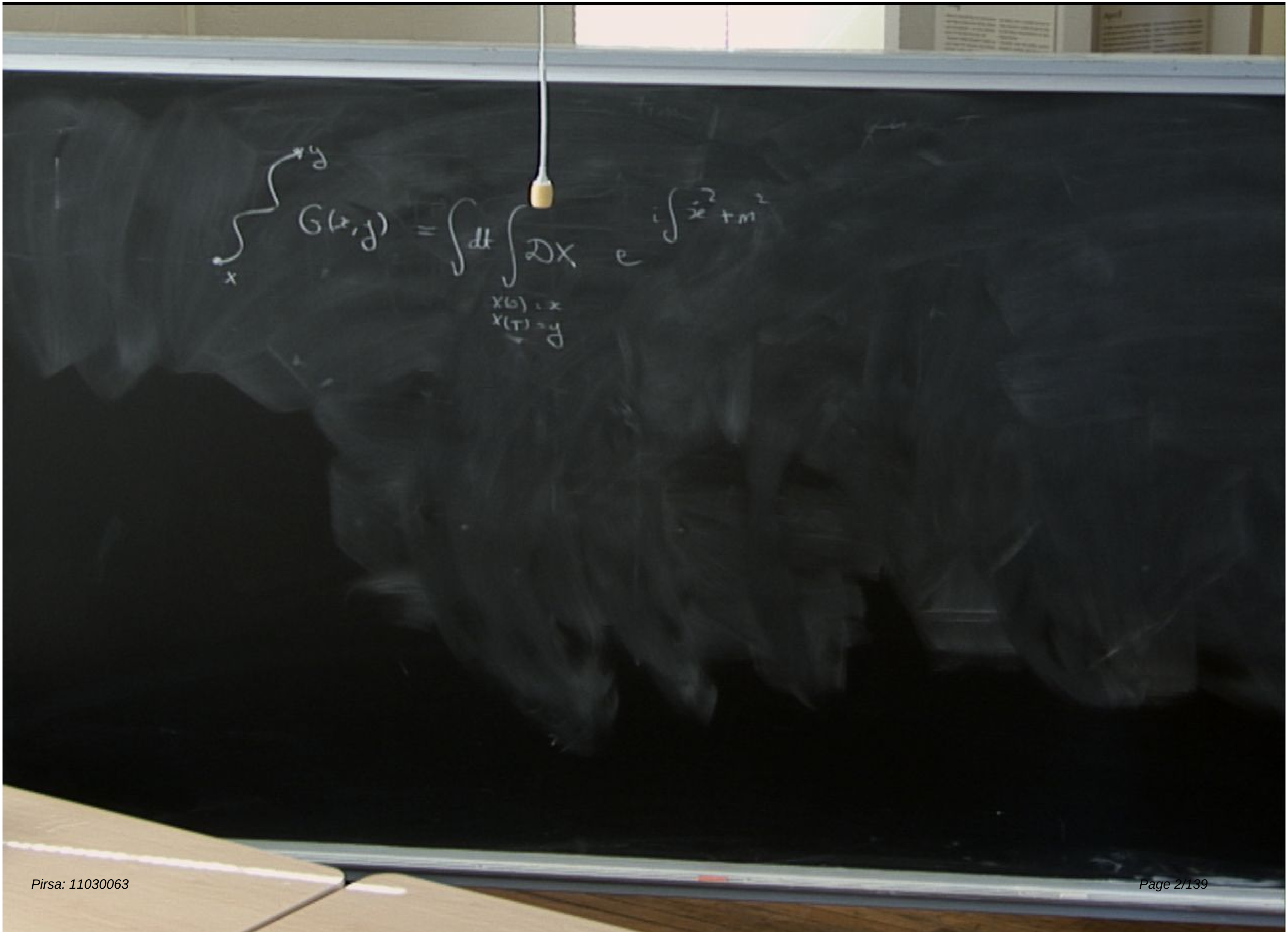
Title: Explorations in String Theory - Lecture 12

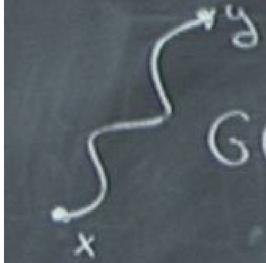
Date: Mar 29, 2011 11:30 AM

URL: http://pirsa.org/11030063

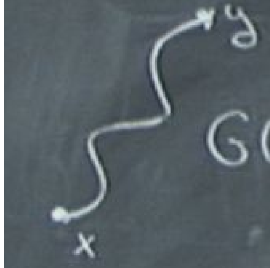
Abstract:



$$G(x,y) = \int dt \int_{\substack{X(0)=x \\ X(T)=y}} \mathcal{D}X \; e^{i\int \dot{x}^2 + m^2}$$



$$G(x,y) = \int dt \int_{\substack{X(0)=x \\ X(T)=y}} \mathcal{D}X \; e^{i\int \dot{x}^2 + m^2} = \int \mathcal{D}x \; e^{im \overbrace{\int \sqrt{\dot{x}^2}\, dt}^{\text{Length}}}$$



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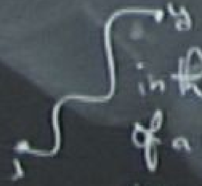
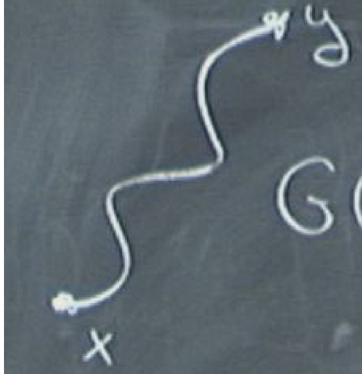
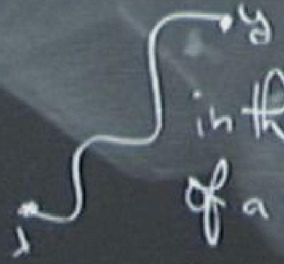


in the presence
of a U(1)
gauge field A_μ

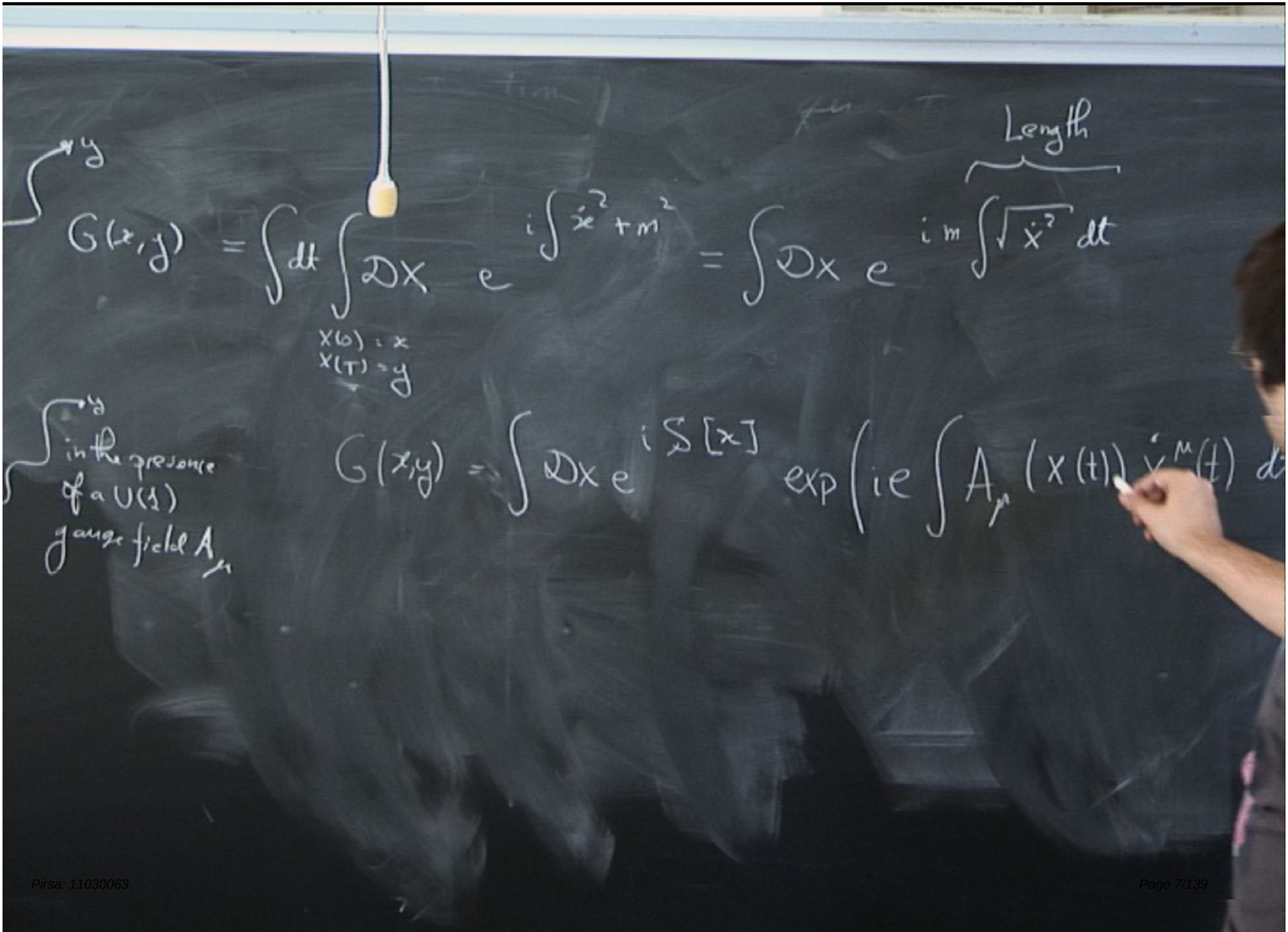


$$G(x,y) = \int dt \int_{\substack{X(0)=x\\ X(T)=y}} \mathcal{D}X \, e^{i\int \dot{x}^2 + m^2} = \int \mathcal{D}x \, e^{i}$$



in the presence of a U(1) gauge field A_μ

$$G(x,y) = \int \mathcal{D}x \, e^{iS[x]}$$



y

$$G(x,y) = \int dt \int_{\substack{X(0)=x \\ X(T)=y}} \mathcal{D}X \; e^{i\int \dot{x}^2 + m^2} = \int \mathcal{D}x \; e^{im \overbrace{\int \sqrt{\dot{x}^2}\, dt}^{\text{Length}}}$$

y

in the presence of a U(1) gauge field A_μ

$$G(x,y) = \int \mathcal{D}x \; e^{iS[x]} \exp\Big(ie \int A_\mu (x(t))\, \dot{x}^\mu(t)\; d$$

$$y) = \int dt \int_{\substack{X(0)=x \\ X(T)=y}} \mathcal{D}X \; e^{i\int \dot{x}^2 + m^2} = \int \mathcal{D}x \; e^{im \overbrace{\int \sqrt{\dot{x}^2}\, dt}^{\text{Length}}}$$

presence
(1)
field A_μ

$$G(x,y) = \int \mathcal{D}x \; e^{iS[x]} \exp\left(ie \int_0^T A_\mu (x(t))\, \dot{x}^\mu(t)\, dt \right)$$

$$^2 + m^2 = \int \mathcal{D}x \; e^{\,i m \overbrace{\int \sqrt{\dot{x}^2}\, dt}^{\text{Length}}}$$

$$e^{iS[x]} \quad \exp\left(ie \int_0^T A_\mu (x(t))\, \dot{x}^\mu (t)\, dt \right) , \quad \oint$$

$$^2 + m^2 = \int \mathcal{D}x \; e^{\, i m \overbrace{\int \sqrt{\dot{x}^2}\, dt}^{\text{Length}}}$$

Δ phases =

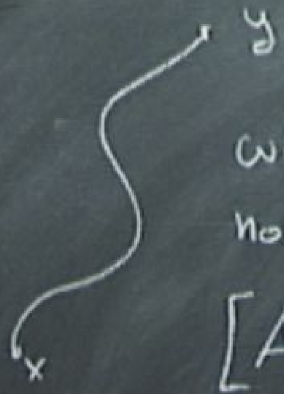
$$e^{iS[x]} \quad \exp\left(ie \int_0^T A_\mu (x(t))\, \dot{x}^\mu(t)\, dt \right) , \quad \oint$$

$${}^2 + m^2 = \int \mathcal{D}x \; e^{im \overbrace{\int \sqrt{\dot{x}^2}\, dt}^{\text{Length}}}$$

Δ phases =

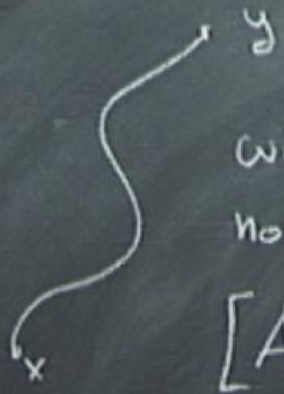
$$x\, e^{iS[x]} \quad \exp\left(ie \int_0^T A_\mu (x(t))\, \dot{x}^\mu(t)\, dt \right) , \quad \oint$$

$$= \iint B = \Phi$$



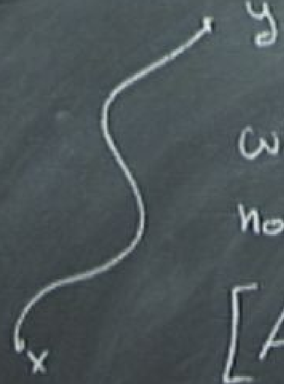
with a
non Abelian gauge field

$[A_\mu]_{ab}$



with a
non Abelian gauge field
$[A_\mu]_{ab}$

$$G(x,y) = \int \mathcal{D}x \, e^{iS[x]} \quad P\exp \int_0^T A_\mu(x(t))\,\dot{x}^\mu(t)\, d$$



with a
non Abelian gauge field

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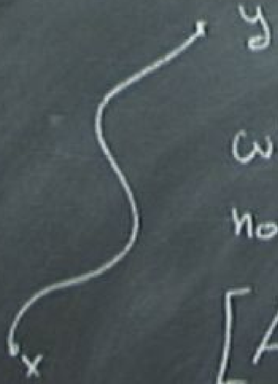
$$G(x,y) = \int \mathcal{D}x\, e^{iS[x]} \quad P\exp\left[ie\int_0^T A_\mu(x(t))\,\dot{x}^\mu(t)\,dt\right]$$



with a
non Abelian gauge field
$[A_\mu]_{ab}$

$$G(x,y) = \int \mathcal{D}x \, e^{iS[x]} \quad P\exp\left[ie\int_0^T A_\mu(x(t))\,\dot{x}^\mu(t)\,dt\right]$$

Slower (on def)

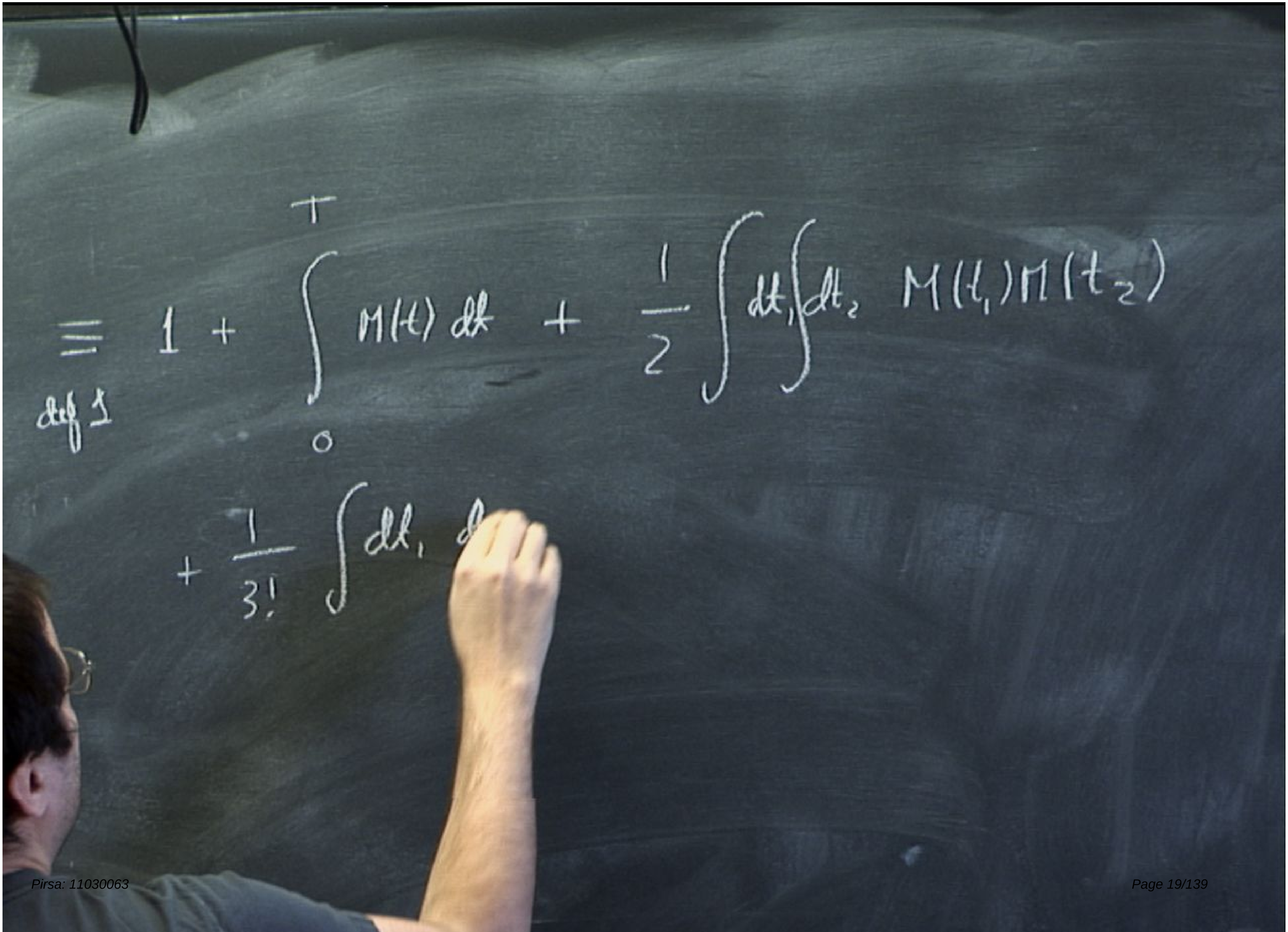
$$\exp \int_0^T ieA_\mu(x(t)) \cdot X^\mu(t)\, dt$$

Slower (on def)

$$\exp \int_0^T \underbrace{ieA_\mu(x(t)) \cdot X^\mu(t)}_{M(t)} \, dt$$

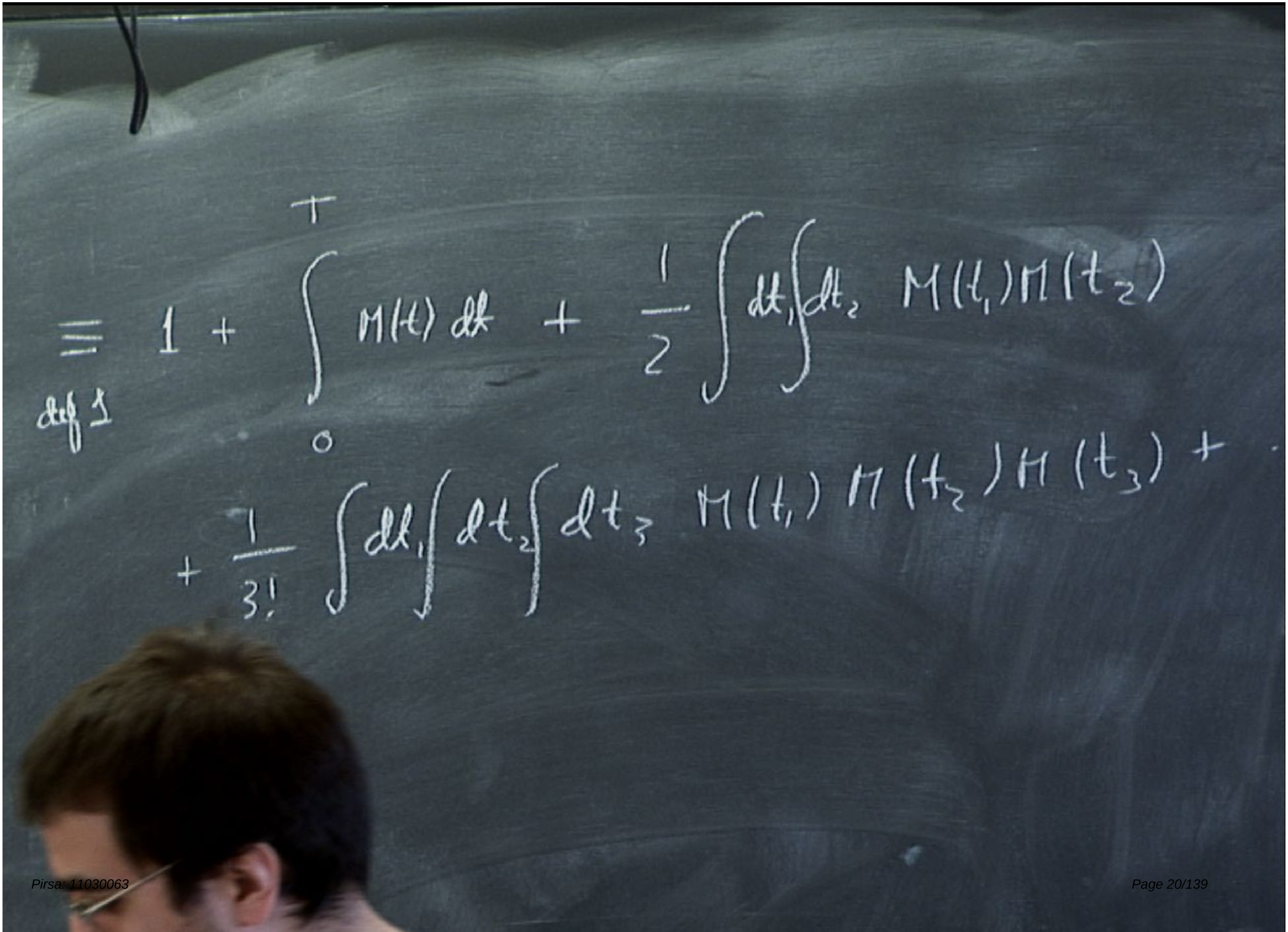
Slower (on def)

$$\exp \int_0^T \underbrace{ieA_\mu(x(t)) \cdot X^\mu(t)}_{M(t)} \, dt \quad \underset{\text{def 1}}{\equiv}$$



$$\underset{\text{def 1}}{\equiv} 1 + \int_0^T M(t)\, dt + \frac{1}{2}\int dt_1 \int dt_2 \; M(t_1) M(t_2)$$

$$+ \frac{1}{3!}\int dt_1 \, d$$



$$\underset{\text{def } 1}{\equiv} 1 + \int_0^T M(t)\,dt + \frac{1}{2}\int dt_1 \int dt_2 \; M(t_1) M(t_2)$$

$$+ \frac{1}{3!}\int dt_1 \int dt_2 \int dt_3 \; M(t_1)\, M(t_2)\, M(t_3) +$$

$$\underset{\text{def 1}}{\equiv} 1 + \int_0^T M(t)\,dt + \frac{1}{2}\int dt_1 \int dt_2 \; M(t_1) M(t_2)$$

$$+ \frac{1}{3!}\int dt_1 \int dt_2 \int dt_3 \; M(t_1)\, M(t_2)\, M(t_3) + \cdot$$

$$\exp \int_0 \underbrace{ieA_\mu(x(t))\cdot X^\mu(t)}_{M(t)}\, dt \underset{\text{def } 1}{\equiv} 1 + \int_0 M(t)$$

$$+ \frac{1}{3!}\int dt_1 \int$$

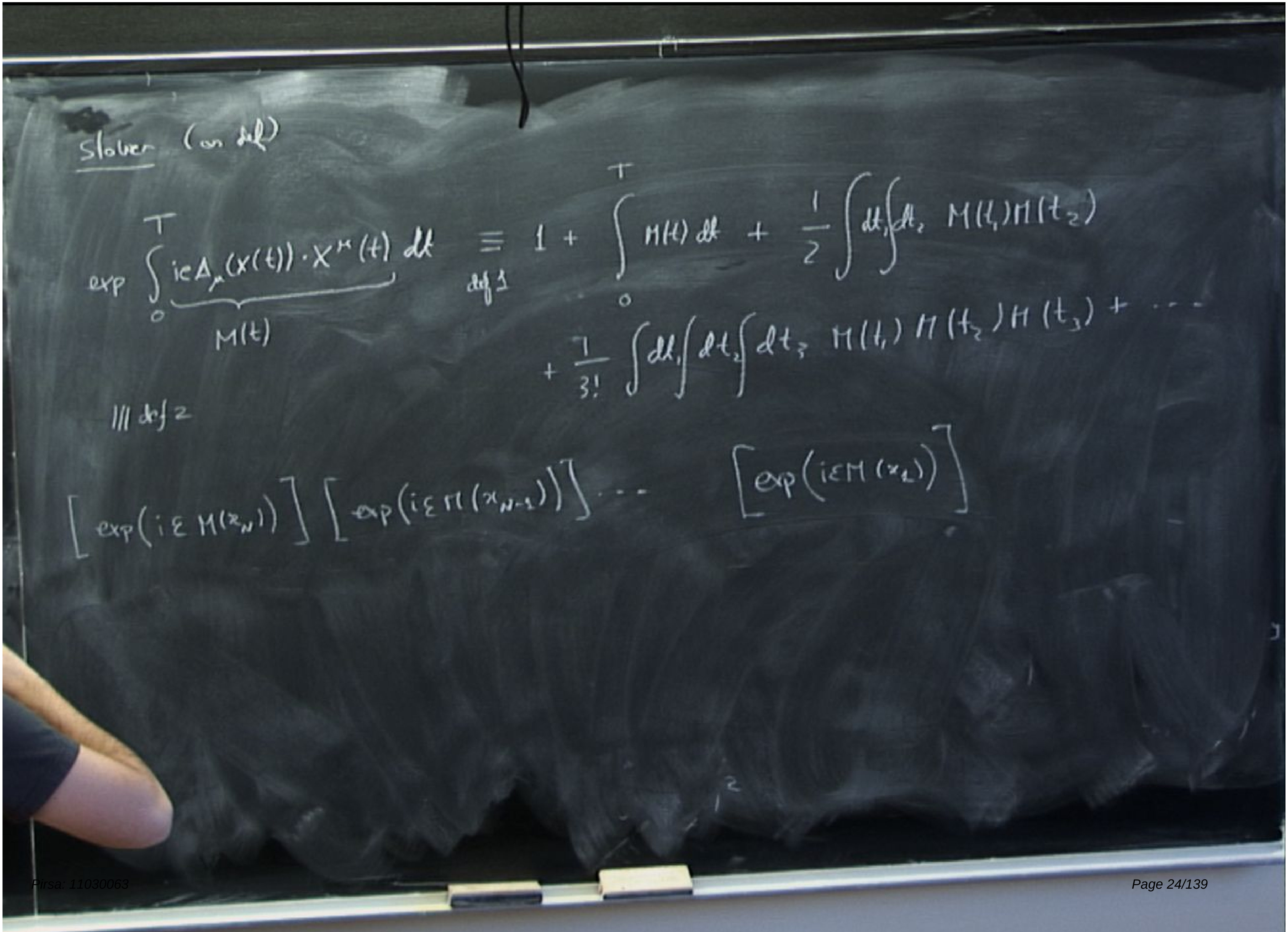
$$\underset{\text{def } 2}{\equiv}$$

$$\left[\exp\left(i\varepsilon M(x_N)\right)\right]\left[\exp\left(i\varepsilon M(x_{N-1})\right)\right]\cdots$$

$$(t)\, dt \underset{\text{def } 1}{\equiv} 1 + \int_0 M(t)\, dt + \frac{}{2} \int \int$$

$$+ \frac{1}{3!} \int dt_1 \int dt_2 \int dt_3 \; M(t_1)\, M(t_2)\, M($$

$$\exp\left(i\varepsilon M(x_{N-1})\right)\Big] \cdots \qquad \Big[\exp\left(i\varepsilon M(x_1)\right)\Big]$$



