Title: Local and Global Properties of Green Functions in Black Hole Space-times

Date: Mar 17, 2011 01:00 PM

URL: http://pirsa.org/11030109

Abstract: The local and global properties of the retarded and Feynman Green functions to the wave equation in curved spacetime are crucial for radiation reaction in the classical theory and for renormalisation in the quantum quantum theory. Building on an insight due to Avramidi, we provide a system of transport equations for determining key fundamental geometrical bitensors determining the local Hadamard singularity structure of these GreenÃ¢ÂÂ€ÂÂ™s functions. We illustrate their use in a semi-recursive approach showing how to determine covariant expansions to high order, for example, calculating the tail term reflecting backscattering by the curvature of spacetime to 20th order in the geodesic separation in a matter of minutes, and as the basis of numerical calculations. We also present an efficient method to construct covariant expansions of the tail term, without using the formal Hadamard light-cone expansion. Finally we discuss the relationship between the geodesic structure, quasi-normal modes with associated excitation factors and the global behaviour of Green functions in black hole space-times.

Local and Global Properties of Green Functions in Black Hole Space-times

Adrian Ottewill

School of Mathematical Sciences

University College Dublin

UCD

17 March, 2011

Collaborators:

- Marc Casals, PI
- Sam Dolan, Southampton
- Anna Heffernan, UCD
- Barry Wardell, AEI-Gölm

Sponsors:

- Science Foundation Ireland
- Irish Research Council for Science Engineering and Technology

The motivation: classical

- EMRIs (BH/BH or BH/NS)
- $m \sim M_\odot, M \sim 10^3 - 10^8 M_\odot$
- LISA will see 10-1000 EMRIs/yr
- Outside range of NR or PN
- Perturb in mass ratio $\mu = m/M$
- Smaller mass treated as point particle
- At 0^{th} order particle follows a geodesic
- 1^{st} order correction can be described as a force - ‘the self force’

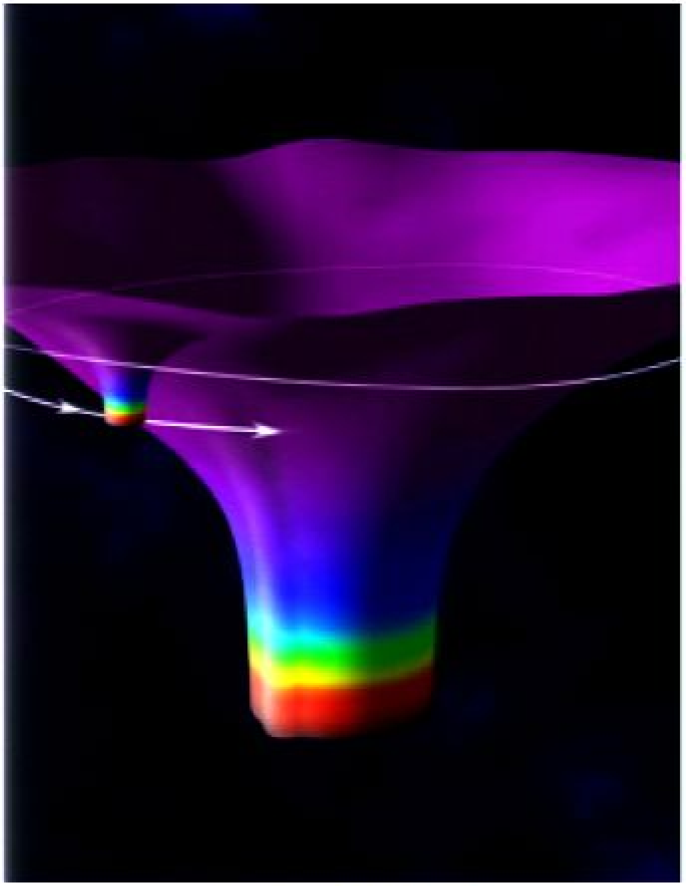


Image credit: NASA JPL

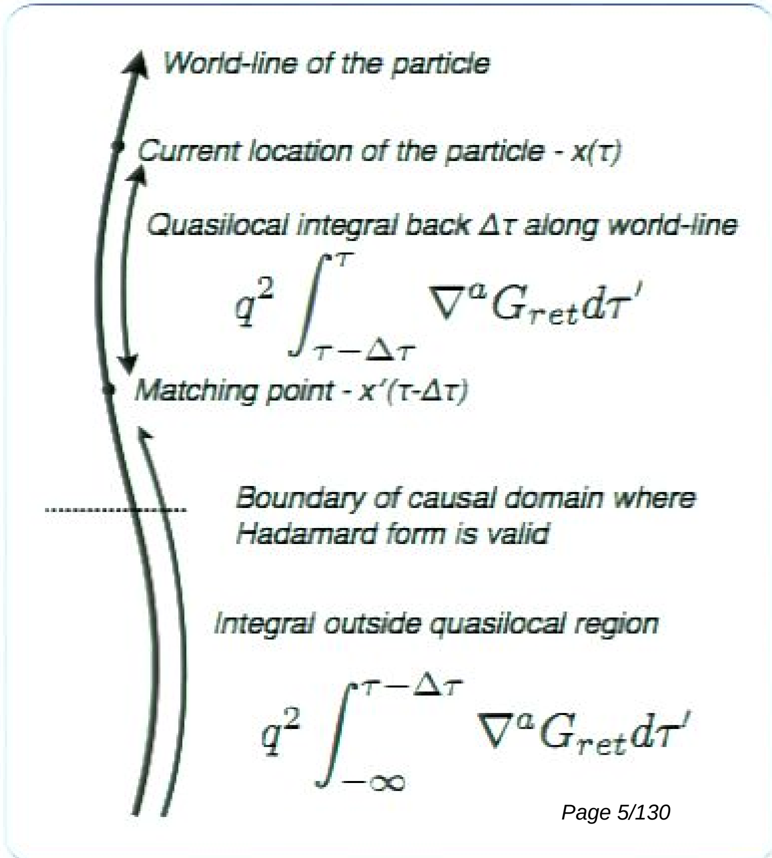
Matched expansions formalism for 'self-force'

$$\Phi_\mu^{\rm tail} = q \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} \nabla_\mu G_{\rm ret}(z(\tau), z(\tau'))d\tau'$$

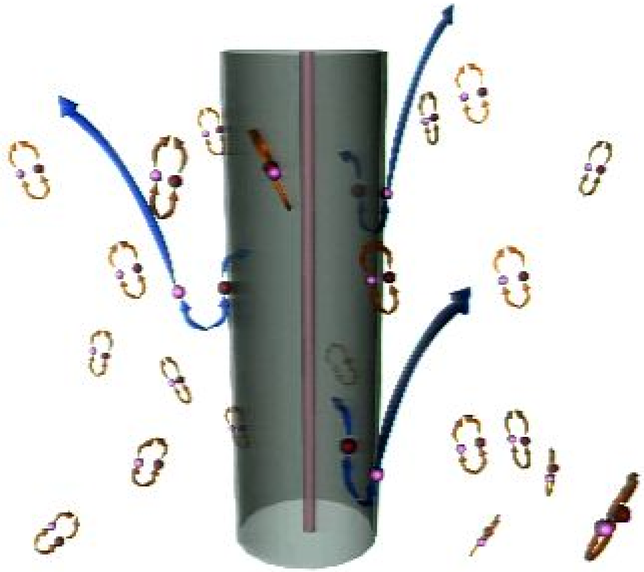
- Wiseman/Poisson/Anderson/...
- Separate integral into two parts:

  1. Quasi-local contribution from recent past

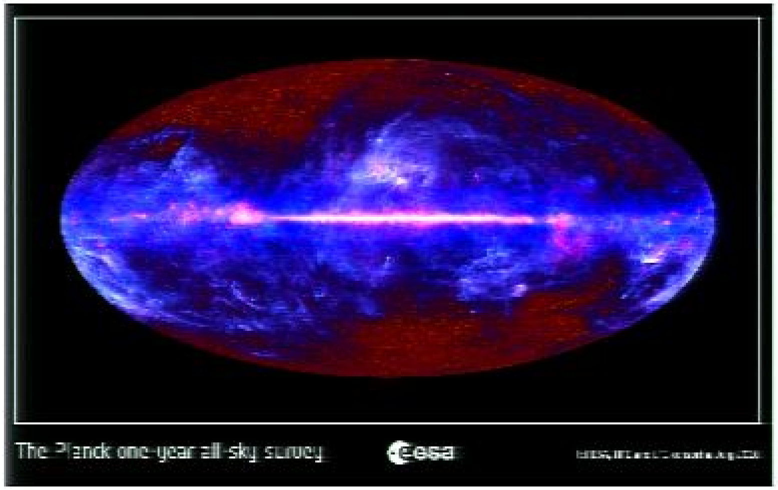
  2. Contribution from 'Distant Past'
- Calculate in each region separately
- Match at intermediate time '$\Delta\tau$'



The motivation: quantum



- Ultraviolet divergences lead to infinities
- Regularization requires a knowledge of singularity structure
- State construction requires a knowledge of global structure
- Renormalisation leads to geometrical anomalies



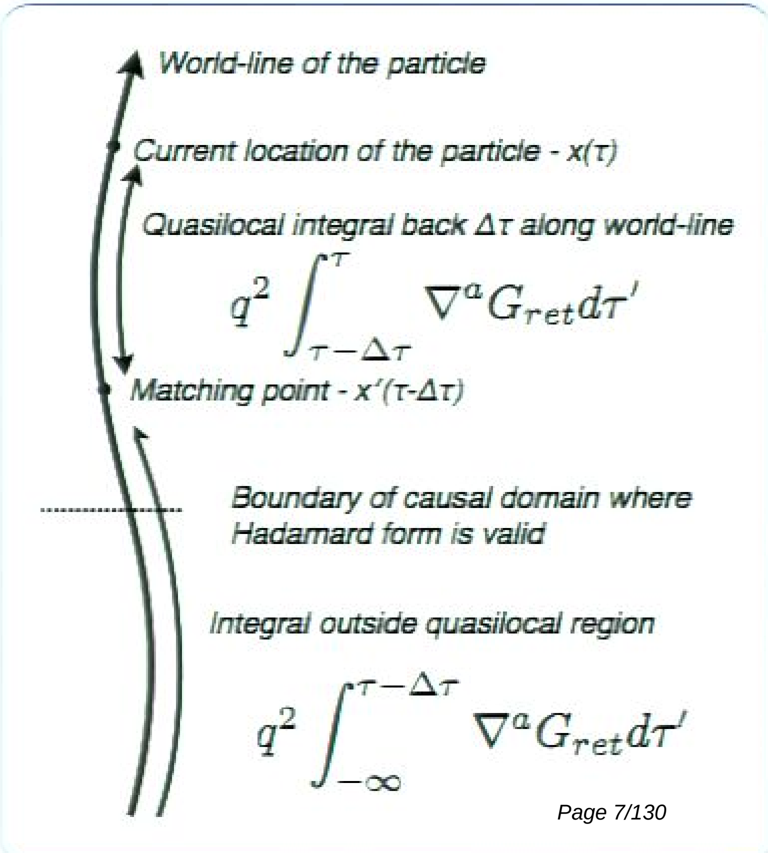
Matched expansions formalism for 'self-force'

$$\Phi_\mu^{\text{tail}} = q \lim_{\epsilon\to 0^+} \int_{-\infty}^{\tau-\epsilon} \nabla_\mu G_{\text{ret}}(z(\tau), z(\tau'))d\tau'$$

- Wiseman/Poisson/Anderson/...
- Separate integral into two parts:

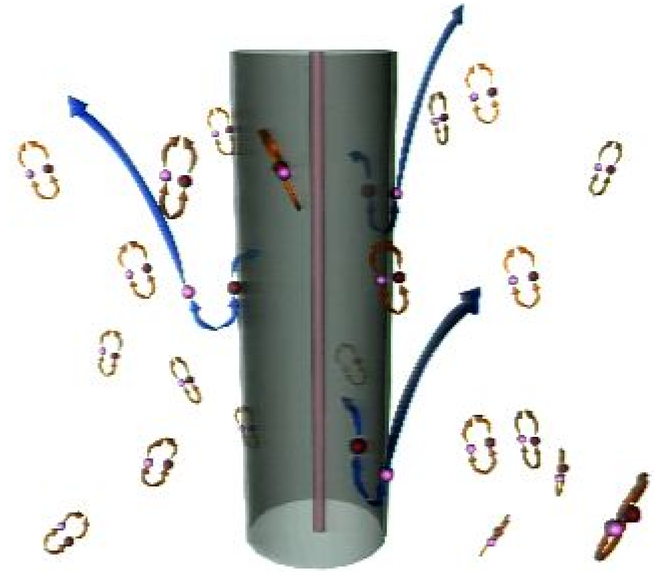
  1. Quasi-local contribution from recent past

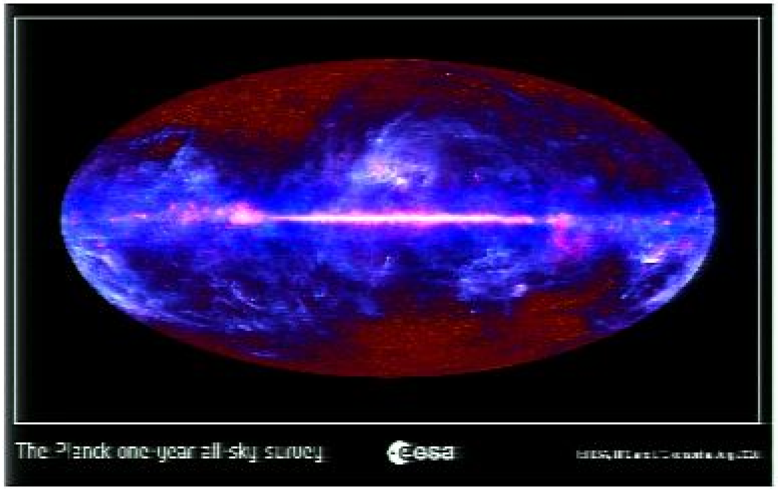
  2. Contribution from 'Distant Past'
- Calculate in each region separately
- Match at intermediate time '$\Delta\tau$'



The motivation: quantum

- Ultraviolet divergences lead to infinities
- Regularization requires a knowledge of singularity structure
- State construction requires a knowledge of global structure
- Renormalisation leads to geometrical anomalies





The fundamental formalism

$$\mathcal{D}^A{}_B = \delta^A{}_B(\Box - m^2) - P^A{}_B$$

$$\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta,$$

∇_α is the covariant derivative defined by a connection $\mathcal{A}^A{}_{B\alpha}$:

$$\nabla_\alpha\varphi^A = \partial_\alpha\varphi^A + \mathcal{A}^A{}_{B\alpha}\varphi^B$$

m is the mass of the field and $P^A{}_B(x)$ is a possible potential term.

The (Feynman and retarded) Green function satisfy the equation

$$\mathcal{D}^A{}_B G^B{}_{C'}(x, x') = -\delta^A{}_{C'}\delta(x, x').$$

The proper-time formalism

The identity $$i\int_0^\infty ds\, e^{-\epsilon s}\exp(isx) = -\frac{1}{x+i\epsilon}, \qquad (\epsilon > 0),$$

allows the Green function to be encapsulated in the formal expression

$$G^A{}_{C'}(x,x') = i\int_0^\infty ds\, e^{-\epsilon s}\exp(is\mathcal{D})^A{}_B\delta^B{}_{C'}\delta(x,x').$$

$$K^A{}_{C'}(x,x';s) = \exp(is\mathcal{D})^A{}_B\delta^B{}_{C'}\delta(x,x')$$

clearly satisfies the Schrödinger/heat equation

$$\frac{1}{i}\frac{\partial K^A{}_{C'}}{\partial s}(x,x';s) = \mathcal{D}^A{}_B K^B{}_{C'}(x,x';s) \qquad (1)$$

together with the initial condition $K^A{}_{B'}(x,x';0) = \delta^A{}_{B'}\delta(x,x')$.

The fundamental formalism

$$\mathcal{D}^A{}_B = \delta^A{}_B(\Box - m^2) - P^A{}_B$$

$$\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta,$$

∇_α is the covariant derivative defined by a connection $\mathcal{A}^A{}_{B\alpha}$:

$$\nabla_\alpha\varphi^A = \partial_\alpha\varphi^A + \mathcal{A}^A{}_{B\alpha}\varphi^B$$

m is the mass of the field and $P^A{}_B(x)$ is a possible potential term.

The (Feynman and retarded) Green function satisfy the equation

$$\mathcal{D}^A{}_B G^B{}_{C'}(x, x') = -\delta^A{}_{C'}\delta(x, x').$$

The proper-time formalism

The identity $$i\int_0^\infty \mathrm{d}s\, e^{-\epsilon s} \exp(isx) = -\frac{1}{x+i\epsilon}, \qquad (\epsilon > 0),$$

allows the Green function to be encapsulated in the formal expression

$$G^A{}_{C'}(x,x') = i\int_0^\infty \mathrm{d}s\, e^{-\epsilon s} \exp(is\mathcal{D})^A{}_B \delta^B{}_{C'} \delta(x,x').$$

$$K^A{}_{C'}(x,x';s) = \exp(is\mathcal{D})^A{}_B \delta^B{}_{C'} \delta(x,x')$$

clearly satisfies the Schrödinger/heat equation

$$\frac{1}{i}\frac{\partial K^A{}_{C'}}{\partial s}(x,x';s) = \mathcal{D}^A{}_B K^B{}_{C'}(x,x';s) \qquad (1)$$

together with the initial condition $K^A{}_{B'}(x,x';0) = \delta^A{}_{B'}\delta(x,x')$.

Relation between the massive and massless theories

The trivial way in which m enters these equations allows us to write

$$K^A{}_{C'}(x, x'; s) = e^{-im^2 s} K_0{}^A{}_{C'}(x, x'; s),$$

with the massless heat kernel satisfying the equation

$$\frac{1}{i}\frac{\partial K_0{}^A{}_{C'}}{\partial s}(x, x'; s) = (\delta^A{}_B \Box - P^A{}_B) K_0{}^B{}_{C'}(x, x'; s)$$

together with the 'initial condition' $K_0{}^A{}_{B'}(x, x'; 0) = \delta^A{}_{B'}\delta(x, x')$.

Relation between the massive and massless theories

The trivial way in which m enters these equations allows us to write

$$K^A{}_{C'}(x, x'; s) = e^{-im^2 s} K_0{}^A{}_{C'}(x, x'; s),$$

with the massless heat kernel satisfying the equation

$$\frac{1}{i}\frac{\partial K_0{}^A{}_{C'}}{\partial s}(x, x'; s) = (\delta^A{}_B \Box - P^A{}_B) K_0{}^B{}_{C'}(x, x'; s)$$

together with the 'initial condition' $K_0{}^A{}_{B'}(x, x'; 0) = \delta^A{}_{B'} \delta(x, x')$.

A suitable representation for $\delta(x, x')$: global version

Sturm-Liouville approach: build a complete set of eigenfunctions by

$$\mathcal{D}^A{}_B u_i^B = -\lambda_i u_i^A$$

normalised so that

$$\int u_i^A u_{jA} \, \mathrm{d}x = \delta_{ij}$$

then

$$\sum_i u_i^A(x) u_{iB'}(x') \, \mathrm{d}x = \delta^A{}_{B'} \delta(x, x')$$

so

$$K^A{}_{B'}(x, x'; s) = \sum_i e^{-i(\lambda_i + m^2)s} u_i^A(x) u_{iB'}(x')$$

$$G^A{}_{B'}(x, x') = \sum_i \frac{u_i^A(x) u_{iB'}(x')}{\lambda_i + m^2}$$

A suitable representation for $\delta(x, x')$: global version

Sturm-Liouville approach: build a complete set of eigenfunctions by

$$\mathcal{D}^A{}_B u_i^B = -\lambda_i u_i^A$$

normalised so that

$$\int u_i^A u_{jA}\, \mathrm{d}x = \delta_{ij}$$

then

$$\sum_i u_i^A(x) u_{iB'}(x')\, \mathrm{d}x = \delta^A{}_{B'}\delta(x, x')$$

so

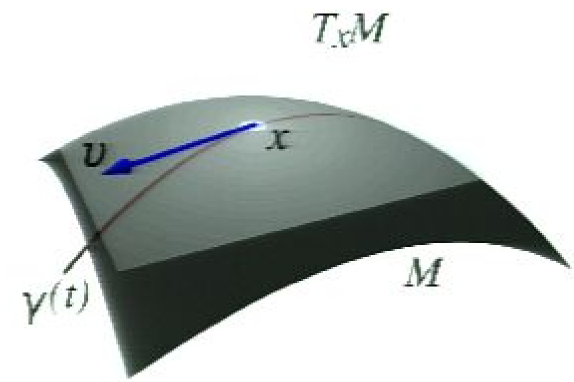
$$K^A{}_{B'}(x, x'; s) = \sum_i e^{-i(\lambda_i + m^2)s} u_i^A(x) u_{iB'}(x')$$

$$G^A{}_{B'}(x, x') = \sum_i \frac{u_i^A(x) u_{iB'}(x')}{\lambda_i + m^2}$$

A suitable representation for $\delta(x, x')$: local version

Cover the region close to x with a geodesics: $P = \mathcal{G}(\lambda, v^a \left(\frac{\partial}{\partial x^a}\right)_x)$

Invariance: $P = \mathcal{G}(\lambda, v) = \mathcal{G}(\lambda/\alpha, \alpha v)$



Two natural choices:

$\mathcal{G}(1, y^a \left(\frac{\partial}{\partial x^a}\right)_x)$ (Riemann normal coordinates)

$\mathcal{G}(\tau, u^a \left(\frac{\partial}{\partial x^a}\right)_x)$ $g_{ab}(x)u^a u^b = -1$ (timelike case)

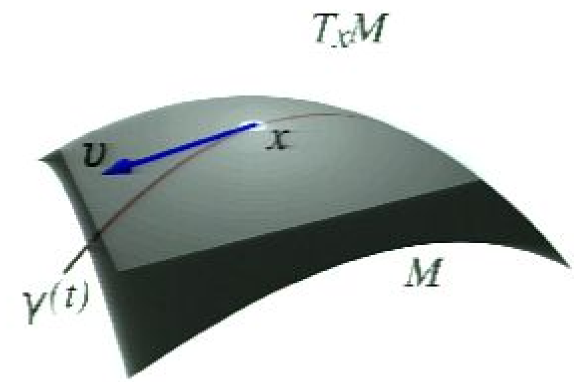
Bunch-Parker: Using Riemann normal coordinates y^a based at x write

$$\delta(x, x') = \frac{1}{(2\pi)^4} \int e^{ik_a y^a} \, \mathrm{d}^4 k$$

A suitable representation for $\delta(x, x')$: local version

Cover the region close to x with a geodesics: $P = \mathcal{G}(\lambda, v^a \left(\frac{\partial}{\partial x^a}\right)_x)$

Invariance: $P = \mathcal{G}(\lambda, v) = \mathcal{G}(\lambda/\alpha, \alpha v)$



Two natural choices:

$\mathcal{G}(1, y^a \left(\frac{\partial}{\partial x^a}\right)_x)$ (Riemann normal coordinates)

$\mathcal{G}(\tau, u^a \left(\frac{\partial}{\partial x^a}\right)_x)$ $g_{ab}(x) u^a u^b = -1$ (timelike case)

Bunch-Parker: Using Riemann normal coordinates y^a based at x write

$$\delta(x, x') = \frac{1}{(2\pi)^4} \int e^{ik_a y^a} \, d^4 k$$

$$\mathcal{D}' = (\eta^{ab}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b} - m^2) + \delta\mathcal{D}' = \mathcal{D}'_\flat + \delta\mathcal{D}'$$

where, for example, for a scalar field with $P = \xi R$

$$\delta\mathcal{D}' = -(\xi - \tfrac{1}{6})R - \tfrac{1}{3}R_a{}^p y^a \frac{\partial}{\partial y^p} + \tfrac{1}{3}R^p{}_a{}^q{}_b y^a y^b \frac{\partial}{\partial y^p}\frac{\partial}{\partial y^q}$$
$$- (\xi - \tfrac{1}{6})R_{;a}y^a + (-\tfrac{1}{3}R_a{}^p{}_{;b} + \tfrac{1}{6}R_{ab}{}^{;p})y^a y^b \frac{\partial}{\partial y^p} \cdots$$

Then by a Zassenhaus-like formula

$$\exp(is\mathcal{D}') = (1 + is\delta\mathcal{D}' + \tfrac{1}{2}(is)^2([\mathcal{D}'_\flat, \delta\mathcal{D}'] + \delta\mathcal{D}'^2) + \cdots)\exp(is\mathcal{D}'_\flat)$$

Then

$$K_\flat(x, x'; s) = \exp(is\mathcal{D}'_\flat)\frac{1}{(2\pi)^4}\int e^{ik_a y^a}\, d^4k$$

$$= \frac{1}{(2\pi)^4}\int e^{-is(k_a k^a + m^2) + ik_a y^a}\, d^4k = \frac{1}{(4\pi s)^2} e^{-im^2 s} e^{-y_a y^a/(4is)}$$

Elementary calculation gives

$$[\mathcal{D}'_\flat, \delta\mathcal{D}'] = -\tfrac{2}{3} R^{pq} \frac{\partial}{\partial y^p}\frac{\partial}{\partial y^q}$$

and so

$$K(x, x'; s) = (1 + (is)(-(\xi - \tfrac{1}{6})R + \tfrac{1}{12}\tfrac{1}{is}R_{ab}y^a y^b + \cdots)) + \cdots$$
$$= (1 + \tfrac{1}{12}R_{ab}y^a y^b + \cdots) + (is)(-(\xi - \tfrac{1}{6})R + \cdots)$$

Thus $K(x, x'; s)$ constructed in this way has the form (DeWitt)

$$K(x, x'; s) = \frac{1}{(4\pi s)^2} e^{-im^2 s} e^{-\frac{y_a y^a}{4is}} \sum_{r=0}^{\infty} a_r(y^a)(is)^r ,$$

where the $a_r(y^a)$ possess Taylor series and $a_0(0) = 1$.

$$\mathcal{D}' = (\eta^{ab}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b} - m^2) + \delta\mathcal{D}' = \mathcal{D}'_\flat + \delta\mathcal{D}'$$

where, for example, for a scalar field with $P = \xi R$

$$\delta\mathcal{D}' = -(\xi - \tfrac{1}{6})R - \tfrac{1}{3}R_a{}^p y^a \frac{\partial}{\partial y^p} + \tfrac{1}{3}R^p{}_a{}^q{}_b y^a y^b \frac{\partial}{\partial y^p}\frac{\partial}{\partial y^q}$$
$$- (\xi - \tfrac{1}{6})R_{;a}y^a + (-\tfrac{1}{3}R_a{}^p{}_{;b} + \tfrac{1}{6}R_{ab}{}^{;p})y^a y^b \frac{\partial}{\partial y^p} \cdots$$

Then by a Zassenhaus-like formula

$$\exp(is\mathcal{D}') = (1 + is\delta\mathcal{D}' + \tfrac{1}{2}(is)^2([\mathcal{D}'_\flat, \delta\mathcal{D}'] + \delta\mathcal{D}'^2) + \cdots)\exp(is\mathcal{D}'_\flat)$$

Then

$$K_\flat(x, x'; s) = \exp(is\mathcal{D}'_\flat)\frac{1}{(2\pi)^4}\int e^{ik_a y^a}\,\mathrm{d}^4 k$$
$$= \frac{1}{(2\pi)^4}\int e^{-is(k_a k^a + m^2) + ik_a y^a}\,\mathrm{d}^4 k = \frac{1}{(4\pi s)^2} e^{-im^2 s} e^{-y_a y^a/(4is)}$$

Elementary calculation gives

$$[\mathcal{D}'_\flat, \delta\mathcal{D}'] = -\tfrac{2}{3} R^{pq} \frac{\partial}{\partial y^p} \frac{\partial}{\partial y^q}$$

and so

$$\begin{aligned} K(x, x'; s) &= (1 + (is)(-(\xi - \tfrac{1}{6})R + \tfrac{1}{12}\tfrac{1}{is} R_{ab} y^a y^b + \cdots)) + \cdots \\ &= (1 + \tfrac{1}{12} R_{ab} y^a y^b + \cdots) + (is)(-(\xi - \tfrac{1}{6})R + \cdots) \end{aligned}$$

Thus $K(x, x'; s)$ constructed in this way has the form (DeWitt)

$$K(x, x'; s) = \frac{1}{(4\pi s)^2} e^{-im^2 s} e^{-\frac{y_a y^a}{4is}} \sum_{r=0}^{\infty} a_r(y^a)(is)^r \ ,$$

where the $a_r(y^a)$ possess Taylor series and $a_0(0) = 1$.

