Title: Regulating Quantum Field Theory with Matrix Models

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Abstract: The standard method to study nonperturbative properties of quantum field theories is to Wick rotate the theory to Euclidean space and regulate it on a Euclidean Lattice. An alternative is "fuzzy field theory". This involves replacing the lattice field theory by a matrix model that approximates the field theory of interest, with the approximation becoming better as the matrix size is increased. The regulated field theory is one on a background noncommutative space. I will describe how this method works and present recent progress and surprises.

Regulating Quantum Field Theory with Matrix Models

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Perimeter Institute, March 9th 2011

Motivation

- Regularization of Quantum Field Theory.
- Physics in noncommutative spacetime.
- Possible new micro structure for spacetime.

Outline

- Divergences in Quantum Field Theory.
- Parallels with statistical systems near a continuous phase transition.
- Reformulating familiar concepts for a non-commutative world.
- Matrix models, the basic phenomena
- Examples of fuzzy spaces.
- Simulations
- Emergent background geometry.

Divergences in Quantum Field Theory

Quantum Electrodynamics

$$\mathcal{L} = -\frac{Z_3}{4} F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\psi}(\gamma^\mu D_\mu + m + \delta m)\psi$$

$$D_\mu = \partial_\mu + ieA_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$e = Z_3^{1/2} e_B \qquad A_\mu = Z_3^{-1/2} A_\mu^B \qquad \psi = Z_2^{-1/2} \psi^B$$

$$Z_i(e, \frac{\mu}{\Lambda_0})$$

The simplest field theory: A Hermitian Scalar Field φ

Wick rotate

- $t -> i\tau$,
- $\varphi(\vec{x}, t) \to \phi(\vec{x}, \tau)$, with ϕ also Hermitian, $\phi^* = \phi$
- $S[\varphi] \to -iS_{\mathrm{E}}[\phi]$
- $\mathrm{e}^{iS[\varphi]} \to \mathrm{e}^{-S_{\mathrm{E}}}$

$$S[\varphi] = \int dt d^3x \{ \frac{1}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{1}{2} \left(\vec{\nabla} \varphi \right)^2 - V(\varphi) \}$$

$$S_{\mathrm{E}}[\phi] = \int d\tau d^3x \{ \frac{1}{2} \left(\frac{\partial \varphi}{\partial \tau} \right)^2 + \frac{1}{2} \left(\vec{\nabla} \phi \right)^2 + V(\phi) \}$$

Universality

K. Wilson explained that renormasibility is the insensitivity to micro-scructure. His renormalization group put the Landau theory of continuous phase transitions on a firmer footing. In Phys. Rev. B 4 3184 (1971) Wilson explained the relation to the Kadanoff block spin transformations deriving scailing at the critical point from block spin transformations via the renormalization group equations. The Wilson renormalization group explained why critical exponents in critical phenomena depended only on space dimension and symmetry of the orderpameter.

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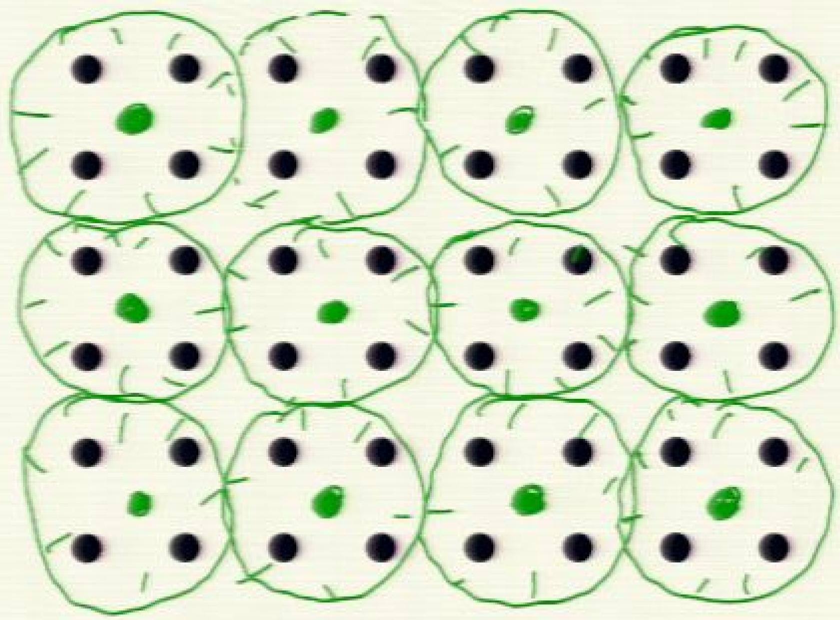
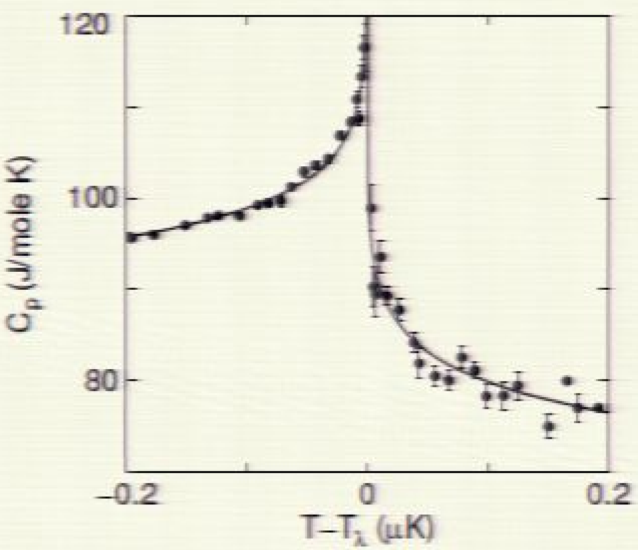
Block Spin Transformations



New effective spins are placed on the contracted lattice sites.

Superfluid Specific Heat

The Specific Heat of Liquid Helium in Zero Gravity very near the Lambda Point from *J. A. Lipa et al* Phys. Rev. **B 68**, 174518 (2003). The specific heat exponent $\alpha = -0.0127 \pm 0.0003$.



Landau-Ginzburg Model

Superfluid helium-4 is very well described by the microscopic 3-dimensional energy density:

$$\mathcal{H}_{XY}(\psi) = \frac{1}{2}|\nabla\psi|^2 + \frac{r}{2}|\psi|^2 + \frac{\lambda}{4!}|\psi|^4$$

From this, using perturbation theory, the renormalization group and orther resummation techniques, the most precise estimates of the specific heat, critical exponents and universal amplitute ratios as well as scaling functions are determined. All are in excellent agreement with experiment, but recent space based experiments are challenging the precision of exponents obtained from 7-loop calculations.

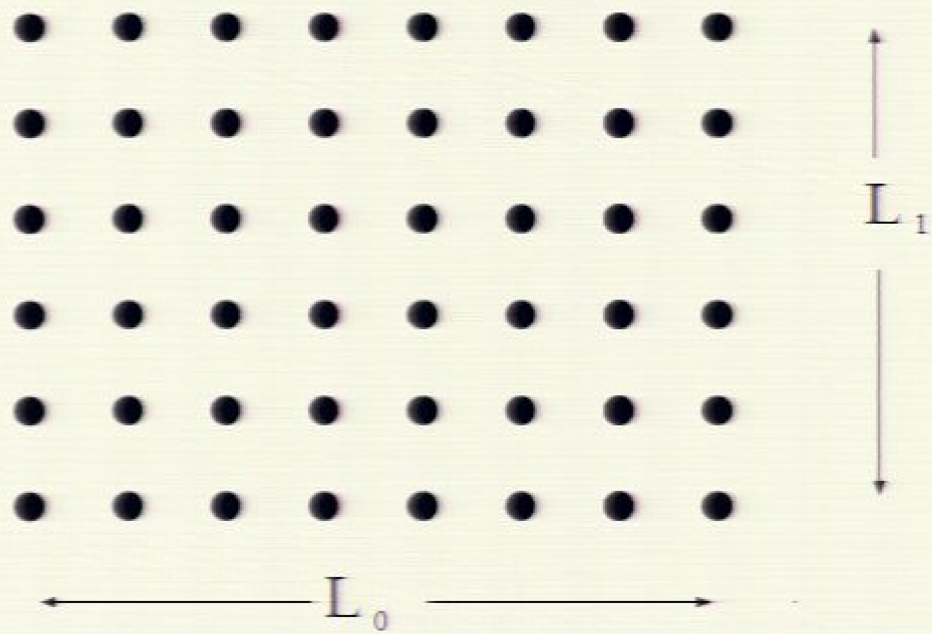
Non-perturbative approach to Field Theory

Use a microscopic lattice structure:
The lattice regularization involves replacing $S_{\mathrm{E}}[\phi]$ with a lattice Hamiltonian so that $S_{\mathrm{E}}[\phi] \to \beta H[s]$ with

$$H[s] = -\frac{1}{2} \sum_{<i,j>} s_i s_j + \sum_i U(s_i)$$

If $s_i \in \{-1, 1\}$ the model reduces to an Ising model.

A 2-dimensional lattice



Scalar and spinor field variables are placed on the lattice sites while gauge fields (group elements) are placed on links.

2-d Ising model

Simple Sampling:

- Choose $s_i = \pm 1$ with equal probability independently at each lattice site.
- Generate N configurations $s^{(k)} = \{s_i^{(k)}\}$.
- Estimate the partition function

$$Z = \frac{2^{L_0 L_1}}{N} \sum_{k=1}^{N} \mathrm{e}^{-S(s^{(k)})}$$

and observables

$$< \mathcal{O} >= \frac{2^{L_0 L_1}}{ZN} \sum_{k=1}^{N} \mathrm{e}^{-S(s^{(k)})} \mathcal{O}(s^{(k)})$$

But very wasteful: For an $L \times L$ lattice the fraction of configurations that is needed for 90% of the partition function in the 2-d Ising model at the phase transition is (Hasenbusch 2002)

L	Fraction
2	0.875
3	0.133
4	0.0343
5	0.00283

I.e. exponentially few configurations needed.

