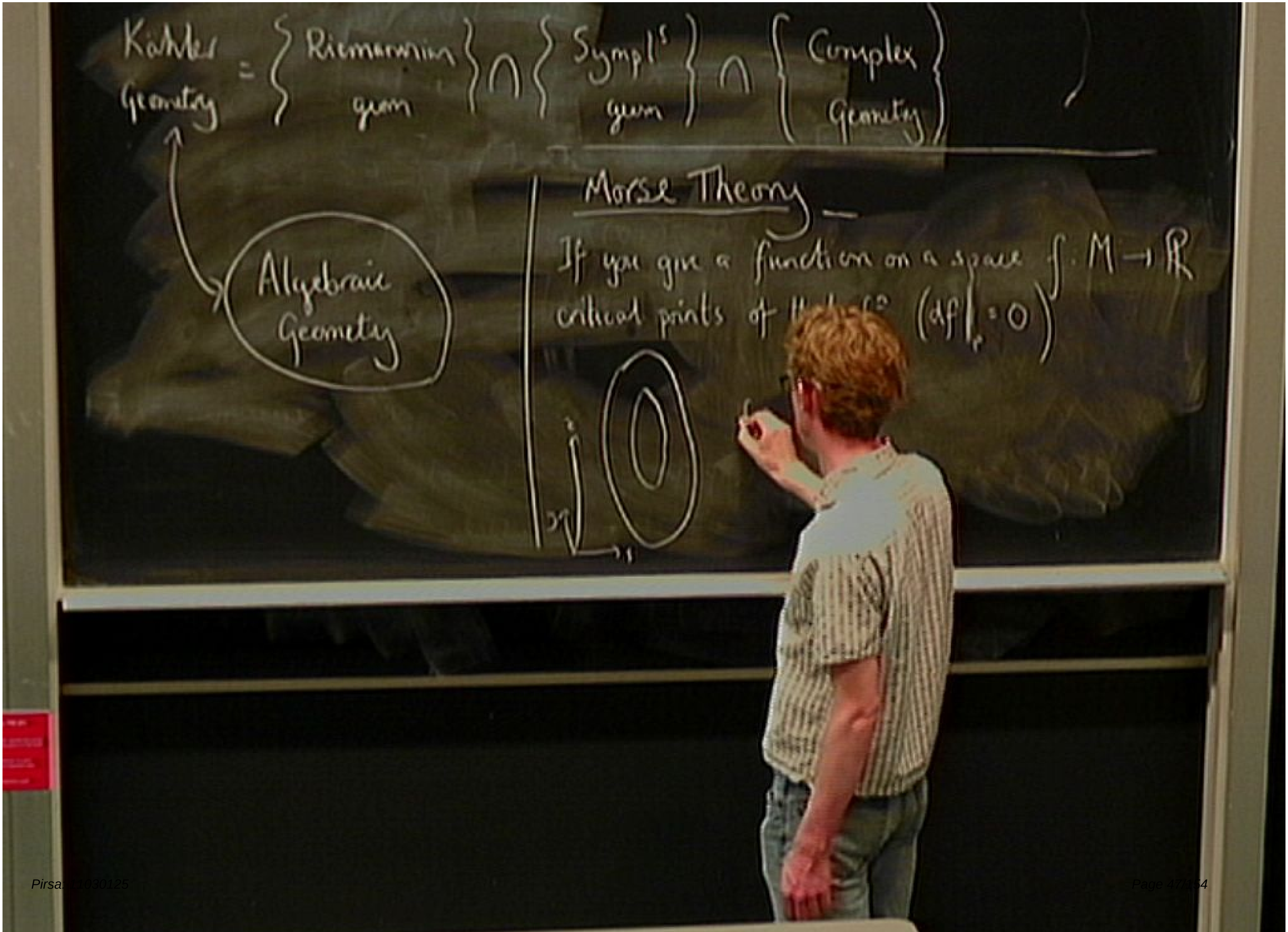

Fibre Bundles
Kähler Geometry = { Riemannian geom } ∩ { Sympl'c geom } ∩ { Complex Geometry }
Algebraic Geometry
Morse Theory
If you give a function on a space f: M → ℝ
critical points of that f° (df|p = 0)


Kähler Geometry
Riemannian geom
Sympl^c geom
Complex Geometry
Algebraic Geometry
Morse Theory
If you give a function on a space f: M → R
critical points of that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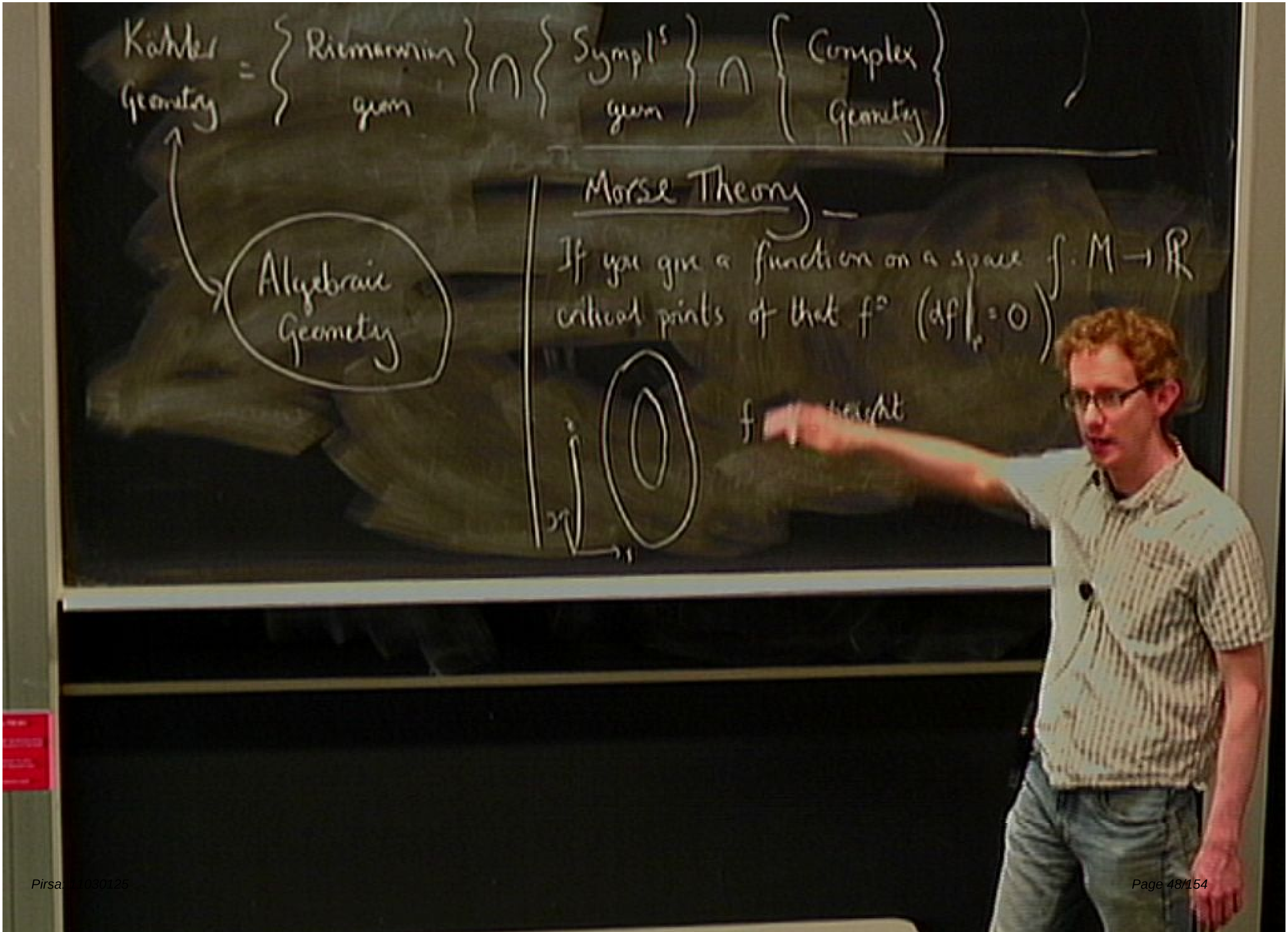
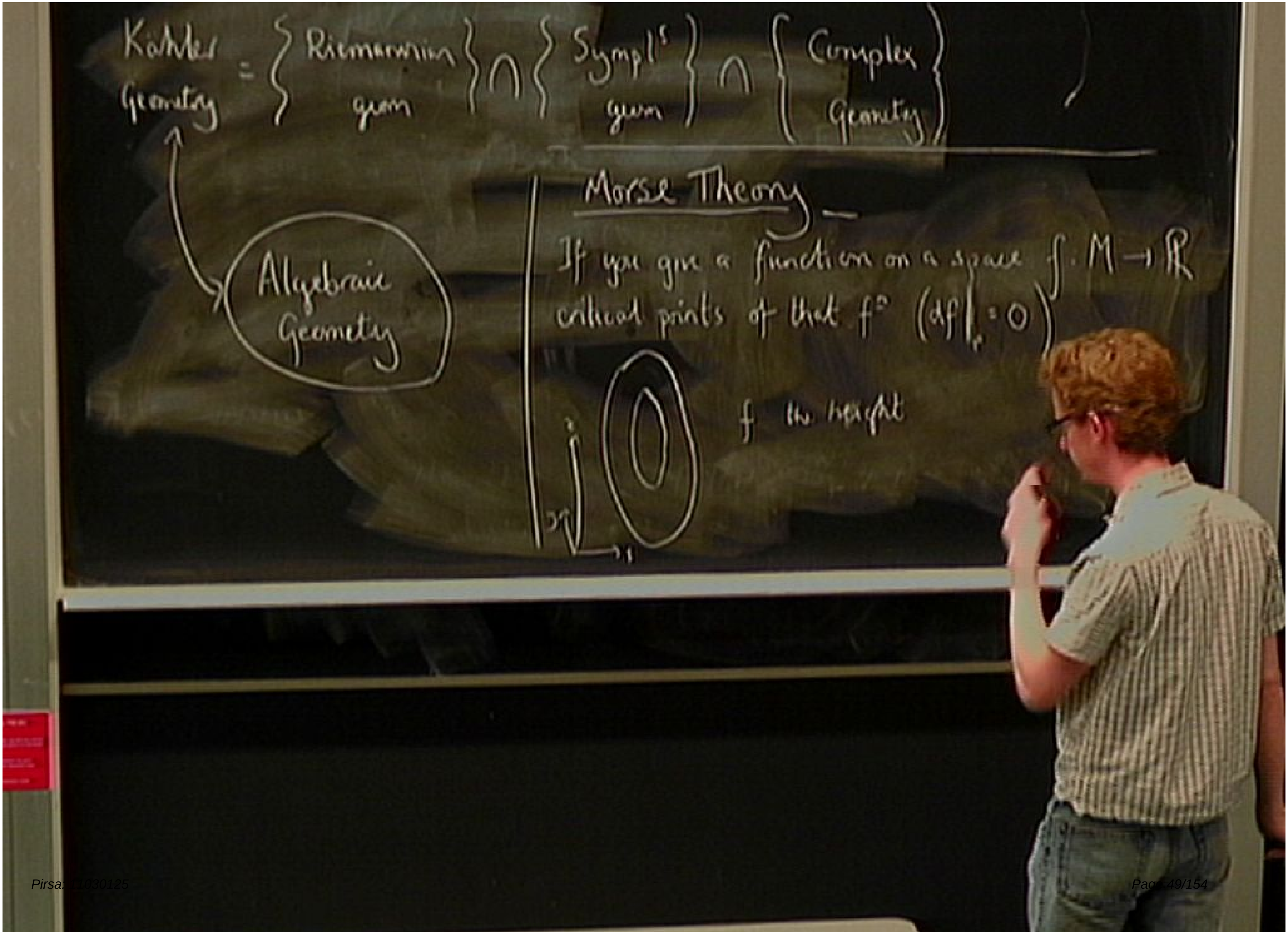
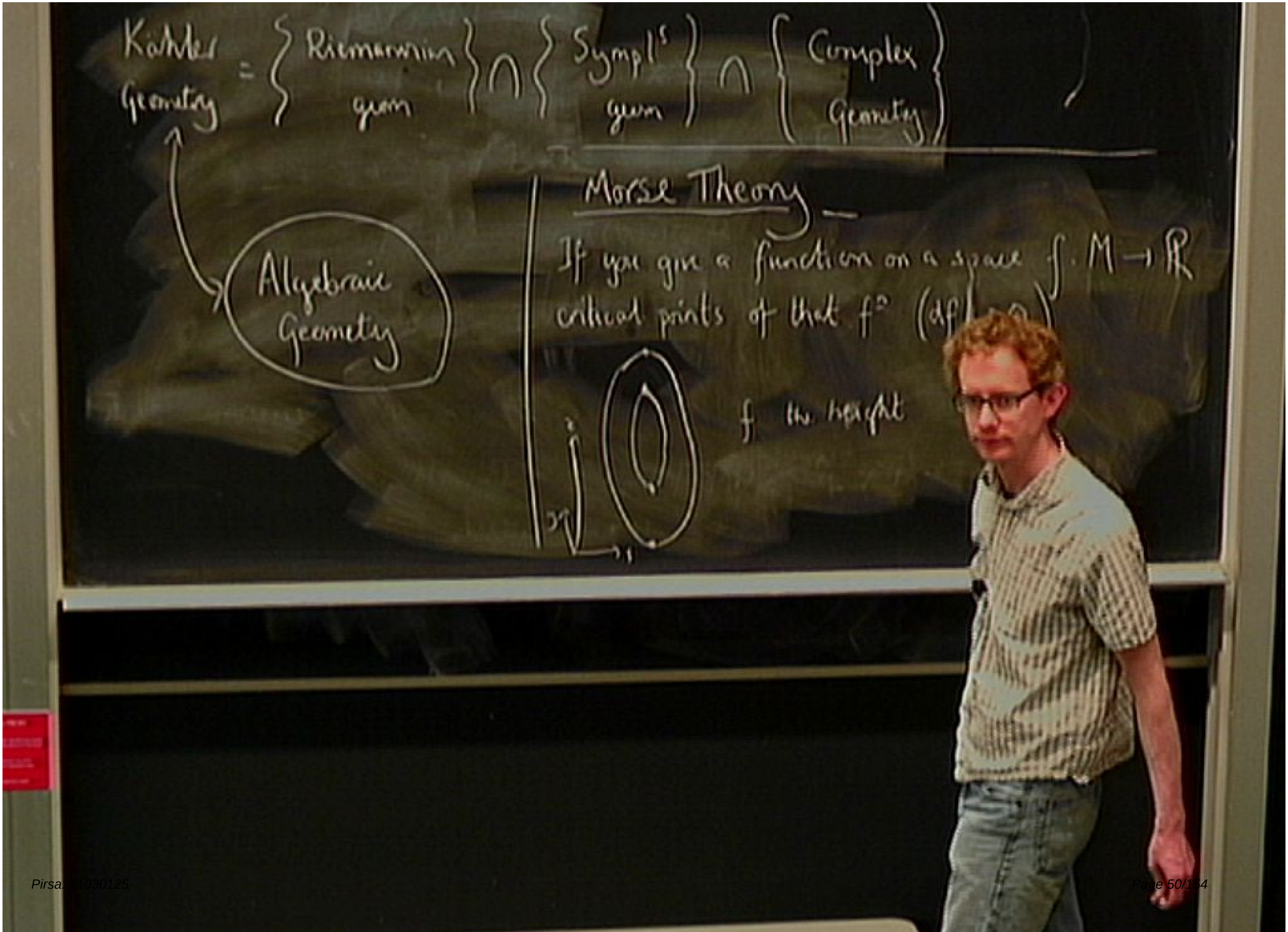
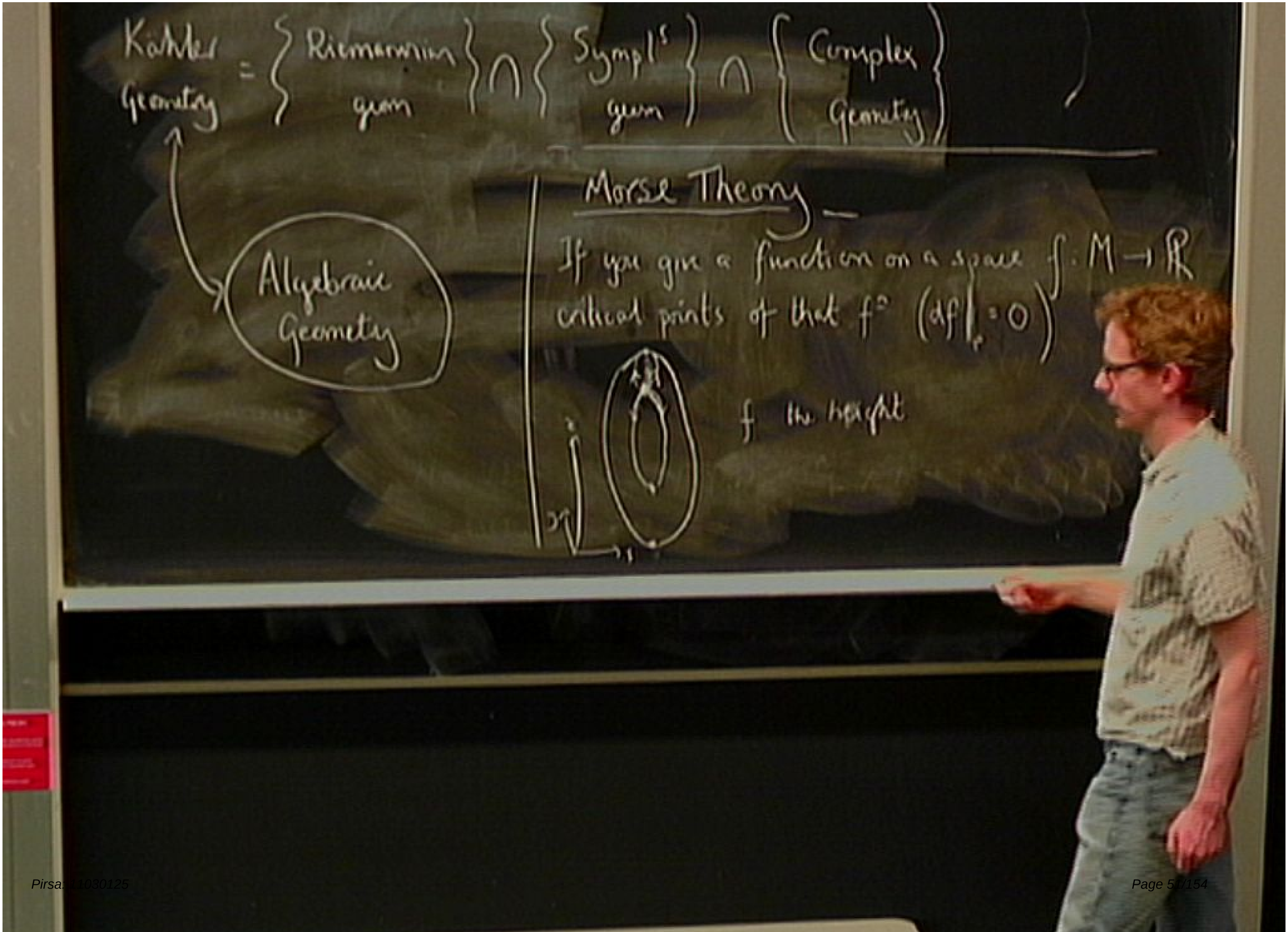
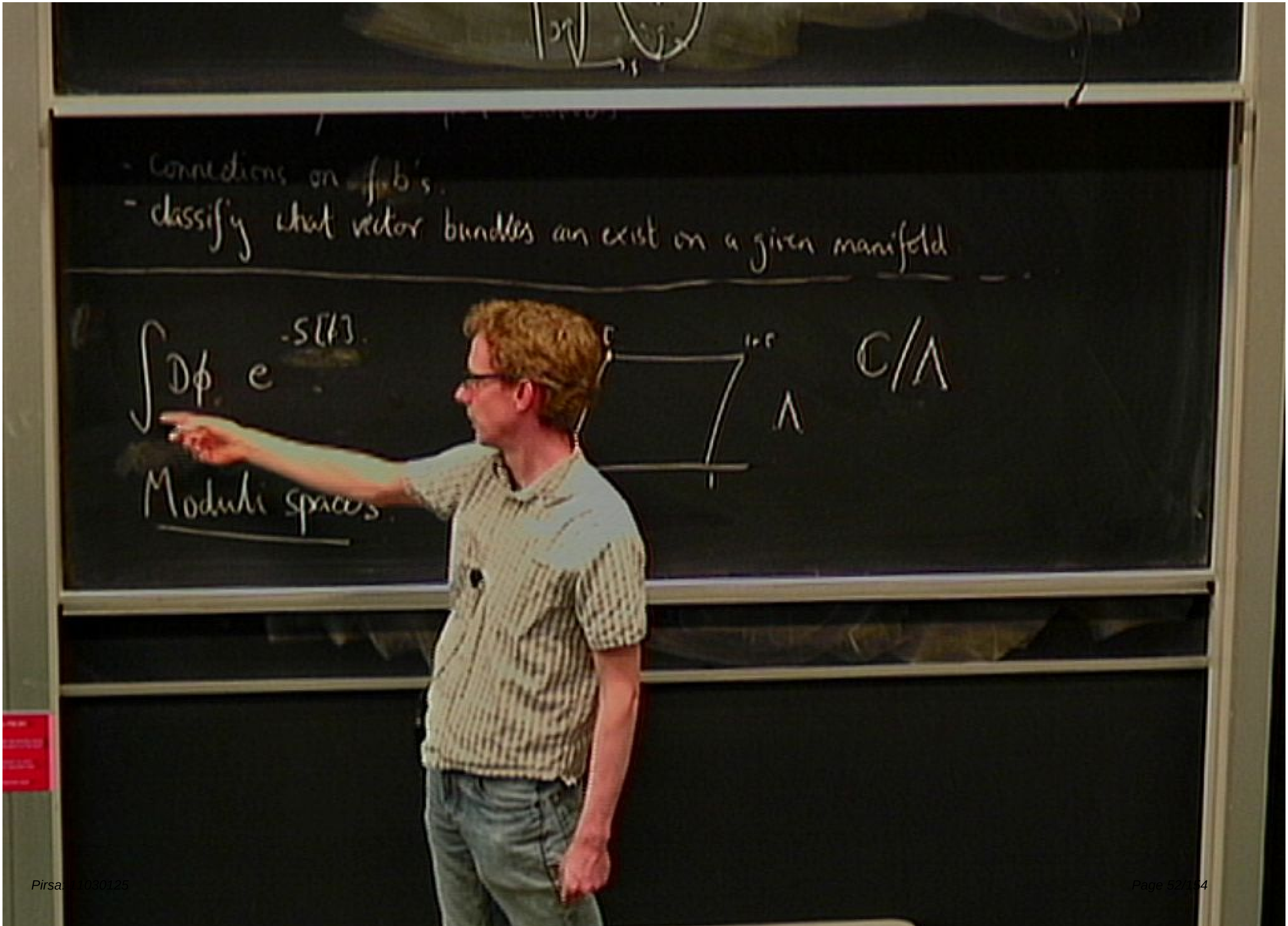

Kähler Geometry = { Riemannian geom } ∩ { Sympl'c geom } ∩ { Complex Geometry }
Algebraic Geometry
Morse Theory
If you give a function on a space f : M → R
critical points of that f (df|p = 0)
f
height

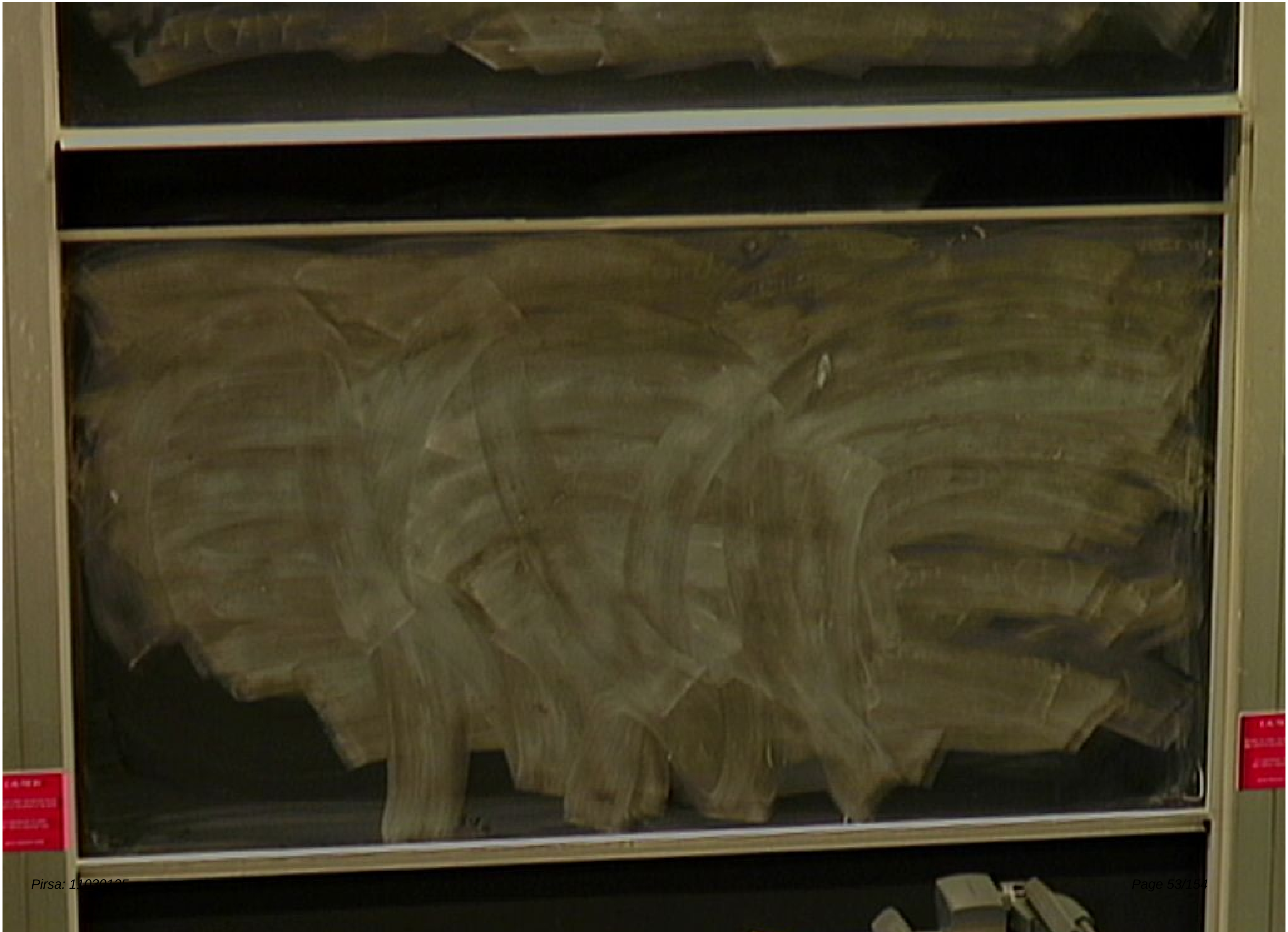

Kähler Geometry = { Riemannian geom } ∩ { Sympl^c geom } ∩ { Complex Geometry }
Algebraic Geometry
Morse Theory
If you give a function on a space f : M → R
critical points of that f^n (df|_p = 0)
f the height

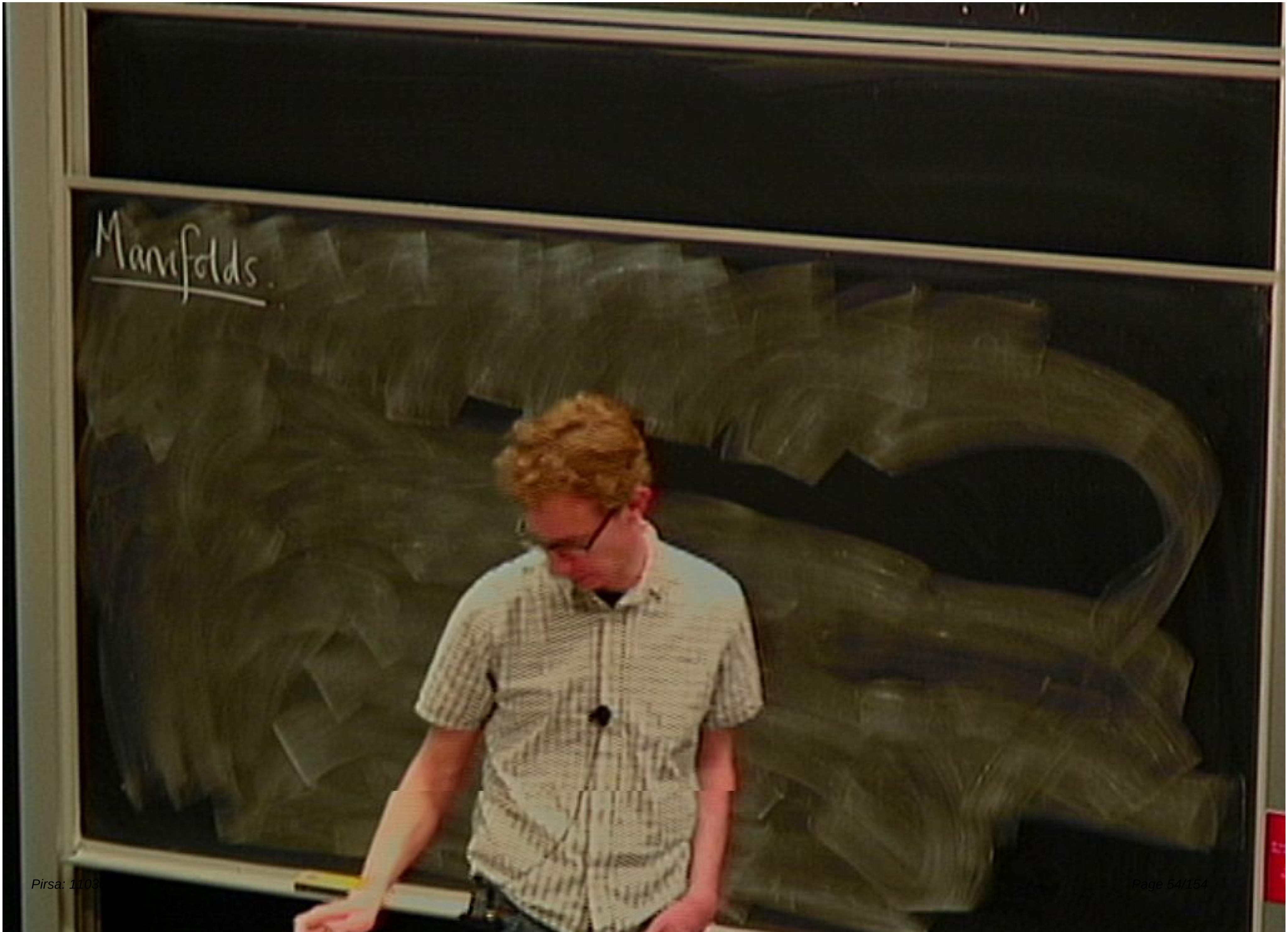

Kähler Geometry = { Riemannian geom } ∩ { Sympl'c geom } ∩ { Complex Geometry }
Algebraic Geometry
Morse Theory
If you give a function on a space f: M → R
critical points of that f? (df|p = 0)
f the height

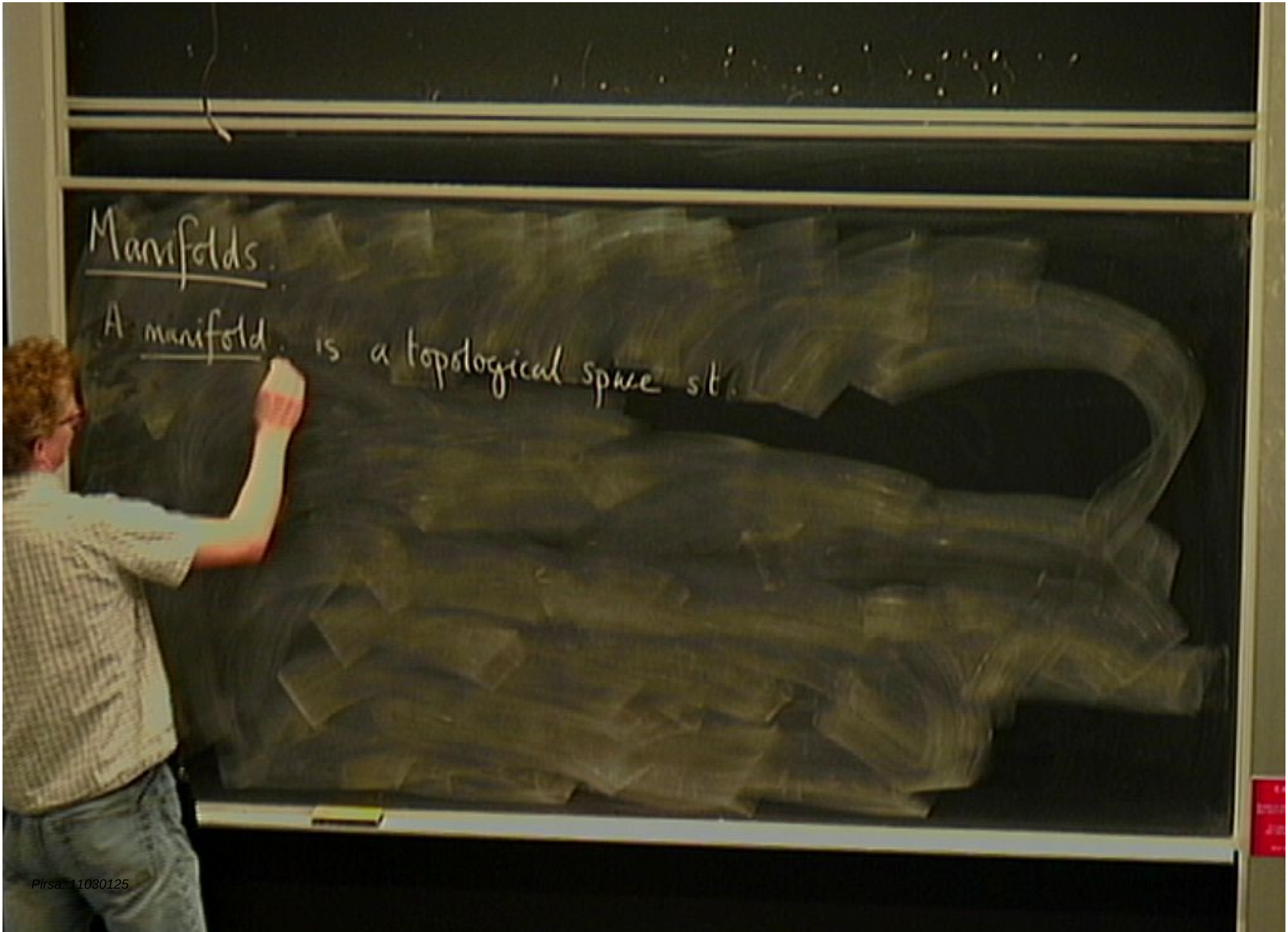


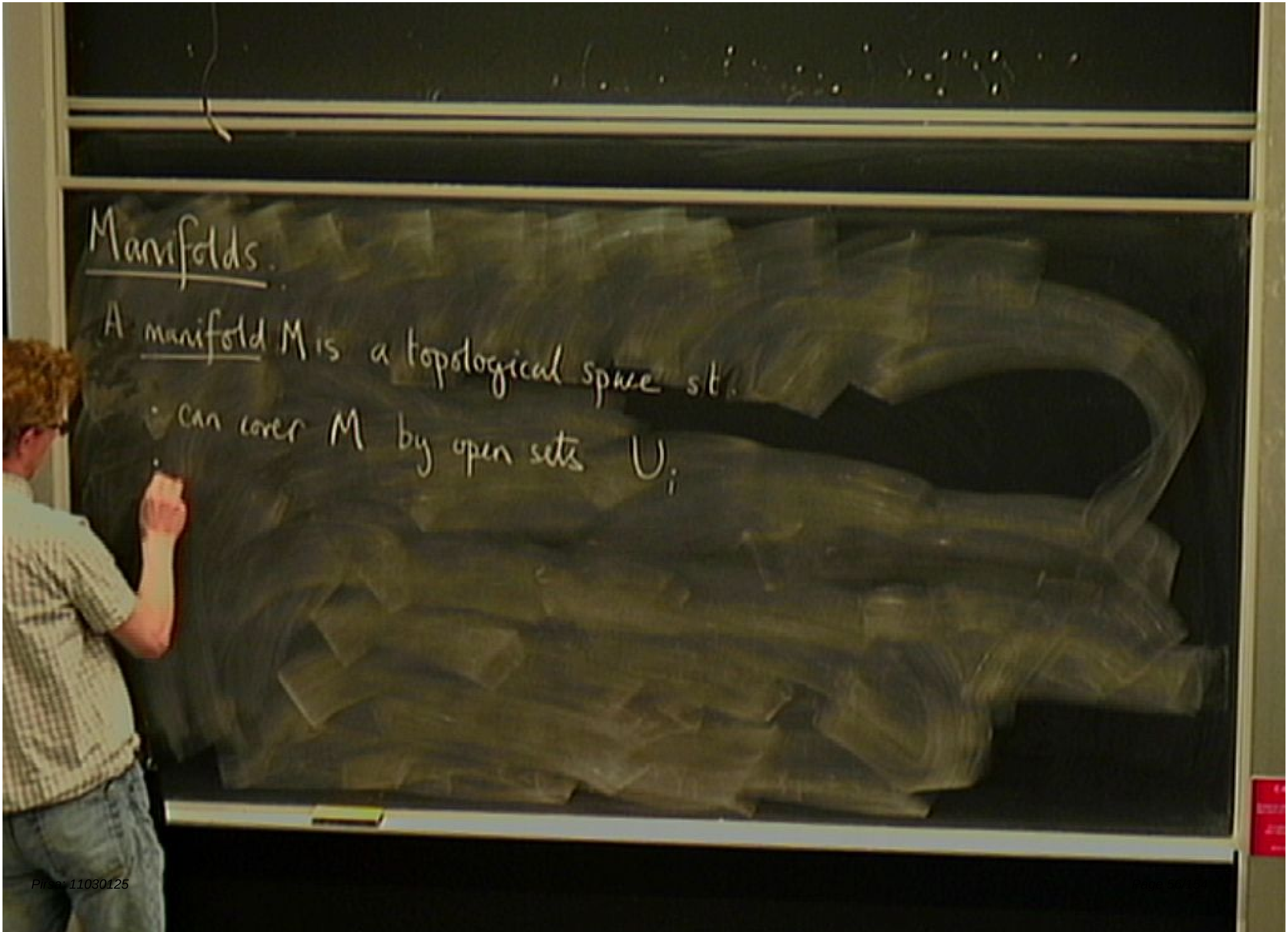
Kähler Geometry = {Riemannian geom} ∩ {Sympl^c geom} ∩ {Complex Geometry}
Algebraic Geometry
Morse Theory
If you give a function on a space f: M → R
critical points of that f (df|p = 0)
f the height

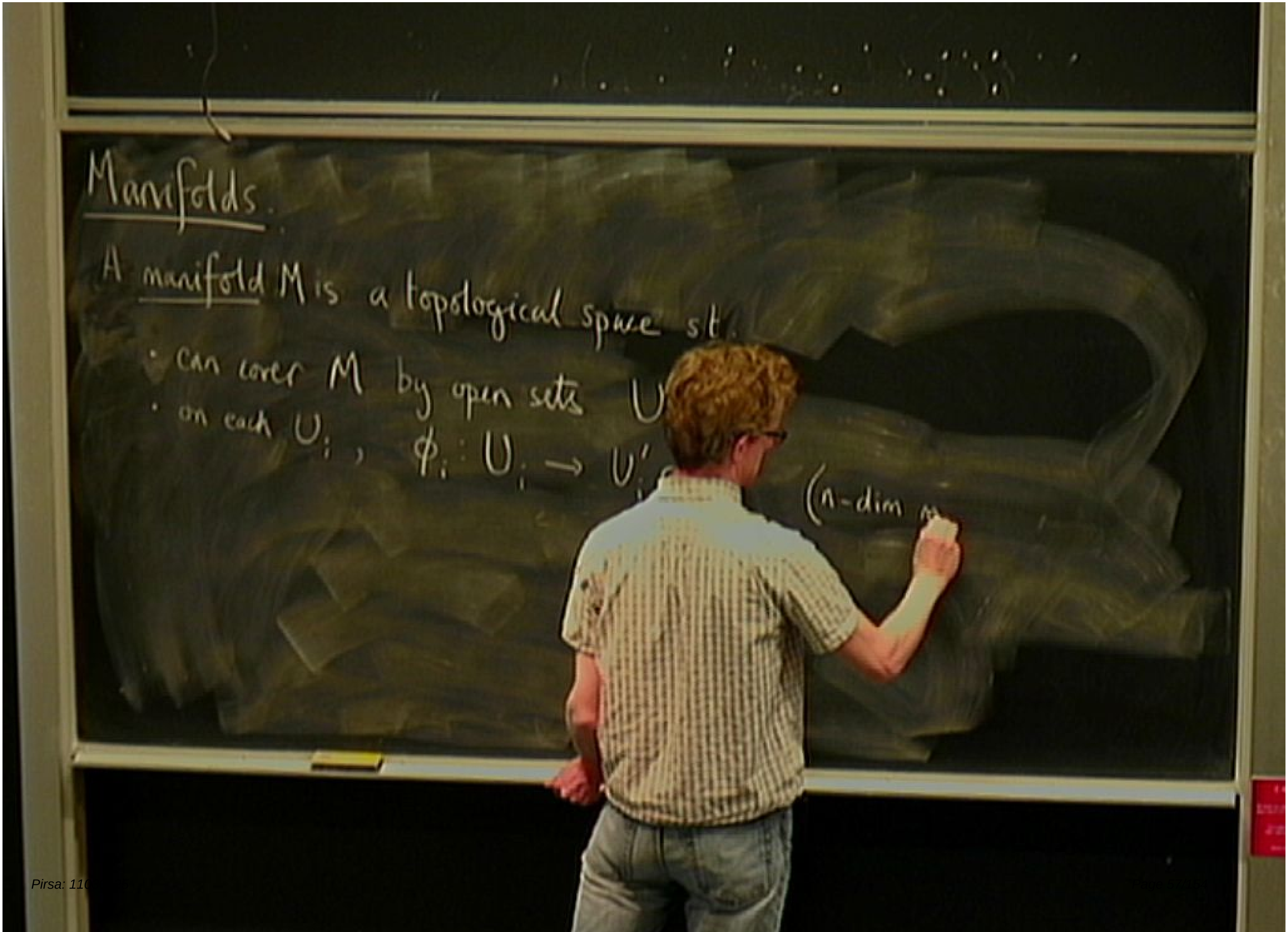

- Connections on f.b's.
- classify what vector bundles can exist on a given manifold
∫Dφ e^{-S[φ]}
C/Λ
Λ
Moduli spaces

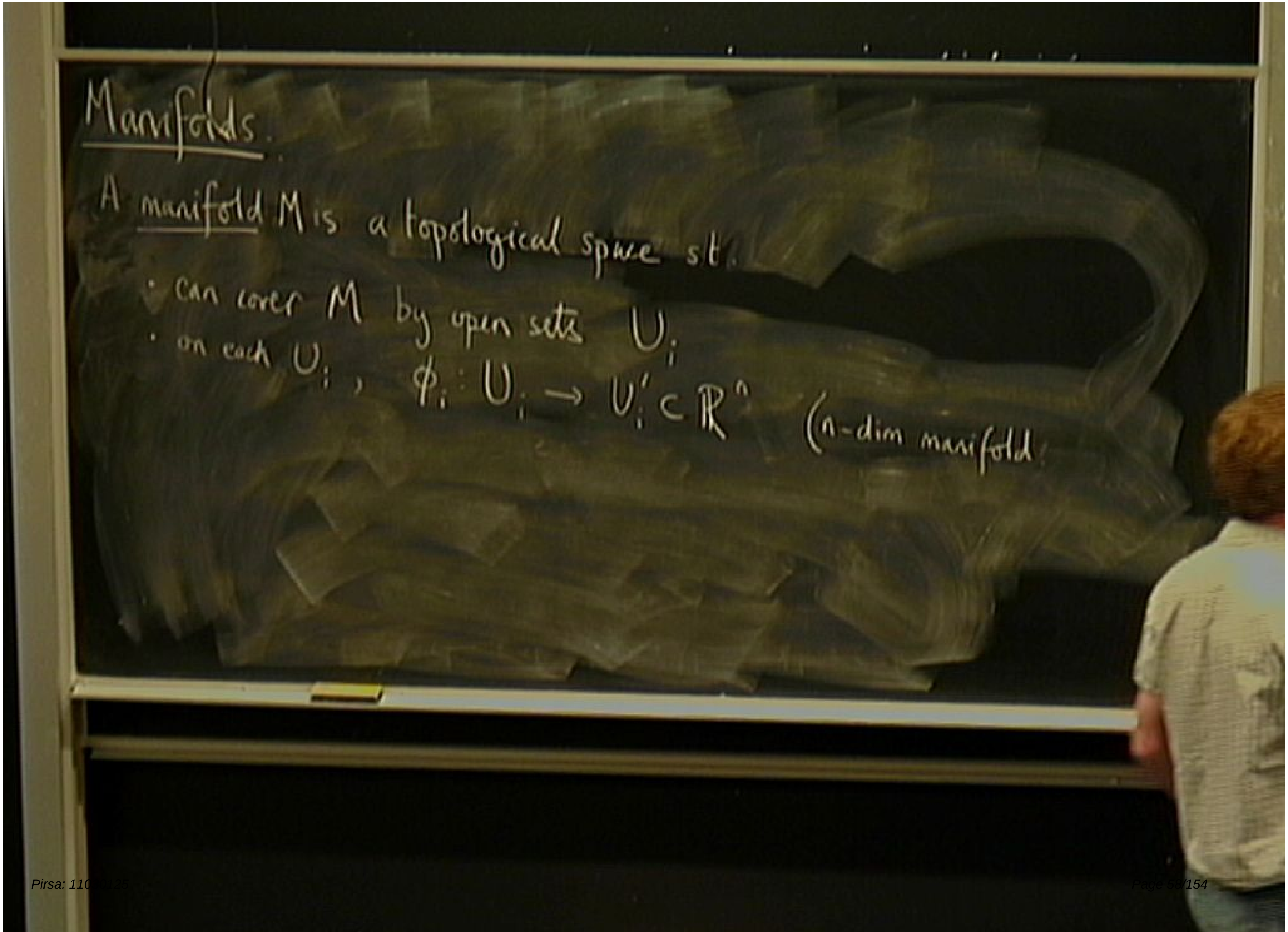



Manifolds


Manifolds.
A manifold is a topological space s.t.