The Synge world function (democracy restored!)

Well known can obtain geodesics from an action principle:

$$S[x(\tau)] \propto \int_{\tau_0}^{\tau_1} g_{ab}(x(\tau)) \frac{\mathrm{d}x^a}{\mathrm{d}\tau} \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \mathrm{d}\tau \qquad x(\tau_0) = x \quad x(\tau_1) = x'$$

where τ is proper time. Independent of affine parameter choosing

$$S[x(\lambda)] = \tfrac{1}{2}(\lambda_1 - \lambda_0) \int_{\lambda_0}^{\lambda_1} g_{ab}(x(\lambda)) \frac{\mathrm{d}x^a}{\mathrm{d}\lambda} \frac{\mathrm{d}x^b}{\mathrm{d}\lambda} \mathrm{d}\lambda \qquad x(\lambda_0) = x \quad x(\lambda_1) = x$$

corresponding to Lagrangian

$$L[x^a, \dot{x}^a] = \tfrac{1}{2} \Delta\lambda g_{ab}(x) \dot{x}^a \dot{x}^b \qquad \text{where } \dot{x}^a = \frac{\mathrm{d}x^a}{\mathrm{d}\lambda}, \quad \Delta\lambda = \lambda_1 - \lambda_0$$

Under a variation $x(\tau) \to x(\tau) + \delta x(\tau)$

$$\delta S = \int_{\lambda_0}^{\lambda_1} \left[\frac{\partial L}{\partial x^a} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^a} \right] \delta x^a \, \mathrm{d}\lambda + \left[\frac{\partial L}{\partial \dot{x}^a} \delta x^a \right]_{\lambda_0}^{\lambda_1}$$

Then

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda} = \frac{\partial L}{\partial x^a} \dot{x}^a + \frac{\partial L}{\partial \dot{x}^a} \ddot{x}^a = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^a} \dot{x}^a \right) = 2 \frac{\mathrm{d}L}{\mathrm{d}\lambda}$$

so L is a constant of the motion and $S = \Delta\lambda \, L$:

- In normal coordinates: $\Delta\lambda = 1$, $L = S = \frac{1}{2} \eta_{ab} y^a y^b$
- Using arc length: $\Delta\lambda = \tau$, $L = \frac{1}{2}\tau$, $S = \frac{1}{2}\tau^2$

Furthermore $\delta x^a(\lambda_1) = \delta x^{a'}$, $\delta x^a(\lambda_0) = \delta x^a$ so

$$\frac{\partial S}{\partial x^{a'}} = \frac{\partial L}{\partial \dot{x}^{a'}} = \Delta\lambda g_{a'b'} \dot{x}^{b'} \qquad \frac{\partial S}{\partial x^a} = -\frac{\partial L}{\partial \dot{x}^a} = -\Delta\lambda g_{ab} \dot{x}^b$$

Under a variation $x(\tau) \to x(\tau) + \delta x(\tau)$

$$\delta S = \int_{\lambda_0}^{\lambda_1} \left[\frac{\partial L}{\partial x^a} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^a} \right] \delta x^a \, d\lambda + \left[\frac{\partial L}{\partial \dot{x}^a} \delta x^a \right]_{\lambda_0}^{\lambda_1}$$

Then

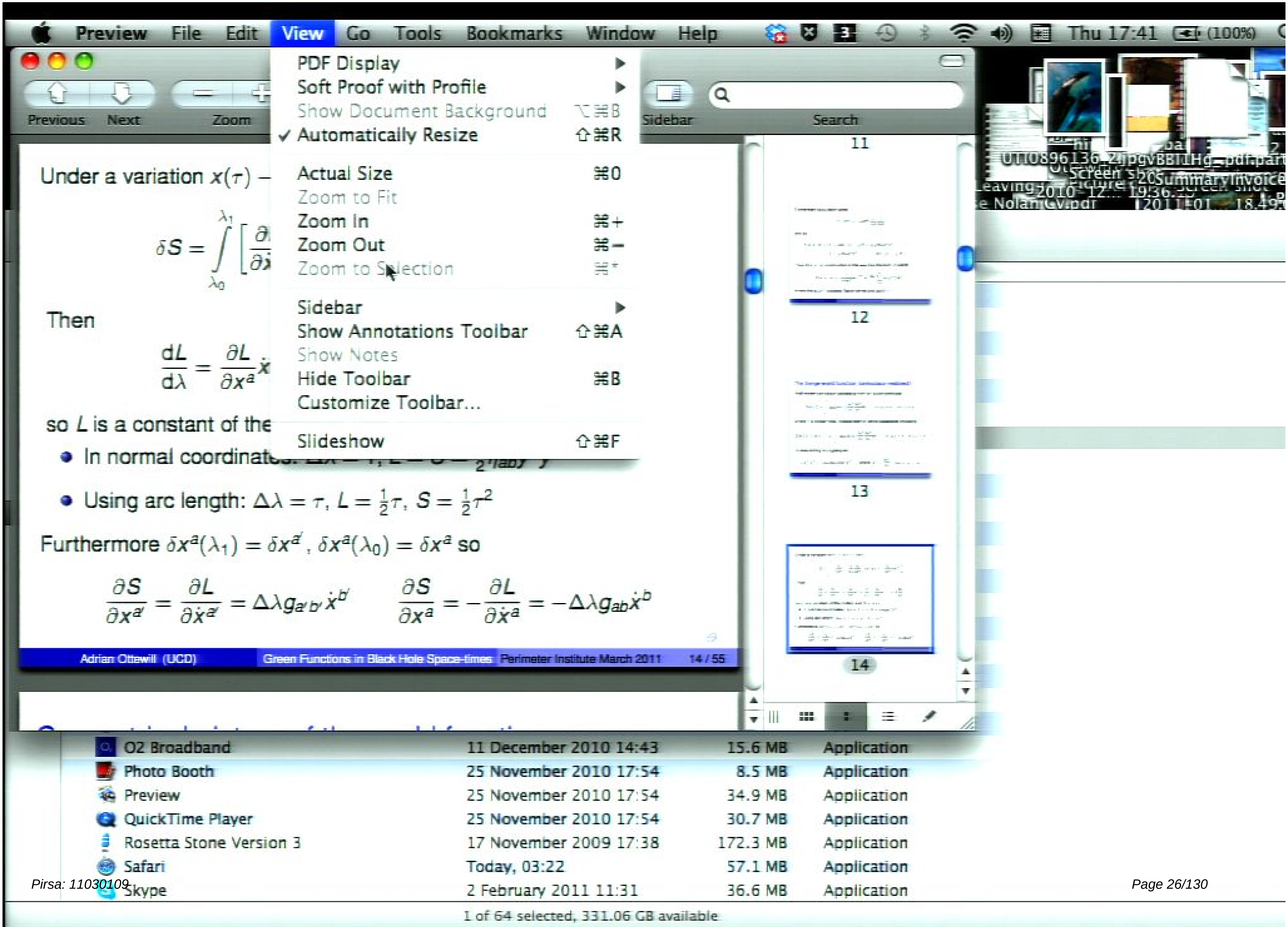
$$\frac{dL}{d\lambda} = \frac{\partial L}{\partial x^a} \dot{x}^a + \frac{\partial L}{\partial \dot{x}^a} \ddot{x}^a = \frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^a} \dot{x}^a \right) = 2 \frac{dL}{d\lambda}$$

so L is a constant of the motion and $S = \Delta\lambda\, L$:

- In normal coordinates: $\Delta\lambda = 1$, $L = S = \frac{1}{2}\eta_{ab} y^a y^b$
- Using arc length: $\Delta\lambda = \tau$, $L = \frac{1}{2}\tau$, $S = \frac{1}{2}\tau^2$

Furthermore $\delta x^a(\lambda_1) = \delta x^{a'}$, $\delta x^a(\lambda_0) = \delta x^a$ so

$$\frac{\partial S}{\partial x^{a'}} = \frac{\partial L}{\partial \dot{x}^{a'}} = \Delta\lambda\, g_{a'b'} \dot{x}^{b'} \qquad \frac{\partial S}{\partial x^a} = -\frac{\partial L}{\partial \dot{x}^a} = -\Delta\lambda\, g_{ab} \dot{x}^b$$


Preview File Edit View Go Tools Bookmarks Window Help
Thu 17:41 (100%)
Previous Next Zoom Sidebar Search
PDF Display
Soft Proof with Profile
Show Document Background
Automatically Resize
Actual Size
Zoom to Fit
Zoom In
Zoom Out
Zoom to Selection
Sidebar
Show Annotations Toolbar
Show Notes
Hide Toolbar
Customize Toolbar...
Slideshow
Under a variation x(τ) –
Then
so L is a constant of the
In normal coordinat
Using arc length: Δλ = τ, L = ½τ, S = ½τ²
Furthermore δxᵃ(λ₁) = δxᵃ′, δxᵃ(λ₀) = δxᵃ so
∂S/∂xᵃ′ = ∂L/∂ẋᵃ′ = Δλ gₐ′b′ ẋᵇ′   ∂S/∂xᵃ = −∂L/∂ẋᵃ = −Δλ gₐb ẋᵇ
Adrian Ottewill (UCD) Green Functions in Black Hole Space-times Perimeter Institute March 2011 14 / 55
11 12 13 14
O2 Broadband 11 December 2010 14:43 15.6 MB Application
Photo Booth 25 November 2010 17:54 8.5 MB Application
Preview 25 November 2010 17:54 34.9 MB Application
QuickTime Player 25 November 2010 17:54 30.7 MB Application
Rosetta Stone Version 3 17 November 2009 17:38 172.3 MB Application
Safari Today, 03:22 57.1 MB Application
Skype 2 February 2011 11:31 36.6 MB Application
1 of 64 selected, 331.06 GB available

Under a variation $x(\tau) \to x(\tau) + \delta x(\tau)$

$$\delta S = \int_{\lambda_0}^{\lambda_1} \left[\frac{\partial L}{\partial x^a} - \frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^a} \right] \delta x^a \, d\lambda + \left[\frac{\partial L}{\partial \dot{x}^a} \delta x^a \right]_{\lambda_0}^{\lambda_1}$$

Then

$$\frac{dL}{d\lambda} = \frac{\partial L}{\partial x^a}\dot{x}^a + \frac{\partial L}{\partial \dot{x}^a}\ddot{x}^a = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\dot{x}^a \right) = 2\frac{dL}{d\lambda}$$

so L is a constant of the motion and $S = \Delta\lambda \, L$:

- In normal coordinates: $\Delta\lambda = 1$, $L = S = \frac{1}{2}\eta_{ab} y^a y^b$
- Using arc length: $\Delta\lambda = \tau$, $L = \frac{1}{2}\tau$, $S = \frac{1}{2}\tau^2$

Furthermore $\delta x^a(\lambda_1) = \delta x^{a'}$, $\delta x^a(\lambda_0) = \delta x^a$ so

$$\frac{\partial S}{\partial x^{a'}} = \frac{\partial L}{\partial \dot{x}^{a'}} = \Delta\lambda g_{a'b'} \dot{x}^{b'} \qquad \frac{\partial S}{\partial x^a} = -\frac{\partial L}{\partial \dot{x}^a} = -\Delta\lambda g_{ab} \dot{x}^b$$

Under a variation $x(\tau) \to x(\tau) + \delta x(\tau)$

$$\delta S = \int_{\lambda_0}^{\lambda_1} \left[\frac{\partial L}{\partial \dot{x}^a} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^a} \right] \delta x^a \, \mathrm{d}\lambda + \left[\frac{\partial L}{\partial \dot{x}^a} \delta x^a \right]_{\lambda_0}^{\lambda_1}$$

Then

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda} = \frac{\partial L}{\partial x^a} \dot{x}^a + \frac{\partial L}{\partial \dot{x}^a} \ddot{x}^a = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^a} \dot{x}^a \right) = 2 \frac{\mathrm{d}L}{\mathrm{d}\lambda}$$

so L is a constant of the motion and $S = \Delta\lambda \, L$:

- In normal coordinates: $\Delta\lambda = 1$, $L = S = \frac{1}{2} \eta_{ab} y^a y^b$
- Using arc length: $\Delta\lambda = \tau$, $L = \frac{1}{2}\tau$, $S = \frac{1}{2}\tau^2$

Furthermore $\delta x^a(\lambda_1) = \delta x^{a'}$, $\delta x^a(\lambda_0) = \delta x^a$ so

$$\frac{\partial S}{\partial x^{a'}} = \frac{\partial L}{\partial \dot{x}^{a'}} = \Delta\lambda g_{a'b'} \dot{x}^{b'} \qquad \frac{\partial S}{\partial x^{a}} = -\frac{\partial L}{\partial \dot{x}^{a}} = -\Delta\lambda g_{ab} \dot{x}^{b}$$

Under a variation $x(\tau) \to x(\tau) + \delta x(\tau)$

$$\delta S = \int_{\lambda_0}^{\lambda_1} \left[\frac{\partial L}{\partial \dot{x}^a} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^a} \right] \delta x^a \, \mathrm{d}\lambda + \left[\frac{\partial L}{\partial \dot{x}^a} \delta x^a \right]_{\lambda_0}^{\lambda_1}$$

Then

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda} = \frac{\partial L}{\partial x^a} \dot{x}^a + \frac{\partial L}{\partial \dot{x}^a} \ddot{x}^a = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^a} \dot{x}^a \right) = 2 \frac{\mathrm{d}L}{\mathrm{d}\lambda}$$

so L is a constant of the motion and $S = \Delta\lambda\, L$:

- In normal coordinates: $\Delta\lambda = 1$, $L = S = \frac{1}{2}\eta_{ab} y^a y^b$
- Using arc length: $\Delta\lambda = \tau$, $L = \frac{1}{2}\tau$, $S = \frac{1}{2}\tau^2$

Furthermore $\delta x^a(\lambda_1) = \delta x^{a'}$, $\delta x^a(\lambda_0) = \delta x^a$ so

$$\frac{\partial S}{\partial x^{a'}} = \frac{\partial L}{\partial \dot{x}^{a'}} = \Delta\lambda g_{a'b'} \dot{x}^{b'} \qquad \frac{\partial S}{\partial x^a} = -\frac{\partial L}{\partial \dot{x}^a} = -\Delta\lambda g_{ab} \dot{x}^b$$

Under a variation $x(\tau) \to x(\tau) + \delta x(\tau)$

$$\delta S = \int_{\lambda_0}^{\lambda_1} \left[\frac{\partial L}{\partial \dot{x}^a} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^a} \right] \delta x^a \, \mathrm{d}\lambda + \left[\frac{\partial L}{\partial \dot{x}^a} \delta x^a \right]_{\lambda_0}^{\lambda_1}$$

Then

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda} = \frac{\partial L}{\partial x^a} \dot{x}^a + \frac{\partial L}{\partial \dot{x}^a} \ddot{x}^a = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial L}{\partial \dot{x}^a} \dot{x}^a \right) = 2 \frac{\mathrm{d}L}{\mathrm{d}\lambda}$$

so L is a constant of the motion and $S = \Delta\lambda \, L$:

- In normal coordinates: $\Delta\lambda = 1$, $L = S = \frac{1}{2} \eta_{ab} y^a y^b$
- Using arc length: $\Delta\lambda = \tau$, $L = \frac{1}{2}\tau$, $S = \frac{1}{2}\tau^2$

Furthermore $\delta x^a(\lambda_1) = \delta x^{a'}$, $\delta x^a(\lambda_0) = \delta x^a$ so

$$\frac{\partial S}{\partial x^{a'}} = \frac{\partial L}{\partial \dot{x}^{a'}} = \Delta\lambda g_{a'b'} \dot{x}^{b'} \qquad \frac{\partial S}{\partial x^a} = -\frac{\partial L}{\partial \dot{x}^a} = -\Delta\lambda g_{ab} \dot{x}^b$$

Under a variation $x(\tau) \to x(\tau) + \delta x(\tau)$

$$\delta S = \int_{\lambda_0}^{\lambda_1} \left[\frac{\partial L}{\partial \dot{x}^a} - \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^a} \right] \delta x^a \, \mathrm{d}\lambda + \left[\frac{\partial L}{\partial \dot{x}^a} \delta x^a \right]_{\lambda_0}^{\lambda_1}$$

Then

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda} = \frac{\partial L}{\partial x^a}\dot{x}^a + \frac{\partial L}{\partial \dot{x}^a}\ddot{x}^a = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x}^a}\dot{x}^a \right) = 2\frac{\mathrm{d}L}{\mathrm{d}\lambda}$$

so L is a constant of the motion and $S = \Delta\lambda \, L$:

- In normal coordinates: $\Delta\lambda = 1$, $L = S = \frac{1}{2}\eta_{ab} y^a y^b$
- Using arc length: $\Delta\lambda = \tau$, $L = \frac{1}{2}\tau$, $S = \frac{1}{2}\tau^2$

Furthermore $\delta x^a(\lambda_1) = \delta x^{a'}$, $\delta x^a(\lambda_0) = \delta x^a$ so

$$\frac{\partial S}{\partial x^{a'}} = \frac{\partial L}{\partial \dot{x}^{a'}} = \Delta\lambda g_{a'b'} \dot{x}^{b'} \qquad \frac{\partial S}{\partial x^a} = -\frac{\partial L}{\partial \dot{x}^a} = -\Delta\lambda g_{ab} \dot{x}^b$$

Geometrical picture of the world function

What about normalisation?

$$S_{;a'} = \frac{\partial S}{\partial x^{a'}} = \Delta\lambda g_{a'b'}\dot{x}^{b'} \qquad S_{;a} = \frac{\partial S}{\partial x^{a}} = -\Delta\lambda g_{ab}\dot{x}^{b}$$

so

$$g^{a'b'}S_{;a'}S_{;b'} = = \Delta\lambda^2 g_{a'b'}\dot{x}^{a'}\dot{x}^{b'} = 2\Delta\lambda L = 2S$$

$$g^{ab}S_{;a}S_{;b} = = \Delta\lambda^2 g_{ab}\dot{x}^{a}\dot{x}^{b} = 2\Delta\lambda L = 2S$$

These equations together with the initial condition

$$S[x(\tau)] \sim \frac{1}{2}(\mathrm{d}\lambda)^2 g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = \frac{1}{2}g_{ab}\mathrm{d}x^a\mathrm{d}x^b$$

totally determine S and hence the local geodesic structure.

- Conventionally use σ not S.

Geometrical picture of the world function

What about normalisation?

$$S_{;a'} = \frac{\partial S}{\partial x^{a'}} = \Delta\lambda g_{a'b'}\dot{x}^{b'} \qquad S_{;a} = \frac{\partial S}{\partial x^{a}} = -\Delta\lambda g_{ab}\dot{x}^{b}$$

so

$$g^{a'b'}S_{;a'}S_{;b'} == \Delta\lambda^2 g_{a'b'}\dot{x}^{a'}\dot{x}^{b'} = 2\Delta\lambda L = 2S$$

$$g^{ab}S_{;a}S_{;b} == \Delta\lambda^2 g_{ab}\dot{x}^{a}\dot{x}^{b} = 2\Delta\lambda L = 2S$$

These equations together with the initial condition

$$S[x(\tau)] \sim \frac{1}{2}(\mathrm{d}\lambda)^2 g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = \frac{1}{2}g_{ab}\mathrm{d}x^a\mathrm{d}x^b$$

totally determine S and hence the local geodesic structure.

- Conventionally use σ not S.

Geometrical picture of the world function

What about normalisation?

$$S_{;a'} = \frac{\partial S}{\partial x^{a'}} = \Delta\lambda g_{a'b'}\dot{x}^{b'} \qquad S_{;a} = \frac{\partial S}{\partial x^{a}} = -\Delta\lambda g_{ab}\dot{x}^{b}$$

so

$$g^{a'b'}S_{;a'}S_{;b'} == \Delta\lambda^2 g_{a'b'}\dot{x}^{a'}\dot{x}^{b'} = 2\Delta\lambda L = 2S$$

$$g^{ab}S_{;a}S_{;b} == \Delta\lambda^2 g_{ab}\dot{x}^{a}\dot{x}^{b} = 2\Delta\lambda L = 2S$$

These equations together with the initial condition

$$S[x(\tau)] \sim \frac{1}{2}(\mathrm{d}\lambda)^2 g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\lambda}\frac{\mathrm{d}x^b}{\mathrm{d}\lambda} = \frac{1}{2}g_{ab}\mathrm{d}x^a\mathrm{d}x^b$$

totally determine S and hence the local geodesic structure.

- Conventionally use σ not S.

World function expression of the DeWitt ansatz

$$K^A{}_{B'}(x,x';s) \sim \frac{1}{(4\pi s)^2} e^{-im^2 s} e^{-\frac{\sigma(x,x')}{2is}} \Omega^A{}_{B'}(x,x';s)$$

where $\Omega^A{}_{B'}(x,x';s)$ possesses the asymptotic expansion as $s \to 0+$

$$\Omega^A{}_{B'}(x,x';s) \sim \sum_{r=0}^{\infty} a_r{}^A{}_{B'}(x,x')(is)^r$$

where $a_0{}^A{}_B(x,x) = \delta^A{}_B$ and $a_r{}^A{}_{B'}(x,x')$ has dimension $(\text{length})^{-2r}$

Notes:

- Effect of topology/boundaries - terms in $\Omega^A{}_{B'}(x,x';s)$ not analytic in s at $s=0$
- Can show $a_0{}^A{}_{B'}(x,x') = \Delta^{1/2}(x,x')\delta^A{}_{B'}$ and write $\bar{a}_r{}^A{}_{B'} = a_r{}^A{}_{B'}/\Delta^{1/2}$

Can one hear the shape of a drum?

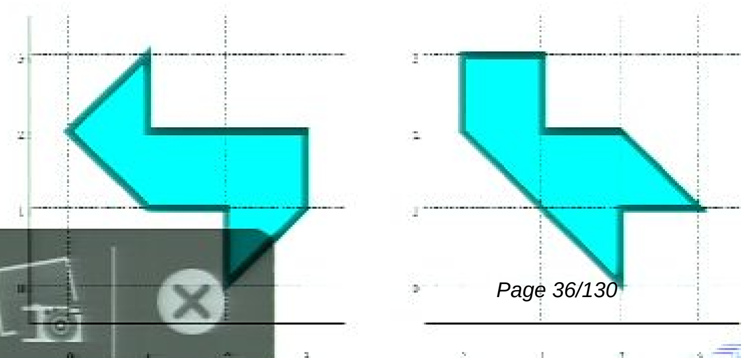
$$K^A{}_{B'}(x, x'; s) = \sum_i e^{-i(\lambda_i + m^2)s} u_i^A(x) u_{iB'}(x')$$

$$\Rightarrow \quad \int dx\, K^A{}_A(x, x; s) = e^{-im^2 s} \sum_i e^{-i\lambda_i s}$$

$$K^A{}_{B'}(x, x'; s) \sim \frac{1}{(4\pi s)^2} e^{-im^2 s} e^{-\frac{\sigma(x,x')}{2is}} \sum_{r=0}^{\infty} a_r^A{}_{B'}(x, x')(is)^r$$

$$\Rightarrow \quad \int dx\, K^A{}_A(x, x; s) \sim e^{-im^2 s} \frac{1}{(4\pi s)^2} \sum_{r=0}^{\infty} \int dx\, a_r^A{}_A(x, x)(is)^r$$

- These ‘diagonal’ coefficients $a_r^A{}_A(x, x)$ are also associated with Hadamard, Minakshisundaram, Seeley, ...



DeWitt recursion relations

Requiring the DeWitt expansion satisfy the wave equation requires the recursion relations

$$\sigma^{;\alpha'} a_{r+1}{}^{AB'}{}_{;\alpha'} + (r+1)\, a_{r+1}{}^{AB'} - (\delta^{B'}{}_{C'}\Box - P^{B'}{}_{C'}) a_r{}^{AC'} = 0$$

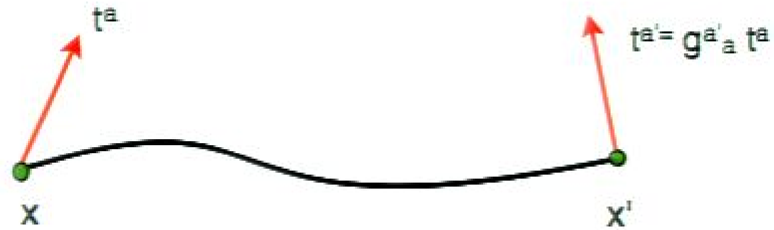
for $r \in \mathbb{N}$ along with the 'initial condition'

$$\sigma^{;\alpha'} a_0{}^{AB'}{}_{;\alpha'} \;=\; \tfrac{1}{2}(4 - \Box'\sigma) a_0{}^{AB'}$$

with the implicit requirement that they be regular as $x' \to x$.