Metropolis Algorithm

Idea: Generate the configurations with the probability distribution

$$P(s) = \frac{1}{Z} e^{-S(s)}$$

Done by creating a discrete dynamics (Markov Chain) to approximate a Boltzmann Equation. The late time equilibrium should be the Boltzmann distribution with energy functional $S(s)/\beta$.
The dynamics is fictional and choosen to best fit the purpose and

$$< \mathcal{O} > = \frac{1}{N} \sum_{k=1}^{N} \mathcal{O}(s^{(k)})$$

Difficulties for Lattice Field Theory

- Complex terms in the action functional
- Fermions
  - Non-local
  - Fermion doubling
  - Chiral symmetry on the lattice
  - Supersymmetry on the lattice
  - Fermion determinant of no definite sign

The fuzzy sphere
Berezin (1975), Hoppe (1982) and Madore (1992)

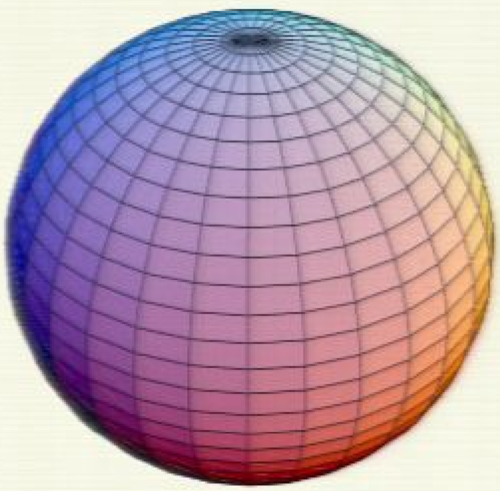
Berezin, Hoppe and Madore treated the sphere as a phase space and quantized this[1].
Take the $su(2)$ generators L_i (familiar generators of angular momentum). $[L_a, L_b] = i\epsilon_{abc}L_c$

$$L_a^2 + L_b^2 + L_c^2 = j(j+1)\mathbf{1}$$

[1] See Lectures on fuzzy and fuzzy SUSY physics by A. P. Balachandran, Seçkin Kürkçüoğlu and S. Vaidya World Scientific (2007)

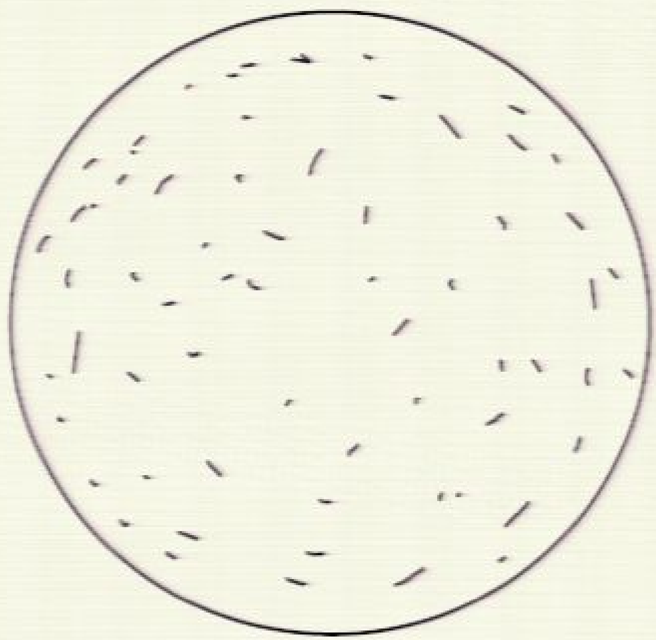
A Sphere in 3-space

A discretized sphere à la Mathematica



A Fuzzy Sphere in 3-space

The fuzzy sphere is naturally round with $SO(3)$ symmetry and represented by an algebra $\mathrm{Mat_N}$ with additional structure.



A sphere from matrices

Let $N_j = \frac{2}{\sqrt{N^2-1}} L_j$

We get a sphere

$N_1^2 + N_2^2 + N_3^2 = \mathbf{1}$. A nice round "sphere".

But it is non-commutative.

$[N_1, N_2] = \frac{2\mathrm{i}}{\sqrt{N^2-1}} N_3$

There is an uncertainty principal for spatial position!
But for $\mathrm{N} \to \infty$ we recover a commutative sphere.

The algebra of functions on S^2 is the algebra of the spherical harmonics Y_{lm}. In the fuzzy sphere the

$$Y_{lm} \to \hat{Y}_{lm}$$

with $\hat{Y}_{lm}$ the polarization tensors of nuclear physics. Simplest example $N = 2$,

$$\vec{x} \to \vec{\sigma}.$$

Using the technology of $\star$-products it is possible to keep the elements of the algebra as the Y_{lm} and only change the multiplication rule so that

$$Y_{lm} \cdot Y_{l'm'} \to Y_{lm} \star_N Y_{lm}$$

and for

$$N \to \infty \qquad \star_N \to \cdot$$

But!

No matter how we dress it up with $\star$ products or suggestive matrices looking like spheres, M_{N} is just a matrix algebra. To capture a geometrical object we must supply more structure.

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No matter how we dress it up with $\star$ products or suggestive matrices looking like spheres, $\mathrm{Mat}_{\mathrm{N}}$ is just a matrix algebra. To capture a geometrical object we must supply more structure.

Can you hear the shape of a drum?

M. Kac, American Mathematical Monthly Vol 73, 1 (1966)

Fuzzy Spaces

The round fuzzy S^2 is specified by the Laplacian

$$\mathcal{L}^2 = [L_i, [L_i, \cdot]]$$

where L_i are the irreducible $SU(2)$ generators of dimension $d_L = L + 1$.
An ellipsoidal geometry can be specified by

$$\Delta_L = \sum_{i=1}^{3} \frac{1}{l_i} [L_i, [L_i, \cdot]].$$

This construction generalizes to $\mathbb{CP}^N$ by choosing the matrix dimension to be $d_L = \frac{(N+L)!}{N!L!}$ and replacing the L_i by the $SU(N+1)$ generators in this representation.
For example $\mathbb{CP}^3$ can be specified by choosing the matrix size to be $d_L = \frac{1}{6}(L+3)(L+2)(L+1)$ and the Laplacian to be

$$\mathcal{L}^2 = [L_a, [L_a, \cdot]]$$

But $\mathbb{CP}^3$ is rather special since it can be also be realised as an $SO(6)$ orbit

$$\mathcal{L}^2_{(6)} = \frac{1}{2}[J_{AB}, [J_{AB}, \cdot]]$$

where J_{AB} are the irreducible $SO(6)$ generators in this representation. The geometry again can be deformed by giving different coefficients for the different generators.
In fact the same orbit of points can also be realised as an $SO(5)$ orbit.

One special family of deformations is that which preserves the $SO(5)$ subgroup of $SO(6)$ and takes advantage of the fact that $\mathbb{CP}^3$ is an S^2 bundle over S^4. In this case the Laplacian can be written

$$\mathcal{L}_h^2 = \mathcal{L}_{(5)}^2 + h\Big(2\mathcal{L}_{(5)}^2 - \mathcal{L}_{(6)}^2\Big)$$

The eigenvalues of $\mathcal{L}_h$ are $n(n+3) + h2m(m+1)$ with $n = 0, \cdots, L$ and $m = 0, \cdots, n$. The term $\mathrm{C}_I = \Big(2\mathcal{L}_{(5)}^2 - \mathcal{L}_{(6)}^2\Big)$ is zero on representations that correspond to functions on S^4 and non-zero and positive on the remaining representations, i.e. representations with $m \neq 0$. When the parameter h is chosen large these non-S^4 representations are suppressed and the low lying spectrum coincides with that of S^4. If we choose $h \gg L(L+3)$ all the modes with $\Lambda^2 \leq L(L+3)$ will correspond to S^4 modes and the others will decouple from the low energy physics.

By choosing the Laplacian to be

$$\mathcal{L}^2_{h,h'} = \mathcal{L}^2_{(4)} + h'\Big(L(L+3) - \mathcal{L}^2_{(5)}\Big) + h\Big(2\mathcal{L}^2_{(5)} - \mathcal{L}^2_{(6)}\Big), \qquad (1)$$

we specify the low energy geometry to be that of a round S^3. The term proportional to h' makes the top representation associated with the fuzzy S^4 the lowest lying representation of this new Laplacian and $\mathcal{L}^2_{(4)} = \frac{1}{2}[J_{\alpha\beta}, [J_{\alpha\beta}, \cdot]]$ breaks the $SO(5)$ symmetry down to $SO(4)$ and ensures that the new low lying spectrum coincides with a cutoff version of that for a round S^3. Thus if $h' \gg L$ all modes with eigenvalues $\leq L(L+2)$ correspond to S^2 modes and the others decouple from the low energy physics.

The unit disc The fuzzy unit disc
(F. Lizzi, P. Vitale, A. Zampini, JHEP 0308 (2003) 057)
is obtained by taking the Laplacian given by

$$\Delta = 4[a^\dagger, [a, \cdot]]$$

where $a^\dagger$ and a are creation and annihilation operators $[a, a^\dagger] = 1$.

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Larger Point of View

Quantize $\mathbb{C}^K$ for some K and find a subsurface in this space. Restrict to this surface in the quantized space. The simplest examples are the Moyal-Groenewold plane.
Many examples are based on coadjoint orbits
[though see J. Arnlind, M. Bordemann, L. Hofer, J. Hoppe, H. Shimada arXiv:hep-th/0602290;
T.R.Govindarajan, Pramod Padmanabhan, T.Shreecharan arXiv:0906.1660] who give an examples of a fuzzy tori not based on coadjoint orbits.