Manifolds.
A manifold M is a topological space s.t.
· can cover M by open sets U_i
·


Manifolds.
A manifold M is a topological space s.t.
· can cover M by open sets U
· on each U_i , φ_i : U_i → U'_i
(n-dim

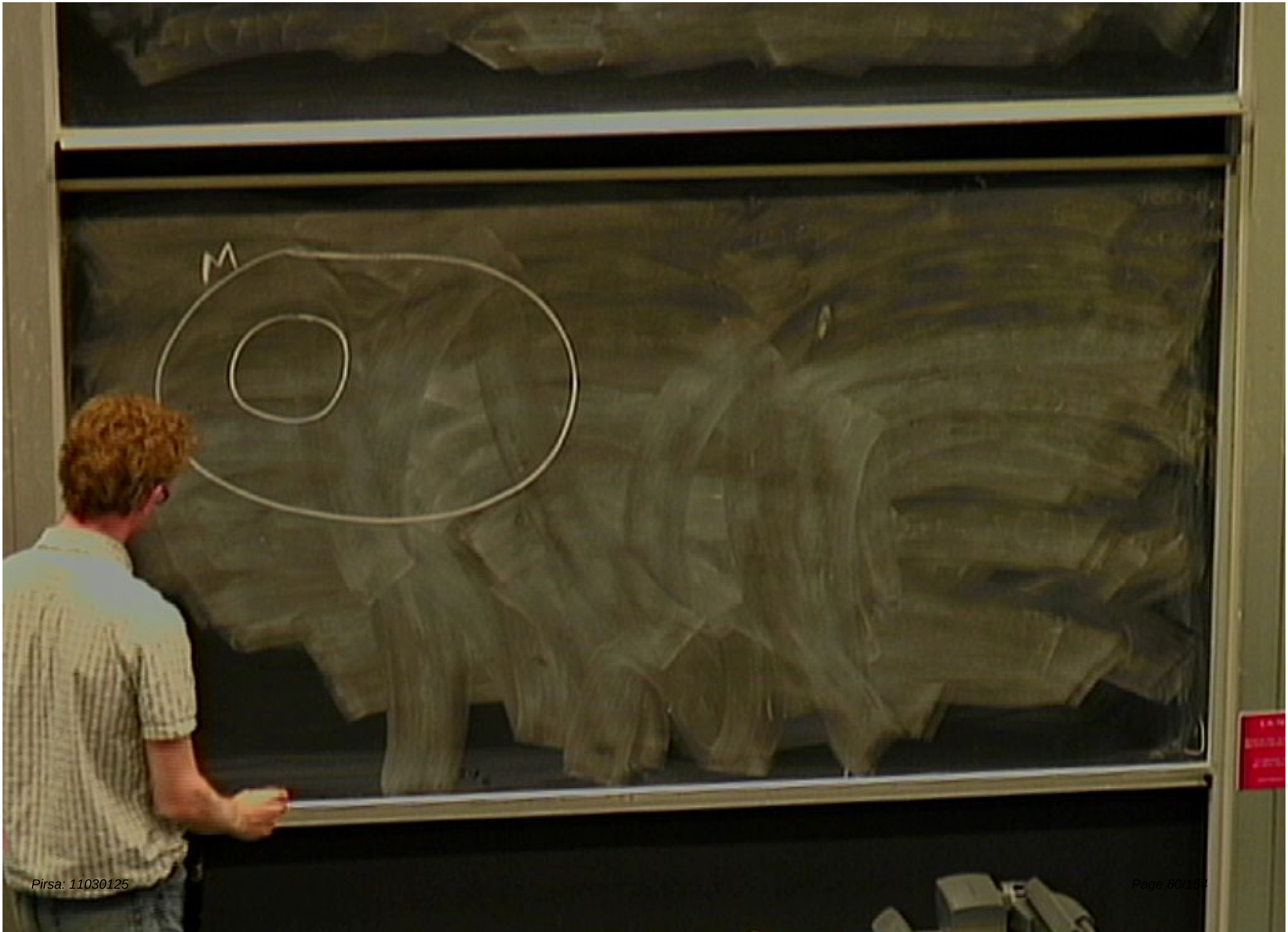


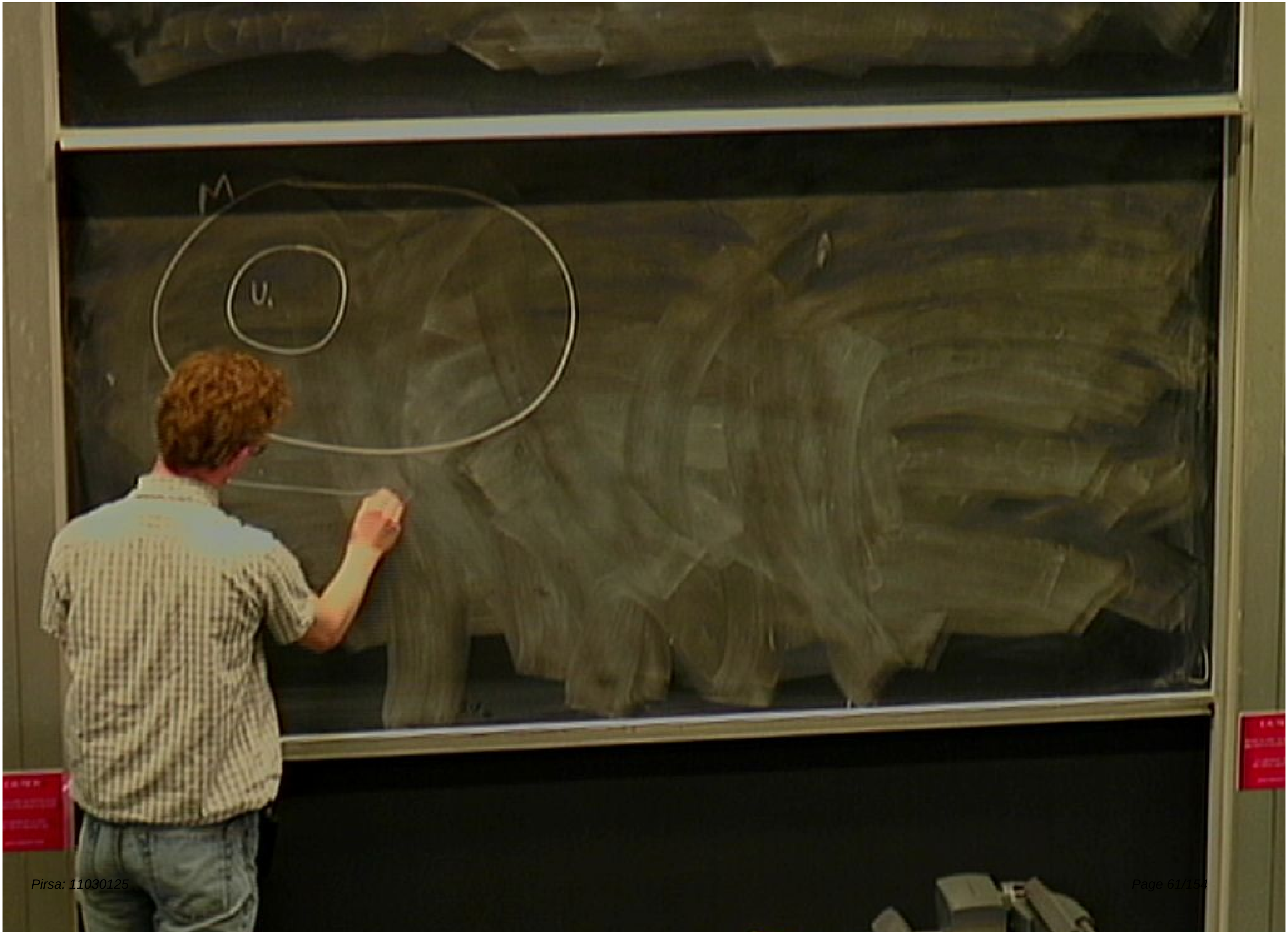
Manifolds.
A manifold M is a topological space s.t.
· can cover M by open sets U_i
· on each U_i , φ_i : U_i → U'_i ⊂ ℝ^n
(n-dim manifold

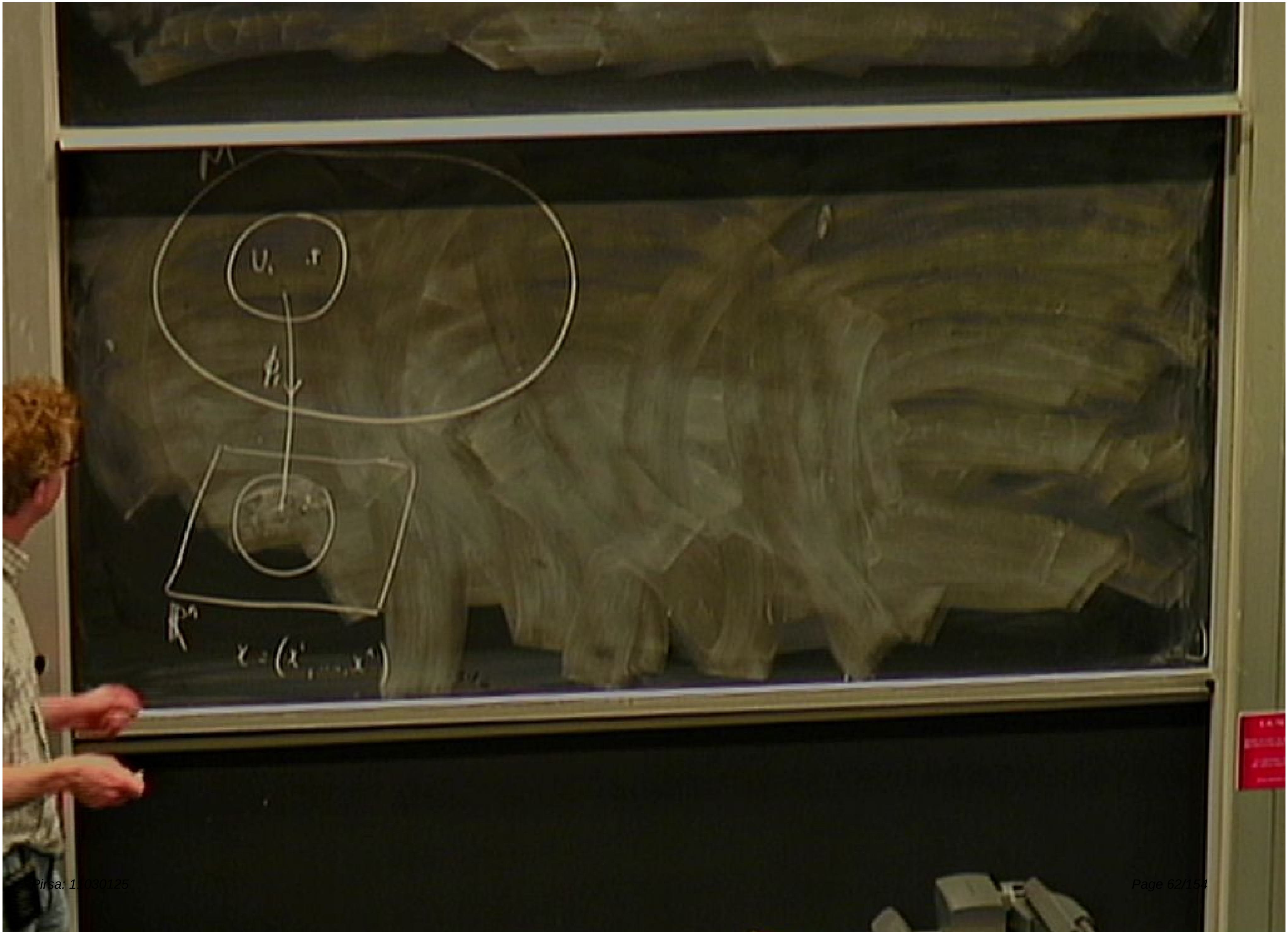
Manifolds.

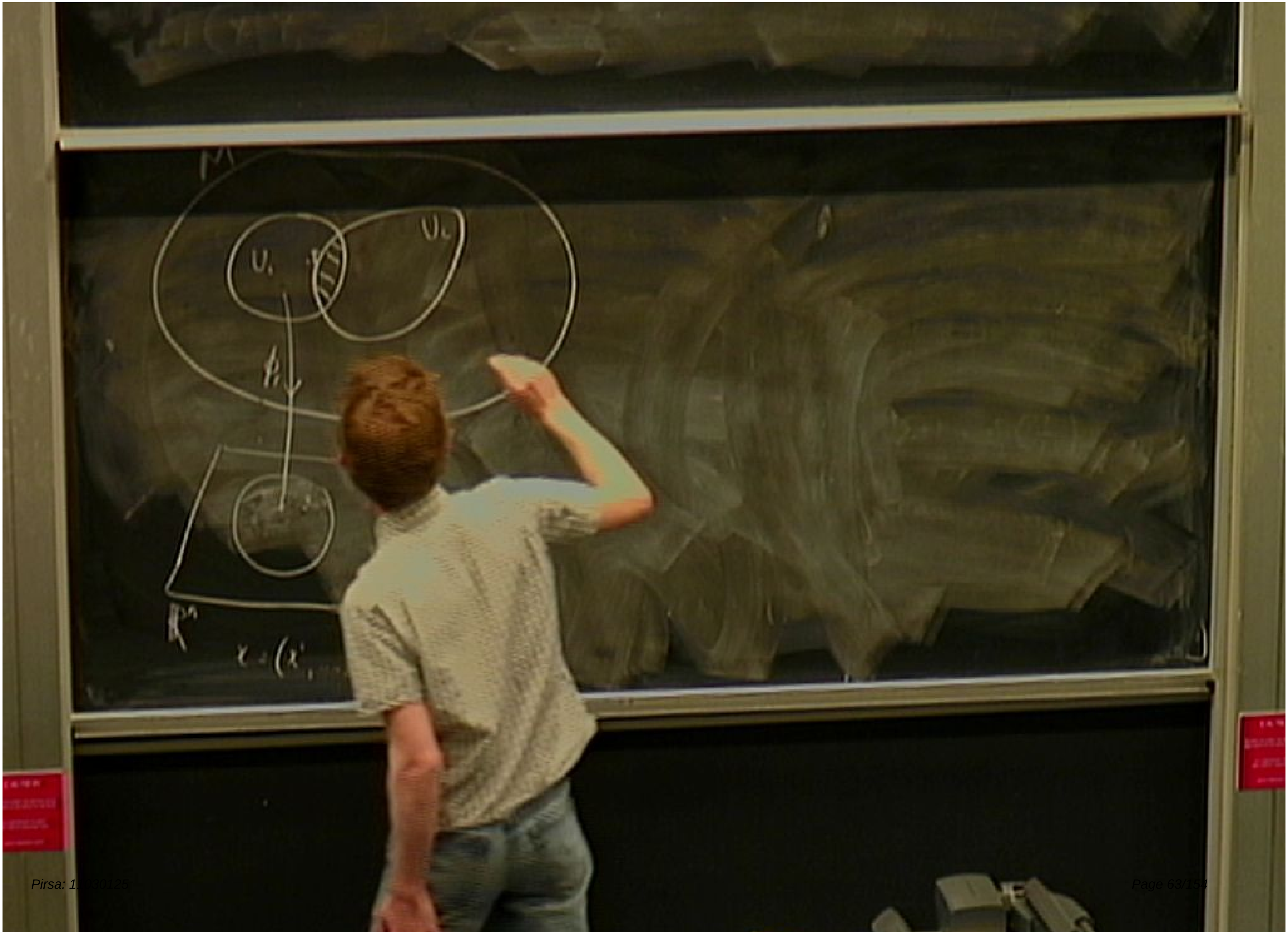
A manifold M is a topological space s.t.

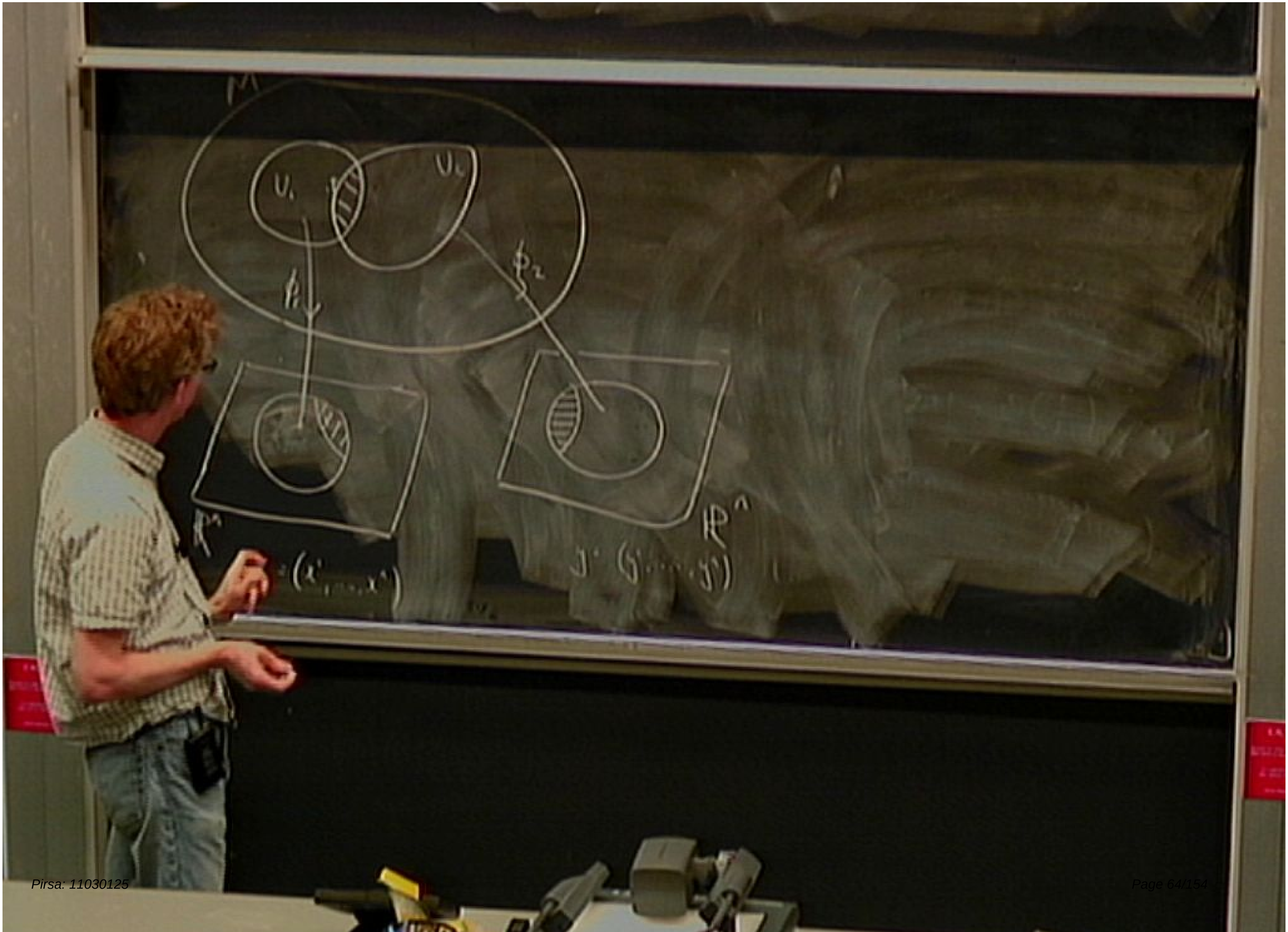
- can cover M by open sets U_i
- on each U_i, $\phi_i : U_i \to U_i' \subset \mathbb{R}^n$ (n-dim manifold)


M


M
U₁


M
U
x
R^n
x = (x^1, ..., x^n)

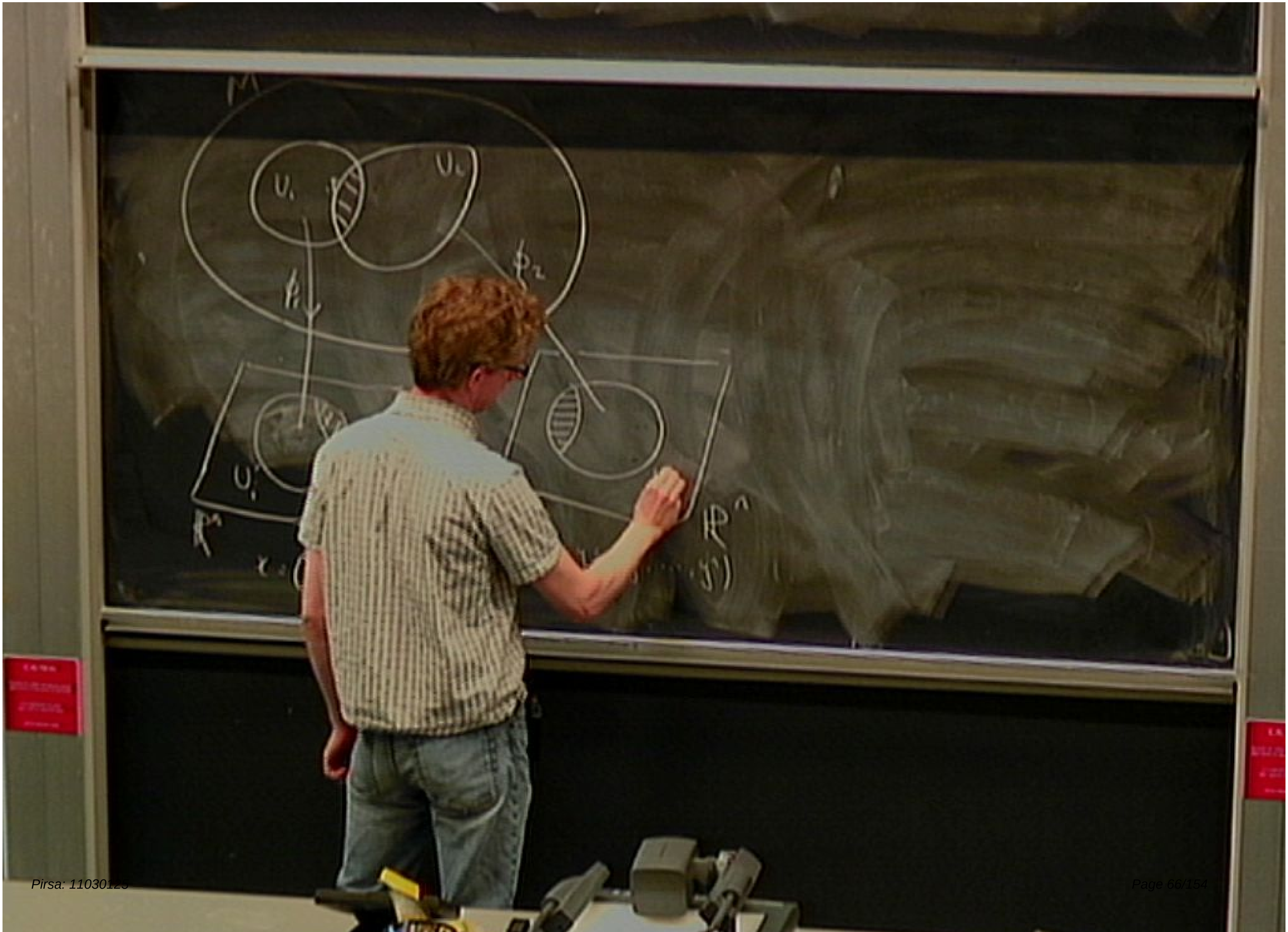

M

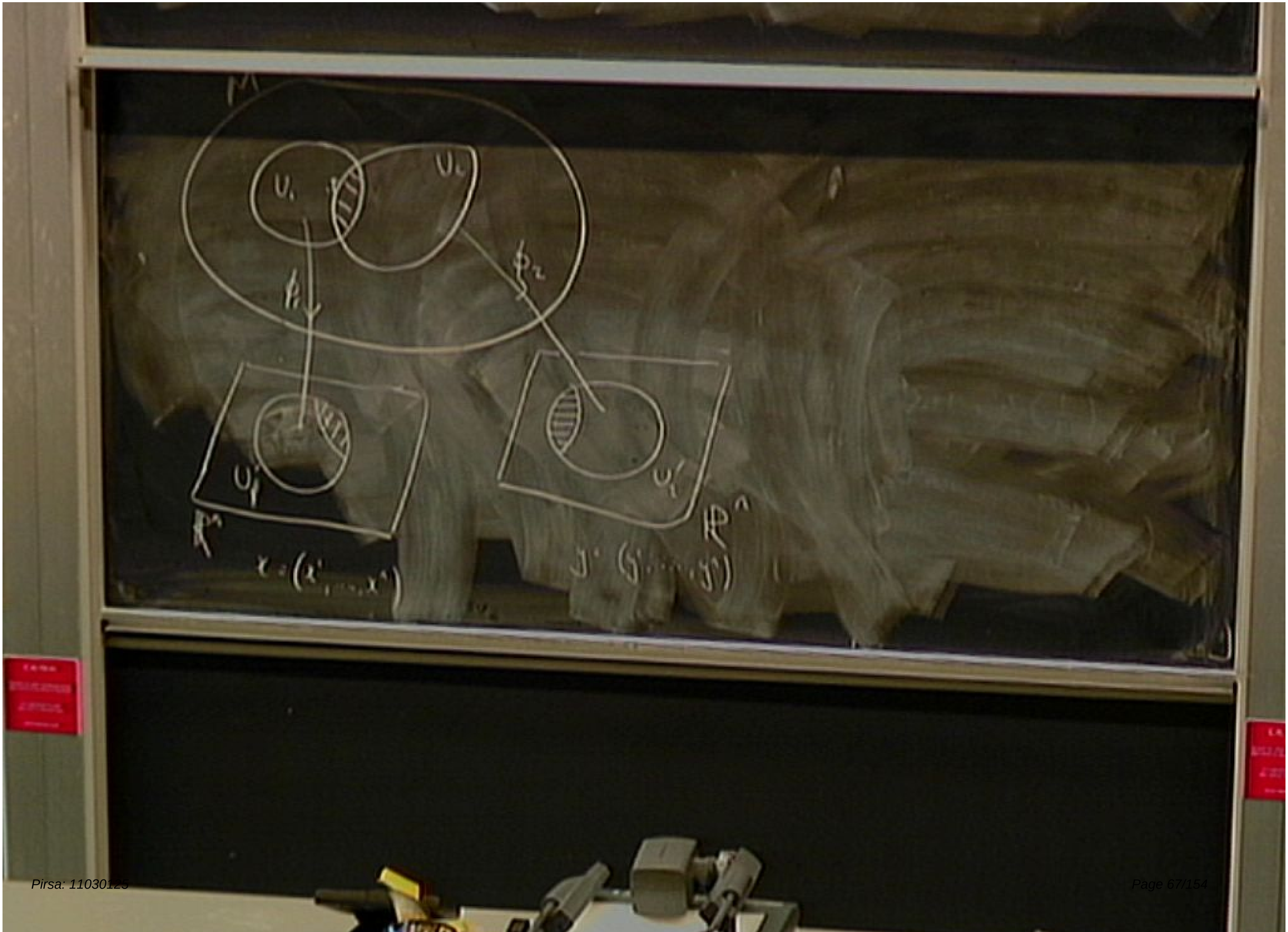


Manifolds.

A manifold M is a topological space s.t.

- can cover M by open sets U_i
- on each U_i, $\phi_i : U_i \to U_i' \subset \mathbb{R}^n$ (n-dim manifold)
- $\phi_i \phi_j^{-1} : U_j \to U_i$

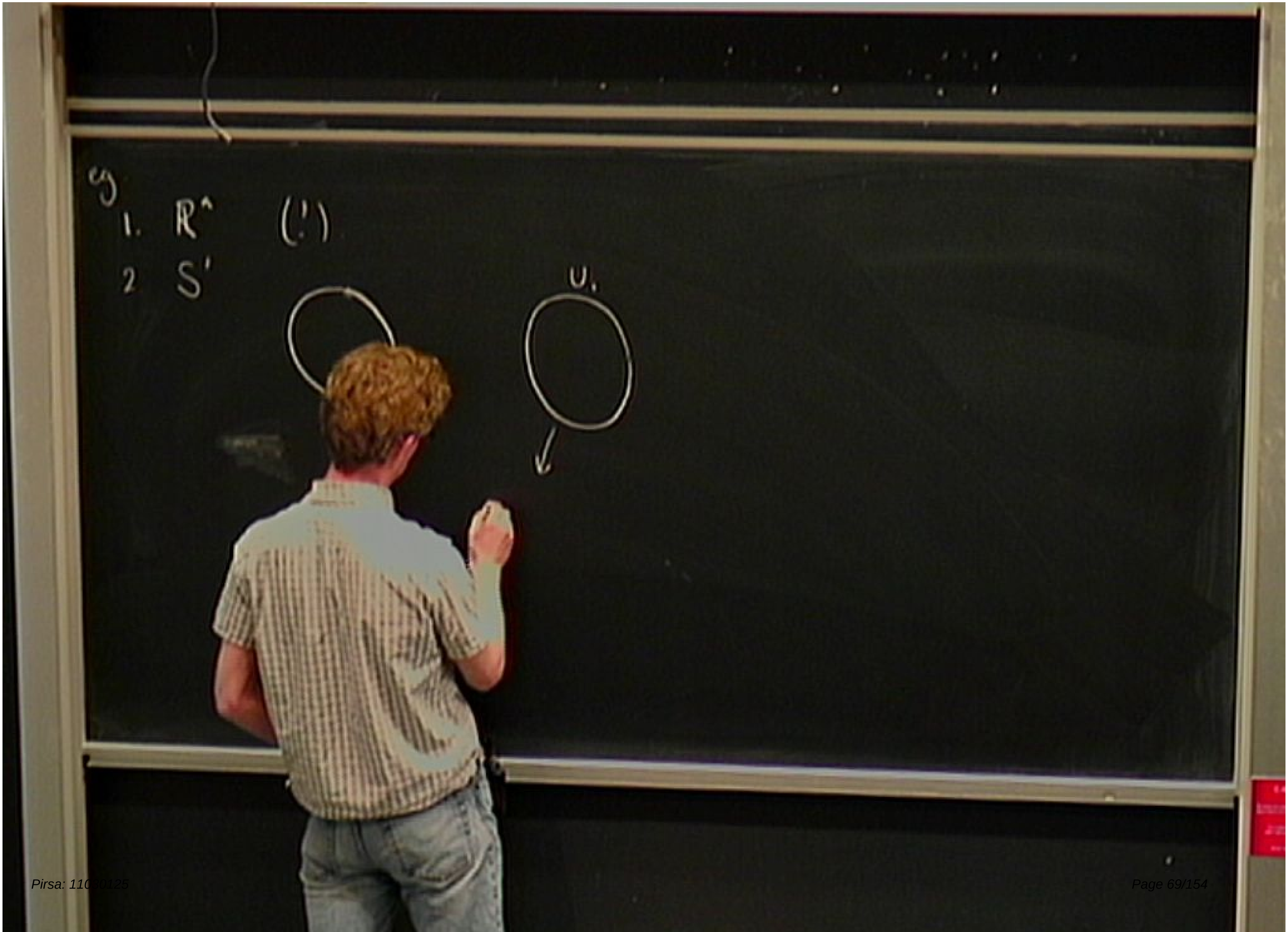


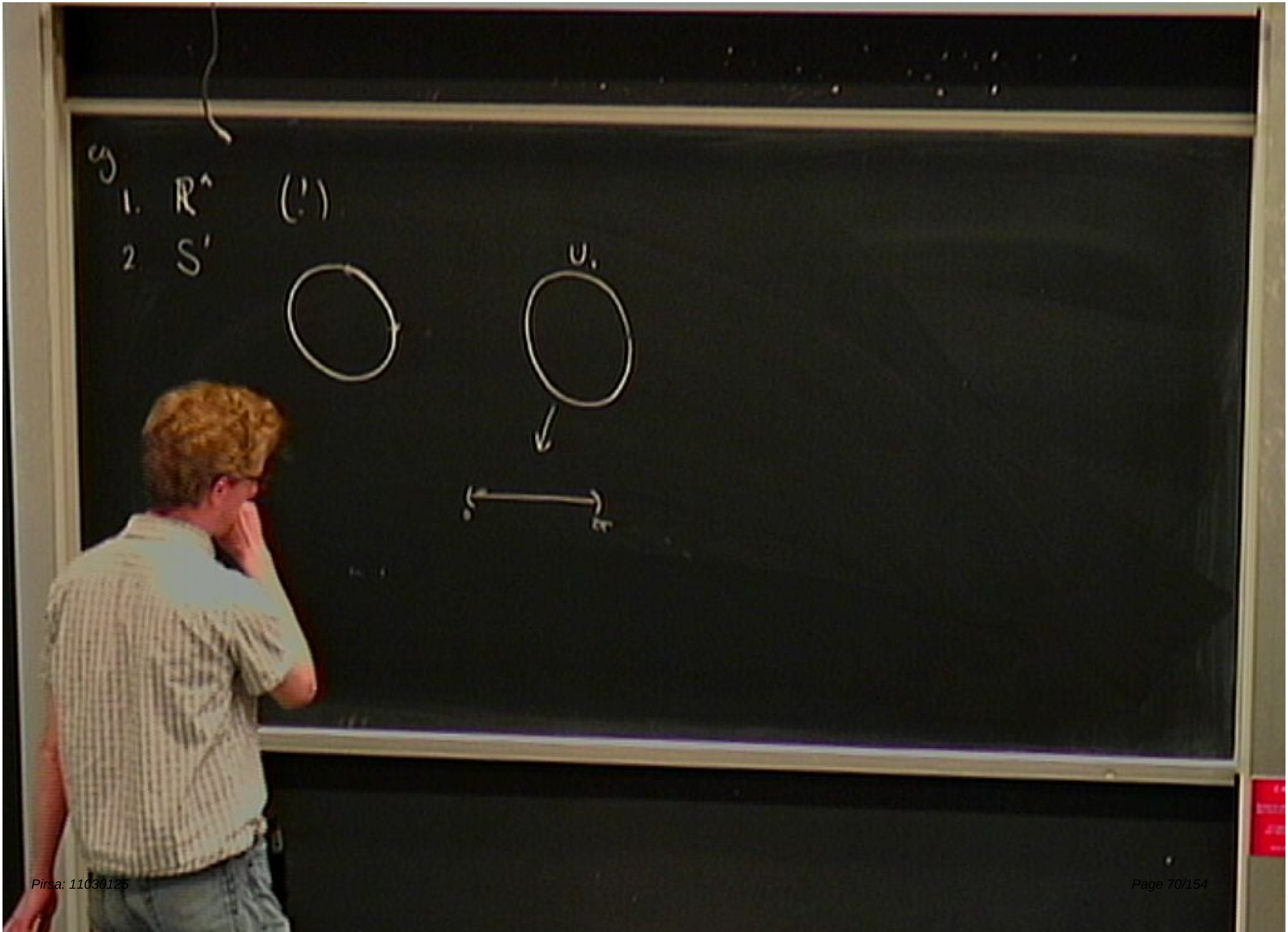

M
U1
U2
φ1
φ2
U1'
U2'
ℝn
ℝn
x = (x1, ..., xn)
y = (y1, ..., yn)

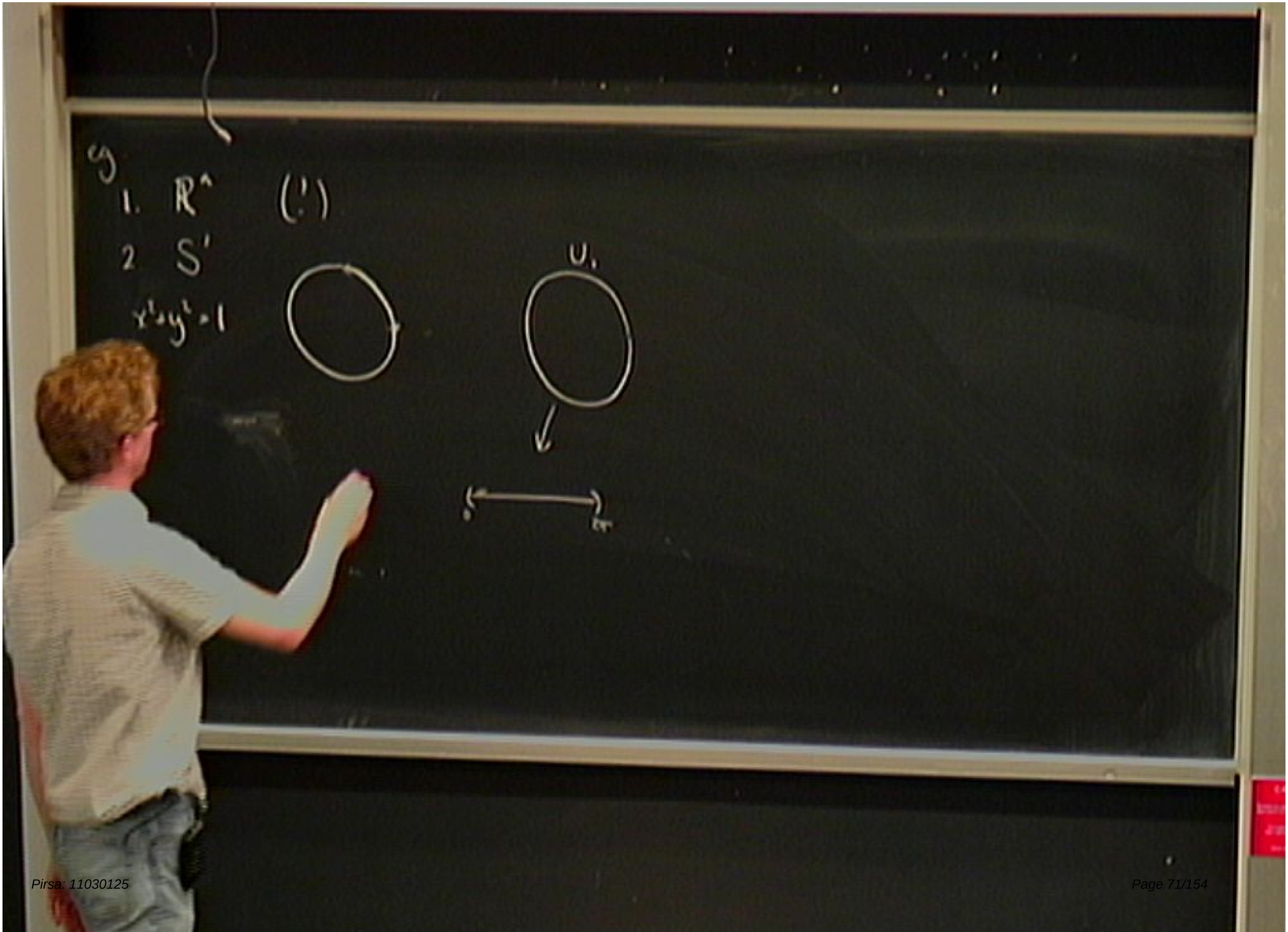
Manifolds.

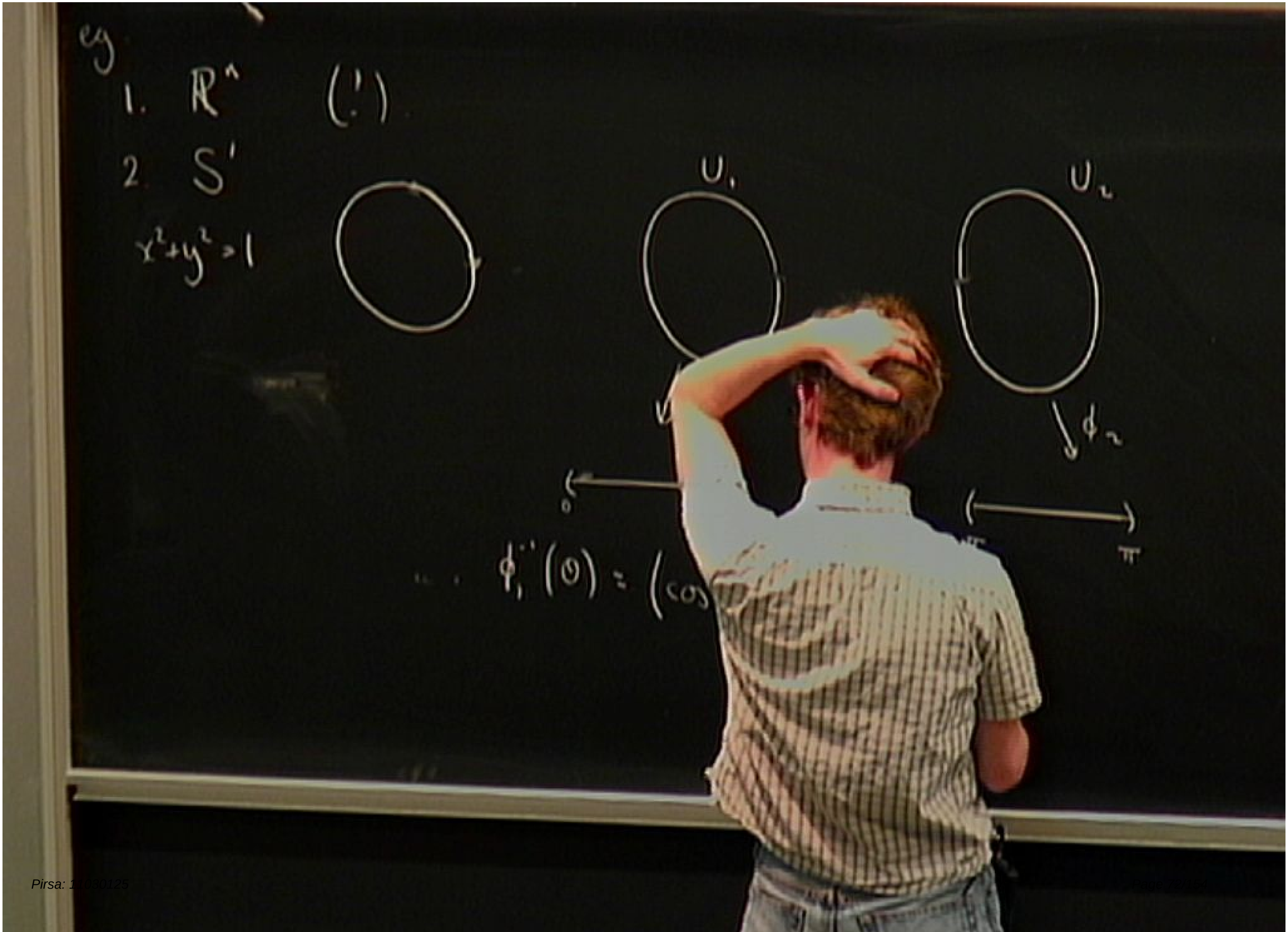
A manifold M is a topological space s.t.

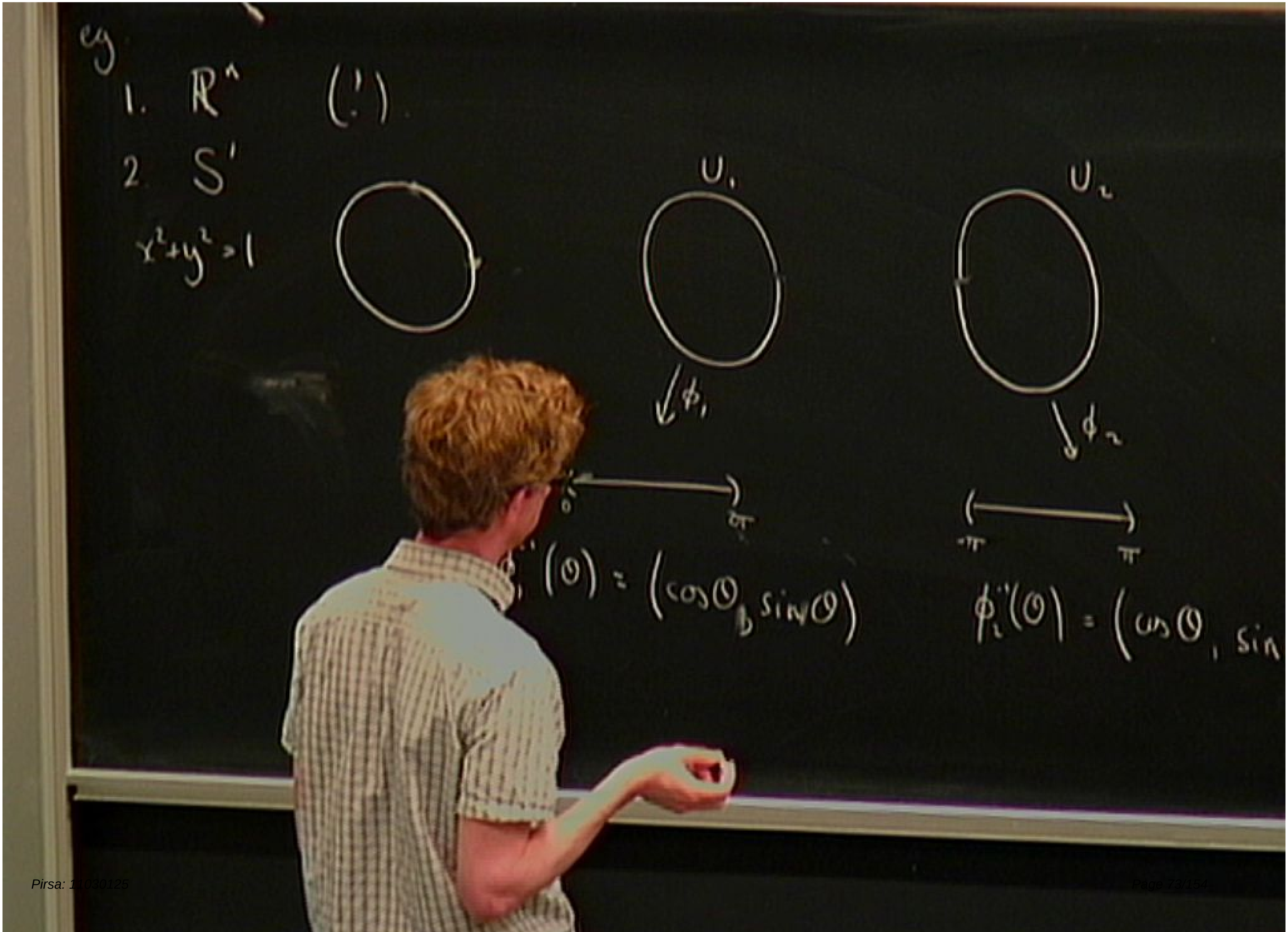
- can cover M by open sets U_i
- on each U_i, $\phi_i : U_i \to U_i' \subset \mathbb{R}^n$ (n-dim manifold)
- $\phi_i \phi_j^{-1} : U_j' \to U_i'$ must be smooth (infinitely differentiable).