Parallel transport

Can parallel transport any tensor along our geodesic



$$\dot{x}^{b'} t^{a'}{}_{;b'} = 0 \Leftrightarrow \sigma^{b'} t^{a'}{}_{;b'} = 0$$

$$t^{a'} = g^{a'}{}_b t^b$$

$$\sigma^{c'} g^{a'}{}_{b;c'} = 0 \qquad \text{with} \qquad g^{a'}{}_b|_{x'=x} = \delta^a{}_b$$

Special case: $\sigma^{;a'} = -g^{a'}{}_b \sigma^{;b}$

The VanVleck-Morette determinant: definition

A key role is played by 2nd derivatives, start with: $\Delta^{a'}{}_{b'} = g^{a'}{}_{c}\sigma^{;c}{}_{;b'}$

$\Delta^{a'}{}_{b'}|_{x'=x} = -\delta^{a}{}_{b}$, how does it change along the geodesic?

$$D'\Delta^{a'}{}_{b'} \equiv \sigma^{;d'}\Delta^{a'}{}_{b';d'} = g^{a'}{}_{c}\sigma^{;d'}\sigma^{;c}{}_{;b'd'}$$

Now $2\sigma = \sigma^{;d'}\sigma_{;d'} \Rightarrow \sigma_{;b'} = \sigma^{;d'}\sigma_{;b'd'} \Rightarrow \sigma^{;c}{}_{;b'} = \sigma^{;c;d'}\sigma_{;b'd'} + \sigma^{;d'}\sigma^{;c}{}_{;b'd'}$

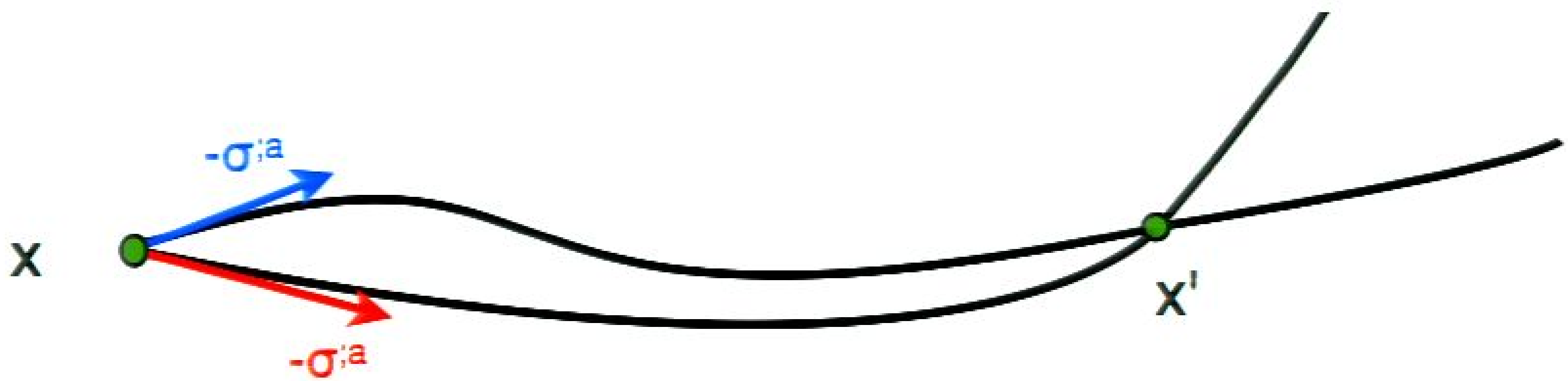
$$D'\Delta^{a'}{}_{b'} = g^{a'}{}_{c}(\sigma^{;c}{}_{;b'} - \sigma^{;c}{}_{;d'}\sigma_{;b'}{}^{d'}) = \Delta^{a'}{}_{b'} - \Delta^{a'}{}_{d'}\sigma_{;b'}{}^{d'}$$

In particular, letting $\Delta = \det \Delta^{a'}{}_{b'}$, and using $\mathrm{d}\det A = \det A\,\mathrm{tr}(A^{-1}\mathrm{d}A)$

$$D'\Delta = \Delta\,\mathrm{tr}(\delta_{b'}{}^{d'} - \sigma_{;b'}{}^{d'}) = \Delta(4 - \Box'\sigma)$$

Conclusion: $D'\Delta^{1/2} = \frac{1}{2}(4 - \Box'\sigma)\Delta^{1/2} \Rightarrow a_0{}^{A}{}_{B'} = \Delta^{1/2}\delta^{A}{}_{B'}$

The VanVleck-Morette determinant: interpretation



Can use x' or $\sigma^{;a}$ as coordinates near x

Jacobian of transformation is $\sigma^{;a}{}_{;b'}$

The Hadamard approach

$$G_{\mathrm{f}}^{AB'}(x,x') = i\langle\Psi|\mathrm{T}\left[\hat{\varphi}^{A}(x)\hat{\varphi}^{B'}(x')\right]|\Psi\rangle$$
$$= \frac{1}{8\pi^2}\left[\frac{U^{AB'}(x,x')}{\sigma+i\epsilon} + V^{AB'}(x,x')\ln(\sigma+i\epsilon)\right]$$

$$G_{\mathrm{f}}{}^{AB'}(x,x') = \frac{1}{8\pi}\left(G_{\mathrm{adv}}^{AB'}(x,x') + G_{\mathrm{ret}}^{AB'}(x,x')\right) + \frac{i}{2}\langle\Psi|\{\hat{\varphi}^{A}(x),\hat{\varphi}^{B'}(x')\}|\Psi$$

$$G_{\mathrm{ret}}{}^{A}{}_{B'}(x,x') = \theta_{-}(x,x')\left\{U^{A}{}_{B'}(x,x')\,\delta(\sigma) - V^{A}{}_{B'}(x,x')\,\theta(-\sigma)\right\},$$

$$U^{A}{}_{B'}(x,x') = \sum_{r=0}^{\infty} U_r(x,x')^{A}{}_{B'}\sigma^{r} \qquad V^{A}{}_{B'}(x,x') = \sum_{r=0}^{\infty} V_r(x,x')^{A}{}_{B'}\sigma^{r}$$

Find $U_0{}^{A}{}_{B'} = \Delta^{1/2}\delta^{A}{}_{B'}$, then write

$$\sum_{r=1}^{\infty} U_r(x,x')\sigma^{r-1} = W(x,x') = \sum_{r=0}^{\infty} W_r(x,x')\sigma^{r}$$

The Hadamard and DeWitt coefficients are related by

$$V_r{}^A{}_{B'} = \frac{\Delta^{1/2}}{2^{r+1} r!} \sum_{k=0}^{r+1} (-1)^k \frac{(m^2)^{r-k+1}}{(r-k+1)!} a_k{}^A{}_{B'}$$

with inverse

$$a_{r+1}{}^A{}_{B'} = \Delta^{-1/2} \sum_{k=0}^{r} (-2)^{k+1} \frac{k!}{(r-k)!} (m^2)^{r-k} V_k{}^A{}_{B'} + \frac{(m^2)^{r+1}}{(r+1)!} \delta^A{}_{B'}$$

In particular,

$$V_r^{(m^2=0)\,A}{}_{B'} = \frac{\Delta^{1/2}}{2^{r+1} r!} (-1)^{r+1} a_{r+1}{}^A{}_{B'}$$

These relate the 'tail term' of the massive and massless theories

$$V^A{}_{B'} = \sum_{r=0}^{\infty} V_r^{(m^2=0)\,A}{}_{B'} \frac{(2\sigma)^r\, r! J_r\left((-2m^2\sigma)^{1/2}\right)}{(-2m^2\sigma)^{r/2}}$$

$$+ m^2 \Delta^{1/2} \frac{J_1\left((-2m^2\sigma)^{1/2}\right)}{(-2m^2\sigma)^{1/2}} \delta^A{}_{B'}$$

Covariant Taylor series

In flat space time $\sigma = \frac{1}{2}\eta_{ab}(x'^a - x^a)(x'^b - x^b)$ so $\sigma^{;a} = (x'^a - x^a)$

$$g_{b_1}{}^{b'_1} \cdots g_{b_n}{}^{b'_n} T_{a_1 \cdots a_m b'_1 \cdots b'_n}(x, x') = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t_{a_1 \cdots a_m b_1 \cdots b_n \alpha_1 \cdots \alpha_k}(x) \sigma^{\alpha_1} \cdots \sigma^{\alpha_k}$$

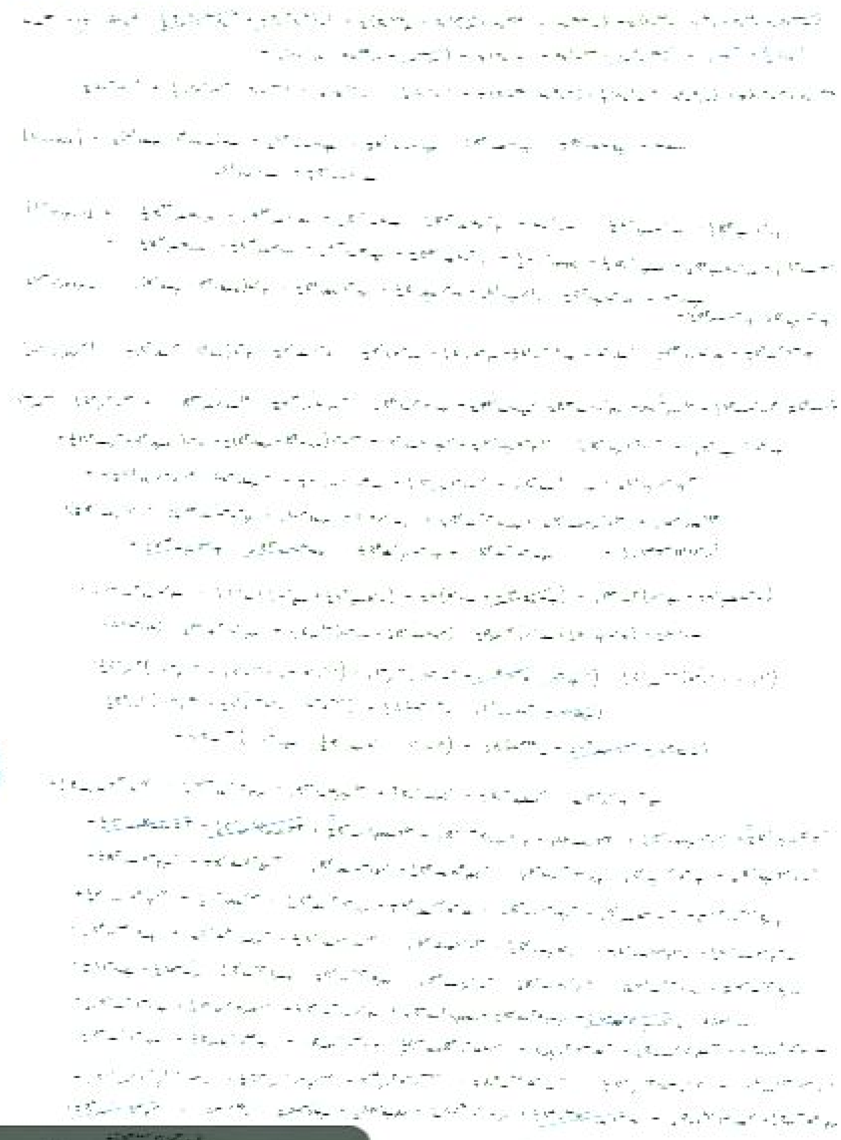
$$\equiv \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} T_{a_1 \cdots a_m b_1 \cdots b_n\,(k)}$$

Standard rules:

$$(S \otimes T)_{(k)} = \sum_{k=0}^{n} \binom{n}{k} S_{(k)} T_{(n-k)}$$

$$(D' T)_{(k)} = k T_{(k)}$$

Computing series: traditional method



- Differentiate $\sigma^{;a}\sigma_{;a} = 2\sigma$

  repeatedly and commute derivatives

  introducing Riemann

- For a_r to order $(\sigma^a)^s$ need

  $2r + s$ derivatives of σ

Fundamental transport equations

$$D' \Delta^{a'}{}_{b'} = \Delta^{a'}{}_{b'} - \Delta^{a'}{}_{d'} \sigma_{;b'}{}^{d'}$$

$$D' \sigma^{;a'}{}_{;b'} = -\sigma^{a'}{}_{d'} \sigma_{;b'}{}^{d'} - R^{a'}{}_{p'b'q'} \sigma^{;p'} \sigma^{;q'}$$

$$D'^2 (\Delta^{-1})^{a'}{}_{b'} = -D(\Delta^{-1})^{a'}{}_{b'} - R^{a'}{}_{p'r'q'} (\Delta^{-1})^{r'}{}_{b'} \sigma^{;p'} \sigma^{;q'}$$

Defining $\quad \mathcal{K}^a{}_{b\,(n)} \equiv R^a{}_{(\alpha_1|b|\alpha_2;\alpha_3\cdots\alpha_n)} \sigma^{\alpha_1} \cdots \sigma^{\alpha_n}$

$$(\Delta^{-1})^a{}_{b\,(n)} = -\left(\frac{n-1}{n+1}\right) \sum_{k=0}^{n-2} \binom{n-2}{k} \mathcal{K}^a{}_{p\,(n-k)} (\Delta^{-1})^p{}_{b\,(k)}$$

started by

$$(\Delta^{-1})^a{}_{b\,(0)} = -\delta^a{}_b, \quad (\Delta^{-1})^a{}_{b\,(1)} = 0$$

$$(\Delta^{-1})^a{}_{b(2)} = \frac{1}{3}\mathcal{K}^a{}_{b(2)}, \qquad (\Delta^{-1})^a{}_{b(3)} = \frac{1}{2}\mathcal{K}^a{}_{b(3)},$$

$$(\Delta^{-1})^a{}_{b(4)} = \frac{3}{5}\mathcal{K}^a{}_{b(4)} - \frac{1}{5}\mathcal{K}^a{}_{p(2)}\mathcal{K}^p{}_{b(2)},$$

$$(\Delta^{-1})^a{}_{b(5)} = \frac{2}{3}\mathcal{K}^a{}_{b(5)} - \frac{2}{3}\mathcal{K}^a{}_{p(3)}\mathcal{K}^p{}_{b(2)} - \frac{1}{3}\mathcal{K}^a{}_{p(2)}\mathcal{K}^p{}_{b(3)},$$

$$\Delta^a{}_{b(n)} = \sum_{k=2}^{n} \binom{n}{k} (\Delta^{-1})^a{}_{p(k)} \Delta^p{}_{b(n-k)}$$

$$\Delta^a{}_{b(0)} = -\delta^a{}_b, \qquad \Delta^a{}_{b(1)} = 0$$

$$\sigma^{a'}{}_{b'(n)} = n\,\Delta^a{}_{b(n)} - \sum_{k=2}^{n-2} \binom{n}{k} k\,(\Delta^{-1})^a{}_{p(n-k)} \Delta^p{}_{b(k)}$$

$$\sigma^{a'}{}_{b'(0)} = \delta^a{}_b, \qquad \sigma^{a'}{}_{b'(1)} = 0$$

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

Order	General				Vacuum, $m = 0$	
	Time	Terms	Can.	Can., $P = 0$	Terms	Can.
a_1	0	2	2	1	0	0
a_2	0.003	10	7	4	2	1
a_3	0.02	91	26	15	7	2
a_4	0.2	1058	113	68	56	5
a_5	3.6	13972	–	–	507	–
a_6	76	199264	–	–	4988	–
a_7	1489	2987366	–	–	51700	–
a_8	–	–	–	–	554715	–
a_9	–	–	–	–	6098069	–

Examples

Leading term for self-force in vacuum:

$$
\begin{aligned}
V_{abcd} = &-\frac{1}{280} R^{p}{}_{(a}{}^{q}{}_{b|;r|} R_{|p|c|q|d)}{}^{;r} - \frac{2}{315} R^{pqrs} R_{p(a|r|b} R_{|q|c|s|d)} \\
&+ \frac{1}{105} R^{p}{}_{(a}{}^{q}{}_{b} R^{rs}{}_{|p|c} R_{rsq|d)} + \frac{1}{840} R^{pqrs} R_{pq}{}^{t}{}_{(a} R_{|rst|b} g_{cd)} \\
&+ \frac{1}{8960} R^{pqrs} \Box R_{pqrs} g_{(ab} g_{cd)} - \frac{1}{40320} R^{pqrs} R_{pq}{}^{tu} R_{rstu} g_{(ab} g_{cd)}
\end{aligned}
$$

In Schwarzschild:

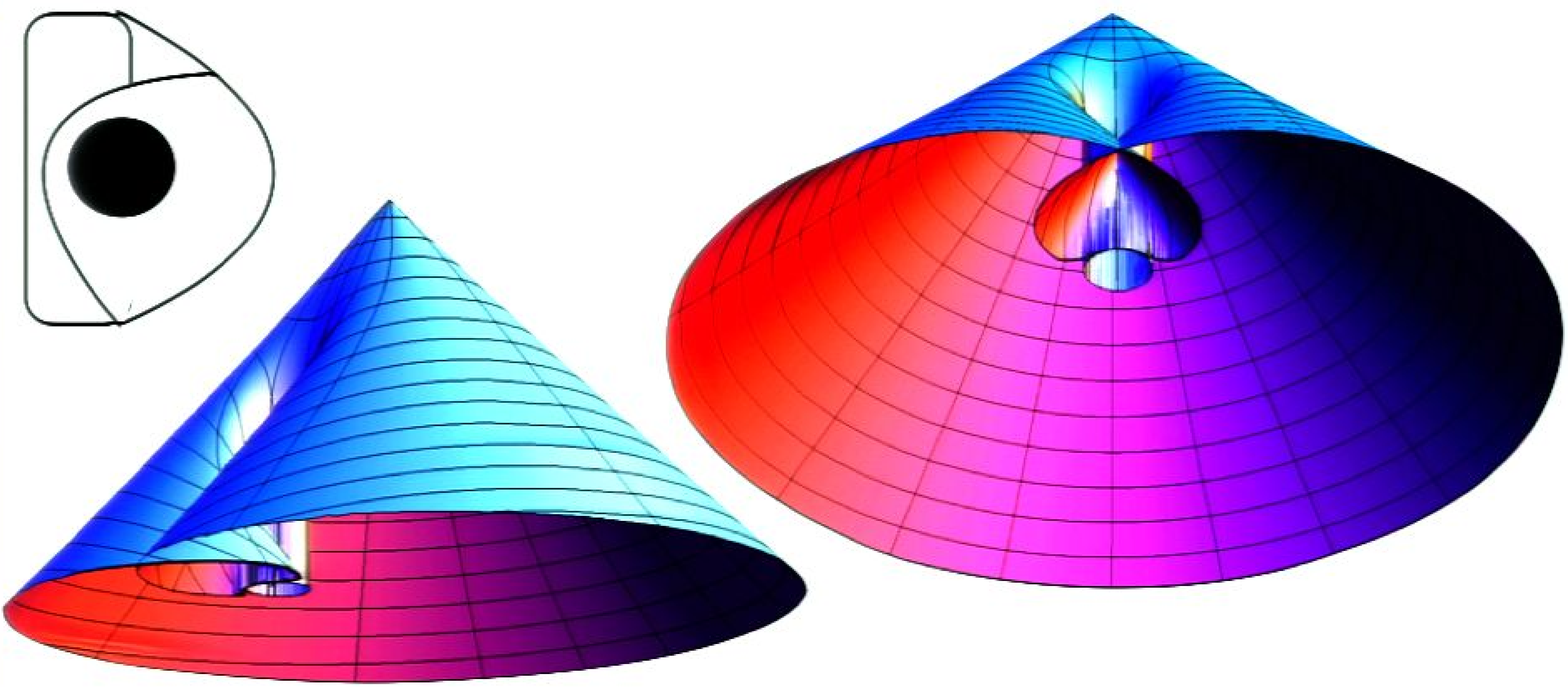
$$
V_{0(0)} = 0, \qquad V_{1(0)} = \frac{M^2}{15r^6}, \qquad V_{2(0)} = \frac{M^2}{1008r^9}(194M - 81r)
$$

$$
V_{3(0)} = \frac{M^2}{3150r^{12}}(210r^2 - 1125rM + 1454M^2),
$$

$$
V_{4(0)} = \frac{M^4}{3819240000r^{15}}(-12867705r + 28164482M).
$$

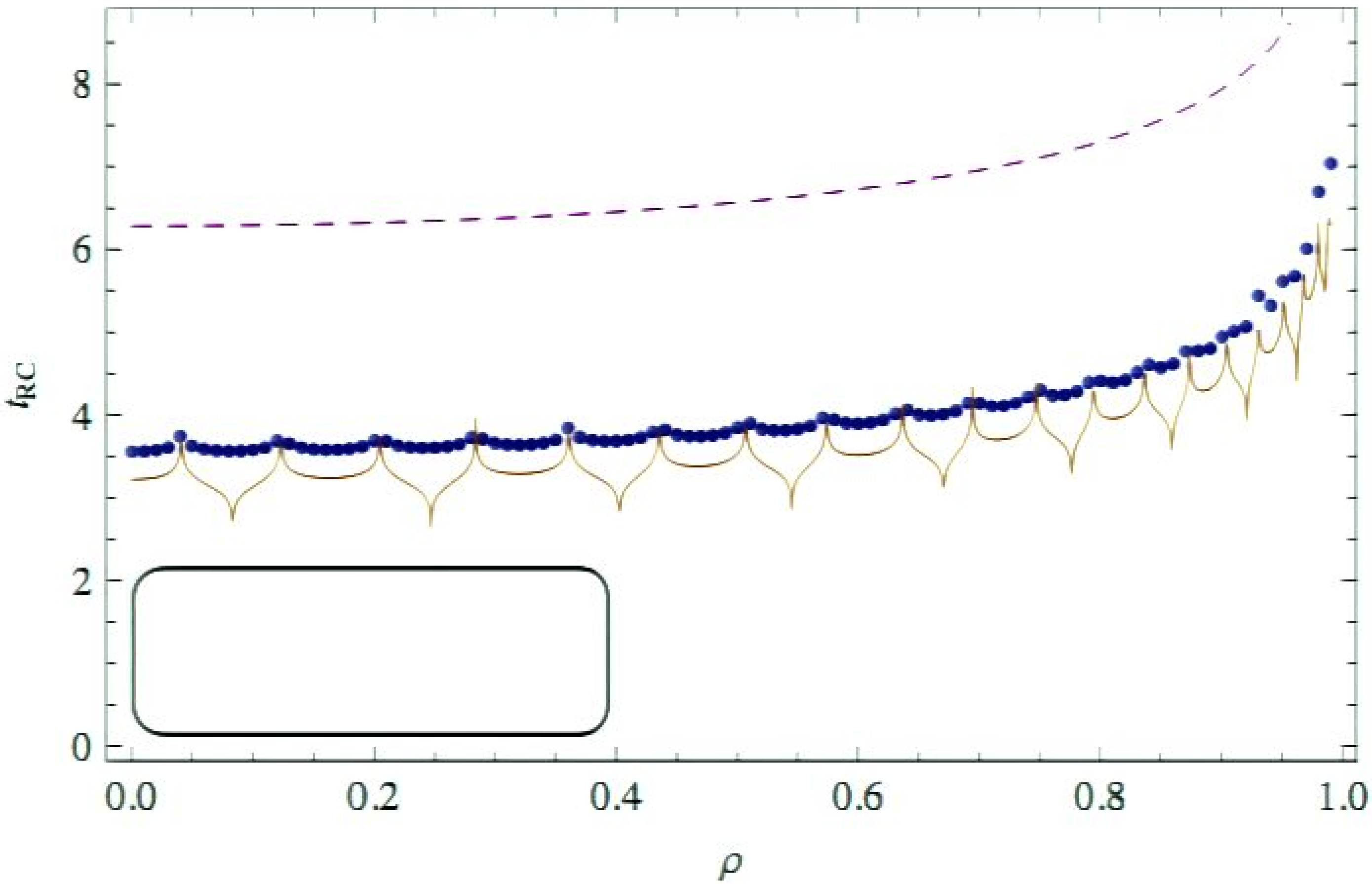
Normal Neighbourhood

Geometrical form only valid with normal neighbourhood



Images inspired by Perlick - Living Rev. Relativity 7, 9

Convergence of series



Radius of convergence for time separated points (i.e. static particle) as a function of radial position

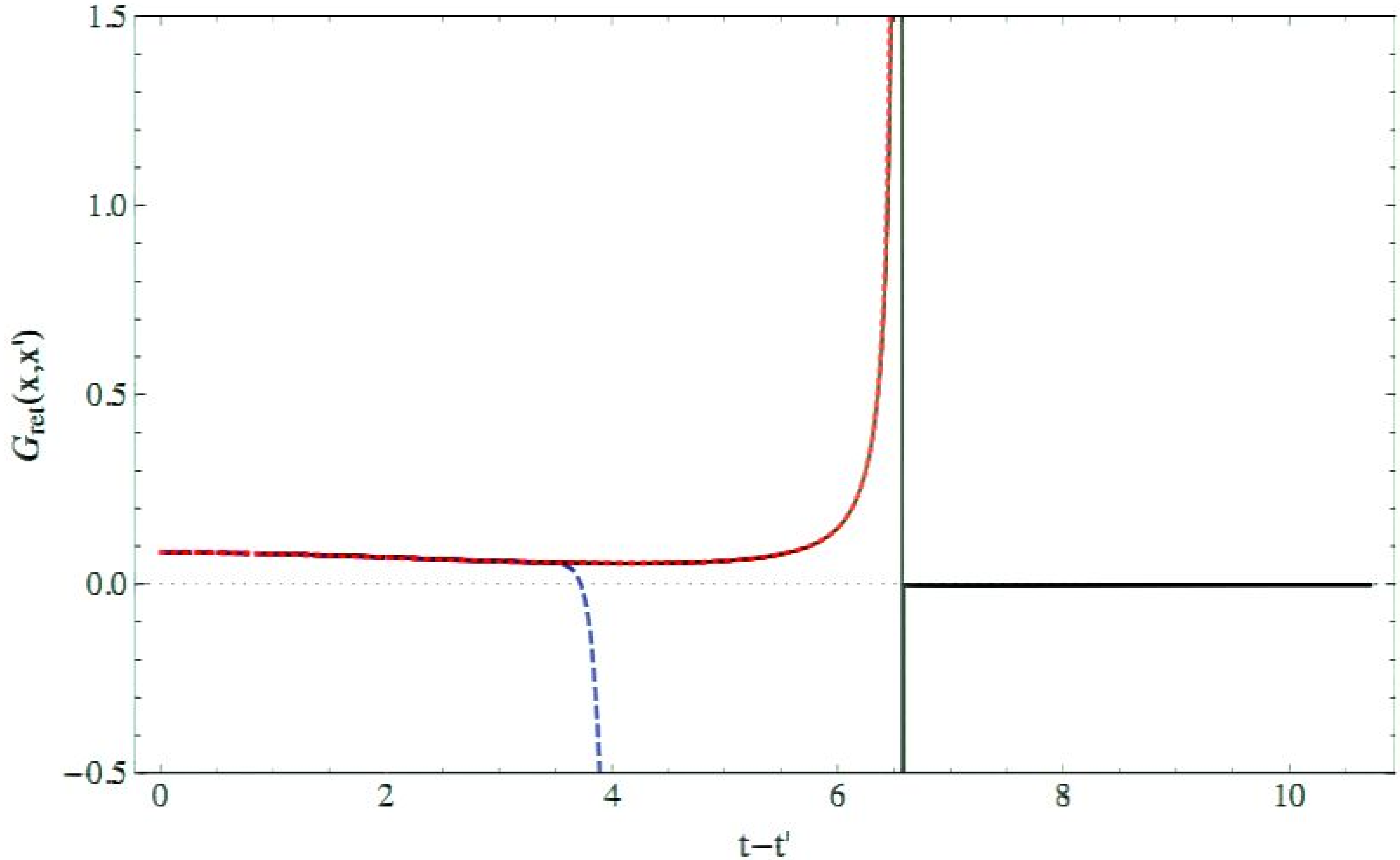
Padé resummation

- Express the truncated series as a rational function

$$V(t-t') \sim P_M^N(t-t') = \frac{\sum_{n=0}^{N} A_n (t-t')^n}{\sum_{m=0}^{M} B_m (t-t')^m} \, .$$

- Captures the behaviour of the singularities in functional form.
- Accelerate convergence of the series.
- Extend domain of series.