Slower (on def)

$$\exp \int_0^T \underbrace{i\epsilon A_\mu(x(t)) \cdot X^\mu(t)}_{M(t)} \, dt \underset{\text{def 1}}{\equiv} 1 + \int_0^T M(t)\, dt + \frac{1}{2}\int dt_1 \int dt_2 \; M(t_1) M(t_2)$$

$$+ \frac{1}{3!} \int dt_1 \int dt_2 \int dt_3 \; M(t_1)\, M(t_2)\, M(t_3) + \dots$$

||| def 2

$$\left[\exp\left(i\varepsilon\, M(x_N)\right)\right] \left[\exp\left(i\varepsilon\, M(x_{N-1})\right)\right] \dots \quad \left[\exp\left(i\varepsilon M(x_1)\right)\right]$$

Slower (on def)

$$\exp \int_0^T \underbrace{ie A_\mu(x(t)) \cdot \dot{X}^\mu(t)}_{M(t)} \, dt \underset{\text{def 1}}{\equiv} 1 + \int_0^T M(t)\, dt + \frac{1}{2} \int dt_1 \int dt_2$$

$$+ \frac{1}{3!} \int dt_1 \int dt_2 \int dt_3 \; M(t_1) \, M$$

$|||$ def 2

$$\left[\exp(i\varepsilon M(x_N))\right] \left[\exp(i\varepsilon M(x_{N-1}))\right] \cdots \quad \left[\exp(i\varepsilon M(x_1))\right.$$

DEF 3

$$\left[\partial_t - M(t)\right] \psi(t) = 0 \quad \text{with b.c } \psi(0) = 1$$

$$M(t)\,dt \underset{\text{def 1}}{\equiv} 1 + \int_0^T M(t)\,dt + \frac{1}{2}\int dt_1\int dt_2\; M(t_1)M(t_2) + \cdots$$

$$+ \frac{1}{3!}\int dt_1\int dt_2\int dt_3\; M(t_1)\,M(t_2)\,M(t_3)$$

$$\left[\exp\left(i\varepsilon M(x_{N-1})\right)\right]\cdots\left[\exp\left(i\varepsilon M(x_1)\right)\right]$$

$t) = 0$ with b.c $\psi(0) = 1$. Then $\psi(T) = \exp\int_0^T M(t)\,dt$

Slower (on def)

$$\exp\int_0^T \underbrace{ie A_\mu(x(t))\cdot X^\mu(t)}_{M(t)}\, dt \underset{\text{def } 1}{\equiv} 1 + \int_0^T M(t)\, dt + \frac{1}{2}\int dt_1 \int dt_2\, M(t_1) M(t_2)$$

$$+ \frac{1}{3!}\int dt_1 \int dt_2 \int dt_3\, M(t_1)\, M(t_2)\, M(t_3) + \cdots$$

$\equiv$ def 2

$$\left[\exp\left(i\varepsilon M(x_N)\right)\right]\left[\exp\left(i\varepsilon M(x_{N-1})\right)\right]\cdots \quad \left[\exp\left(i\varepsilon M(x_1)\right)\right]$$

DEF 3

$\left[\partial_t - M(t)\right]\psi(t) = 0$ with b.c $\psi(0) = 1$. Then $\psi(T) = \exp\int_0^T M(t)\, dt$

Slower (on def)

$=$ non-Abelian exponential P exp

$$P\exp\int_0^T \underbrace{i\epsilon A_\mu(x(t))\cdot X^\mu(t)}_{M(t)}\,dt \underset{\text{def 1}}{\equiv} 1+\int_0^T M(t)\,dt+\frac{1}{2}\int dt_1\int dt_2\; M(t_1)M(t_2)$$

$$+\frac{1}{3!}\int dt_1\int dt_2\int dt_3\; M(t_1)\,M(t_2)\,M(t_3)+\cdots$$

$\equiv$ def 2

$$\left[\exp\left(i\epsilon M(x_N)\right)\right]\left[\exp\left(i\epsilon M(x_{N-1})\right)\right]\cdots\left[\exp\left(i\epsilon M(x_1)\right)\right]$$

DEF 3

$$\left[\partial_t-M(t)\right]\psi(t)=0$$

with b.c $\psi(0)=1$. Then $\psi(T)=\exp\int_0^T M(t)\,dt$

Slower (on def)

= non-Abelian exponential Pexp

$$P\exp \int_0^T \underbrace{i\epsilon A_\mu(x(t))\cdot X^\mu(t)}_{M(t)}\,dt \underset{\text{def 1}}{\equiv} 1 + \int_0^T M(t)\,dt + \frac{1}{2}\int dt_1 \int dt_2\; M(t_1)M(t_2)$$

$$+ \frac{1}{3!}\int dt_1 \int dt_2 \int dt_3\; M(t_1)\,M(t_2)\,M(t_3) + \cdots$$

$\equiv$ def 2 ✓

$$\left[\exp(i\varepsilon M(x_N))\right]_{a_1a_2}\left[\exp(i\varepsilon M(x_{N-1}))\right]_{a_2a_3}\cdots\quad\left[\exp(i\varepsilon M(x_1))\right]_{a_N a_{N+1}}$$

DEF 3

$$\left[\partial_t - M(t)\right]\psi(t) = 0$$

with b.c $\psi(0)=1$. Then $\psi(T) = P\exp\int_0^T M(t)\,dt$

Slower (on def) $\qquad = $ non-Abelian exponential Pexp

$$P\exp\int_0^T \underbrace{i\epsilon A_\mu(x(t))\cdot X^\mu(t)}_{M(t)}\,dt \underset{\text{def 1}}{\equiv} 1 + \int_0^T M(t)\,dt + \frac{1}{2}\int dt_1\int dt_2\, M(t_1)M(t_2)$$

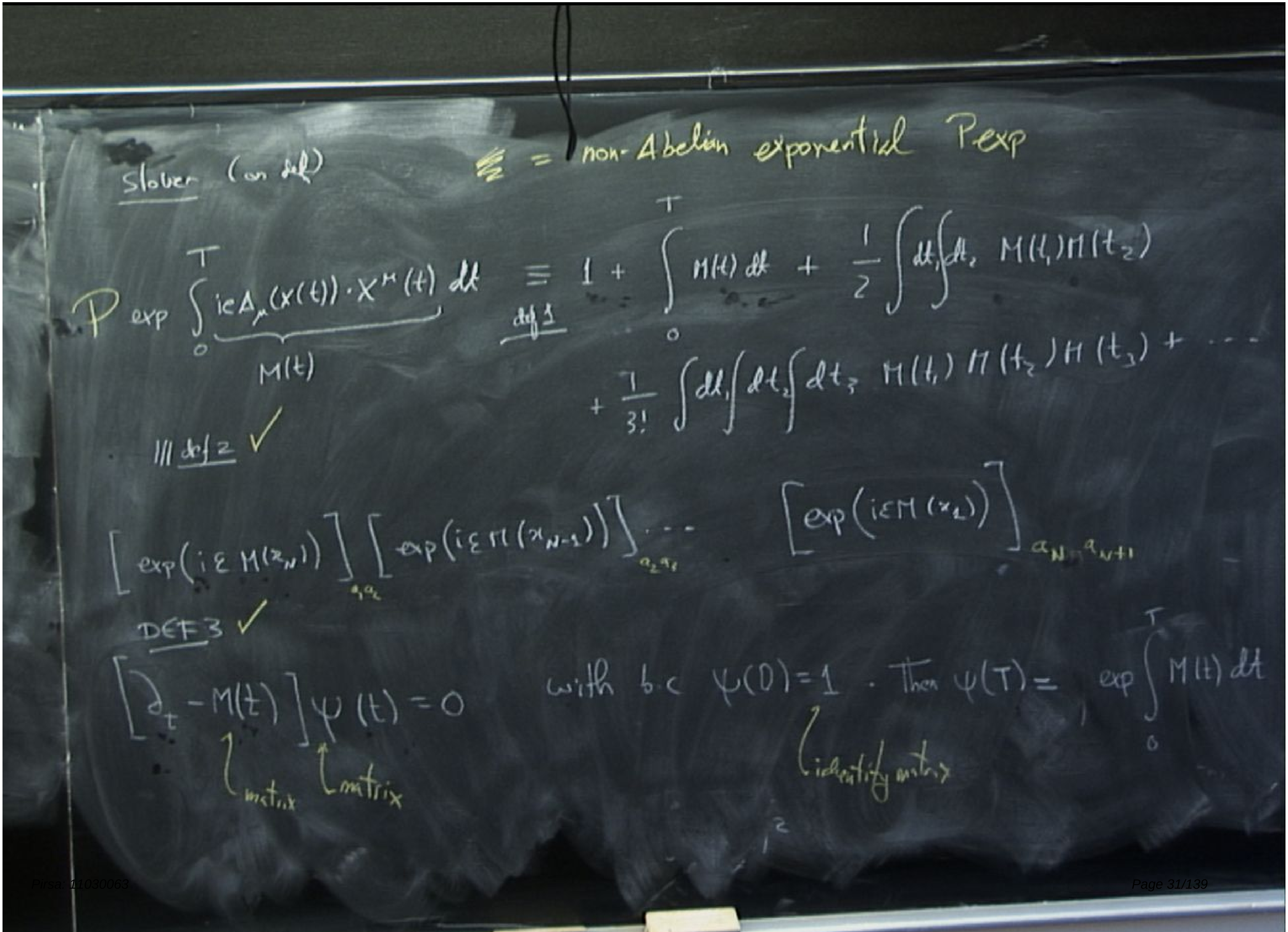
$$+\frac{1}{3!}\int dt_1\int dt_2\int dt_3\, M(t_1)\,M(t_2)\,M(t_3) + \cdots$$

||| def 2 ✓

$$\left[\exp\left(i\varepsilon M(x_N)\right)\right]_{a_1a_2}\left[\exp\left(i\varepsilon M(x_{N-1})\right)\right]_{a_2a_3}\cdots\quad\left[\exp\left(i\varepsilon M(x_1)\right)\right]_{a_Na_{N+1}}$$

DEF 3 ✓

$$\left[\underset{\text{matrix}}{\partial_t - M(t)}\right]\underset{\text{matrix}}{\psi}(t) = 0 \quad \text{with b.c } \psi(0)=\underset{\text{identity matrix}}{1}\text{. Then } \psi(T) = \exp\int_0^T M(t)\,dt$$



Slower (on def)

= non-Abelian exponential Pexp

$$P\exp\int_0^T \underbrace{ie A_\mu(x(t))\cdot \dot{x}^\mu(t)}_{M(t)}\,dt \underset{\text{def 1}}{\equiv} 1+\int_0^T M(t)\,dt + \frac{1}{2}\int dt_1\int dt_2\, M(t_1)M(t_2) + \frac{1}{3!}\int dt_1\int dt_2\int dt_3\, M(t_1)M(t_2)M(t_3)+\dots$$

$\equiv$ def 2 ✓

$$\left[\exp(i\varepsilon M(x_N))\right]_{a_1a_2}\left[\exp(i\varepsilon M(x_{N-1}))\right]_{a_2a_3}\cdots\left[\exp(i\varepsilon M(x_1))\right]_{a_Na_{N+1}}$$

DEF 3 ✓

$$\underbrace{[\partial_t - M(t)]}_{\text{matrix}}\underbrace{\psi(t)}_{\text{matrix}}=0$$

with b.c. $\underbrace{\psi(0)=1}_{\text{identity matrix}}$. Then $\psi(T)= \exp\int_0^T M(t)\,dt$

$$P\exp\int_0^T \underset{\uparrow\ \text{matrix}}{M(t)}\,dt \; =$$

$$P\exp\int_{0}^{T}\underset{\uparrow\ \text{matrix}}{M(t)}\,dt \;=\; \mathbb{1} + \int M(t)\,dt$$

$$P\exp\int_0^T M(t)\,dt = \underline{\mathbb{1}} + \int M(t)\,dt$$

matrix

$$+ \qquad M(t_2)\,M(t_1)$$

$$\mathrm{Pexp}\int_0^T \underset{\uparrow\ \text{matrix}}{M(t)}\,dt = \mathbb{1} + \int M(t)\,dt$$

$$+\int_0^T dt_1 \int_{t_1}^T dt_2 \quad M(t_2)\,M(t_1)$$

$$= \mathbb{1} + \int M(t)\,dt$$

ix

$$+ \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2}\; M(t_2)\,M(t_1) + \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2 \int_{t_2}^T dt_3}\; M(t_3)\,M(t_2)\,M(t_1)$$

$\frac{1}{2!}$ of the total int. region

$\frac{1}{3!}$ of the total region

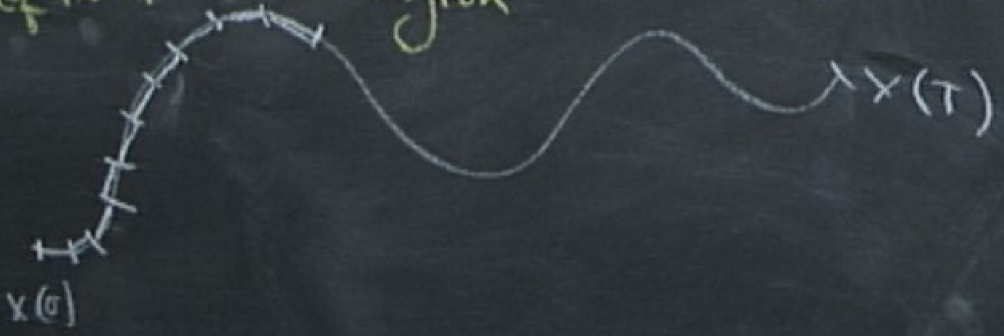
$$\overset{\text{DEF!}}{\equiv} \mathbb{1} + \int M(t)\,dt$$

ix

$$+\underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2}\; M(t_2)\,M(t_1) + \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2 \int_{t_2}^T dt_3}_{\frac{1}{3!}\text{ of the total region}}\; M(t_3)\,M(t_2)\,M(t_1)$$

$$+ \ldots$$

$\frac{1}{2!}$ of the total int. region

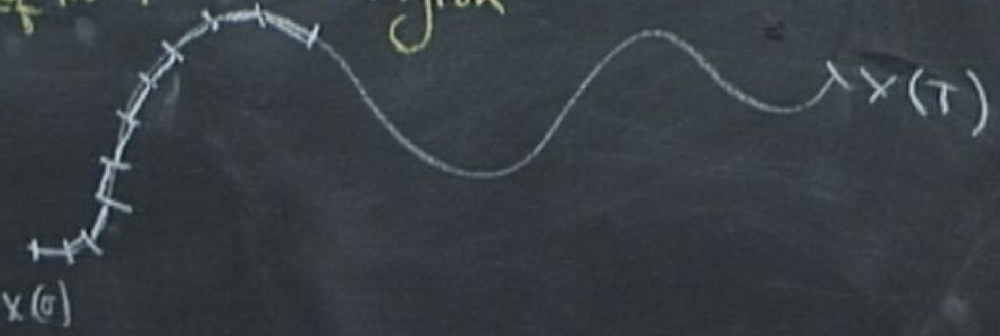


$$\overset{\text{DEF}}{\equiv} \mathbb{1} + \int M(t)\,dt$$

ix

$$+\underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2}\ M(t_2)\,M(t_1) + \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2 \int_{t_2}^T dt_3}_{\frac{1}{3!}\text{ of the total region}} M(t_3)\,M(t_2)\,M(t_1) + \dots$$

$\frac{1}{2!}$ of the total int. region



$$\overset{\text{DEF}}{\equiv} \mathbb{1} + \int M(t)\,dt$$

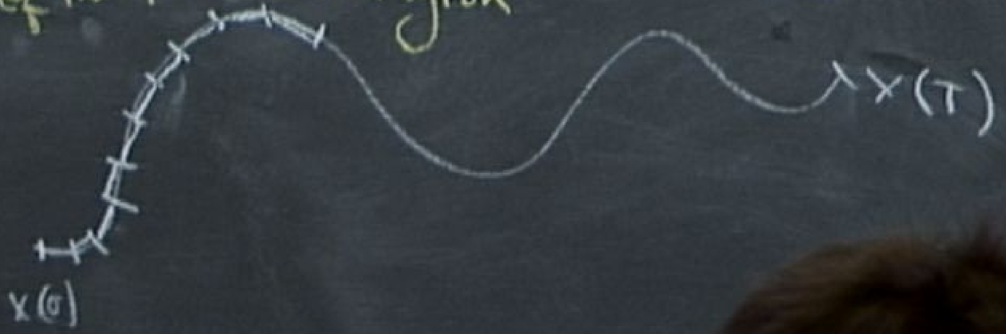
works for U(1) or non-Abelian

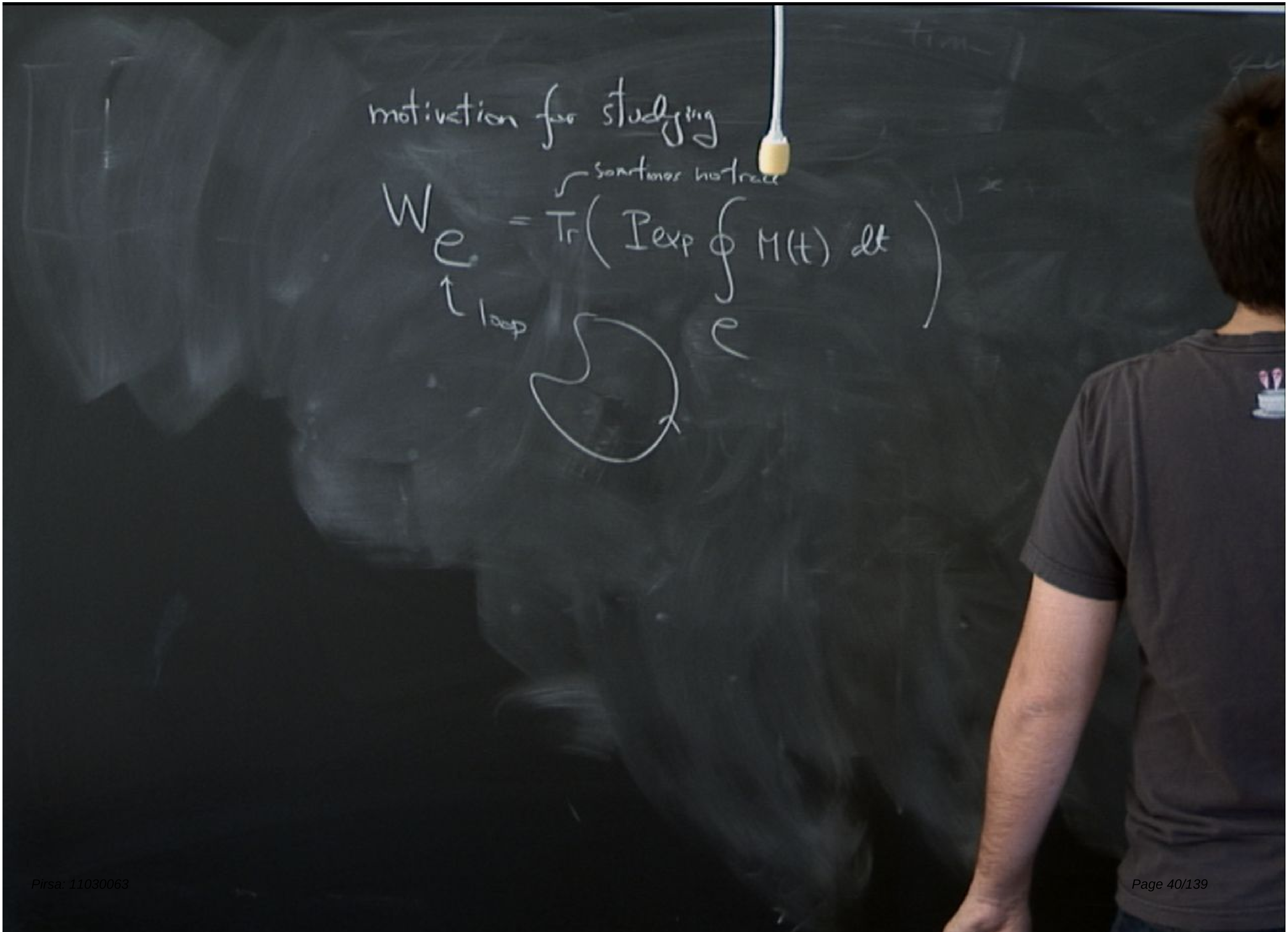
ix

$$+ \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2 \quad M(t_2)\,M(t_1)} + \underbrace{\int_0^T dt_1 \int_{t_1}^T dt_2 \int_{t_2}^T dt_3 \; M(t_3)\,M(t_2)\,M(t_1)} + \ldots$$

$\frac{1}{2!}$ of the total int. region

$\frac{1}{3!}$ of the total region





motivation for studying

sometimes no trace

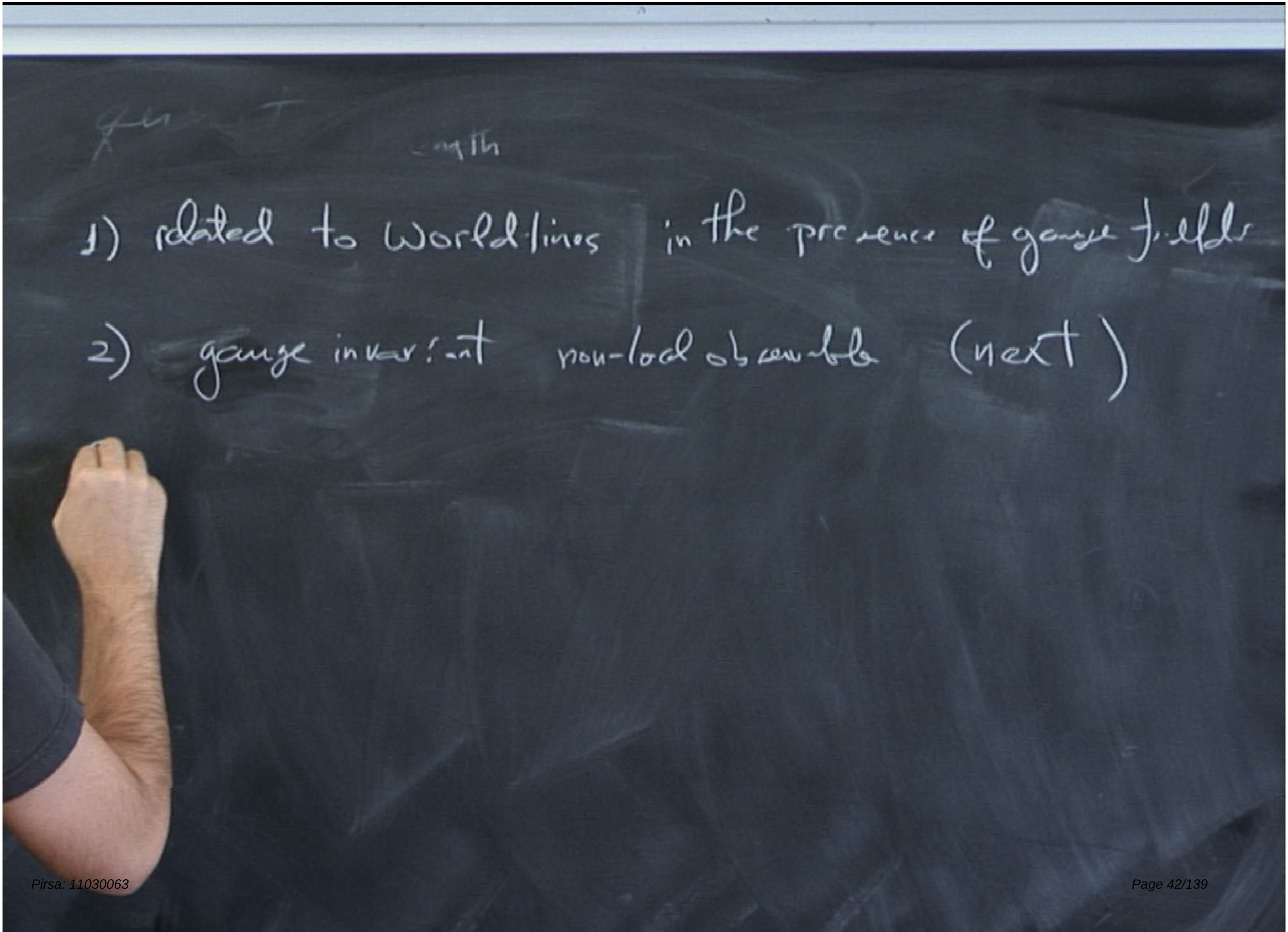
$$W_e = \mathrm{Tr}\left(P\exp \oint_e M(t)\, dt \right)$$

↑ loop

motivation for studying

$$W_{\mathcal{C}} = \overset{\text{sometimes no trace}}{\mathrm{Tr}}\left(P\exp \oint_{\mathcal{C}} M(t)\, dt \right)$$