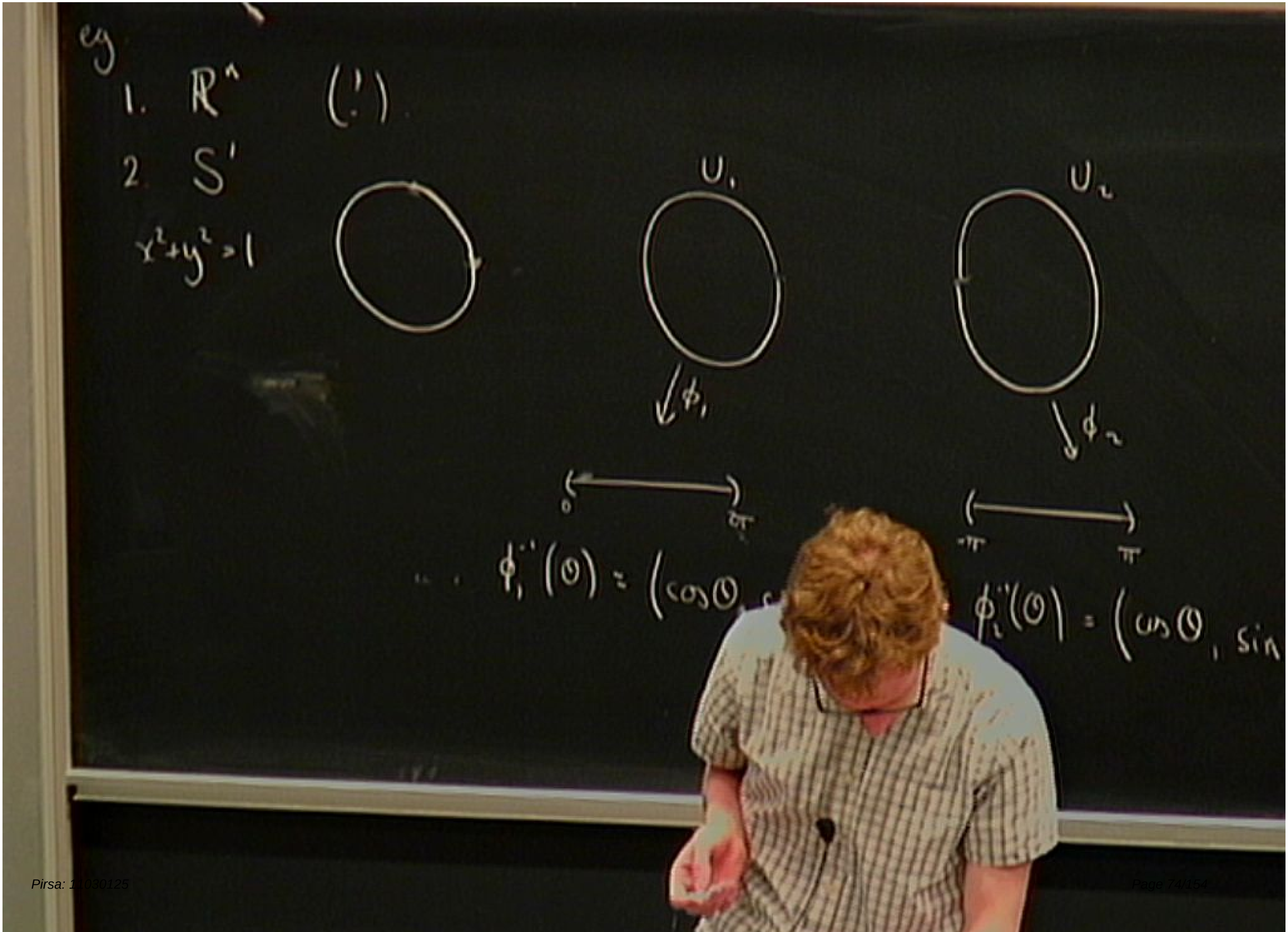

1. ℝ^n (!)
2 S^1
U_1

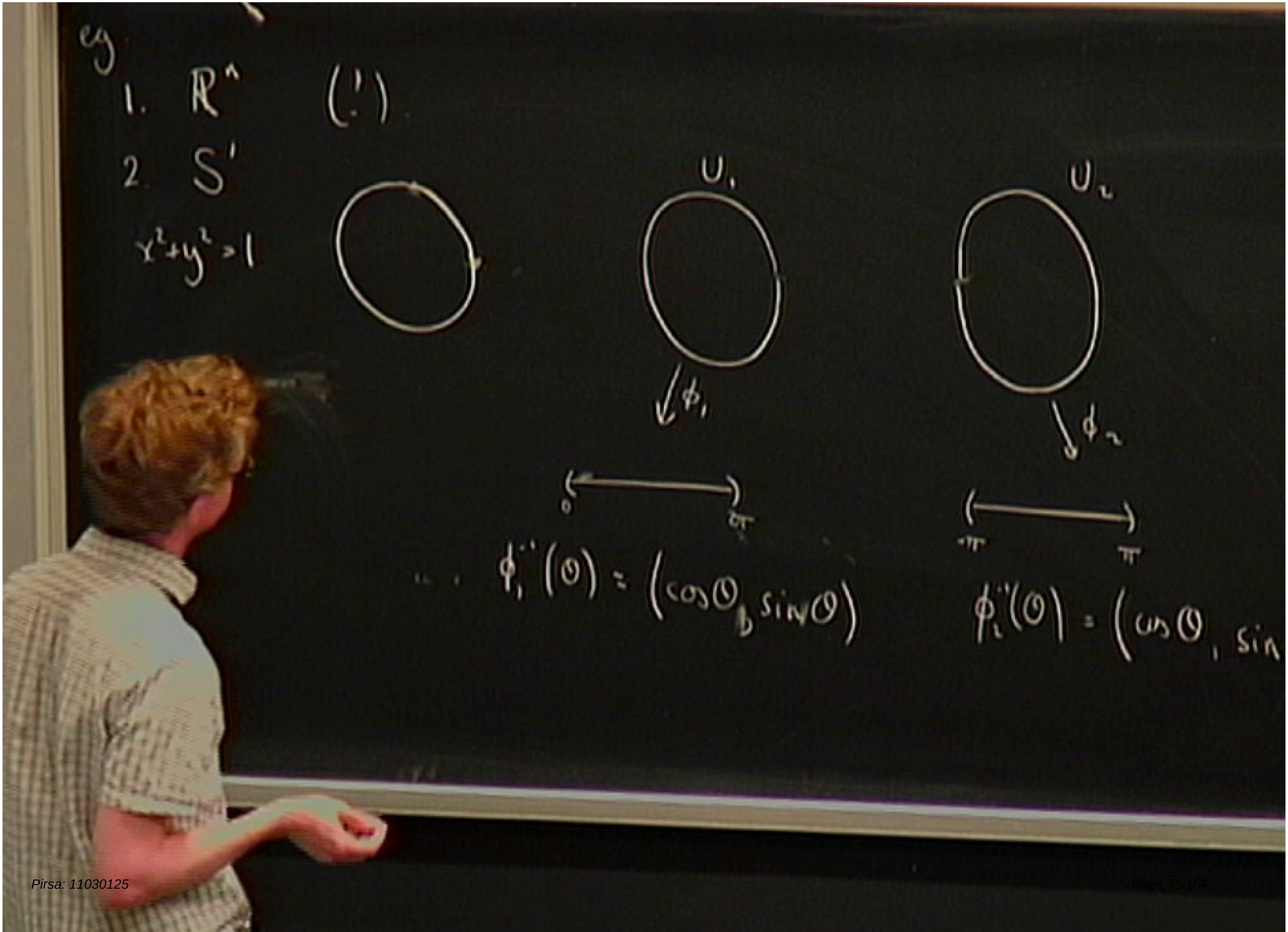



eg
1. R^n
2 S'
x²+y²=1
U₁


eg
1. ℝ^n (!)
2. S'
x²+y²=1
U₁
U₂
φ₂
0
π
φ₁⁻¹(θ) = (cos


eg
1. ℝ^n (!)
2 S'
x²+y²=1
U₁
U₂
φ₁
φ₂
(θ) = (cosθ, sinθ)
φ₂⁻¹(θ) = (cosθ, sin

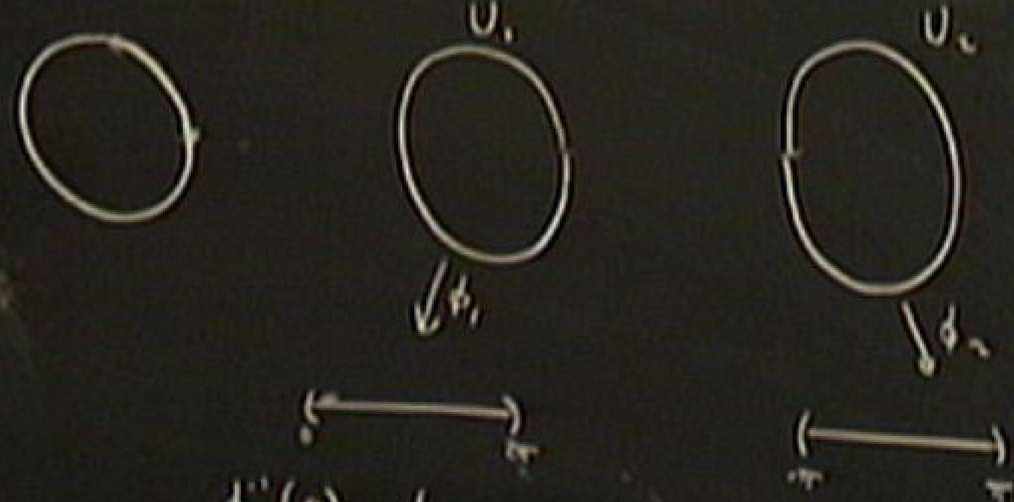

eg
1. ℝ^n (!)
2. S'
x²+y²=1
U₁
U₂
φ₁
φ₂
0
2π
-π
π
φ₁⁻¹(θ) = (cos θ,
φ₂⁻¹(θ) = (cos θ, sin


eg
1. ℝⁿ (!)
2. S¹
x²+y²=1
U₁
U₂
φ₁
φ₂
0
2π
-π
π
φ₁⁻¹(θ) = (cosθ, sinθ)
φ₂⁻¹(θ) = (cosθ, sin

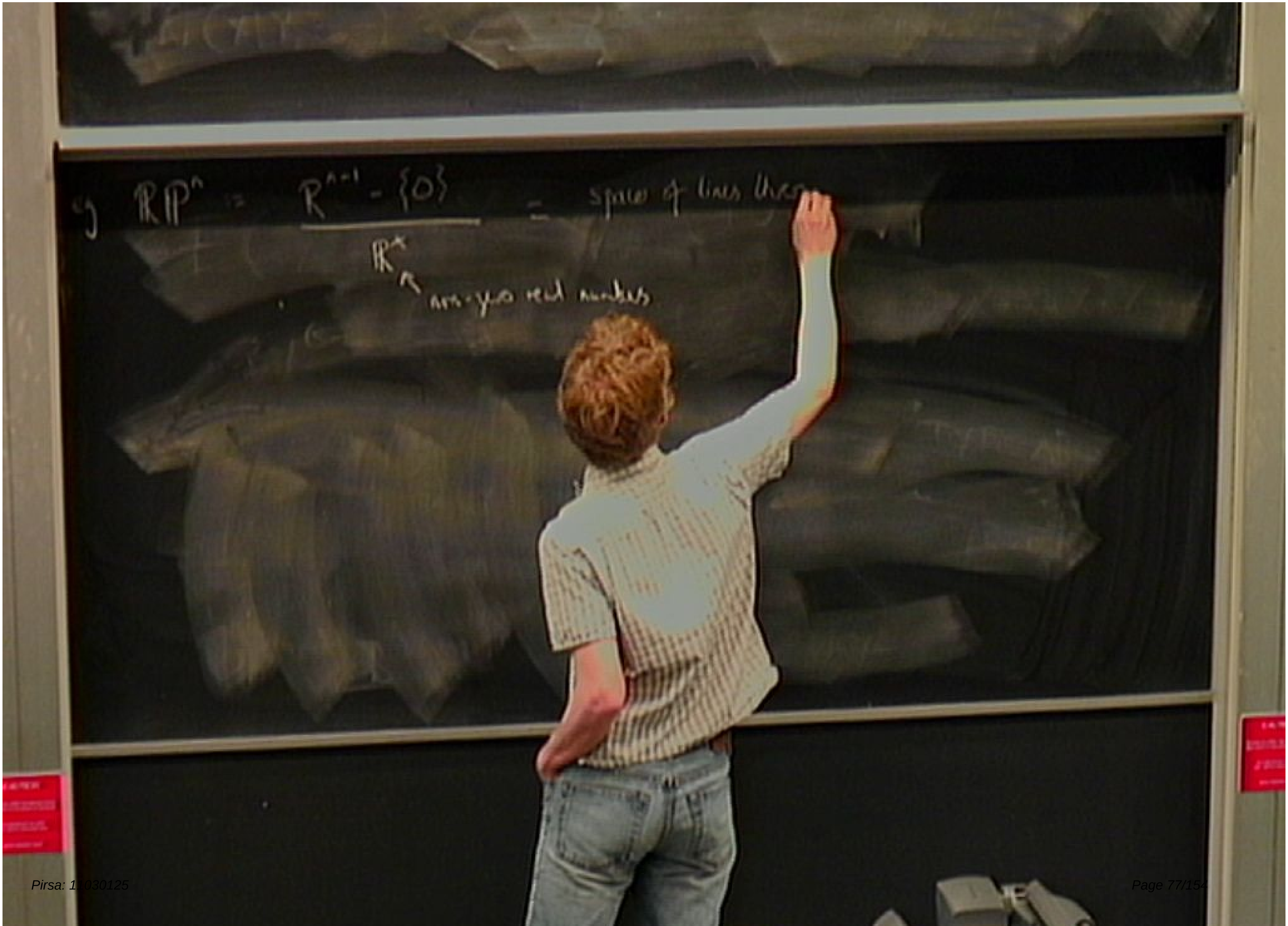
eg

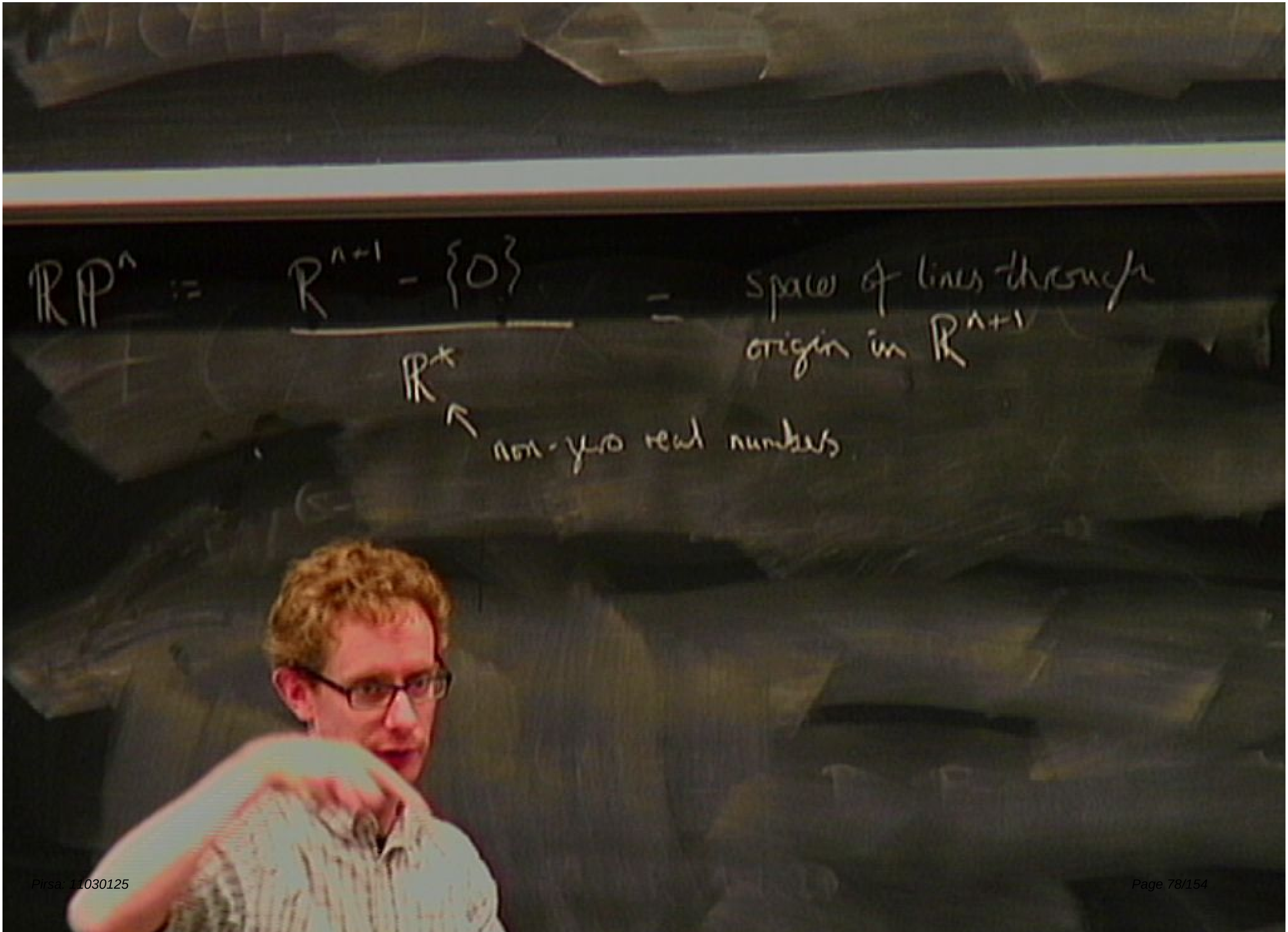
1. $\mathbb{R}^n$ (!)
2. S^1

$x^2+y^2=1$



$\phi_1^{-1}(\theta) = (\cos\theta, \sin\theta)$ $\qquad$ $\phi_2^{-1}(\theta) = (\cos\theta, \sin\theta)$

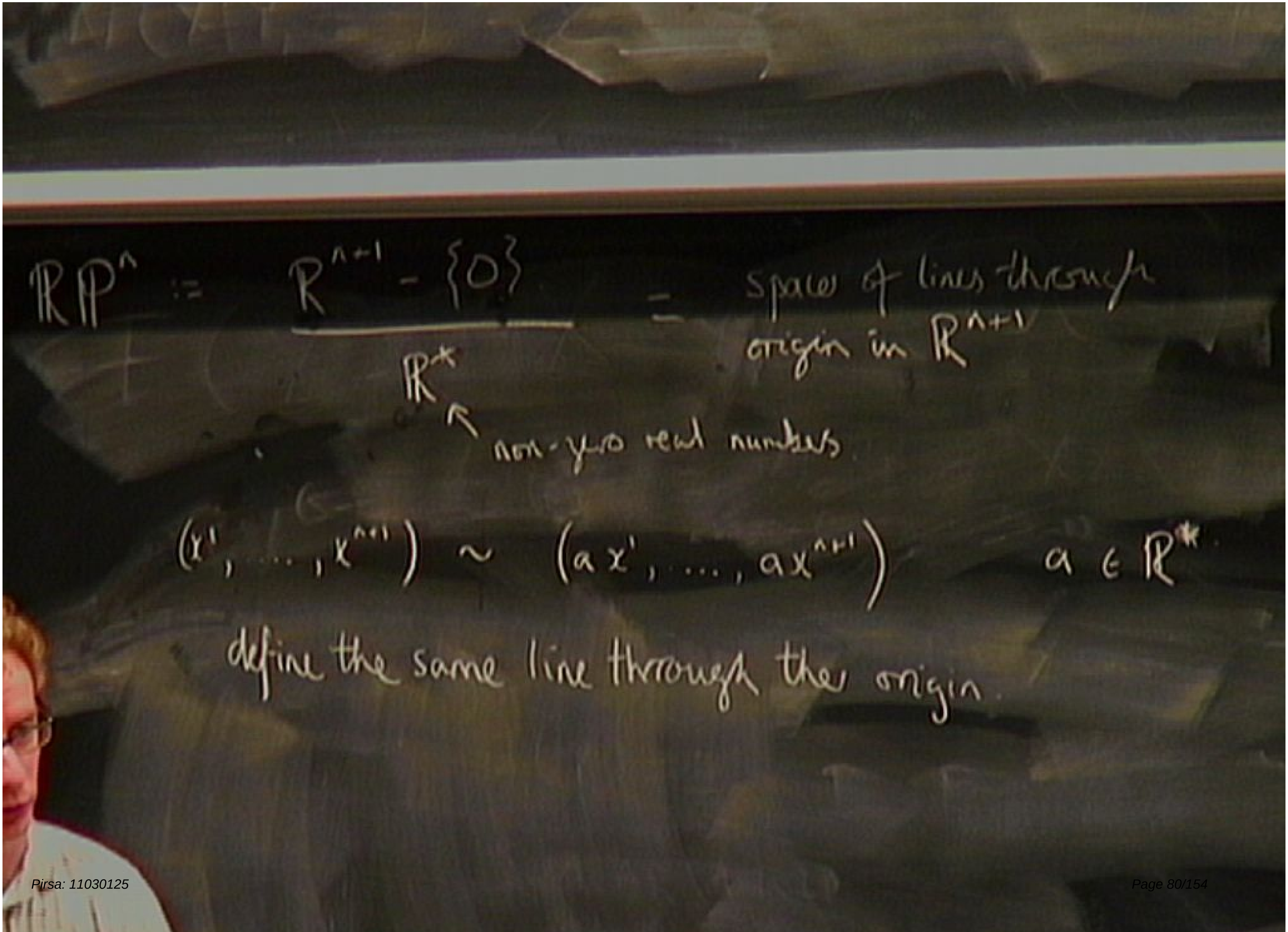

eg $\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}$ / $\mathbb{R}^*$ = space of lines through
non-zero real numbers


RP^n := R^{n+1} - {0} / R* = space of lines through origin in R^{n+1}
non-zero real numbers

$$\mathbb{RP}^n := \frac{\mathbb{R}^{n+1} - \{0\}}{\mathbb{R}^*} = \text{space of lines through origin in } \mathbb{R}^{n+1}$$

$\uparrow$ non-zero real numbers

$$(x^1, \ldots, x^{n+1}) \sim (a x^1, \ldots, a x^{n+1}) \qquad a \in \mathbb{R}^*$$

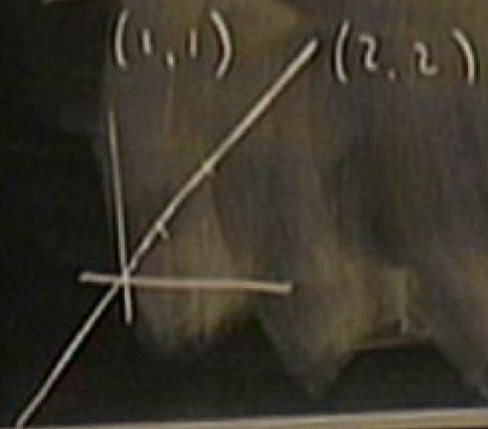

$\mathbb{RP}^n := \frac{\mathbb{R}^{n+1} - \{0\}}{\mathbb{R}^*}$ = space of lines through origin in $\mathbb{R}^{n+1}$
non-zero real numbers
$(x^1, \ldots, x^{n+1}) \sim (a x^1, \ldots, a x^{n+1})$ $a \in \mathbb{R}^*$
define the same line through the origin.

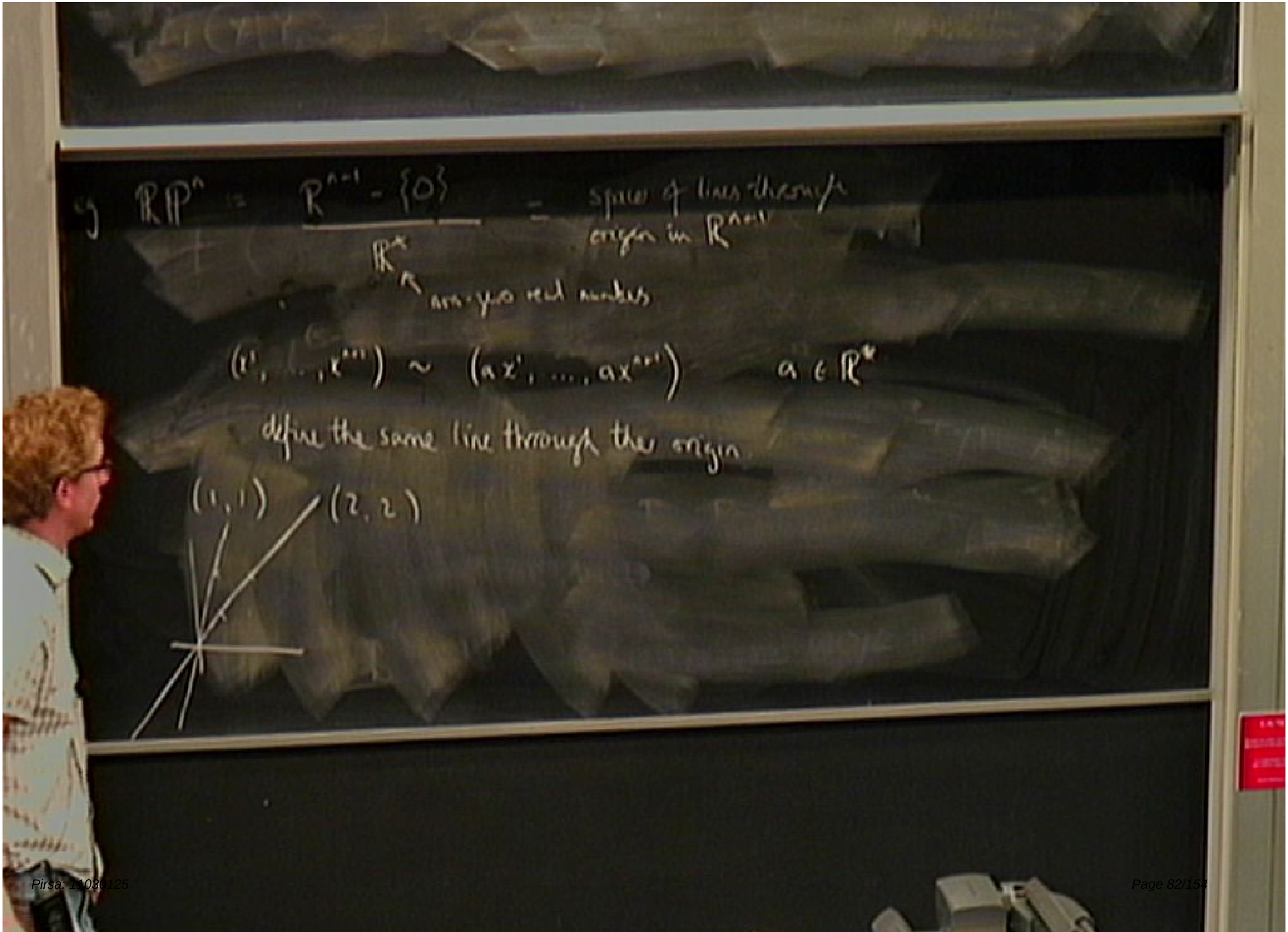
eg $\mathbb{RP}^n = \dfrac{\mathbb{R}^{n+1} - \{0\}}{\mathbb{R}^*}$ = space of lines through origin in $\mathbb{R}^{n+1}$

$\mathbb{R}^*$ ↖ non-zero real numbers

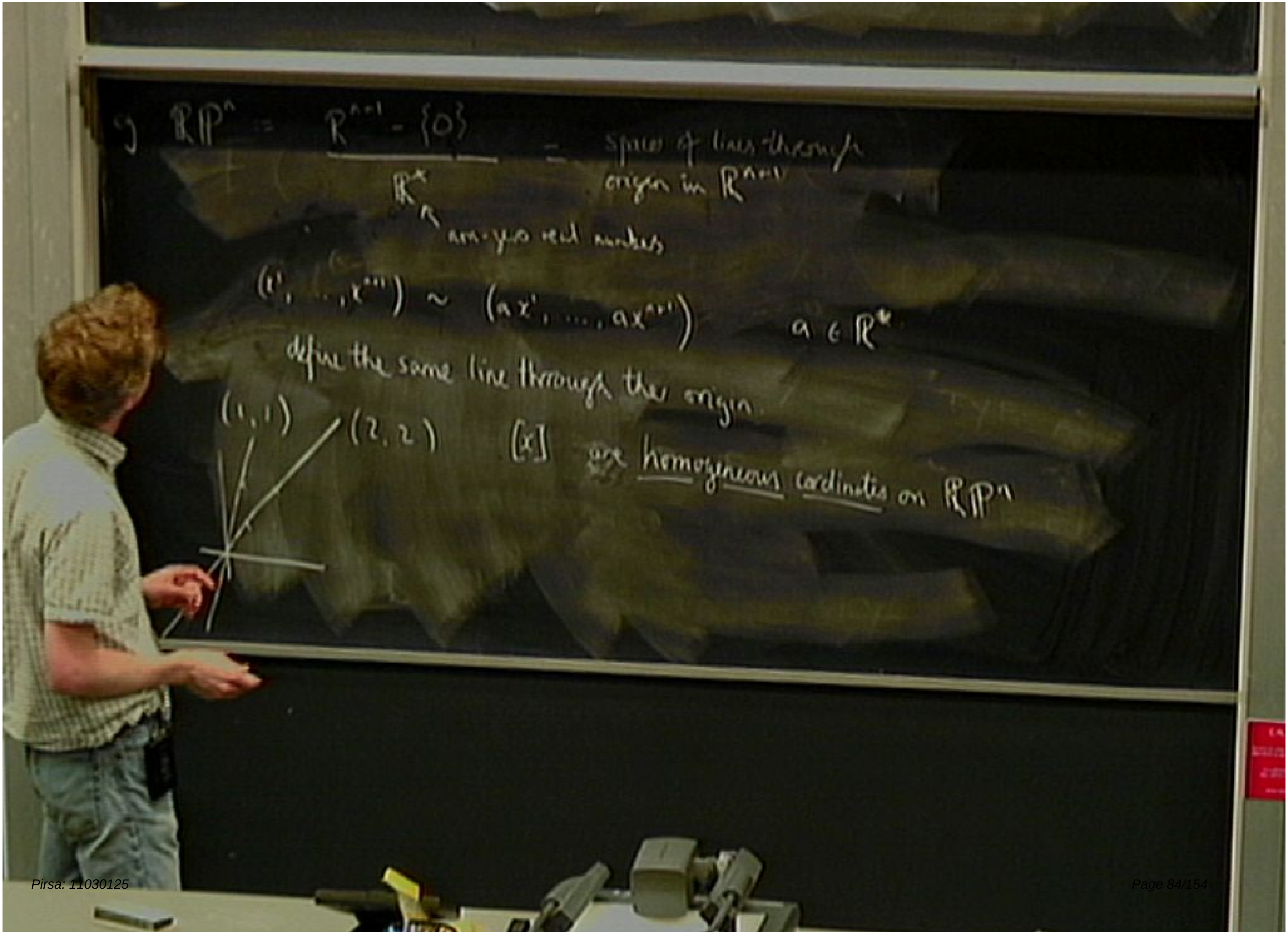
$$(x^1, \ldots, x^{n+1}) \sim (a x^1, \ldots, a x^{n+1}) \qquad a \in \mathbb{R}^*$$

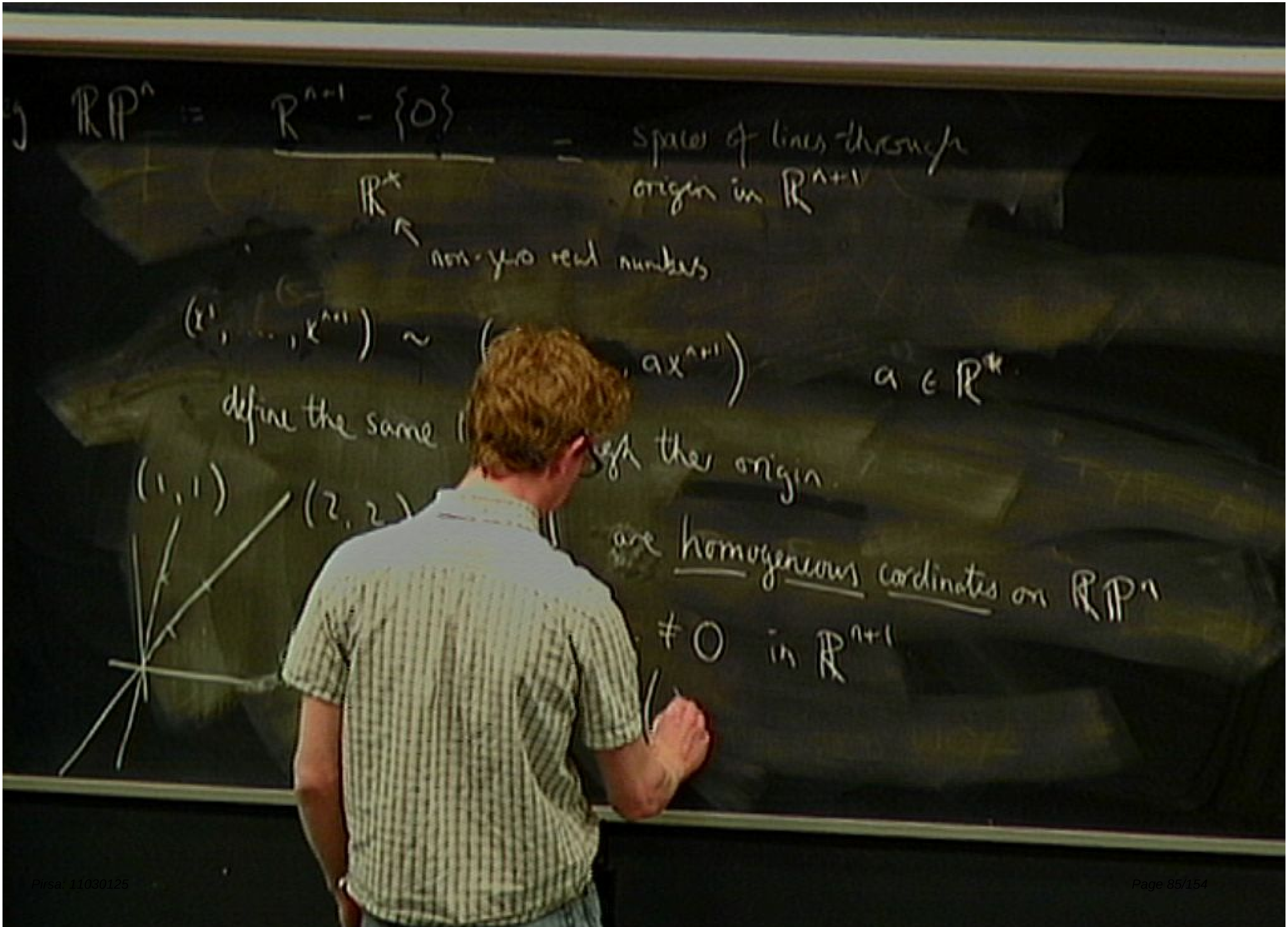
define the same line through the origin

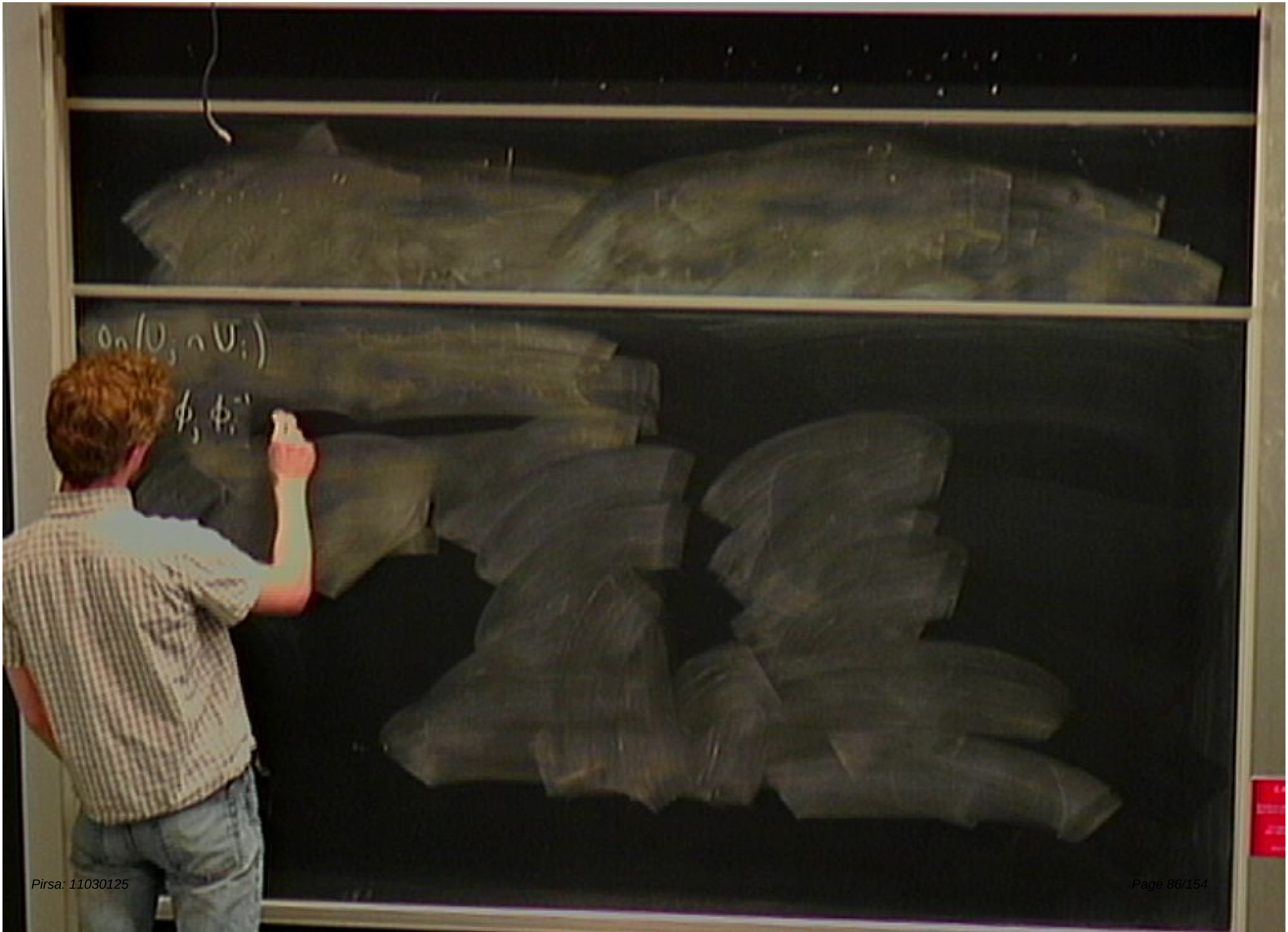


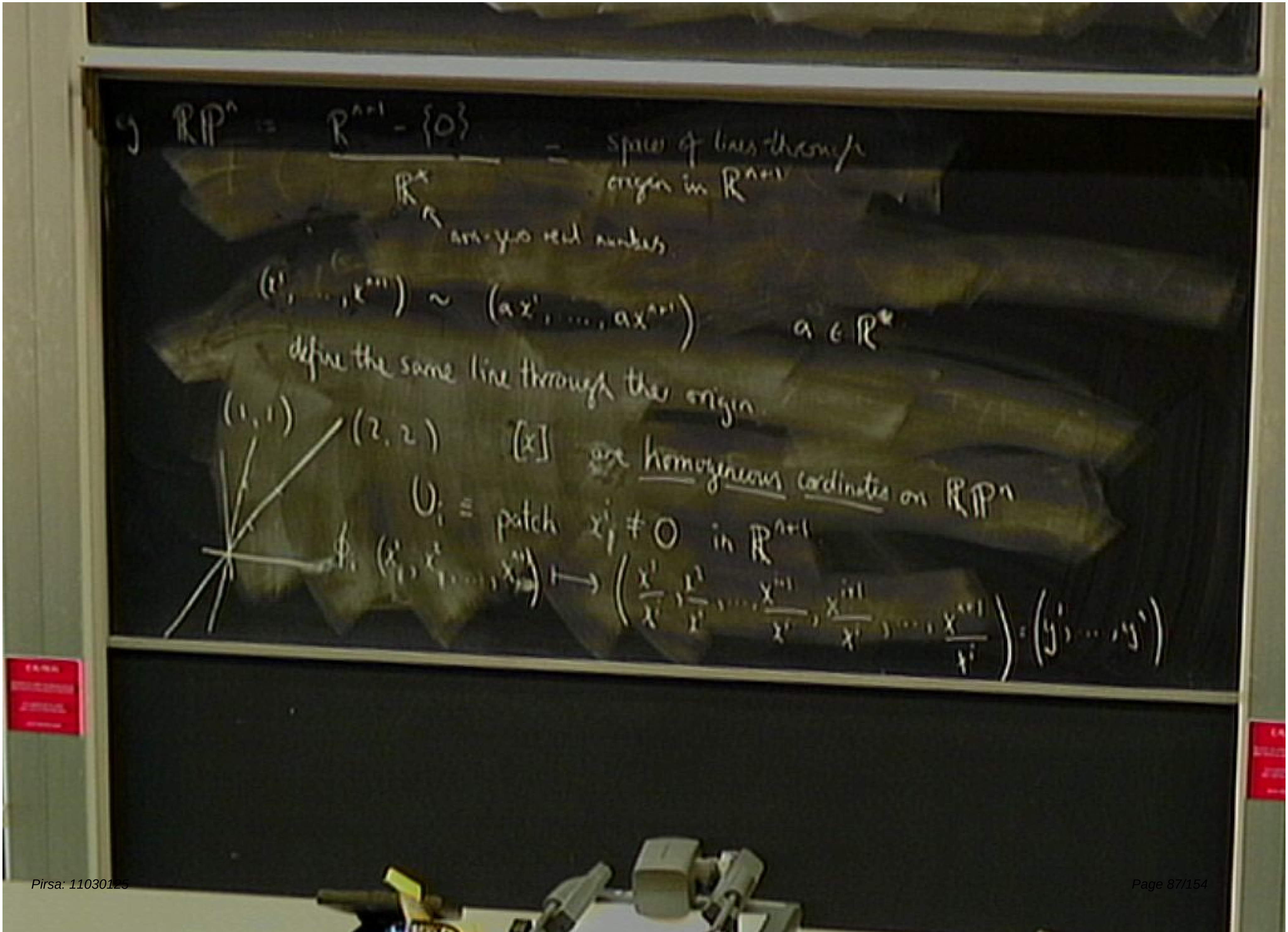

eg RP^n = R^{n+1} - {0} / R^* = space of lines through origin in R^{n+1}
R^* ← non-zero real numbers
(x^1, ..., x^{n+1}) ~ (ax^1, ..., ax^{n+1}) a ∈ R^*
define the same line through the origin
(1, 1)
(2, 2)

eg $G_k(\mathbb{R}^n)$ = space of k-planes through origin in $\mathbb{R}^n$


eg RP^n = (R^{n+1} - {0}) / R^* = space of lines through origin in R^{n+1}
non-zero real numbers
(x^1, ..., x^{n+1}) ~ (ax^1, ..., ax^{n+1})   a ∈ R^*
define the same line through the origin.
(1,1)
(2,2)
[x] are homogeneous coordinates on RP^n


RP^n := R^{n+1} - {0} / R*
= space of lines through origin in R^{n+1}
non-zero real numbers
(x^1, ..., x^{n+1}) ~ (..., ax^{n+1})
a ∈ R*
define the same l... the origin.
(1, 1)
(2, 2)
are homogeneous cordinates on RP^n
≠ 0 in R^{n+1}