Convergence of series



Numerical Calculation along Geodesics

Numerically integrate the transport equations (ODEs) for $V_r(x, x')$ along geodesics.

$$D' \equiv \sigma^{;\alpha'} \nabla_{\alpha'} = s' \left(\frac{d}{ds'} + \Gamma^{a'}_{b'\gamma'} u^{\gamma'} + \cdots \right)$$

$$D' \ln \Delta = (4 - \sigma^{\alpha'}{}_{\alpha'})$$

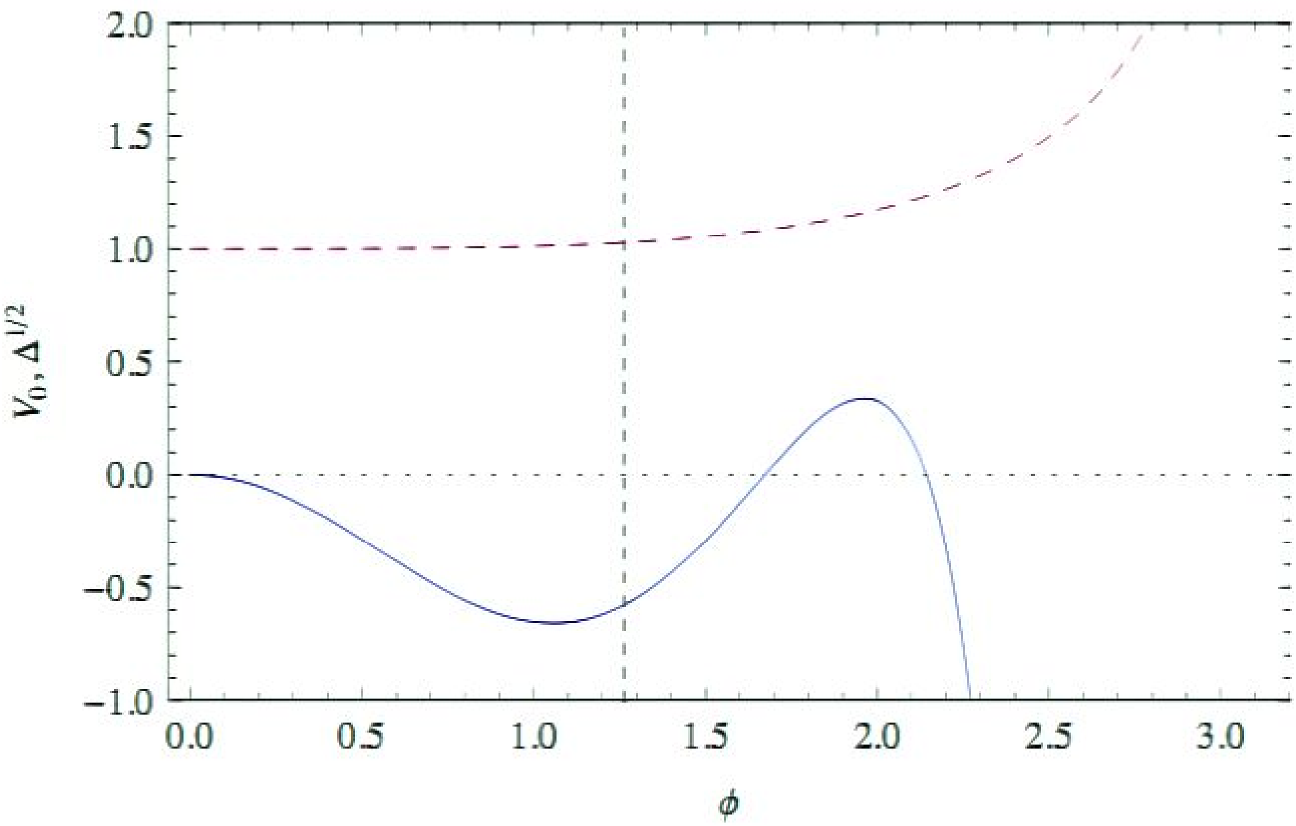
$$D' \sigma^{a'}{}_{b'} = -\sigma^{a'}{}_{\alpha'} \sigma^{\alpha'}{}_{b'} + \sigma^{a'}{}_{b'} - R^{a'}{}_{\alpha' b' \beta'} \sigma^{\alpha'} \sigma^{\beta'}$$

$$(D' + 1) V_0 + \frac{1}{2} V_0 \left(\sigma^{\mu'}{}_{\mu'} - 4 \right) + \frac{1}{2} (\Box' - \xi R') \Delta^{1/2} = 0$$

$$(D' + r + 1) V_r + \frac{1}{2} V_r \left(\sigma^{\mu'}{}_{\mu'} - 4 \right) + \frac{1}{2r} (\Box' - \xi R') V_{r-1} = 0$$

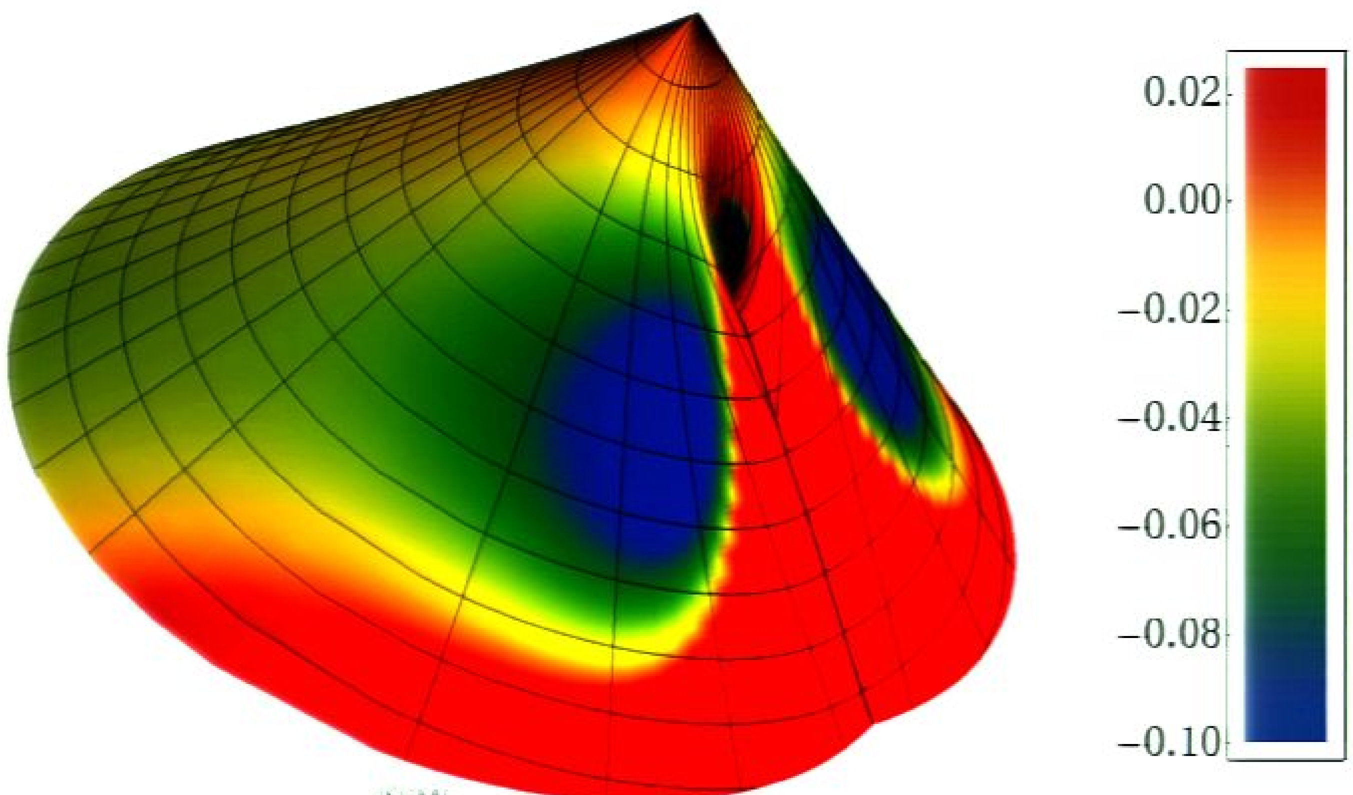
$$\vdots$$

Numerical integration of the transport equations for $\Delta^{1/2}(x, x')$ and $V_0(x, x')$ along a geodesic.



Numerical Calculation on-light cone

Numerically integrate the transport equations (ODEs) for $V_0(x, x')$ along null geodesics to get $V(x, x')$ on the light-cone.



Numerical Calculation on-light cone

- $V(x, x)$ satisfies the homogeneous wave equation:

$$(\Box - m^2 - P)V(x, x') = 0$$

- Characteristic initial data given by values on the light-cone.
- Numerically solve wave equation for $V(x, x')$ within the light-cone.

Is Quasi-Local enough?

- No!
- Can calculate the Hadamard Green function everywhere in the normal neighborhood
- But, that's not enough for the self force - the Distant Part Green function is crucial

Mode-sum decomposition

For a spherically symmetric space-time of the form

$$ds^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + h(r)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

$$G_{\mathrm{ret}} = \frac{1}{2\pi[h(r)h(r')]^{1/2}} \int_{-\infty+ic}^{+\infty+ic} d\omega \sum_{l=0}^{+\infty} (2l+1) P_l(\cos\gamma) e^{-i\omega(t-t')} \frac{u_{l\omega}(r_<) v_{l,\omega}(r_>}{f(r) W[u,v]}$$

Introducing r_* by $\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{1}{f(r)}$, u, v satisfy

$$\frac{\mathrm{d}^2 u}{\mathrm{d}{r_*}^2} + \left[\omega^2 - \frac{f(r)}{h(r)}\left((l+1/2)^2 - \frac{1}{4}\right) - h^{-1/2}\frac{\mathrm{d}^2 h^{1/2}}{\mathrm{d}{r_*}^2}\right] u = 0$$

Asymptotic behaviour of solutions

Assume existence of an horizon $f(r_h) = 0$ where $r_* \to -\infty$ and asymptotically flat/cosmological horizon so $f(r)/h(r) \to 0$ as $r_* \to \infty$.

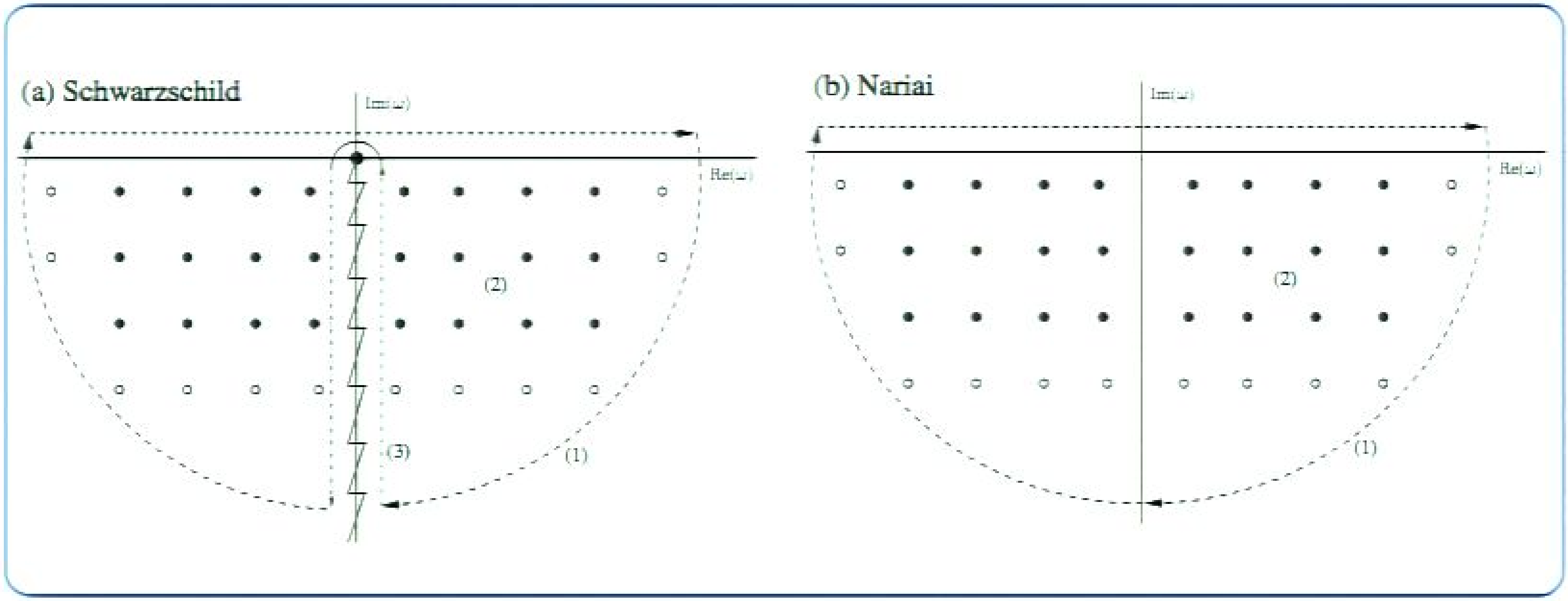
Then

$$u^{\text{in}}(r) \sim \begin{cases} e^{-i\omega r_*} & r_* \to -\infty \\ A^{\text{in}}(\omega) e^{-i\omega r_*} + B^{\text{in}}(\omega) e^{+i\omega r_*} & r_* \to \infty \end{cases}$$

$$u^{\text{out}}(r) \sim \begin{cases} A^{\text{out}}(\omega) e^{+i\omega r_*} + B^{\text{out}}(\omega) e^{-i\omega r_*} & r_* \to -\infty \\ e^{+i\omega r_*} & r_* \to \infty \end{cases}$$

$$f(r) W[u^{\text{in}}, u^{\text{out}}] = 2i\omega A^{\text{in}}(\omega)$$

The Green function in the complex plane



- Quasi Normal Modes: Zeroes of the Wronskian $A^{\rm in}_{lm}(\omega) = 0$

Asymptotic behaviour of solutions

Assume existence of an horizon $f(r_h) = 0$ where $r_* \to -\infty$ and asymptotically flat/cosmological horizon so $f(r)/h(r) \to 0$ as $r_* \to \infty$.

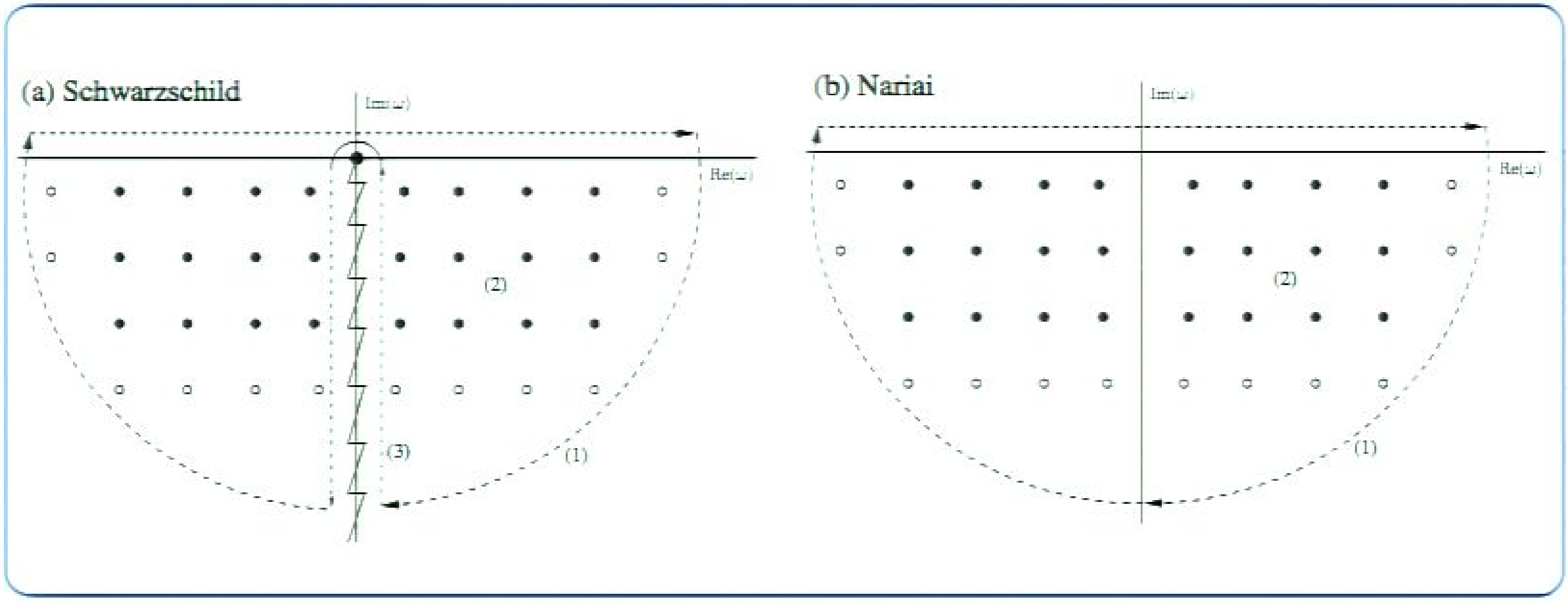
Then

$$u^{\text{in}}(r) \sim \begin{cases} e^{-i\omega r_*} & r_* \to -\infty \\ A^{\text{in}}(\omega)e^{-i\omega r_*} + B^{\text{in}}(\omega)e^{+i\omega r_*} & r_* \to \infty \end{cases}$$

$$u^{\text{out}}(r) \sim \begin{cases} A^{\text{out}}(\omega)e^{+i\omega r_*} + B^{\text{out}}(\omega)e^{-i\omega r_*} & r_* \to -\infty \\ e^{+i\omega r_*} & r_* \to \infty \end{cases}$$

$$f(r)W[u^{\text{in}}, u^{\text{out}}] = 2i\omega A^{\text{in}}(\omega)$$

The Green function in the complex plane



- Quasi Normal Modes: Zeroes of the Wronskian $A^{\text{in}}_{lm}(\omega) = 0$

Contribution from poles

$$G_{\rm QNM} = 2\,{\rm Re}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)B_{ln}\tilde{u}_{ln}(r)\tilde{u}_{ln}(r')e^{-i\omega_{ln}(t-t'-r_*-r'_*)}$$

B_{ln} are the *excitation factors*

$$B_{ln} \equiv \frac{A^{\rm out}_{l\omega_{ln}}}{2\omega_{ln}\frac{dA^{\rm in}_{l\omega}}{d\omega}}$$

and $\tilde{u}_{ln}(r)$ are the QNM radial functions

$$\tilde{u}_{ln}(r_*) = \frac{u^{\rm in}_{l\omega_{ln}}(r_*)}{A^{\rm out}_{l\omega_{ln}}e^{i\omega_{ln}r_*}}$$

Critical orbits

Null geodesics: $\frac{1}{h(r)^2}\left(\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)} \equiv k^2(r,b)$ $b = L/E$ where

$L = h(r)\dot{\phi}$ and $E = f(r)\dot{t}$ are constants of the motion

Suppose there are critical values r_c and b_c such that

$$k^2(r_c, b_c) = 0 \qquad \text{and} \qquad \frac{\partial k^2(r_c, b_c)}{\partial r} = 0$$

Example: Schwarzschild $r_c = 3M$, $b_c = \sqrt{27}M$

Then defining $k_c(r) = \text{sign}(r - r_c)\sqrt{k^2(r, b_c)}$

$$\exp\left(i\omega \int^{r_*} b_c k_c(r)\, dr_*\right) v(r)$$

obeys the QNM boundary conditions.

Critical orbits

Null geodesics: $\frac{1}{h(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)} \equiv k^2(r,b)$ $b = L/E$ where

$L = h(r)\dot{\phi}$ and $E = f(r)\dot{t}$ are constants of the motion

Suppose there are critical values r_c and b_c such that

$$k^2(r_c, b_c) = 0 \qquad \text{and} \qquad \frac{\partial k^2(r_c, b_c)}{\partial r} = 0$$

Example: Schwarzschild $r_c = 3M$, $b_c = \sqrt{27}M$

Then defining $k_c(r) = \mathrm{sign}(r - r_c)\sqrt{k^2(r, b_c)}$

$$\exp\left(i\omega \int^{r*} b_c k_c(r)\,\mathrm{d}r_*\right) v(r)$$

obeys the QNM boundary conditions.

Critical orbits

Null geodesics: $\frac{1}{h(r)^2}\left(\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)} \equiv k^2(r,b)$ $b = L/E$ where

$L = h(r)\dot{\phi}$ and $E = f(r)\dot{t}$ are constants of the motion

Suppose there are critical values r_c and b_c such that

$$k^2(r_c, b_c) = 0 \qquad \text{and} \qquad \frac{\partial k^2(r_c, b_c)}{\partial r} = 0$$

Example: Schwarzschild $r_c = 3M$, $b_c = \sqrt{27}M$

Then defining $k_c(r) = \mathrm{sign}(r - r_c)\sqrt{k^2(r, b_c)}$

$$\exp\left(i\omega\int^{r_*} b_c k_c(r)\, dr_*\right) v(r)$$

obeys the QNM boundary conditions.

Contribution from poles

$$G_{\rm QNM} = 2\,{\rm Re}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)B_{ln}\tilde{u}_{ln}(r)\tilde{u}_{ln}(r')e^{-i\omega_{ln}(t-t'-r_*-r'_*)}$$

B_{ln} are the *excitation factors*

$$B_{ln} \equiv \frac{A^{\rm out}_{l\omega_{ln}}}{2\omega_{ln}\frac{dA^{\rm in}_{l\omega}}{d\omega}}$$

and $\tilde{u}_{ln}(r)$ are the QNM radial functions

$$\tilde{u}_{ln}(r_*) = \frac{u^{\rm in}_{l\omega_{ln}}(r_*)}{A^{\rm out}_{l\omega_{ln}}e^{i\omega_{ln}r_*}}$$

Critical orbits

Null geodesics: $\frac{1}{h(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)} \equiv k^2(r,b)$ $b = L/E$ where

$L = h(r)\dot{\phi}$ and $E = f(r)\dot{t}$ are constants of the motion

Suppose there are critical values r_c and b_c such that

$$k^2(r_c, b_c) = 0 \qquad \text{and} \qquad \frac{\partial k^2(r_c, b_c)}{\partial r} = 0$$

Example: Schwarzschild $r_c = 3M$, $b_c = \sqrt{27}M$

Then defining $k_c(r) = \mathrm{sign}(r - r_c)\sqrt{k^2(r, b_c)}$

$$\exp\left(i\omega \int^{r*} b_c k_c(r)\,\mathrm{d}r_*\right) v(r)$$

obeys the QNM boundary conditions.

The equation for $v(r)$ yields a natural expansion in $L = l + 1/2$:

$$\omega = L\omega_{-1} + \omega_0 + L^{-1}\omega_1 + \cdots$$

$$v(r) = \exp(S_0(r) + L^{-1}S_1(r) + L^{-2}S_2(r) + \cdots)$$

and imposing a continuity condition on $S'_k(r)$ at $r = r_c$

	$l = 2, n = 0$	$l = 3, n = 0$
Ctd. Frac.	$0.373672 - i0.088962$	$0.599444 - i0.092703$
12th order	$0.373679 - i0.088955$	$0.599443 - i0.092703$
6th order	$0.373642 - i0.088967$	$0.599439 - i0.092684$
WKB (6)	$0.3736 \quad - i0.0890$	$0.5994 \quad - i0.0927$

Critical orbits

Null geodesics: $\frac{1}{h(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)} \equiv k^2(r,b)$ $b = L/E$ where

$L = h(r)\dot{\phi}$ and $E = f(r)\dot{t}$ are constants of the motion

Suppose there are critical values r_c and b_c such that

$$k^2(r_c, b_c) = 0 \qquad \text{and} \qquad \frac{\partial k^2(r_c, b_c)}{\partial r} = 0$$

Example: Schwarzschild $r_c = 3M$, $b_c = \sqrt{27}M$

Then defining $k_c(r) = \text{sign}(r - r_c)\sqrt{k^2(r, b_c)}$

$$\exp\left(i\omega \int^{r_*} b_c k_c(r)\,\mathrm{d}r_*\right) v(r)$$

obeys the QNM boundary conditions.

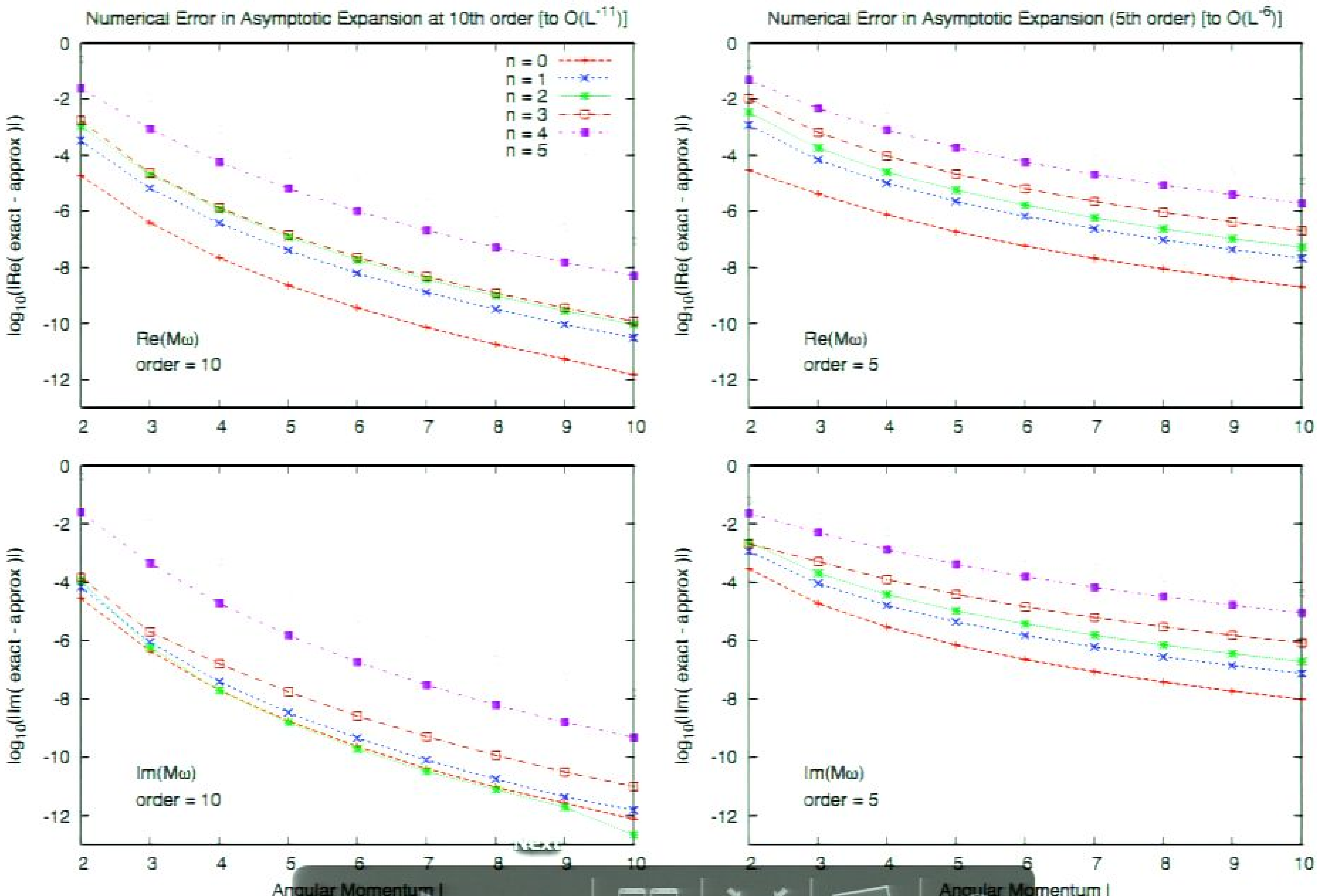
The equation for $v(r)$ yields a natural expansion in $L = l + 1/2$:

$$\omega = L\omega_{-1} + \omega_0 + L^{-1}\omega_1 + \cdots$$

$$v(r) = \exp(S_0(r) + L^{-1}S_1(r) + L^{-2}S_2(r) + \cdots)$$

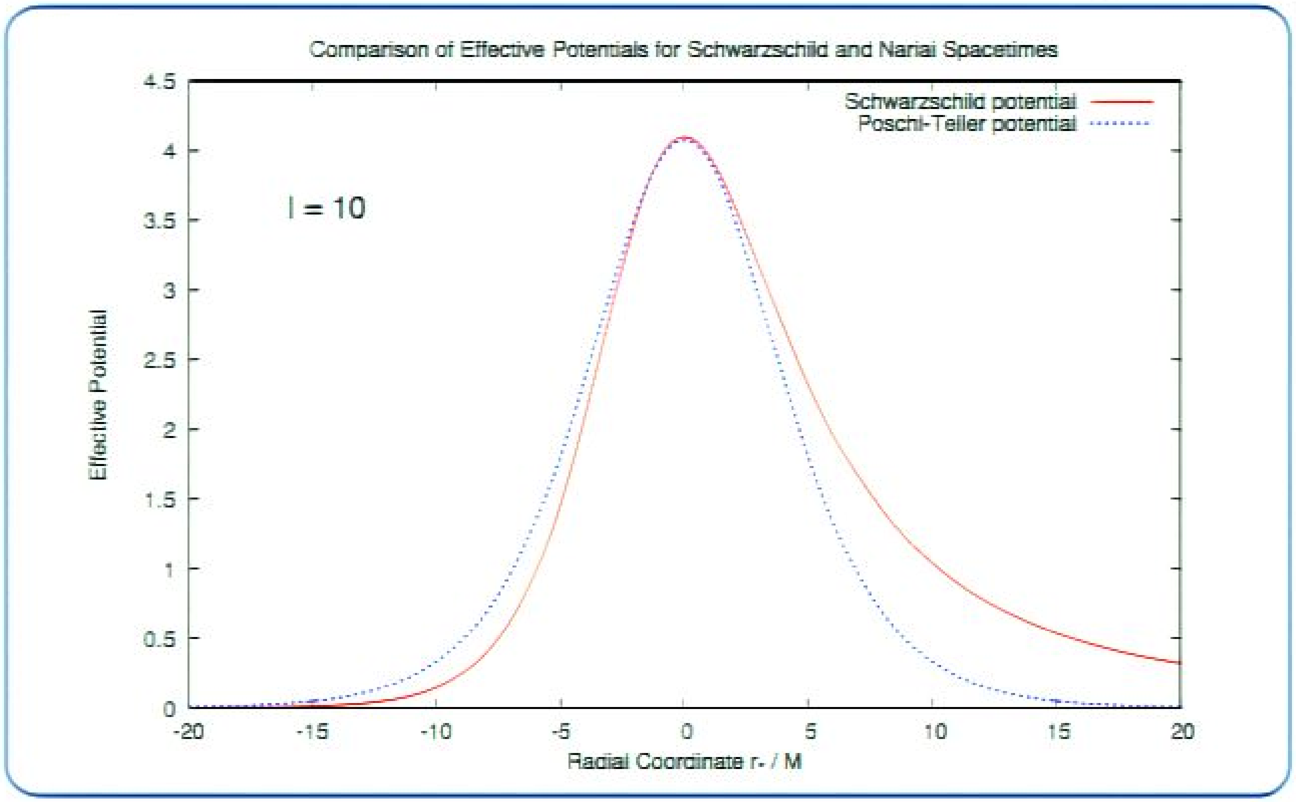
and imposing a continuity condition on $S'_k(r)$ at $r = r_c$

	$l = 2, n = 0$	$l = 3, n = 0$
Ctd. Frac.	$0.373672 - i0.088962$	$0.599444 - i0.092703$
12th order	$0.373679 - i0.088955$	$0.599443 - i0.092703$
6th order	$0.373642 - i0.088967$	$0.599439 - i0.092684$
WKB (6)	$0.3736 \quad - i0.0890$	$0.5994 \quad - i0.0927$