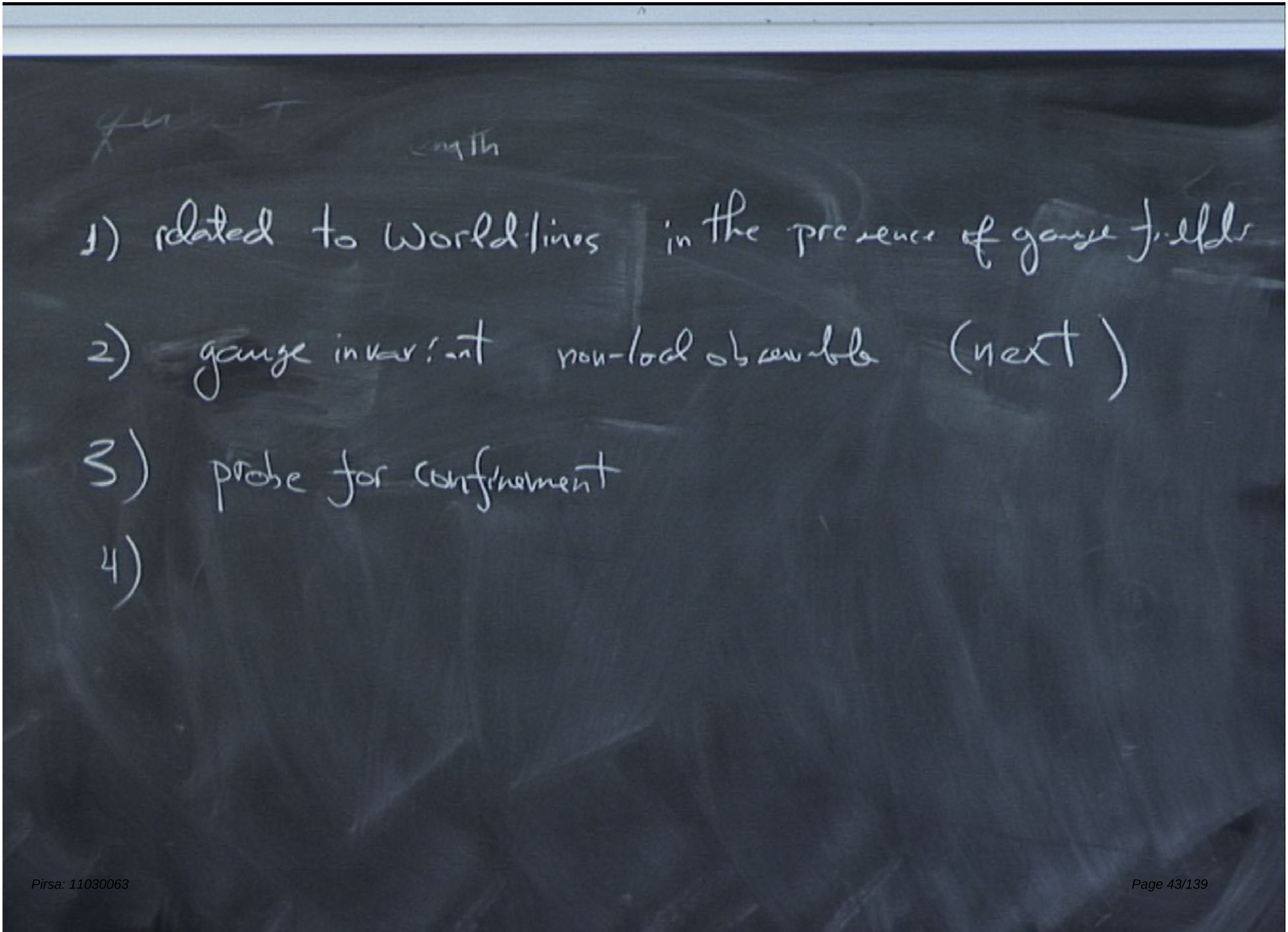
↑ loop



ngth

1) related to world-lines in the presence of gauge fields

2) gauge invariant non-local observable (next)



ngth

1) related to Worldlines in the presence of gauge fields

2) gauge invariant non-local observable (next)

3) probe for confinement

4)

ength

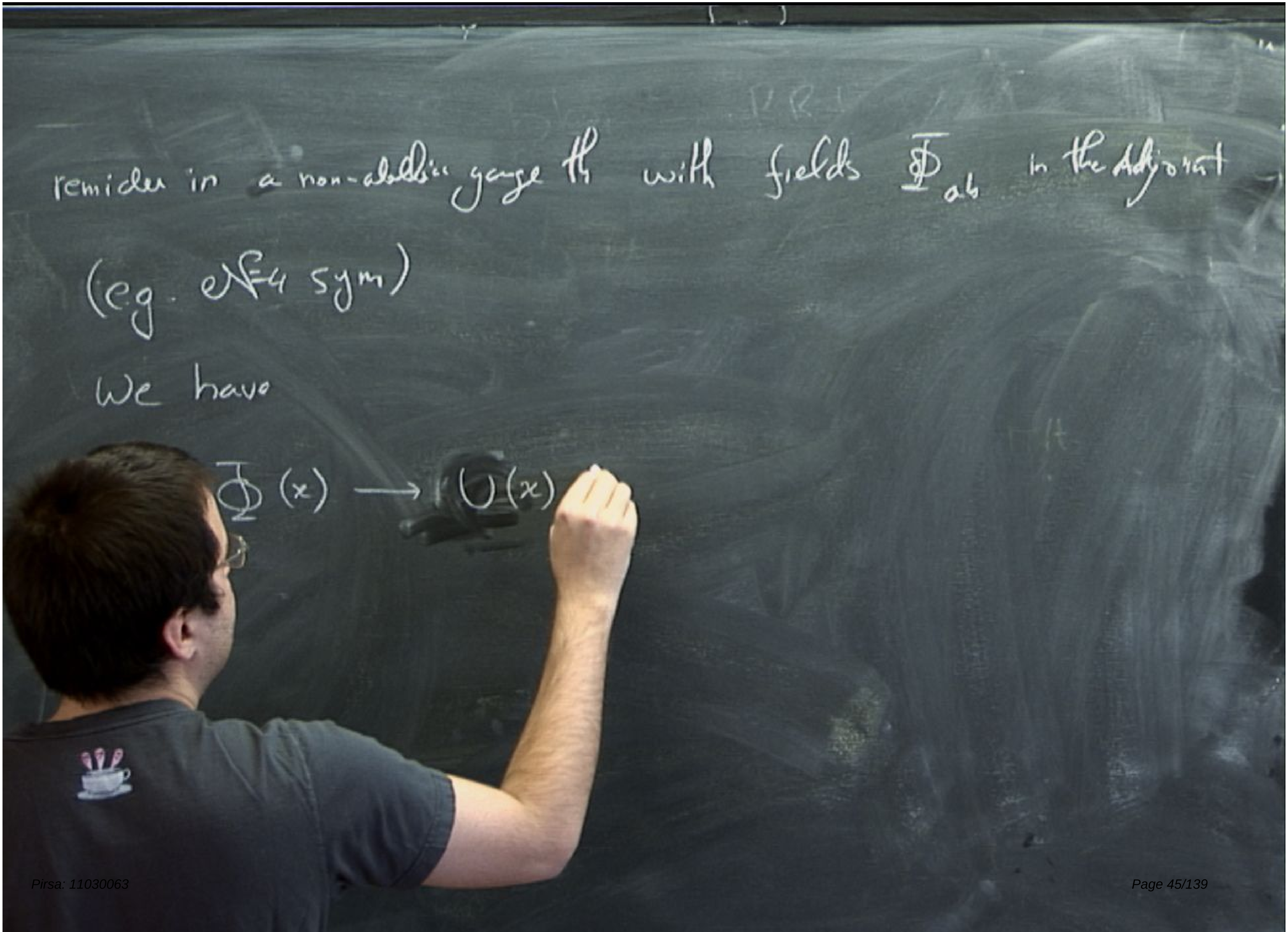
1) related to Worldlines in the presence of gauge fields

2) gauge invariant non-local observable (next)

3) probe for confinement

4) related to scattering amplitudes

5) nice qty in AdS/CFT and in lattice QFT



reminder in a non-abelian gauge th with fields Φ_{ab} in the Adjoint

(eg. $\mathcal{N}=4$ sym)

We have

$$\Phi(x) \longrightarrow U(x)$$

reminder in a non-abelian gauge th with fields Φ_{ab} in the Adjoint

(eg. $\mathcal{N}=4$ sym)

We have

$$\Phi(x) \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

reminder in a non-abelian gauge th with fields Φ_{ab} in the Adjoint

(e.g. $\mathcal{N}=4$ sym)

We have

$$\underbrace{\Phi(x)}_{\text{not a gauge inv. op.}} \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

reminder in a non-abelian gauge th with fields Φ_{ab} in the Adjoint

(eg. $\mathcal{N}=4$ sym)

We have

$$\underbrace{\Phi(x)}_{\underline{\text{not}}\text{ a gauge inv. op'}} \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

reminder in a non-abelian gauge th with fields Φ_{ab} in the Adjoint

(eg. $\mathcal{N}=4$ sym)

We have

$$\underbrace{\Phi(x)}_{\text{\underline{not} a gauge inv. op.}} \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

$$\Phi(x)\,\Phi(x) \longrightarrow U(x)\,\Phi\,\Phi\,U^{-1}(x)$$

reminder in a non-abelian gauge th with fields Φ_{ab} in the Adjoint

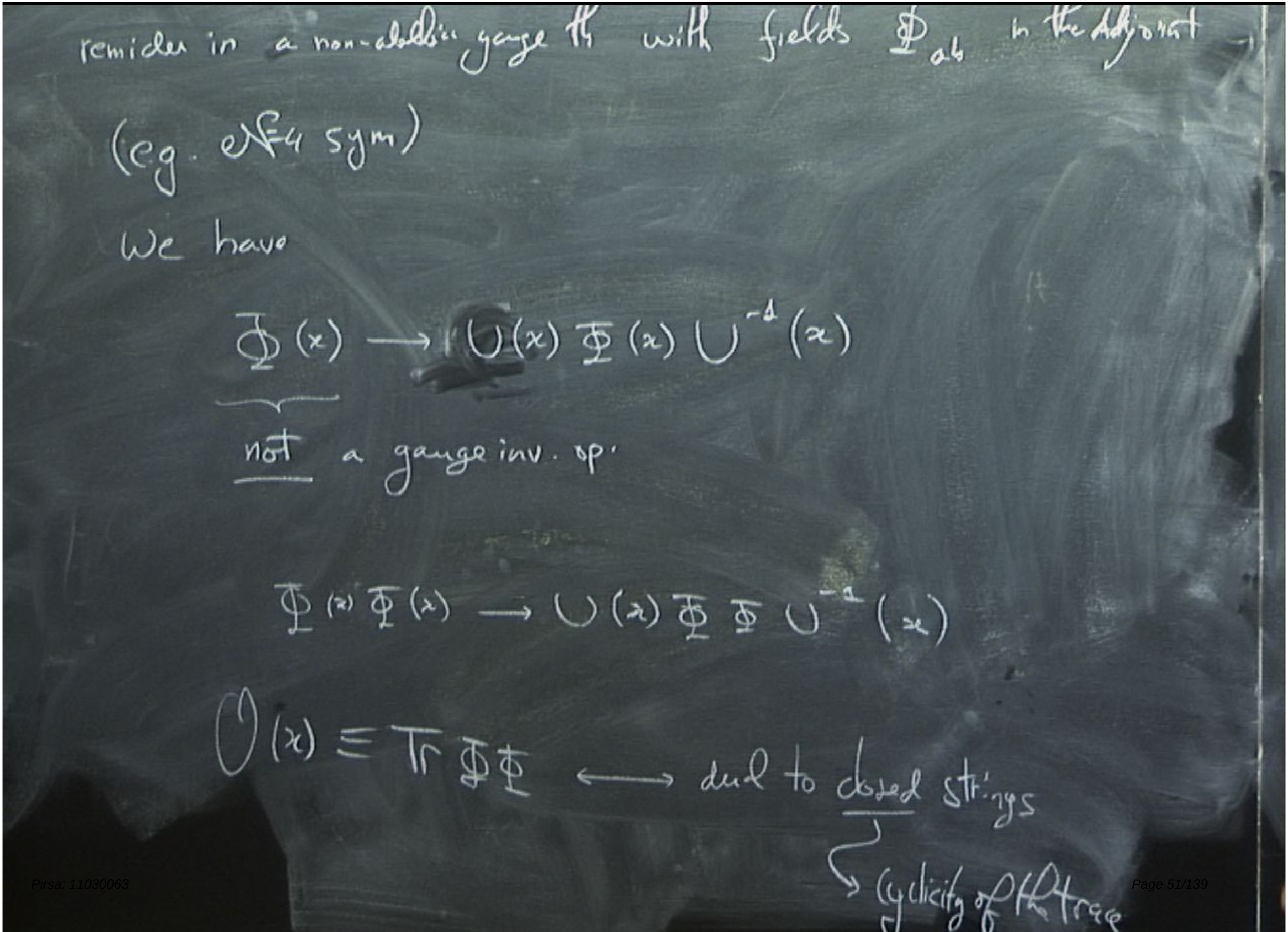
(eg. $\mathcal{N}=4$ sym)

We have

$$\underbrace{\Phi(x)}_{\underline{\text{not}}\text{ a gauge inv. op}} \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

$$\Phi(x)\,\Phi(x) \longrightarrow U(x)\,\Phi\,\Phi\,U^{-1}(x)$$

$$\mathcal{O}(x) \equiv \mathrm{Tr}\,\Phi\,\Phi$$



reminder in a non-abelian gauge th. with fields Φ_{ab} in the adjoint

(e.g. $\mathcal{N}=4$ SYM)

We have

$$\underbrace{\Phi(x)}_{\underline{\text{not}}\text{ a gauge inv. op.}} \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

$$\Phi(x)\,\Phi(x) \longrightarrow U(x)\,\Phi\,\Phi\,U^{-1}(x)$$

$$\mathcal{O}(x) \equiv \mathrm{Tr}\,\Phi\Phi \longleftrightarrow \text{dual to } \underline{\text{closed}} \text{ strings}$$

↳ cyclicity of the trace

what if we insist on constructing something for $\Phi(x)\,\Phi(y)$

(t_3)

$$+\frac{1}{3!}\int dt_1\int dt_2\int dt_3\; M(t_1)\,\cdots$$

$$\cdot(i\varepsilon M(x_N))\Big]_{a_1a_2}\Big[\exp\big(i\varepsilon M(x_{N-1})\big)\Big]_{a_2a_3}\cdots\Big[\exp\big(i\varepsilon M(x_1)\big)\Big]_{a_N a_{N+1}}$$

$=3$ ✓

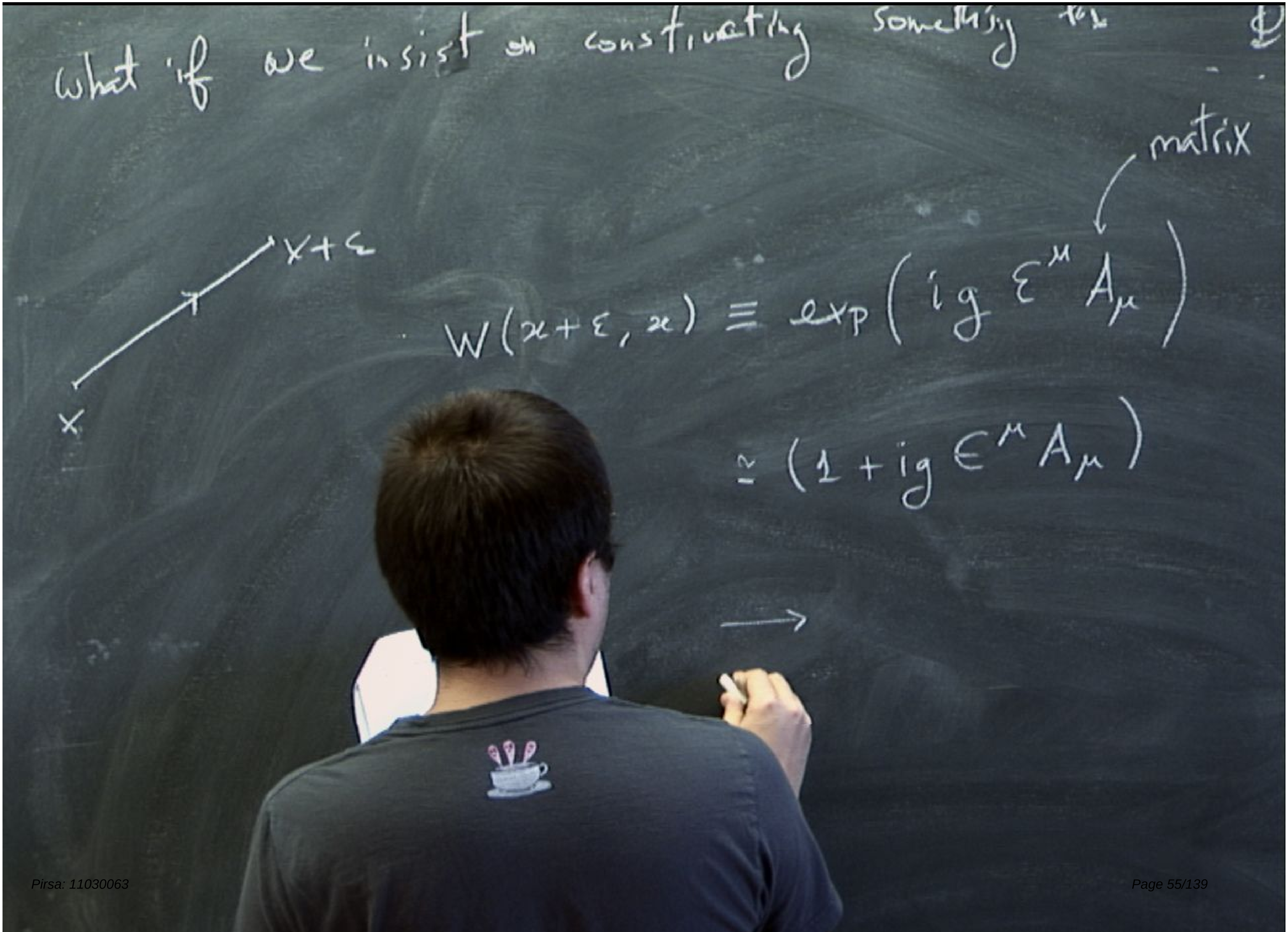
$M(t)\,\Big]\,\psi(t)=0$ with b.c $\psi(0)=1$. Then $\psi(T)=\exp$

matrix matrix identity matrix

what if we insist on constructing something for $\Phi(x)\,\Phi(y)$?

What if we insist on constructing something the D - .

$$W(x+\varepsilon, x) \equiv \exp\left(i g \, \varepsilon^{\mu} \overset{\text{matrix}}{A_{\mu}}\right)$$



what if we insist on constructing something for

$$W(x+\varepsilon, x) \equiv \exp\left(ig\,\varepsilon^\mu A_\mu\right)$$

$$\simeq \left(1 + ig\,\epsilon^\mu A_\mu\right)$$

$$\longrightarrow$$

matrix

$$W(x+\varepsilon, x) \equiv \exp\left(i g\, \varepsilon^{\mu} A_{\mu} \right)$$

$$\simeq \left(1 + i g\, \epsilon^{\mu} A_{\mu} \right)$$

$$\xrightarrow[\text{gauge transf}]{} \left(1 + i g\, \epsilon^{\mu} \left(U A_{\mu} U^{-1} - U \partial_{\mu} U^{-1}\right.\right.$$

$$W(x+\varepsilon, x) \equiv \exp\left(i g\, \varepsilon^{\mu} \overset{\text{matrix}}{A_{\mu}} \right)$$

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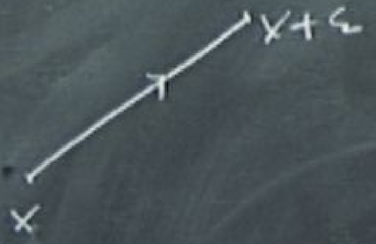
$$\simeq \left(1 + i g\, \epsilon^{\mu} A_{\mu} \right)$$

$$\xrightarrow[\text{gauge transf}]{} \left(1 + i g\, \epsilon^{\mu} \left(U A_{\mu} U^{-1} - U \partial_{\mu} U^{-1} \right) \right)$$

$$\simeq U(x+\epsilon)\, W(x+\varepsilon, x)\, U^{-1}(x)$$

What if we insist on constructing something like $\Phi(x)\,\Phi(y)$?



matrix

$$W(x+\varepsilon, x) \equiv \exp\left(ig\, \varepsilon^\mu A_\mu\right)$$

$$\simeq \left(1 + ig\, \epsilon^\mu A_\mu\right)$$

$$\xrightarrow[\text{transf}]{}\left(1 + ig\, \epsilon^\mu \left(U A_\mu U^{-1} - U \partial_\mu U^{-1}\right)\right)$$

gauge transf

$$\simeq U(x+\epsilon)\, W(x+\varepsilon, x)\, U^{-1}(x)$$

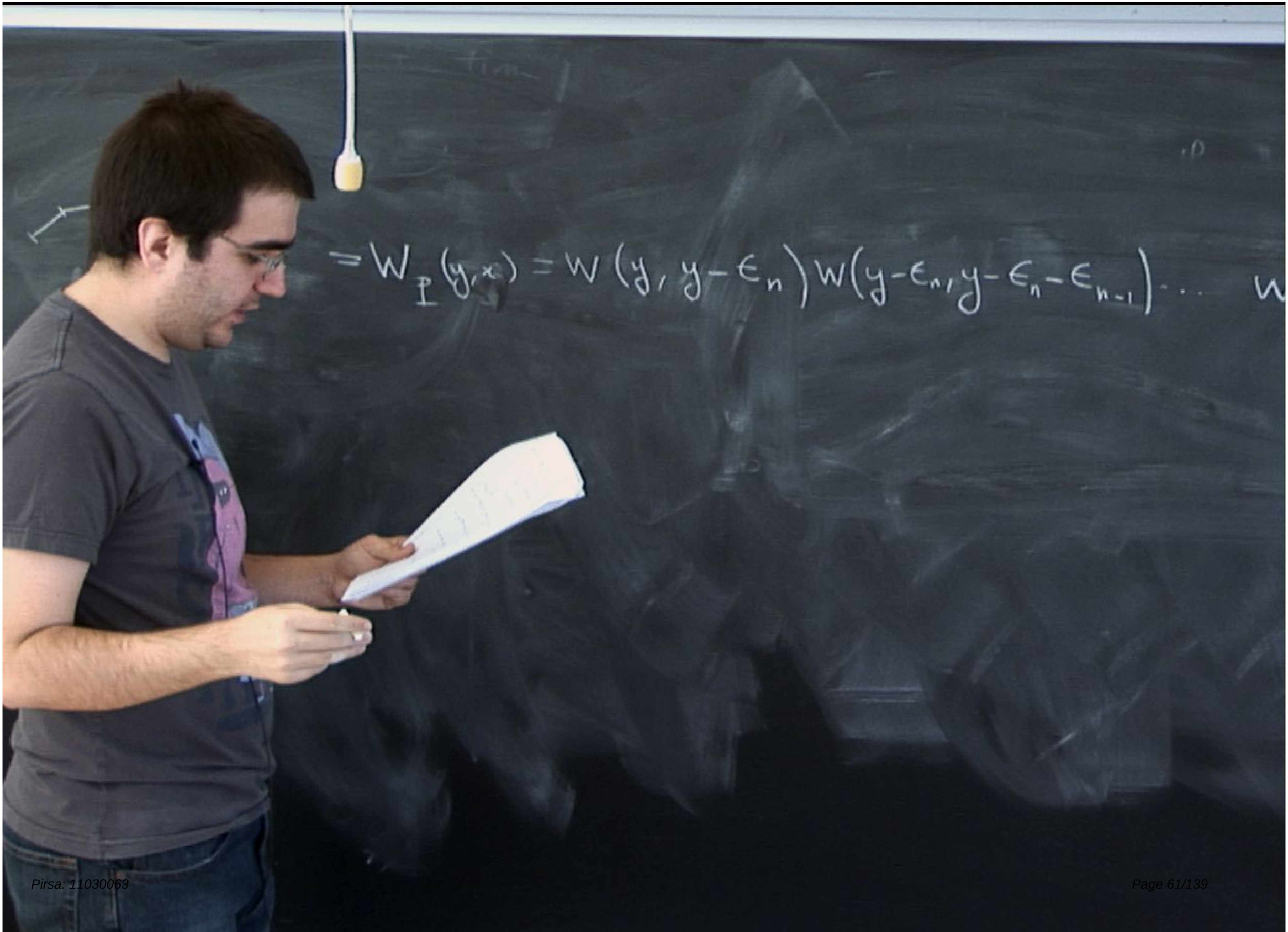
in a non-abelian gauge th with fields in the Adjoint,

e$\mathcal{N}=4$ sym)

have

$$\underbrace{\Phi(x)} \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

not a gauge inv. op'



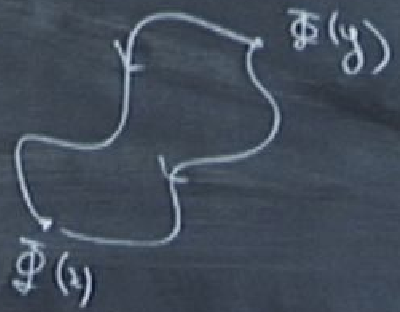
$$= W_{P}(y,x) = W(y, y-\epsilon_n)\, W(y-\epsilon_n, y-\epsilon_n-\epsilon_{n-1}) \dots \quad W$$

$$= W_P(y,x) = W(y, y-\epsilon_n)\, W(y-\epsilon_n, y-\epsilon_n-\epsilon_{n-1}) \cdots W(x_1+\epsilon_1, x)$$

$$\xrightarrow[\text{gauge transf}]{} U(y)\, W_P(y,x)\, U^{-1}(x)$$

$$= W_{\Phi}(y,x) = W(y, y-\epsilon_n)\, W(y-\epsilon_n, y-\epsilon_n-\epsilon_{n-1}) \cdots W(x_1+\epsilon_1, x)$$