$\mathcal{O}_n(U_j \cap U_i)$
$\phi_j \phi_i^{-1}$


eg RP^n = R^{n+1} - {0} / R* = space of lines through origin in R^{n+1}
non-zero real numbers
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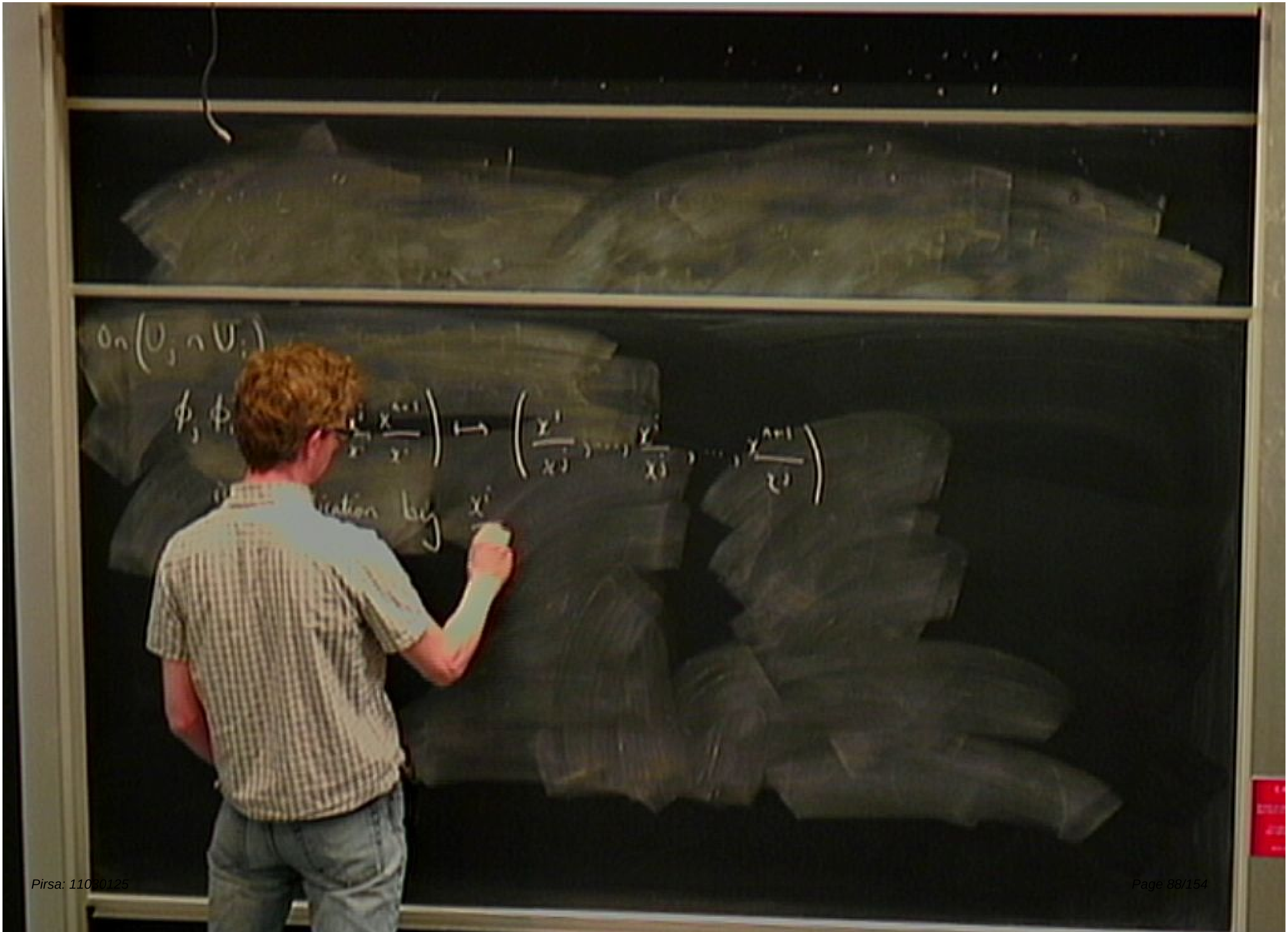
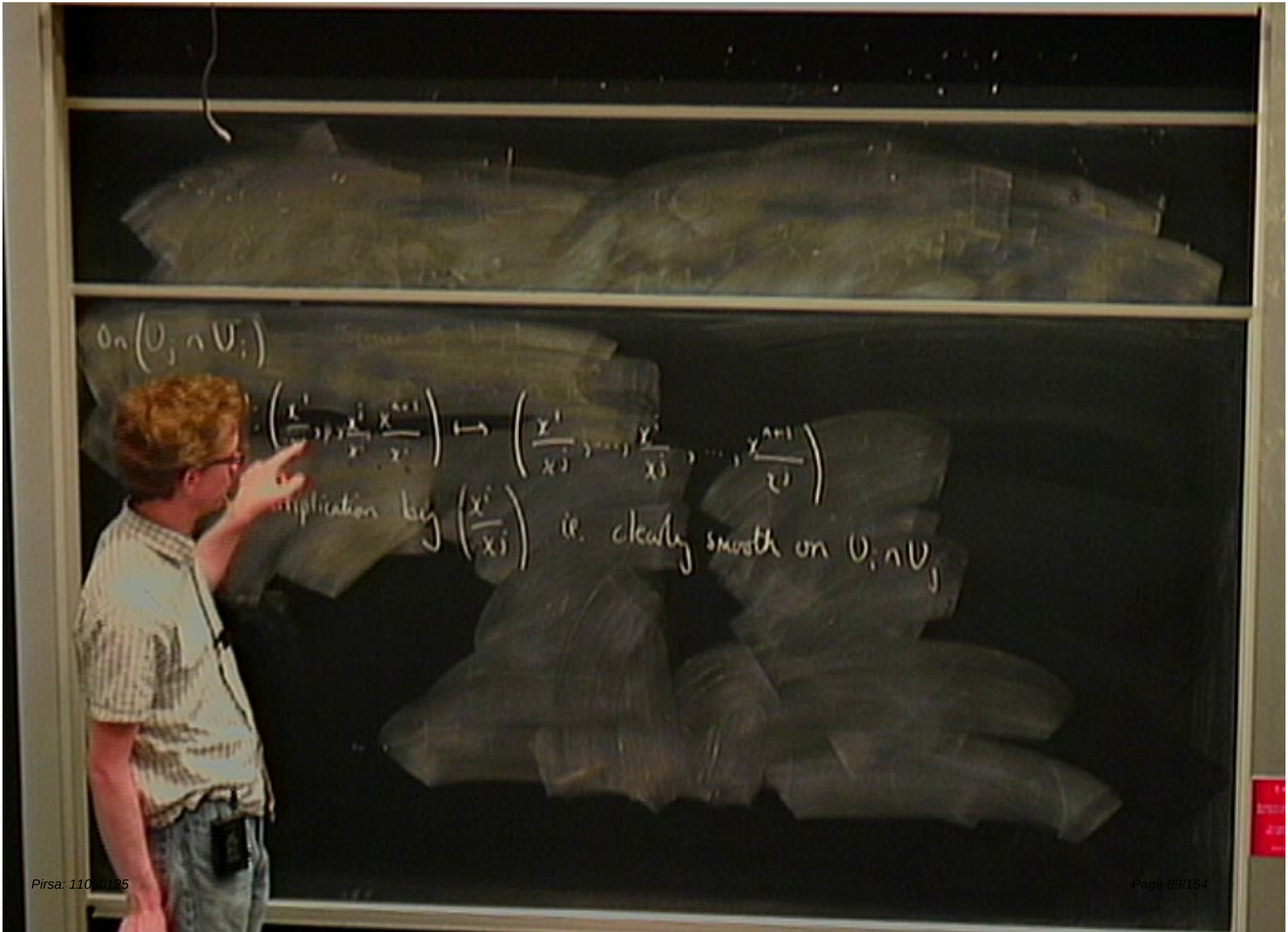
define the same line through the origin.
(1, 1)
(2, 2)
[x] are homogeneous coordinates on RP^n
U_i = patch x^i ≠ 0 in R^{n+1}


On (U_j ∩ U_i)


On (U_j ∩ U_i)
iplication by
ie. clearly smooth on U_i ∩ U_j

On $(U_j \cap U_i)$

$$\phi_j \phi_i^{-1} : \left(\frac{x^1}{x^i}, \dots, \frac{x^j}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right) \mapsto \left(\frac{x^1}{x^j}, \dots, \frac{x^i}{x^j}, \dots, \frac{x^{n+1}}{x^j}\right)$$

is multiplication by $\left(\frac{x^i}{x^j}\right)$ i.e. clearly smooth on $U_i \cap U_j$

On $(U_j \cap U_i)$

$$\phi_j \phi_i^{-1} : \left(\frac{x^1}{x^i}, \dots, \frac{x^i}{x^i}, \frac{x^{n+1}}{x^i}\right) \mapsto \left(\frac{x^1}{x^j}, \dots, \frac{x^i}{x^j}, \dots, \frac{x^{n+1}}{x^j}\right)$$

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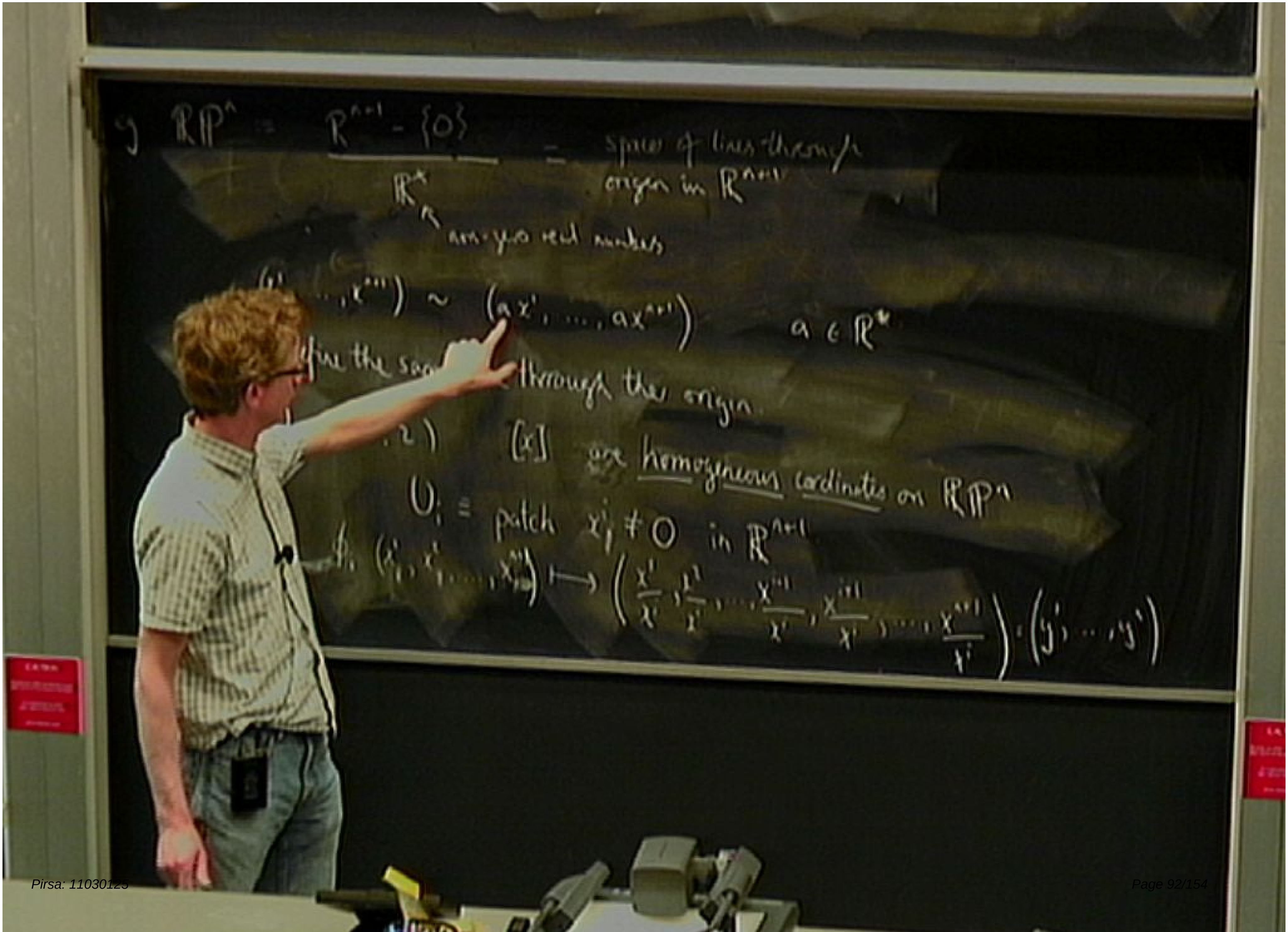
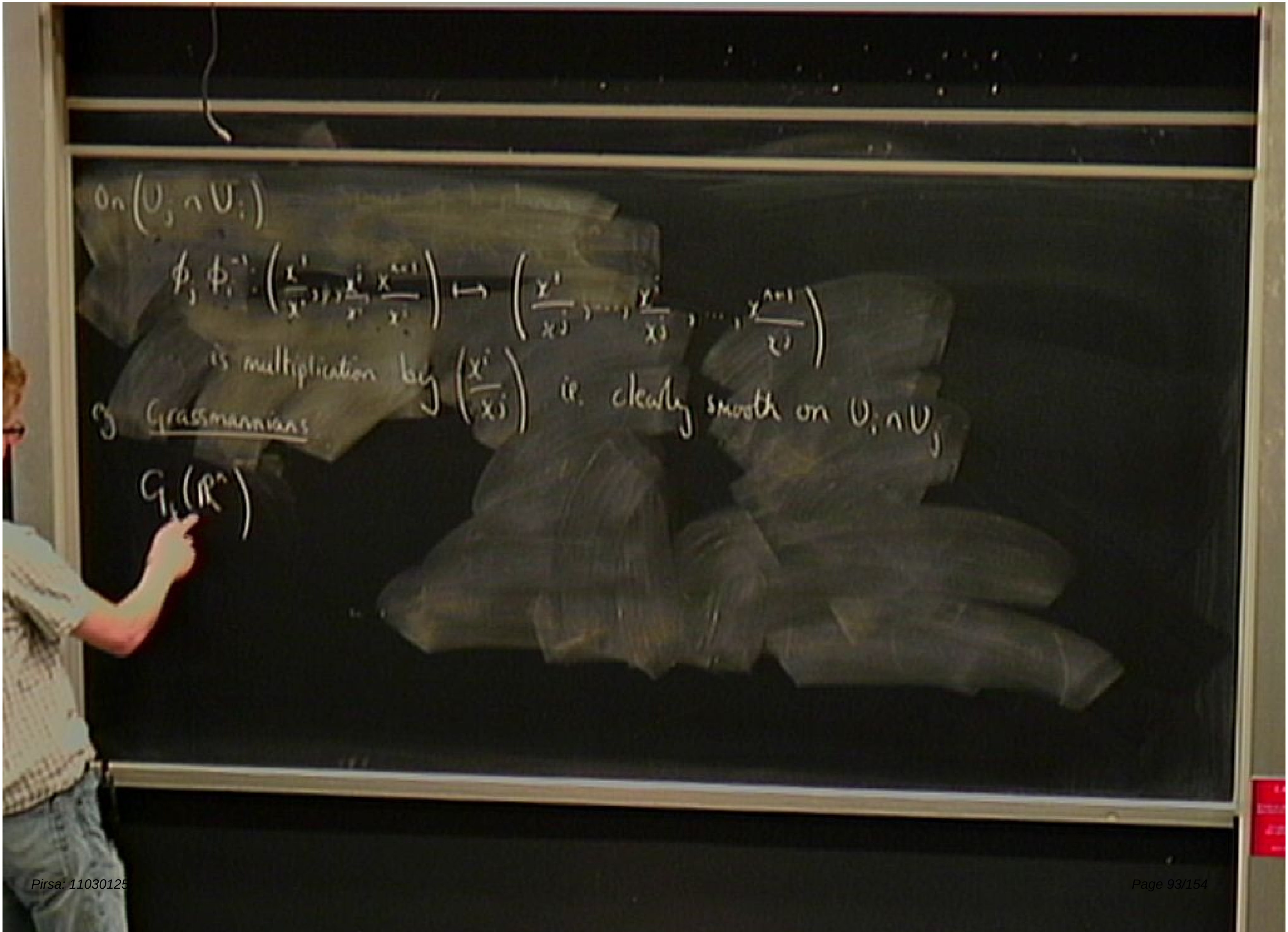
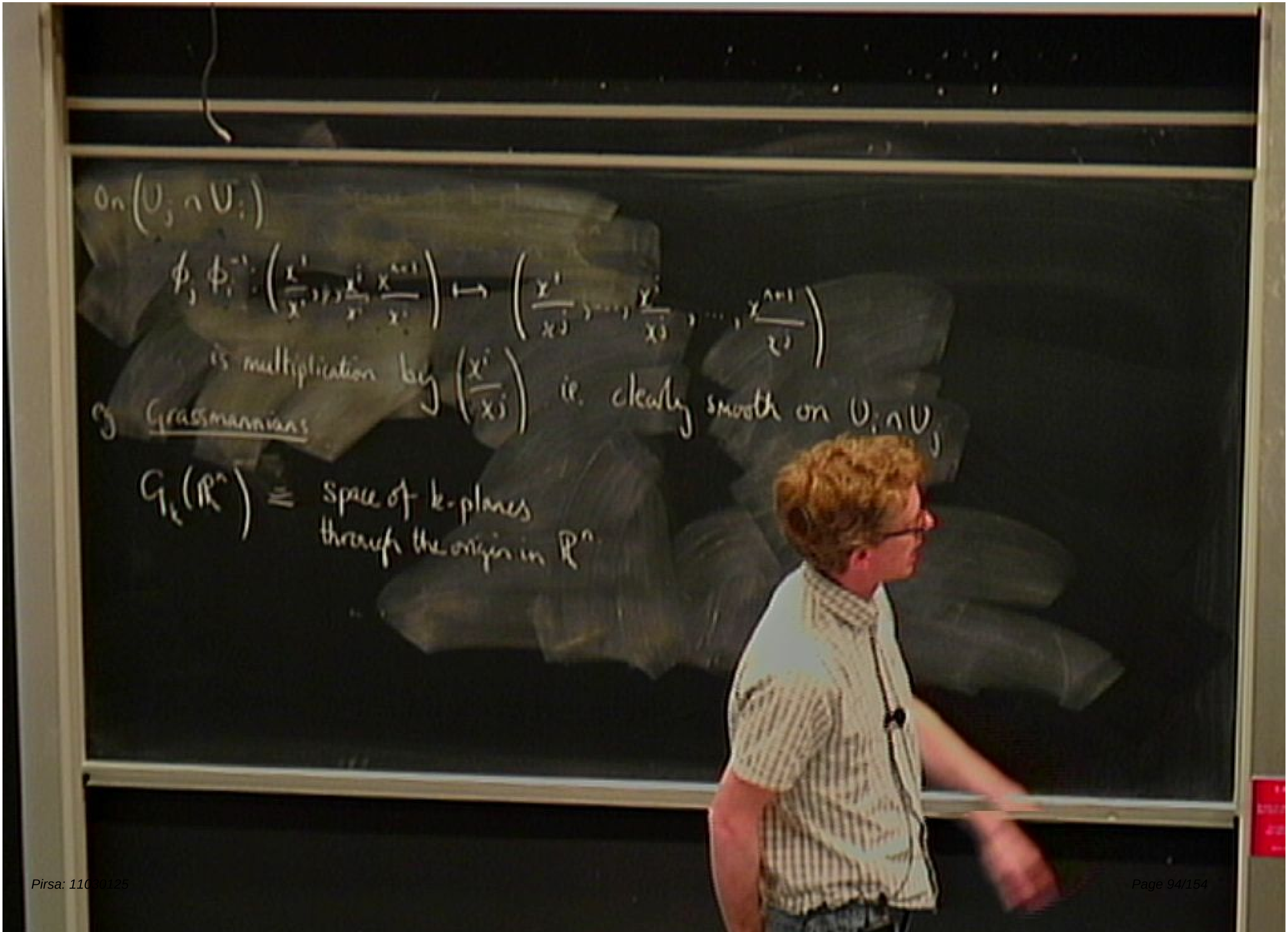
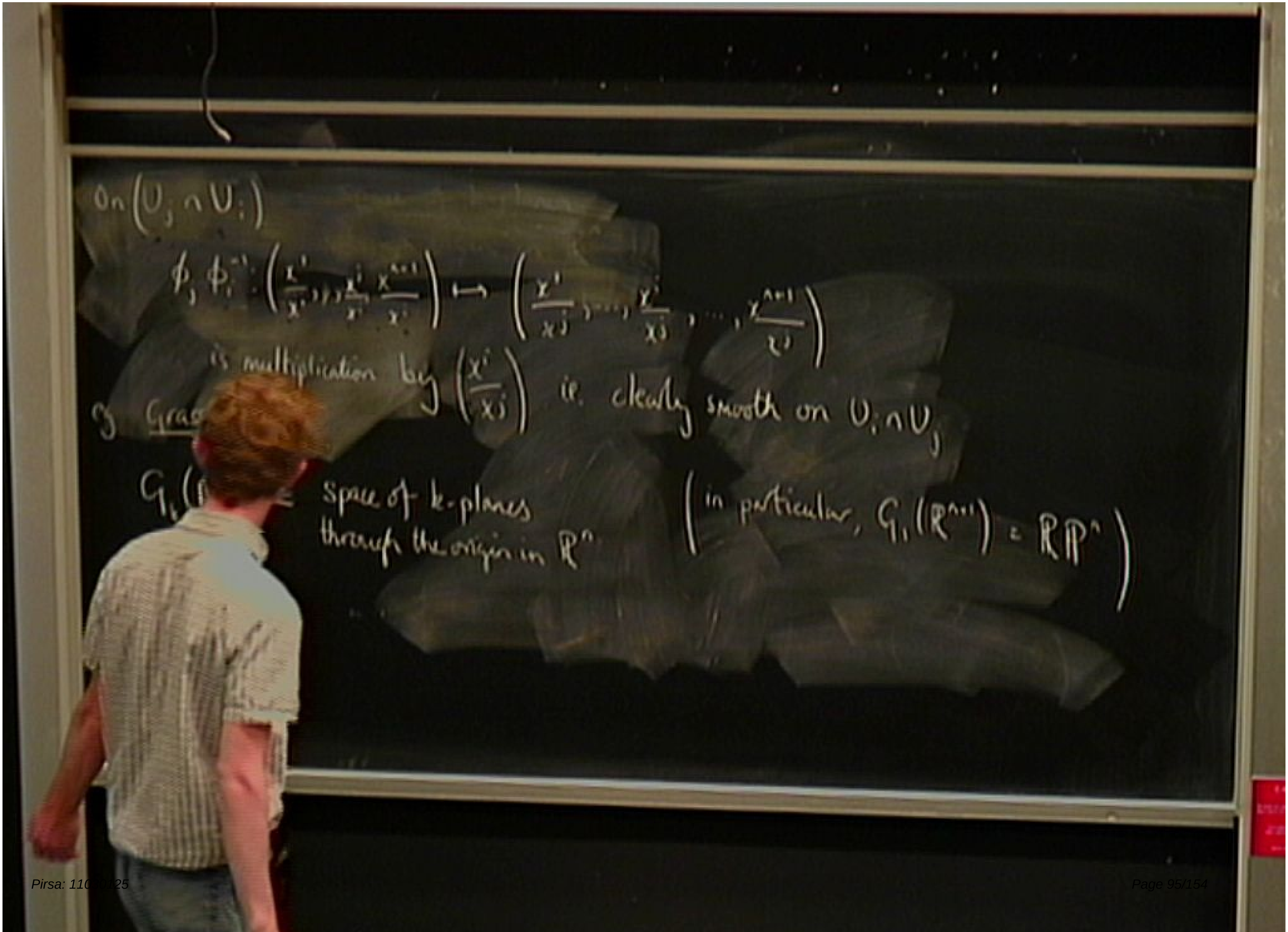

RP^n = R^{n+1} - {0} / R*
= space of lines through origin in R^{n+1}
non-zero real numbers
~ (ax^1, ..., ax^{n+1})
a ∈ R*
through the origin.
[x] are homogeneous coordinates on RP^n
U_i = patch x^i ≠ 0 in R^{n+1}


On $(U_j \cap U_i)$
$\phi_j \phi_i^{-1} : \left(\frac{x^1}{x^i}, \ldots, \frac{x^j}{x^i}, \frac{x^{n+1}}{x^i}\right) \mapsto \left(\frac{x^1}{x^j}, \ldots, \frac{x^i}{x^j}, \ldots, \frac{x^{n+1}}{x^j}\right)$
is multiplication by $\left(\frac{x^i}{x^j}\right)$ ie. clearly smooth on $U_i \cap U_j$
eg Grassmannians
$G_k(\mathbb{R}^n)$


On $(U_j \cap U_i)$
$\phi_j \phi_i^{-1} : \left(\frac{x^1}{x^i}, \dots, \frac{x^j}{x^i}, \frac{x^{n+1}}{x^i}\right) \mapsto \left(\frac{x^1}{x^j}, \dots, \frac{x^i}{x^j}, \dots, \frac{x^{n+1}}{x^j}\right)$
is multiplication by $\left(\frac{x^i}{x^j}\right)$ i.e. clearly smooth on $U_i \cap U_j$
eg Grassmannians
$G_k(\mathbb{R}^n) =$ space of k-planes through the origin in $\mathbb{R}^n$


On $(U_j \cap U_i)$
$\phi_j \phi_i^{-1} : \left(\frac{x^1}{x^i}, \dots, \frac{x^j}{x^i}, \frac{x^{n+1}}{x^i}\right) \longmapsto \left(\frac{x^1}{x^j}, \dots, \frac{x^i}{x^j}, \dots, \frac{x^{n+1}}{x^j}\right)$
is multiplication by $\left(\frac{x^i}{x^j}\right)$ i.e. clearly smooth on $U_i \cap U_j$
eg Gras
$G_k($
space of k-planes through the origin in $\mathbb{R}^n$
(in particular, $G_1(\mathbb{R}^{n+1}) = \mathbb{R}\mathbb{P}^n$)

On $(U_j \cap U_i)$

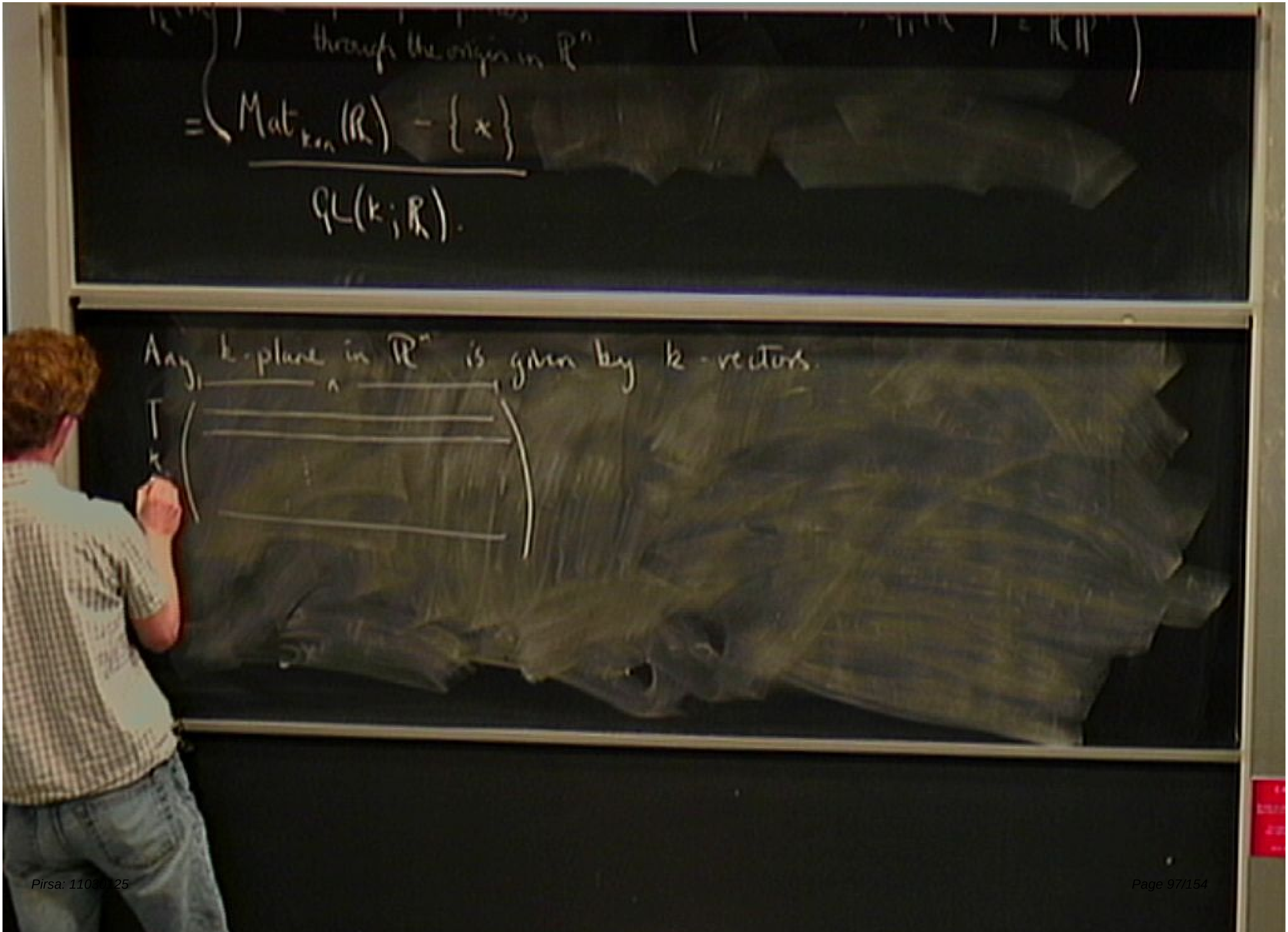
$$\phi_j \phi_i^{-1} : \left(\frac{x^1}{x^i}, \dots, \frac{x^j}{x^i}, \frac{x^{n+1}}{x^i}\right) \mapsto \left(\frac{x^1}{x^j}, \dots, \frac{x^i}{x^j}, \dots, \frac{x^{n+1}}{x^j}\right)$$

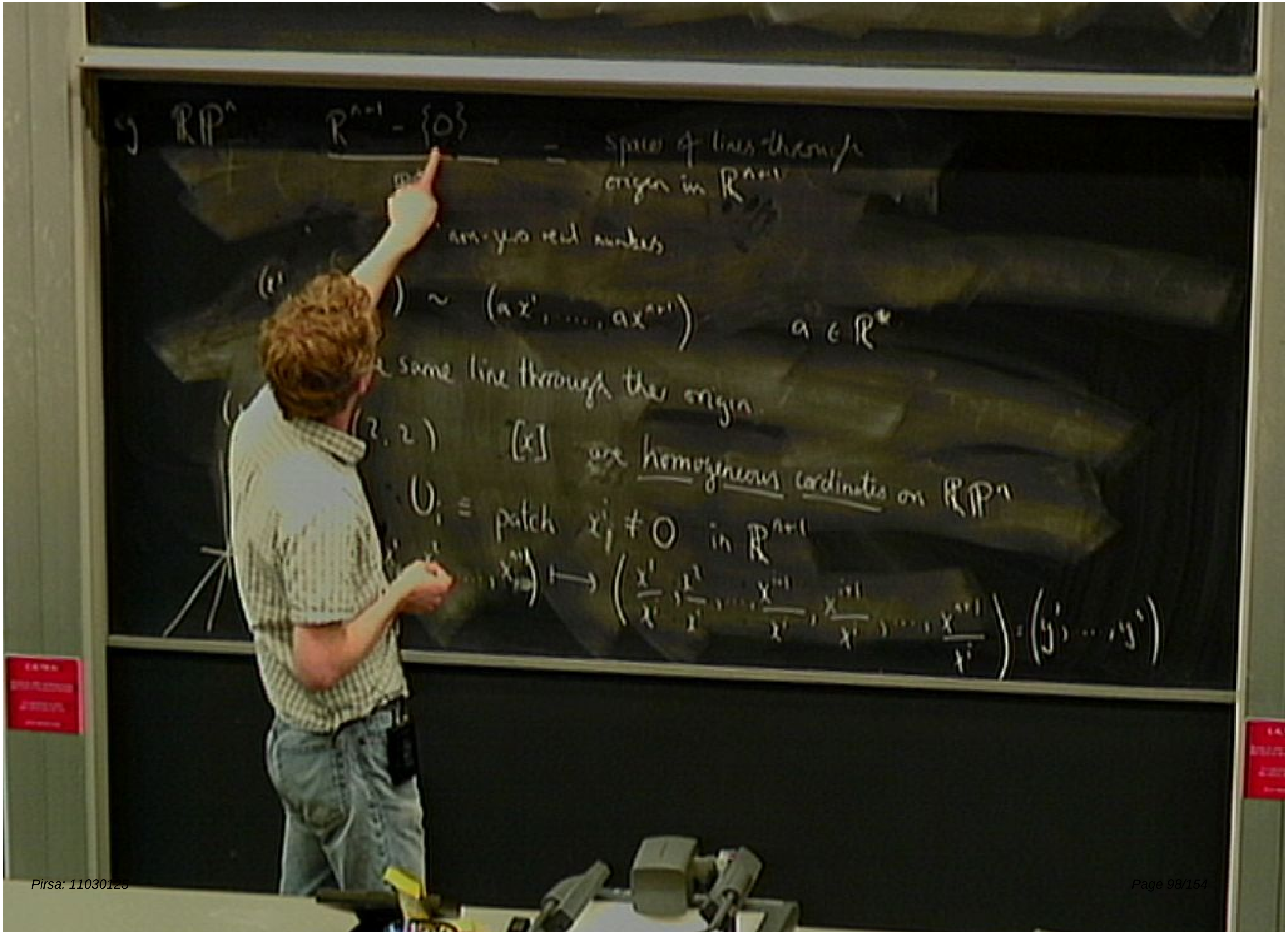
is multiplication by $\left(\frac{x^i}{x^j}\right)$ i.e. clearly smooth on $U_i \cap U_j$

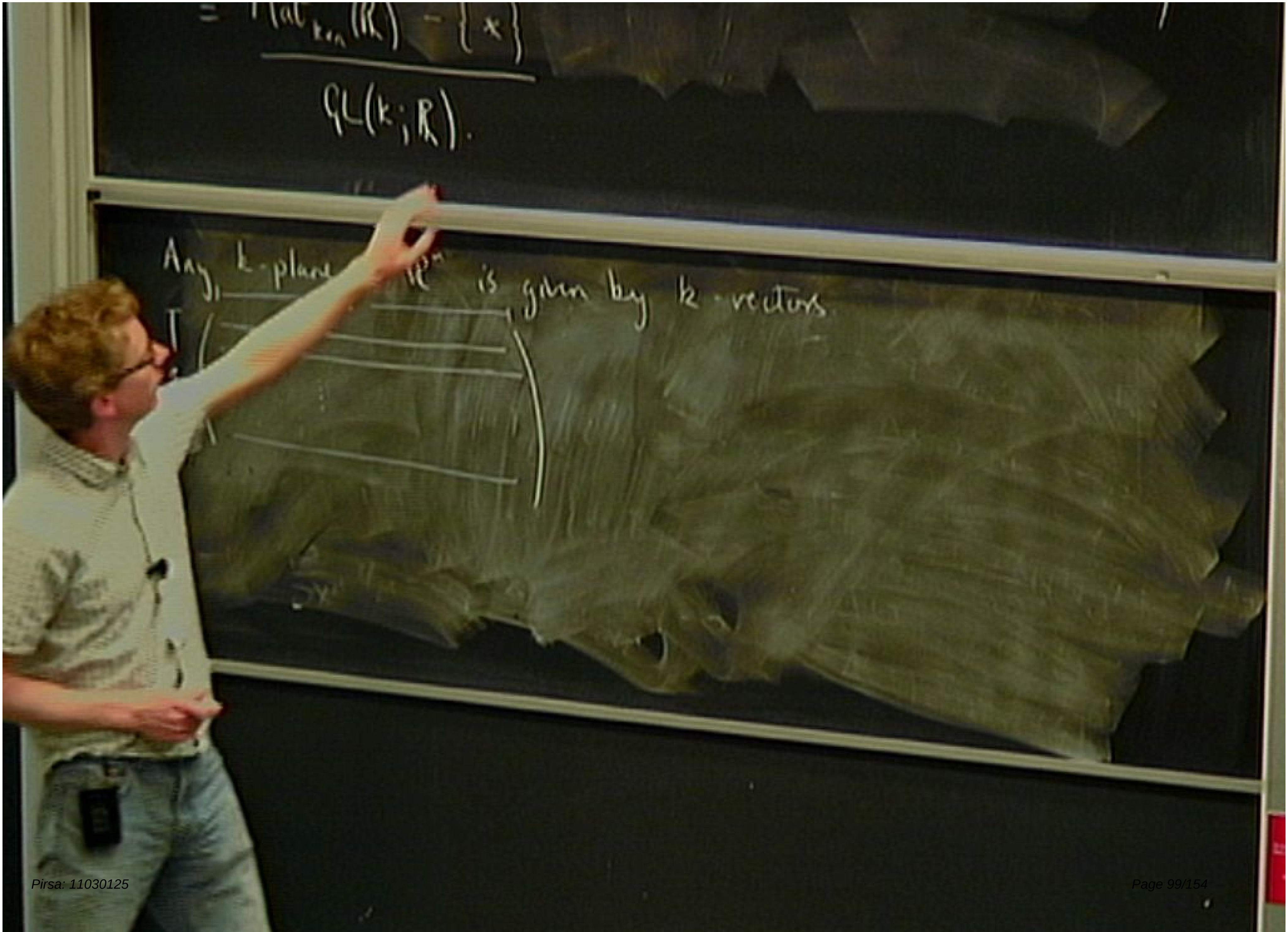
eg Grassmannians

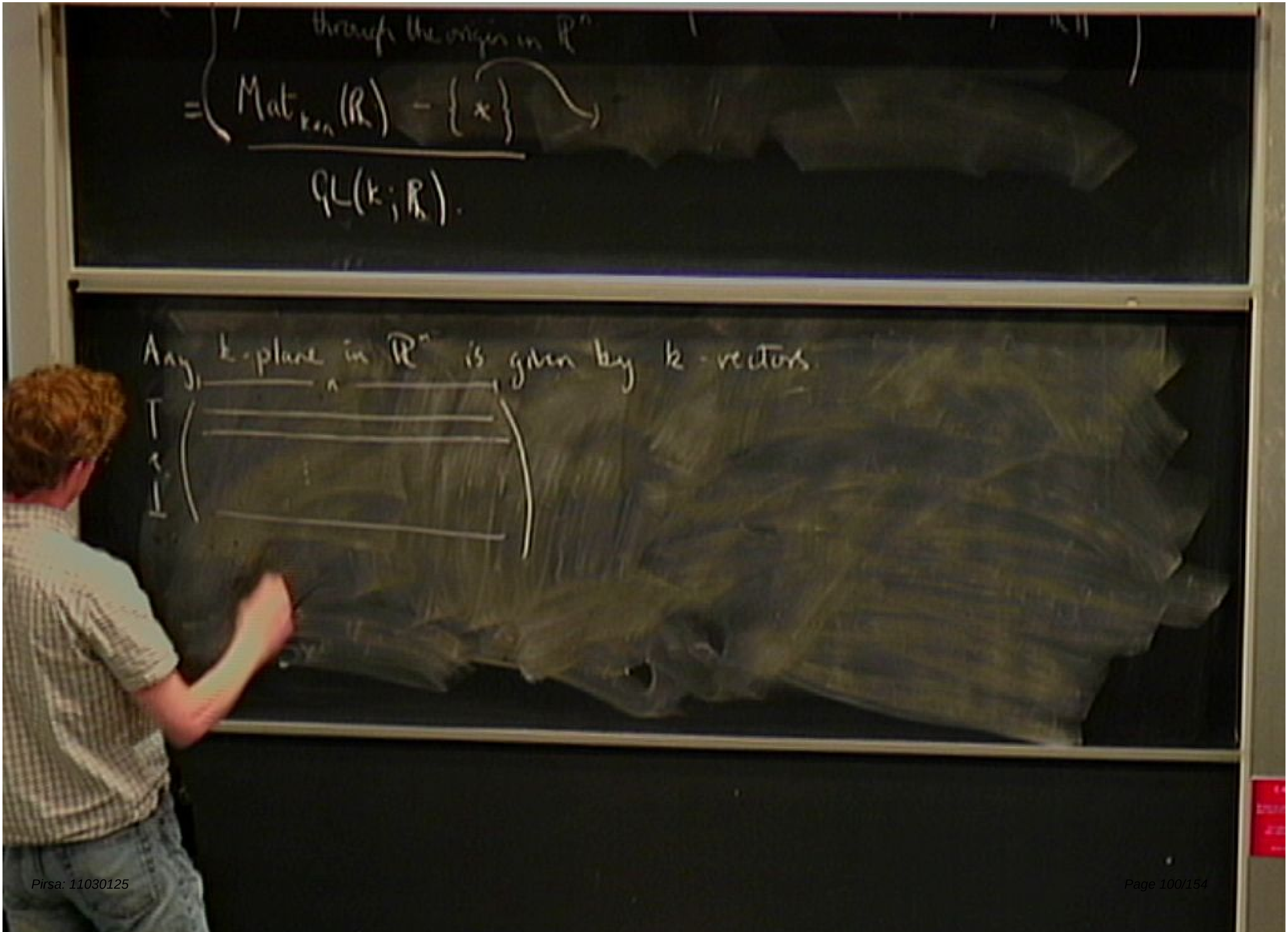
$G_k(\mathbb{R}^n) \equiv$ space of k-planes through the origin in $\mathbb{R}^n$. (in particular, $G_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n$)

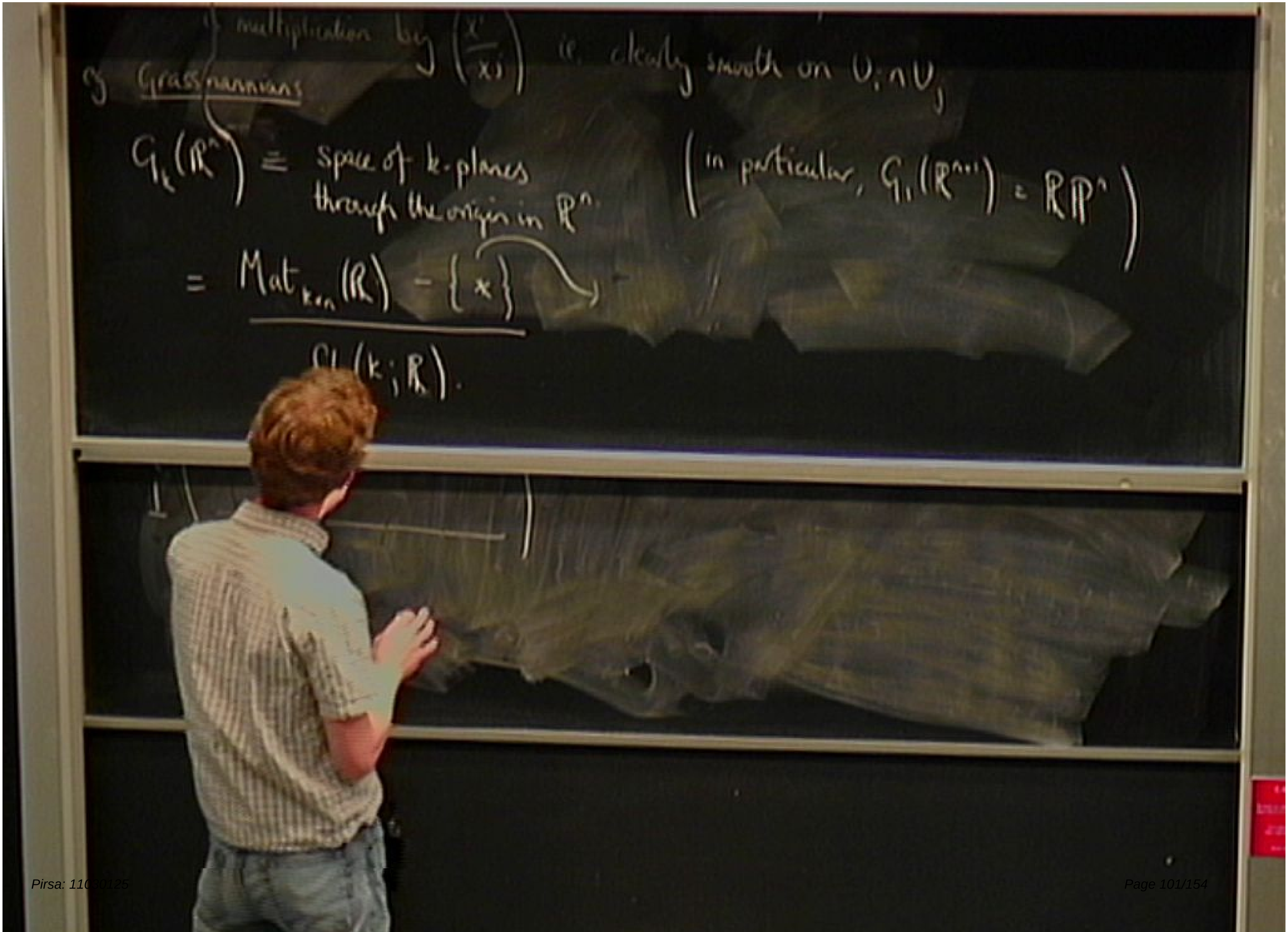
$$= \frac{\mathrm{Mat}_{k \times n}(\mathbb{R}) - \{*\}}{GL(k;\mathbb{R})}.$$


through the origin in ℝ^n
= Mat_{k×n}(ℝ) − {*} / GL(k;ℝ).
Any k-plane in ℝ^n is given by k-vectors.
k