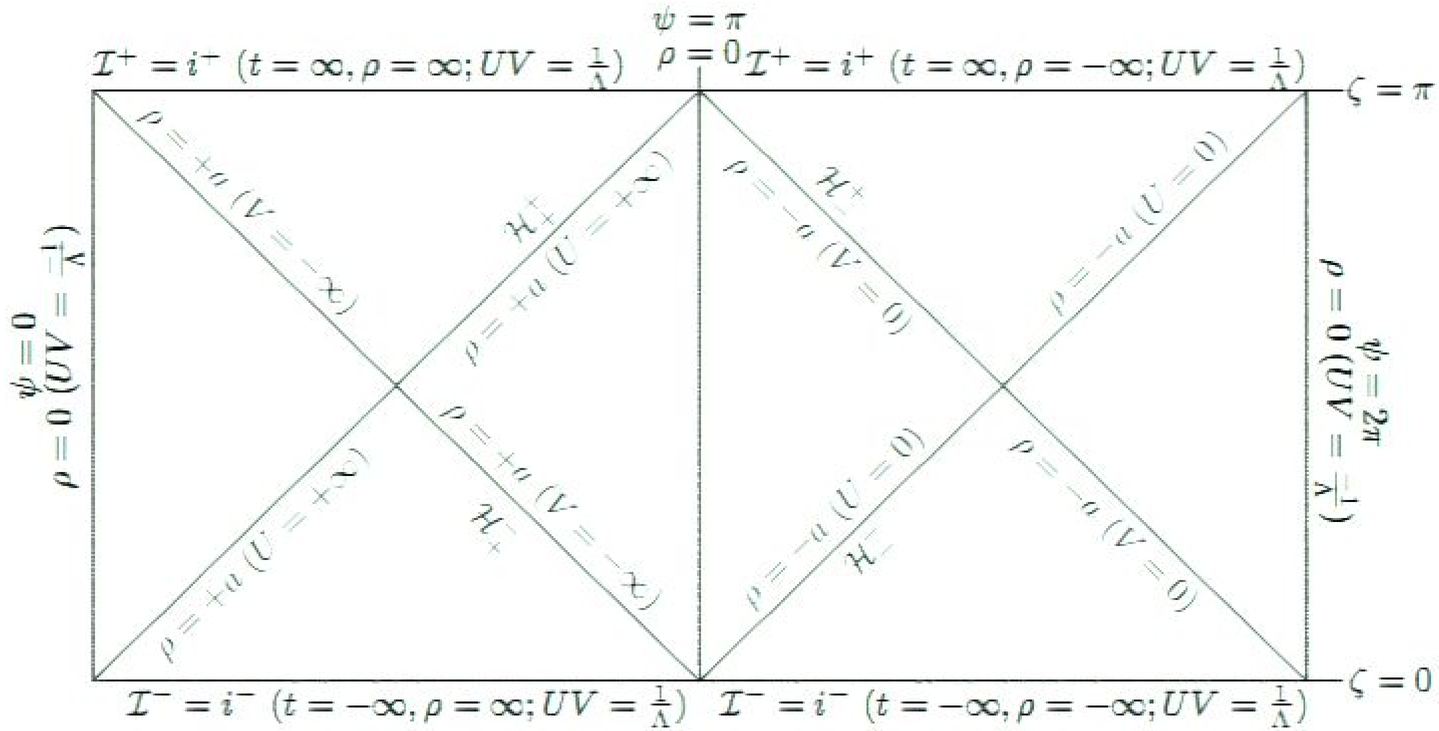
Numerical Error in Asymptotic Expansion at 10th order [to O(L^-11)]
Numerical Error in Asymptotic Expansion (5th order) [to O(L^-6)]
n = 0
n = 1
n = 2
n = 3
n = 4
n = 5
log10(|Re(exact - approx)|)
log10(|Im(exact - approx)|)
Re(Mω)
order = 10
Re(Mω)
order = 5
Im(Mω)
order = 10
Im(Mω)
order = 5
Angular Momentum l
Angular Momentum l

A simpler potential: Pöschl-Teller



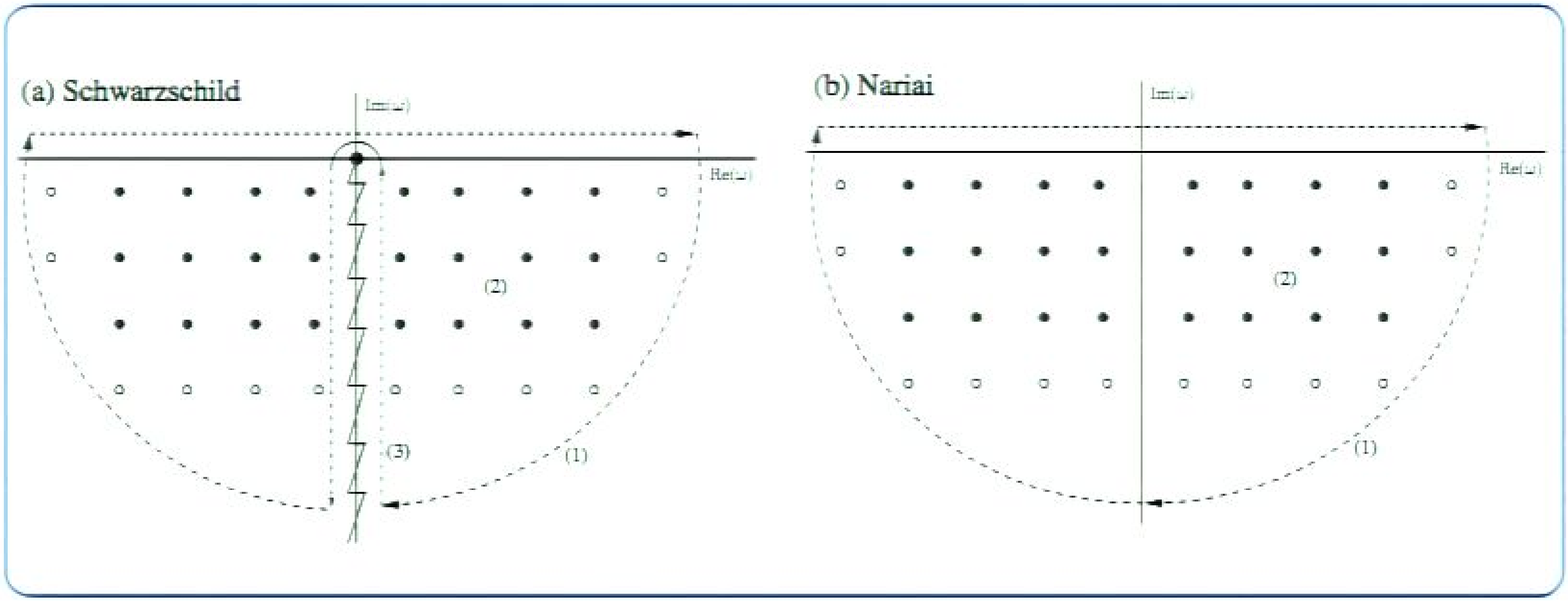
A simpler space-time: Nariai

Does the Poschl-Teller potential arise in any spacetime? yes!



$$ds^2 = -(1-\rho^2)dt^2 + \frac{d\rho^2}{1-\rho^2} + d\theta^2 + \sin^2\theta d\phi^2$$

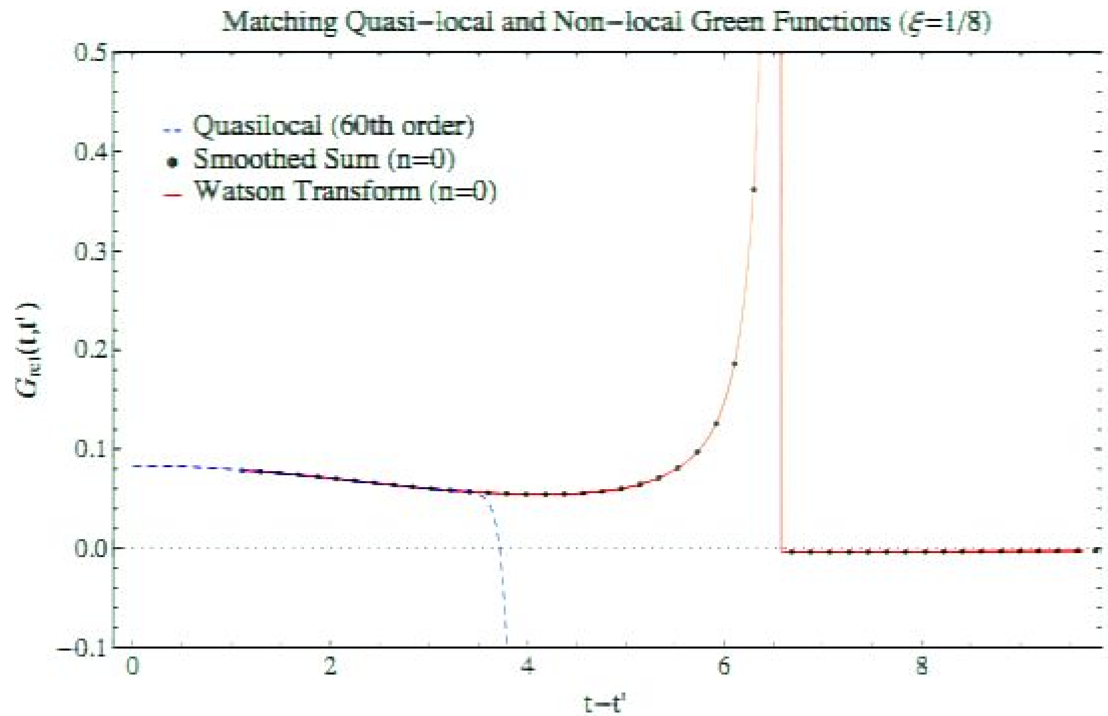
The Green function in the complex plane

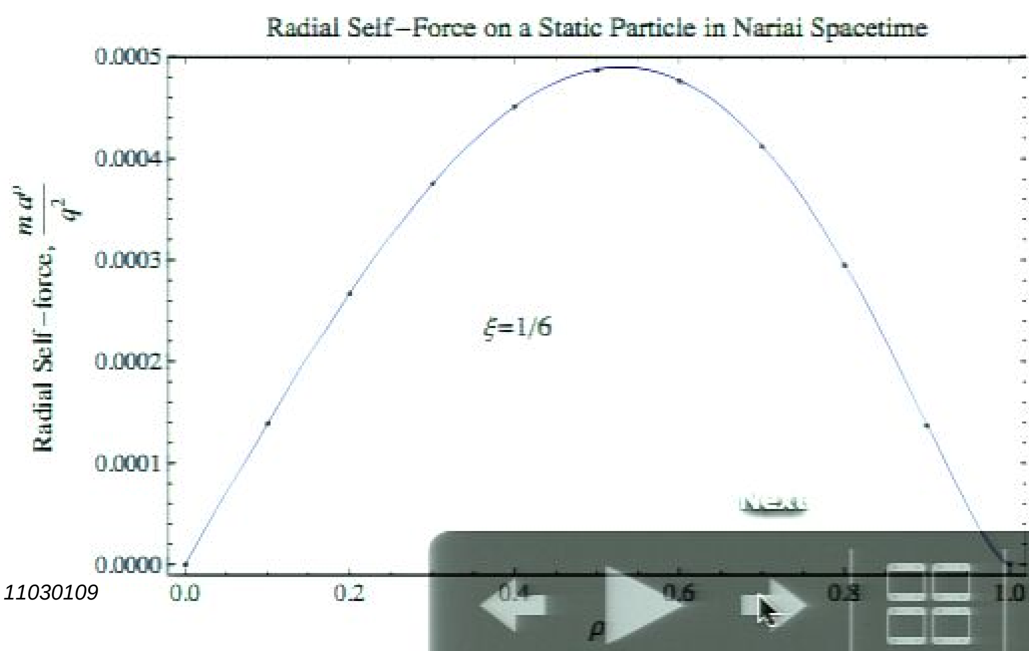


- QNMs: poles/residues - excitation factors
- QNM n-sum divergent[convergent] for $T = t - t' - r_* - r'_* < 0$ [> 0]
- $T \sim 0$ time for light ray to go from r'_* to 0 and out to r_*

Matching QL and QNM in Nariai:

$$\gamma = 0$$



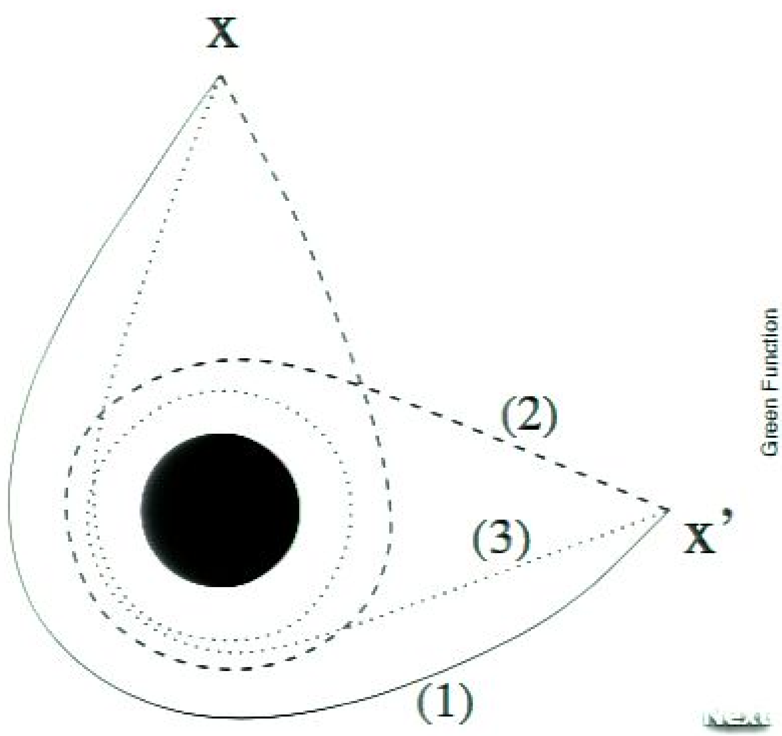


Self-force for static scalar particle

Singularities arising in the QNM sum

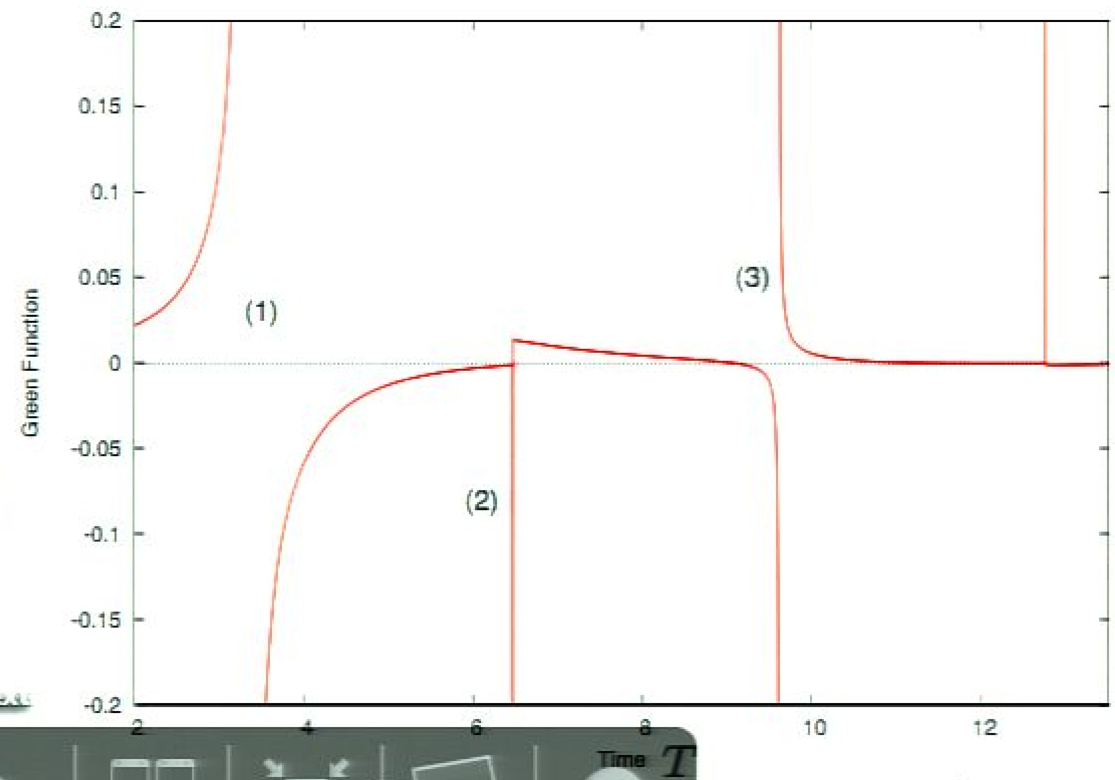
singular when phases are coherent $\Leftrightarrow$ connecting null geodesic

Null geodesics joining x & x'



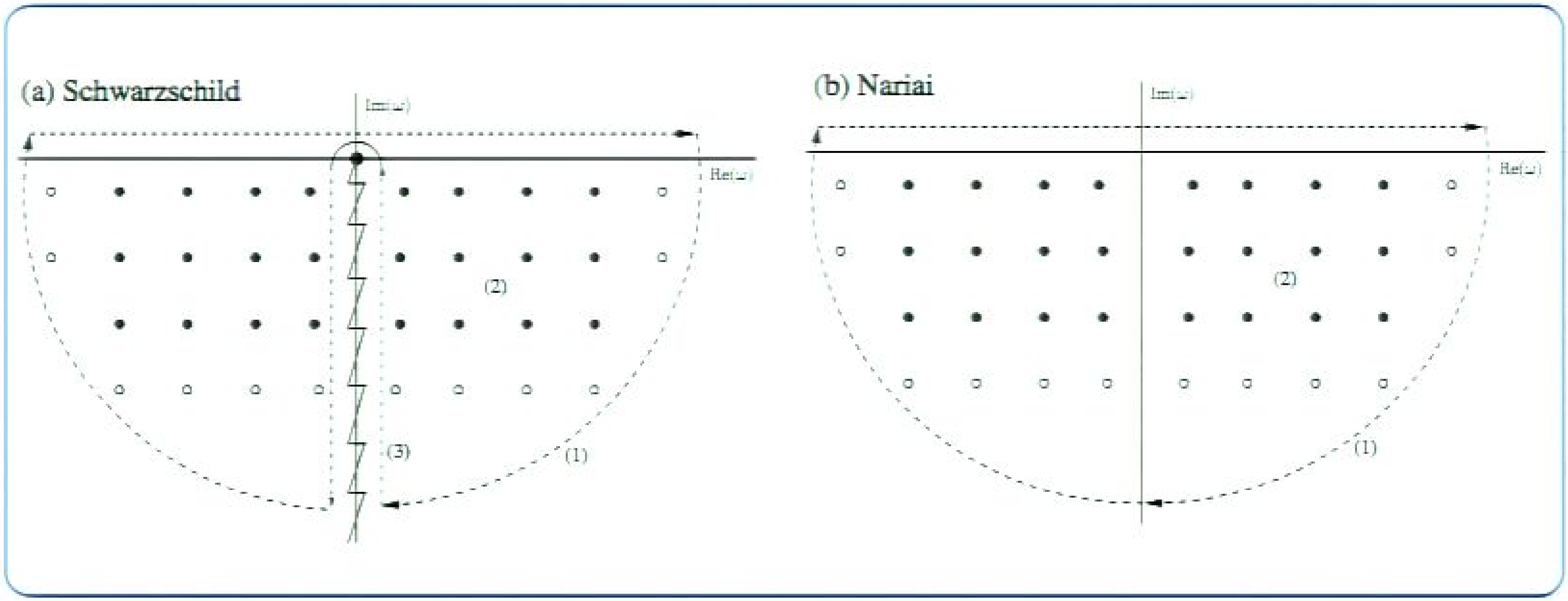
$\gamma = \pi/2$

QNM Green function in Nariai



$\xi = 1/6$

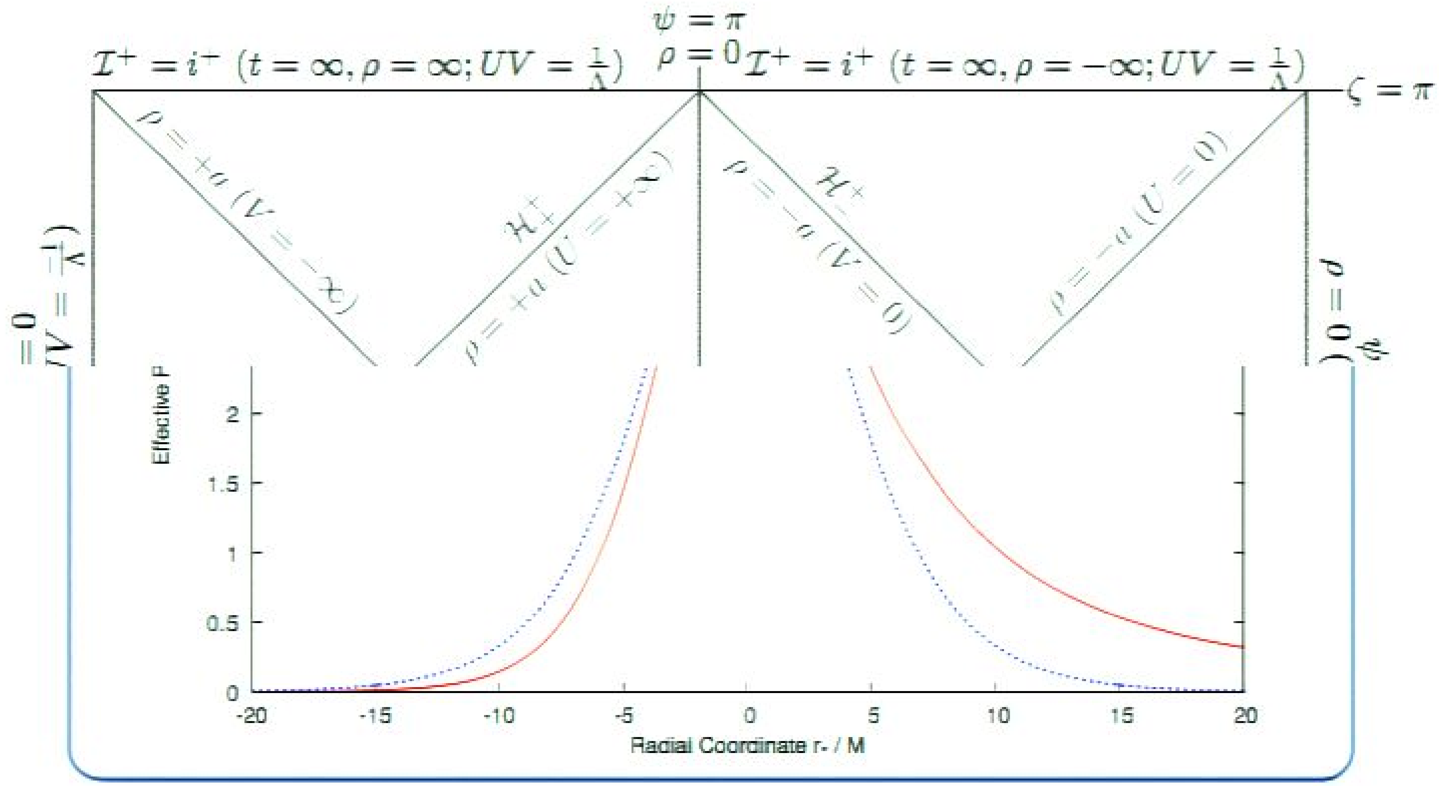
The Green function in the complex plane



- QNMs: poles/residues - excitation factors
- QNM n-sum divergent[convergent] for $T = t - t' - r_* - r'_* < 0$ $[> 0]$
- $T \sim 0$ time for light ray to go from r'_* to 0 and out to r_*

A simpler space-time: Nariai

Does the Poschl-Teller potential arise in any spacetime? yes!



Contribution from poles

$$G_{\text{QNM}} = 2\,\text{Re}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)B_{ln}\tilde{u}_{ln}(r)\tilde{u}_{ln}(r')e^{-i\omega_{ln}(t-t'-r_*-r'_*)}$$

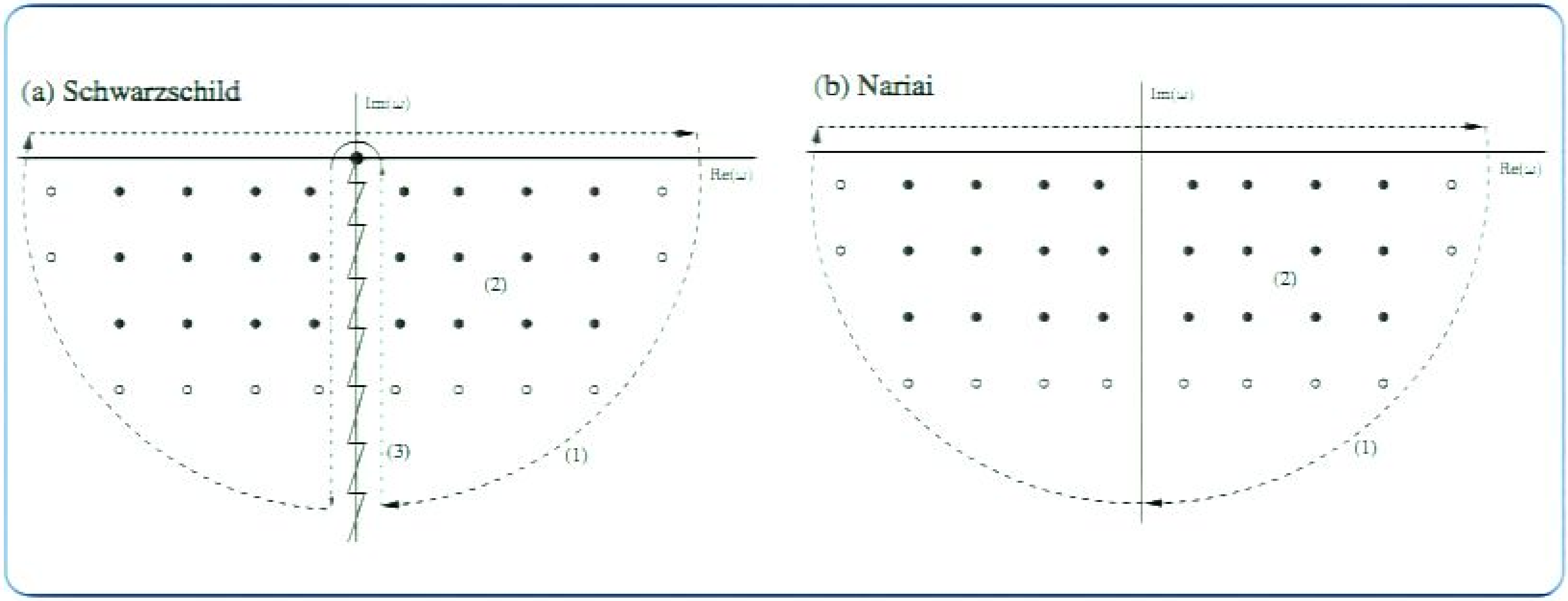
B_{ln} are the *excitation factors*

$$B_{ln} \equiv \frac{A^{\text{out}}_{l\omega_{ln}}}{2\omega_{ln}\frac{dA^{\text{in}}_{l\omega}}{d\omega}}$$

and $\tilde{u}_{ln}(r)$ are the QNM radial functions

$$\tilde{u}_{ln}(r_*) = \frac{u^{\text{in}}_{l\omega_{ln}}(r_*)}{A^{\text{out}}_{l\omega_{ln}}e^{i\omega_{ln}r_*}}$$

The Green function in the complex plane



- Quasi Normal Modes: Zeroes of the Wronskian $A^{\text{in}}_{lm}(\omega) = 0$

Asymptotic behaviour of solutions

Assume existence of an horizon $f(r_h) = 0$ where $r_* \to -\infty$ and asymptotically flat/cosmological horizon so $f(r)/h(r) \to 0$ as $r_* \to \infty$.