However, the ability to retain rotational and higher symmetries is of significant advantage in applications especially in the non-perturbative study of field theories.

The action for fields other than scalar fields involves more than just the geometry of the underlying space. In the commutative setting there is typically some bundle structure associated with the construction.

In the case of spinor fields when the manifold $\mathcal{M}$ admits a spin-structure the fuzzy spinor action can be taken as

$$S[\Psi] = \frac{Tr}{d_L}(\bar{\Psi}(D_{\mathcal{M}} + m)\Psi)$$

where $D_{\mathcal{M}}$ is the Dirac operator and Ψ is a suitably defined spinor field. On S_F^2 one can take Ψ to be a two component column vector with entries from $\mathrm{Mat_N}$ and the Dirac operator to be

$$D_{S^2} = \sigma_i[L_i, \cdot] + 1.$$

The spectrum is an exact cutoff version of the commutative one.

A Dirac operator for S^4

In the case of S^4 one can take Ψ to be a four component spinor with matrix valued entries from $\mathrm{Mat}_{\mathrm{d_L}}$ and the Dirac operator given by

$$D_{S^4} = \sigma_{AB}[J_{AB}, \cdot] + 2 + h\mathrm{C}_I$$

with σ_{AB} the generators of $Spin(6)$ in the four dimensional representation.

$$\mathrm{C}_I = \left(2\mathcal{L}^2_{(5)} - \mathcal{L}^2_{(6)}\right)$$

This construction does not correspond to the fuzzy Kaluza-Klein construction of the scalar case, but for sufficiently large h the low lying spectrum is identical to that of a truncated version of the commutative theory. More generally the spinor field will require many more components than the minimum.

Yang-Mills Fields

$F_{jk} = i[L_j, A_k] - i[L_j, A_k] + \epsilon_{jkl} A_l + i[A_j, A_b k]$.
This includes a "normal" scalar field,

$$\Phi = \frac{1}{\sqrt{N^2-1}}(D_j - L_j)^2 = \frac{1}{2}(N_j A_j + A_j N_j + \frac{A_j^2}{\sqrt{c_2}}) \; .$$

A natural action for Yang-Mills is then

$$S[A] = \frac{Tr}{N}\left(-\frac{1}{4}F_{ij}^2 + \frac{m^2}{2}\Phi^2\right)$$

.

The Simplest Matrix Model

Consider the Gaussian probability distribution

$$\mathcal{P}(\Phi) = \frac{e^{-\beta Tr(\Phi^2)}}{Z} \qquad \text{where} \qquad Z = \int [d\Phi] e^{-\beta Tr(\Phi^2)} .$$

This distribution splits into the uniform distribution on $\mathrm{M} = SU(N)/U(1)^N$ and *a probability distribution* for the eigenvalues of Φ:

$$\mu(\{\lambda\}) = \prod_{i<j} (\lambda_i - \lambda_j)^2 \frac{e^{-\beta \sum_k \lambda_k^2}}{Z/\mathrm{Vol(M)}} .$$

which for large N converges to the Wigner semi-circle distribution $\rho(\lambda) = \frac{\beta}{N\pi}\sqrt{\frac{2N}{\beta} - \lambda^2}$.

$$< \frac{Tr}{N}(\Phi^2) >= \int_{-\sqrt{\frac{2N}{\beta}}}^{\sqrt{\frac{2N}{\beta}}} \lambda^2 \rho(\lambda) d\lambda = \frac{N}{2\beta} .$$

Generic Features of Random Matrix models.

Random matrix models, of a single random matrix, are typically characterised by the eigenvalue distribution of the random matrix. They have the generic features:

- The eigenvalues repell one another.
- The eigenvalues all fall within a finite domain. The domain may not be connected— the distribution is concentrated on "cuts". For the Wigner semi-circle the cut is $[-\sqrt{\frac{2N}{\beta}}, \sqrt{\frac{2N}{\beta}}]$.
- The spread in eigenvalues grows (typically) as $\sqrt{N}$.
- Phase transitions occur when cuts merge or separate.

Example: The Φ^4 matrix model.

$V(\Phi) = Tr(b\Phi^2 + c\Phi^4)$ with Φ an $N \times N$ matrix.

- The model is characterized by the distribution of the eigenvalues of Φ.
- For $c = 0$ the eigenvalues have a Wigner semi-circle distribution.
- For $c > 0$ and $b << 0$ the eigenvalues fall into two disconnected regions, i.e. they have a "two cut" distribution.
- The partition is not analytic at $b = -2\sqrt{Nc}$, only the first two derivatives of $\ln Z$ are continuous and the phase transition of 3rd order.
- The random matrix "gravity" transition occurs for $c < 0$ and $b > 0$.

A fuzzy field theory model.

Fuzzy field theories are matrix models with fixed background matrices. The scalar field theory of the fuzzy sphere has:
$S_N(\Phi, a, b, c) = Tr(-a[L_j, \Phi]^2 + b\Phi^2 + c\Phi^4)$
L_j are the generators of $su(2)$ in the N dimensional representation, again with Φ an $N \times N$ matrix.

The action $S_N(\Phi, a, b, c)$ converges for $N \to \infty$ to the action of a scalar field ϕ on the round commutative sphere.

$$\lim_{N\to\infty} \left| S(\phi, r, \lambda) - S_N(\Phi, \frac{1}{2N}, \frac{r}{2N}, \frac{\lambda}{4!N}) \right| \to 0 \,.$$

The matrix $\Phi = \int_{S^2} \omega \rho_N \phi$, with ω the unit volume form on S^2 and ρ_N is a particular matrix valued function on S^2.

$$\rho_N = \sum_{lm} Y_{lm} \hat{Y}_{lm}$$

where Y_{lm} are the spherical harmonics and $\hat{Y}_{lm}$ are polarization tensors satisfying

$$[L_3, \hat{Y}_{lm}] = m\hat{Y}_{lm} \qquad \text{and} \qquad [L_j, [L_j, \hat{Y}_{lm}] = l(l+1)\hat{Y}_{lm} \, .$$

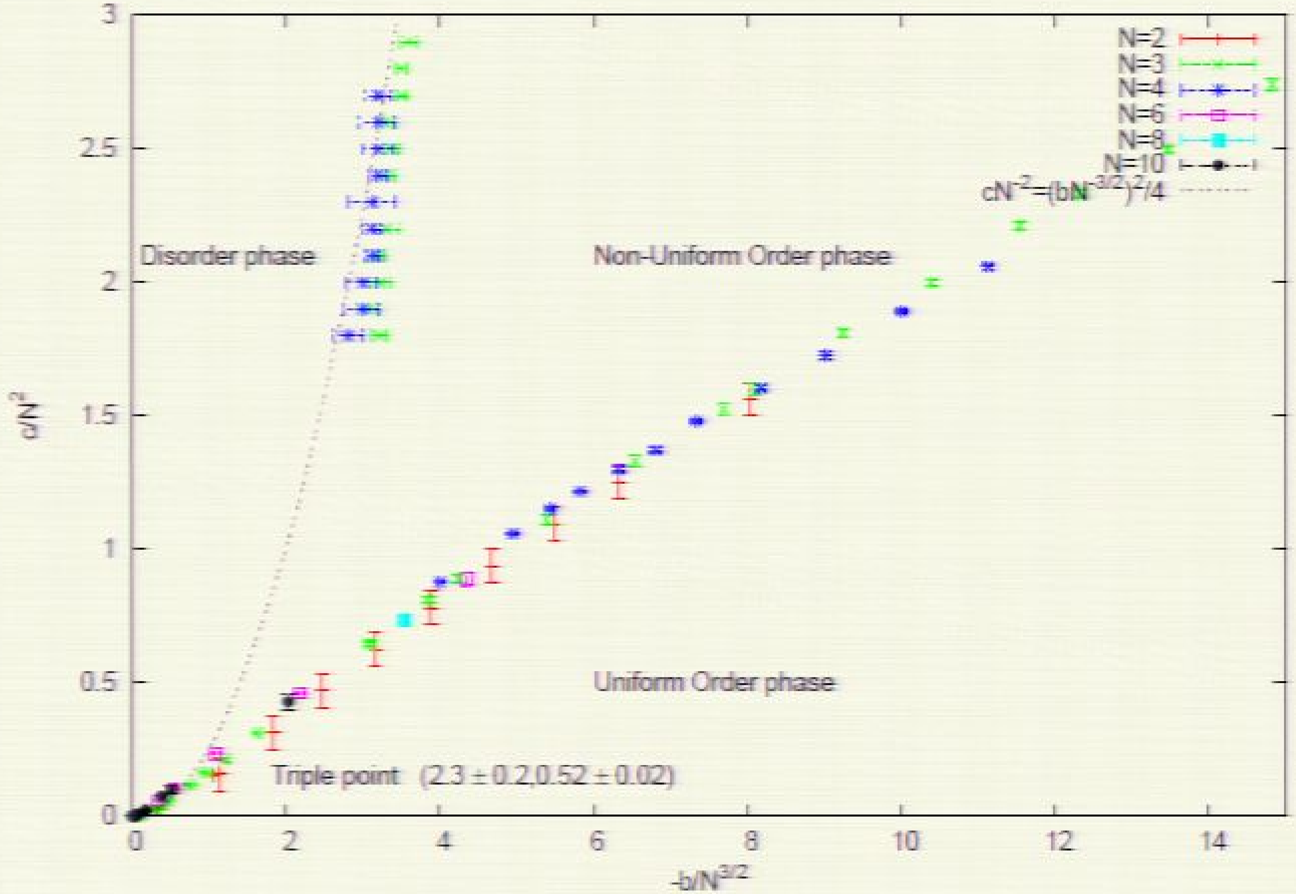
So that if

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} Y_{lm}$$

then

$$\Phi = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} c_{lm} \hat{Y}_{lm}$$

The phase diagram of fuzzy Φ^4



The phase diagram was generated by Monte Carlo simulations [X. Martin, F. Garcia Flores, D.O'C. 2006; Pamero 2007]. Recent perturbative efforts to calculate the transition lines [Saemann 2009]. It is controlled by the triple point.