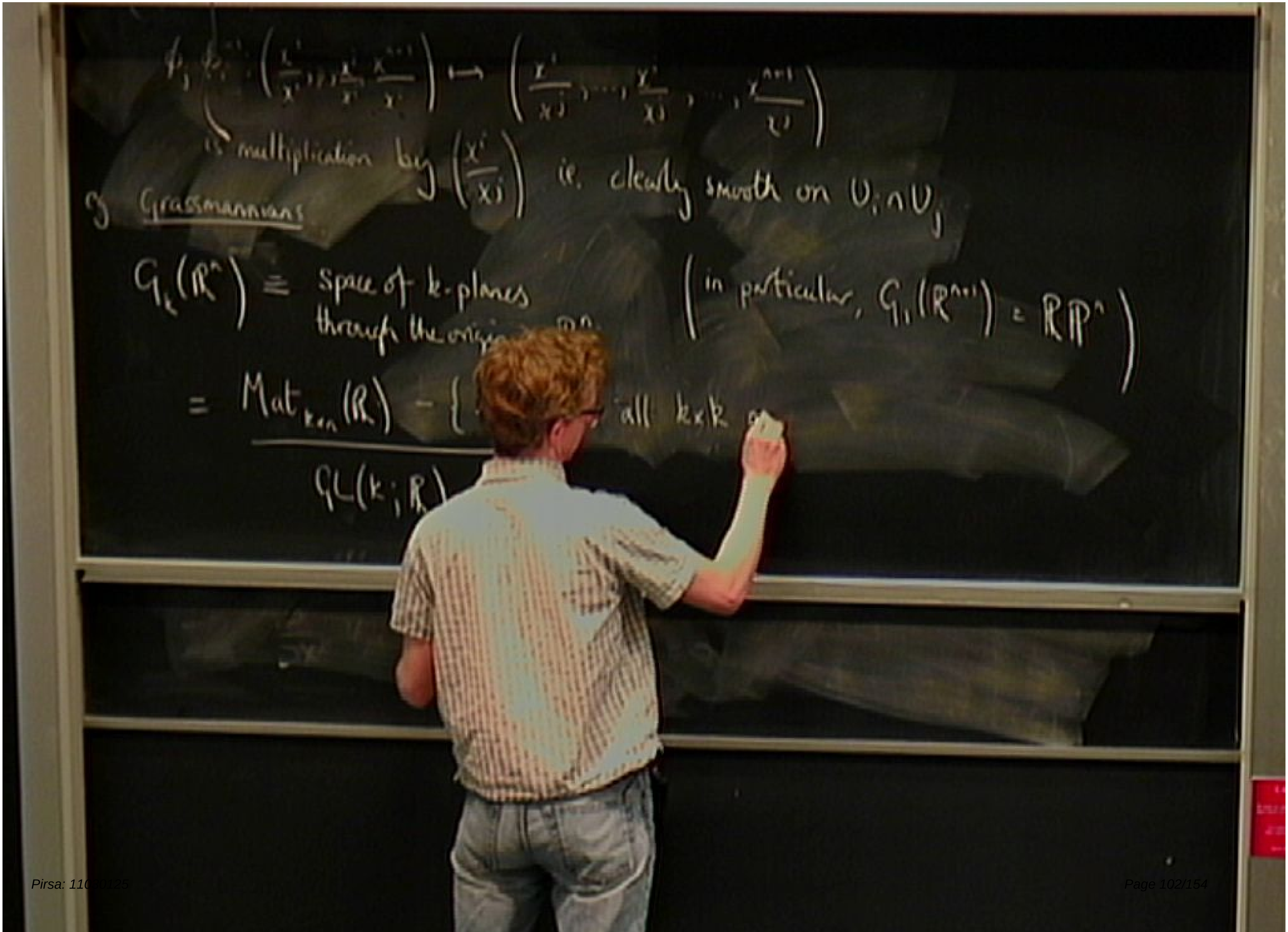

RP^n
R^{n+1} - {0}
space of lines through
origin in R^{n+1}
non-zero real numbers
~ (ax', ..., ax^{n+1})
a ∈ R*
same line through the origin
[x] are homogeneous coordinates on RP^n
U_i = patch x^i ≠ 0 in R^{n+1}

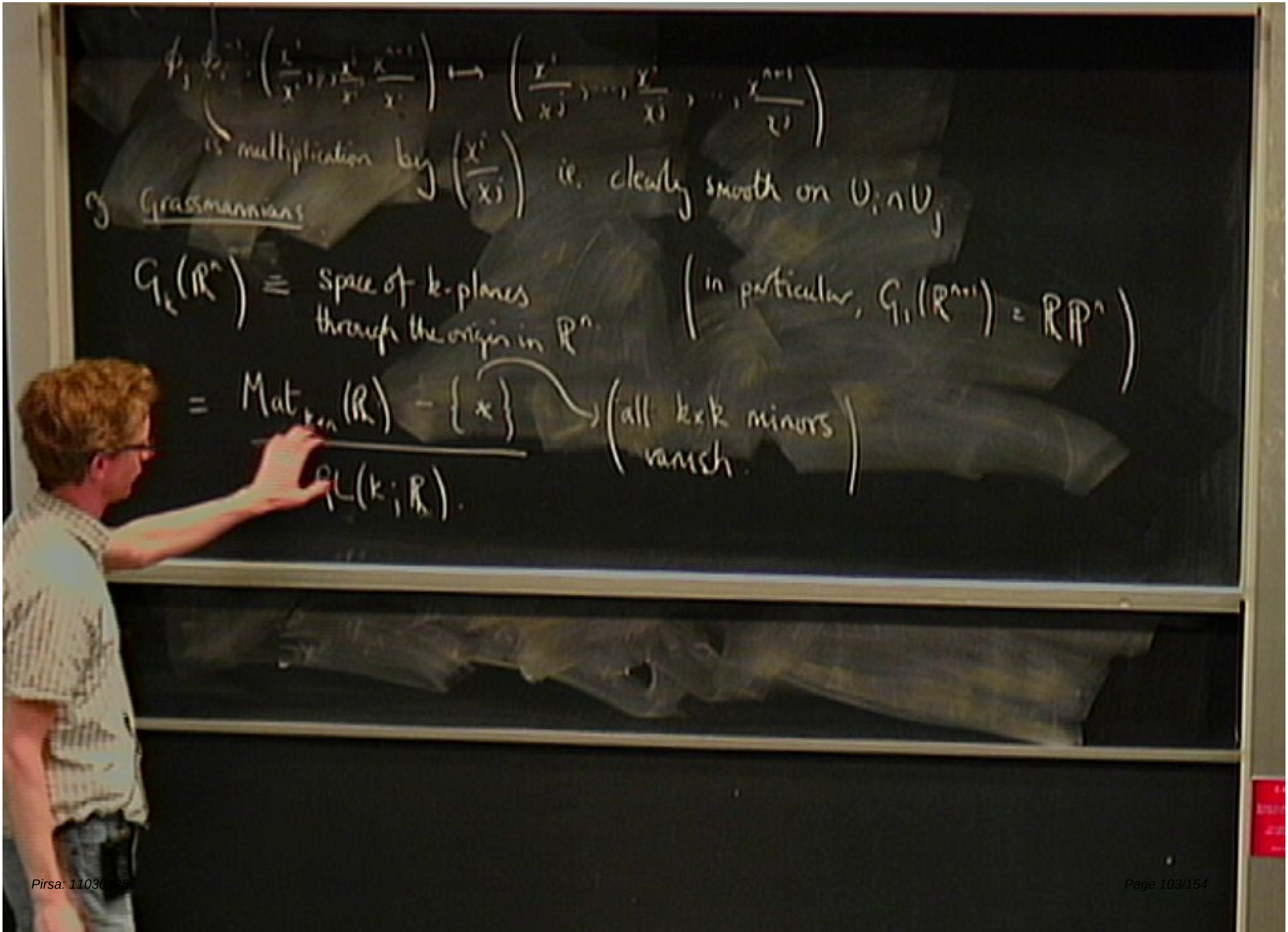

= Mat k×n (R) − {*}
GL(k; R).
Any k-plane in R^n is given by k-vectors.

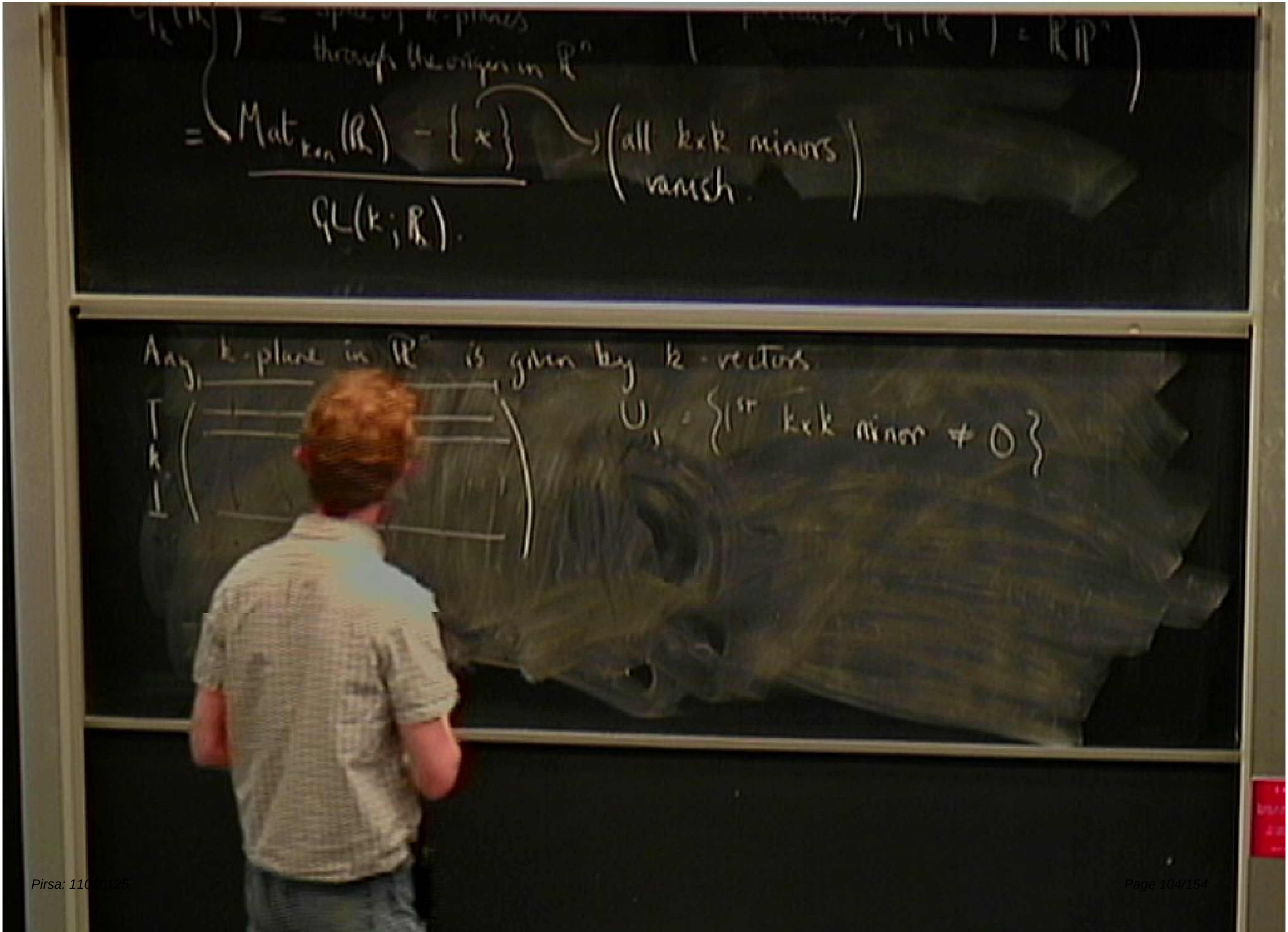

through the origin in ℝⁿ
= (Mat k×n (ℝ) − {*}) / GL(k;ℝ).
Any k-plane in ℝⁿ is given by k-vectors
n
k

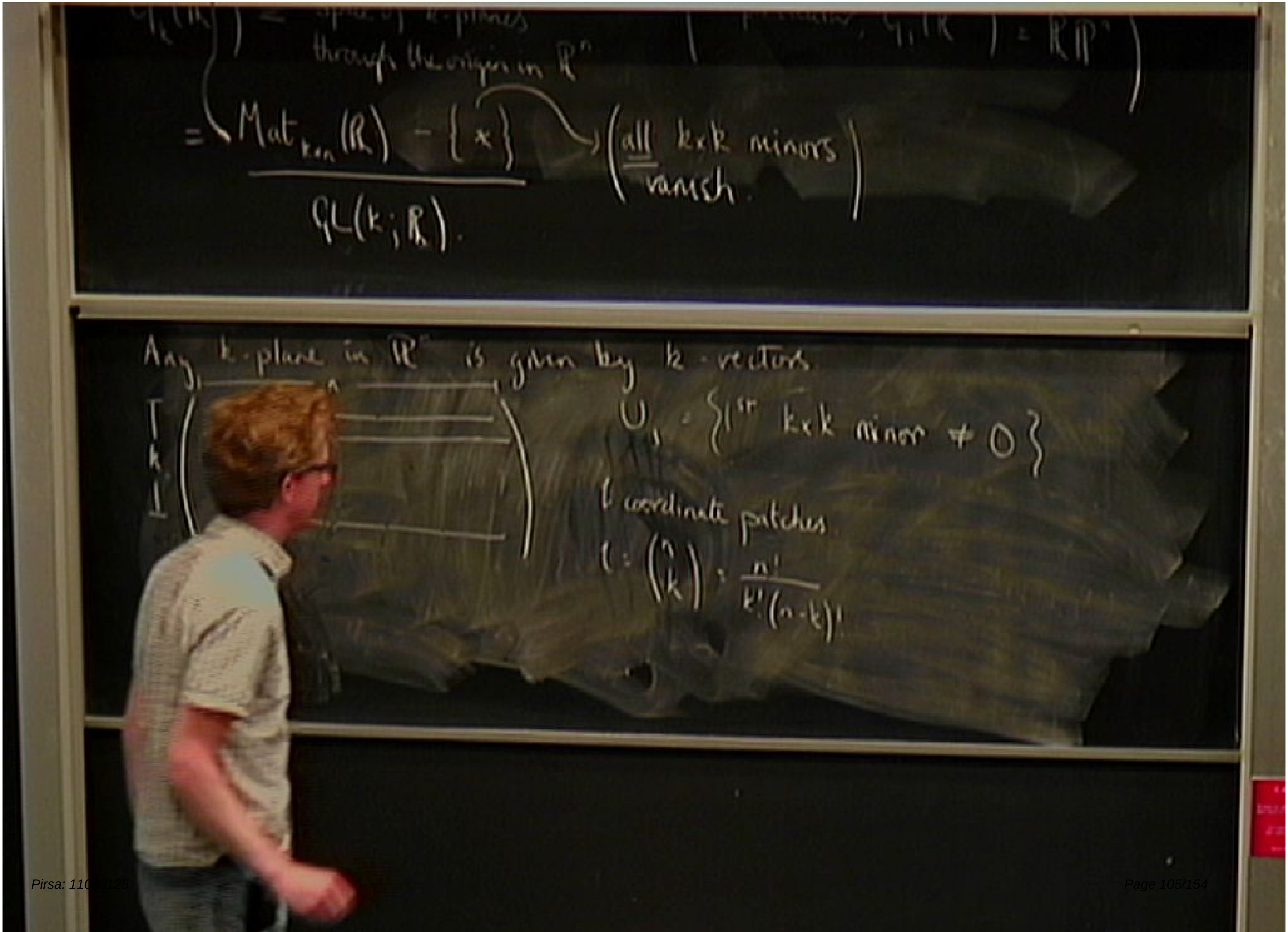


multiplication by $\left(\frac{x^i}{x^j}\right)$ is clearly smooth on $U_i \cap U_j$
Grassmannians
$G_k(\mathbb{R}^n) \equiv$ space of k-planes through the origin in $\mathbb{R}^n$.
(in particular, $G_1(\mathbb{R}^{n+1}) = \mathbb{R}P^n$)
$= \frac{Mat_{k \times n}(\mathbb{R}) - \{*\}}{GL(k;\mathbb{R})}$.


is multiplication by $\left(\frac{x^i}{x^j}\right)$ i.e. clearly smooth on $U_i \cap U_j$
eg Grassmannians
$G_k(\mathbb{R}^n)$ = space of k-planes through the origin
(in particular, $G_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$)
$= \frac{Mat_{k \times n}(\mathbb{R})}{GL(k;\mathbb{R})}$
all k×k


is multiplication by
i.e. clearly smooth on U_i ∩ U_j
Grassmannians
G_k(R^n) = space of k-planes through the origin in R^n.
(in particular, G_1(R^(n+1)) = RP^n)
= Mat_(k×n)(R) − {*} / GL(k; R).
(all k×k minors vanish)


through the origin in ℝ^n
= Mat_{k×n}(ℝ) − {×} / GL(k; ℝ).
(all k×k minors vanish)
Any k-plane in ℝ^n is given by k-vectors
U_1 = {1st k×k minor ≠ 0}


through the origin in ℝⁿ
= Mat_{k×n}(ℝ) − {*} / GL(k;ℝ).
(all k×k minors vanish)
Any k-plane in ℝⁿ is given by k-vectors.
U₁ = {1st k×k minor ≠ 0}
l coordinate patches
l = (n k) = n!/(k!(n−k)!)

$= \mathrm{Mat}_{k\times n}(\mathbb{R}) - \{*\}$ ↪ (all $k\times k$ minors vanish)

$GL(k;\mathbb{R})$.

k-plane in $\mathbb{R}^n$ is given by k-vectors

$U_1 = \{ 1^{st}\ k\times k \text{ minor} \neq 0 \}$

l coordinate patches

$$l = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \mathrm{Mat}_{k\times n}(\mathbb{R}) - \{*\} \Big/ GL(k;\mathbb{R}).$$

$\hookrightarrow$ (all $k\times k$ minors vanish.)

k-plane in $\mathbb{R}^n$ is given by k-vectors.

$$U_1 = \{1^{st}\ k\times k \text{ minor} \neq 0\}$$

ℓ coordinate patches.

$$\ell = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$G_k(\mathbb{R}^n)$ = space of k-planes through the origin in $\mathbb{R}^n$ (in particular, $G_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$)

$$= \frac{\mathrm{Mat}_{k\times n}(\mathbb{R}) - \{*\}}{GL(k;\mathbb{R})}.$$

$\{*\}$ = (all k×k minors vanish)

Any k-plane in $\mathbb{R}^n$ is given by k-vectors

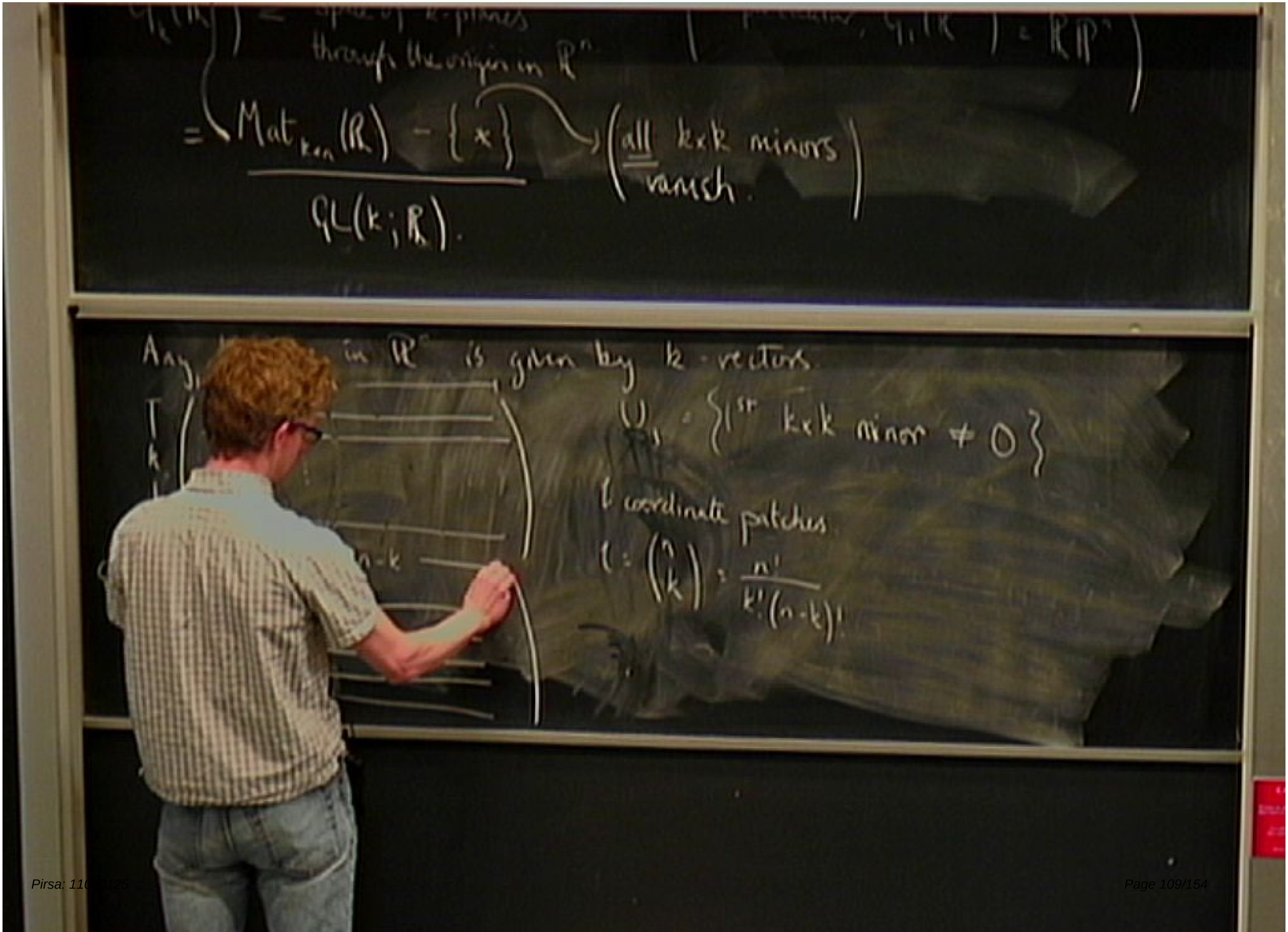
$$k\left\{\begin{pmatrix} \text{———} & \overset{n}{\text{———}} \\ \text{———} & \text{———} \\ \vdots & \\ \text{———} & \text{———} \end{pmatrix}\right.$$

$U_1 = \{1^{st}\ k\times k \text{ minor} \neq 0\}$

ℓ coordinate patches

$$\ell = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$GL(k)\backslash U_1 \cong \begin{pmatrix} 1 & 0 & \cdots & 0 & \text{———} \\ 0 & 1 & & \vdots & \text{———} \\ \vdots & & \ddots & 0 & \text{———} \\ 0 & \cdots & 0 & 1 & \text{———} \end{pmatrix}$$


through the origin in R^n
= Mat_{k×n}(R) − {*} / GL(k; R).
(all k×k minors vanish)
Any in R^n is given by k-vectors.
U_1 = {1st k×k minor ≠ 0}
ℓ coordinate patches
ℓ = (n k) = n!/(k!(n-k)!)

$G_k(\mathbb{R}^n)$ = space of k-planes through the origin in $\mathbb{R}^n$ (in particular, $G_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}$)

$$= \frac{Mat_{k\times n}(\mathbb{R}) - \{*\}}{GL(k;\mathbb{R})}.$$

$\{*\}$ = (all k×k minors vanish)

Any k-plane in $\mathbb{R}^n$ is given by k vectors.

$$k\left\{\begin{pmatrix} \text{———} & n & \text{———} \\ \text{———} & & \text{———} \\ \vdots & & \vdots \\ \text{———} & & \text{———} \end{pmatrix}\right.$$

$U_1 = \{1^{st}\ k \times k \text{ minor} \neq 0\}$

$$GL(k) \curvearrowright U_1 \quad \begin{pmatrix} 1 & 0 & \cdots & 0 & \text{———} \\ 0 & 1 & & \vdots & \text{———} \\ \vdots & & \ddots & 0 & \text{———} \\ 0 & \cdots & & 0\ 1 & \text{———} \end{pmatrix} \quad \begin{matrix} \longleftarrow n-k \longrightarrow \\ k \end{matrix}$$

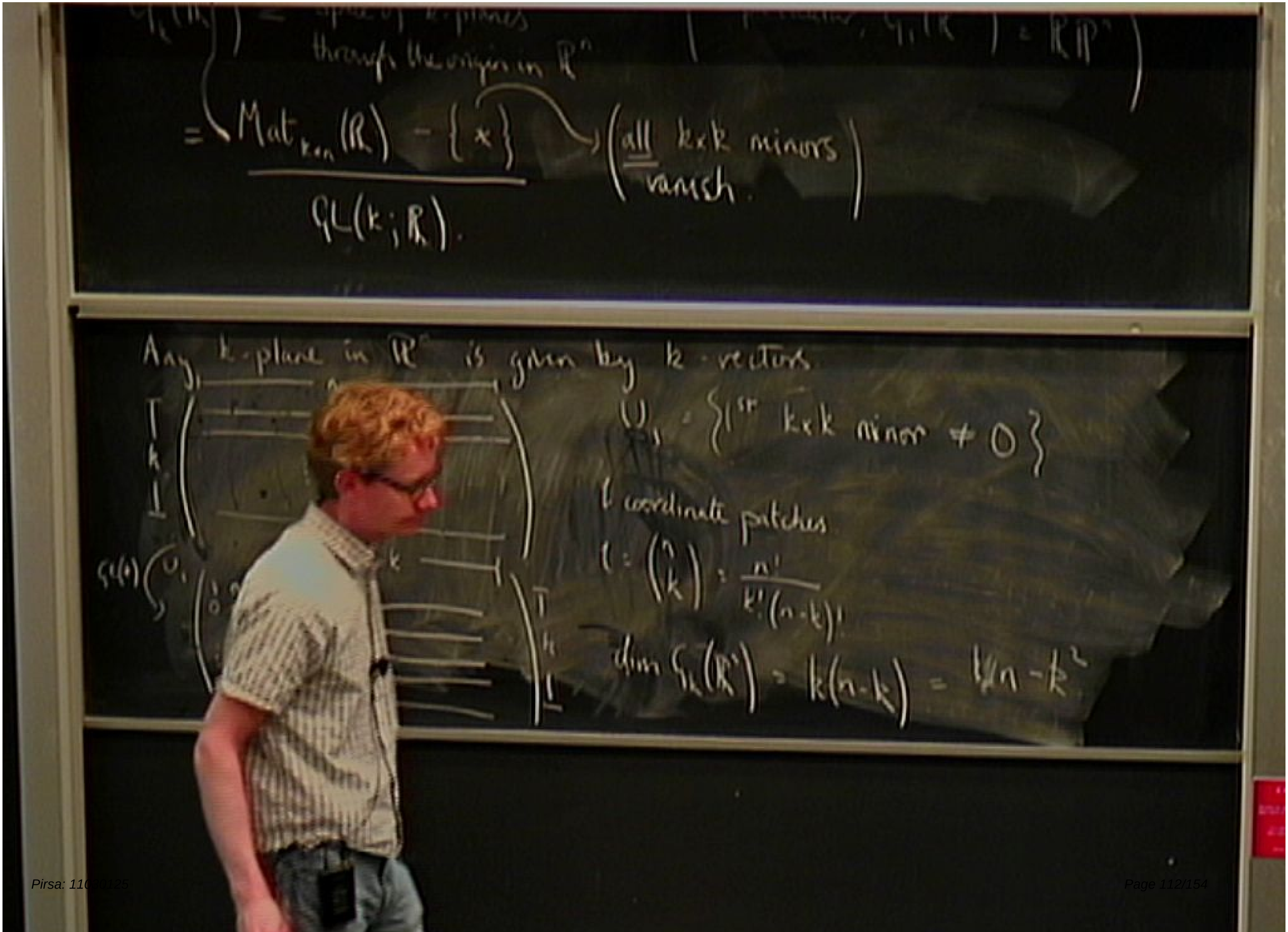
ℓ coordinate patches

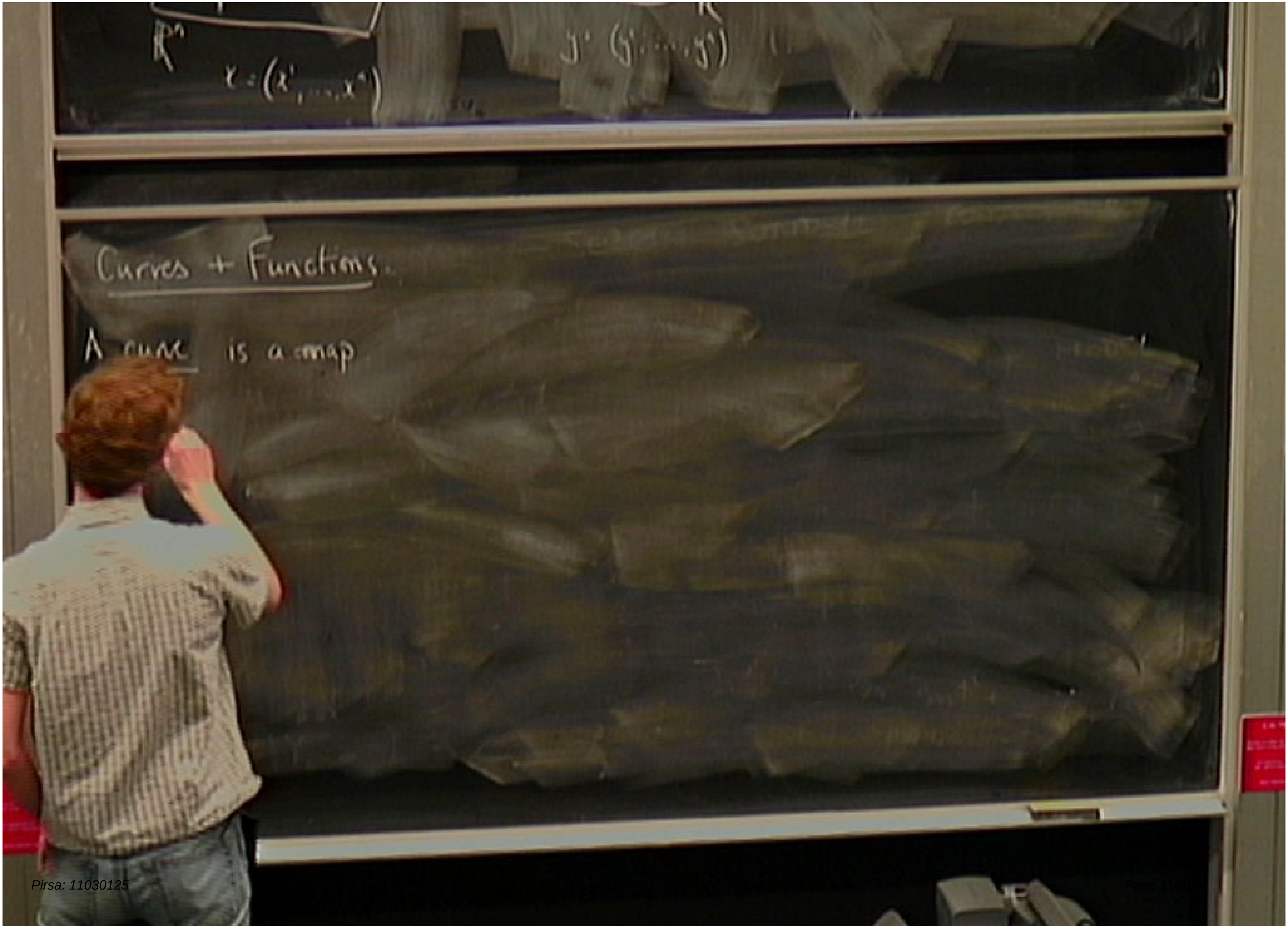
$$\ell = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

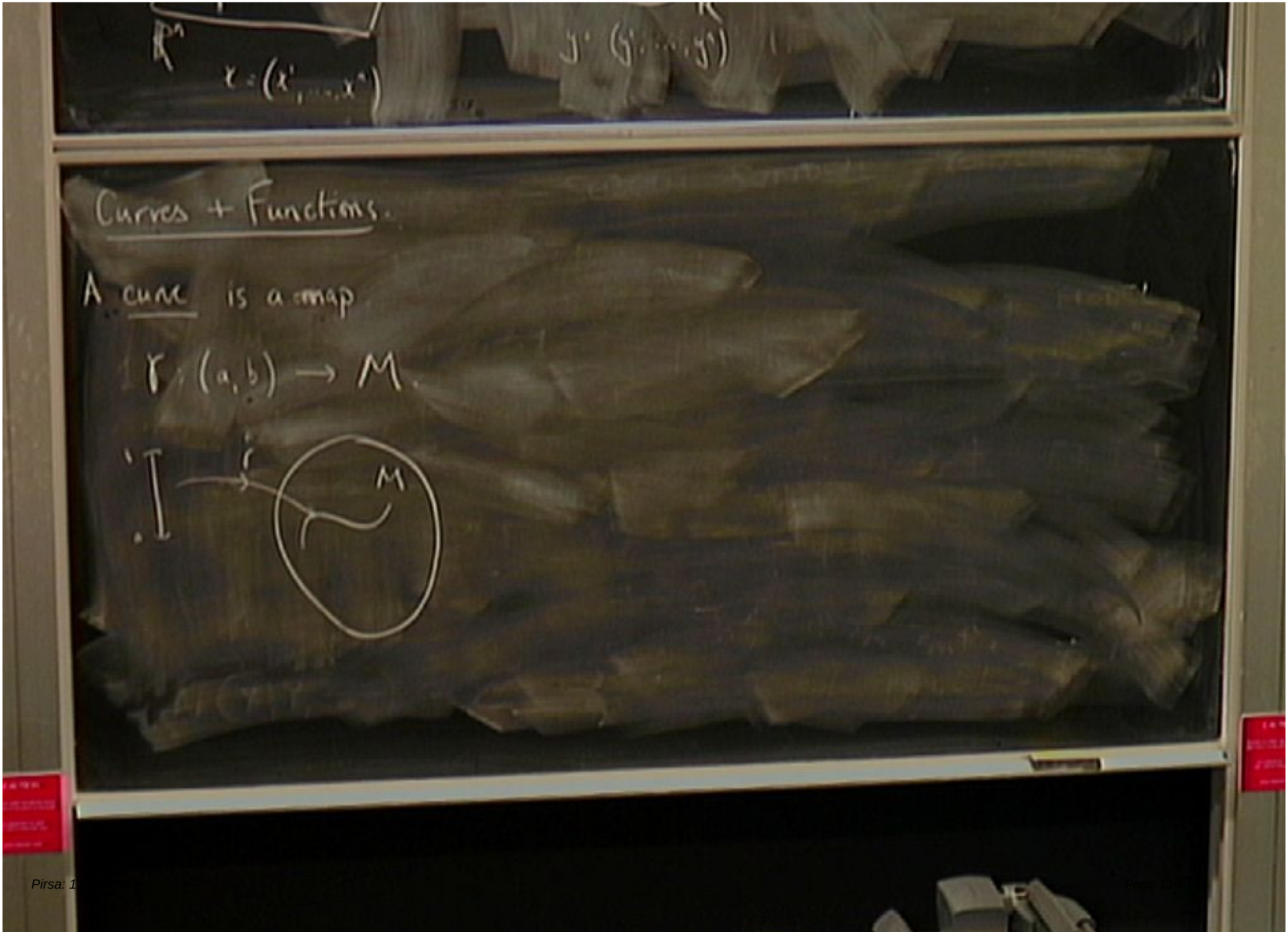
$$\dim G_k(\mathbb{R}^n) = k(n-k)$$

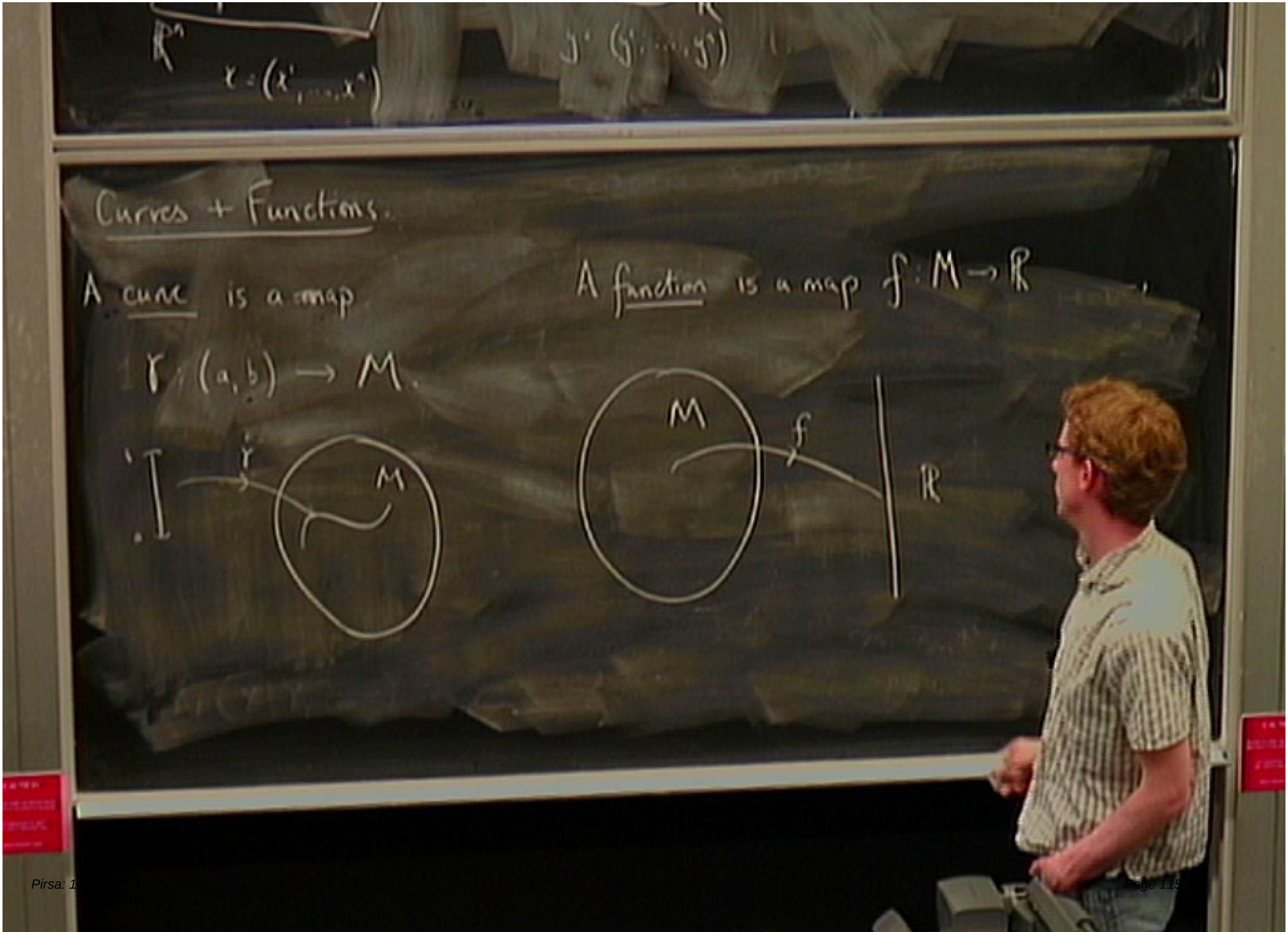
$\mathbb{R}^n$ $\quad x = (x^1, \dots, x^n)$ $\quad y = (y^1, \dots, y^n)$

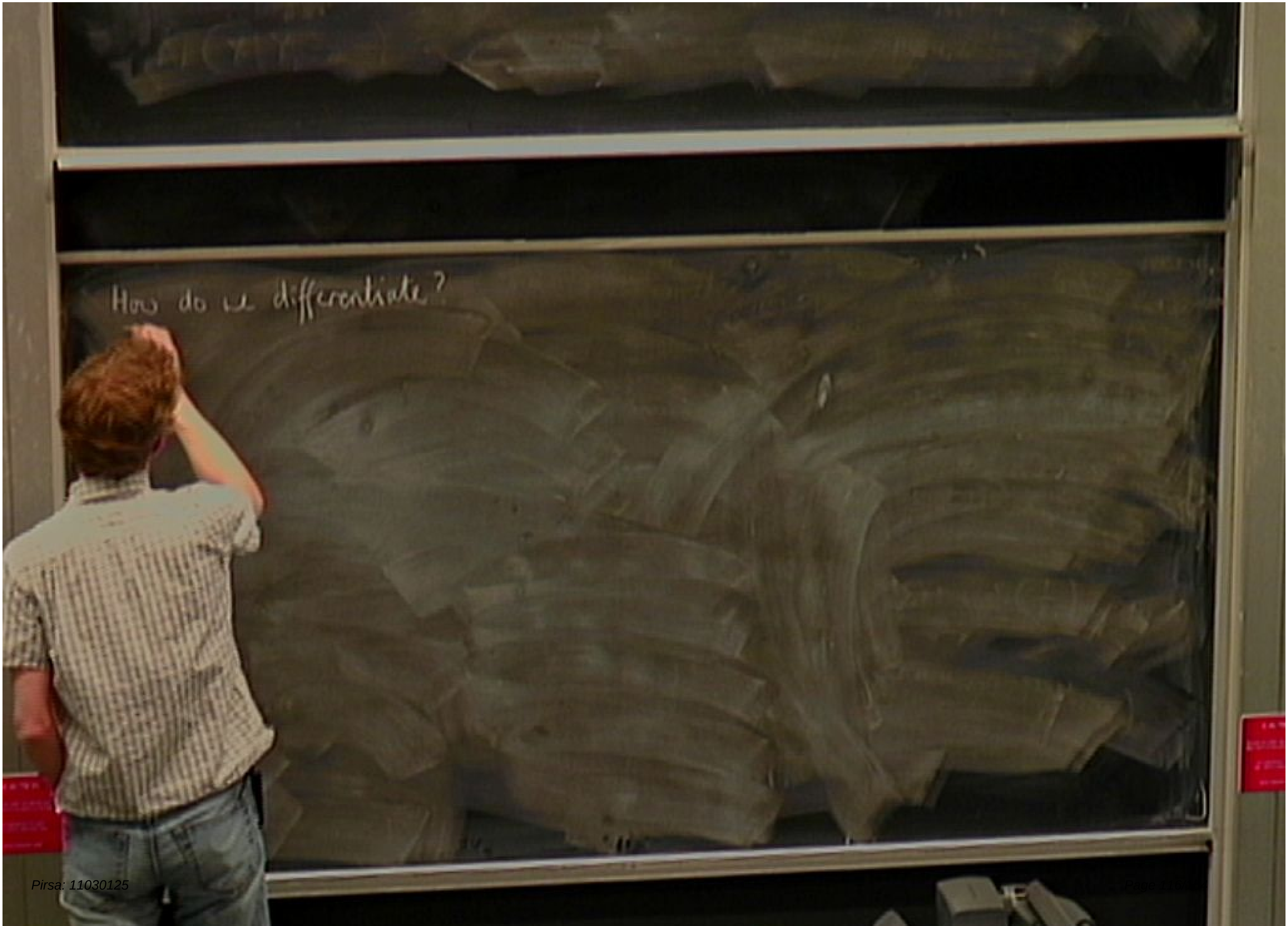
Curves + Functions.