Then

$$u^{\text{in}}(r) \sim \begin{cases} e^{-i\omega r_*} & r_* \to -\infty \\ A^{\text{in}}(\omega)e^{-i\omega r_*} + B^{\text{in}}(\omega)e^{+i\omega r_*} & r_* \to \infty \end{cases}$$

$$u^{\text{out}}(r) \sim \begin{cases} A^{\text{out}}(\omega)e^{+i\omega r_*} + B^{\text{out}}(\omega)e^{-i\omega r_*} & r_* \to -\infty \\ e^{+i\omega r_*} & r_* \to \infty \end{cases}$$

$$f(r)W[u^{\text{in}}, u^{\text{out}}] = 2i\omega A^{\text{in}}(\omega)$$

Contribution from poles

$$G_{\rm QNM} = 2\,{\rm Re}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)B_{ln}\tilde{u}_{ln}(r)\tilde{u}_{ln}(r')e^{-i\omega_{ln}(t-t'-r_*-r'_*)}$$

B_{ln} are the *excitation factors*

$$B_{ln} \equiv \frac{A^{\rm out}_{l\omega_{ln}}}{2\omega_{ln}\frac{dA^{\rm in}_{l\omega}}{d\omega}}$$

and $\tilde{u}_{ln}(r)$ are the QNM radial functions

$$\tilde{u}_{ln}(r_*) = \frac{u^{\rm in}_{l\omega_{ln}}(r_*)}{A^{\rm out}_{l\omega_{ln}}e^{i\omega_{ln}r_*}}$$

Critical orbits

Null geodesics: $\frac{1}{h(r)^2}\left(\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)} \equiv k^2(r,b)$ $b = L/E$ where

$L = h(r)\dot\phi$ and $E = f(r)\dot t$ are constants of the motion

Suppose there are critical values r_c and b_c such that

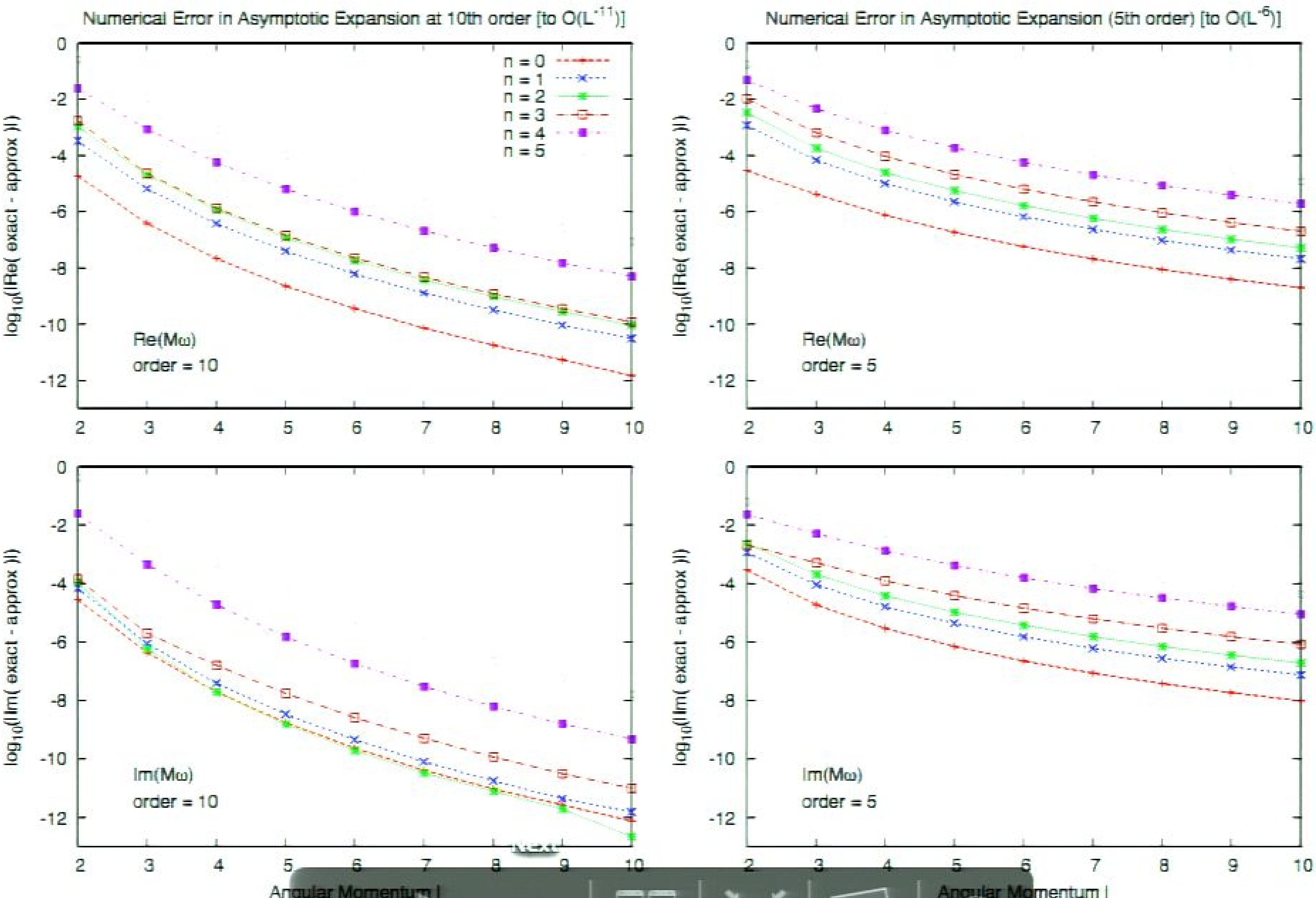
$$k^2(r_c, b_c) = 0 \qquad \text{and} \qquad \frac{\partial k^2(r_c, b_c)}{\partial r} = 0$$

Example: Schwarzschild $r_c = 3M$, $b_c = \sqrt{27}M$

Then defining $k_c(r) = \text{sign}(r - r_c)\sqrt{k^2(r, b_c)}$

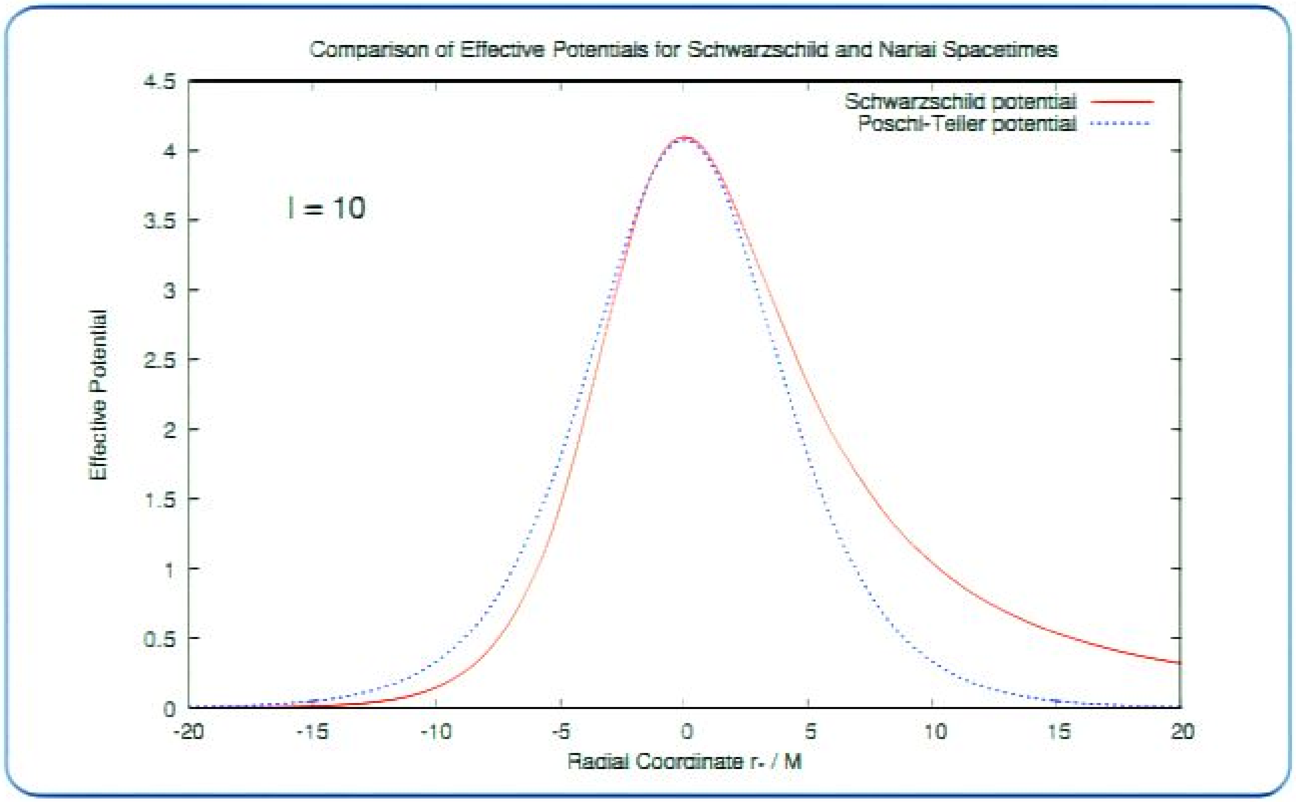
$$\exp\left(i\omega\int^{r_*} b_c k_c(r)\, dr_*\right) v(r)$$

obeys the QNM boundary conditions.



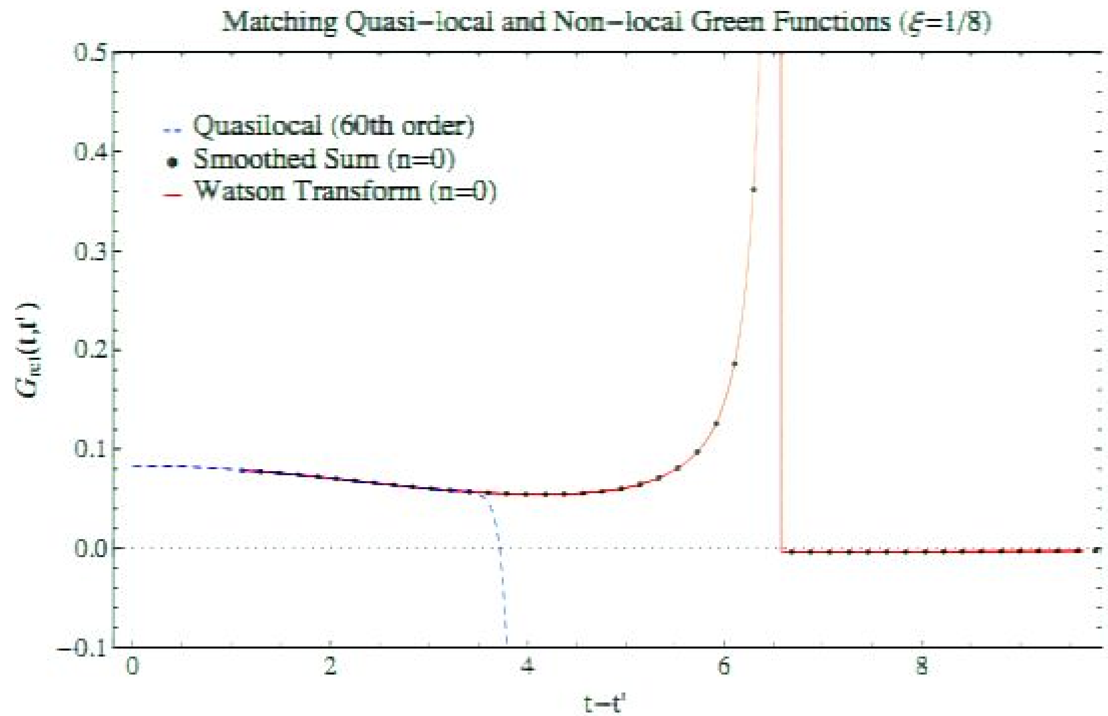
Numerical Error in Asymptotic Expansion at 10th order [to O(L^-11)]
Numerical Error in Asymptotic Expansion (5th order) [to O(L^-6)]
n = 0
n = 1
n = 2
n = 3
n = 4
n = 5
log10(|Re(exact - approx)|)
log10(|Im(exact - approx)|)
Re(Mω)
order = 10
Re(Mω)
order = 5
Im(Mω)
order = 10
Im(Mω)
order = 5
Angular Momentum l
Angular Momentum l

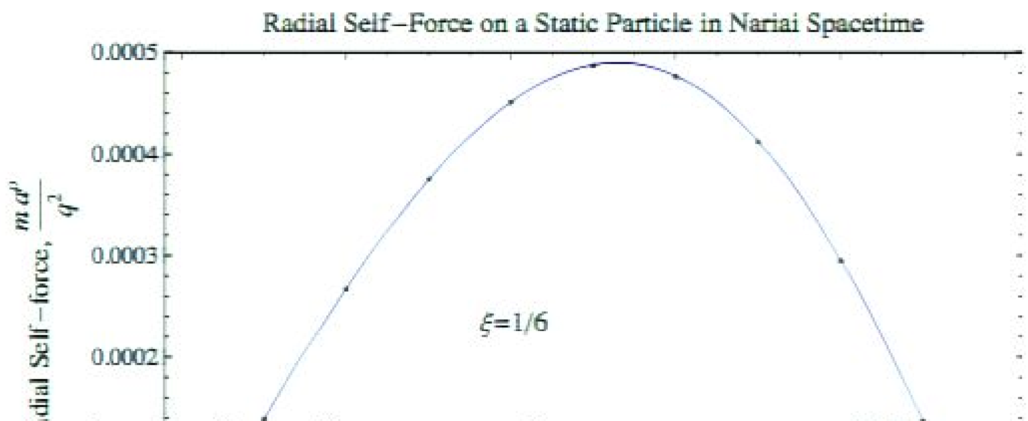
A simpler potential: Pöschl-Teller



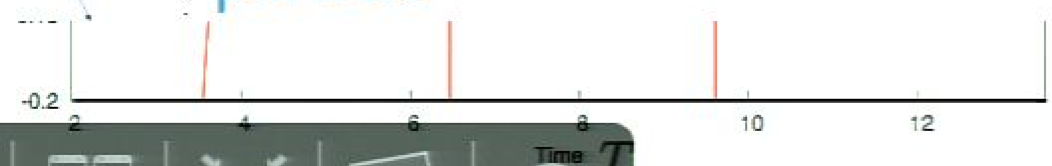
Matching QL and QNM in Nariai:

$$\gamma = 0$$





Self-force for static scalar particle



(1)

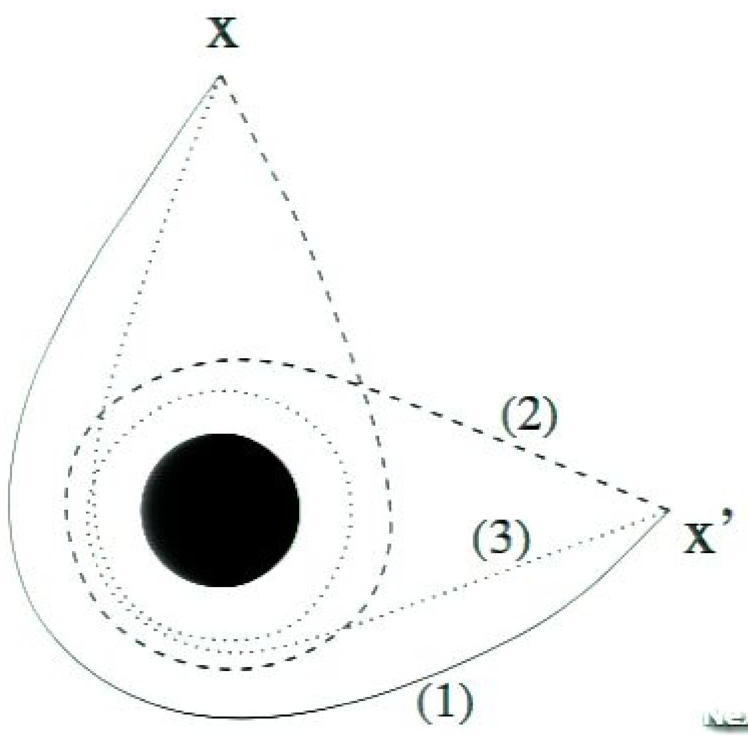
$\gamma = \pi/2$

$\xi = 1/6$

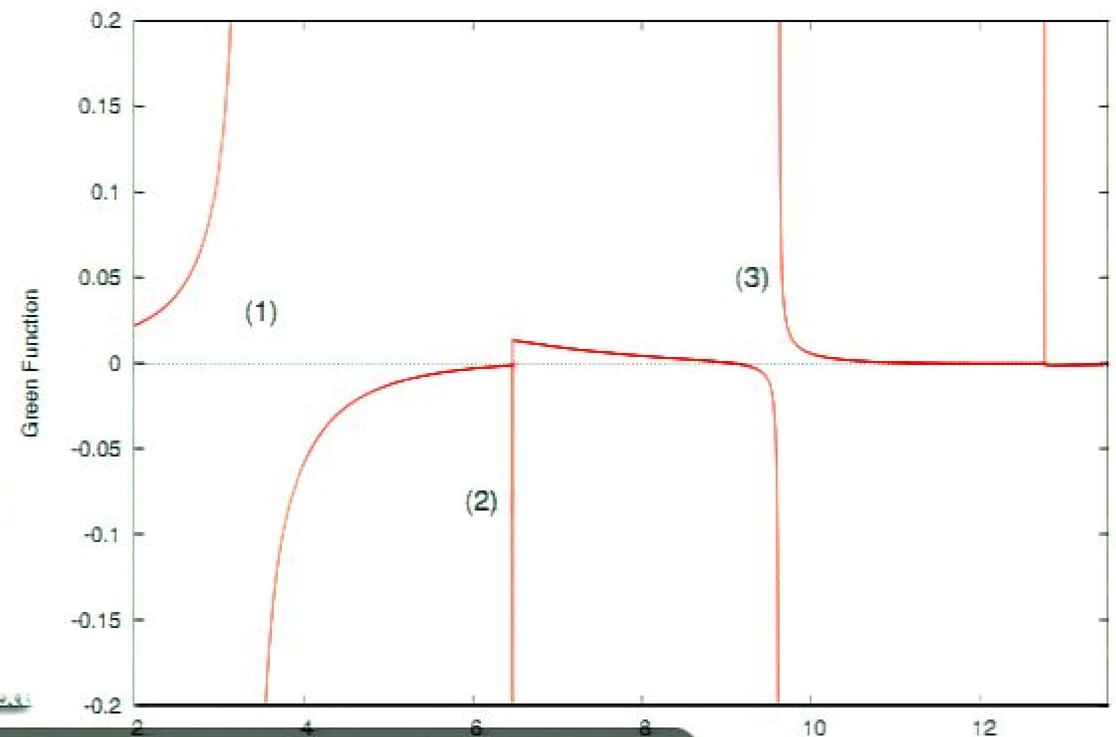
Singularities arising in the QNM sum

singular when phases are coherent ⇔ connecting null geodesic

Null geodesics joining x & x'



QNM Green function in Nariai

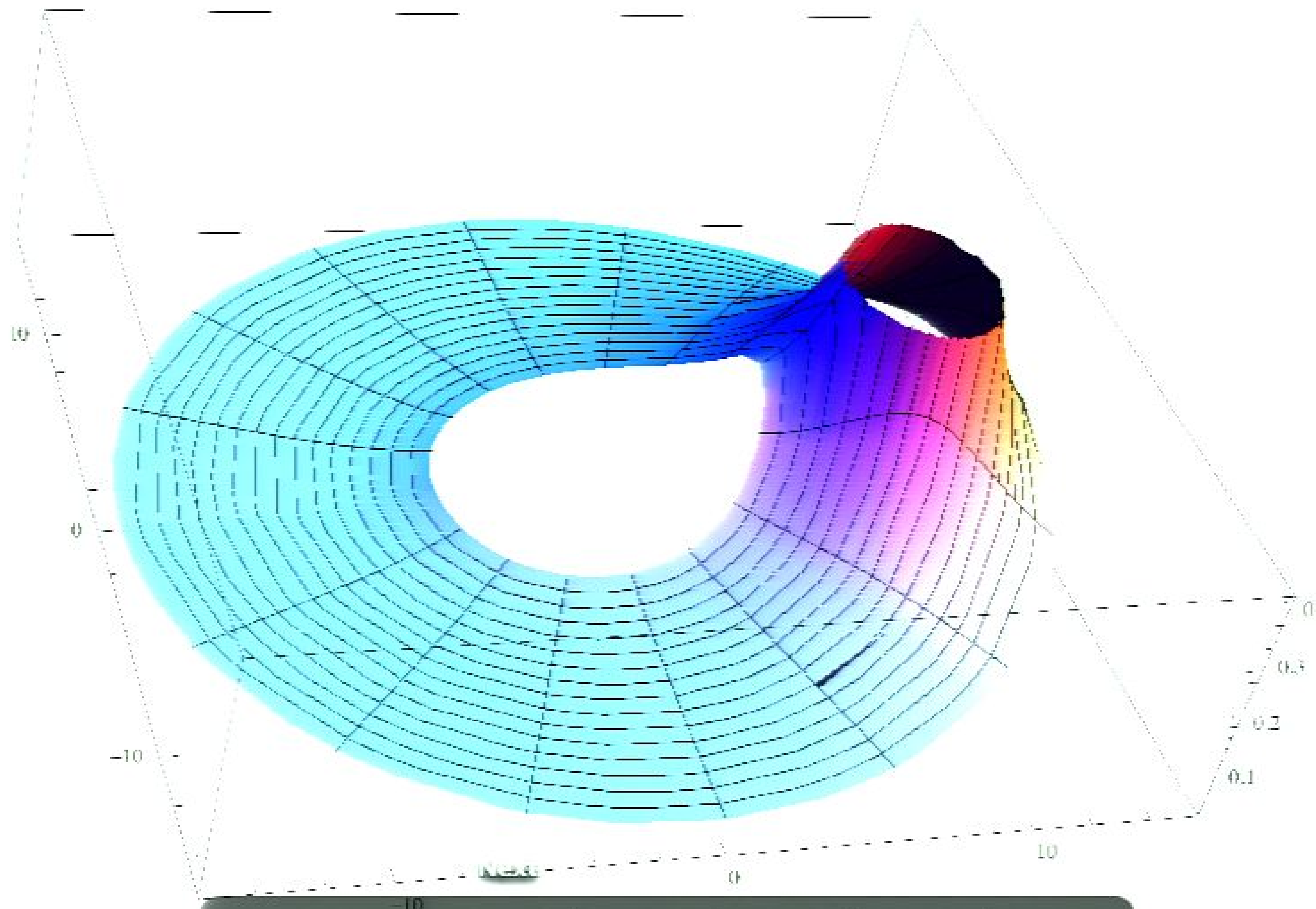


T

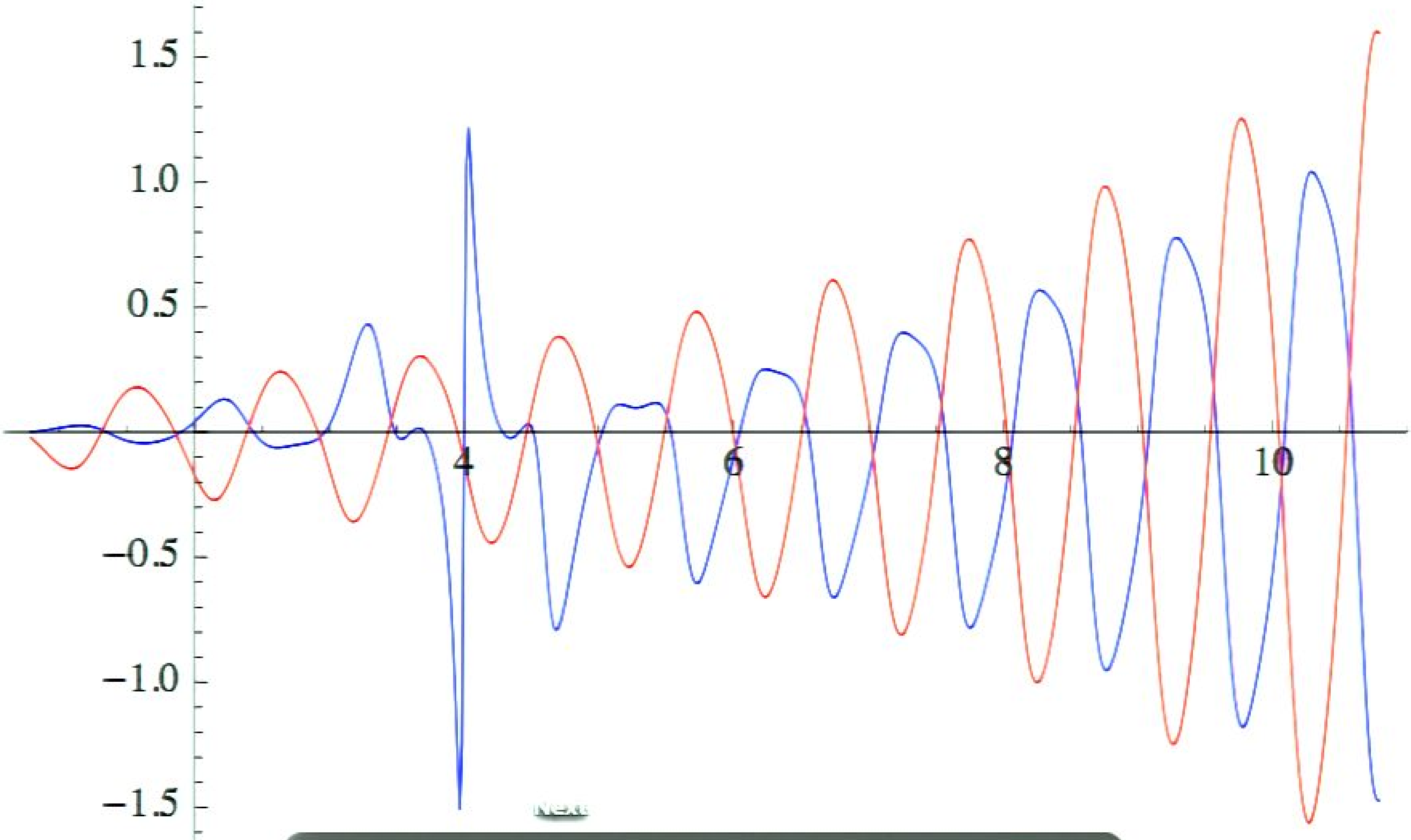
$\gamma = \pi/2$

$\xi = 1/6$

Current research: The singular field approach



Current research: The cut contribution



Other issues

- Alternative approach to Hadamard $V(x,x') = \sum V_k(x,x')\sigma^k(x,x')$ expansion.
- What happens when we try to numerically evolve $\Delta^{1/2}(x,x')$, $V_0(x,x')$ outside the normal neighbourhood?
- Can we relate $V(x,x')$ to geodesics outside the normal neighbourhood - add up contributions? (Certainly sometimes.)
- Have seen importance of the critical orbit to QNM's, what about to the Hawking effect? Note: black hole mining highly effective in 4 dimensions, does nothing in 2 dimensions (no potential barrier)

Other issues

- Alternative approach to Hadamard $V(x, x') = \sum V_k(x, x')\sigma^k(x, x')$ expansion.
- What happens when we try to numerically evolve $\Delta^{1/2}(x, x')$, $V_0(x, x')$ outside the normal neighbourhood?
- Can we relate $V(x, x')$ to geodesics outside the normal neighbourhood - add up contributions? (Certainly sometimes.)
- Have seen importance of the critical orbit to QNM's, what about to the Hawking effect? Note: black hole mining highly effective in 4 dimensions, does nothing in 2 dimensions (no potential barrier)