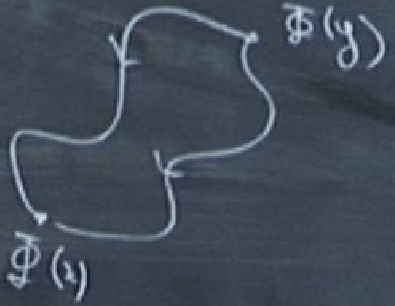
$$\xrightarrow[\text{gauge transf}]{} U(y)\, W_p(y,x)\, U^{-1}(x)$$



$$W_p(x,y)\, \Phi(y)\, W(y,x)$$

$$= W_P(y,x) = W(y, y-\epsilon_n)\, W(y-\epsilon_n, y-\epsilon_n-\epsilon_{n-1}) \cdots W(x_1+\epsilon_1, x)$$

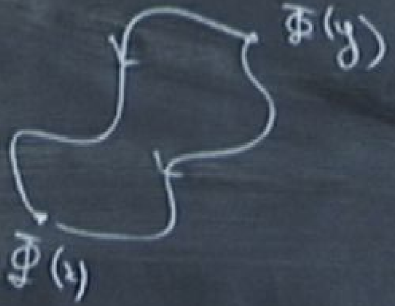
$$\xrightarrow[\text{gauge transf}]{} U(y)\, W_P(y,x)\, U^{-1}(x)$$



$$\Phi(x)\, W_P(x,y)\, \Phi(y)\, W(y,x)$$

$$= W_P(y,x) = W(y, y-\epsilon_n)\,W(y-\epsilon_n, y-\epsilon_n-\epsilon_{n-1}) \cdots W(x_1+\epsilon_1, x)$$

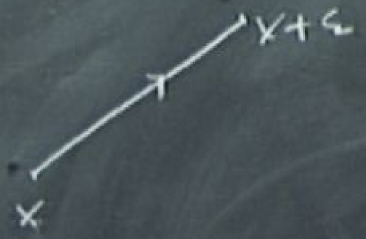
$$\xrightarrow[\text{gauge transf}]{} U(y)\, W_P(y,x)\, U^{-1}(x)$$



$$\mathrm{Tr}\left[\Phi(x)\, W_P(x,y)\, \Phi(y)\, W(y,x)\right]$$

↑ non-local gauge inv. op!

what if we insist on constructing something for $\Phi(x)\,\Phi(y)$?



matrix

$$W(x+\varepsilon, x) \equiv \exp\left(ig\,\varepsilon^\mu A_\mu\right)$$

$$U U^{-1} = \mathbb{1}$$

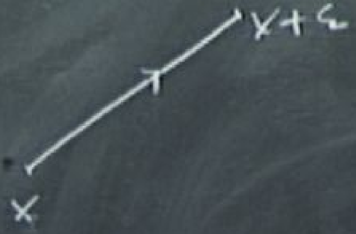
$$\partial U U^{-1} + U \partial U^{-1} = 0$$

$$\simeq \left(\mathbb{1} + ig\,\epsilon^\mu A_\mu\right)$$

$$\xrightarrow[\text{gauge transf}]{} \left(\mathbb{1} + ig\,\epsilon^\mu \left(U A_\mu U^{-1} - U \partial_\mu\right.\right.$$ [illegible] $$1))$$

$$\simeq U(x+\epsilon)\, W(x+\varepsilon, x)\, U^{-1}(x)$$

What if we insist on constructing something for $\Phi(x)\,\Phi(y)$?



matrix

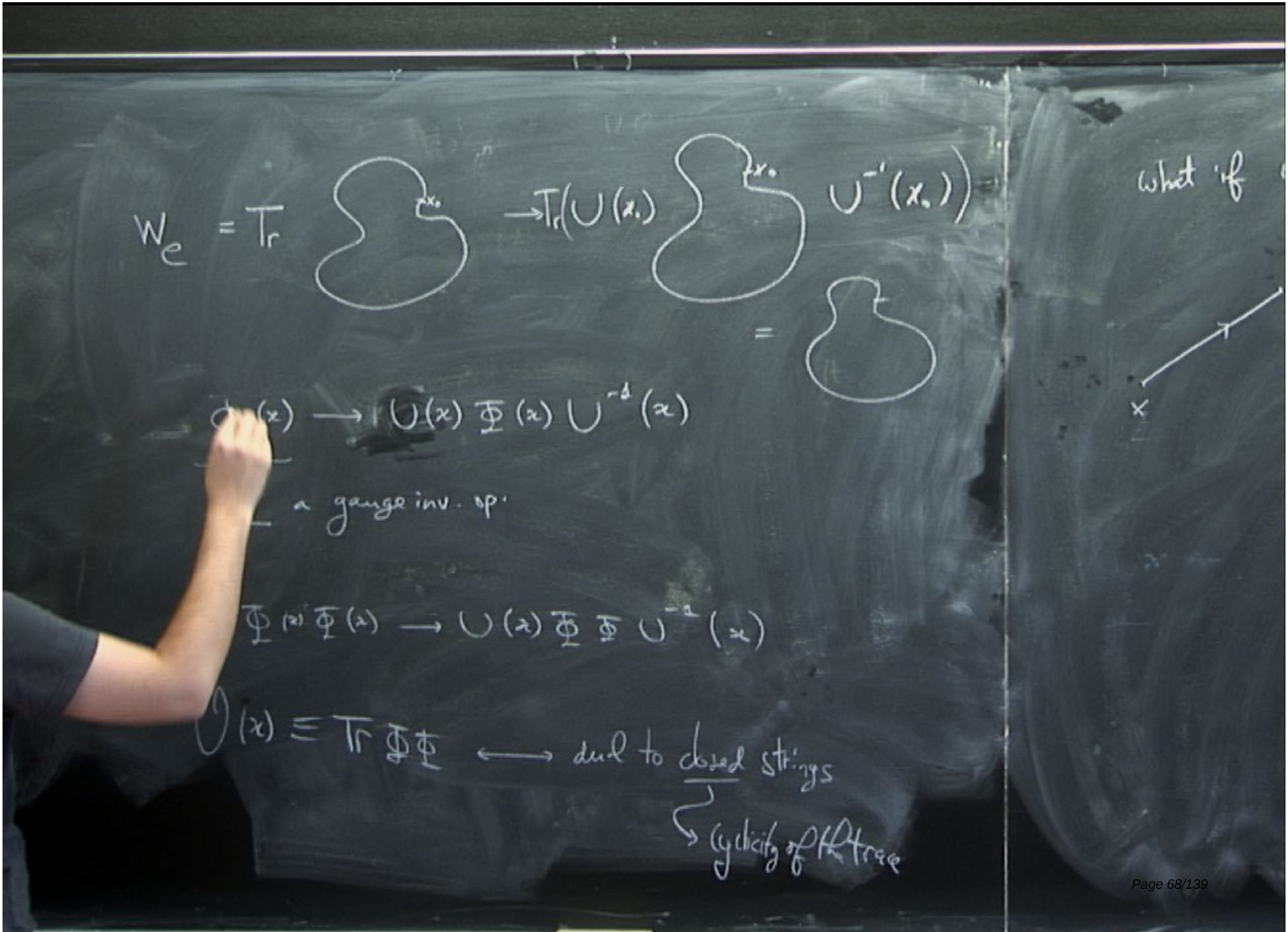
$$W(x+\varepsilon, x) \equiv \exp\left(ig\,\varepsilon^\mu A_\mu\right)$$

$$UU^{-1} = \mathbb{1}$$
$$\partial U U^{-1} + U\partial U^{-1} = 0$$

$$\simeq \left(1 + ig\,\epsilon^\mu A_\mu\right)$$

$$\xrightarrow[\text{gauge transf}]{} \left(1 + ig\,\epsilon^\mu\left(UA_\mu U^{-1} - U\partial_\mu U^{-1}\right)\right)$$

$$\simeq U(x+\epsilon)\,W(x+\varepsilon, x)\,U^{-1}(x)$$



$W_e = \mathrm{Tr}$ ⟶ $\mathrm{Tr}\big(U(x_0)$ $U^{-1}(x_0)\big)$ =

$\Phi(x) \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$

a gauge inv. op.

$\Phi(x)\Phi(x) \longrightarrow U(x)\,\Phi\,\Phi\,U^{-1}(x)$

$\mathcal{O}(x) \equiv \mathrm{Tr}\,\Phi\Phi \longleftrightarrow$ dual to closed strings

↳ cyclicity of the trace

what if

$W_e = \mathrm{Tr}$ 

$\longrightarrow \mathrm{Tr}\big(U(x_0)$ 

$U^{-1}(x_0)\big)$

$=$ 

x_0 is not important

$\Phi(x) \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$

$\underbrace{\Phi(x)}$

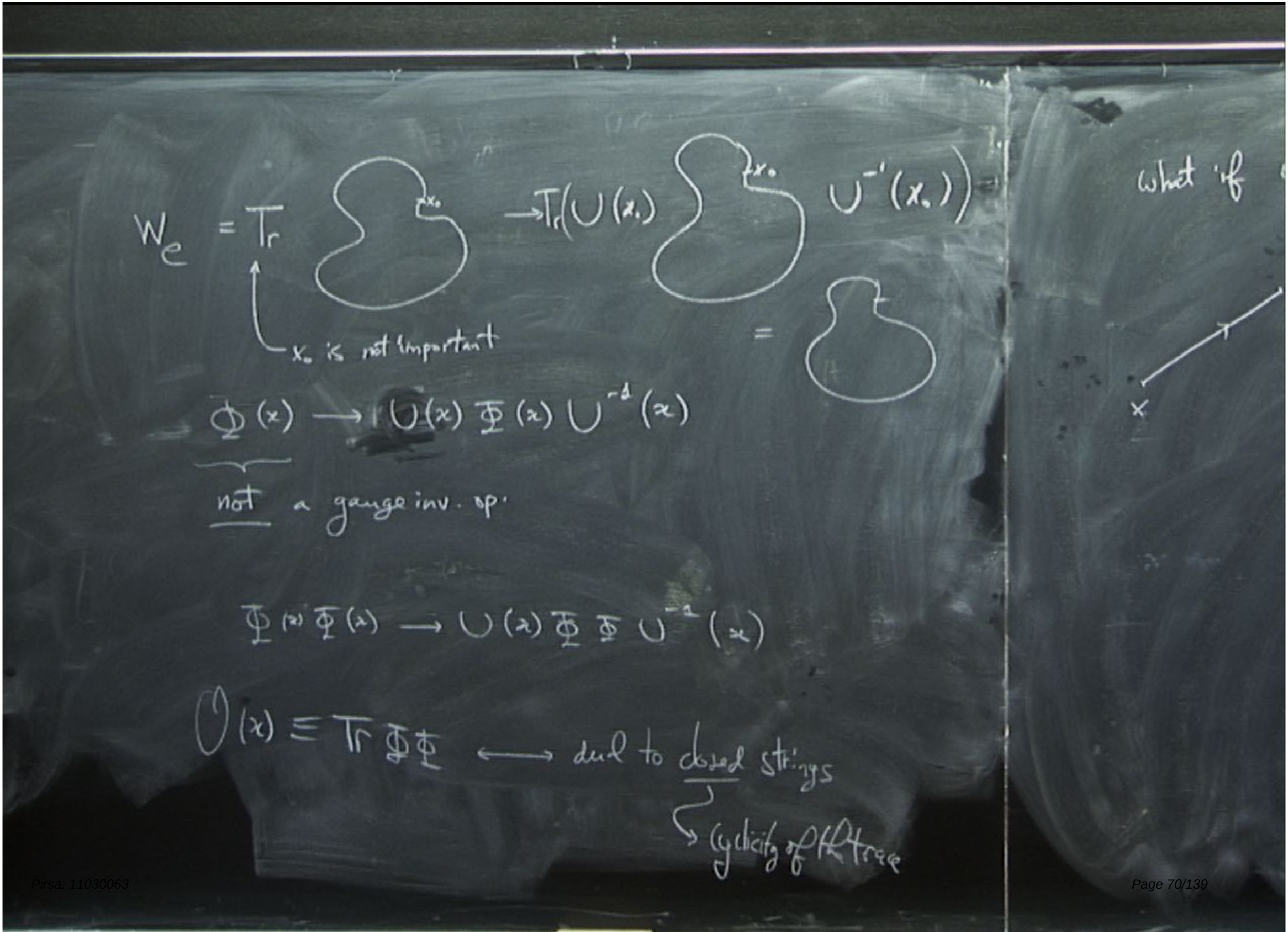
not a gauge inv. op.

$\Phi(x)\,\Phi(x) \longrightarrow U(x)\,\Phi\,\Phi\,U^{-1}(x)$

$O(x) \equiv \mathrm{Tr}\,\Phi\Phi \longleftrightarrow$ dual to closed strings

cyclicity of the trace

what if



$W_e = \mathrm{Tr}$ x_0 $\longrightarrow \mathrm{Tr}\left(U(x_0)\right.$ x_0 $\left.U^{-1}(x_0)\right)$

$=$

x_0 is not important

$$\underbrace{\Phi(x)}_{\underline{\text{not}}\text{ a gauge inv. op.}} \longrightarrow U(x)\,\Phi(x)\,U^{-1}(x)$$

$$\Phi(x)\,\Phi(x) \longrightarrow U(x)\,\Phi\,\Phi\,U^{-1}(x)$$

$\mathcal{O}(x) \equiv \mathrm{Tr}\,\Phi\Phi \longleftrightarrow$ dual to closed strings

cyclicity of the trace

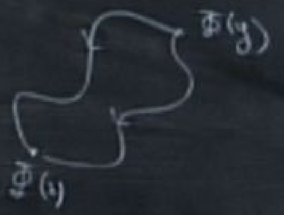
what if

x

$$= W_P(y,x) = W(y, y-\epsilon_n)\, W(y-\epsilon_n, y-\epsilon_n-\epsilon_{n-1}) \cdots \quad W(x_1+\epsilon_1, x)$$

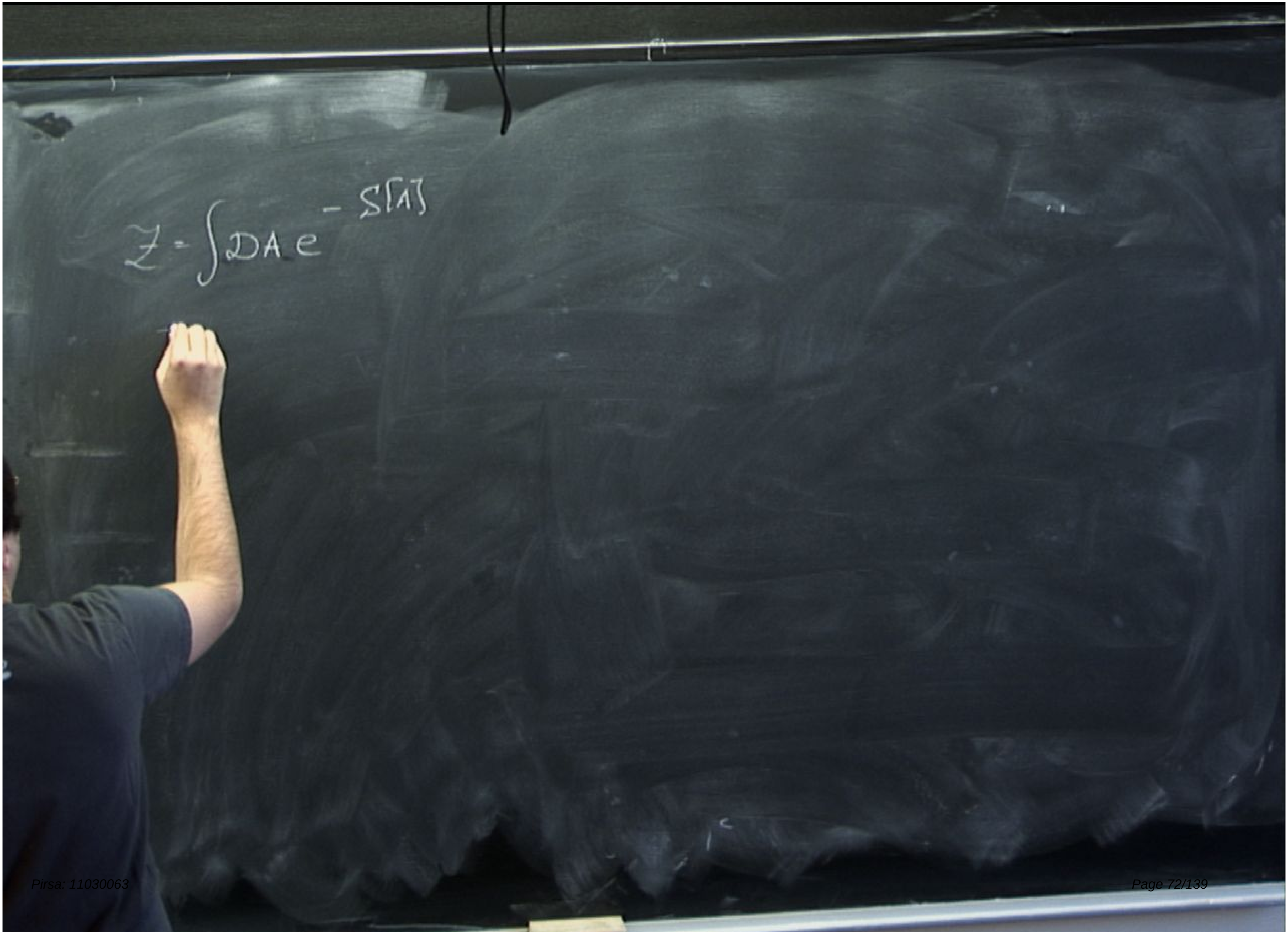
$$\xrightarrow[\text{gauge transf}]{} U(y)\, W_p(y,x)\, U^{-1}(x)$$

$$A \xrightarrow{\text{gauge}} U A U^{-1}$$



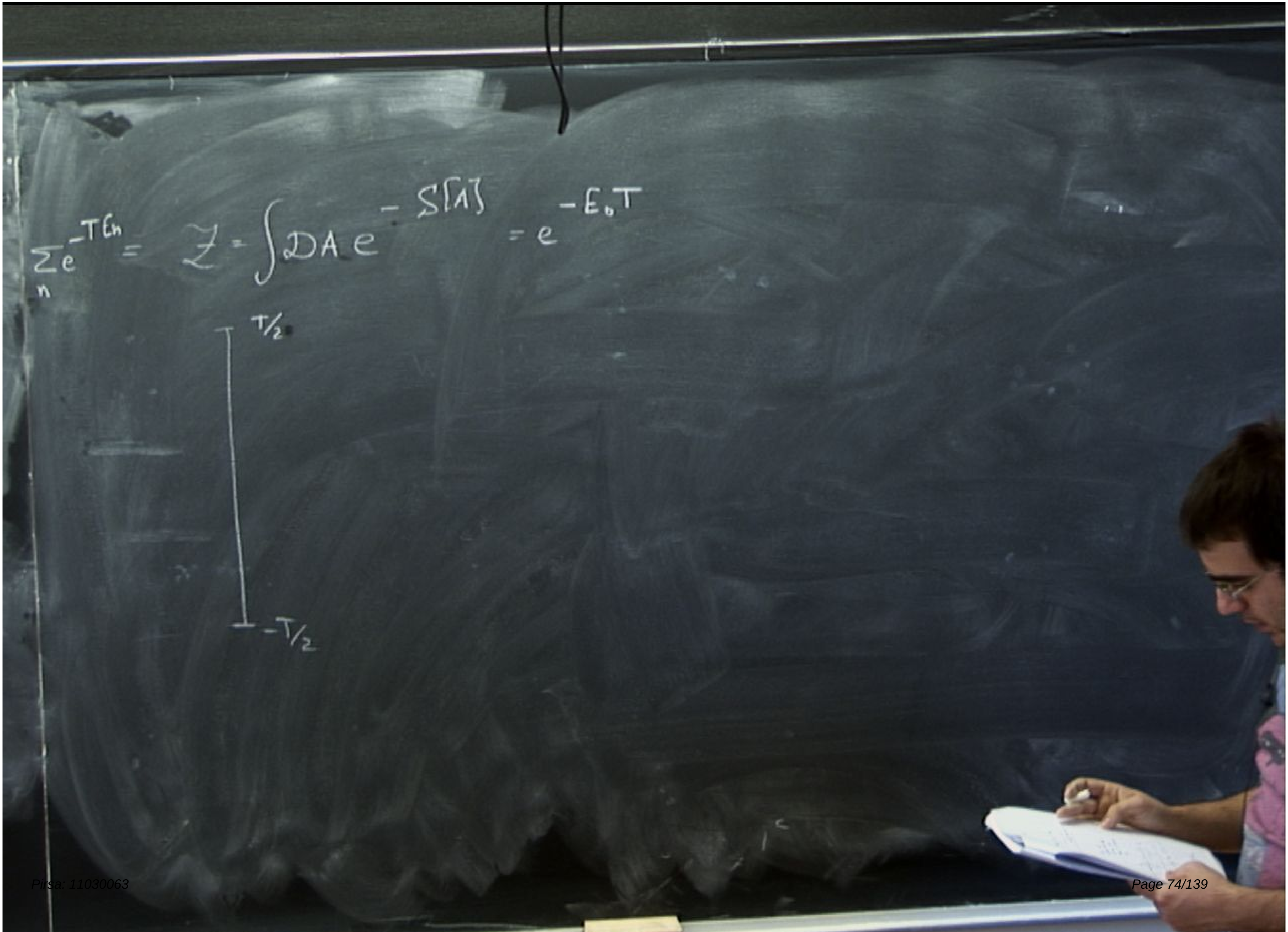
$$\mathrm{Tr}\left[\overbrace{\Phi(x)\, W_p(x,y)\, \Phi(y)\, W(y,x)}\right]$$

non-local gauge inv op!



$$Z = \int \mathcal{D}A \, e^{-S[A]}$$

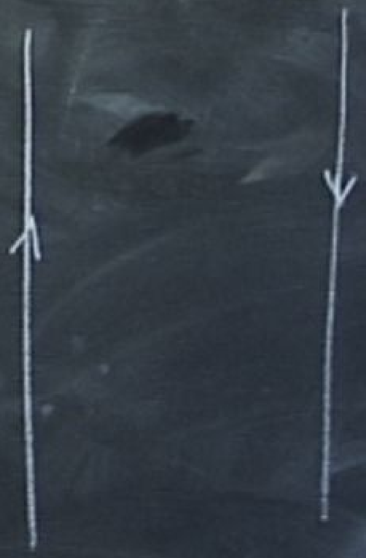
$$Z = \int \mathcal{D}A \, e^{-S[A]}$$



$$\sum_n e^{-TE_n} = Z = \int \mathcal{D}A\, e^{-S[A]} = e^{-E_0 T}$$

$$\sum_n e^{-TE_n} = Z = \int \mathcal{D}A\, e^{-S[A]} = e^{-E_0 T}$$

lets add two charges
(one e, $-e$
separated by R)



$= e^{-E_0 T}$

lets add two charges

(one e, $-e$

separated by R)

$$Z[A] = e^{-E_0 T}$$

$$\downarrow$$

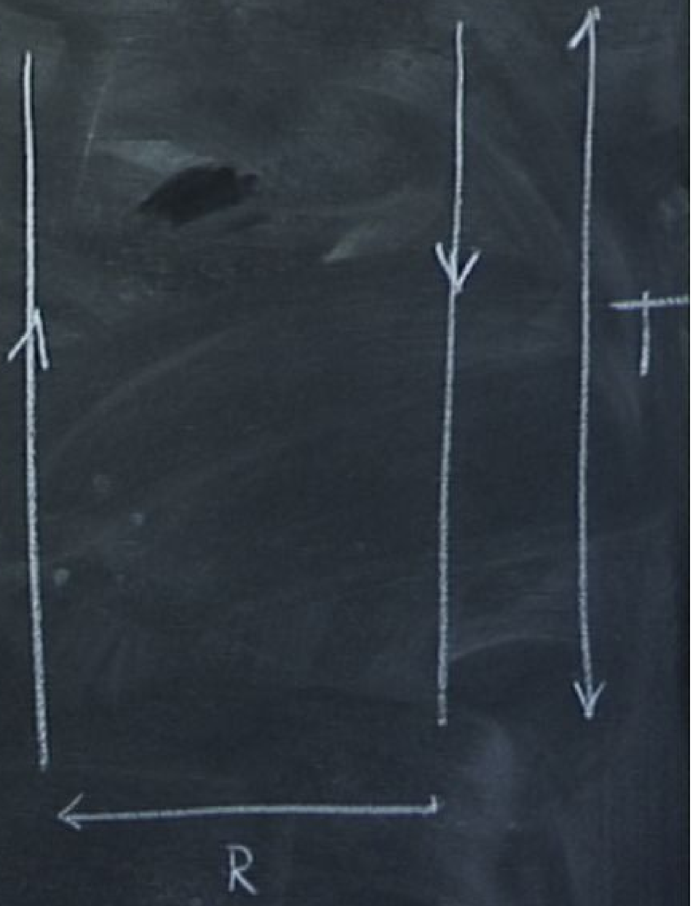
$$e^{-(E_0 + V(R))T}$$

$\nwarrow$ we c

lets add two charges

(one e, $-e$

separated by R)



$$Z[A] = e^{-E_0 T}$$

$$\downarrow$$

$$e^{-(E_0 + V(R))T}$$

↖ we can then read $V(R)$

lets add two charges

(one e, $-e$

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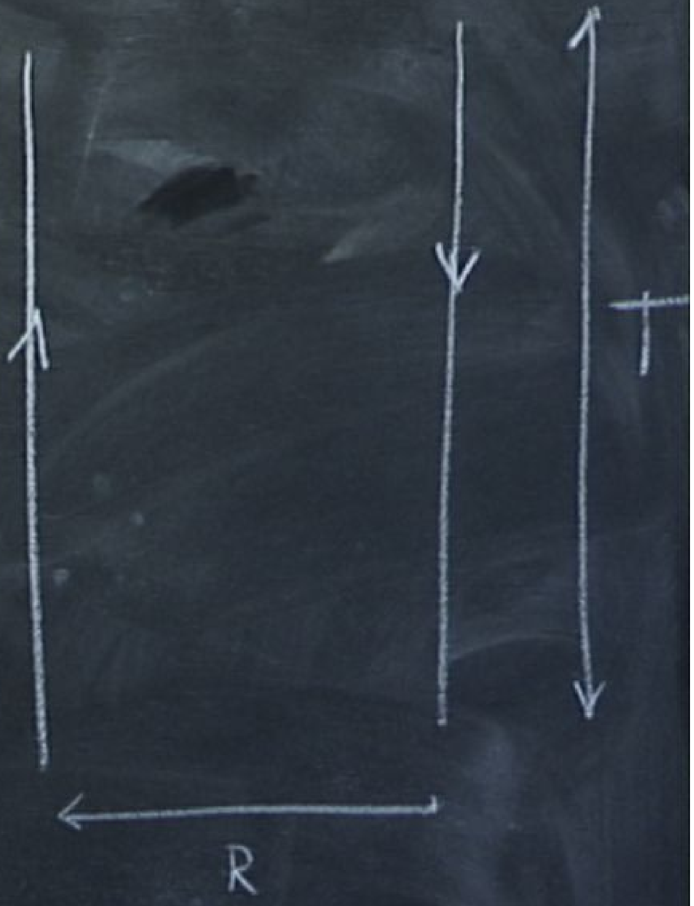

$$^{TE_n} = \mathcal{Z} = \int \mathcal{D}A \, e^{-S[A]} = e^{-E_0 T}$$

$$\downarrow$$

$$e^{-(E_0 + V(R))T}$$

↑ we can then read $V(R)$

lets add two charges

(one e, $-e$

separated by R)