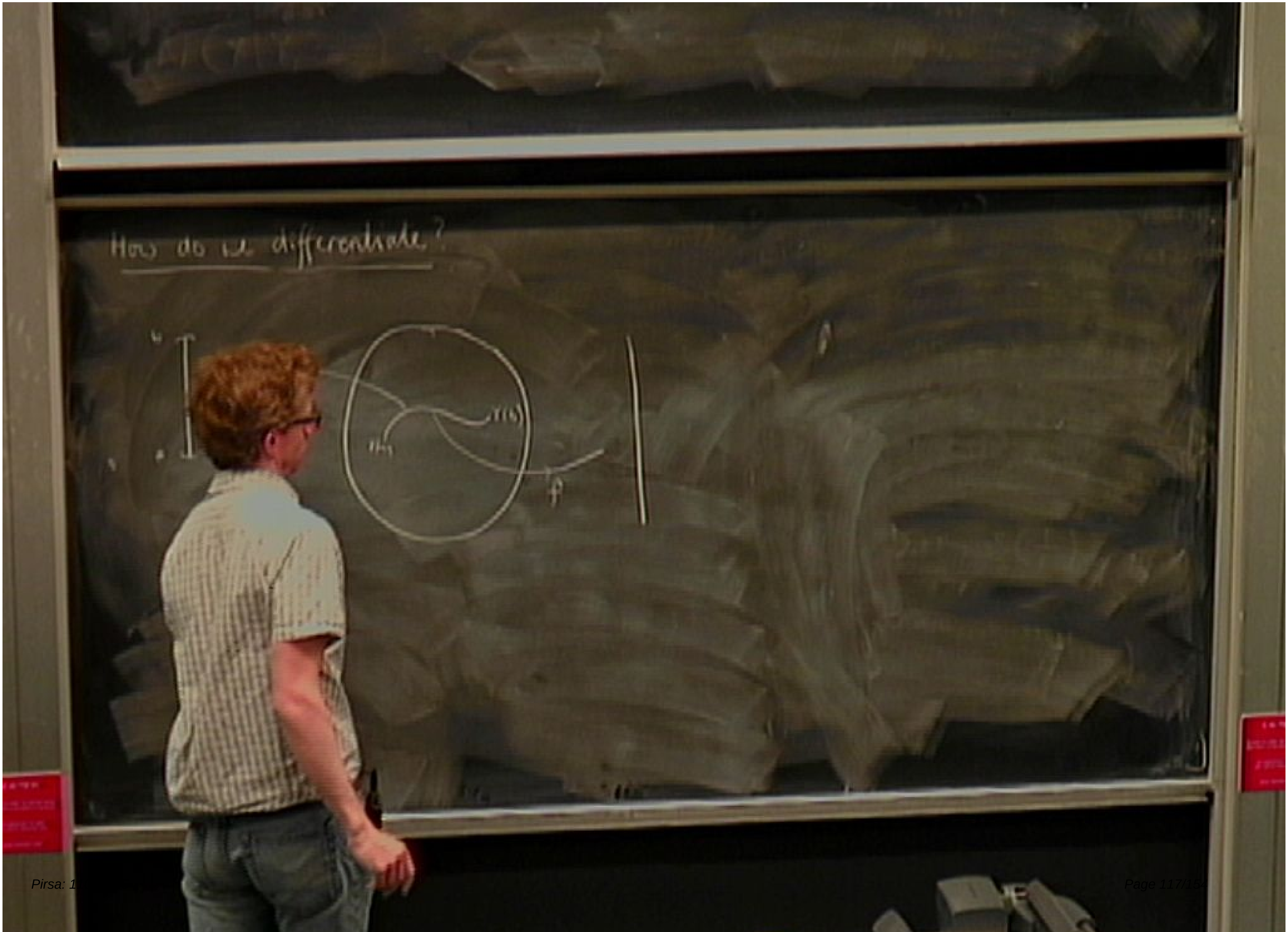

through the origin in R^n
Mat_{k×n}(R) − {*}
GL(k; R)
all k×k minors vanish
Any k-plane in R^n is given by k vectors
U_1 = {1st k×k minor ≠ 0}
coordinate patches
dim G_k(R^n) = k(n−k) = kn − k^2

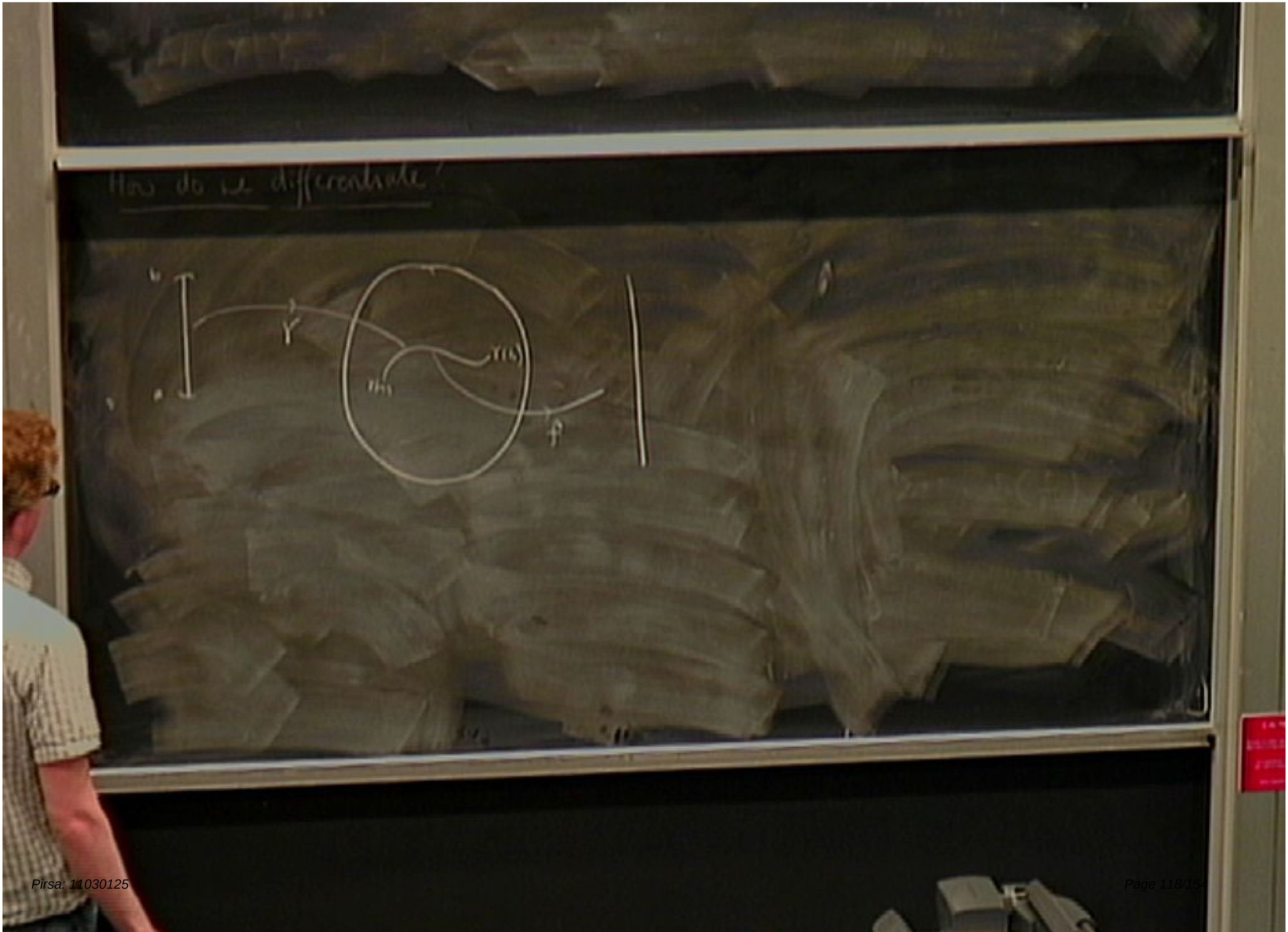

Curves + Functions.
A curve is a map

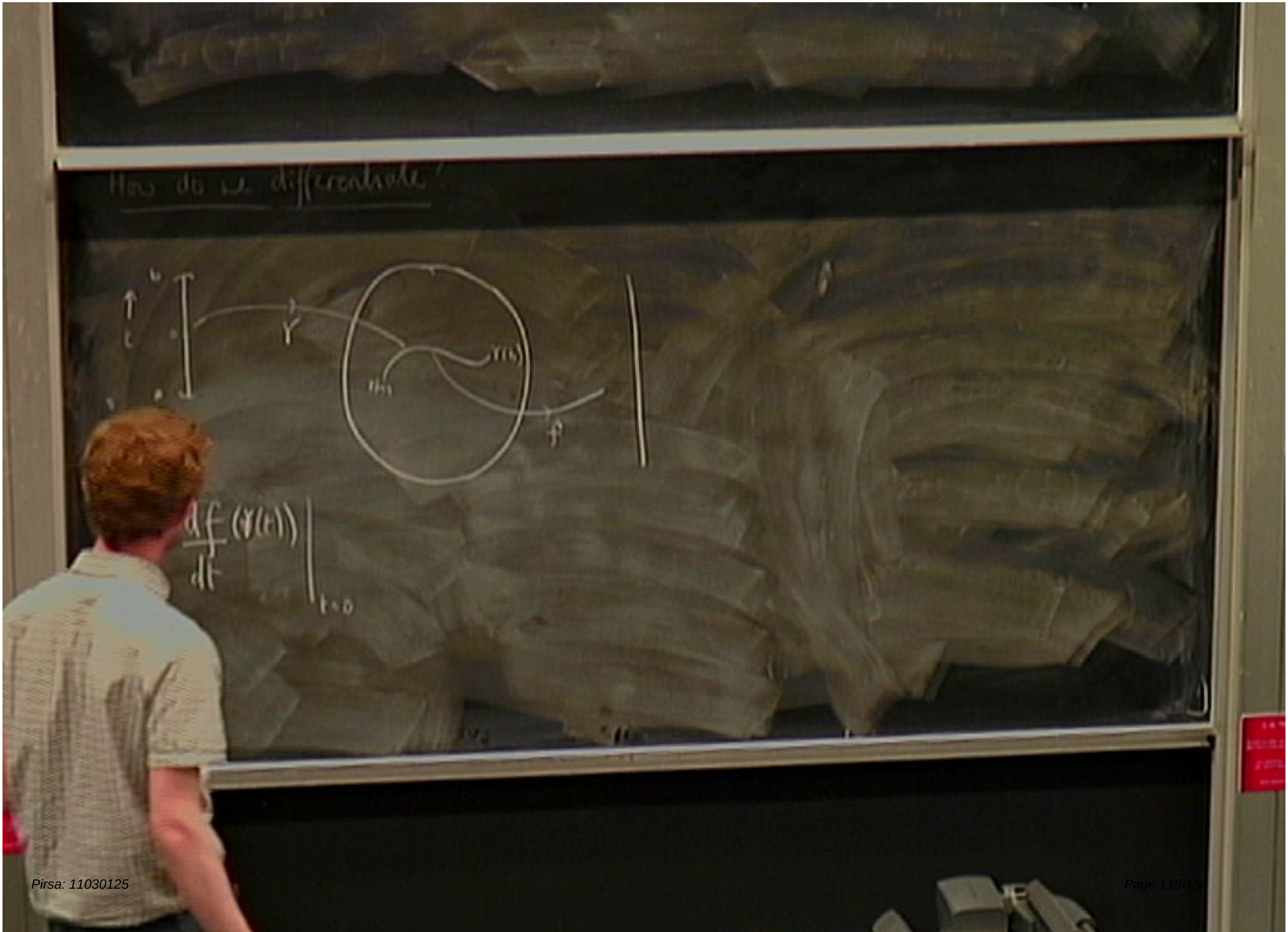

Curves + Functions.
A curve is a map
γ : (a, b) → M.
I
γ
M


Curves + Functions.
A curve is a map
γ : (a, b) → M.
A function is a map f : M → ℝ
M
f
ℝ
γ
M
I

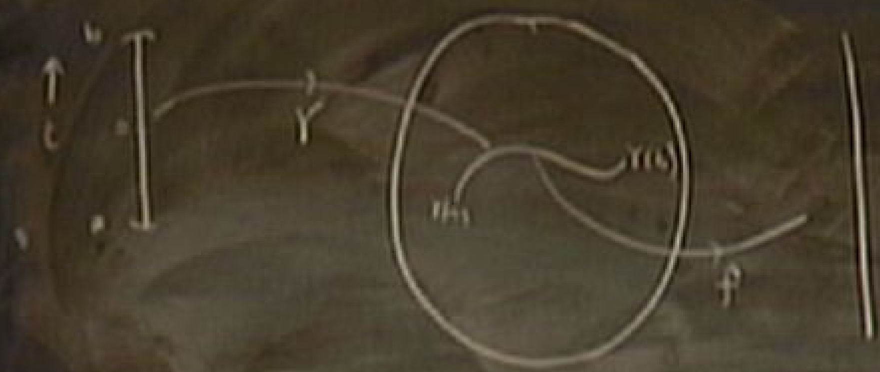

How do we differentiate?


How do we differentiate?

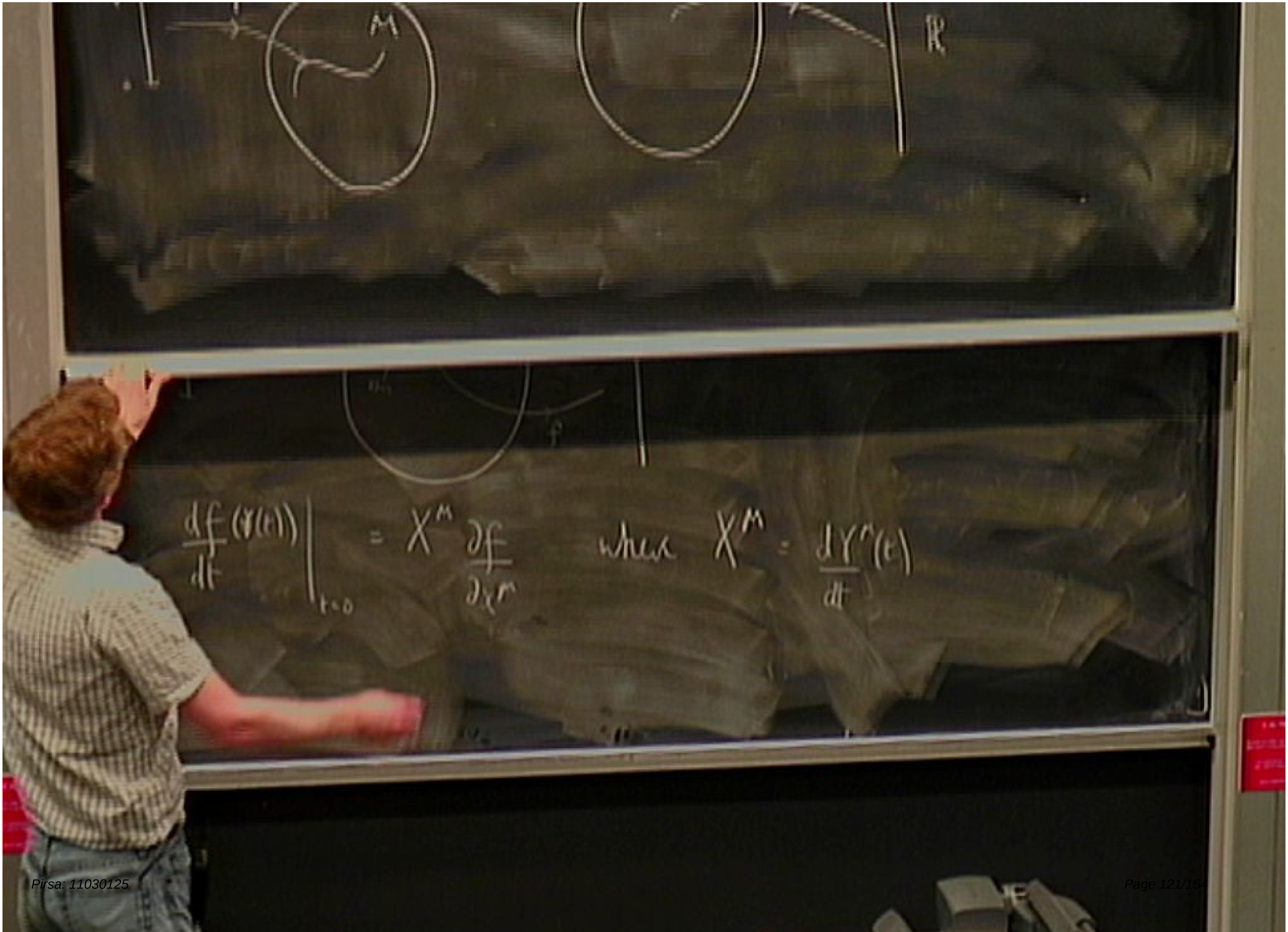

How do we differentiate?

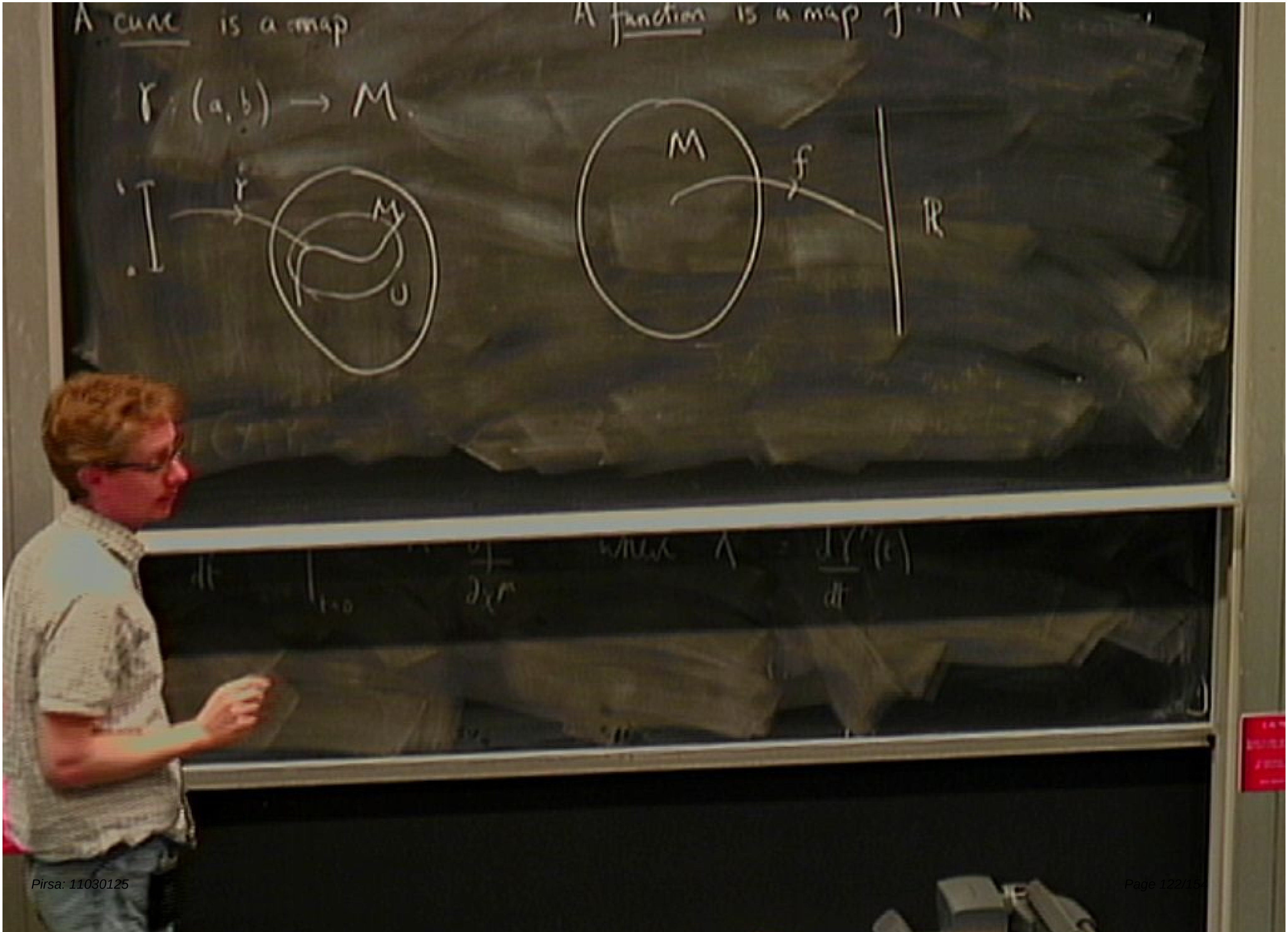

How do we differentiate?

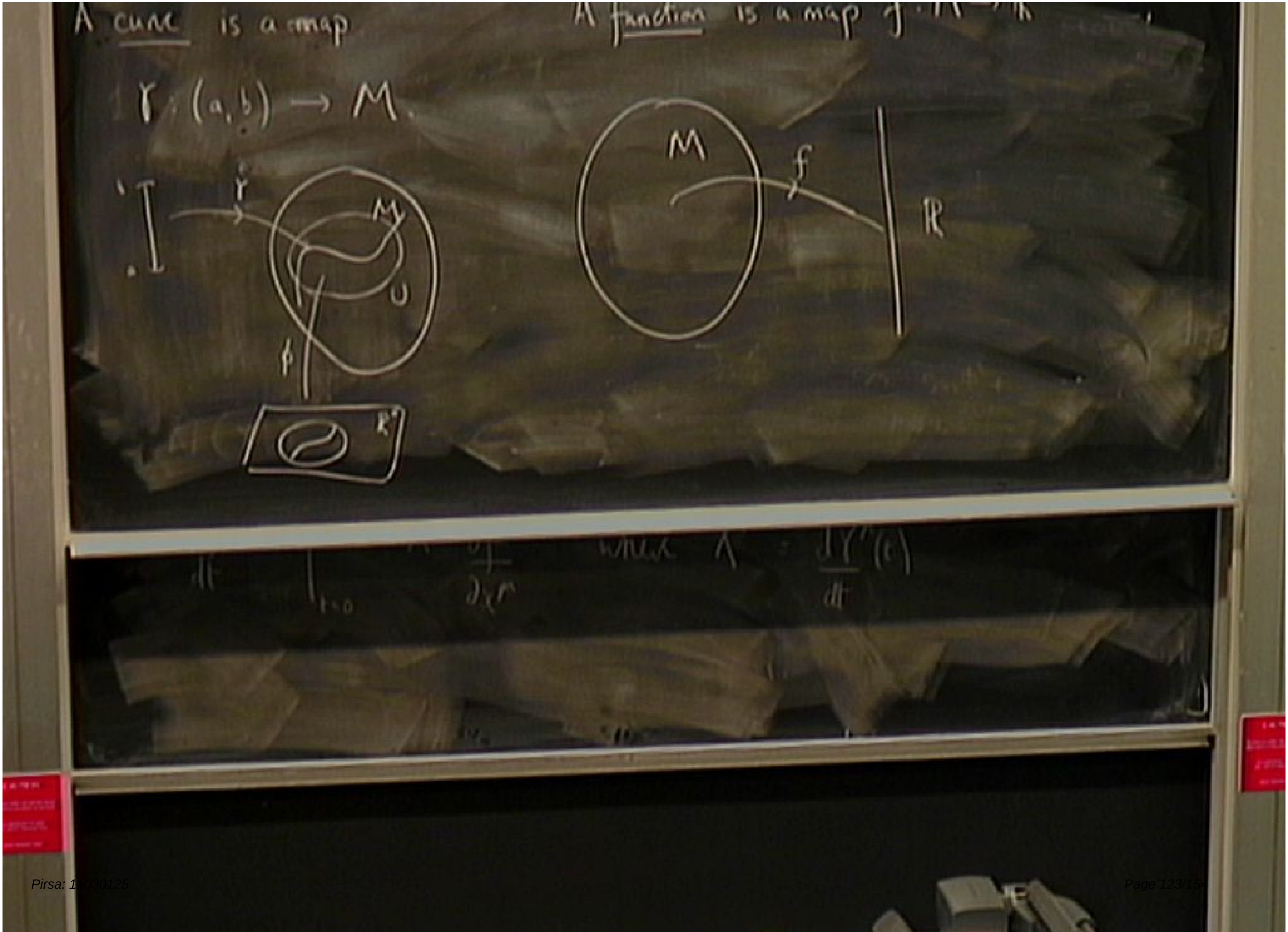
How do we differentiate?

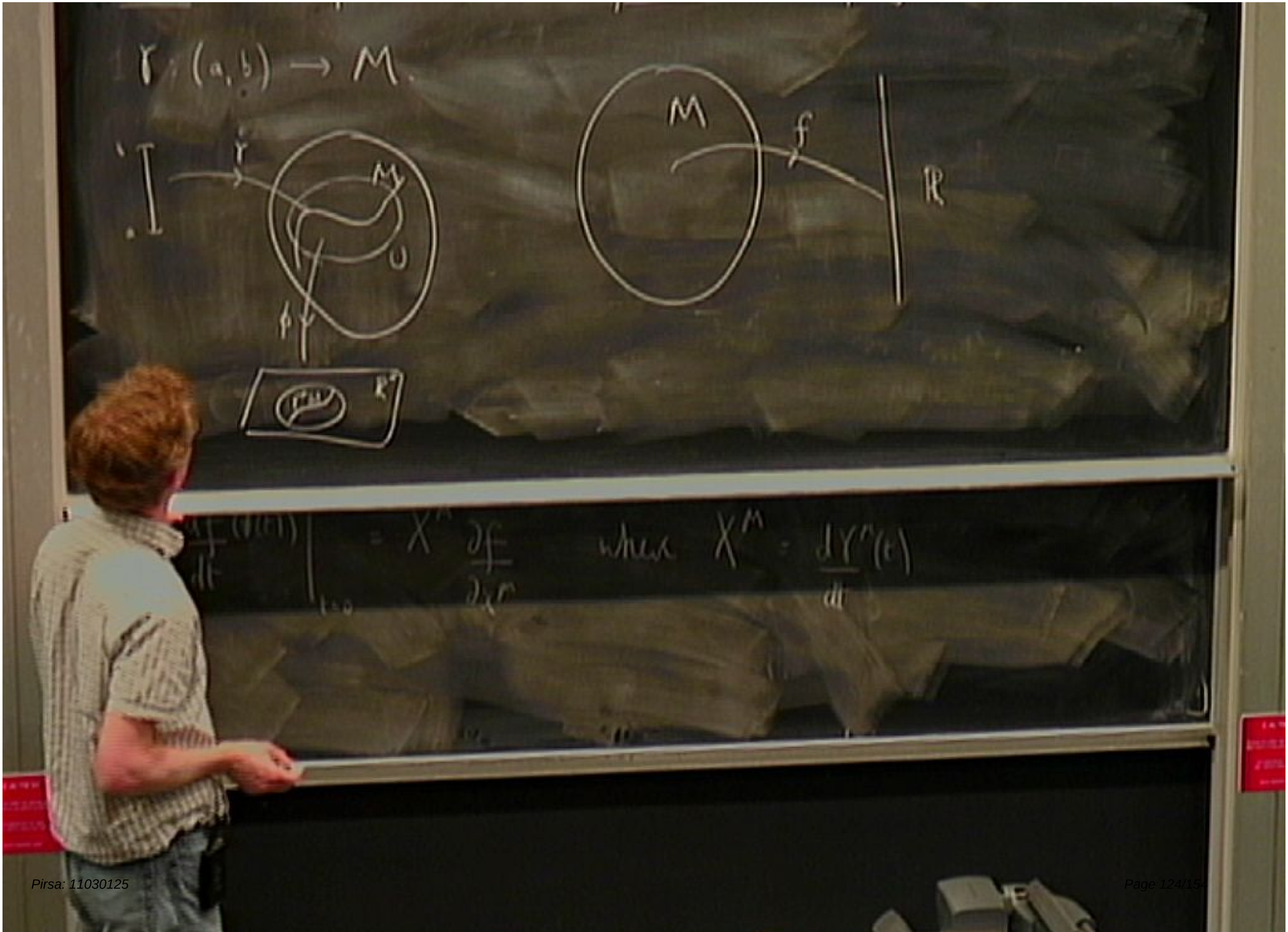


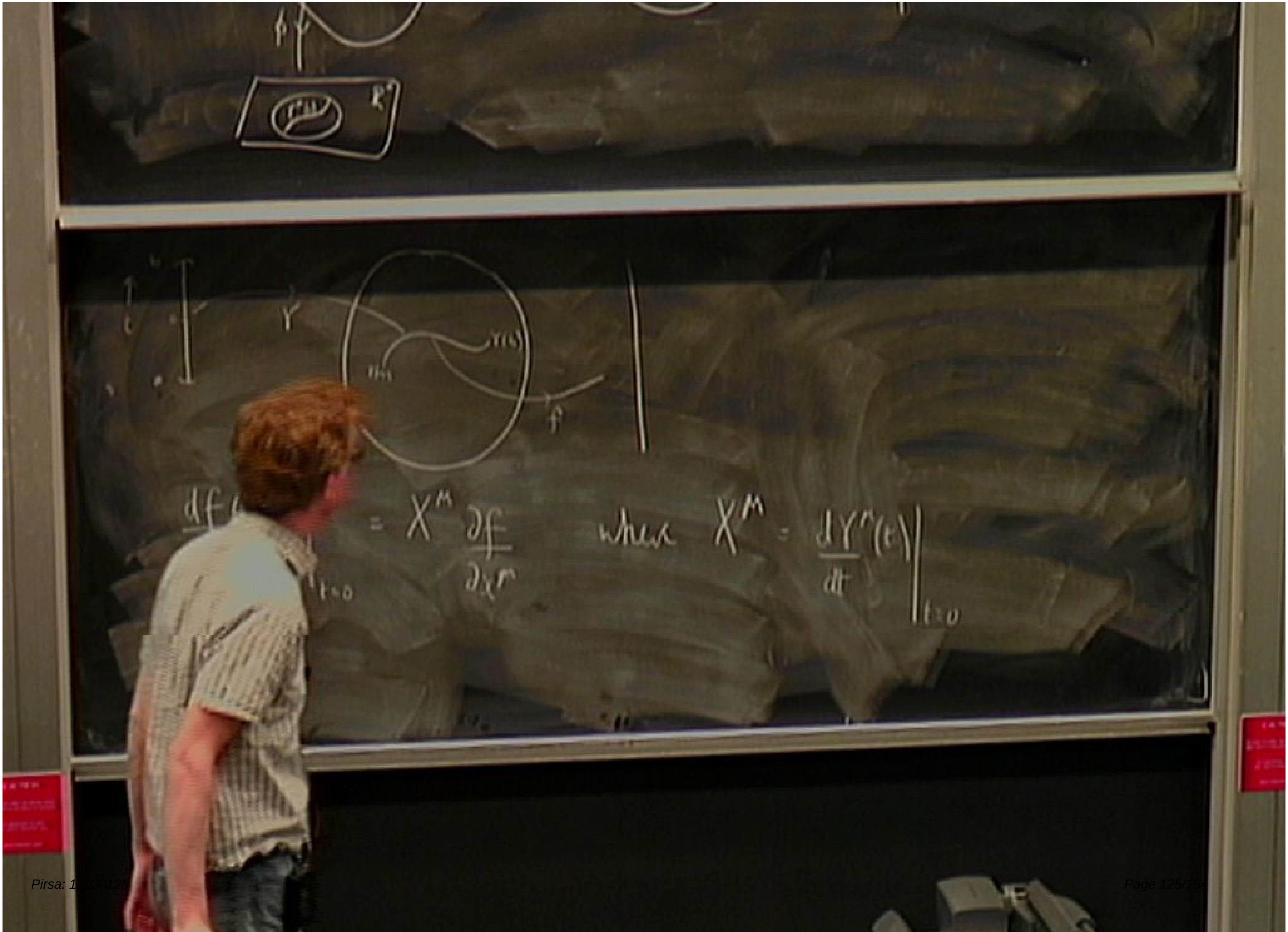
$$\frac{d f(\gamma(t))}{dt}\bigg|_{t=0} = X^\mu \frac{\partial f}{\partial x^\mu}$$

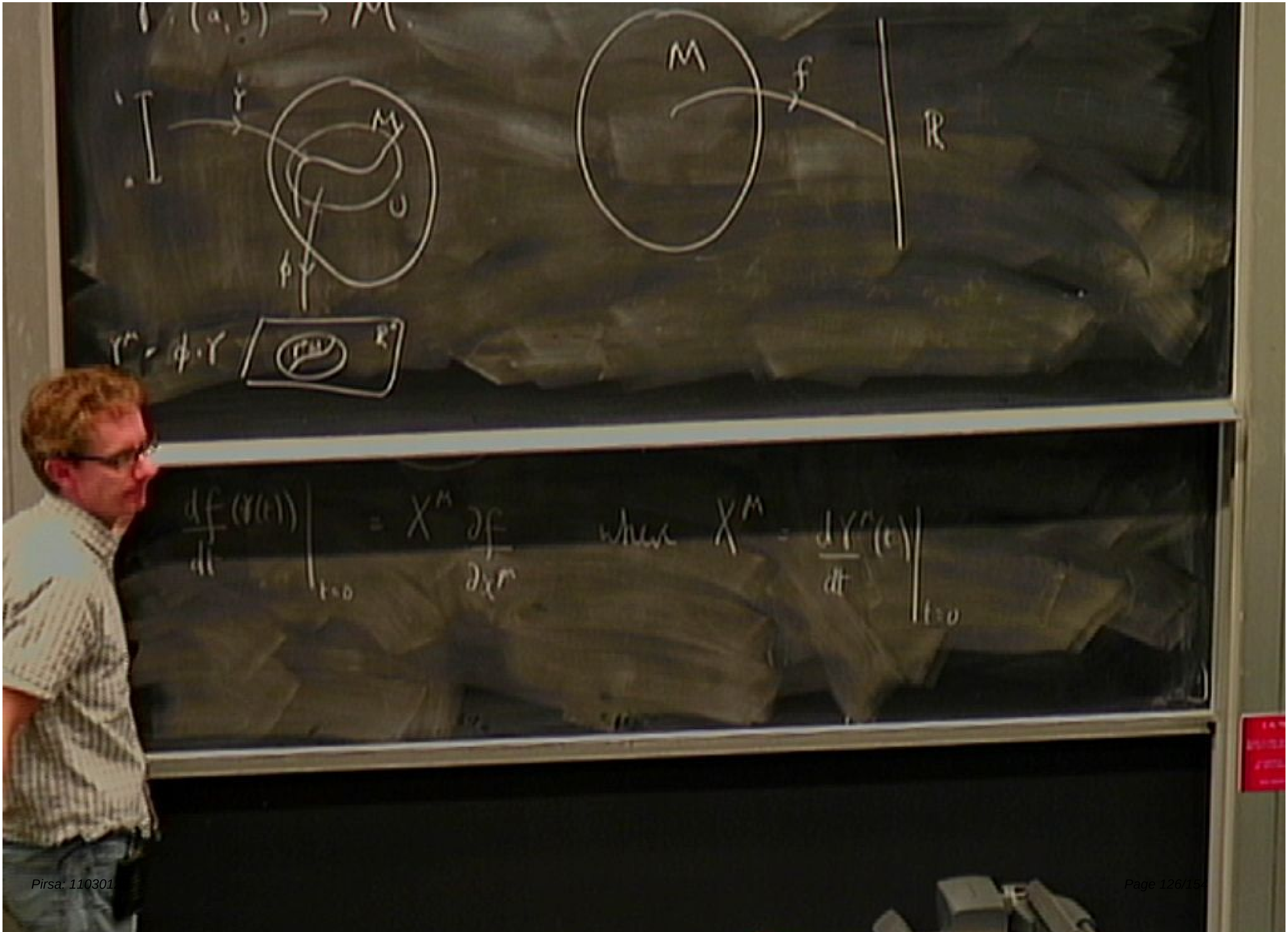

M
ℝ
df/dt (γ(t)) |t=0 = X^μ ∂f/∂x^μ where X^μ = dγ^μ(t)/dt

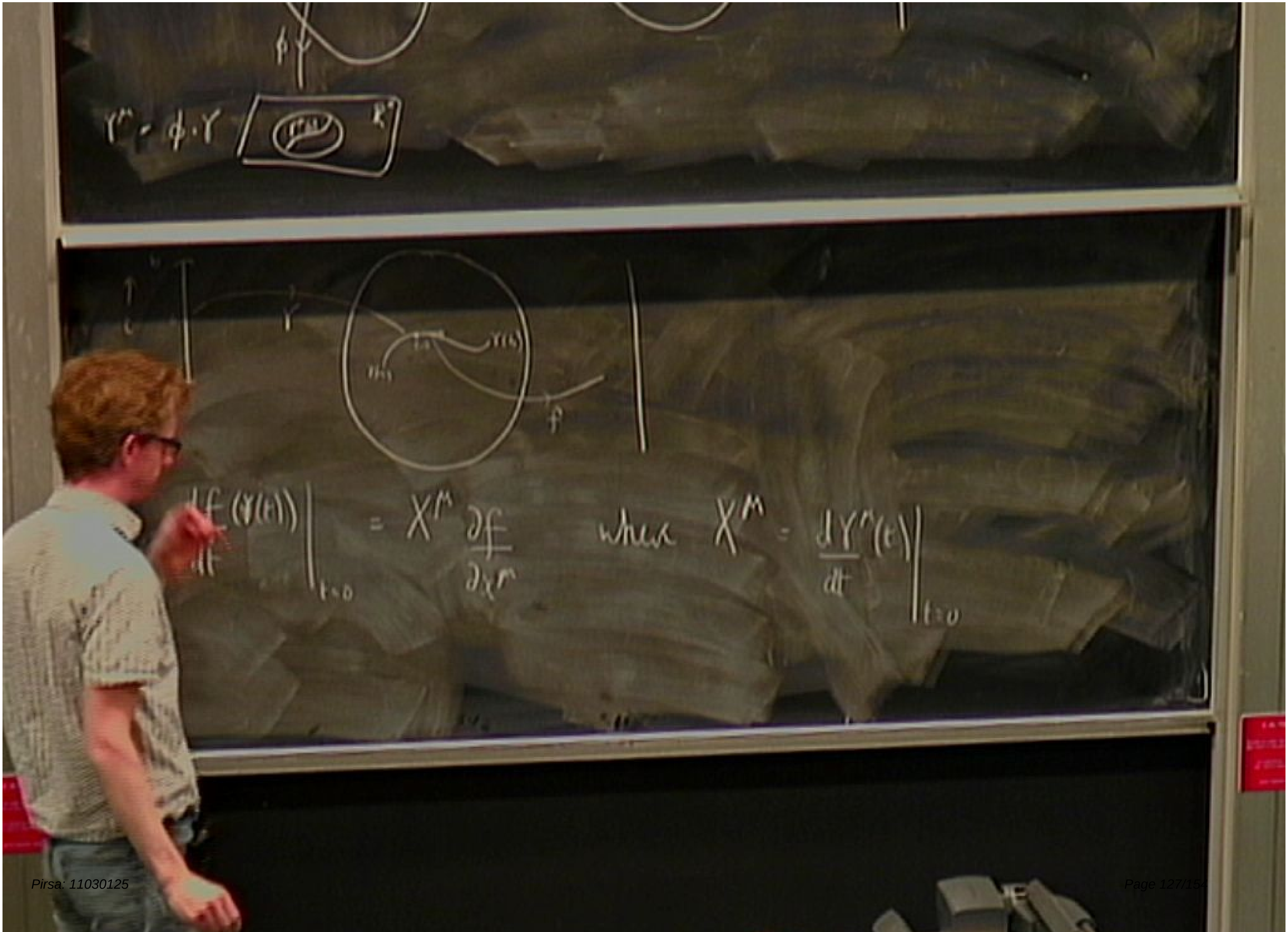

A curve is a map
γ: (a, b) → M.
A function is a map
M
U
I
M
f
R

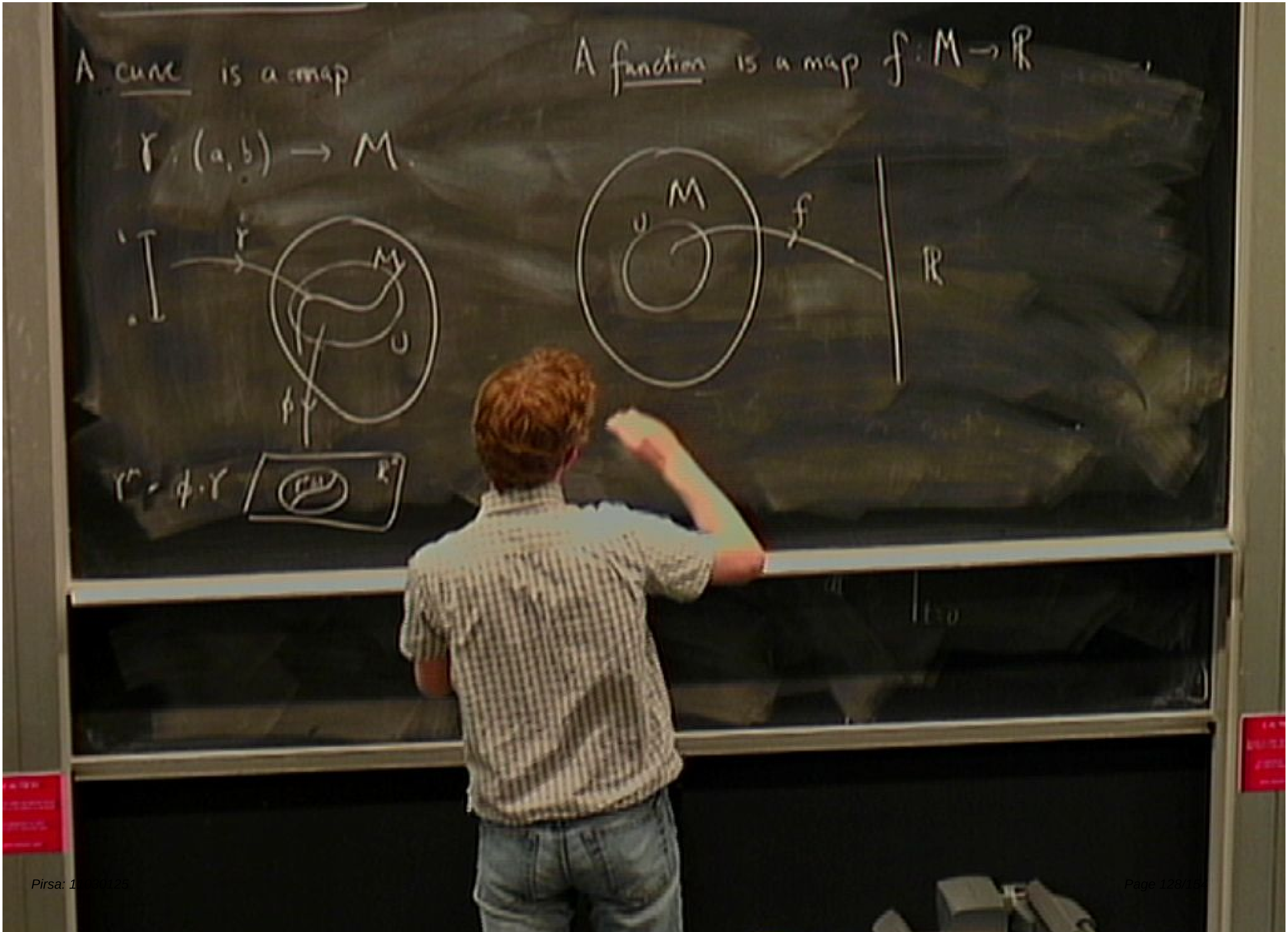

A curve is a map
γ : (a, b) ⟶ M.
A function is a map f
M
U
φ
f
R

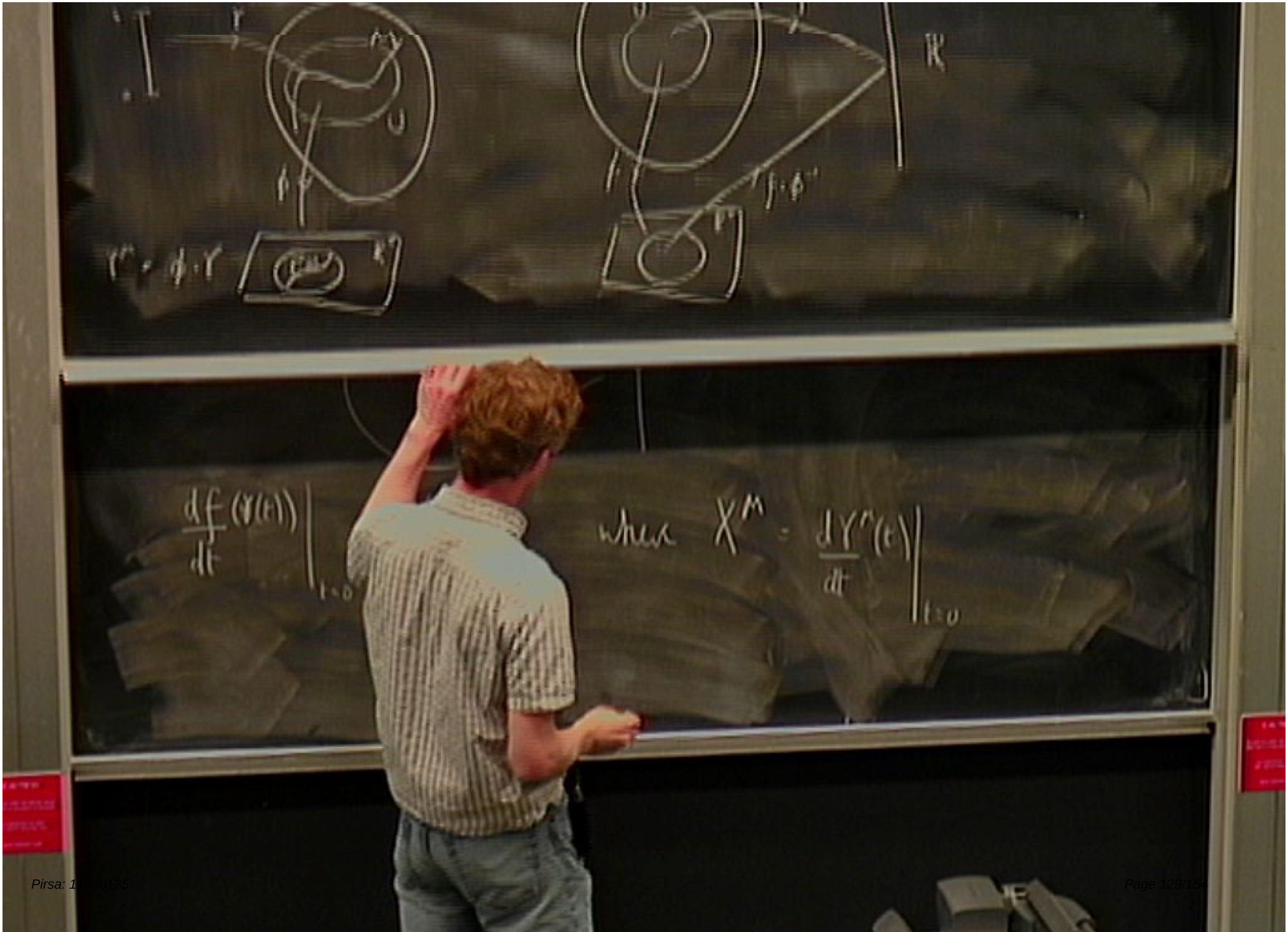

γ : (a, b) → M.
M
f
ℝ
U
ℝ^n
where

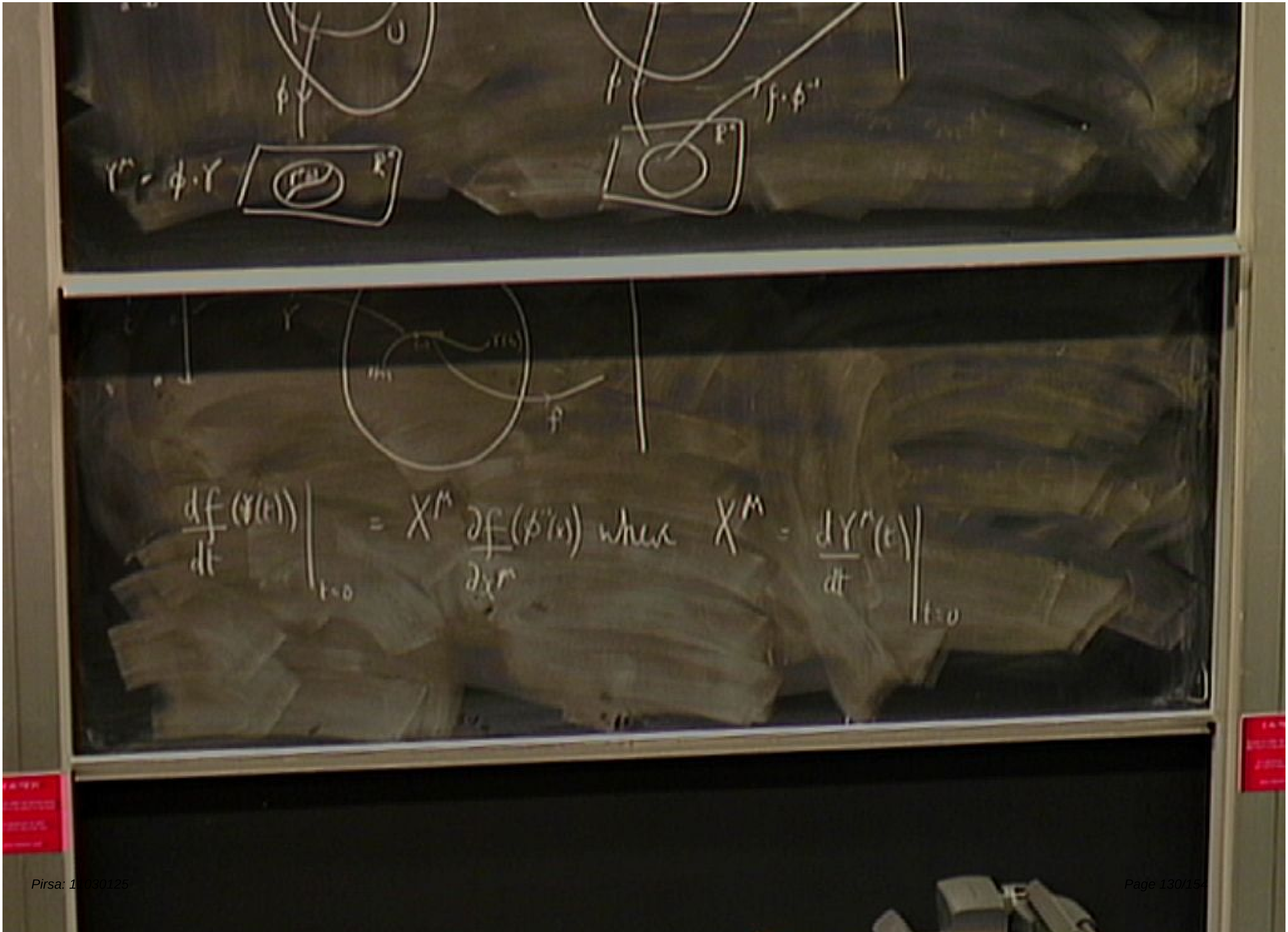

$= X^\mu \frac{\partial f}{\partial x^\mu}$
where $X^\mu = \frac{d\gamma^\mu(t)}{dt}\Big|_{t=0}$