To recover the commutative theory we must move the triple point off to infinity, preferably along a diagonal.

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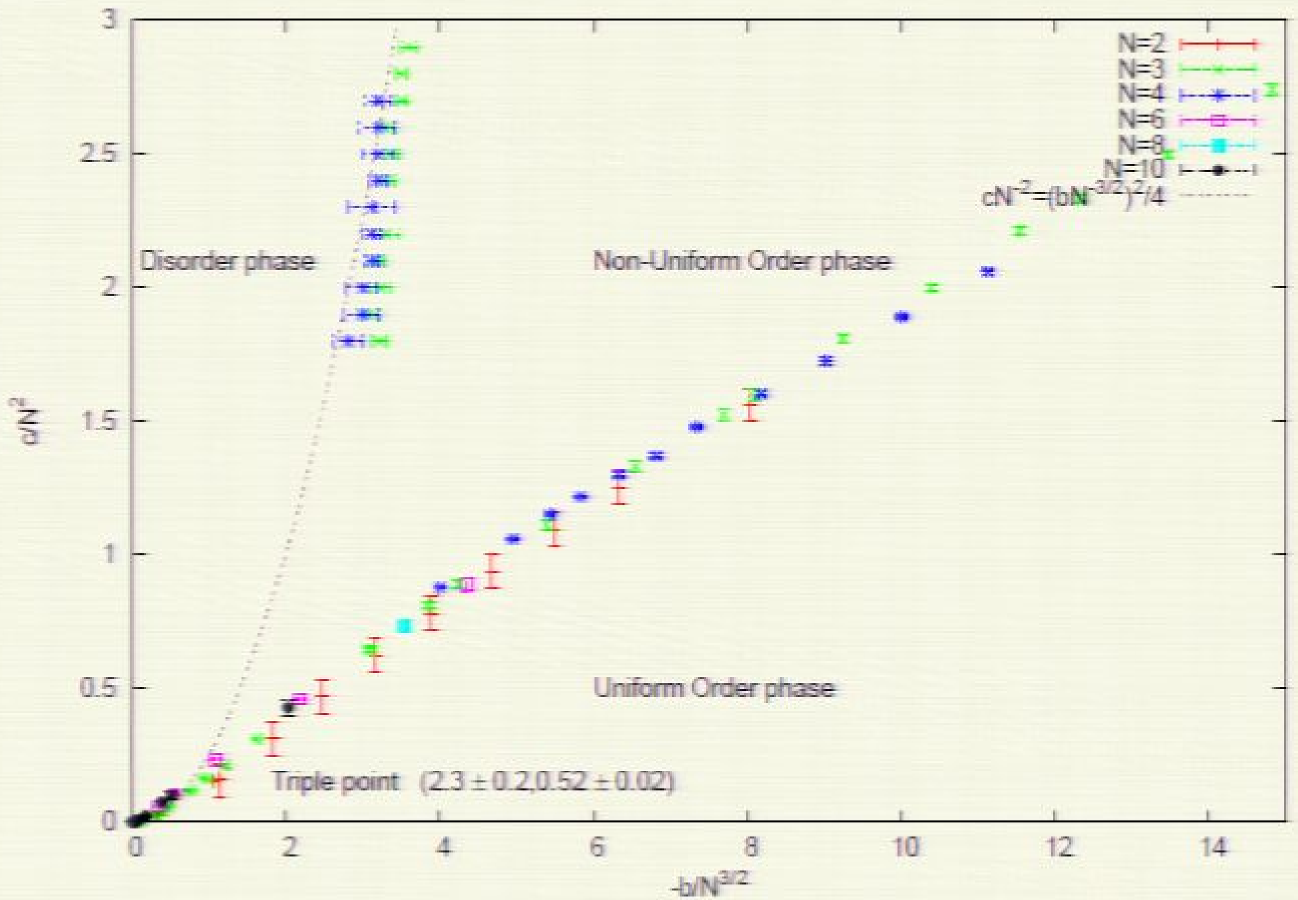

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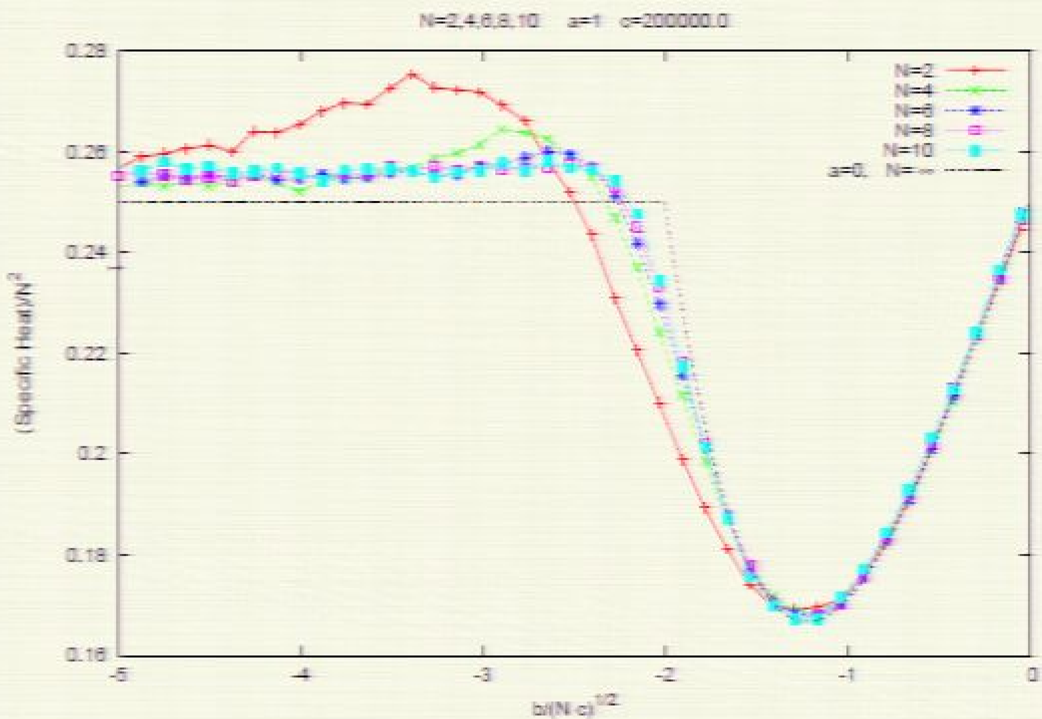
The specific heat

$$C_v = < S^2 > - < S >^2$$
$$S(\Phi) = -aTr([L_a, \Phi]^2) + bTr(\Phi^2) + cTr(\Phi^4)$$



A 3-matrix model with $SO(3)$ symmetry. Yang-Mills revisited.

The most general quartic single trace 3-matrix model with global $SO(3)$ symmetry has energy

$$E = \frac{Tr}{N}\left(-\frac{1}{4}[D_j, D_k]^2 + \frac{2i}{3}\epsilon_{jkl}D_jD_kD_l + bD_j^2 + c(D_j^2)^2\right)$$

The Potential $V(D) = Tr(bD_j^2 + c(D_j^2)^2)$
breaks $D_j \to D_j + d_j\mathbf{1}$ symmetry.

We are left with zero dimensional Yang Mills with **a Myers term**.

Partition Function

$$Z(\beta, g, b, c) = \int [dD_j] e^{-S(D)} \qquad \text{where} \qquad S(D) = -\beta E(D)$$

Ground State

The critical points of the model with $V = 0$ are given by

$$[D_k, ([D_j, D_k] - \mathrm{i}\epsilon_{jkl} D_l)] = 0.$$

So representations of the Lie algebra of $SU(2)$ are critical points with energy $E_{saddle} = -\frac{1}{6}\frac{Tr}{N}(D_j^2)$.

The minimum energy configuration is

$D_j = L_j$ with $E_0 = -\frac{N^2-1}{24}$.

The L_j satisfy

$[L_j, L_j] = \mathrm{i}\epsilon_{jkl} L_l$ and $L_j L_j = \frac{N^2-1}{4}\mathbf{1}$.

These are the familiar commutation relations of angular momentum.

Geometry

Consider the Dirac operator

$$\mathcal{D} = \sigma_j [D_j, \cdot] + 1 \,,$$

with $\mathcal{D}\Psi = \sigma_j [D_j, \Psi] + \Psi$.
Then one can see the ground state geometry via the "spectral triple" $(\mathcal{H}, Mat_N, \mathcal{D}_0)$, where the algebra is Mat_N with trace norm and

$$\mathcal{D}_0 = \sigma_a [L_a, \cdot] + 1.$$

This Dirac operator has the same spectrum as that of the commutative sphere but with a cutoff at high energies.
The ground state geometry is that of a fuzzy sphere.

Small fluctuations

The zero temperature ground state of the model
$E = \frac{Tr}{N}(-\frac{1}{4}[D_j, D_k]^2 + \frac{2i}{3}\epsilon_{jkl} D_j D_k D_l)$
is a round fuzzy sphere with $D_j = L_j$ and $E_0 = -\frac{L_j^2}{6}$.