Adding charges:

$$\int \mathcal{D}A \, \exp\Big(-S[A]$$

$$e^{-S[A]} = e^{-E_0 T}$$

$$\downarrow$$

$$e^{-(E_0 + V(R))T}$$

↖ we can then read $V(R)$

lets add two charges

(one e, $-e$

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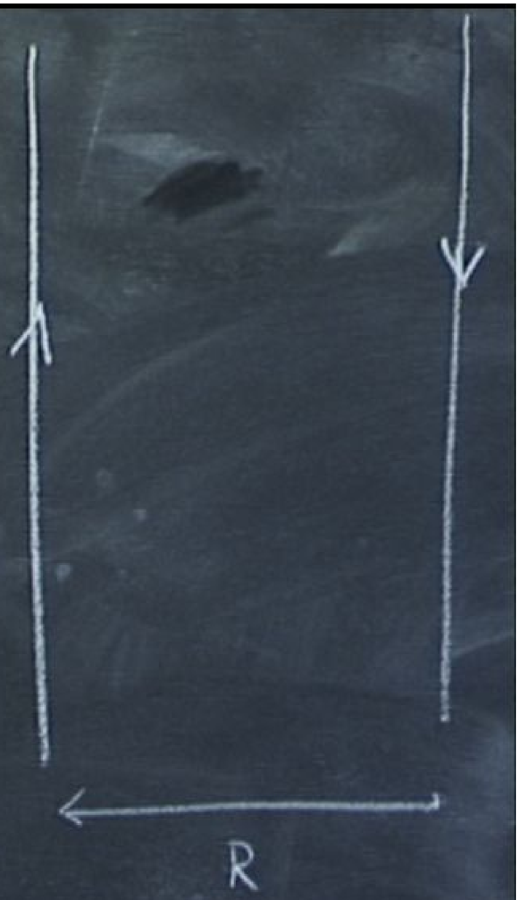


Adding charges:

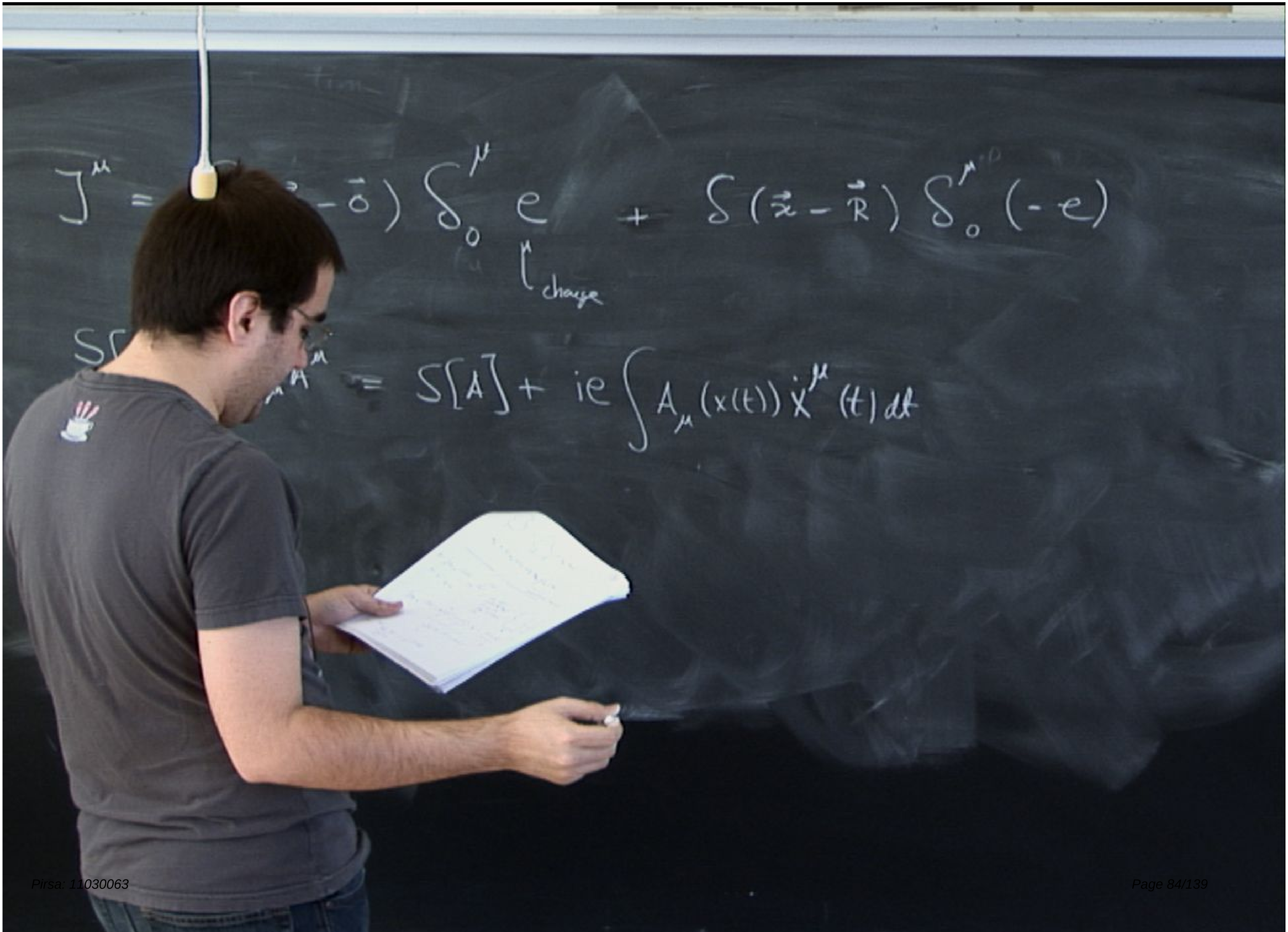
$$\int \mathcal{D}A \exp\left(-S[A] + \int dt\, d^3x\, J_\mu(t,x) A^\mu(t,x)\right)$$

$$J^{\mu} = \delta(\vec{x} - \vec{0})\, \delta^{\mu}_{0}\, e$$

$$J^\mu = \delta(\vec{x}-\vec{0})\,\delta^\mu_0\,\underbrace{e}_{\text{charge}} + \delta(\vec{x}-\vec{R})\,\delta^\mu_0\,(-e)$$

$$J^\mu = \delta(\vec{x}-\vec{0})\,\delta^\mu_0\, \underset{\uparrow\,\text{charge}}{e} + \delta(\vec{x}-\vec{R})\,\delta^\mu_0\,(-e)$$

$$S[A] + i\int J_\mu A^\mu = S[A] +$$


J^μ = …−0⃗) δ^μ_0 e + δ(x⃗ − R⃗) δ^μ_0 (−e)
charge
S[… x^μ] = S[A] + ie ∫ A_μ (x(t)) ẋ^μ (t) dt

$$J^{\mu} = \delta(\vec{x} - \vec{0})\, \delta^{\mu}_{0}\, \underset{\uparrow_{\text{charge}}}{e} + \delta(\vec{x} - \vec{R})\, \delta^{\mu}_{0}\, (-e)$$

$$S[A] + i\int J_{\mu} A^{\mu} = S[A] + ie\int A_{\mu}(x_L(t))\, \dot{X}^{\mu}_{L}(t)\, dt + ie\int A_{\mu}(x_R(t))\, \dot{X}^{\mu}_{R}\, dt$$

$$X^{\mu}_{L} = (t, 0, 0, 0) \qquad X^{\mu}_{R} = (-t, 0, 0, R)$$

$$J^\mu = \delta(\vec{x}-\vec{0})\,\delta^\mu_0\,\underset{\uparrow_{\text{charge}}}{e} + \delta(\vec{x}-\vec{R})\,\delta^\mu_0\,(-e)$$

$$S[A] + i\int J_\mu A^\mu = S[A] + ie\int A_\mu(x_L(t))\,\dot{x}^\mu_L(t)\,dt + ie\int A_\mu(x_R(t))\,\dot{x}^\mu_R\,dt$$

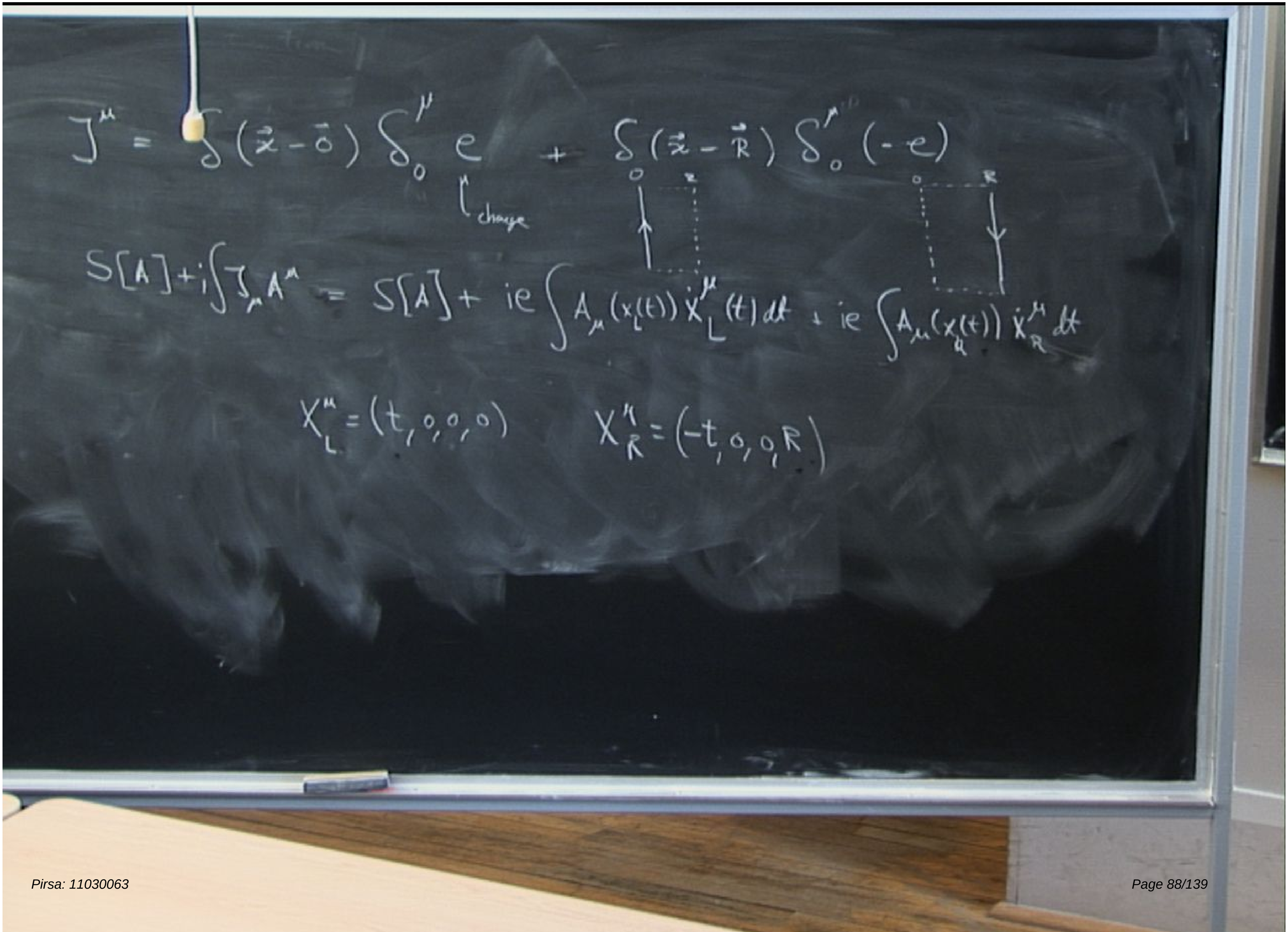
$$X^\mu_L = (t,0,0,0) \qquad X^\mu_R = (-t,0,0,R)$$

lets add two charges

(one e, -e

separated by R)

n then read V(R)



$$J^\mu = \delta(\vec{x}-\vec{0})\,\delta^\mu_0\,\underset{\uparrow\,\text{charge}}{e} + \delta(\vec{x}-\vec{R})\,\delta^\mu_0\,(-e)$$

$$S[A] + i\int J_\mu A^\mu = S[A] + ie\int A_\mu(x_L(t))\,\dot{X}^\mu_L(t)\,dt + ie\int A_\mu(x_R(t))\,\dot{X}^\mu_R\,dt$$

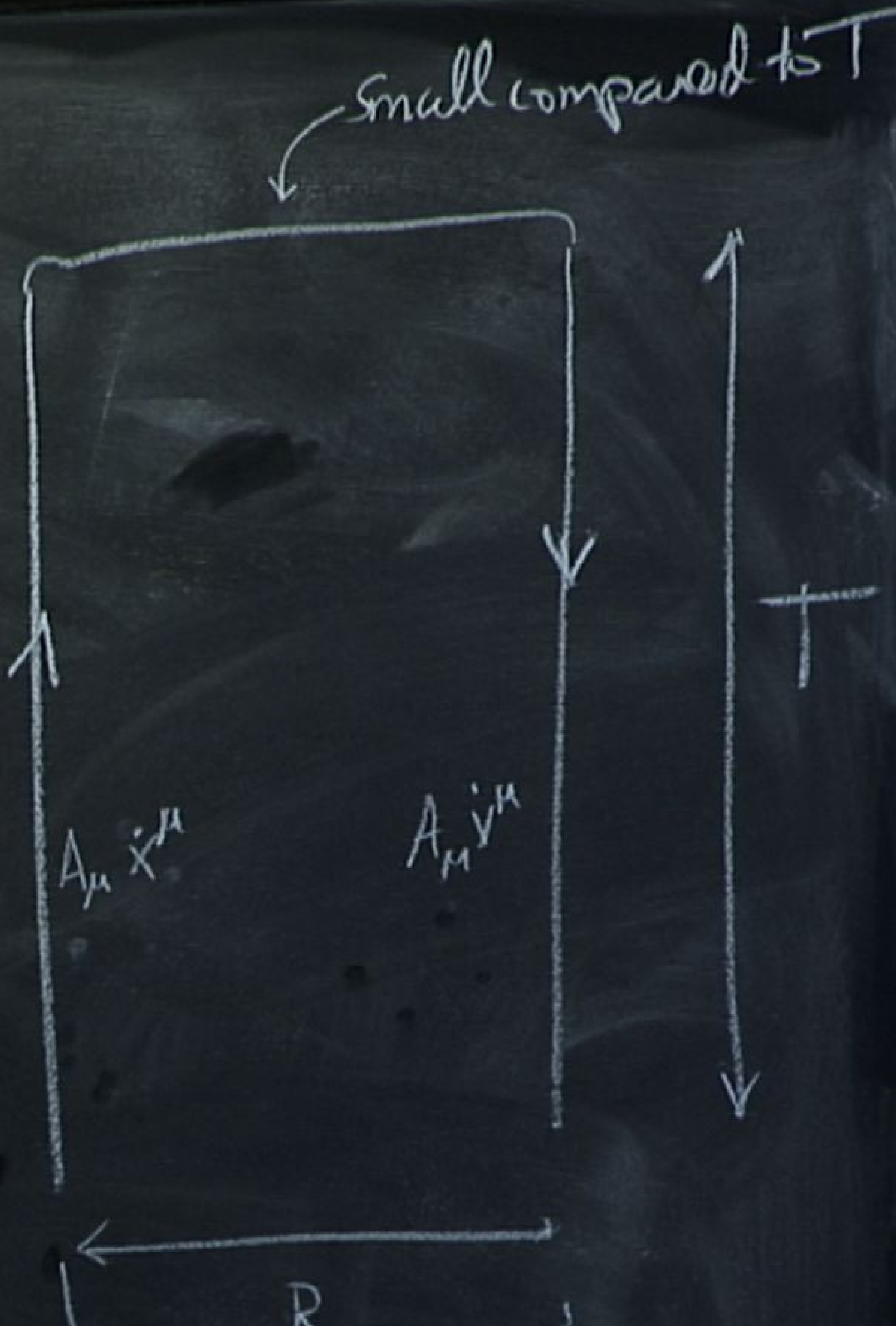
$$X^\mu_L = (t,0,0,0) \qquad X^\mu_R = (-t,0,0,R)$$

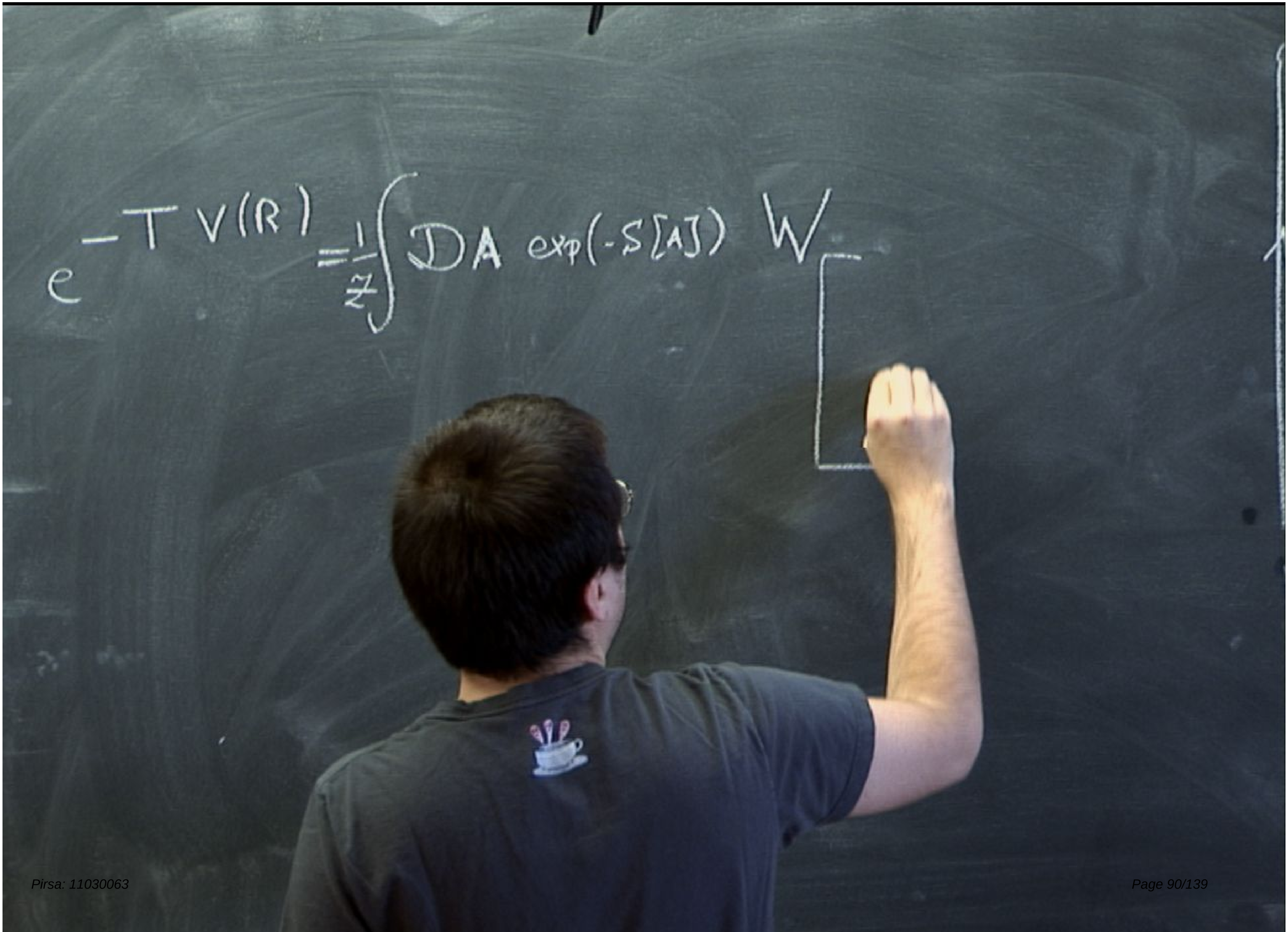
ll two charges

e, $-e$

ated by R)

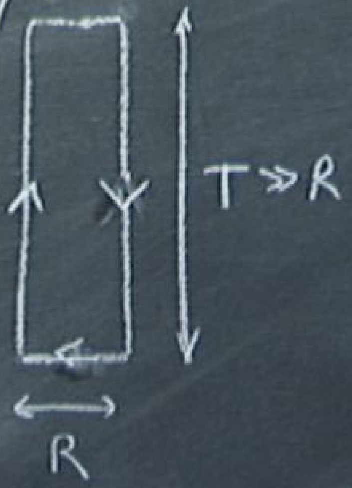
read $V(R)$



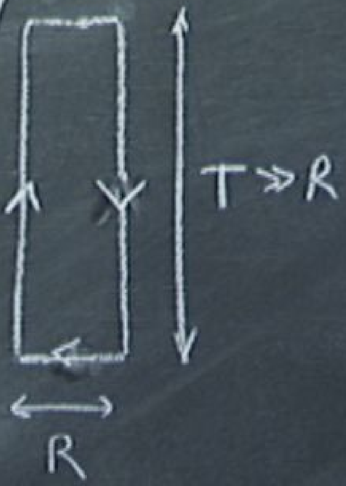


$$e^{-TV(R)} = \frac{1}{Z}\int \mathcal{D}A \, \exp(-S[A]) \, W$$

$$e^{-T\,V(R)} = \frac{1}{Z}\int \mathcal{D}A \; \exp(-S[A]) \;\; W$$

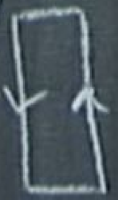


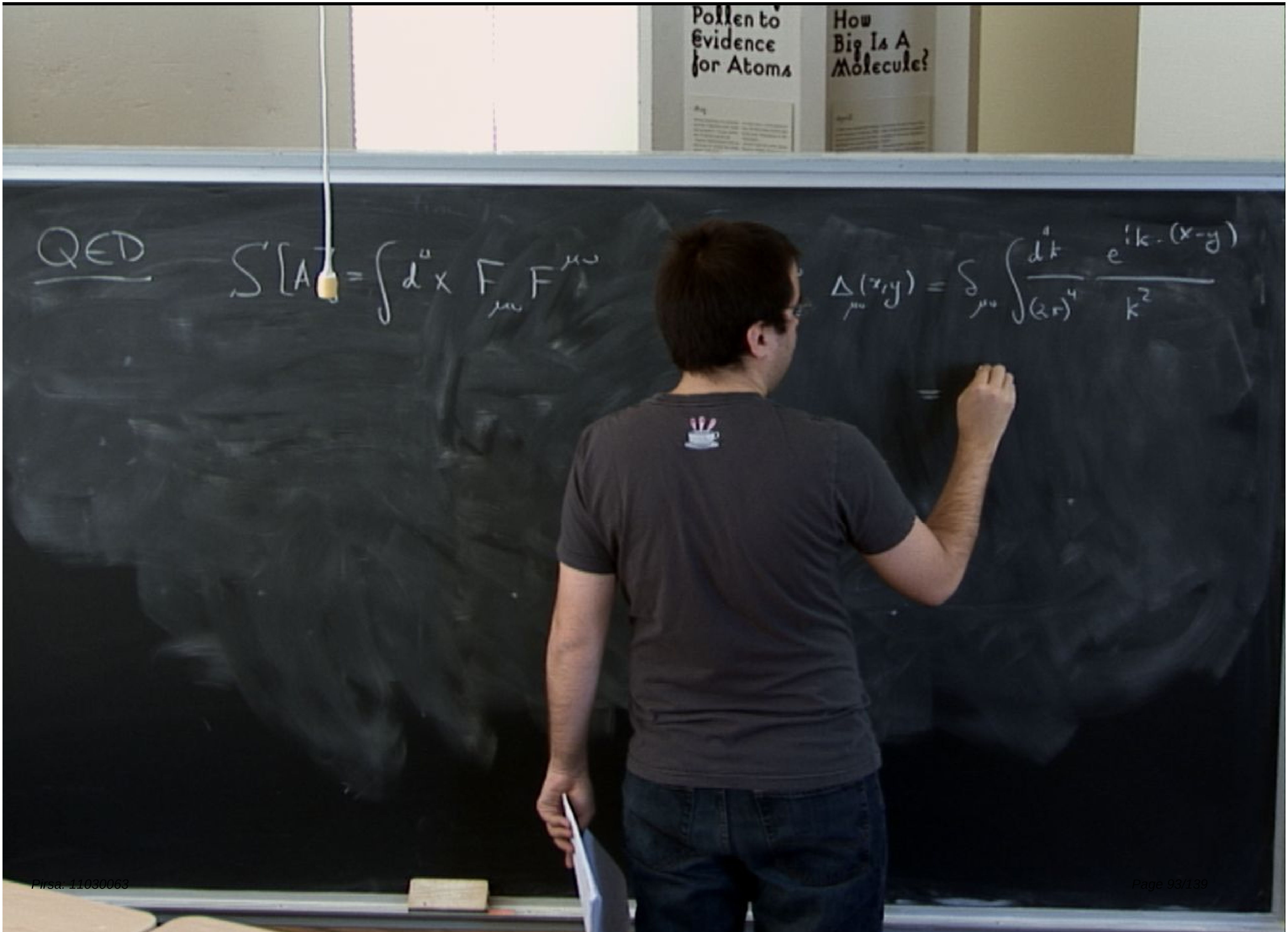
$$e^{-T\,V(R)} = \frac{1}{Z}\int \mathcal{D}A \; \exp(-S[A]) \; W$$