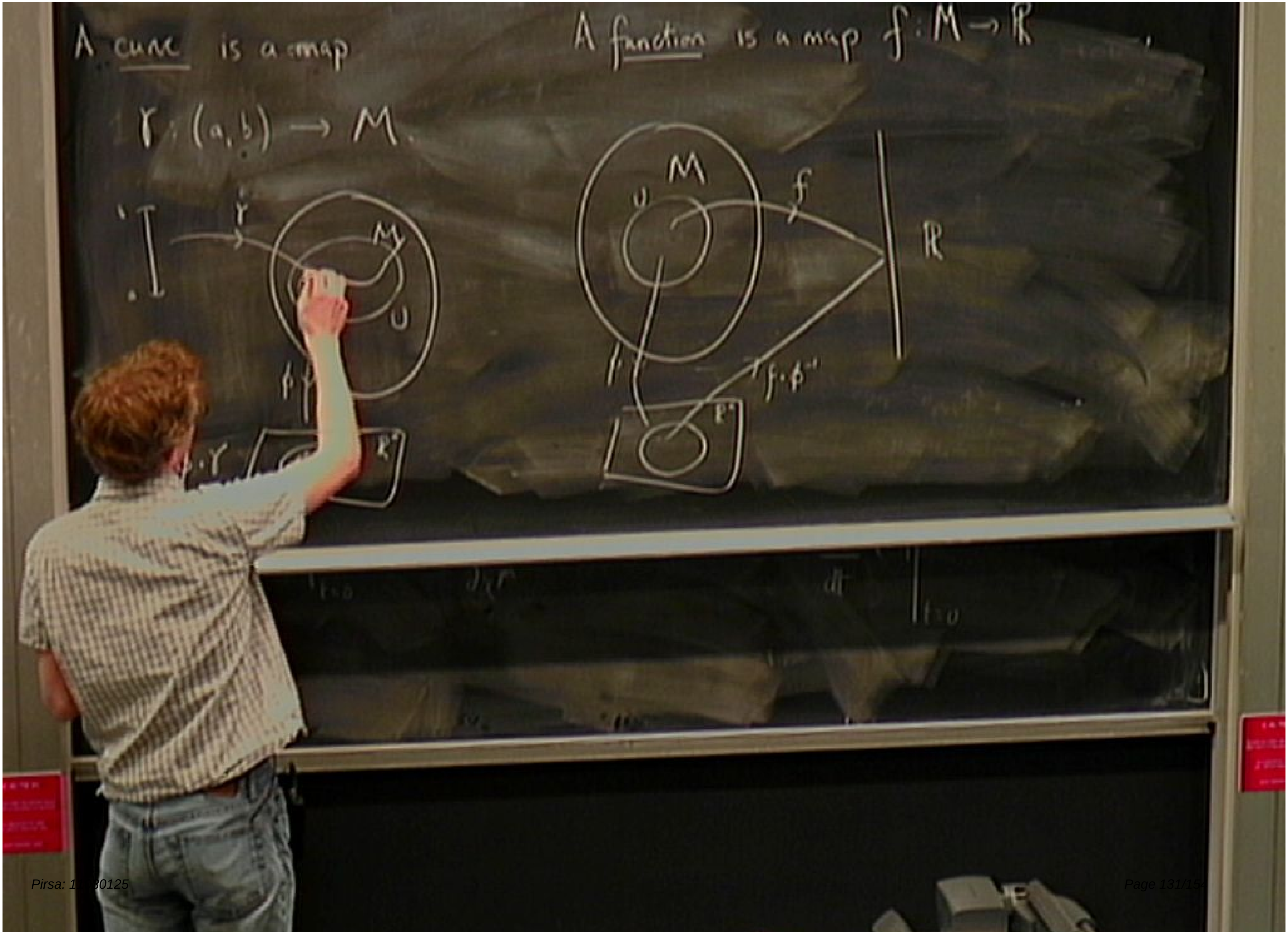

M
f
ℝ
U
where

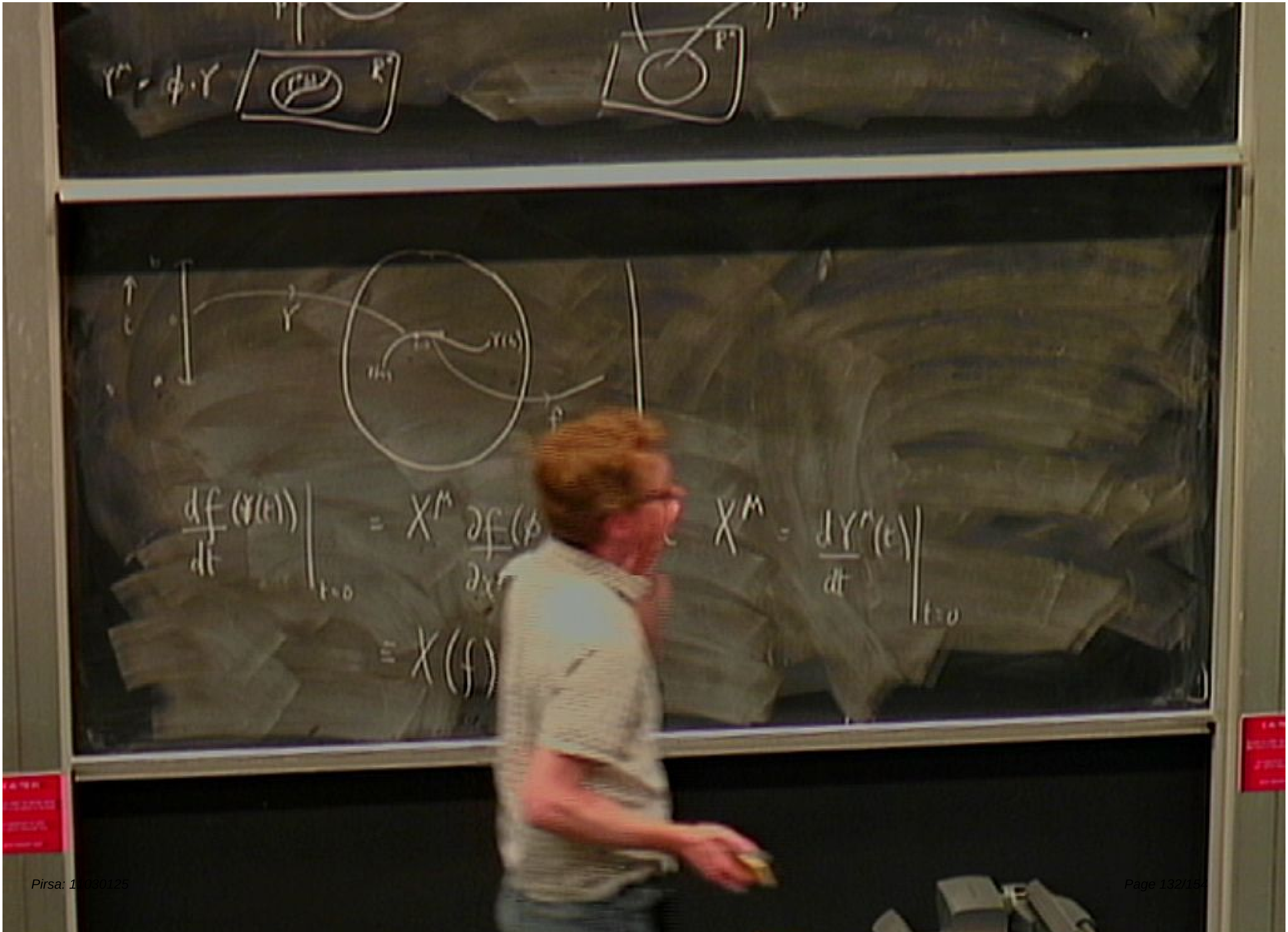

γ^μ = φ ∘ γ
R^n
γ(t)
f
df(γ(t))/dt |t=0 = X^μ ∂f/∂x^μ
where X^μ = dγ^μ(t)/dt |t=0


A curve is a map
γ: (a, b) ⟶ M.
A function is a map f: M → ℝ
M
U
f
ℝ
φ
γ^m = φ ∘ γ

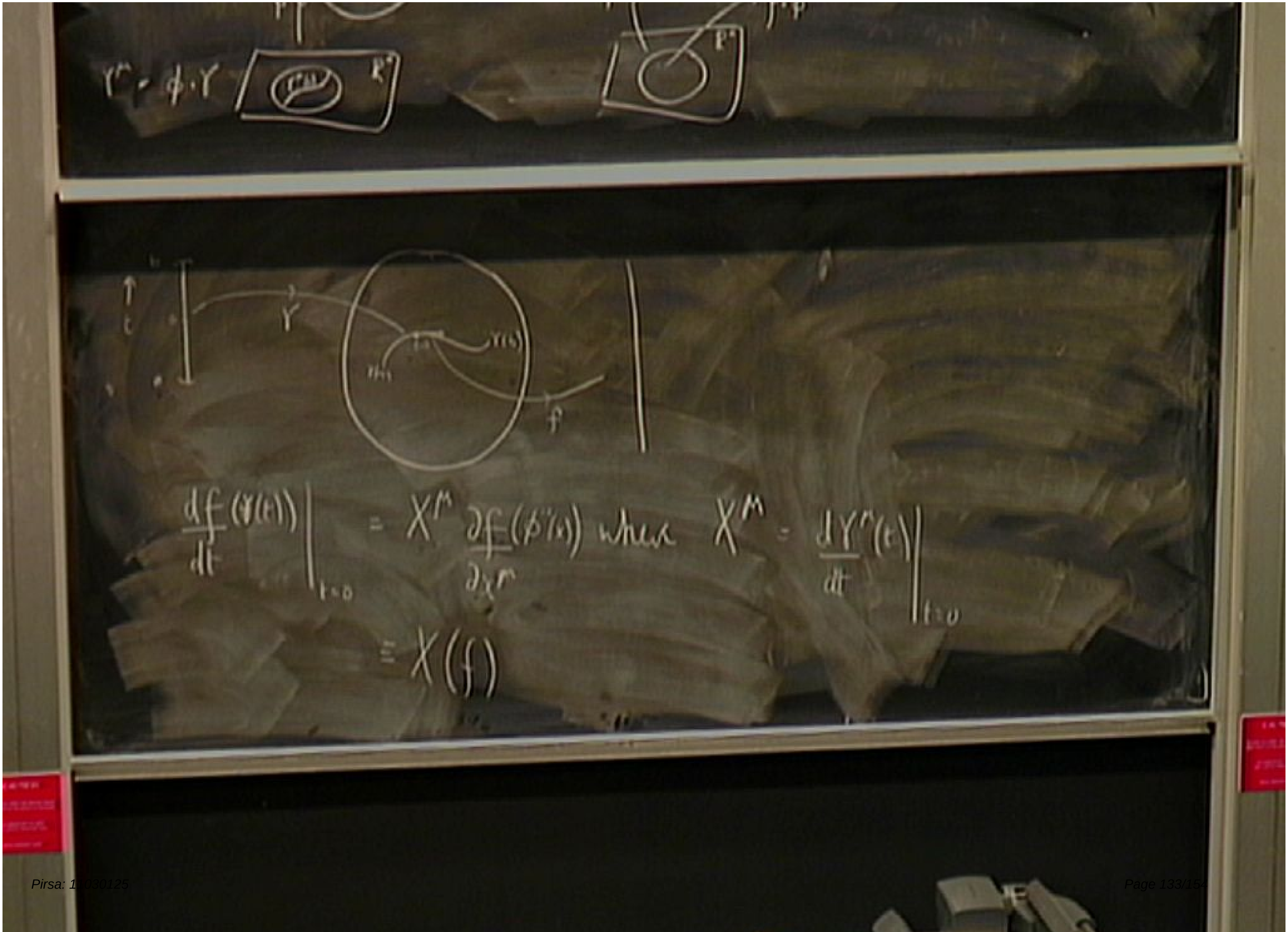

df(γ(t))/dt |t=0
where X^μ = dγ^μ(t)/dt |t=0

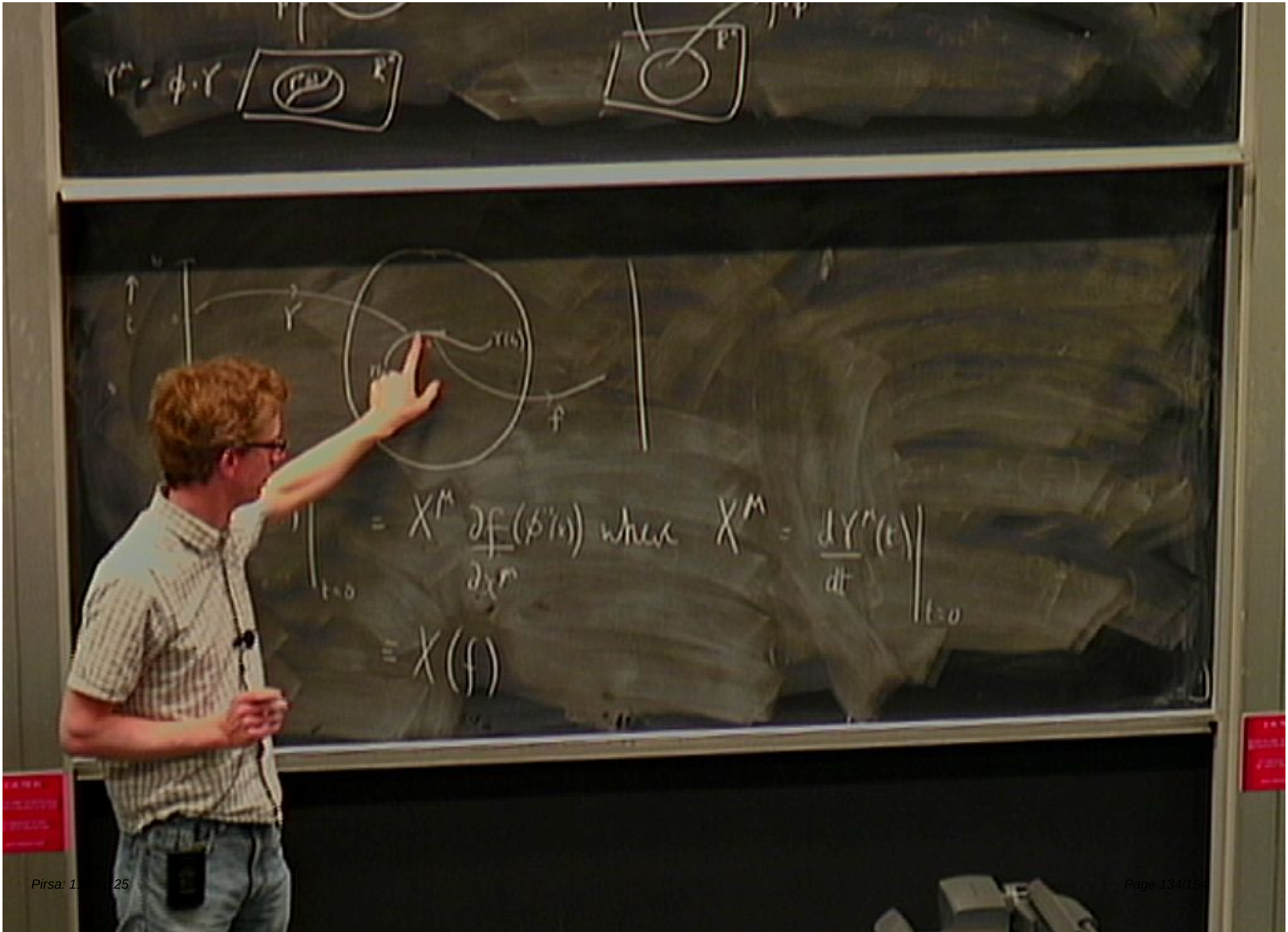

γ^μ = φ∘γ
R^n
φ
U
f∘φ^-1
f
γ
df/dt (γ(t)) |t=0 = X^μ ∂f/∂x^μ (φ^-1(x)) where X^μ = dγ^μ(t)/dt |t=0

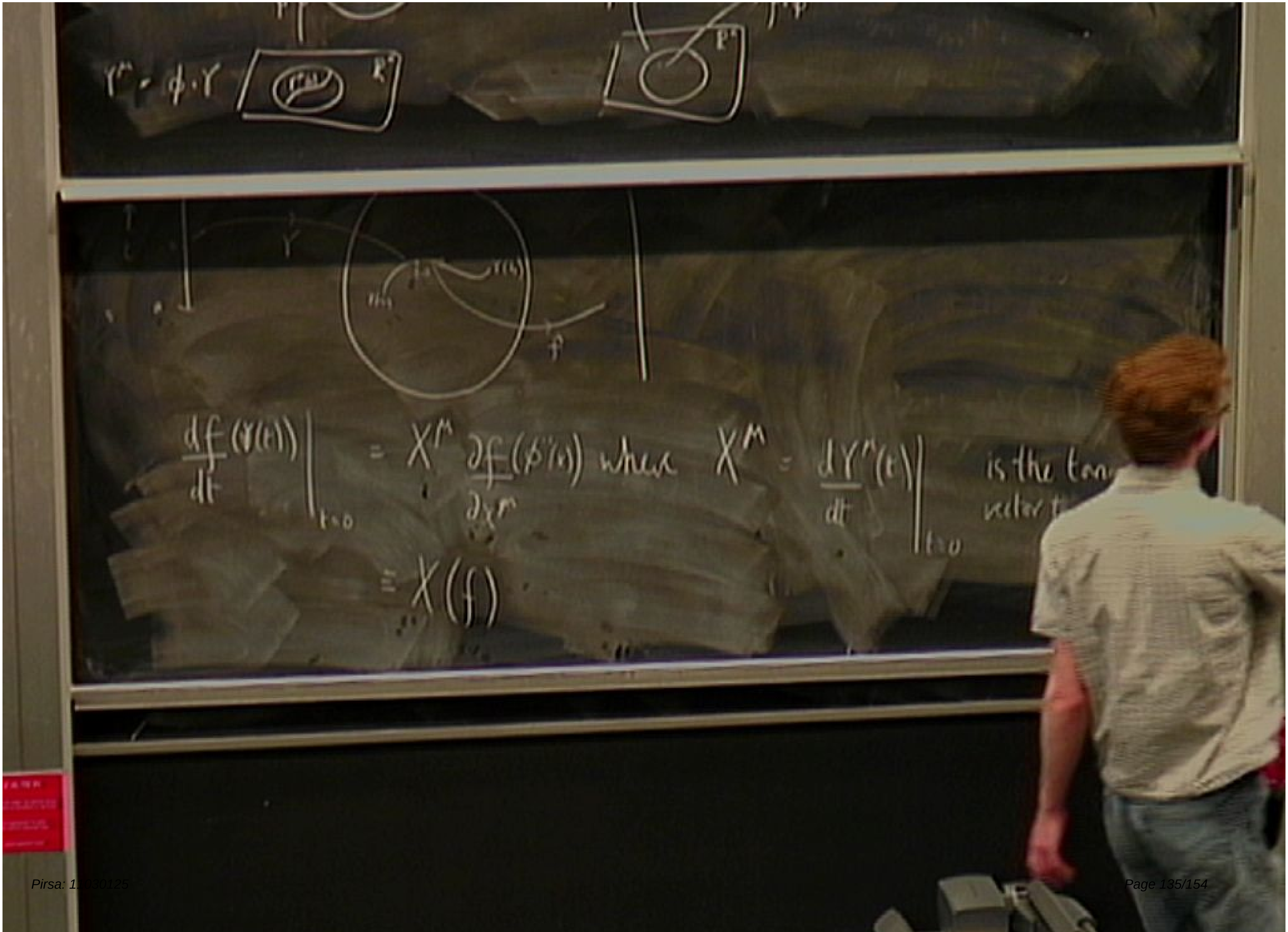

A curve is a map
γ : (a, b) → M.
A function is a map f : M → R
M
U
f
R
f∘φ⁻¹



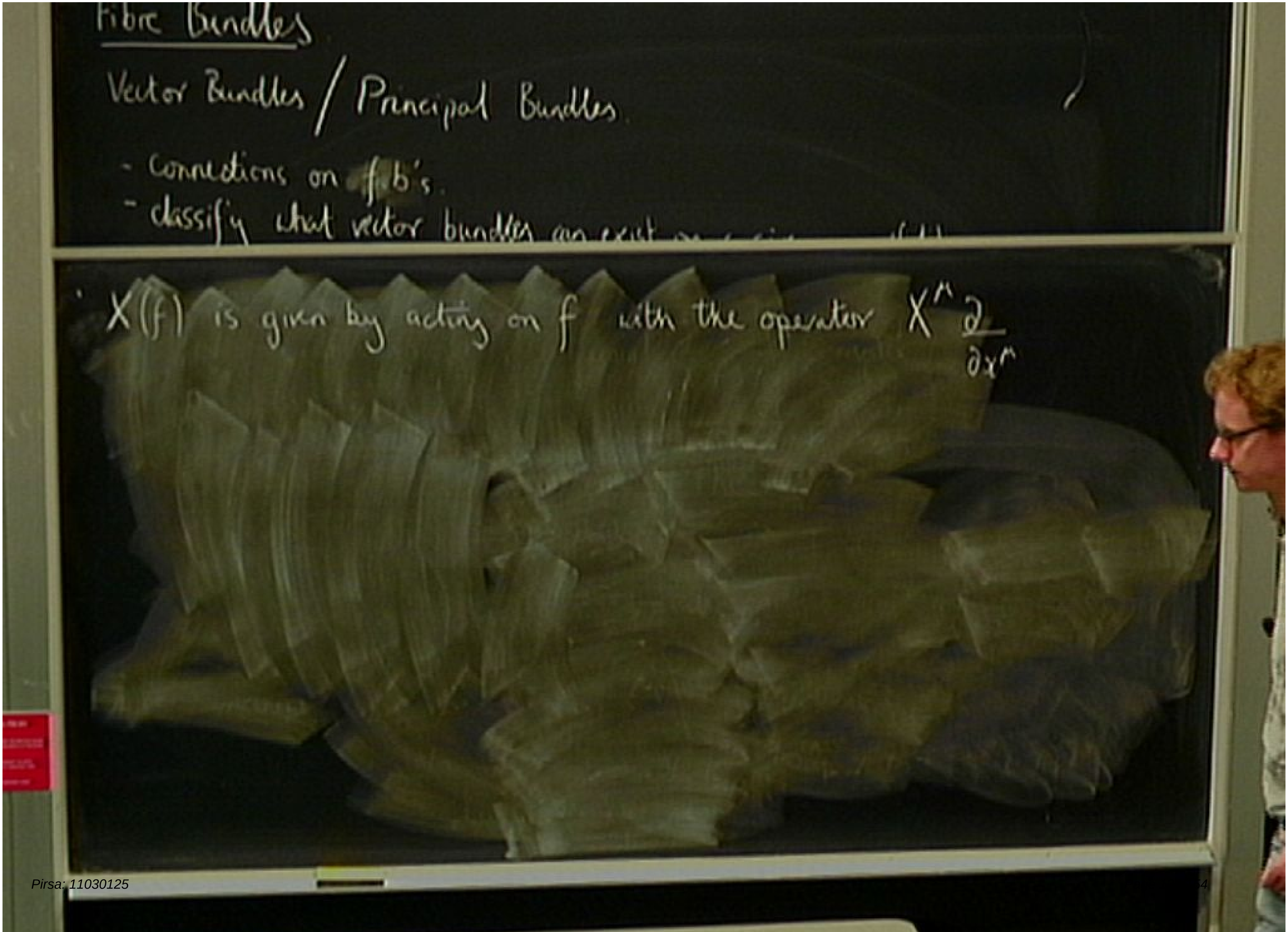
X^μ = dγ^μ(t)/dt |t=0
= X(f)

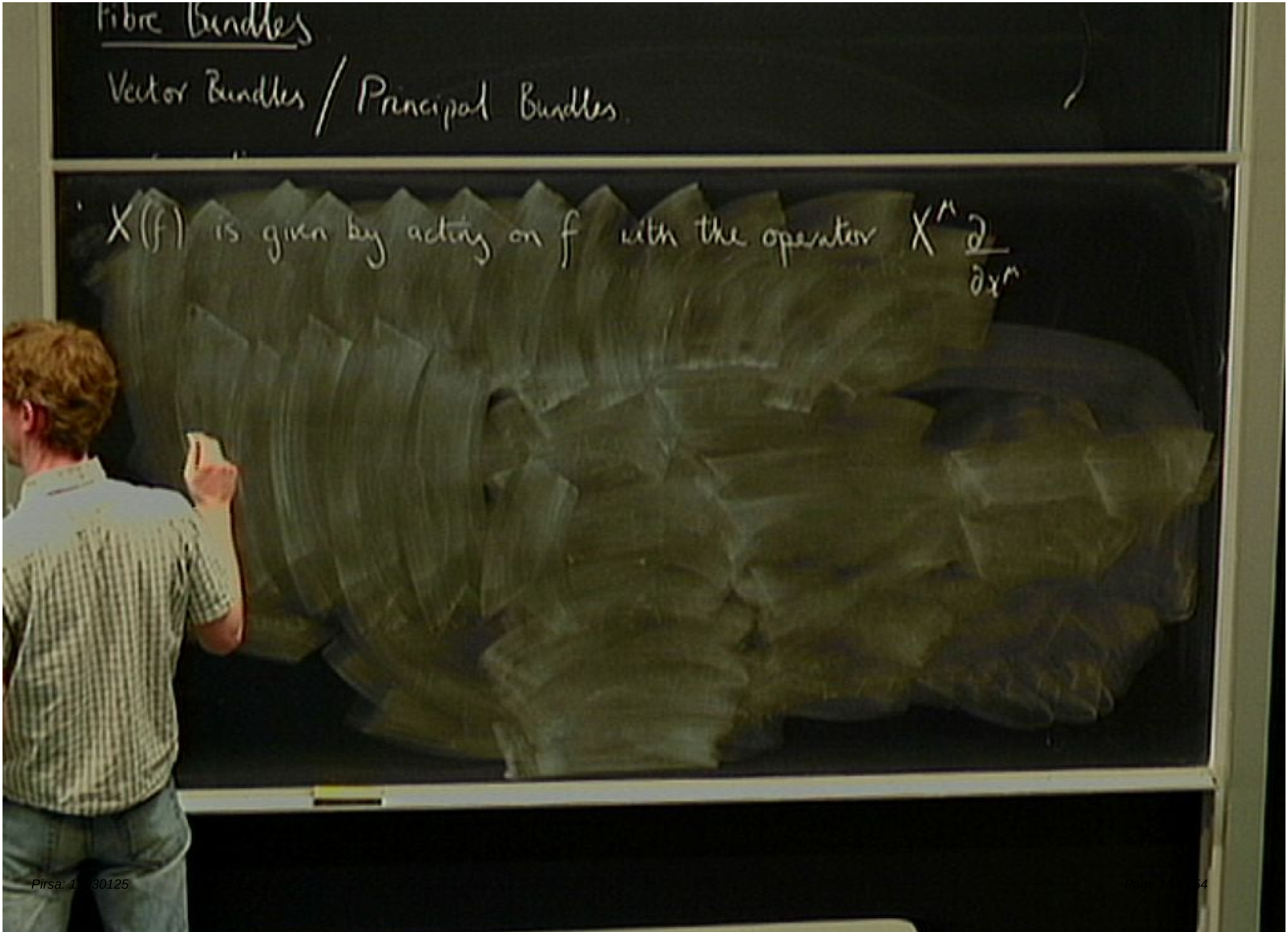

$\frac{df}{dt}(\gamma(t))\Big|_{t=0} = X^\mu \frac{\partial f}{\partial x^\mu}(\phi^{-1}(x))$ where $X^\mu = \frac{d\gamma^\mu(t)}{dt}\Big|_{t=0}$
$\equiv X(f)$

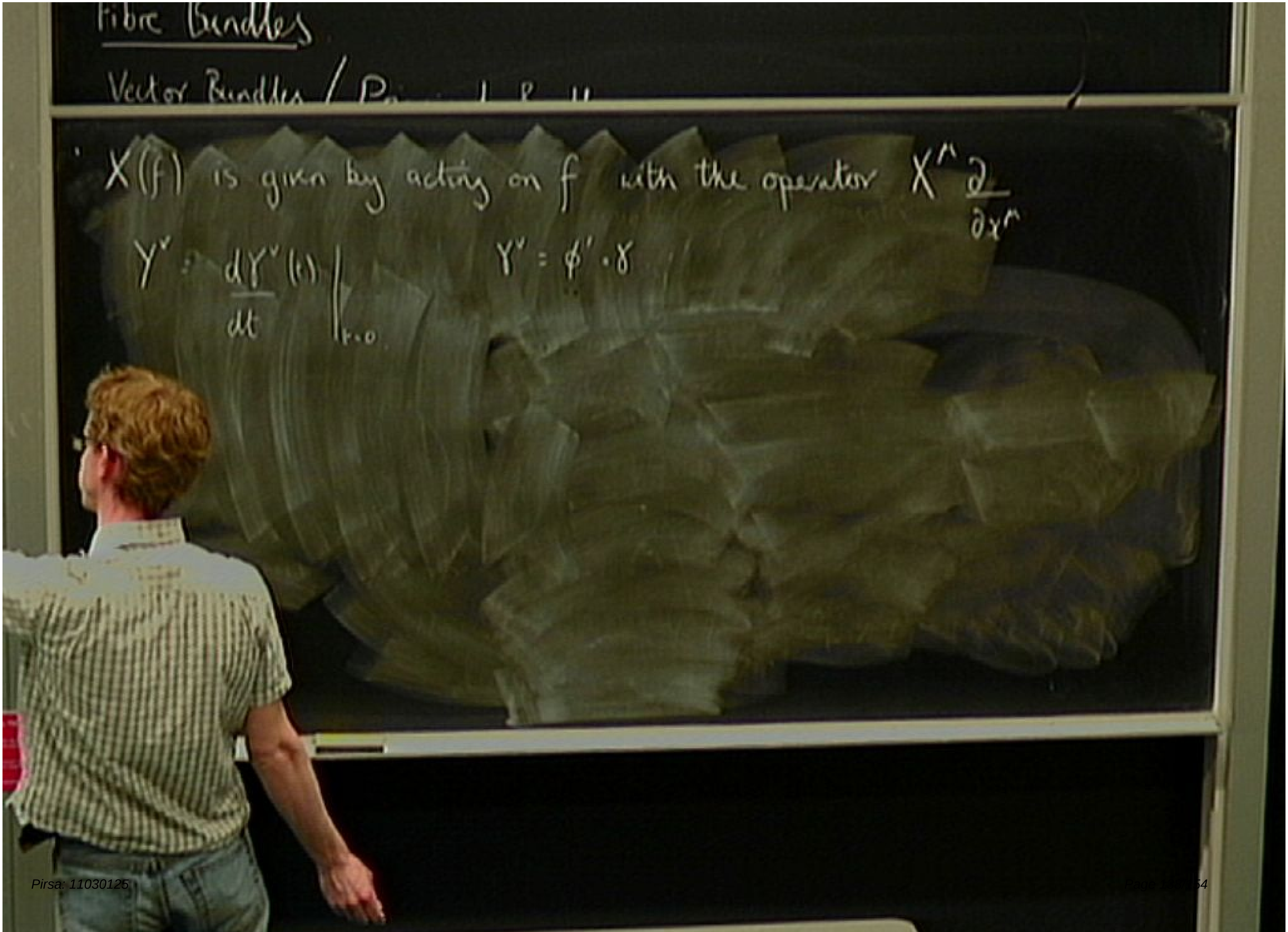

γ^μ = φ·γ
= X^μ ∂f/∂x^μ (φ^{-1}(x)) where X^μ = dγ^μ(t)/dt |_{t=0}
= X(f)



$\frac{df}{dt}(\gamma(t))\Big|_{t=0} = X^\mu \frac{\partial f}{\partial x^\mu}(\phi^{-1}(x))$ where $X^\mu = \frac{d\gamma^\mu(t)}{dt}\Big|_{t=0}$
$= X(f)$


Fibre Bundles
Vector Bundles / Principal Bundles
- connections on f.b's.
- classify what vector bundles
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ


Fibre Bundles
Vector Bundles / Principal Bundles
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ


Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator $X^\mu \frac{\partial}{\partial x^\mu}$
$\gamma^\nu = \frac{d\gamma^\nu(t)}{dt}\Big|_{t=0}$
$\gamma^\nu = \phi^\nu \circ \gamma$

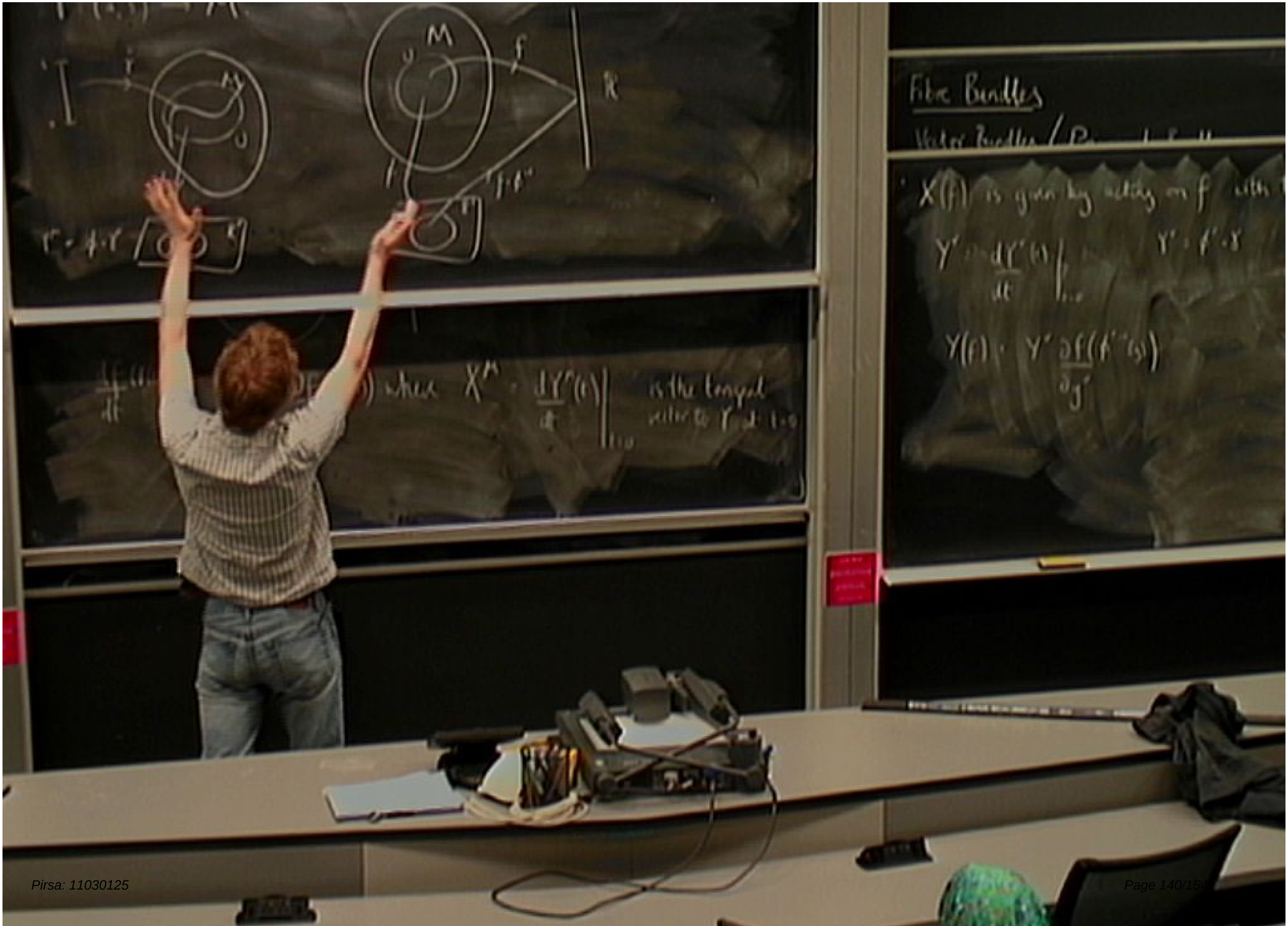
Fibre Bundles

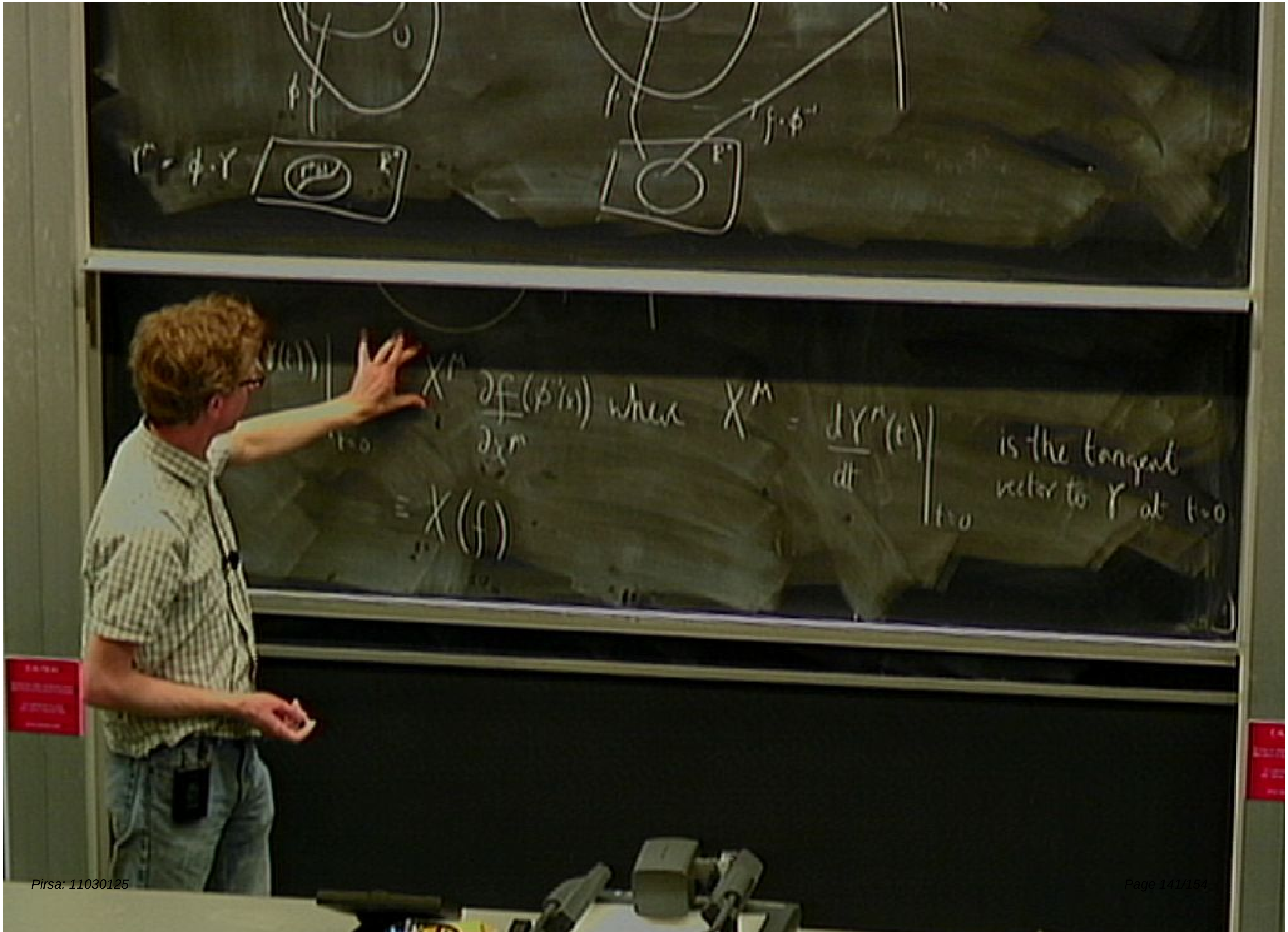
Vector Bundles / ...