Happy

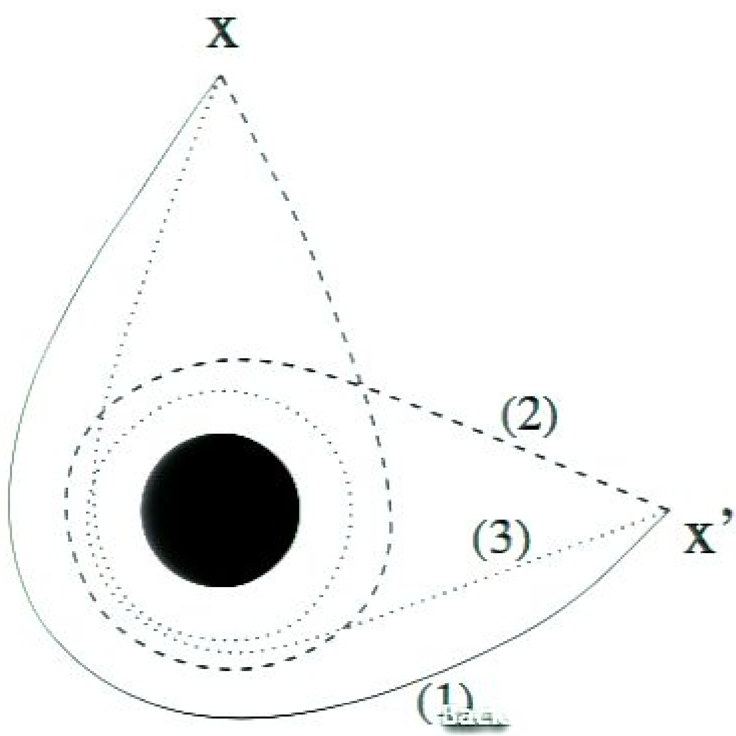
St Patrick's

Day

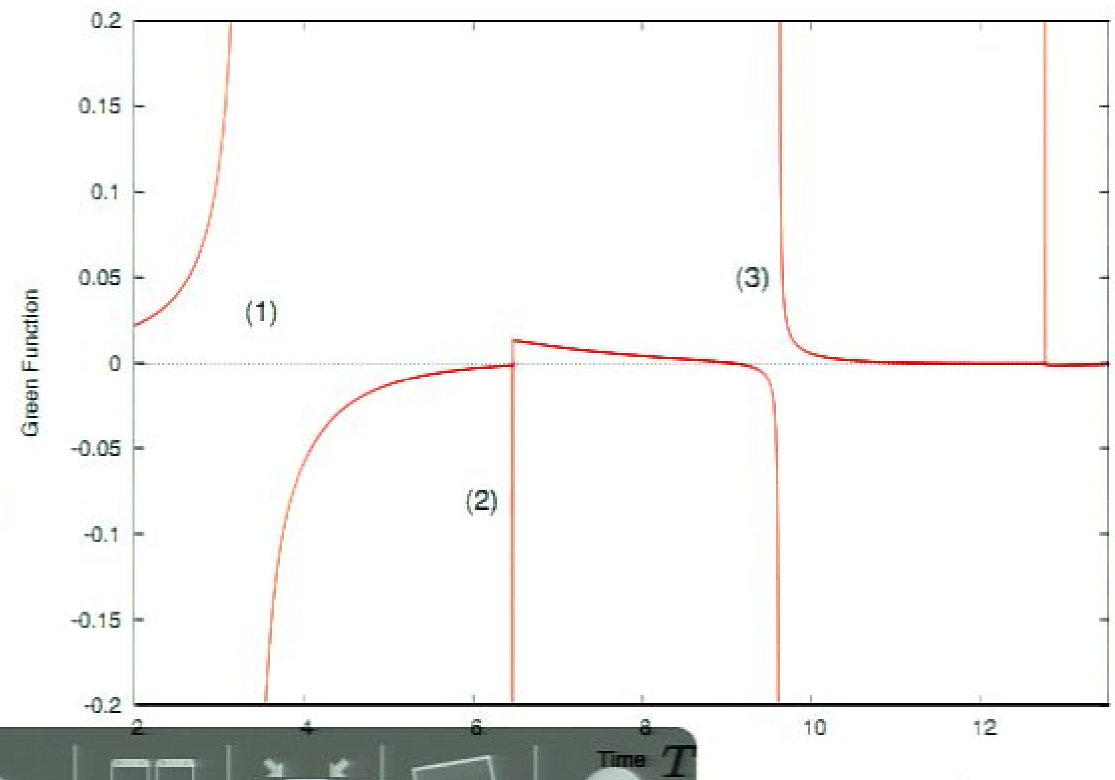
Singularities arising in the QNM sum

singular when phases are coherent ⇔ connecting null geodesic

Null geodesics joining x & x'



QNM Green function in Nariai

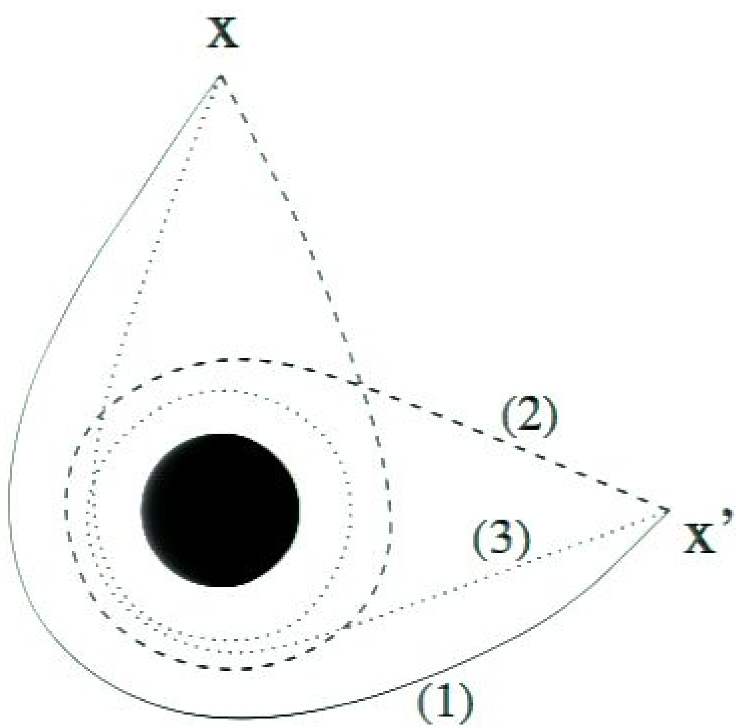


$\gamma = \pi/2$ $\xi = 1/6$

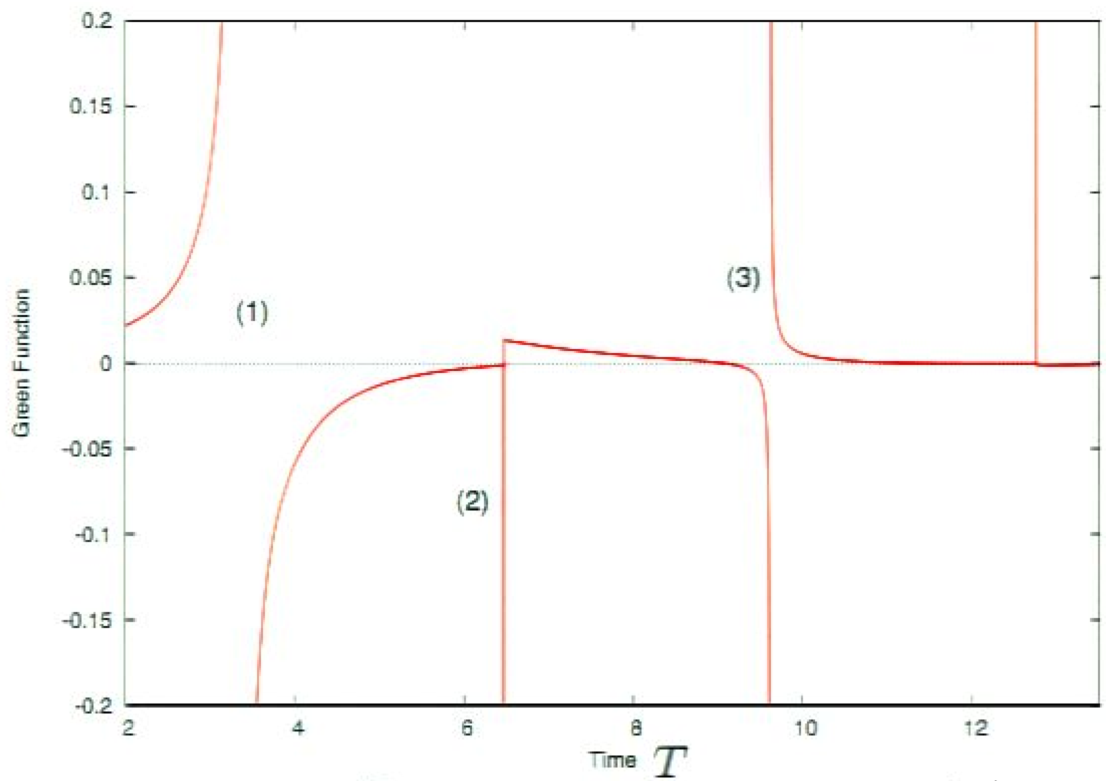
Singularities arising in the QNM sum

singular when phases are coherent ⇔ connecting null geodesic

Null geodesics joining x & x'



QNM Green function in Nariai

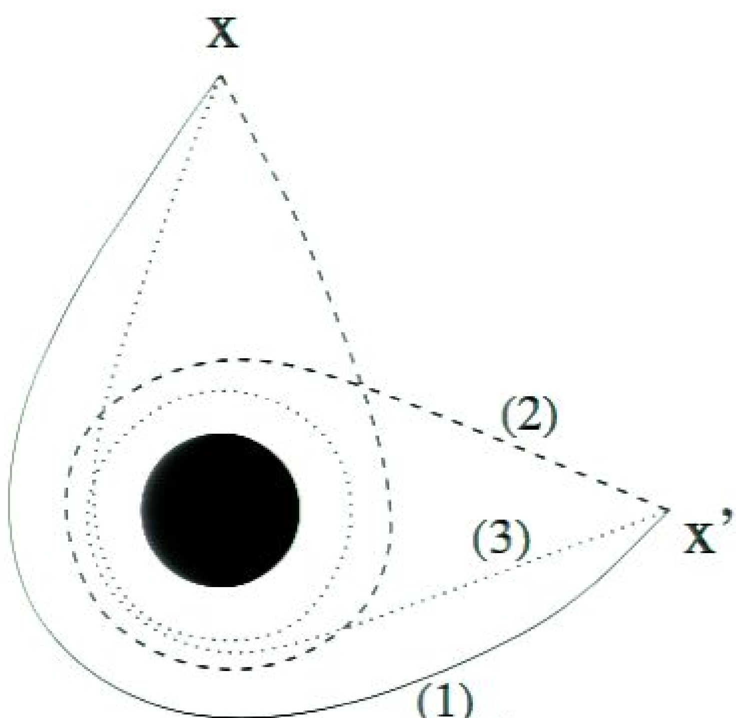


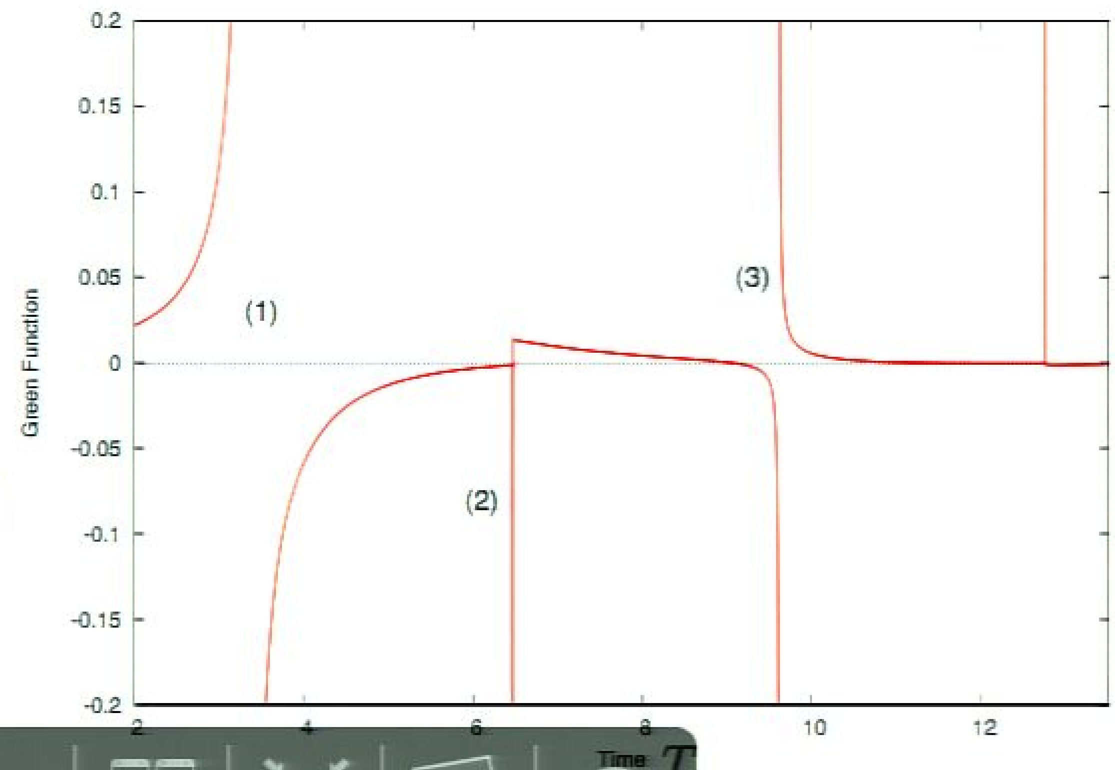
$\gamma = \pi/2$ $\xi = 1/6$

Singularities arising in the QNM sum

singular when phases are coherent ⇔ connecting null geodesic

Null geodesics joining x & x'   QNM Green function in Nariai

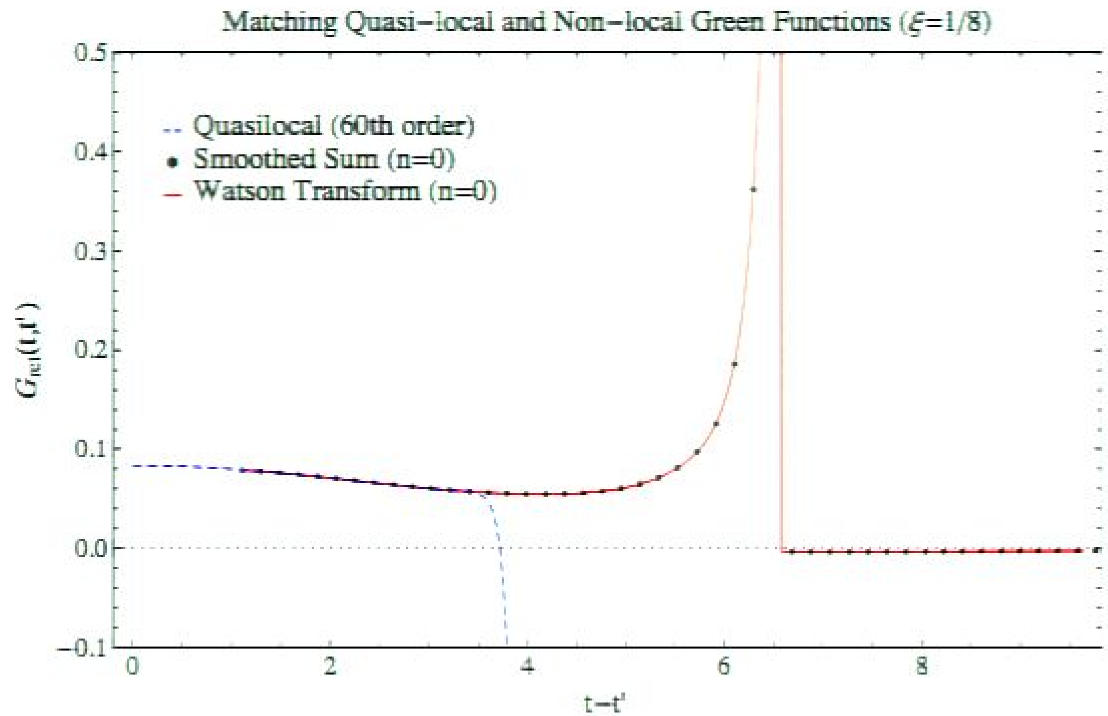


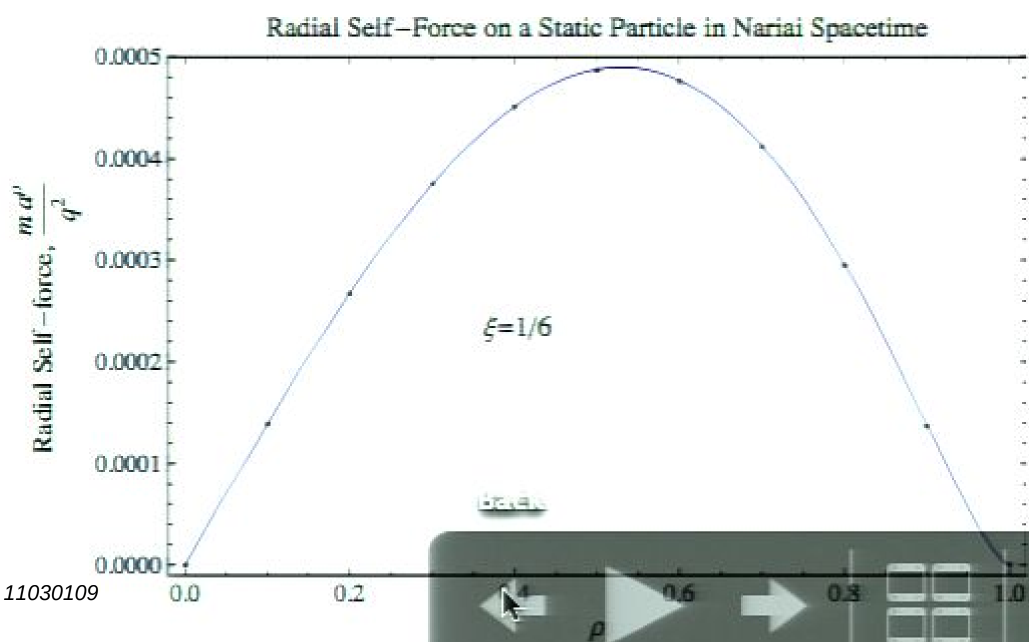


$\gamma = \pi/2$ $\xi = 1/6$

Matching QL and QNM in Nariai:

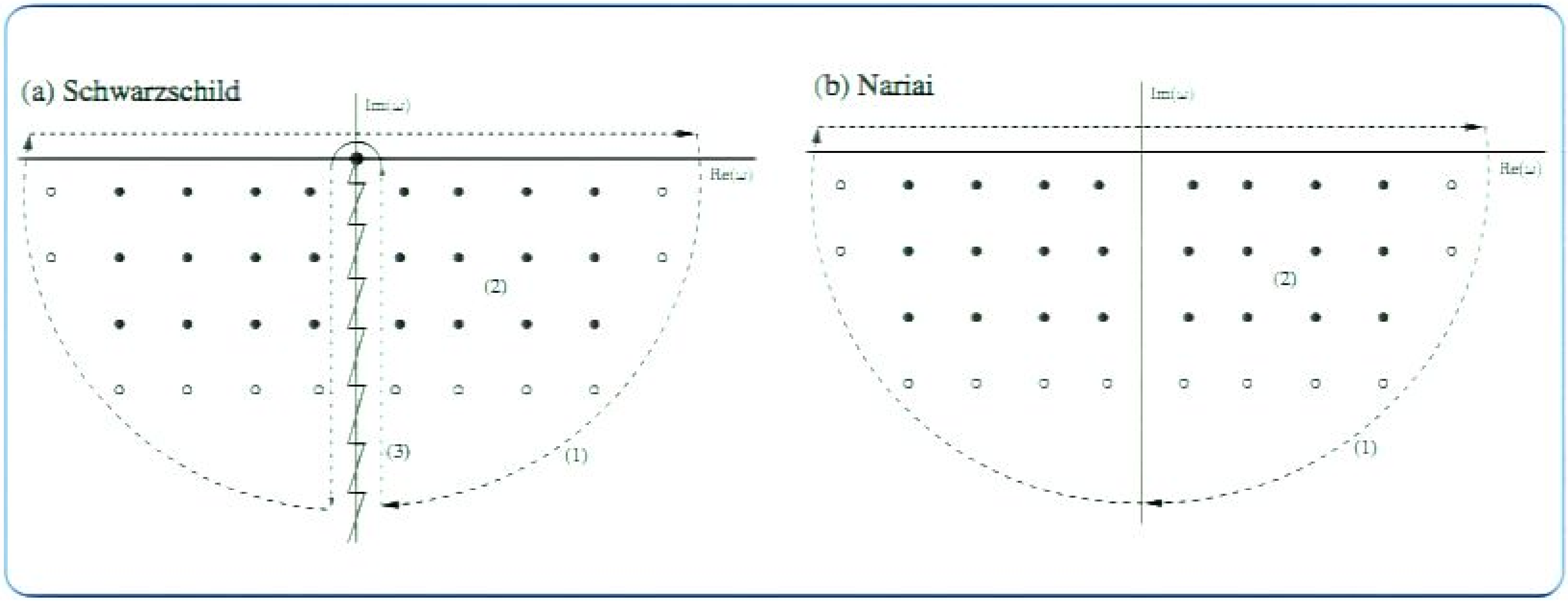
$$\gamma = 0$$





Self-force for static scalar particle

The Green function in the complex plane

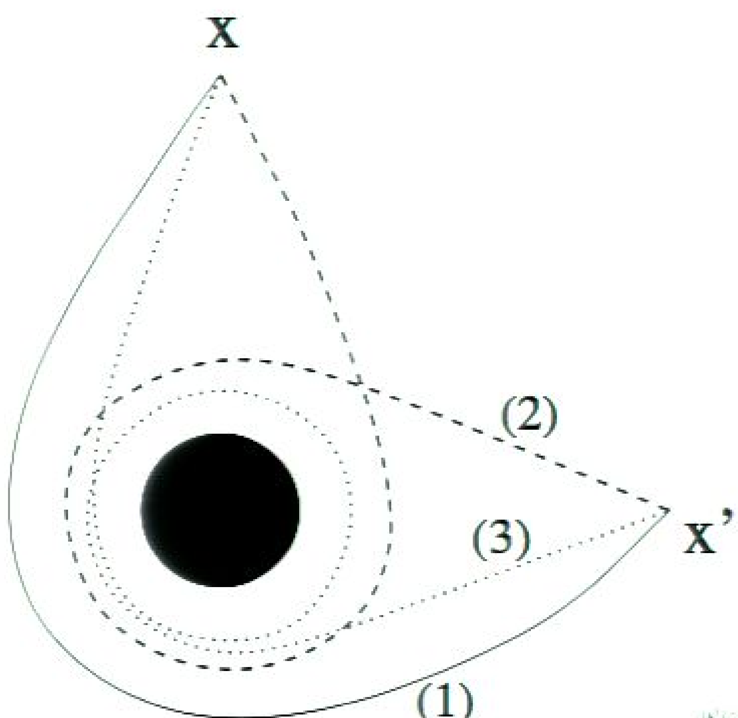


- QNMs: poles/residues - excitation factors
- QNM n-sum divergent[convergent] for $T = t - t' - r_* - r'_* < 0$ [> 0]
- $T \sim 0$ time for light ray to go from r'_* to 0 and out to r_*

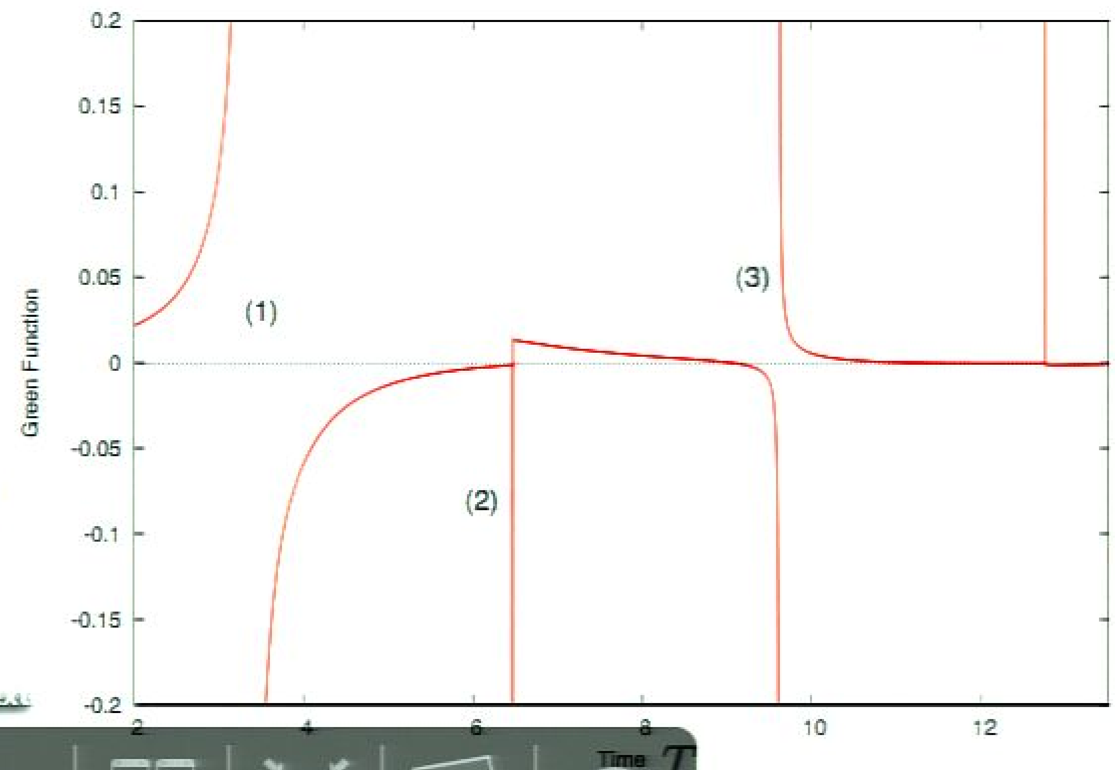
Singularities arising in the QNM sum

singular when phases are coherent ⇔ connecting null geodesic

Null geodesics joining x & x'