$$\Vert$$

$$\langle W \rangle$$



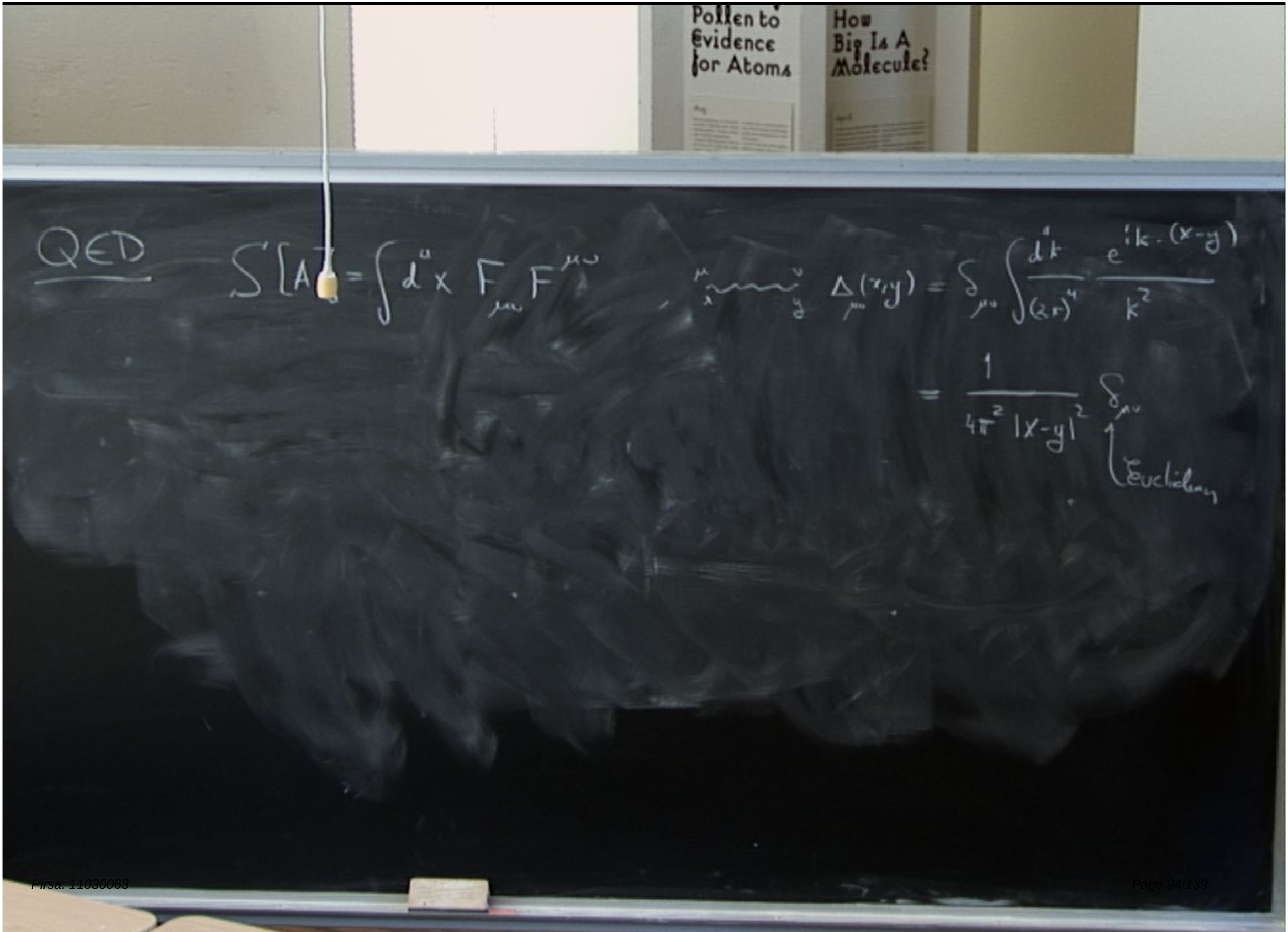


QED

$$S[A] = \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

$$\Delta_{\mu\nu}(x,y) = \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2}$$

$$=$$



QED

$$S[A] = \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

$$\overset{\mu}{\underset{x}{\bullet}}\!\sim\!\sim\!\sim\!\overset{\nu}{\underset{y}{\bullet}} \quad \Delta_{\mu\nu}(x,y) = \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2}$$

$$= \frac{1}{4\pi^2 |x-y|^2} \delta_{\mu\nu}$$

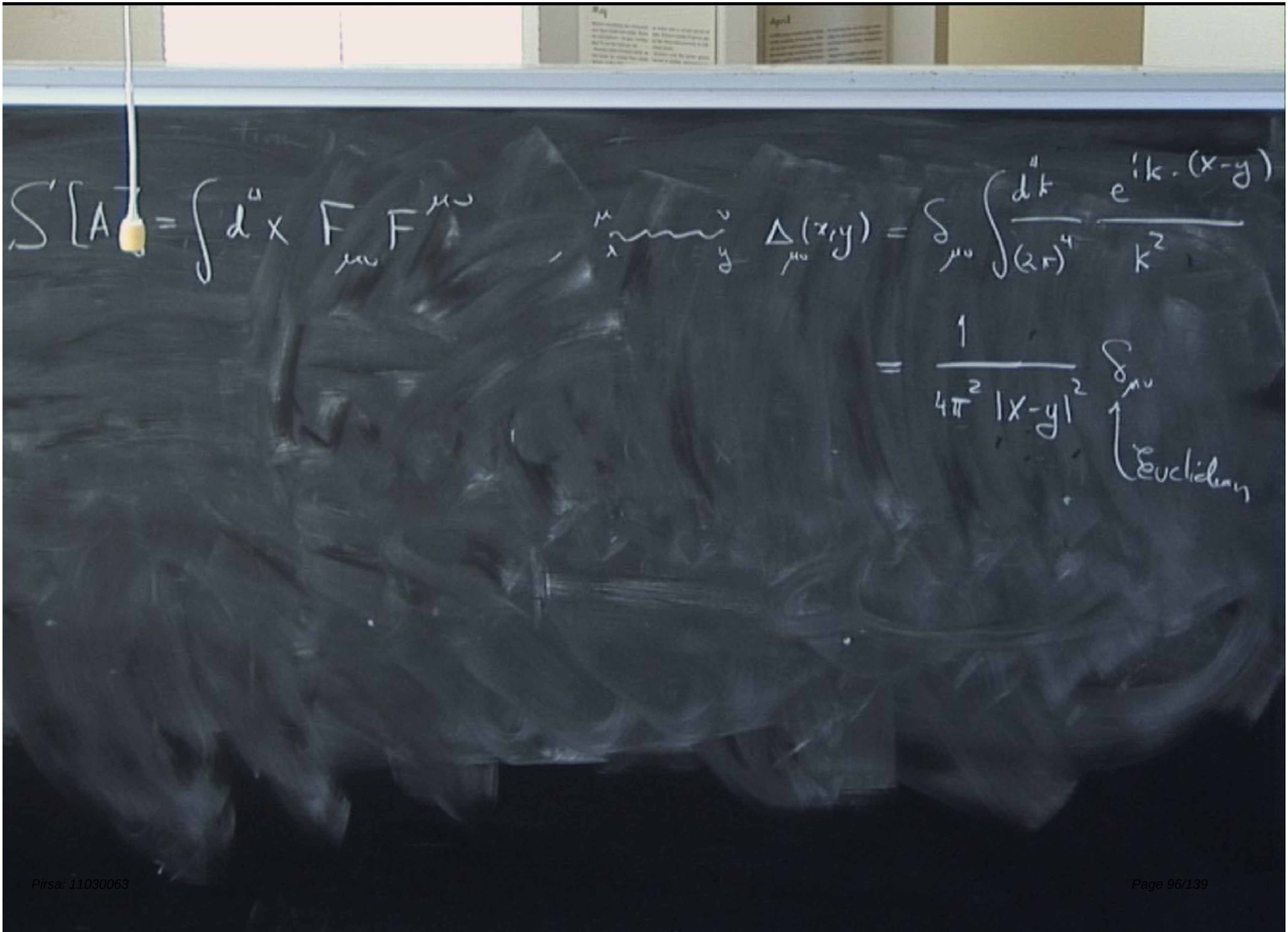
$\uparrow$ Euclidean

$$[A] = \int d^4x\, F_{\mu\nu} F^{\mu\nu}$$

$${}^{\mu}_{x}\sim\!\sim\!\sim\!\sim{}^{\nu}_{y} \quad \Delta_{\mu\nu}(x,y) = \delta_{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik\cdot(x-y)}}{k^2}$$

$$= \frac{1}{4\pi^2\,|x-y|^2}\,\delta_{\mu\nu}$$

$\uparrow$ Euclidean



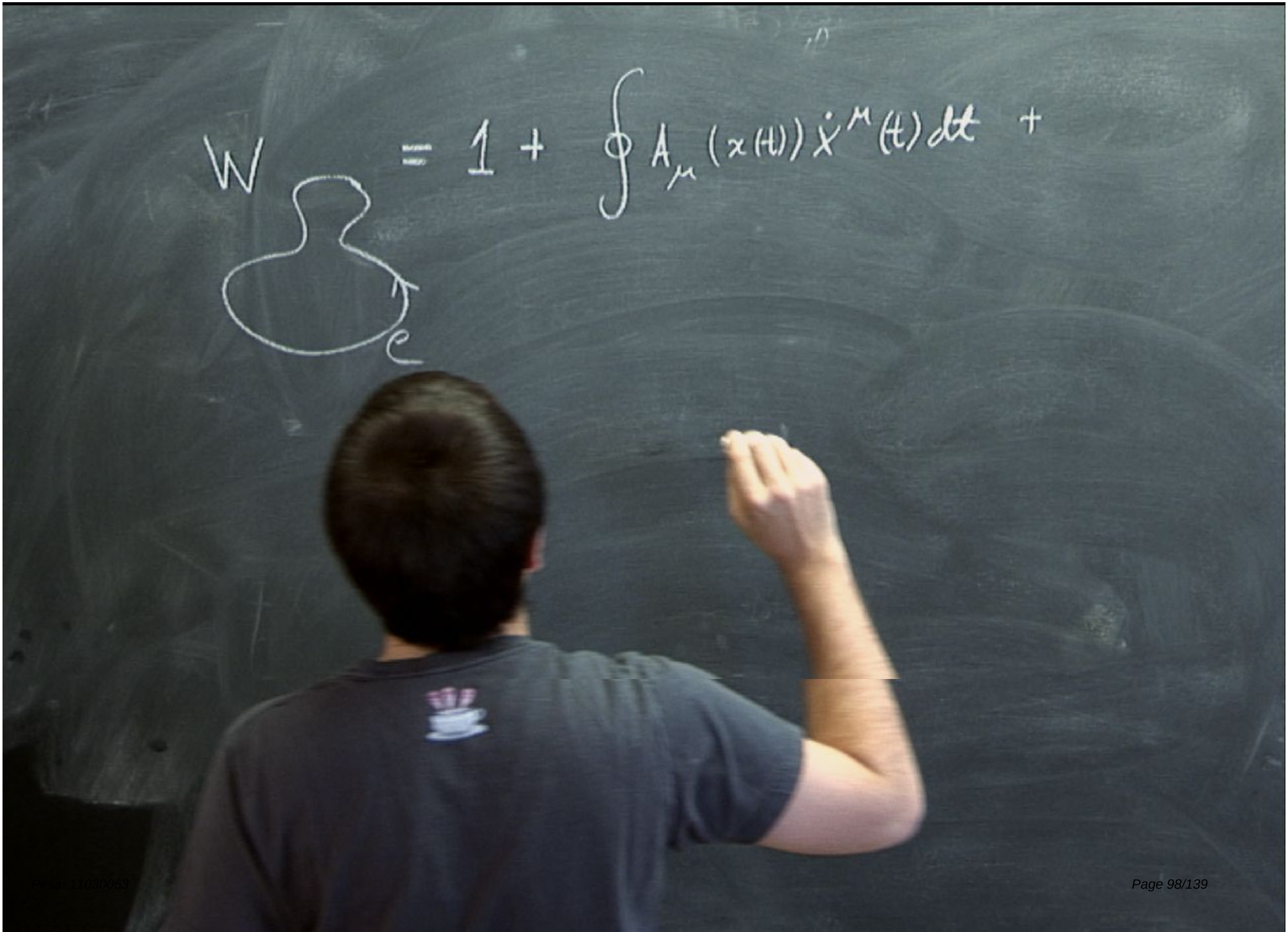
$$S[A] = \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

$$\underset{x}{\overset{\mu}{\bullet}}\sim\!\sim\!\sim\!\sim\underset{y}{\overset{\nu}{\bullet}} \quad \Delta_{\mu\nu}(x,y) = \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2}$$

$$= \frac{1}{4\pi^2 |x-y|^2} \delta_{\mu\nu}$$

↑ Euclidean

W  = 1 +



$$W = 1 + \oint A_{\mu} (x(t)) \dot{x}^{\mu} (t) dt \quad +$$

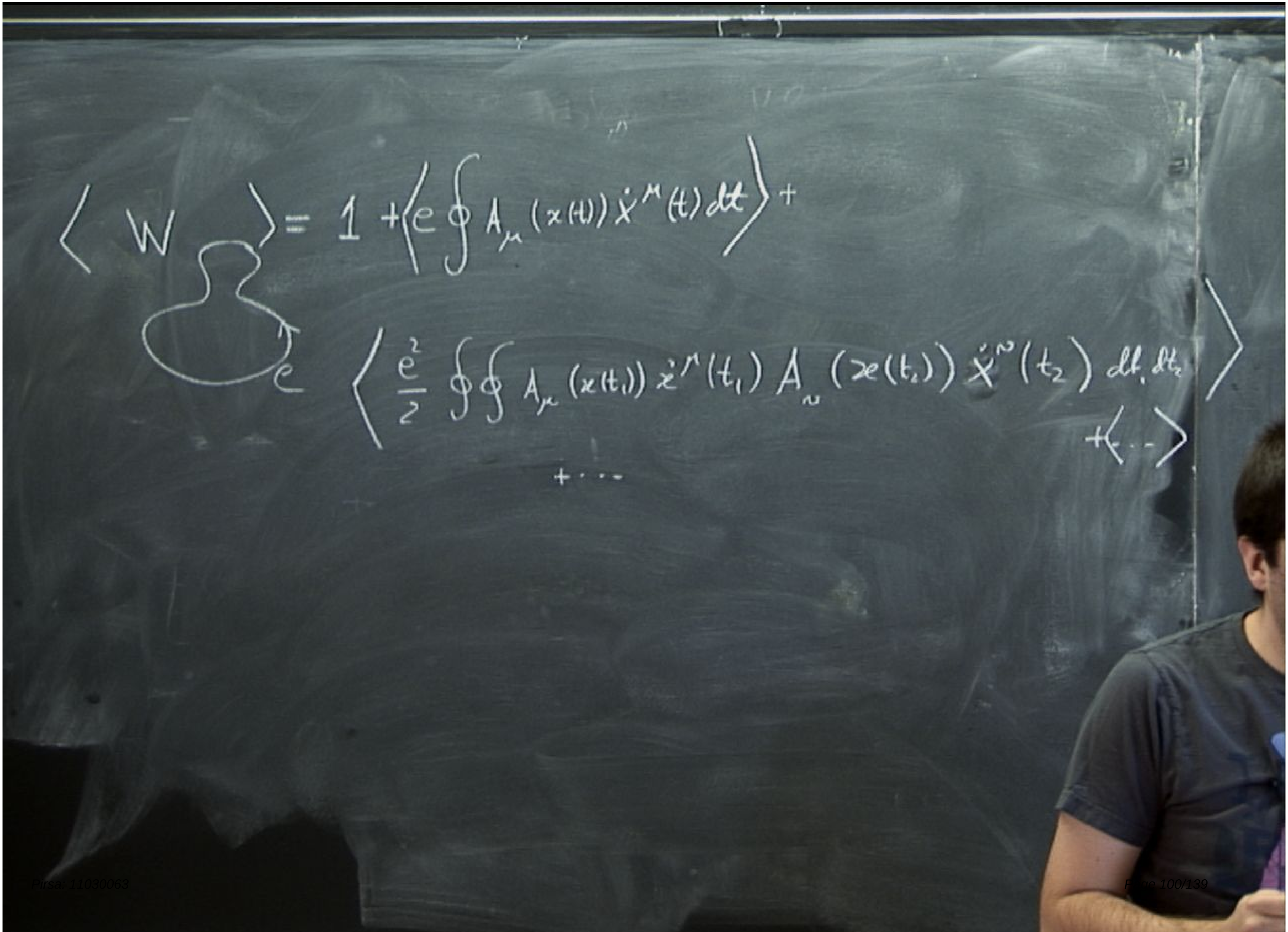
W 

$$= 1 + e \oint A_\mu (x(t)) \dot{x}^\mu (t) dt \ +$$

$$\frac{e^2}{2} \oint \oint A_\mu (x(t_1)) \dot{x}^\mu (t_1) A_\nu (x(t_2)) \dot{x}^\nu (t_2) dt_1 dt_2$$

$$+ \ldots$$

$$+ \ldots$$



$$\langle W_{C} \rangle = 1 + \left\langle e \oint A_\mu (x(t))\, \dot{x}^\mu (t)\, dt \right\rangle +$$

$$\left\langle \frac{e^2}{2} \oint \oint A_\mu (x(t_1))\, \dot{x}^\mu (t_1)\, A_\nu (x(t_2))\, \dot{x}^\nu (t_2)\, dt_1\, dt_2 \right\rangle$$

$$+ \langle \ldots \rangle$$

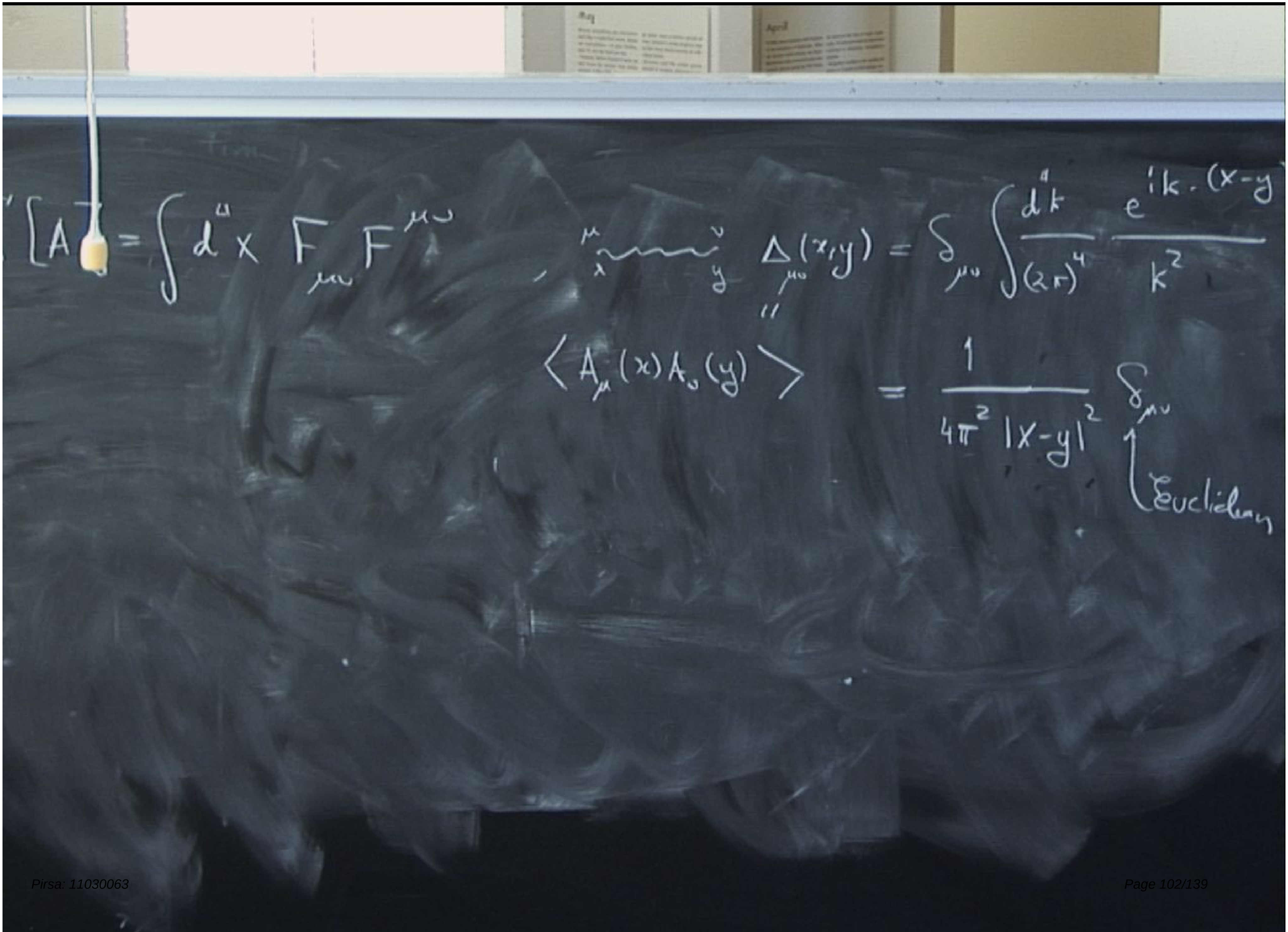
$$+ \cdots$$

$$\langle W \rangle = 1 + \overbrace{\left\langle e \oint A_\mu (x(t))\, \dot{x}^\mu (t)\, dt \right\rangle}^{=0} +$$

$$\left\langle \frac{e^2}{2} \oint \oint A_\mu (x(t_1))\, \dot{x}^\mu (t_1)\, A_\nu (x(t_2))\, \dot{x}^\nu (t_2)\, dt_1\, dt_2 \right\rangle$$

$$+ \langle \dots \rangle$$

$$+ \dots$$

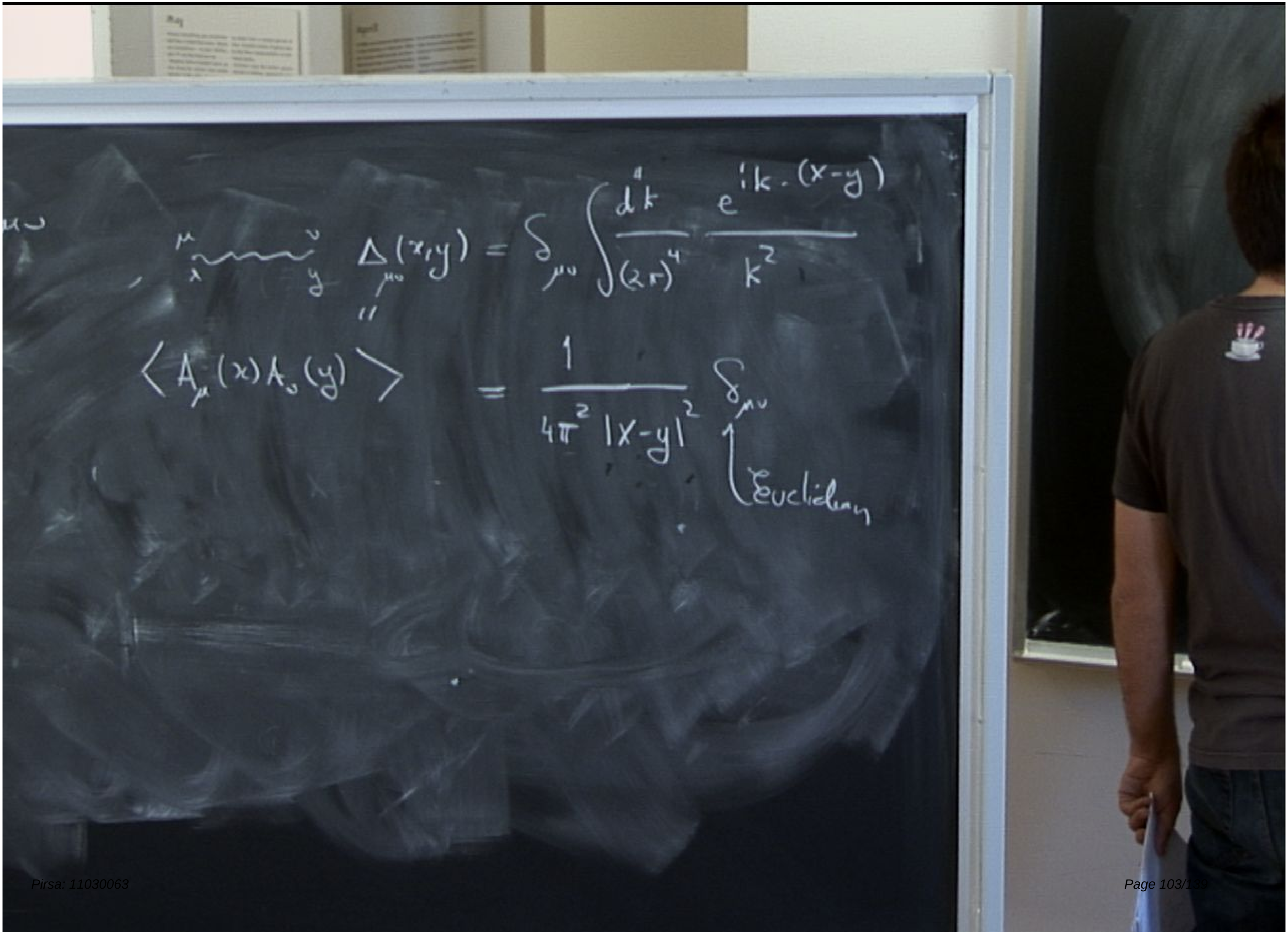


$$'[A] = \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

$$\overset{\mu}{\underset{x}{}} \sim\!\sim\!\sim\!\sim \overset{\nu}{\underset{y}{}} \quad \Delta_{\mu\nu}(x,y) = \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y}}{k^2}$$

$$\| \qquad \langle A_\mu(x) A_\nu(y) \rangle = \frac{1}{4\pi^2 |x-y|^2} \delta_{\mu\nu}$$

$\uparrow$ Euclidean



$$\Delta_{\mu\nu}(x,y) = \delta_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{ik\cdot(x-y)}}{k^2}$$

$$\langle A_\mu(x)A_\nu(y)\rangle = \frac{1}{4\pi^2|x-y|^2}\,\delta_{\mu\nu}$$

Euclidean

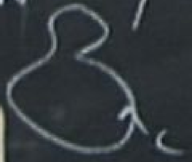
$$\langle W \rangle = 1 + \langle \overbrace{e \oint A_\mu(x(t))\,\dot{x}^\mu(t)\,dt}^{=0} \rangle +$$

$$\langle \frac{e^2}{2} \oint\oint A_\mu(x(t_1))\,\dot{x}^\mu(t_1)\,A_\nu(x(t_2))\,\dot{x}^\nu(t_2)\,dt_1\,dt_2 \rangle$$