Expanding around the minimum solution, $D_j = L_j + A_j$ yields a noncommutative Yang-Mills action with field strength
$F_{jk} = i[L_j, A_k] - i[L_j, A_k] + \epsilon_{jkl} A_l + i[A_j, A_b k]$.
As written the gauge field includes a scalar field,

$$\Phi = \frac{1}{\sqrt{N^2-1}}(D_j - L_j)^2 = \frac{1}{2}(N_j A_j + A_j N_j + \frac{A_j^2}{\sqrt{c_2}}) .$$

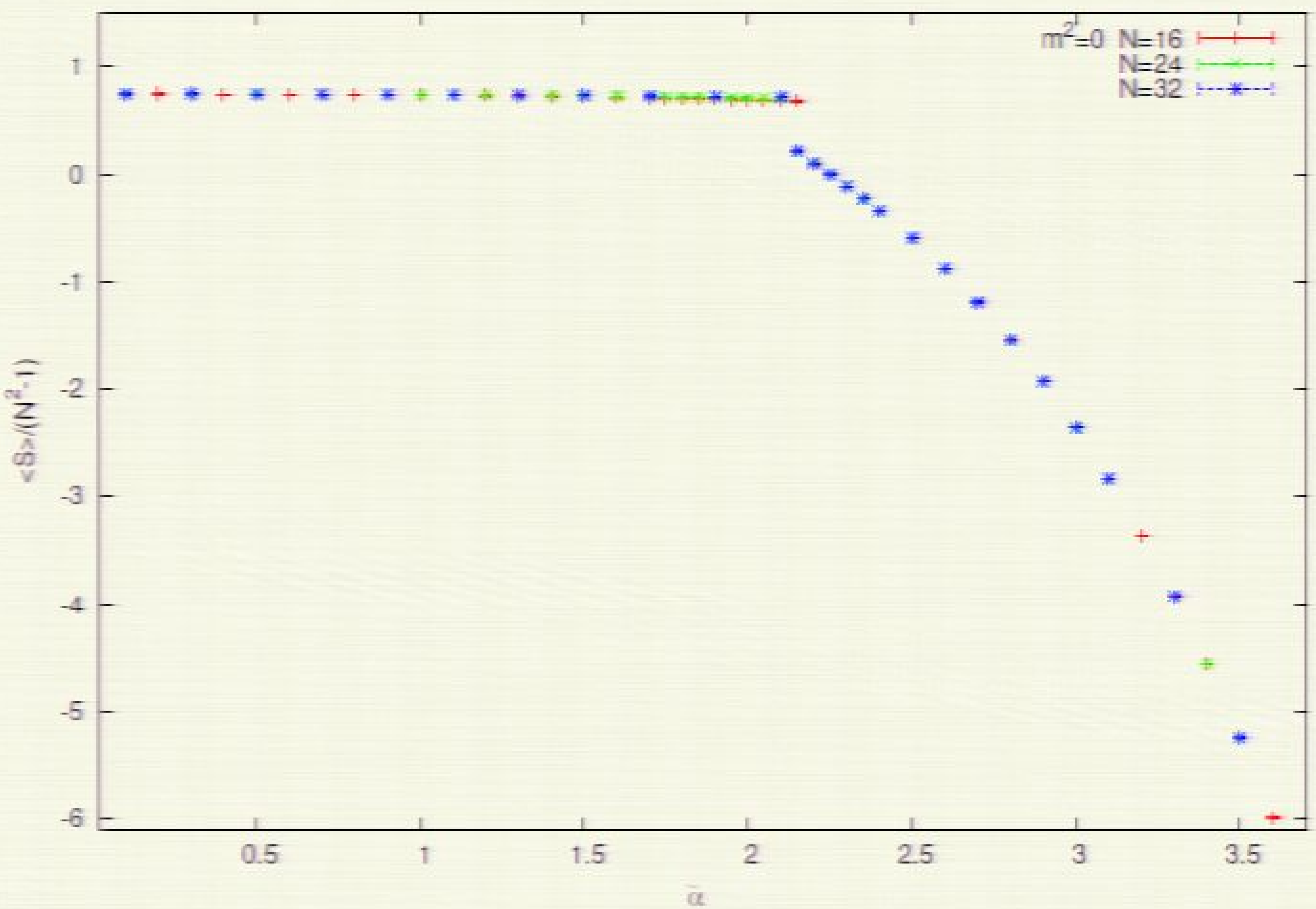
It is the component of the gauge field normal to the sphere when viewed as imbeded in $\mathbf{R}^3$ with $N_j = \frac{L_j}{\sqrt{c_2}}$ and $c_2 = L_j^2 = (N^2 - 1)/4$.

Variations of the model have arisen in work by H.Steinacker, *Nucl.Phys.B*679,66 (2004) and Presnajder *Mod.Phys.Lett.* A18 (2003) 2415. And a close relative (without the scalar field) has been solved exactly by H.Steinacker, R.J. Szabo, *hep-th/0701041*.

The model can be thought of as the low energy dynamics of open strings moving on S^3. The minimum energy configuration corresponds to a stack of N $D0$ branes wrapping a fuzzy sphere centered at the origin.
A. Y. Alekseev, A. Recknagel, V. Schomerus, JHEP **010** 0005 (2000).

Increasing the temperature. Monte Carlo Simulations

The singular part of the entropy is given by $\mathcal{S}/N^2$ where $\mathcal{S} = < S >$ and $\beta = \tilde{\alpha}^4$



The entropy jump

$\mathcal{S} = \frac{5}{12}$ as the transition is approached from the fuzzy sphere side,

and jumps to $\mathcal{S} = \frac{3}{4}$ in the high temperature phase.

The infinite temperature entropy does not contribute $\frac{1}{2}$ but $\frac{1}{4}$ per degree of freedom.

So the model remains highly interacting at high temperatures.

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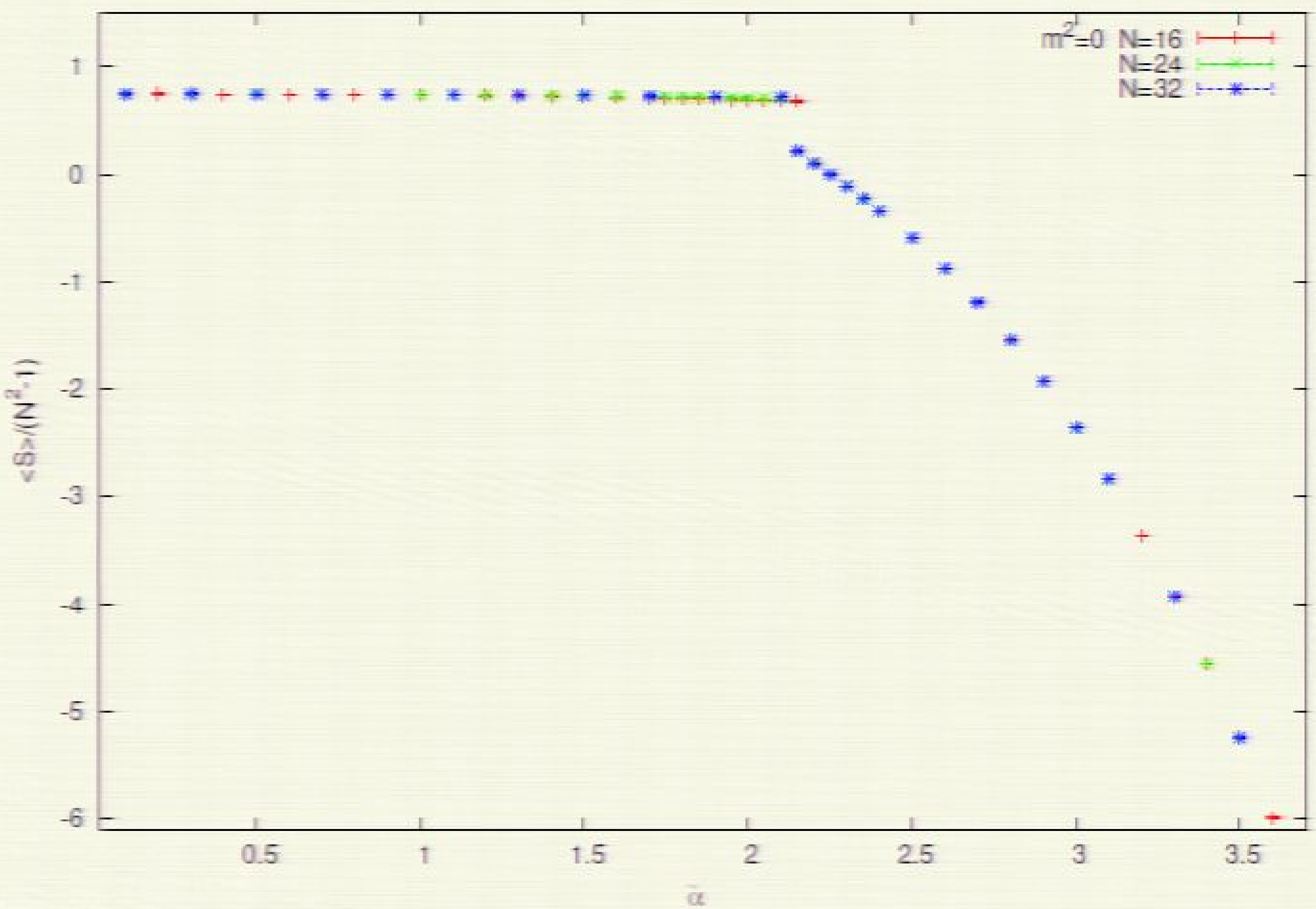