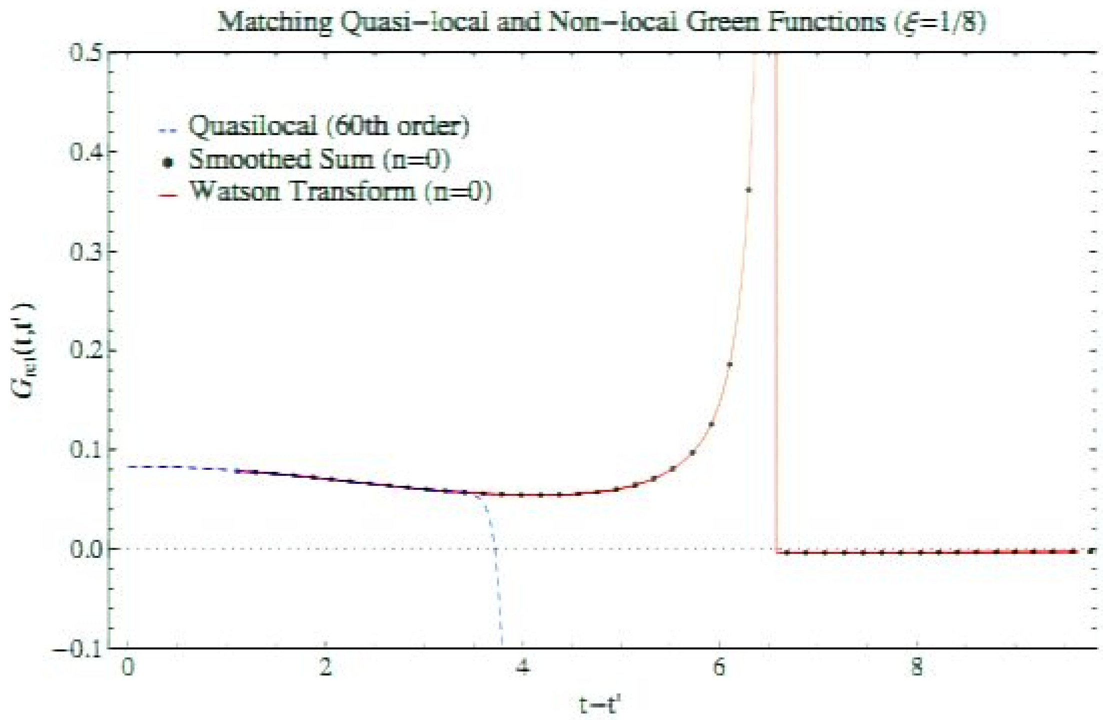
QNM Green function in Nariai

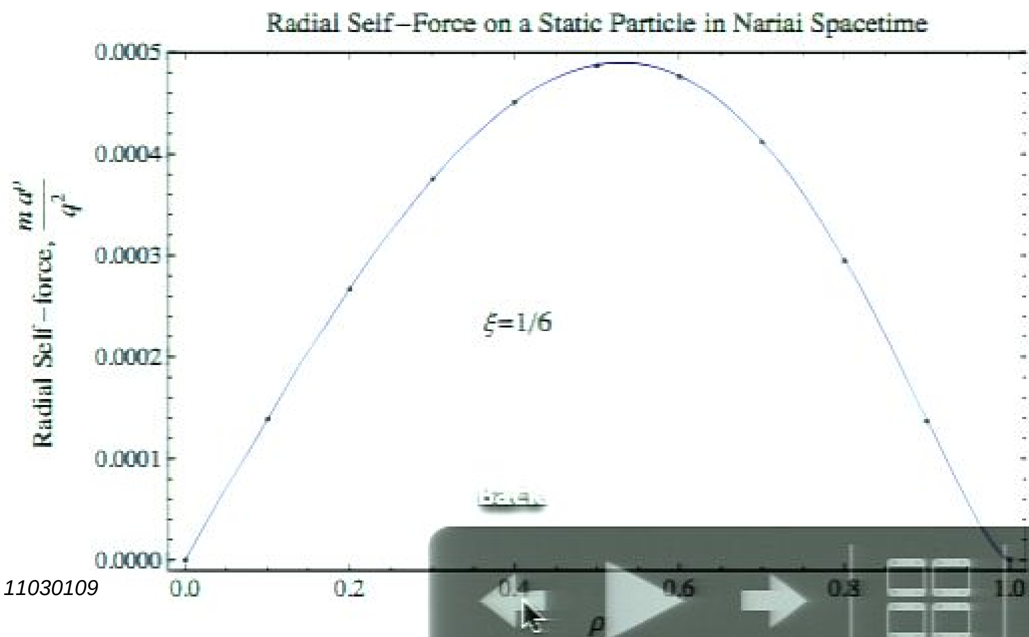


$\gamma = \pi/2$ $\xi = 1/6$

Matching QL and QNM in Nariai:

$$\gamma = 0$$

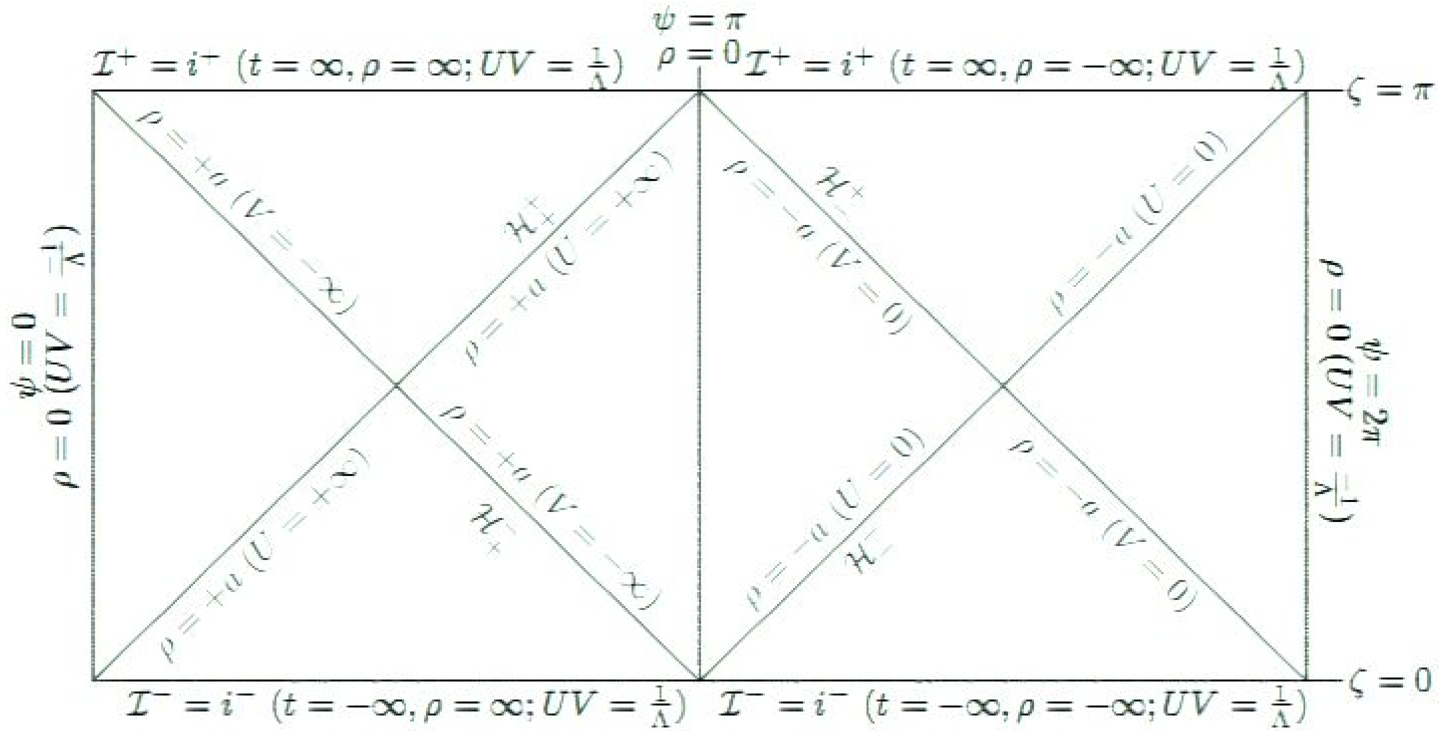




Self-force for static scalar particle

A simpler space-time: Nariai

Does the Poschl-Teller potential arise in any spacetime? yes!



$$ds^2 = -(1-\rho^2)dt^2 + \frac{d\rho^2}{1-\rho^2} + d\theta^2 + \sin^2\theta d\phi^2$$

The equation for $v(r)$ yields a natural expansion in $L = l + 1/2$:

$$\omega = L\omega_{-1} + \omega_0 + L^{-1}\omega_1 + \cdots$$

$$v(r) = \exp(S_0(r) + L^{-1}S_1(r) + L^{-2}S_2(r) + \cdots)$$

and imposing a continuity condition on $S_k'(r)$ at $r = r_c$

	$l = 2, n = 0$	$l = 3, n = 0$
Ctd. Frac.	$0.373672 - i0.088962$	$0.599444 - i0.092703$
12th order	$0.373679 - i0.088955$	$0.599443 - i0.092703$
6th order	$0.373642 - i0.088967$	$0.599439 - i0.092684$
WKB (6)	$0.3736 \quad - i0.0890$	$0.5994 \quad - i0.0927$

Critical orbits

Null geodesics: $\frac{1}{h(r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{1}{b^2} - \frac{f(r)}{h(r)} \equiv k^2(r,b)$ $b = L/E$ where

$L = h(r)\dot{\phi}$ and $E = f(r)\dot{t}$ are constants of the motion

Suppose there are critical values r_c and b_c such that

$$k^2(r_c, b_c) = 0 \qquad \text{and} \qquad \frac{\partial k^2(r_c, b_c)}{\partial r} = 0$$

Example: Schwarzschild $r_c = 3M$, $b_c = \sqrt{27}M$

Then defining $k_c(r) = \mathrm{sign}(r - r_c)\sqrt{k^2(r, b_c)}$

$$\exp\left(i\omega\int^{r_*} b_c k_c(r)\,\mathrm{d}r_*\right) v(r)$$

obeys the QNM boundary conditions.

Contribution from poles

$$G_{\rm QNM} = 2\,{\rm Re}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)B_{ln}\tilde{u}_{ln}(r)\tilde{u}_{ln}(r')e^{-i\omega_{ln}(t-t'-r_*-r'_*)}$$

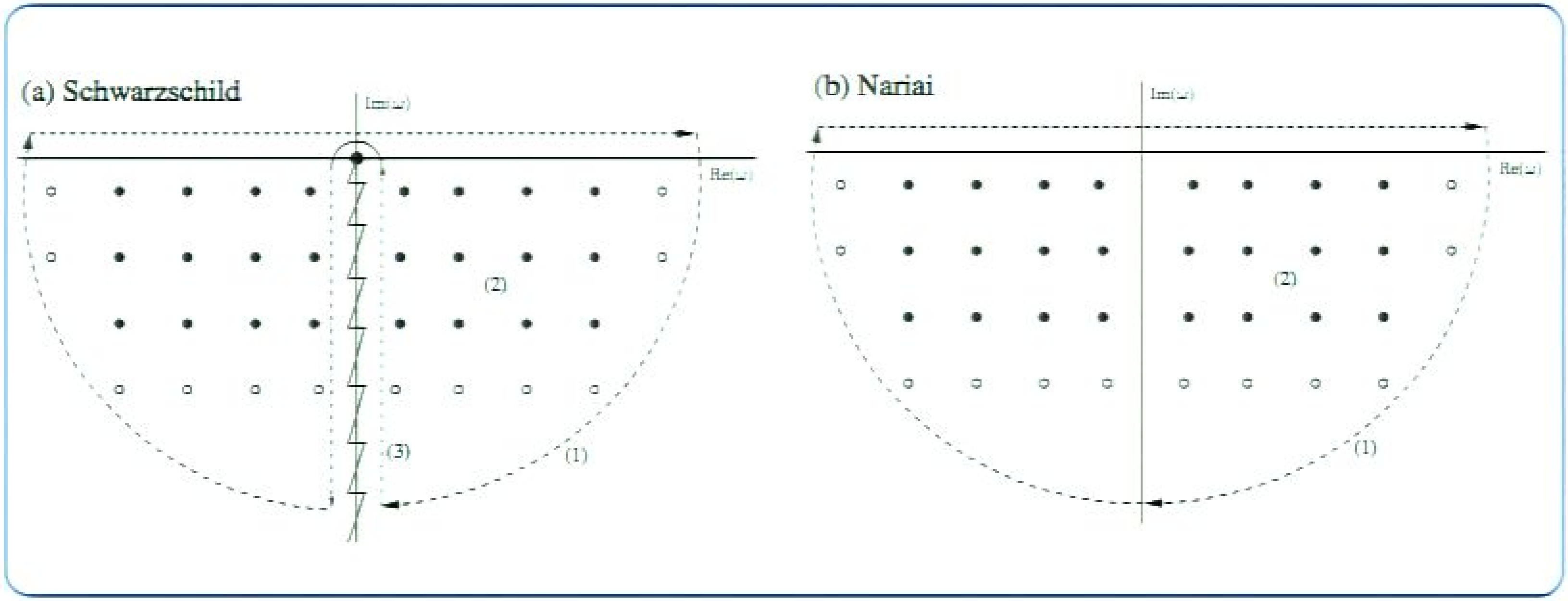
B_{ln} are the *excitation factors*

$$B_{ln} \equiv \frac{A^{\rm out}_{l\omega_{ln}}}{2\omega_{ln}\frac{dA^{\rm in}_{l\omega}}{d\omega}}$$

and $\tilde{u}_{ln}(r)$ are the QNM radial functions

$$\tilde{u}_{ln}(r_*) = \frac{u^{\rm in}_{l\omega_{ln}}(r_*)}{A^{\rm out}_{l\omega_{ln}}e^{i\omega_{ln}r_*}}$$

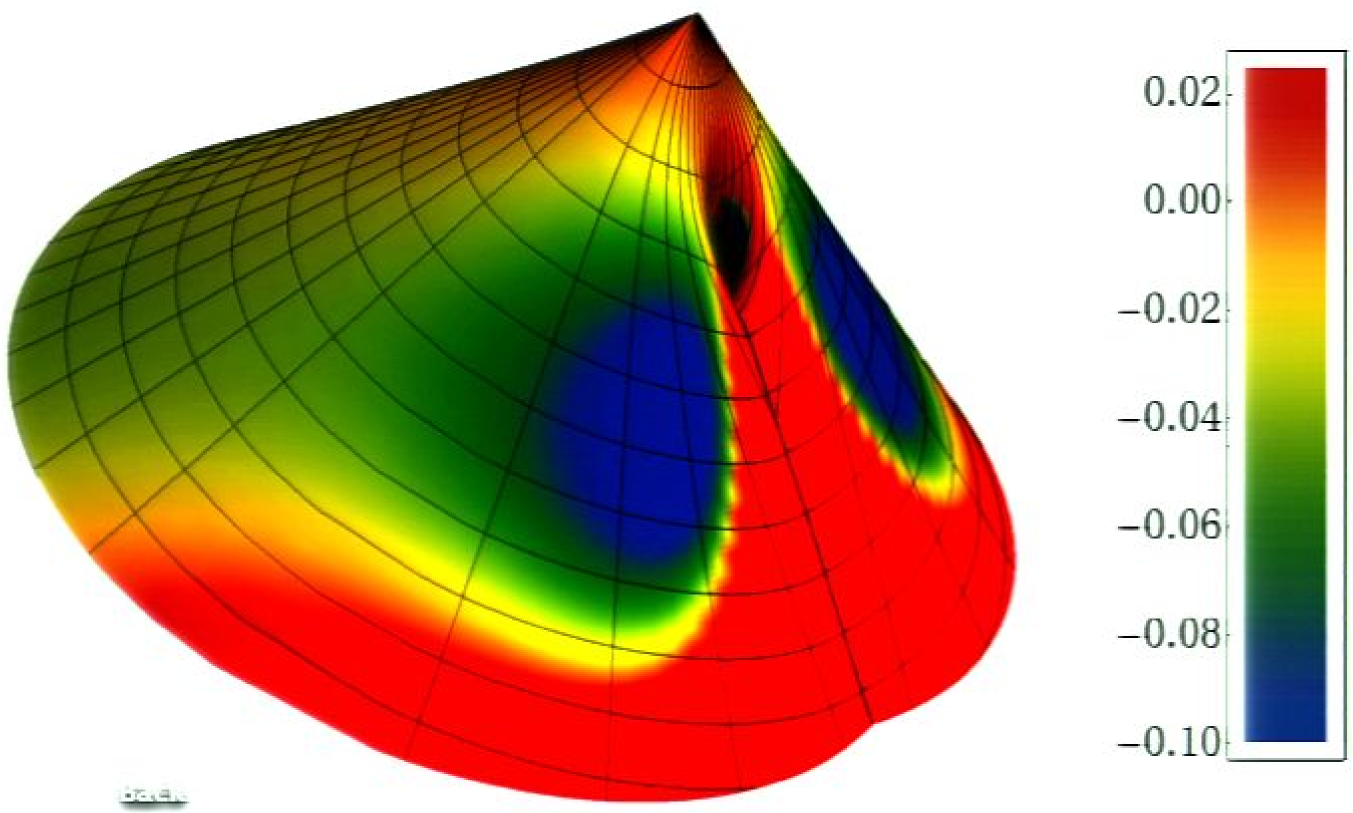
The Green function in the complex plane



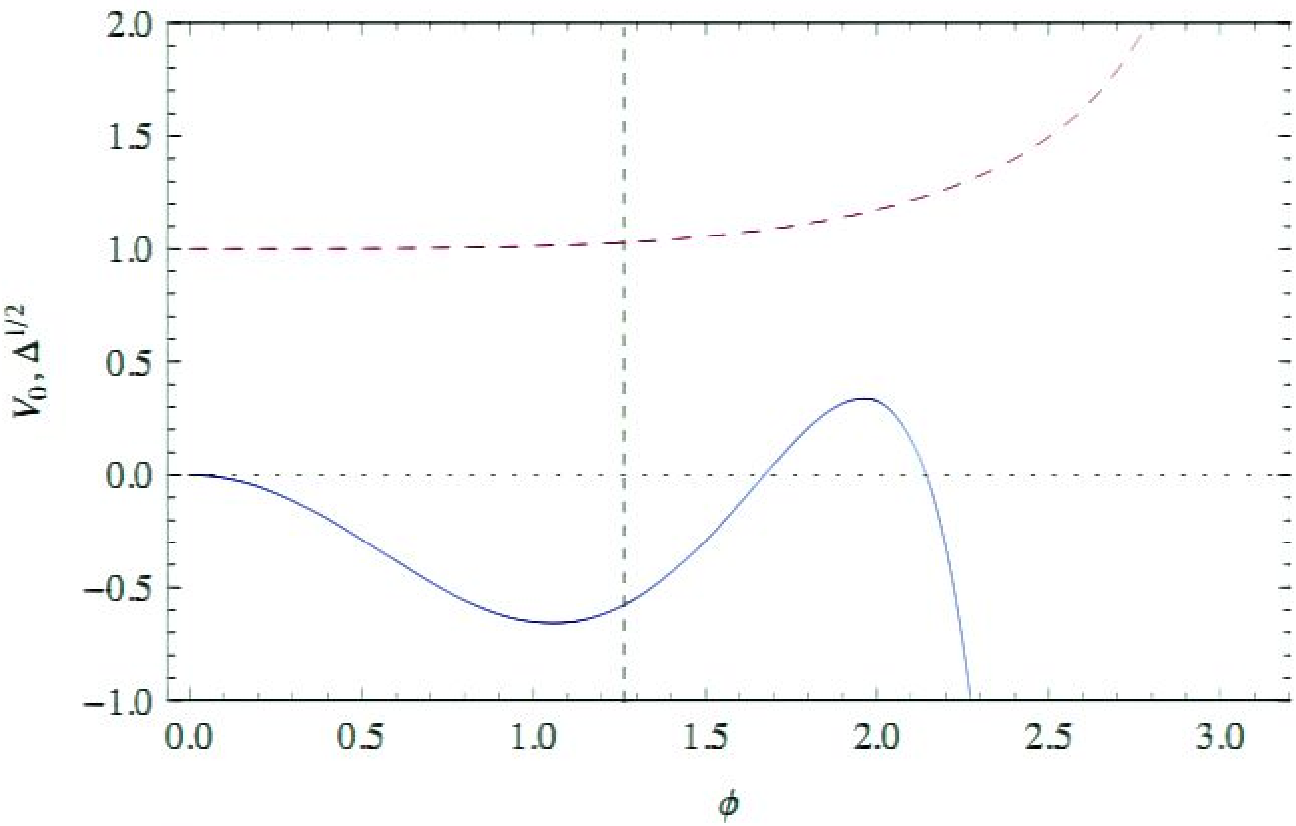
- Quasi Normal Modes: Zeroes of the Wronskian $A^{\text{in}}_{lm}(\omega) = 0$

Numerical Calculation on-light cone

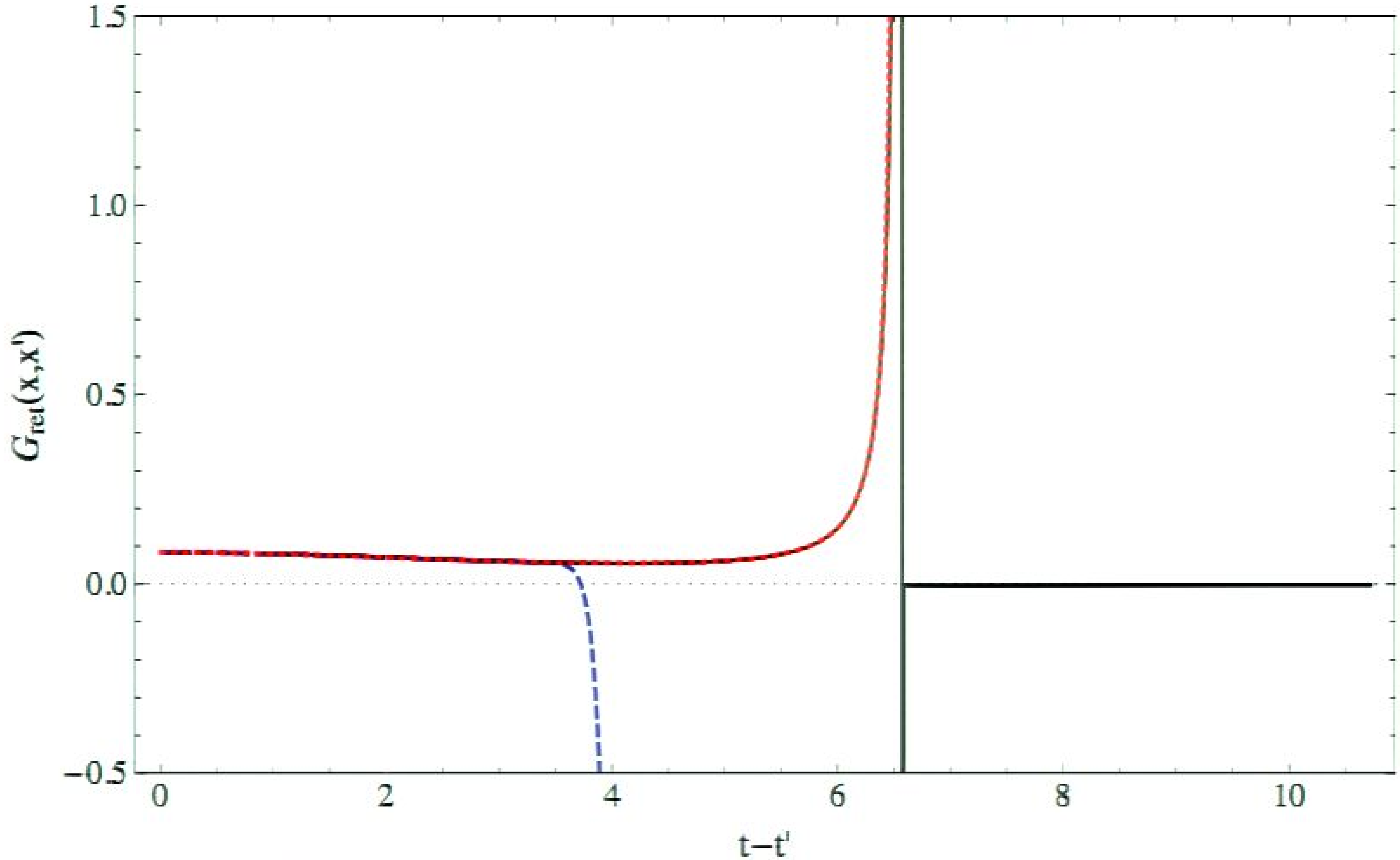
Numerically integrate the transport equations (ODEs) for $V_0(x, x')$ along null geodesics to get $V(x, x')$ on the light-cone.



Numerical integration of the transport equations for $\Delta^{1/2}(x, x')$ and $V_0(x, x')$ along a geodesic.



Convergence of series



Examples

Leading term for self-force in vacuum:

$$
\begin{aligned}
V_{abcd} = &-\frac{1}{280} R^{p}{}_{(a}{}^{q}{}_{b|;r|} R_{|p|c|q|d)}{}^{;r} - \frac{2}{315} R^{pqrs} R_{p(a|r|b} R_{|q|c|s|d)} \\
&+ \frac{1}{105} R^{p}{}_{(a}{}^{q}{}_{b} R^{rs}{}_{|p|c} R_{rsq|d)} + \frac{1}{840} R^{pqrs} R_{pq}{}^{t}{}_{(a} R_{|rst|b} g_{cd)} \\
&+ \frac{1}{8960} R^{pqrs} \Box R_{pqrs} g_{(ab} g_{cd)} - \frac{1}{40320} R^{pqrs} R_{pq}{}^{tu} R_{rstu} g_{(ab} g_{cd)}
\end{aligned}
$$

In Schwarzschild:

$$
V_{0(0)} = 0, \qquad V_{1(0)} = \frac{M^2}{15 r^6}, \qquad V_{2(0)} = \frac{M^2}{1008 r^9}(194M - 81r)
$$

$$
V_{3(0)} = \frac{M^2}{3150 r^{12}}(210 r^2 - 1125 rM + 1454 M^2),
$$

$$
V_{4(0)} = \frac{M^4}{38192400000 r^{15}}(-12867705 r + 28164482 M),
$$

Order	General				Vacuum, $m = 0$	
	Time	Terms	Can.	Can., $P = 0$	Terms	Can.
a_1	0	2	2	1	0	0
a_2	0.003	10	7	4	2	1
a_3	0.02	91	26	15	7	2
a_4	0.2	1058	113	68	56	5
a_5	3.6	13972	–	–	507	–
a_6	76	199264	–	–	4988	–
a_7	1489	2987366	–	–	51700	–
a_8	–	–	–	–	554715	–
a_9	–	–	–	–	6098069	–

How far can one go: V_0?

Order	General			Vacuum, $m = 0$		
	Time	Terms	Memory	Time	Terms	Memory
4	0.005	47	22.0 kB	0.003	5	2.5 kB
6	0.014	206	112 kB	0.009	22	12 kB
8	0.047	856	526 kB	0.019	94	59 kB
10	0.16	3 414	2.25 MB	0.05	384	260 kB
12	0.58	13 064	9.34 MB	0.19	1480	1.1 MB
14	2.1	48 167	37.1 MB	0.61	5485	4.2 MB
16	7.8	172 214	141 MB	2.1	19 637	16 MB
18	28	599 460	522 MB	6.8	68 295	58 MB
20	99	2 039 285	1.81 GB	23	231 837	208 MB

The Hadamard and DeWitt coefficients are related by

$$V_r{}^A{}_{B'} = \frac{\Delta^{1/2}}{2^{r+1} r!} \sum_{k=0}^{r+1} (-1)^k \frac{(m^2)^{r-k+1}}{(r-k+1)!} a_k{}^A{}_{B'}$$

with inverse

$$a_{r+1}{}^A{}_{B'} = \Delta^{-1/2} \sum_{k=0}^{r} (-2)^{k+1} \frac{k!}{(r-k)!} (m^2)^{r-k} V_k{}^A{}_{B'} + \frac{(m^2)^{r+1}}{(r+1)!} \delta^A{}_{B'}$$

In particular,

$$V_r^{(m^2=0)\,A}{}_{B'} = \frac{\Delta^{1/2}}{2^{r+1} r!} (-1)^{r+1} a_{r+1}{}^A{}_{B'}$$

These relate the ‘tail term’ of the massive and massless theories

$$V^A{}_{B'} = \sum_{r=0}^{\infty} V_r^{(m^2=0)\,A}{}_{B'} \frac{(2\sigma)^r \, r! J_r\left((-2m^2\sigma)^{1/2}\right)}{(-2m^2\sigma)^{r/2}}$$

$$+ m^2 \Delta^{1/2} \frac{J_1\left((-2m^2\sigma)^{1/2}\right)}{(-2m^2\sigma)^{1/2}} \delta^A{}_{B'}$$

The Hadamard approach

$$G_{\mathrm{f}}^{AB'}(x,x') = i\langle\Psi|\mathrm{T}\left[\hat{\varphi}^{A}(x)\hat{\varphi}^{B'}(x')\right]|\Psi\rangle$$
$$= \frac{1}{8\pi^2}\left[\frac{U^{AB'}(x,x')}{\sigma+i\epsilon} + V^{AB'}(x,x')\ln(\sigma+i\epsilon)\right]$$

$$G_{\mathrm{f}}{}^{AB'}(x,x') = \frac{1}{8\pi}\left(G_{\mathrm{adv}}^{AB'}(x,x') + G_{\mathrm{ret}}^{AB'}(x,x')\right) + \frac{i}{2}\langle\Psi|\{\hat{\varphi}^{A}(x),\hat{\varphi}^{B'}(x')\}|\Psi$$

$$G_{\mathrm{ret}}{}^{A}{}_{B'}(x,x') = \theta_{-}(x,x')\left\{U^{A}{}_{B'}(x,x')\,\delta(\sigma) - V^{A}{}_{B'}(x,x')\,\theta(-\sigma)\right\},$$

$$U^{A}{}_{B'}(x,x') = \sum_{r=0}^{\infty} U_r(x,x')^{A}{}_{B'}\sigma^{r} \qquad V^{A}{}_{B'}(x,x') = \sum_{r=0}^{\infty} V_r(x,x')^{A}{}_{B'}\sigma^{r}$$

Find $U_0{}^{A}{}_{B'} = \Delta^{1/2}\delta^{A}{}_{B'}$, then write

$$\sum_{r=1}^{\infty} U_r(x,x')\sigma^{r-1} = W(x,x') = \sum_{r=0}^{\infty} W_r(x,x')\sigma^{r}$$

The Hadamard approach

$$G_{\mathrm{f}}^{AB'}(x,x') = i\langle\Psi|\mathrm{T}\left[\hat{\varphi}^{A}(x)\hat{\varphi}^{B'}(x')\right]|\Psi\rangle$$
$$= \frac{1}{8\pi^2}\left[\frac{U^{AB'}(x,x')}{\sigma+i\epsilon} + V^{AB'}(x,x')\ln(\sigma+i\epsilon)\right]$$

$$G_{\mathrm{f}}{}^{AB'}(x,x') = \frac{1}{8\pi}\left(G_{\mathrm{adv}}^{AB'}(x,x') + G_{\mathrm{ret}}^{AB'}(x,x')\right) + \frac{i}{2}\langle\Psi|\{\hat{\varphi}^{A}(x),\hat{\varphi}^{B'}(x')\}|\Psi$$

$$G_{\mathrm{ret}}{}^{A}{}_{B'}(x,x') = \theta_{-}(x,x')\left\{U^{A}{}_{B'}(x,x')\,\delta(\sigma) - V^{A}{}_{B'}(x,x')\,\theta(-\sigma)\right\},$$

$$U^{A}{}_{B'}(x,x') = \sum_{r=0}^{\infty} U_r(x,x')^{A}{}_{B'}\sigma^{r} \qquad V^{A}{}_{B'}(x,x') = \sum_{r=0}^{\infty} V_r(x,x')^{A}{}_{B'}\sigma^{r}$$

Find $U_0{}^{A}{}_{B'} = \Delta^{1/2}\delta^{A}{}_{B'}$, then write

$$\sum_{r=1}^{\infty} U_r(x,x')\sigma^{r-1} = W(x,x') = \sum_{r=0}^{\infty} W_r(x,x')\sigma^{r}$$