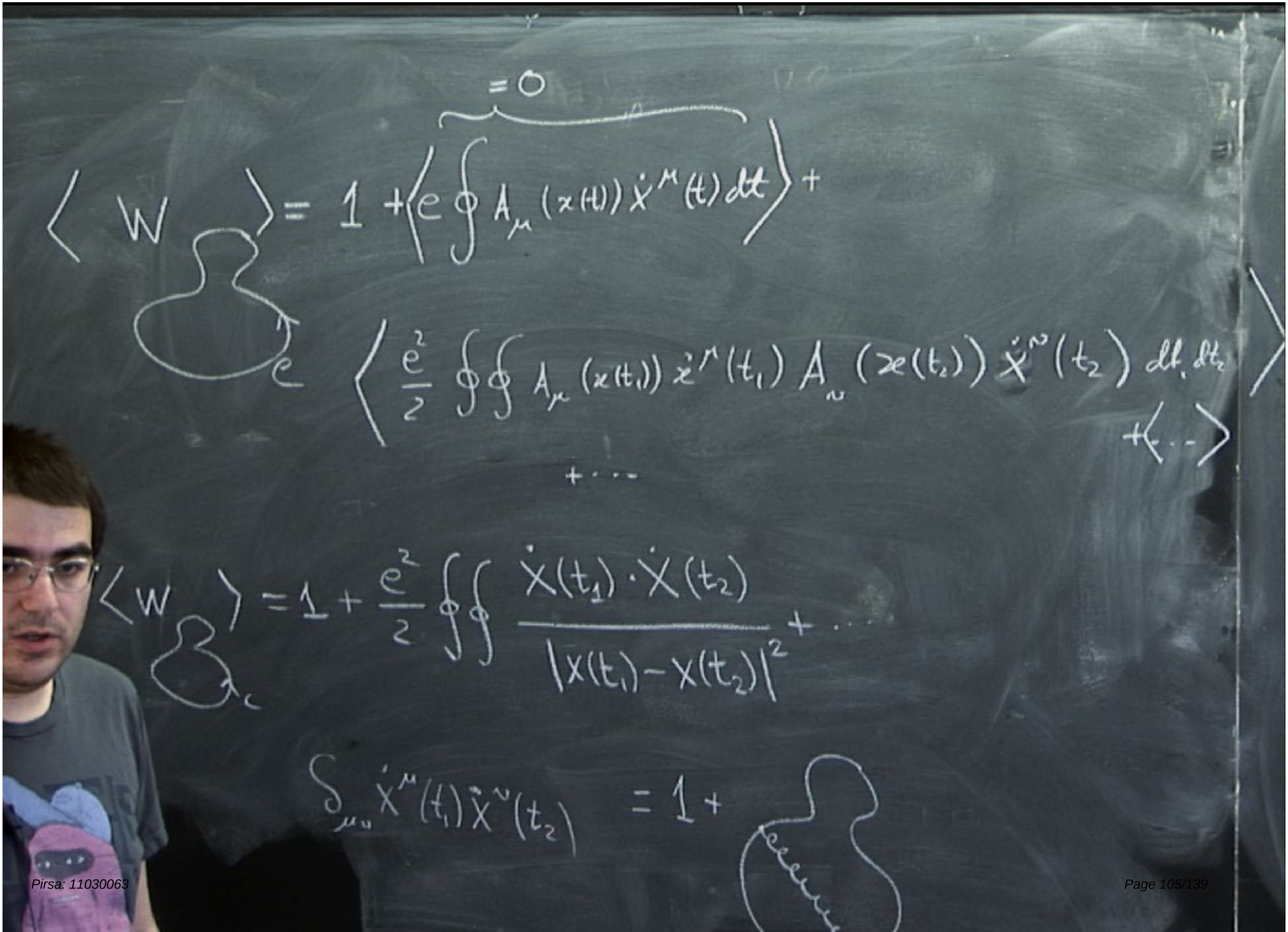
$$+\langle \dots \rangle$$

$$+ \dots$$

$$\langle W \rangle = 1 + \frac{e^2}{2} \oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2} + \dots$$



$$\delta_{\mu\nu}\,\dot{X}^\mu(t_1)\,\dot{X}^\nu(t_2)$$



$$\langle W_C \rangle = 1 + \overbrace{\left\langle e \oint A_\mu(x(t))\,\dot{x}^\mu(t)\,dt \right\rangle}^{=0} +$$

$$\left\langle \frac{e^2}{2} \oint\oint A_\mu(x(t_1))\,\dot{x}^\mu(t_1)\,A_\nu(x(t_2))\,\dot{x}^\nu(t_2)\,dt_1\,dt_2 \right\rangle + \langle \ldots \rangle$$

$$+ \cdots$$

$$\langle W_C \rangle = 1 + \frac{e^2}{2} \oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2} + \cdots$$

$$\delta_{\mu\nu}\,\dot{x}^\mu(t_1)\,\dot{x}^\nu(t_2) \quad = 1 +$$

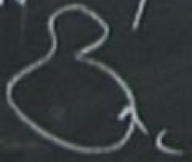
$$\left\langle W \right\rangle = 1 + \left\langle \overbrace{e \oint A_\mu(x(t))\, \dot{x}^\mu(t)\, dt}^{=0} \right\rangle +$$

$$\left\langle \frac{e^2}{2} \oint\oint A_\mu(x(t_1))\, \dot{x}^\mu(t_1)\, A_\nu(x(t_2))\, \dot{x}^\nu(t_2)\, dt_1\, dt_2 \right\rangle$$

$$+\langle \dots \rangle$$

$$+ \cdots$$

$$\langle W \rangle = 1 + \frac{e^2}{2} \oint\oint \frac{\dot{X}(t_1) \cdot \dot{X}(t_2)}{|X(t_1) - X(t_2)|^2} + \cdots$$



$$\delta_{\mu\nu}\, \dot{X}^\mu(t_1)\, \dot{X}^\nu(t_2) \qquad = 1 +$$

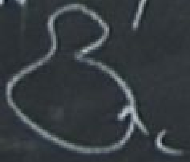
$$\left\langle W \right\rangle = 1 + \overbrace{\left\langle e \oint A_\mu(x(t))\,\dot{x}^\mu(t)\,dt \right\rangle}^{=0} +$$

$$\left\langle \frac{e^2}{2} \oint\oint A_\mu(x(t_1))\,\dot{x}^\mu(t_1)\,A_\nu(x(t_2))\,\dot{x}^\nu(t_2)\,dt_1\,dt_2 \right\rangle$$

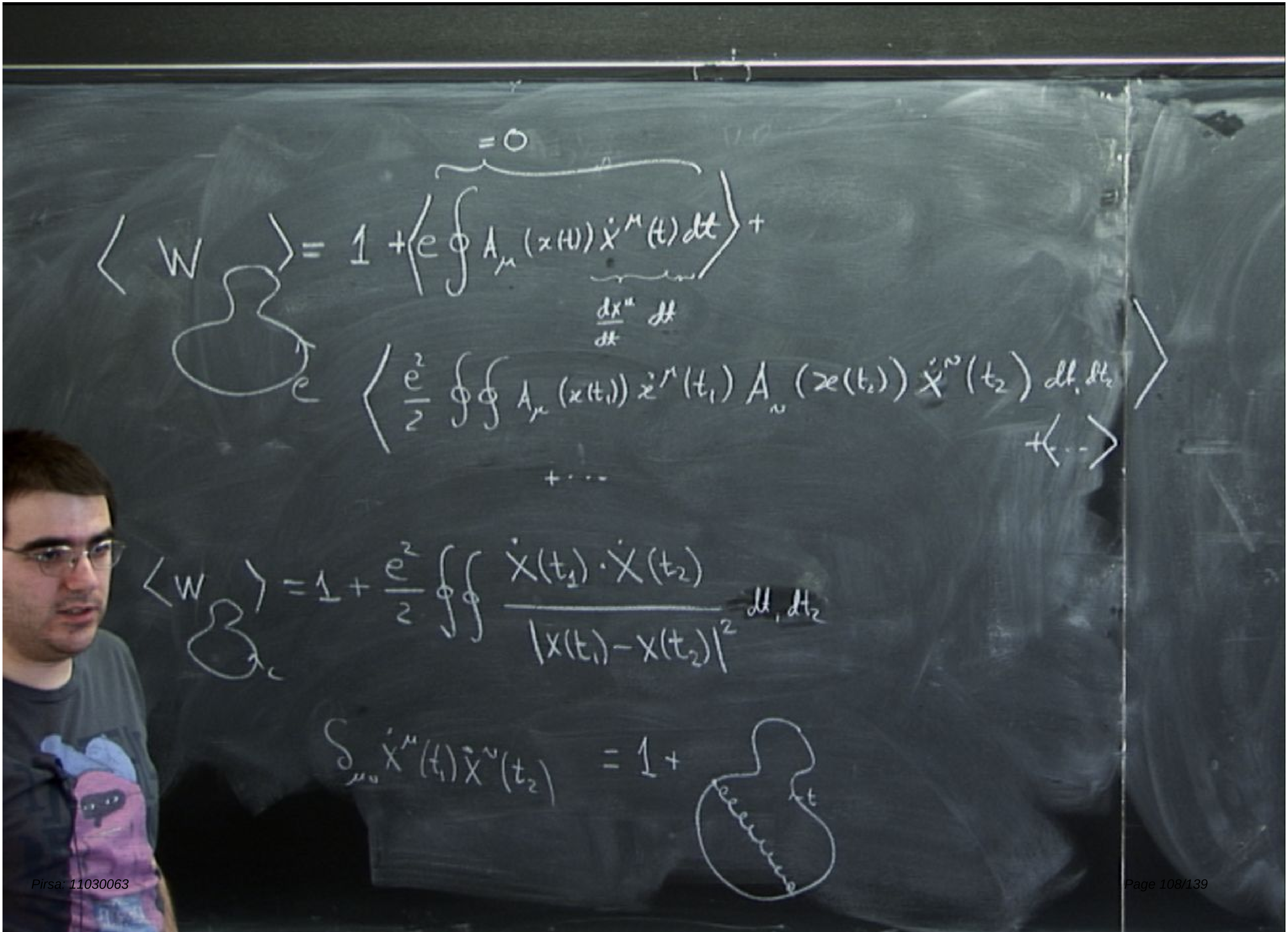
$$+\left\langle \dots \right\rangle$$

$$+ \cdots$$

$$\left\langle W \right\rangle = 1 + \frac{e^2}{2}\oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2} + \cdots$$



$$\delta_{\mu\nu}\,\dot{X}^\mu(t_1)\,\dot{X}^\nu(t_2) \qquad = 1 +$$



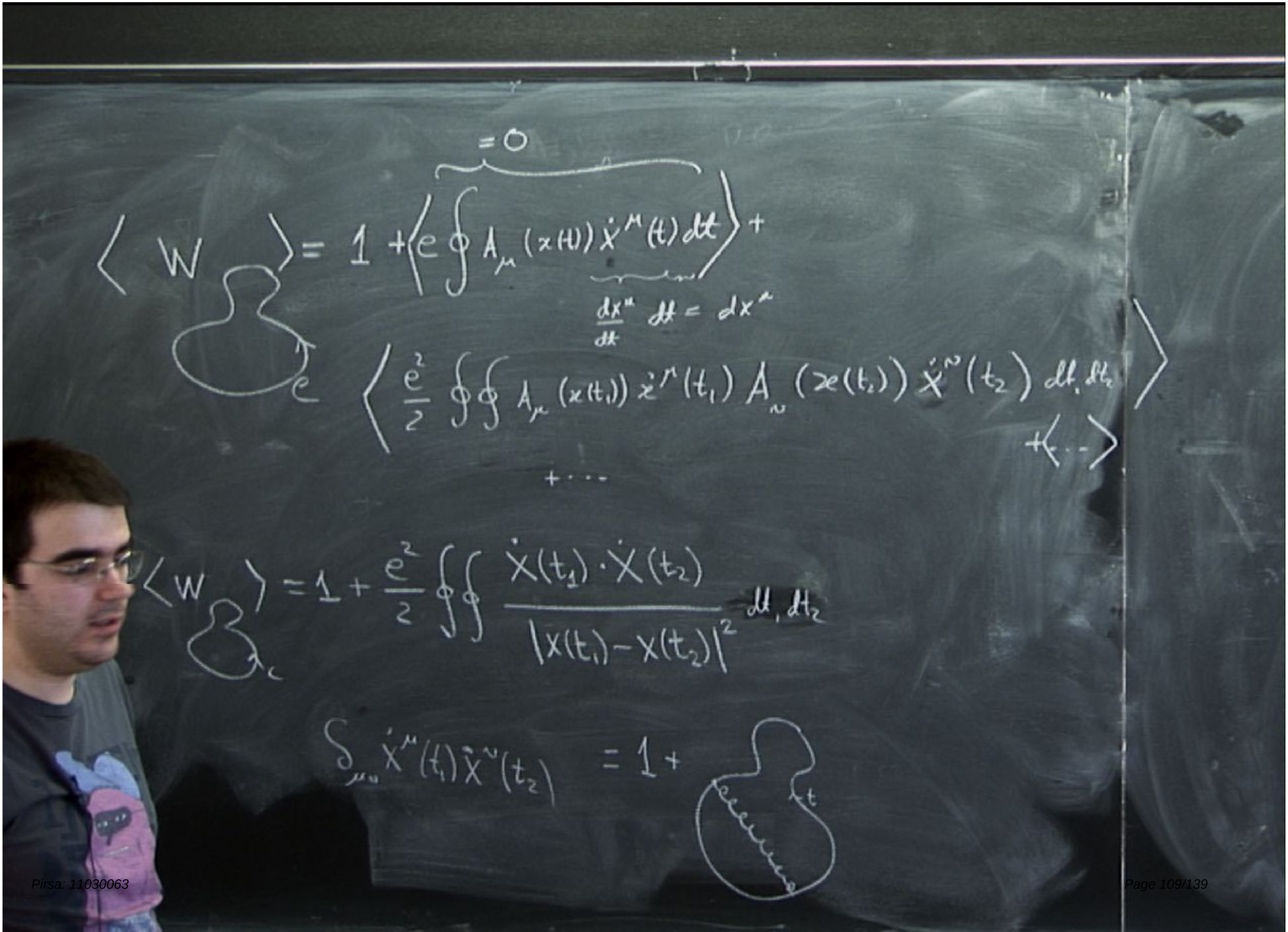
$$\langle W_C \rangle = 1 + \overbrace{\left\langle e \oint A_\mu(x(t))\, \underbrace{\dot{x}^\mu(t)\, dt}_{\frac{dx^\mu}{dt}\, dt} \right\rangle}^{=0} +$$

$$\left\langle \frac{e^2}{2} \oint\oint A_\mu(x(t_1))\, \dot{x}^\mu(t_1)\, A_\nu(x(t_2))\, \dot{x}^\nu(t_2)\, dt_1\, dt_2 \right\rangle + \langle \dots \rangle$$

$$+ \cdots$$

$$\langle W_C \rangle = 1 + \frac{e^2}{2} \oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2}\, dt_1\, dt_2$$

$$\delta_{\mu\nu}\, \dot{X}^\mu(t_1)\, \dot{X}^\nu(t_2) \qquad = 1 +$$



$$\langle W_{C} \rangle = 1 + \Big\langle e \oint \overbrace{A_\mu(x(t))\,\underbrace{\dot{x}^\mu(t)\,dt}_{\frac{dx^\mu}{dt}dt = dx^\mu}}^{=0} \Big\rangle +$$

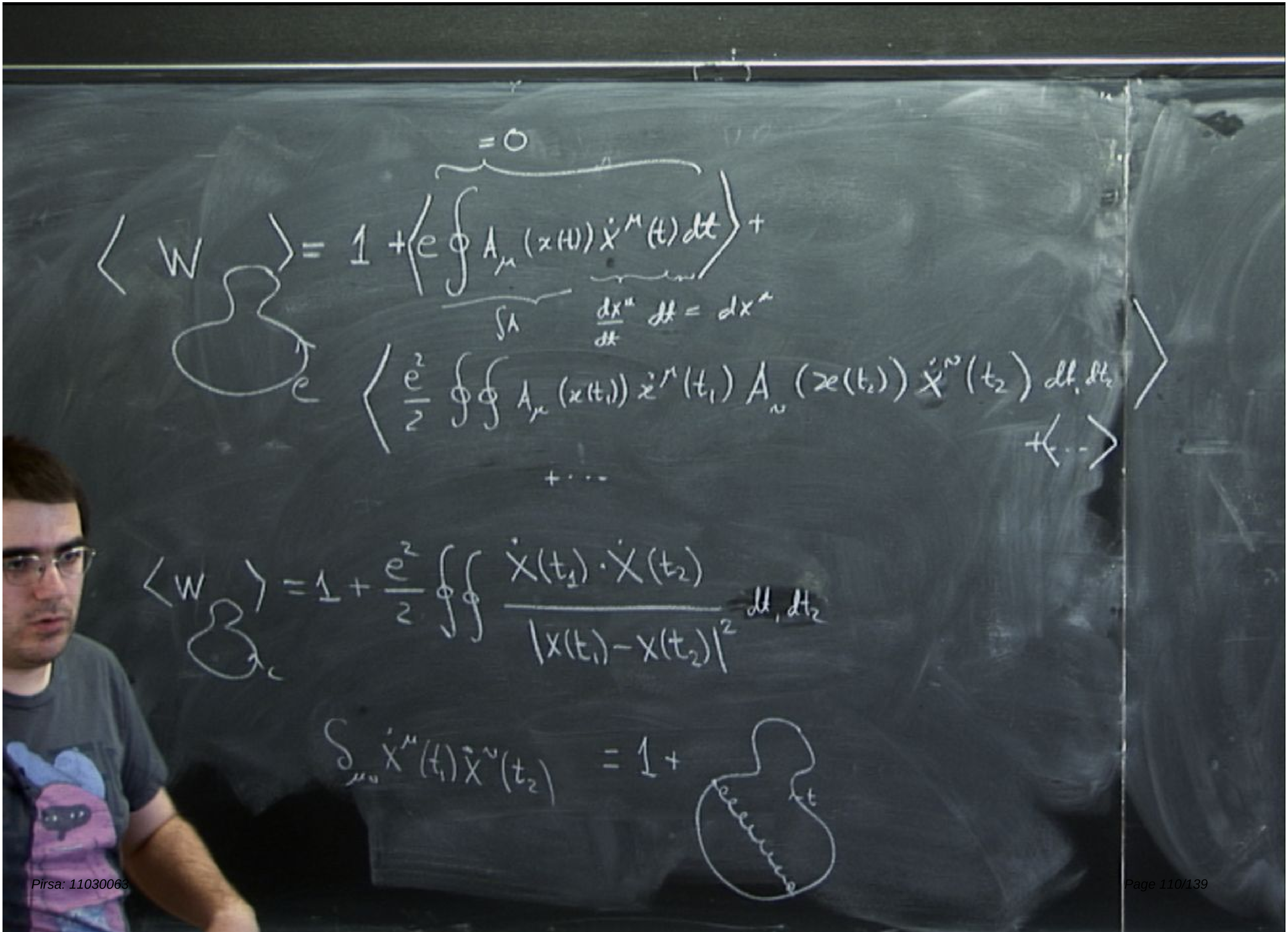
$$\Big\langle \frac{e^2}{2} \oint\oint A_\mu(x(t_1))\,\dot{x}^\mu(t_1)\,A_\nu(x(t_2))\,\dot{x}^\nu(t_2)\,dt_1\,dt_2 \Big\rangle$$

$$+\langle \ldots \rangle$$

$$+ \cdots$$

$$\langle W_{C} \rangle = 1 + \frac{e^2}{2}\oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2}\,dt_1\,dt_2$$

$$\delta_{\mu\nu}\dot{X}^\mu(t_1)\dot{X}^\nu(t_2) \qquad = 1 +$$



$$\langle W \rangle = 1 + \langle \underbrace{e \oint}_{\int A} \overbrace{A_\mu (x(t))\, \underbrace{\dot{x}^\mu(t)\, dt}_{\frac{dx^\mu}{dt} dt = dx^\mu}}^{=0} \rangle +$$

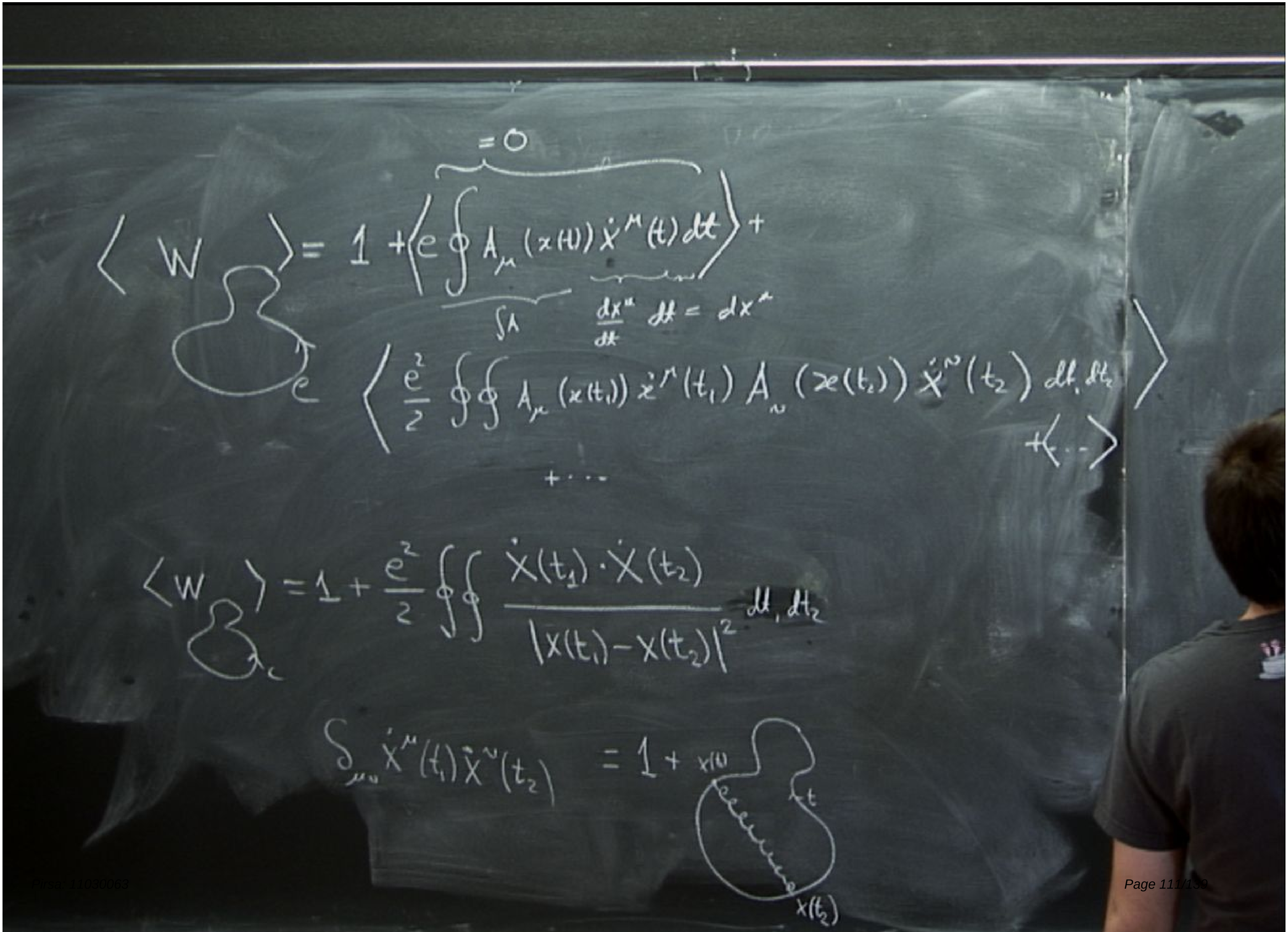
$$\langle \frac{e^2}{2} \oint\oint A_\mu (x(t_1))\, \dot{x}^\mu(t_1)\, A_\nu (x(t_2))\, \dot{x}^\nu(t_2)\, dt_1 dt_2 \rangle$$

$$+ \langle \ldots \rangle$$

$$+ \cdots$$

$$\langle W \rangle = 1 + \frac{e^2}{2} \oint\oint \frac{\dot{X}(t_1) \cdot \dot{X}(t_2)}{|X(t_1) - X(t_2)|^2}\, dt_1\, dt_2$$

$$\delta_{\mu\nu}\, \dot{X}^\mu(t_1)\, \dot{X}^\nu(t_2) \qquad = 1 +$$



$$\langle W_C \rangle = 1 + \langle \overbrace{\underbrace{e \oint A_\mu}_{\int A} (x(t)) \underbrace{\dot{x}^\mu(t)\, dt}_{\frac{dx^\mu}{dt} dt = dx^\mu}}^{=0} \rangle +$$

$$\langle \frac{e^2}{2} \oint\oint A_\mu(x(t_1))\, \dot{x}^\mu(t_1)\, A_\nu(x(t_2))\, \dot{x}^\nu(t_2)\, dt_1 dt_2 \rangle + \langle \ldots \rangle$$