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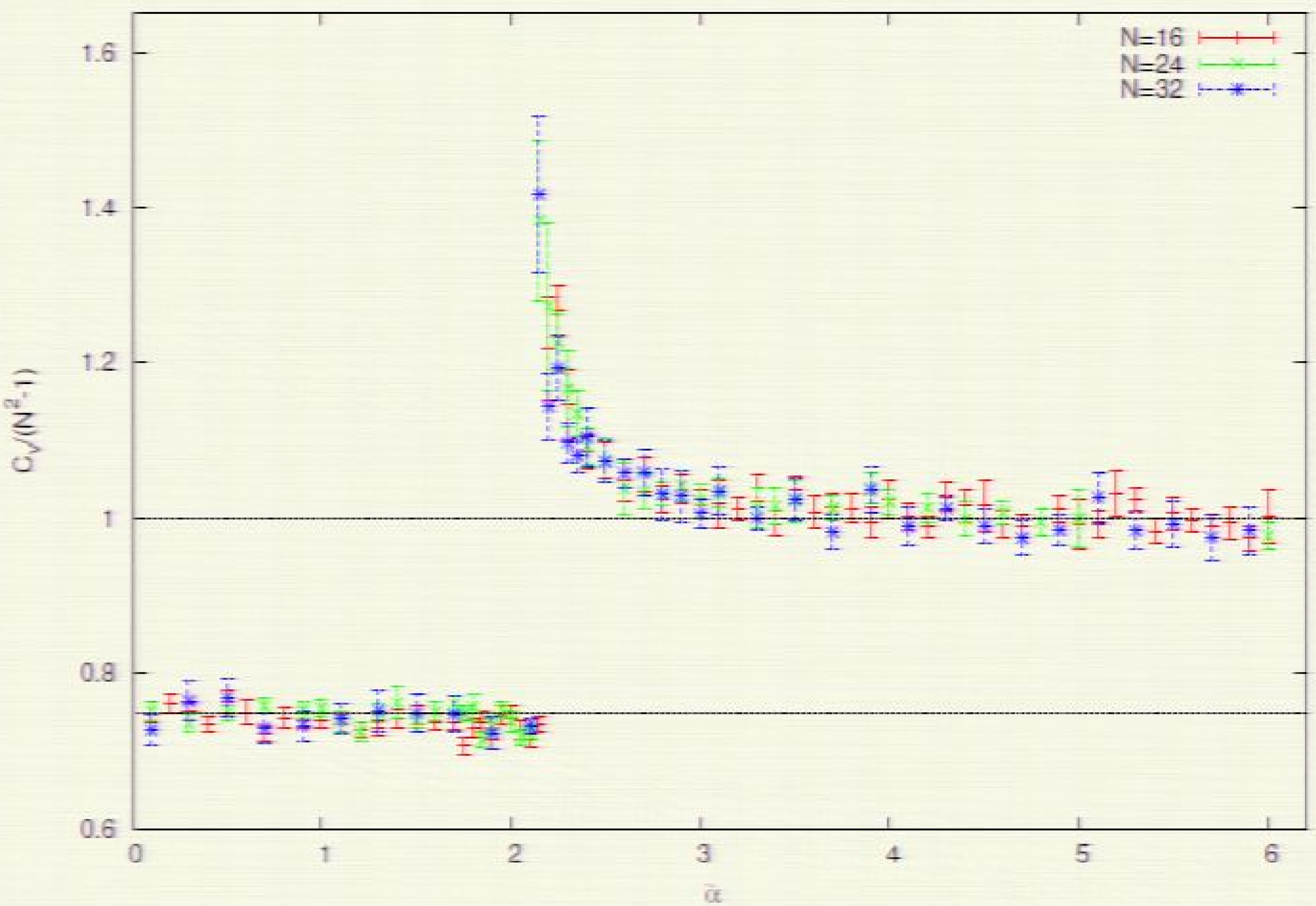
Specific Heat

The specific heat C_v/N^2 where $C_v = < S^2 > - < S >^2$ and



$\beta = \tilde{\alpha}^4$

Specific Heat Exponent

Entropy Jump

The transition is unusual in that it has a jump in the entropy. $\Delta \mathcal{S} = \frac{1}{3}$ indicating a 1st order transition.

Divergent Specific Heat

But it has a divergent specific heat $C = A_{-}(T_c - T)^{-\alpha}$ typical of a continuous (or second order) transition. We find the specific heat exponent $\alpha = \frac{1}{2}$.

Our analysis gives the critical point $\beta_c = (\frac{8}{3})^3$ and a critical exponent $\alpha = \frac{1}{2}$ for the divergence of the specific heat.

Highlights

We have seen that in a very simple model where a two-sphere, S^2, describes the cold phase of the system but the geometry evaporates as the system is heated.

The Dirac operator for S^2 emerge, ***abruptly****, as the system cools.*

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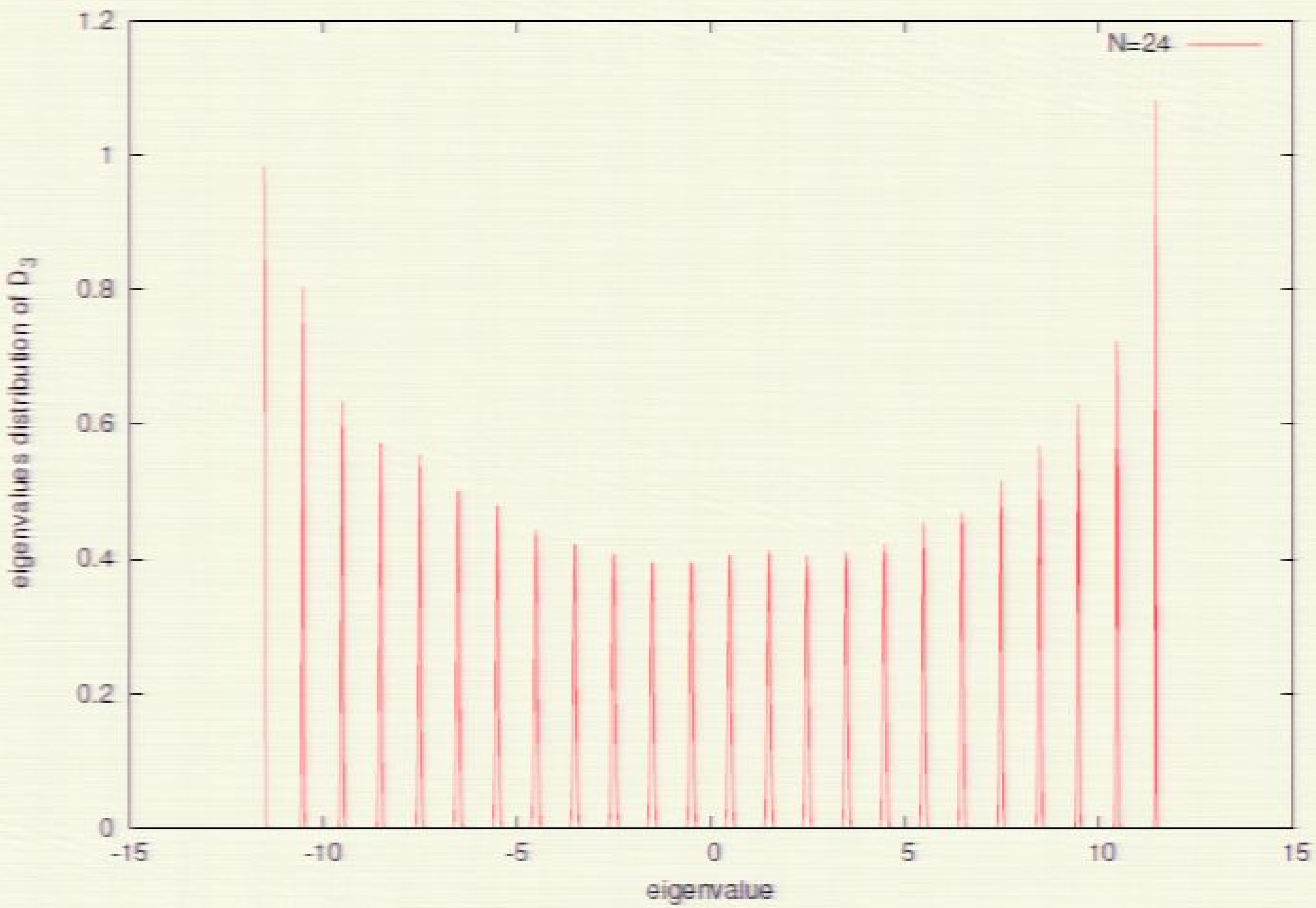
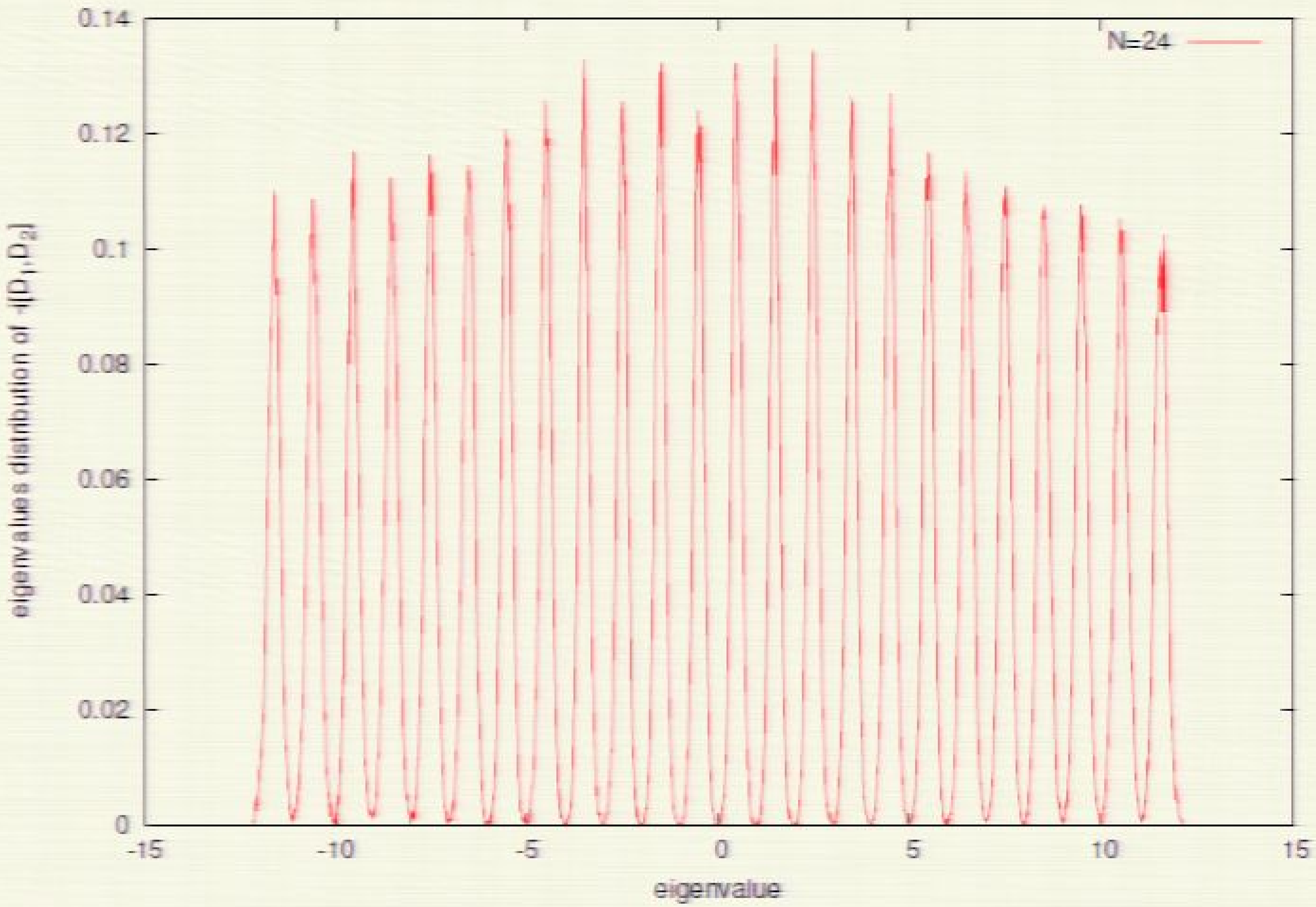
Eigenvalues in the low temperature phase