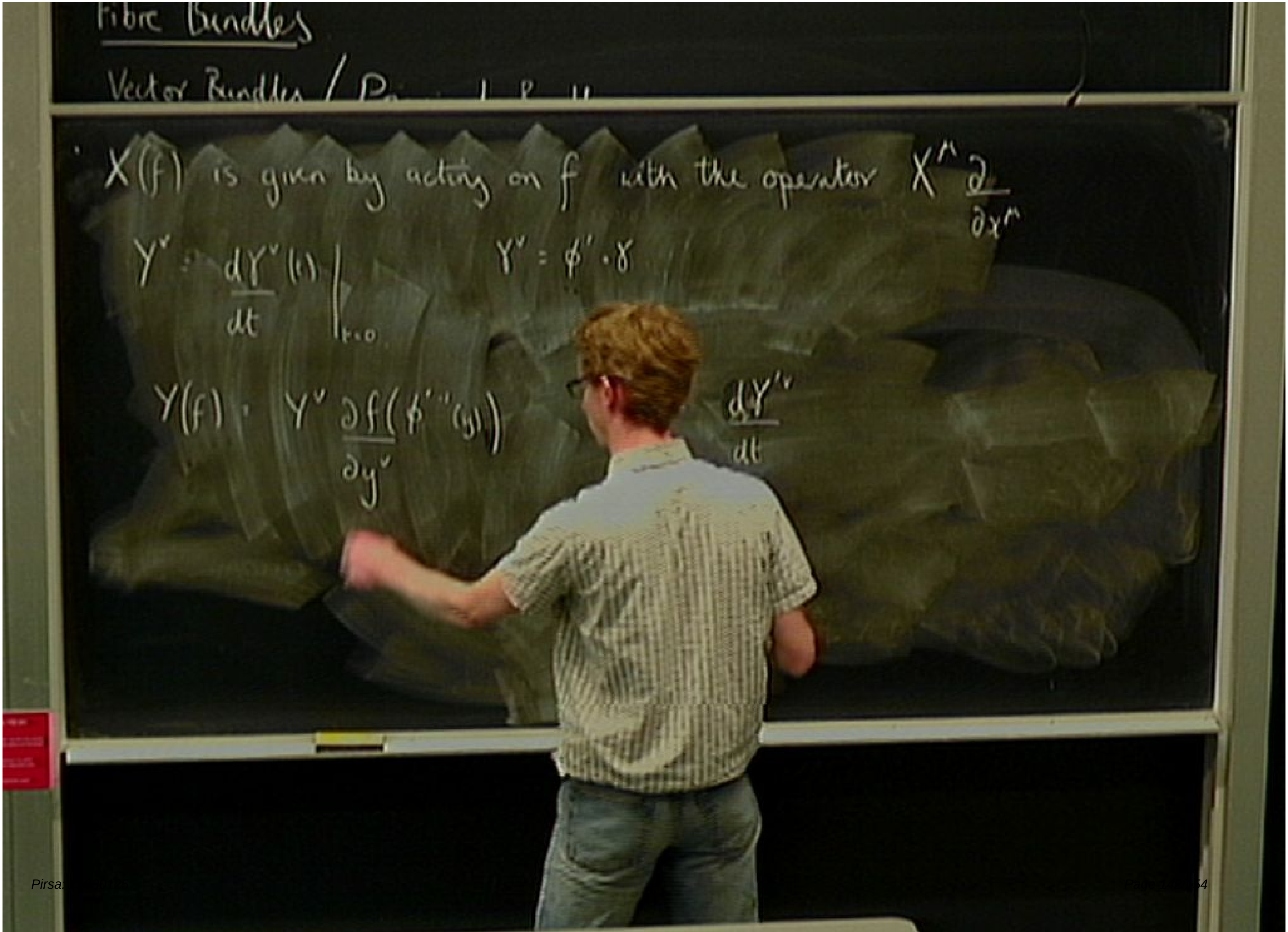
$X(f)$ is given by acting on f with the operator $X^\mu \frac{\partial}{\partial x^\mu}$

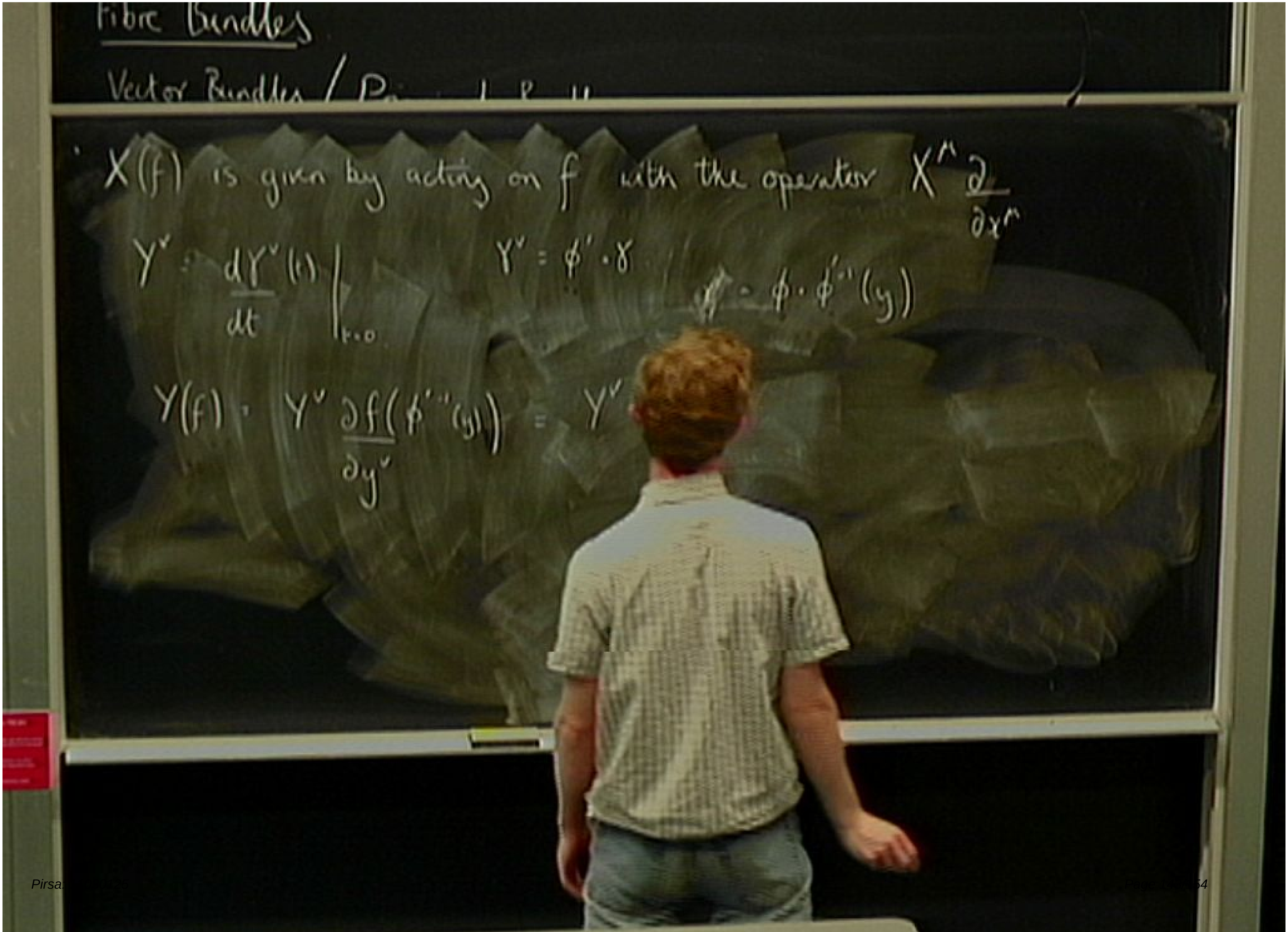
$$Y^\nu = \frac{d\gamma^\nu(t)}{dt}\bigg|_{t=0} \qquad \gamma^\nu = \phi^\nu \circ \gamma$$

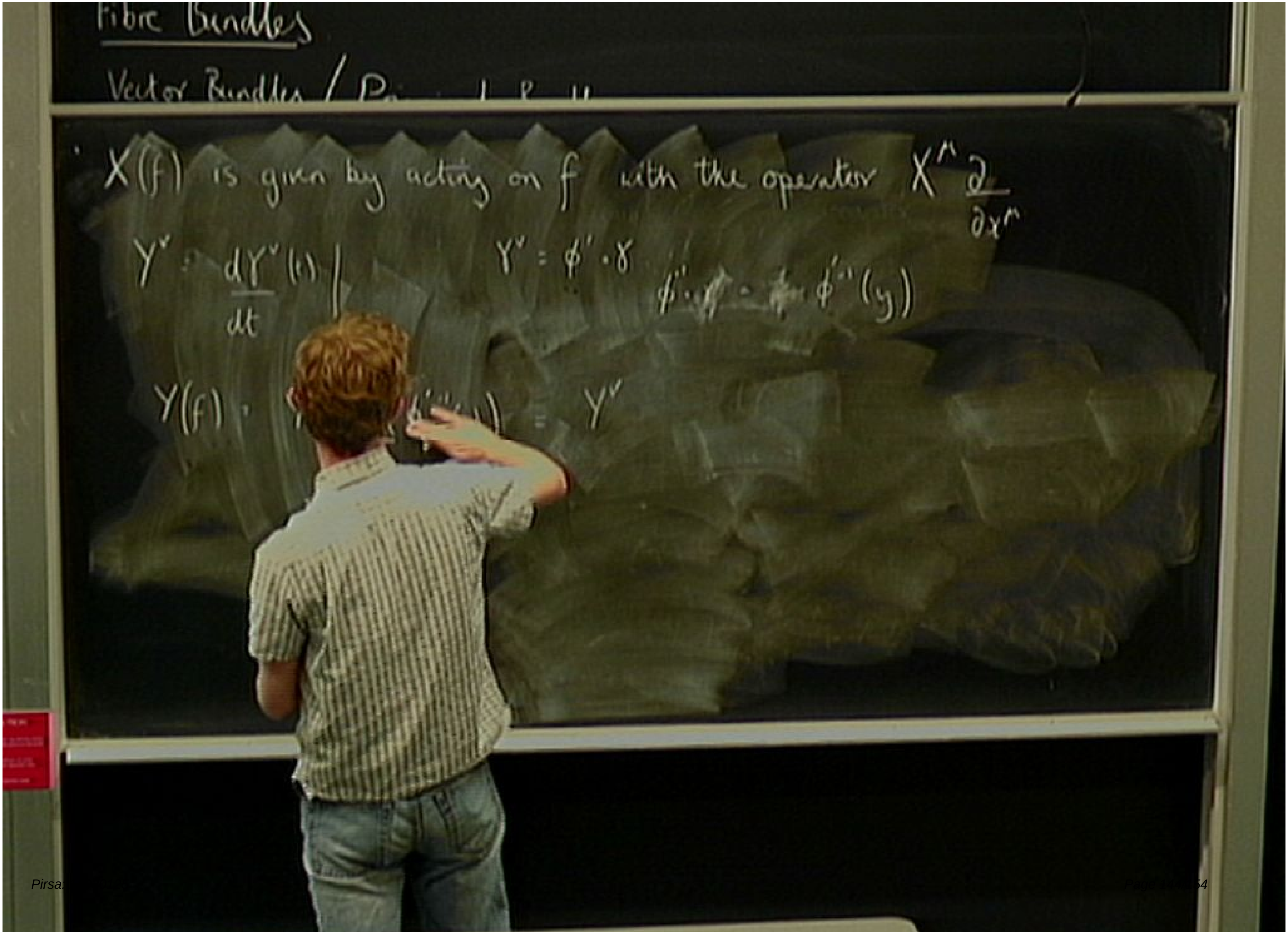
$$Y(f) = Y^\nu \frac{\partial f(\phi^{-1}(y))}{\partial y^\nu}$$


Fibre Bundles

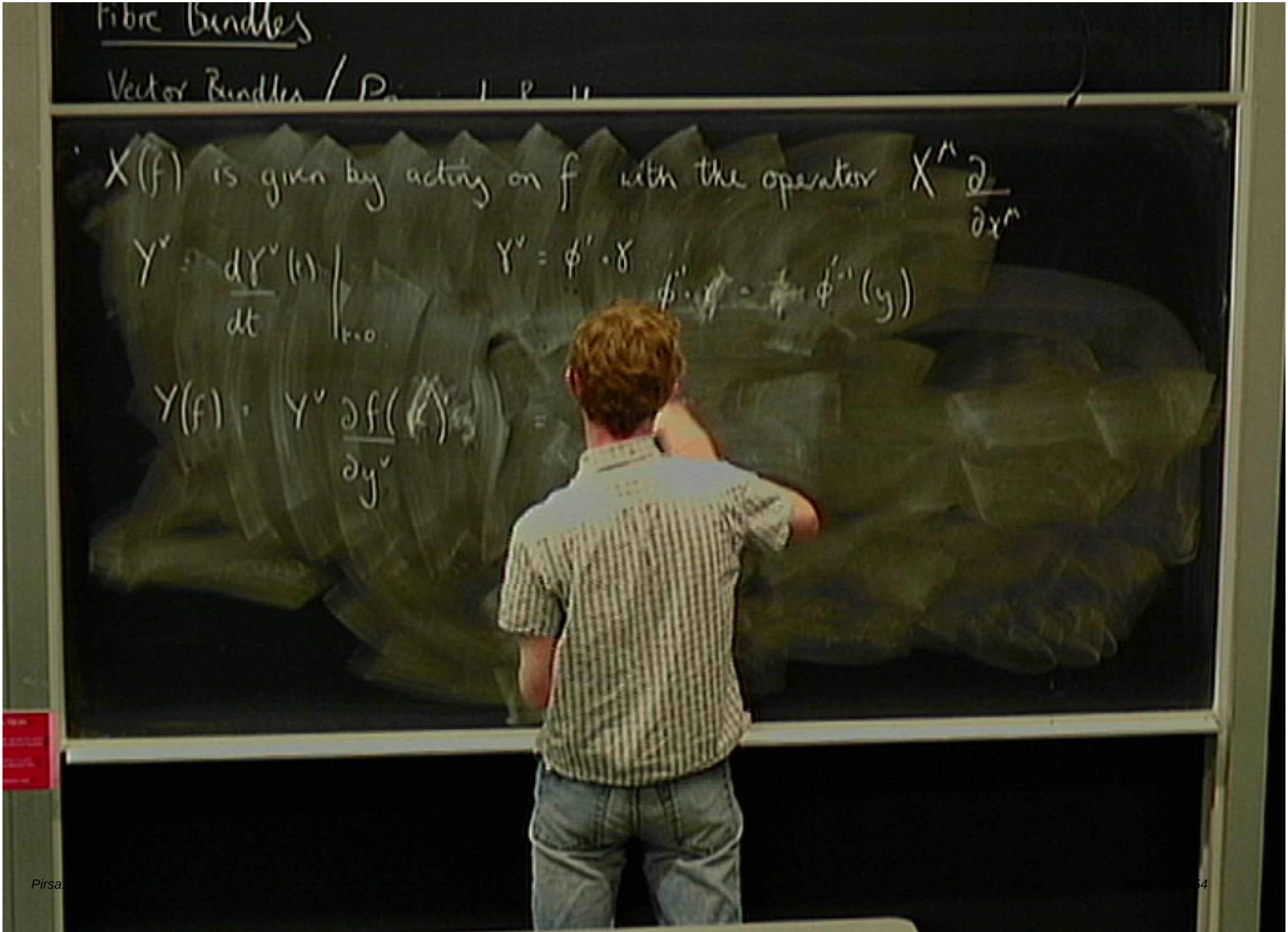

where
is the tangent
vector to γ at t=0

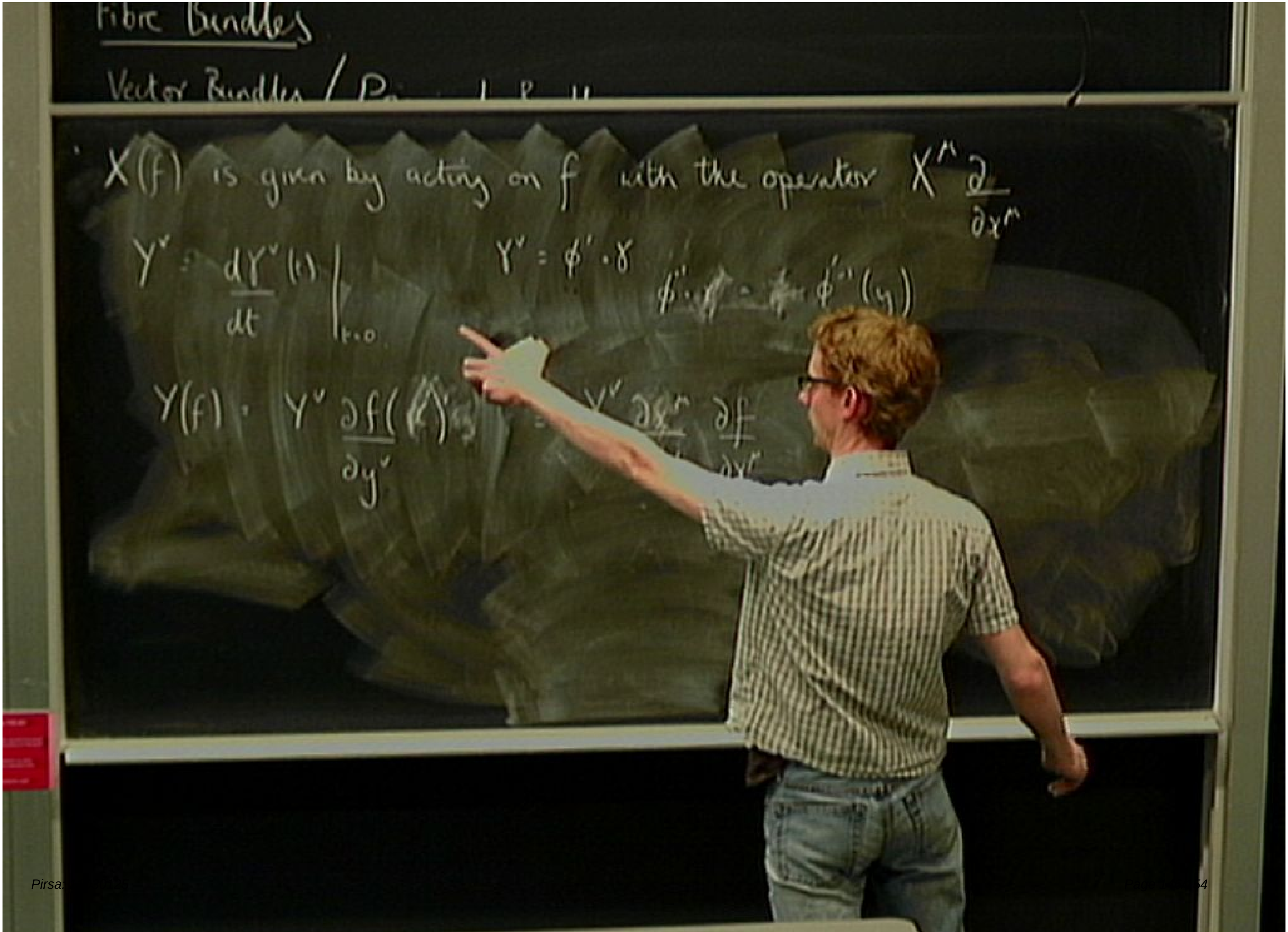

Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dγ^ν(t)/dt |_{t=0}
γ^ν = φ^ν ∘ γ
Y(f) = Y^ν ∂f(φ^{-1}(y))/∂y^ν
dγ^ν/dt

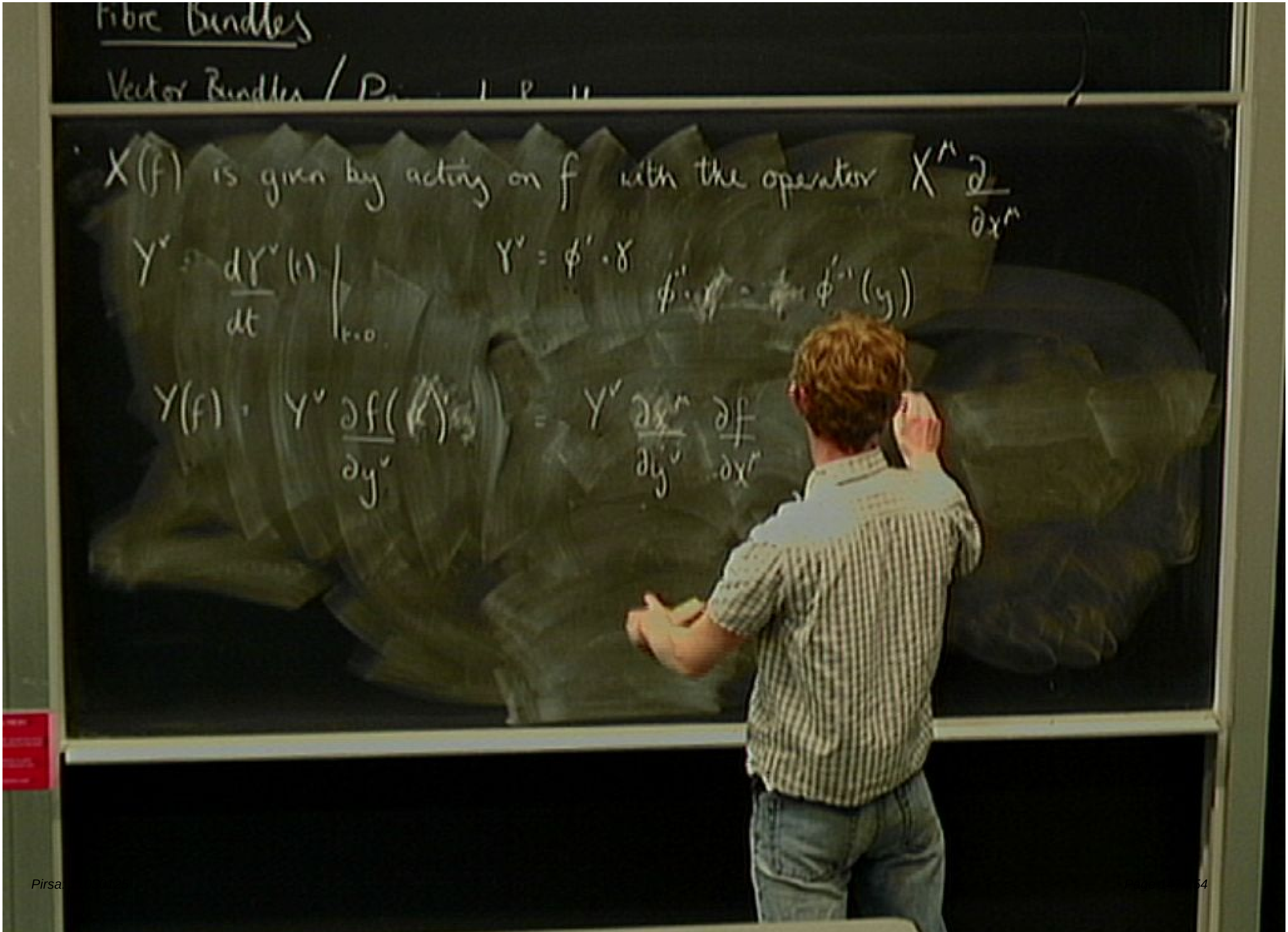

Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dγ^ν(t)/dt |t=0
γ^ν = φ^ν ∘ γ
Y(f) = Y^ν ∂f(φ^-1(y))/∂y^ν = Y^ν

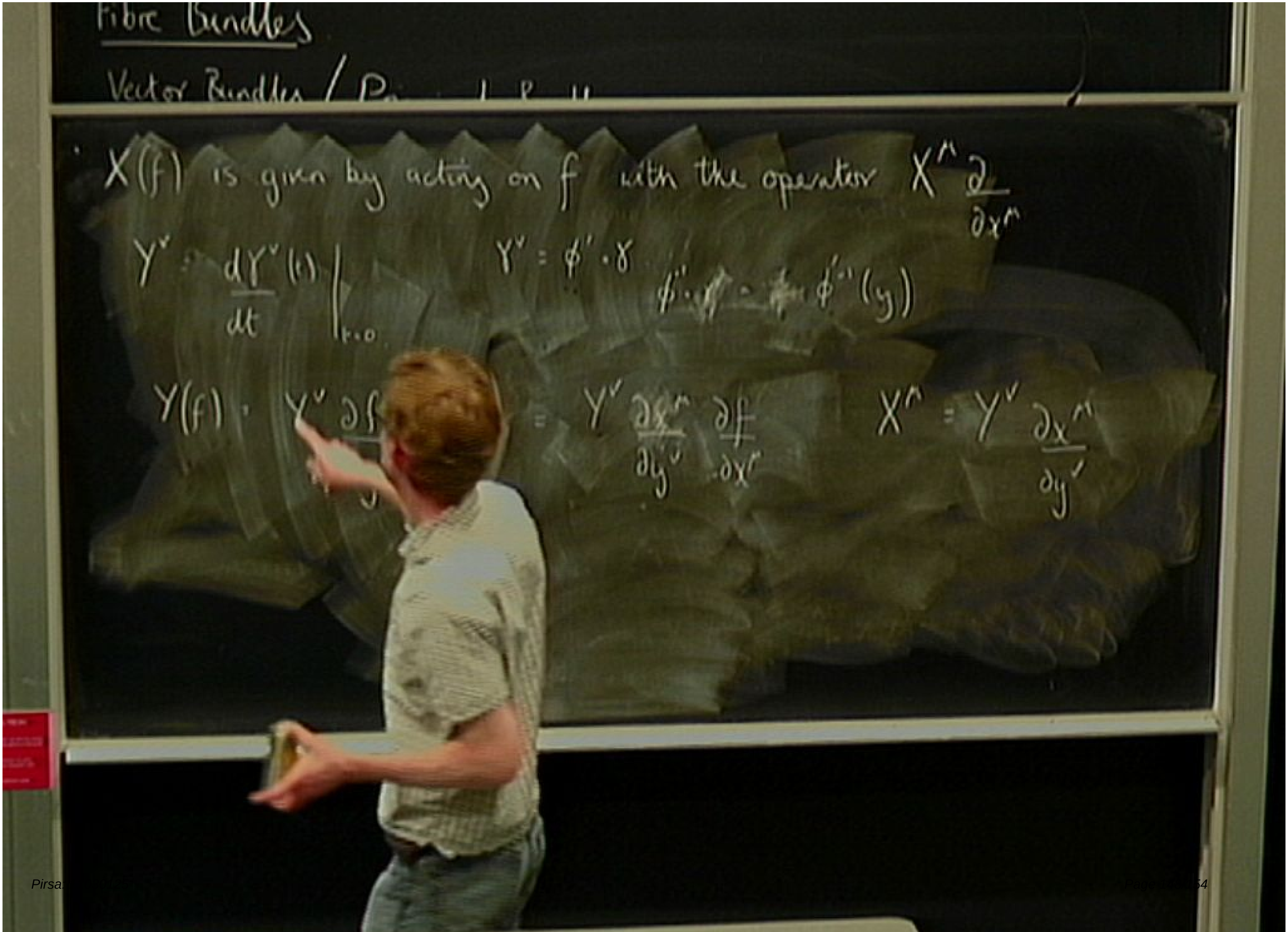


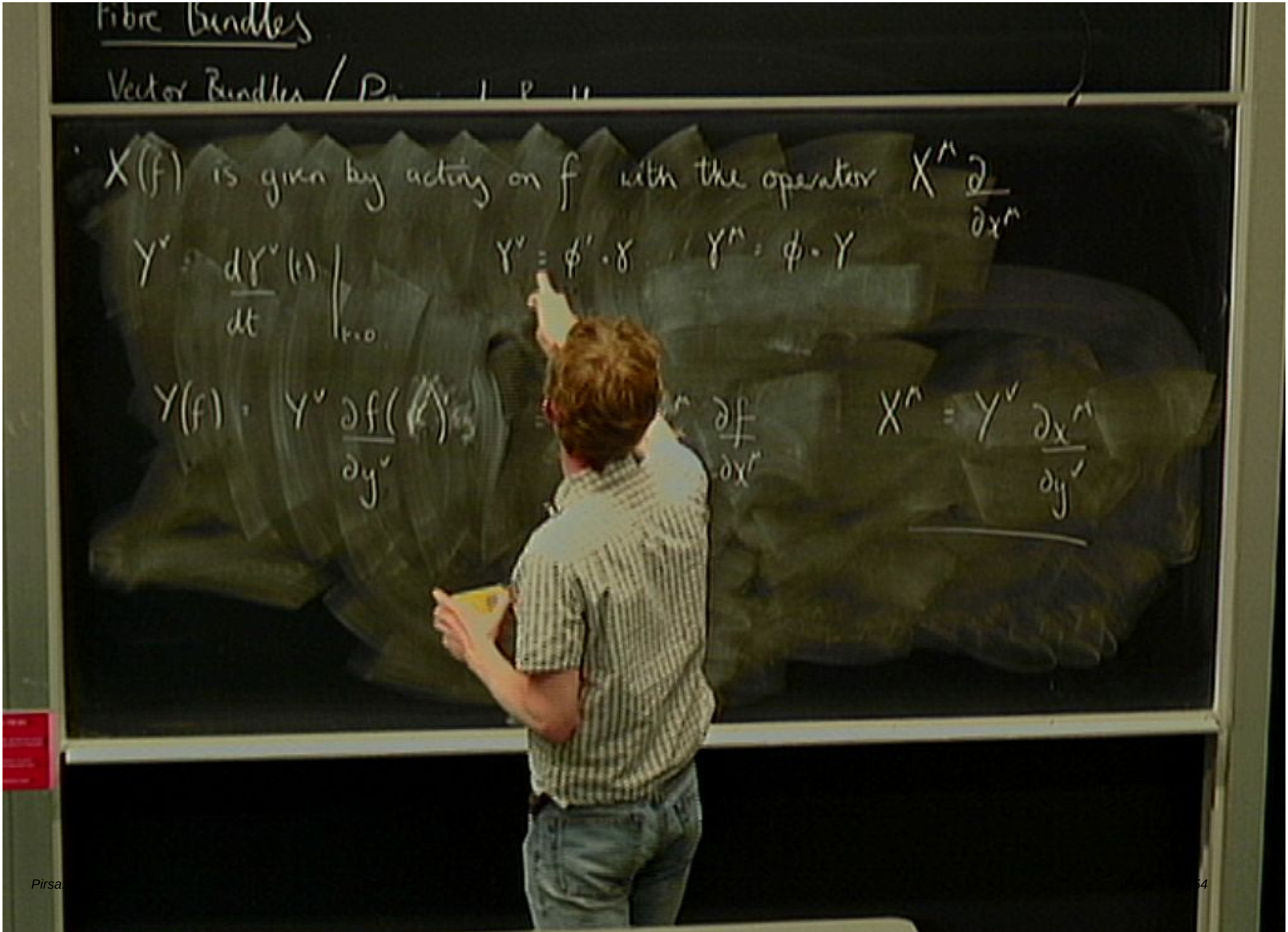
Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dγ^ν(t)/dt |
γ^ν = φ^ν ∘ γ
Y(f) = Y^ν

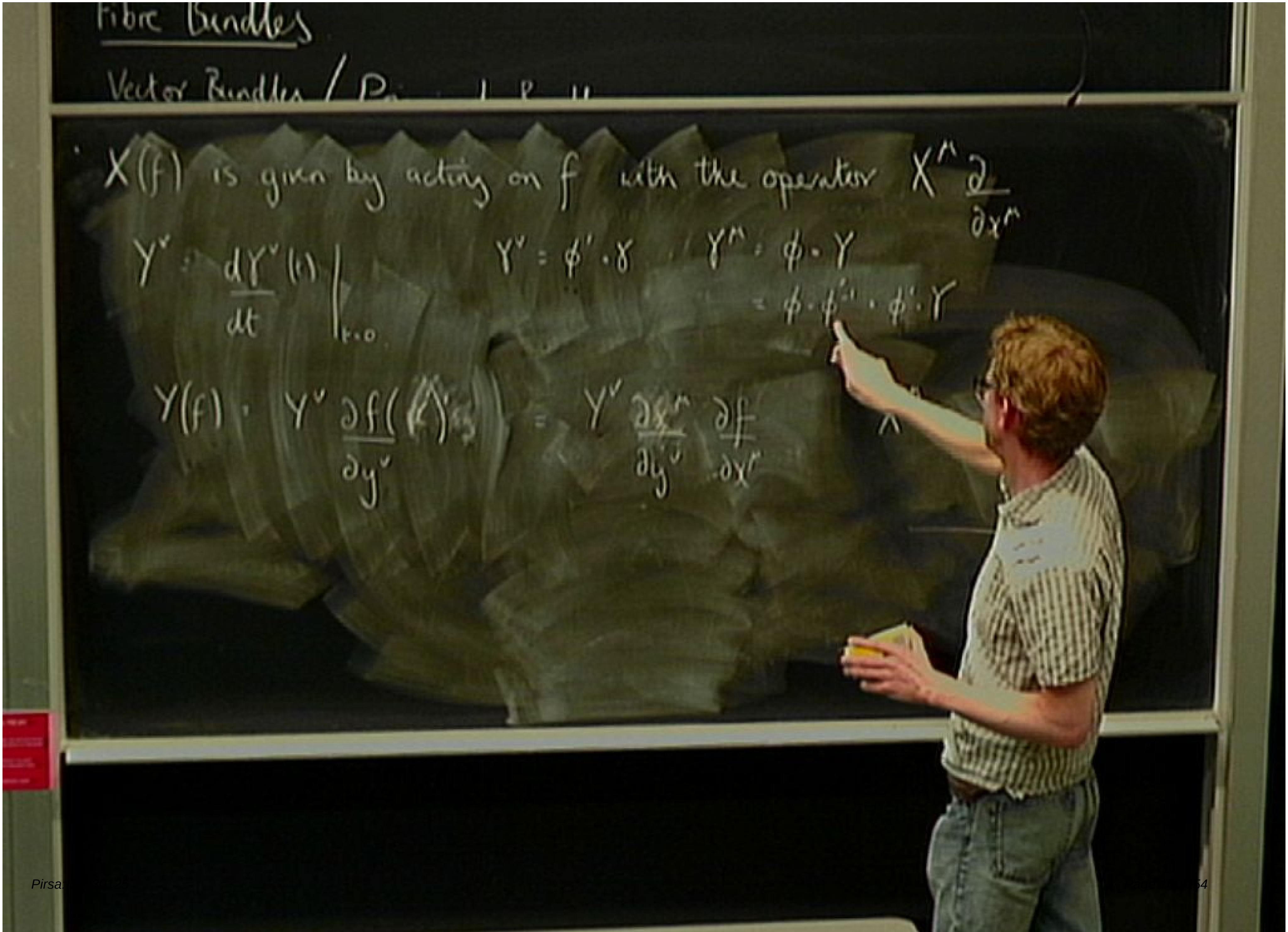

Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dγ^ν(t)/dt |t=0
Y^ν = φ^ν ∘ γ
Y(f) = Y^ν ∂f/∂y^ν =

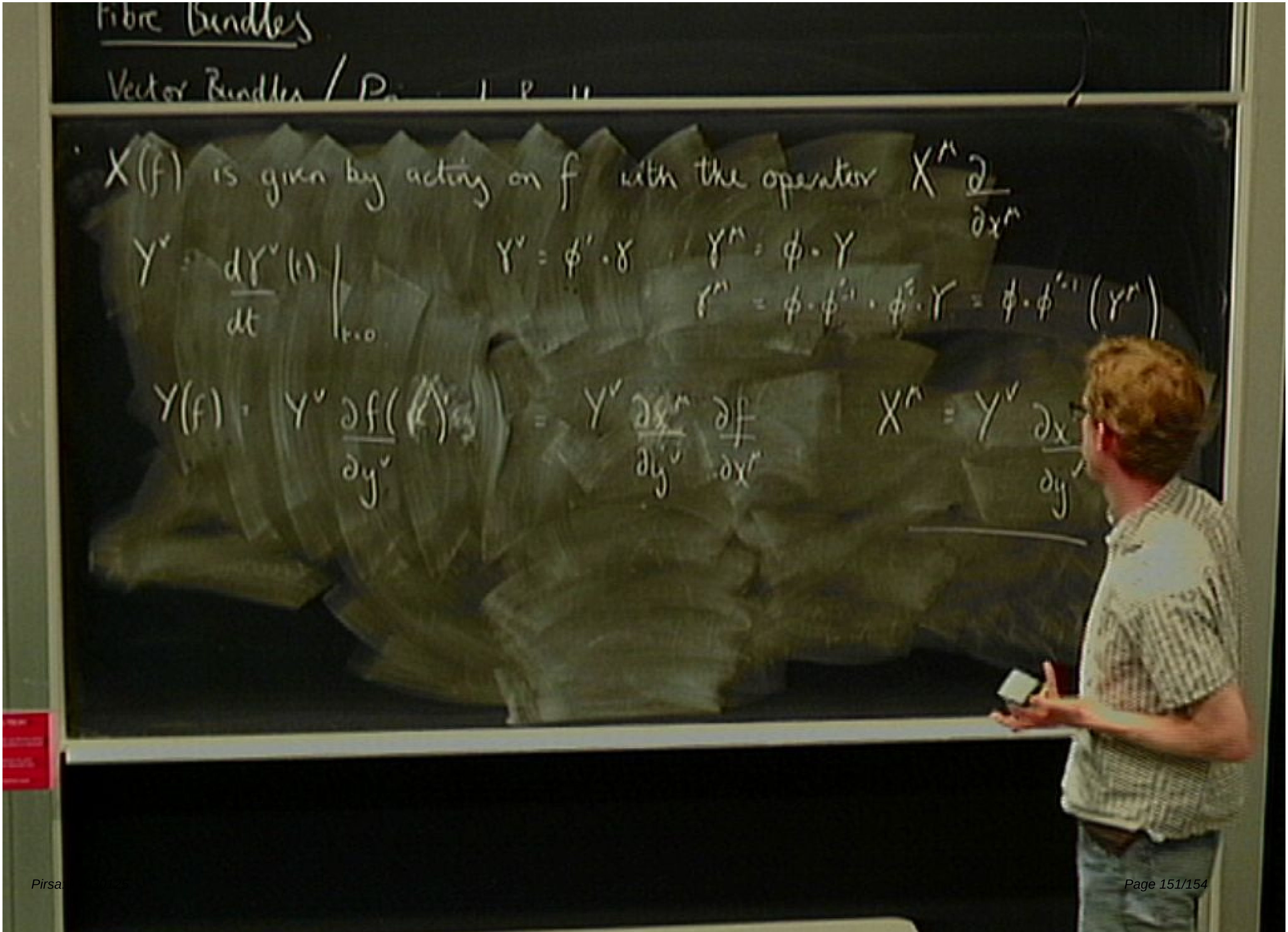

Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator $X^\mu \frac{\partial}{\partial x^\mu}$

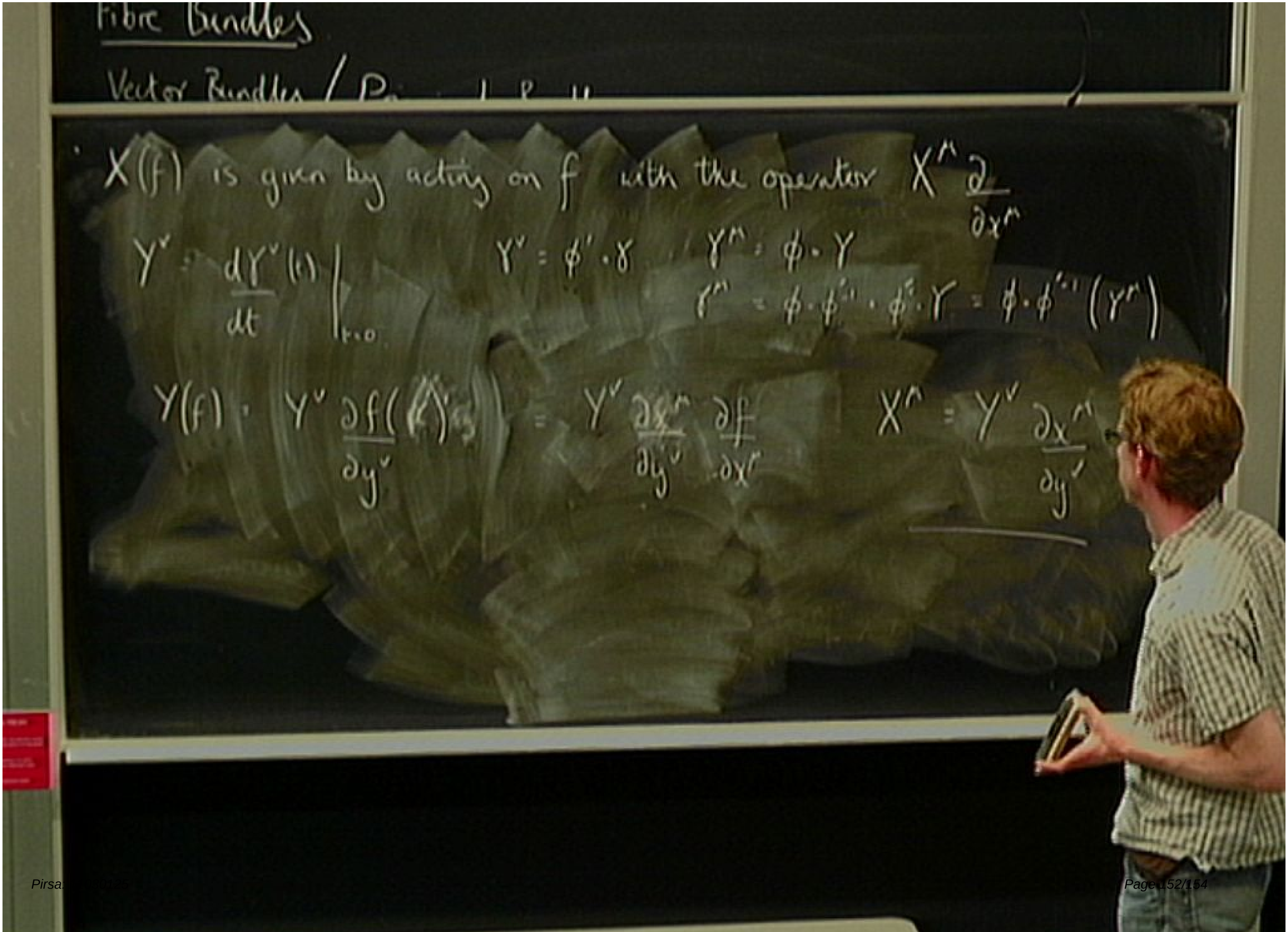

Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator $X^\mu \frac{\partial}{\partial x^\mu}$
$Y^\nu = \frac{d\gamma^\nu(t)}{dt}\Big|_{t=0}$
$Y^\nu = \phi^\nu \circ \gamma$
$Y(f) = Y^\nu \frac{\partial f}{\partial y^\nu} = Y^\nu \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial f}{\partial x^\mu}$

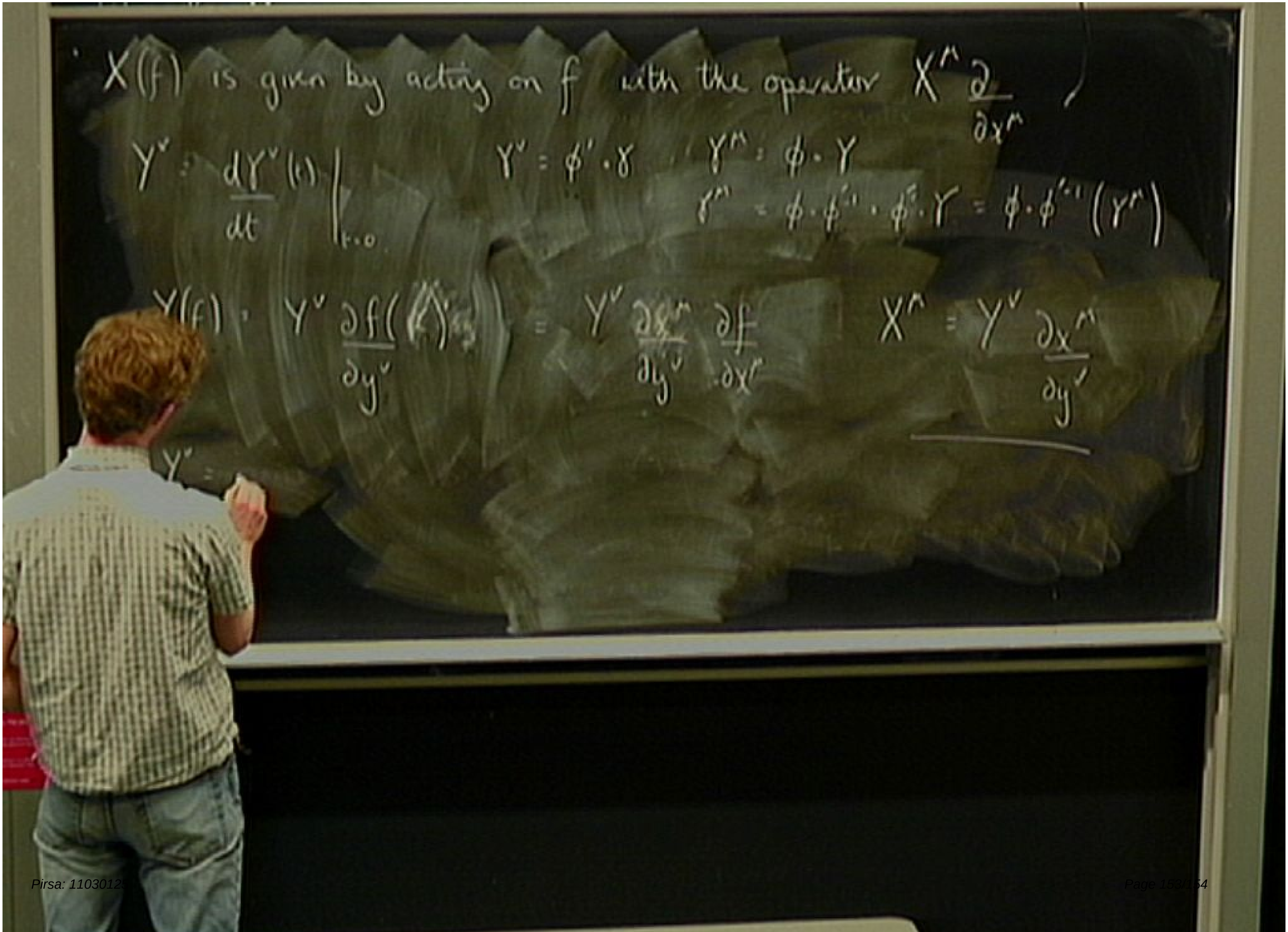

Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dγ^ν(t)/dt |_{t=0}
γ^ν = φ^ν ∘ γ
Y(f) = Y^ν ∂f/∂y^ν = Y^ν ∂x^μ/∂y^ν ∂f/∂x^μ
X^μ = Y^ν ∂x^μ/∂y^ν


Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator $X^\mu \frac{\partial}{\partial x^\mu}$
$Y^\nu = \frac{d\gamma^\nu(t)}{dt}\Big|_{t=0}$
$\gamma^\nu = \phi^{\prime} \circ \gamma$
$\gamma^\mu = \phi \circ \gamma$
$Y(f) = Y^\nu \frac{\partial f}{\partial y^\nu}$
$\frac{\partial f}{\partial x^\mu}$
$X^\mu = Y^\nu \frac{\partial x^\mu}{\partial y^\nu}$


Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dγ^ν(t)/dt |t=0
Y(f) = Y^ν ∂f/∂y^ν = Y^ν ∂x^μ/∂y^ν ∂f/∂x^μ


Fibre Bundles
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dγ^ν(t)/dt |t=0
γ^ν = φ'' ∘ γ
γ^μ = φ ∘ γ
γ^μ = φ ∘ φ'^{-1} ∘ φ' ∘ γ = φ ∘ φ'^{-1}(γ^ν)
Y(f) = Y^ν ∂f/∂y^ν = Y^ν ∂x^μ/∂y^ν ∂f/∂x^μ
X^μ = Y^ν ∂x^μ/∂y^ν


Fibre Bundles
Vector Bundles /
X(f) is given by acting on f with the operator X^μ ∂/∂x^μ
Y^ν = dY^ν(t)/dt |t=0
Y^ν = φ^-1 ∘ γ
γ^μ = φ ∘ Y
γ^μ = φ ∘ φ'^-1 ∘ φ' ∘ Y = φ ∘ φ'^-1 (γ^μ)
Y(f) = Y^ν ∂f/∂y^ν = Y^ν ∂x^μ/∂y^ν ∂f/∂x^μ
X^μ = Y^ν ∂x^μ/∂y^ν


X(f) is given by acting on f with the operator

X(f) is given by acting on f with the operator $X^\mu \frac{\partial}{\partial x^\mu}$

$$Y^\nu = \frac{d\gamma^\nu(t)}{dt}\Big|_{t=0} \qquad \gamma^\nu = \phi' \circ \gamma \qquad \gamma^\mu = \phi \circ \gamma$$

$$\gamma^\mu = \phi \circ \phi'^{-1} \circ \phi' \circ \gamma = \phi \circ \phi'^{-1}(\gamma^\nu)$$

$$Y(f) = Y^\nu \frac{\partial f(x(y))}{\partial y^\nu} = Y^\nu \frac{\partial x^\mu}{\partial y^\nu} \frac{\partial f}{\partial x^\mu} \qquad X^\mu = Y^\nu \frac{\partial x^\mu}{\partial y^\nu}$$

$$Y^\nu = \frac{\partial \gamma^\nu(y(t))}{\partial t} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial \gamma^\mu(x(t))}{\partial t} = \boxed{\frac{\partial y^\nu}{\partial x^\mu}} X^\mu$$