Eigenvalue distribution of D_3 for $N = 24$.



Eigenvalues in the low temperature phase

Eigenvalue distribution of $[D1, D_2]$ for $N = 24$.



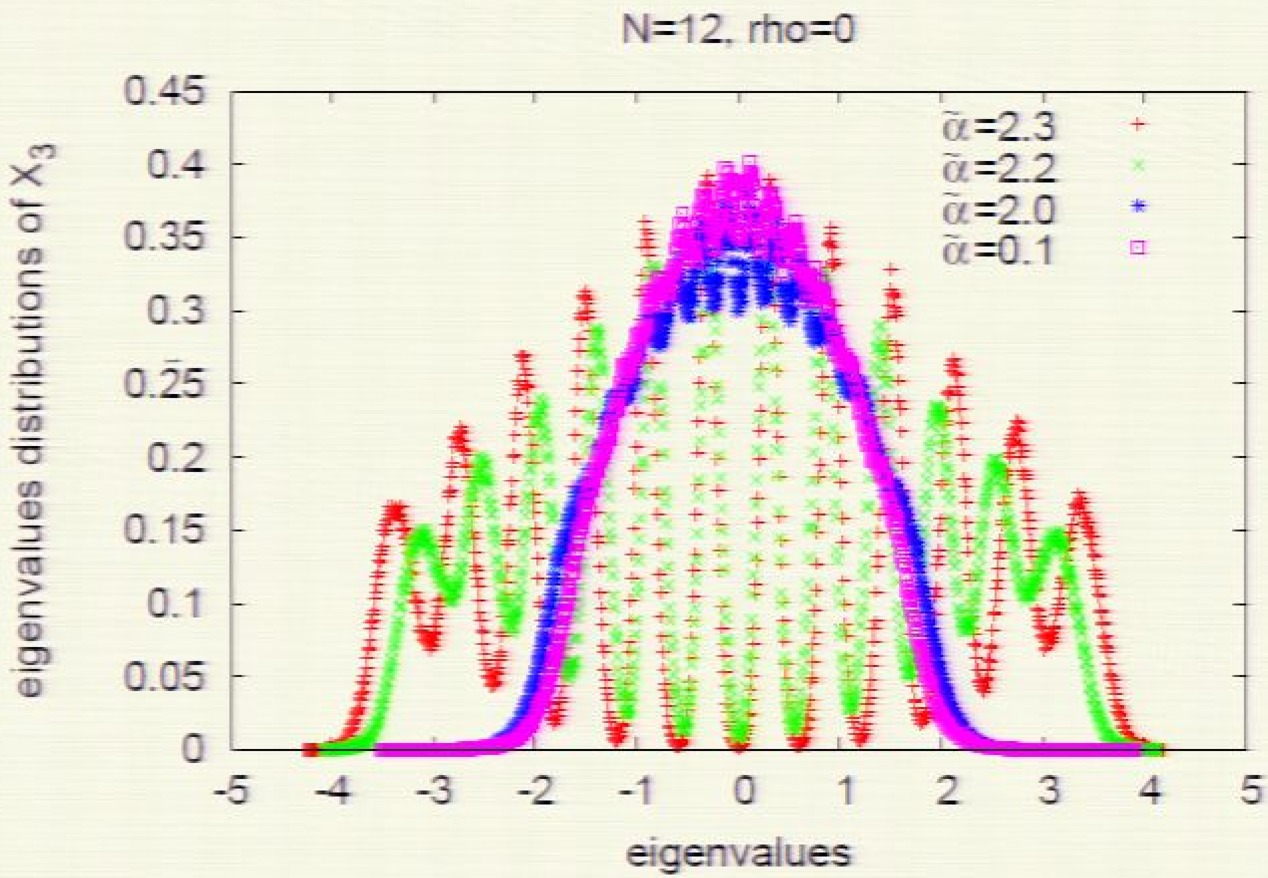
A closer look at the transition

- In the fuzzy sphere phase the eigenvalues fluctuate around the discrete values corresponding to $D_a = L_a$, the irreducible representation of $SU(2)$ of dimension N.
- In the matrix phase, the distribution of eigenvalues of

$$X_a = (\frac{\beta}{N^2})^{1/4} D_a = \frac{\tilde{\alpha}}{N^{1/2}} D_a$$

is largely independent of $\tilde{\alpha}$ and of N.

- In fact fluctuations are around commuting matrices with a uniform distribution in a ball of radius 2. E.g for $N = 12$, the distribution for X_3 ranges from -2 to 2.

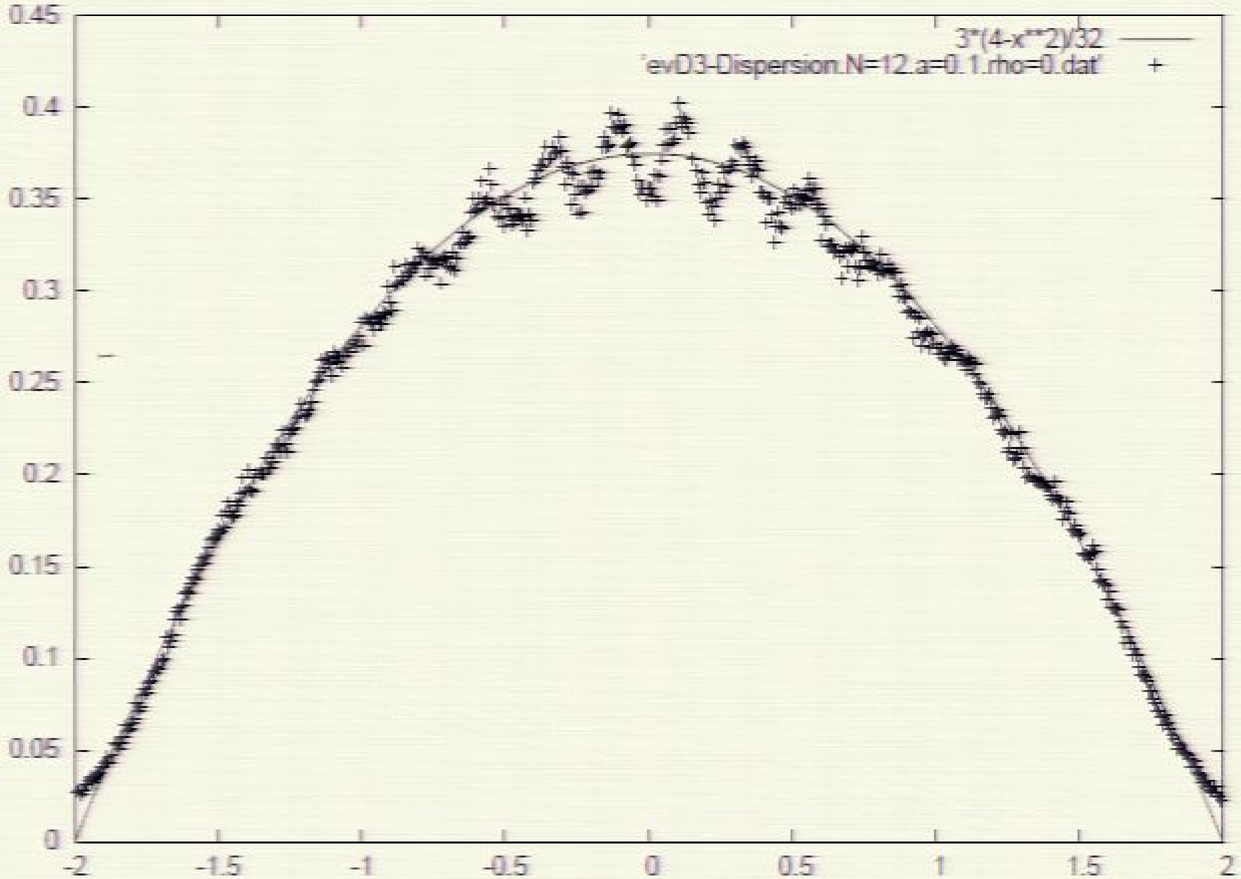

N=12, rho=0
eigenvalues distributions of X_3
eigenvalues
$\tilde{\alpha}$=2.3
$\tilde{\alpha}$=2.2
$\tilde{\alpha}$=2.0
$\tilde{\alpha}$=0.1
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-5
-4
-3
-2
-1
0
1
2
3
4
5

- Following Berenstein et al. (arXiv:0805.4658) one can expand small fluctuations around commuting diagonal matrices. This leads to the conclusion that the eigenvalues form a solid ball of radius R.

The distribution of eigenvalues of X_3 is then:

$$\rho(x) = \frac{3}{4R^3}(R^2 - x^2)$$

This implies $\langle x^2 \rangle = \frac{R^2}{5}$. Numerically, $R \approx 2$.



3*(4-x**2)/32
'evD3-Dispersion.N=12.a=0.1.rho=0.dat'
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

From solid ball of eigenvalues to fuzzy S^2.

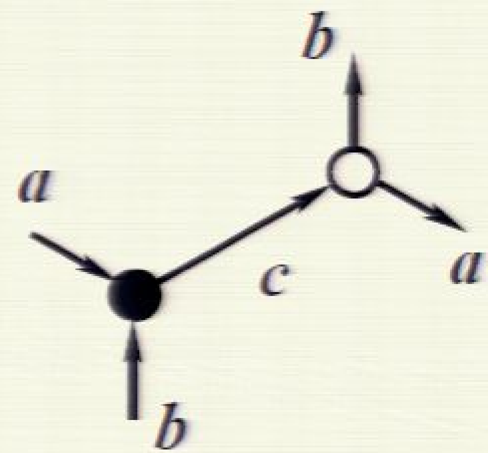
As the system cools a fuzzy S^2 emerges from the ball corresponding the the eigenvalues of the commutating matrices at high temperature.

In passing through the transition the eigenvalue ball of radius 2 expands to a fuzzy sphere of radius $\frac{\sqrt{N\tilde{\alpha}}}{2}$.

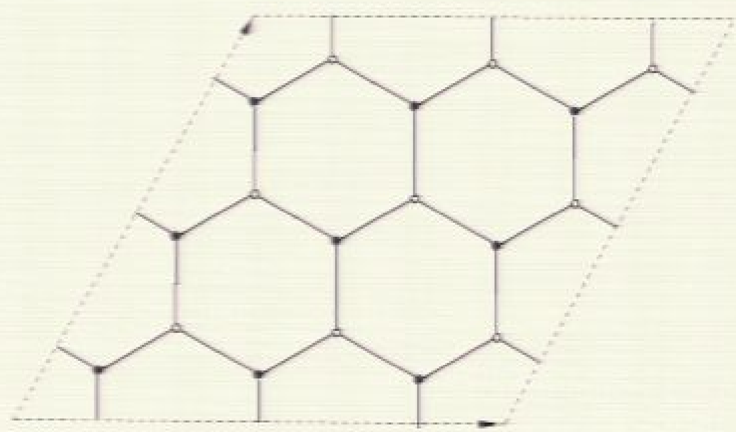
The Dimer Model

Curiously a very classical model called the dimer model has very similar thermodynamic properties.
See Nash and O'Connor J. Phys **A41** (2009) 012002[arXiv:0809.2960].
Mathematician study it as a model to count tilings. E.g. one can easily establish that there 12988816 domino tilings of a chess board.
For a physicist it has many faces but it can be thought of as a lattice model for a two dimensional Fermion.

Hexagonal Tilings of the Torus



A 3×3 tiling of the torus.



Activities $a = e^{-\beta\epsilon_a}$, $b = e^{-\beta\epsilon_b}$ and $c = e^{-\beta\epsilon_c}$ are assigned to the bonds.
These determine the probability of a bond being active and of a rombus tiling of dual triangular lattice.

The partition function is

$Z(N, M, a, b, c) = \sum_{tilings} a^{N_a} b^{N_b} c^{N_c}$

where N_i is the number of active bonds of type i and $N_a + N_b + N_c = NM$, since the lattice must be completely covered.
When the activities are set to one Z counts the number of lozenge tilings of the dual triangular lattice.

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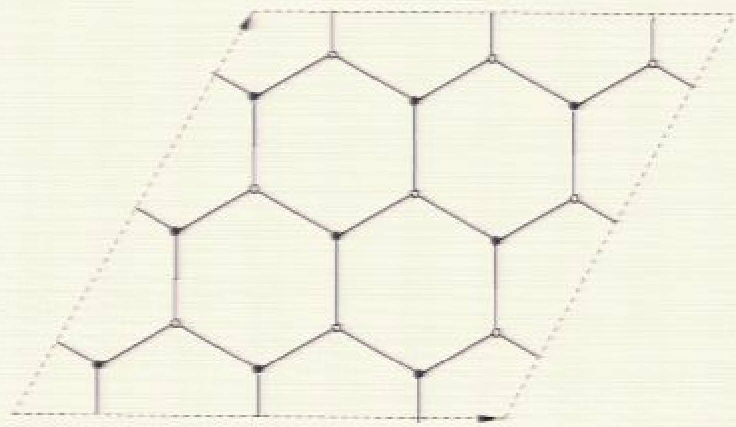

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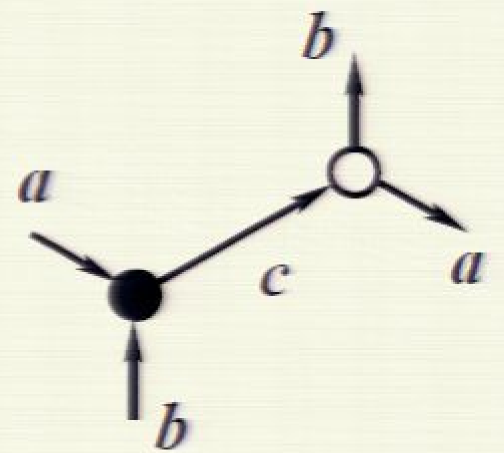

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It turns out (Kasteleyn, Fisher and Temperley 1961) that on a simply connected domain

$$Z = Pfaff(K) = \left| \sqrt{Det(K)} \right|$$

.

And on a torus $Z = \frac{1}{2}\left(-Z_{00} + Z_{\frac{1}{2}0} + Z_{0\frac{1}{2}} + Z_{\frac{1}{2}\frac{1}{2}}\right)$.

In the thermodynamic limit the logarithm of the bulk partition function per dimer has a phase transition at $\beta_c = \ln 2$, with $\ln 2 - \beta \geq 0$ we have

$$W(\beta) \simeq \frac{4\sqrt{2}}{3\pi}(\ln 2 - \beta)^{\frac{3}{2}}.$$

The transition is continuous with no latent heat. The specific heat is zero in the low temperature frozen phase; there is a phase transition at $\beta = \ln 2$, and the specific heat diverges with critical exponent $\alpha = \frac{1}{2}$ as the transition is approached from the high temperature side.

Finite size effects.

For large M and N we have

$$\lim_{N,M\to\infty} \frac{Z(N,M)}{\mathrm{e}^{NMW(a,b,c)}} = Z_{Dirac}(\tau,\theta,\phi)$$

$$= \frac{1}{2}\sum_{u,v=0}^{1/2} \left| \frac{\theta[^{\theta+u}_{\phi+v}](0|\tau)}{\eta(\tau)} \right| \qquad (2)$$

$\tau = \frac{Nb}{Ma}e^{i(\Theta+\Phi)}$ with Θ and Φ determinde by β (or more generally the fugacities). $Z_{Dirac}(\tau,\theta,\phi)$ is the partition function for a Dirac Fermion propagating on the continuum torus with modular parameter τ in the presence of a gauge potential with zero field strength, i.e. a flat connection, but with holonomies $\mathrm{e}^{2\pi i\theta}$ and $\mathrm{e}^{2\pi i\phi}$ round the cycles of the torus.

As the edge of the transition is approached $\tau \to \tau_0$ and the continuum geometry collapses.

When the transition is apporached the specific heat diverges with exponent $\alpha = \frac{1}{2}$. and is constant in the new phase.

Here there is no latent heat (or jump in the entropy).

Conclusions

- Matrix models provide a new arena for the regularization of field theories.
- New physical effects not discussed, e.g. UV/IR mixing.
- The models are intrinsically non-local, but not as bad as Fermion determinants.
- Yang-Mills type models blurr the distinction between background geometry and the gauge fields.

- I described in detail a 3-matrix model which provides a concrete model where one can track the geometry as it passes through a phase transition and dissapears.
  Such transitions belong to a new universality class of topological phase transitions.
- The transition is from one where the underlying geometry at a microscopic level is non-commutative, and described by a fuzzy sphere with matter fluctuations to one a commutative sphere of much smaller radius.
- The geometrical phase emerges as the system cools. This is suggestive of a geometrical phase emerging as the universe cools, or perhaps as the relevant coupling runs to a larger scale.
- The fluctuations around the fuzzy sphere phase are consistent with being $U(1)$ gauge fields in the large mass limit.

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