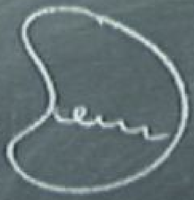
$$+ \cdots$$

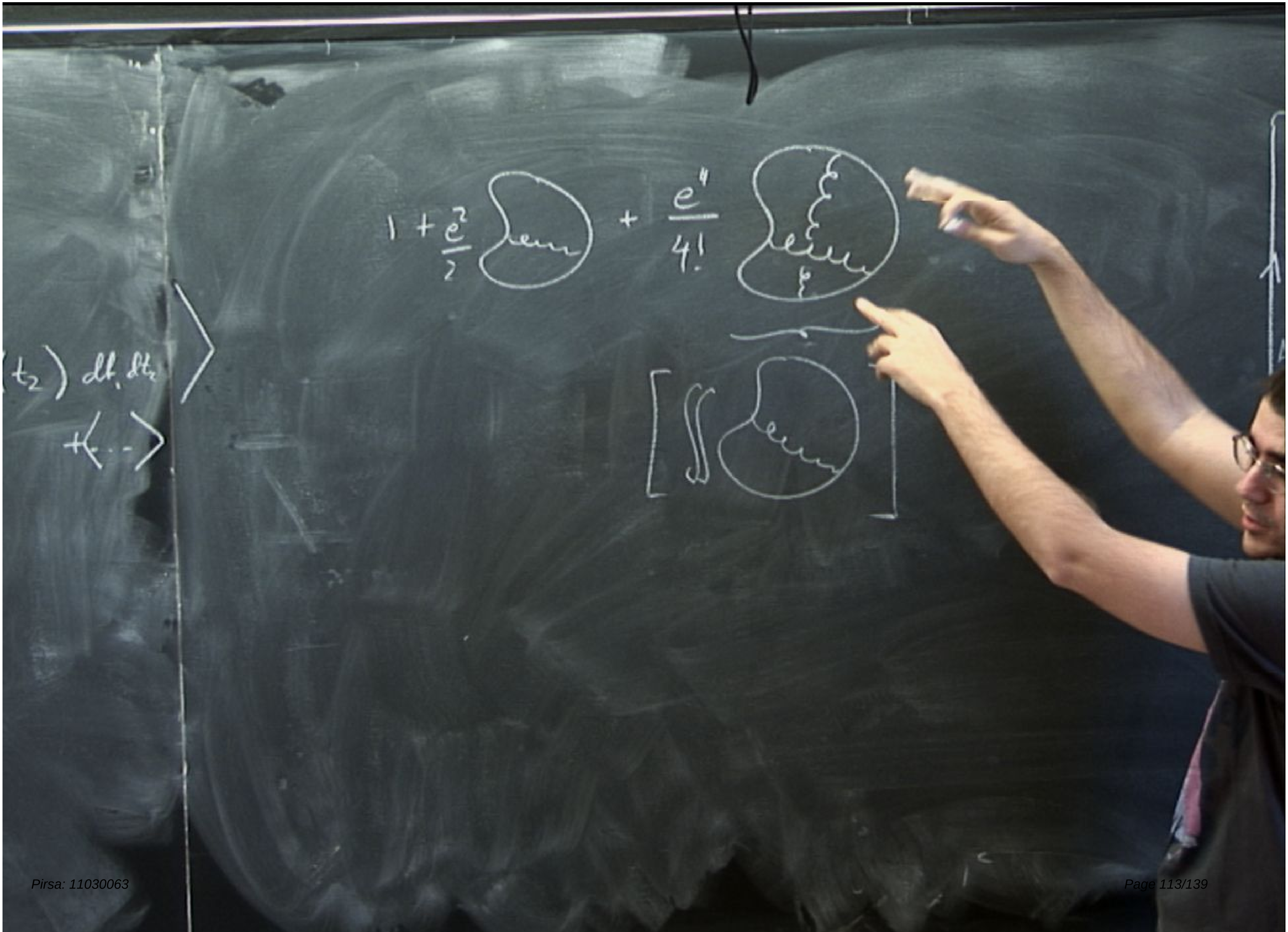
$$\langle W_C \rangle = 1 + \frac{e^2}{2} \oint\oint \frac{\dot{X}(t_1) \cdot \dot{X}(t_2)}{|X(t_1) - X(t_2)|^2}\, dt_1 dt_2$$

$$\delta_{\mu\nu} \dot{X}^\mu(t_1) \dot{X}^\nu(t_2) \qquad = 1 +$$

$(t_2)\, dt_1\, dt_2 \Big\rangle$

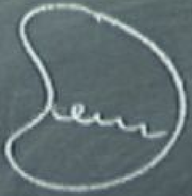
$+\langle \ldots \rangle$

$1 + \frac{e^2}{2}$  $+ \frac{e^4}{4!}$ 

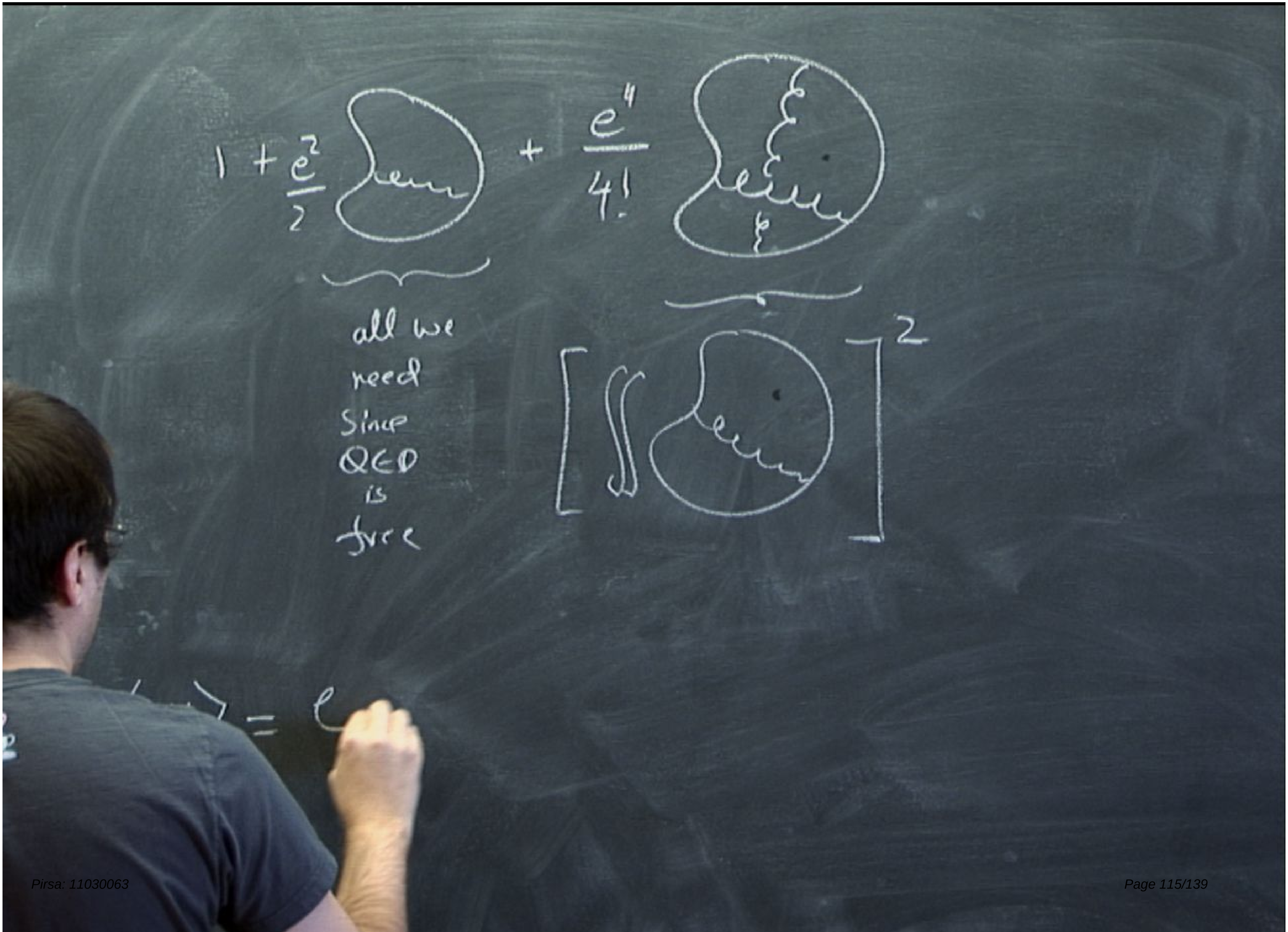

$(t_2)\, dt_1\, dt_2 \rangle$
$+\langle \ldots \rangle$
$1 + \frac{e^2}{2}$
$+ \frac{e^4}{4!}$
$\Big[\iint$
$\Big]$

$(t_2)\, dt_1\, dt_2 \rangle$

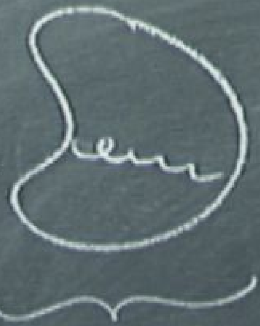
$+\langle \ldots \rangle$

$1 + \frac{e^2}{2}$  $+ \frac{e^4}{4!}$ 

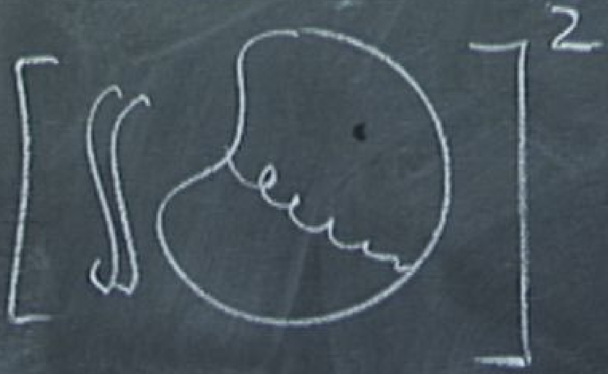
$\Big[\iint$  $\Big]^2$



all we need since QED is free

$1 + \frac{e^2}{2}$  $+ \frac{e^4}{4!}$ 

all we need since QED is free



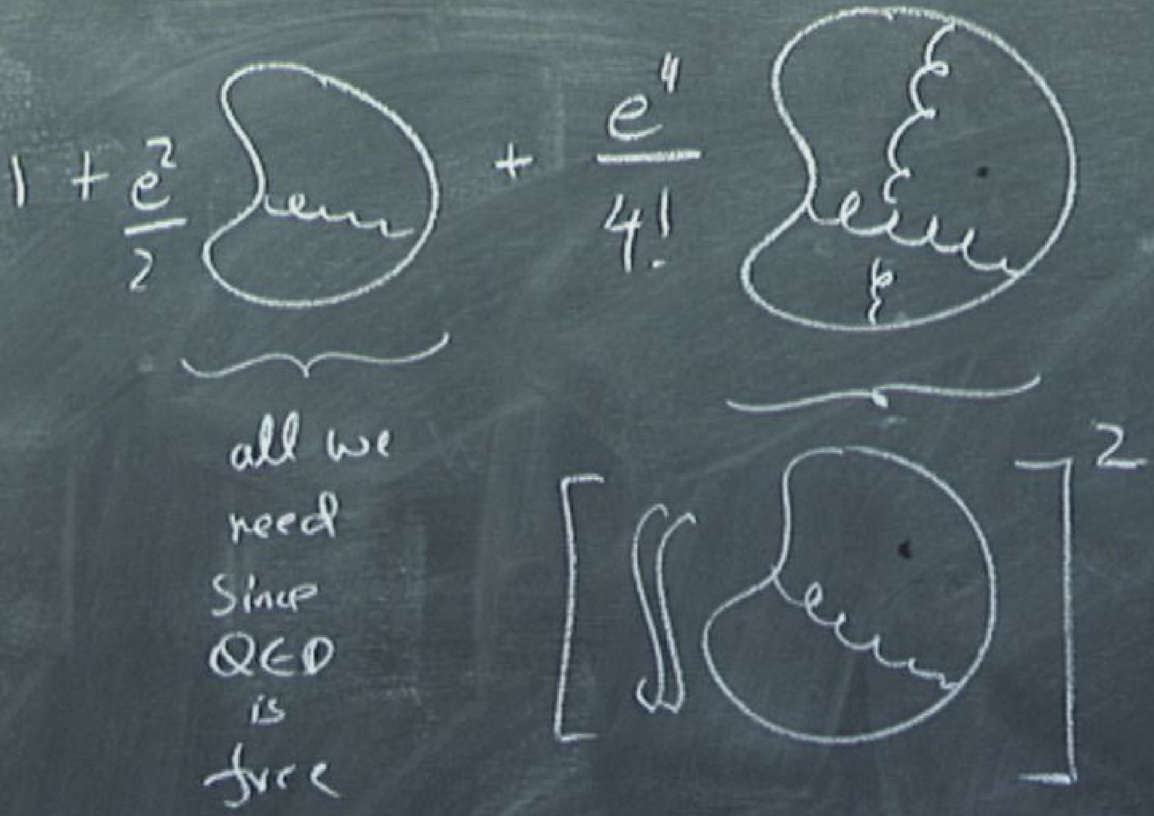
$\langle W \rangle = \exp$

$1 + \frac{e^2}{2}$  $+ \frac{e^4}{4!}$ 

all we need since QED is free

$\Big[\iint$  $\Big]^2$

$$\langle W \rangle_{QED} = \exp\left(e^2 \oint\oint dt_1 dt_2 \frac{\dot{x}(t_1)\cdot\dot{x}(t_2)}{|x(t_1)-x(t_2)|^2} \right)$$



$1 + \frac{e^2}{2}$ … $+ \frac{e^4}{4!}$ …

all we need since QED is free

$\left[\iint \; \cdot \;\right]^2$

$$\langle W \rangle_{\substack{QED\\ e}} = \exp\left(e^2 \oint_e \oint_e dt_1 dt_2 \; \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|X(t_1)-X(t_2)|^2} \right)$$

$1 + \frac{e^2}{2}$

$+ \frac{e^4}{4!}$

all we need since QED is free

$\Big[\iint$

$\Big]^2$

$$\boxed{\underset{e}{\langle W \rangle_{QED}} = \exp\left(e^2 \oint_e \oint_e dt_1\, dt_2 \, \frac{\dot{X}(t_1) \cdot \dot{X}(t_2)}{|X(t_1) - X(t_2)|^2} \right)}$$

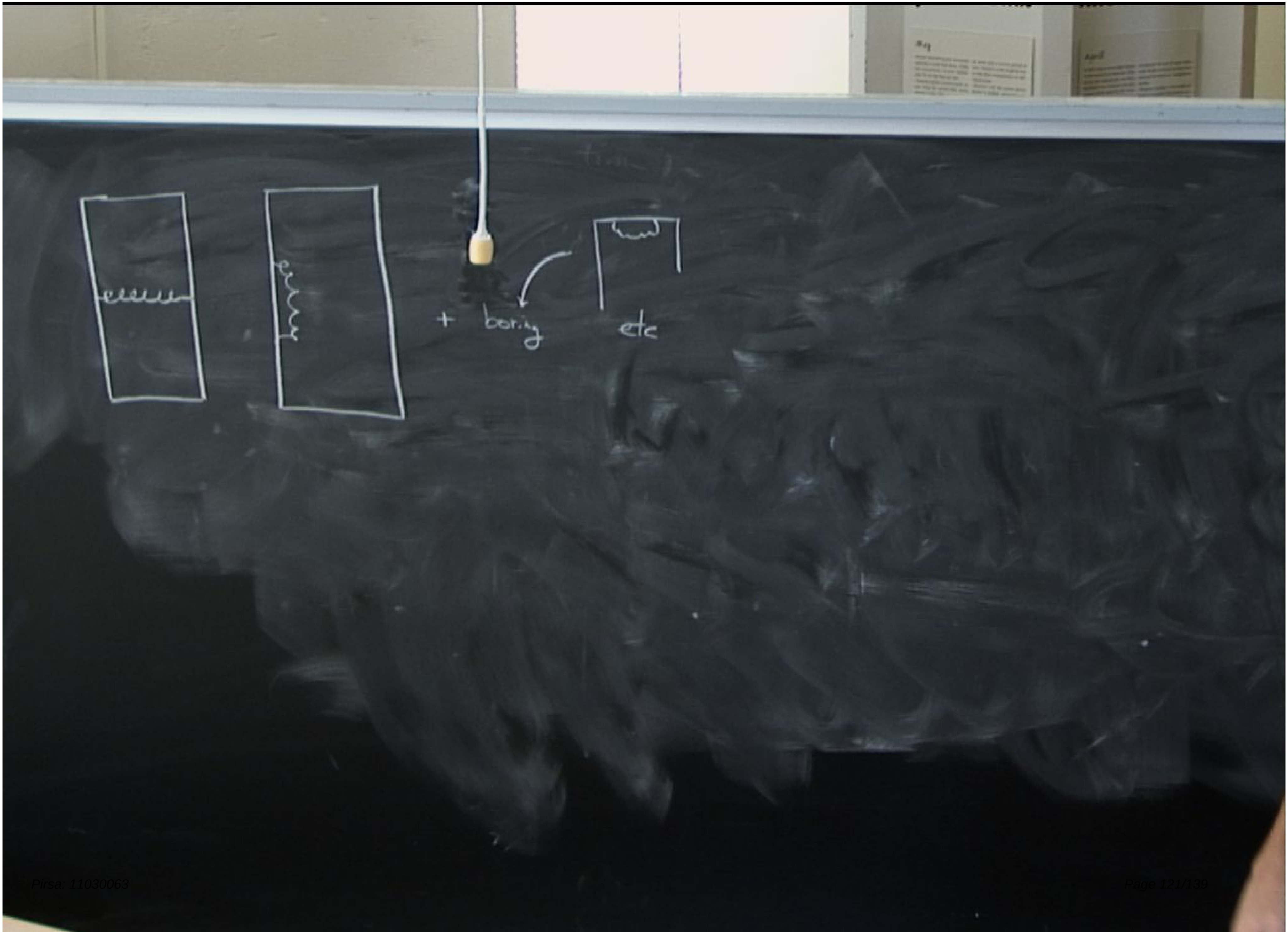
$1 + \frac{e^2}{2}$  $+ \frac{e^4}{4!}$ 

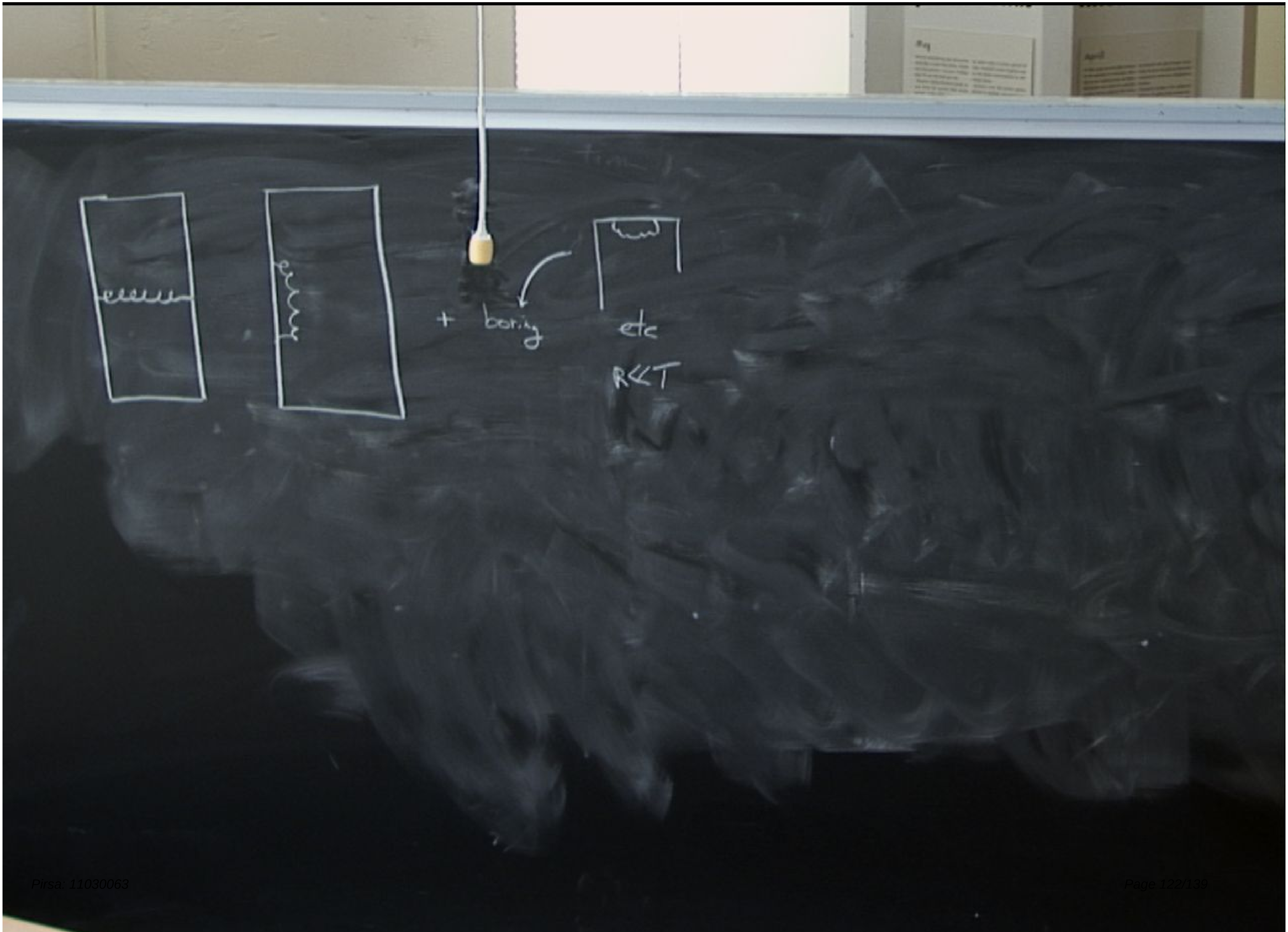
all we need since QED is free

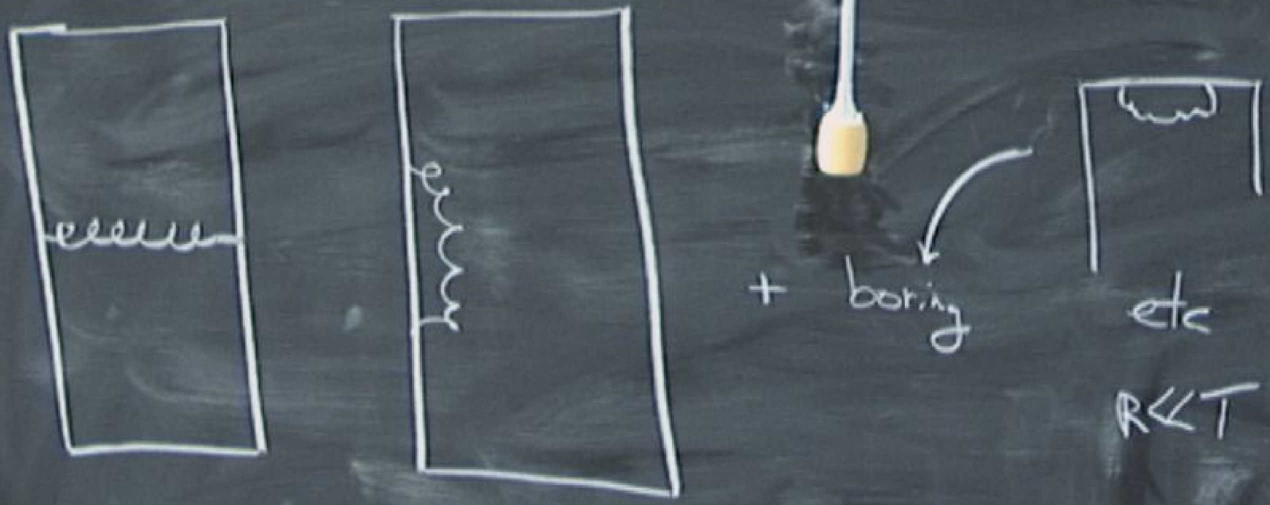
$\Big[\iint$  $\Big]^2$

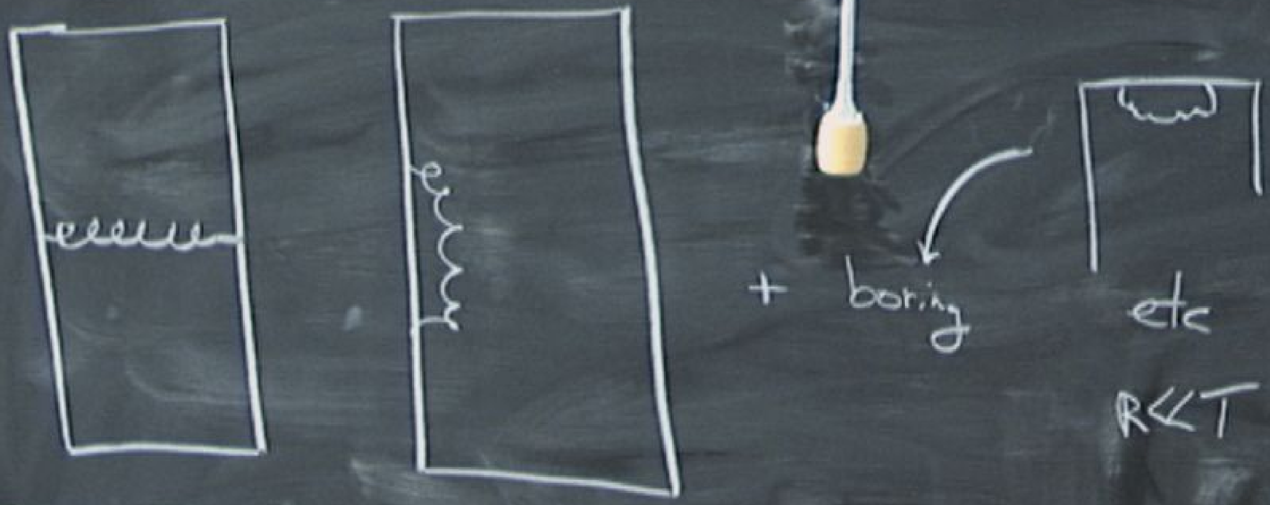
$$\boxed{\langle W\rangle_{\substack{QED\\ e}} = \exp\left(e^2 \oint_e \oint_e dt_1\, dt_2 \, \frac{\dot{x}(t_1)\cdot\dot{x}(t_2)}{|x(t_1)-x(t_2)|^2}\right)}$$

$\langle W\rangle_{U} = e^{-V(R)T}$


+ boring
etc

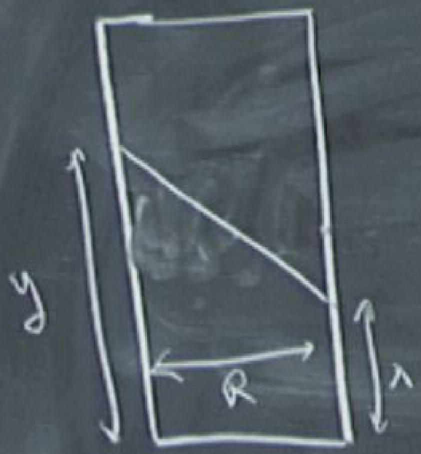

+ boring
etc
R<<T

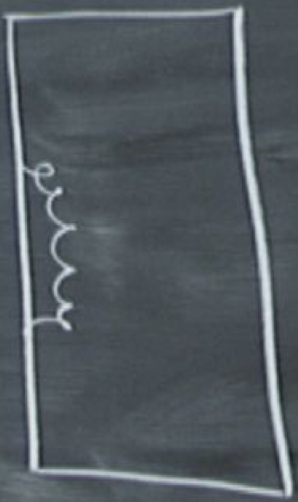

+
boring
etc
R<<T



$R \ll T$

$$\int_0^T\int_0^T \frac{dx\,dy}{(x-y)^2+R^2}$$



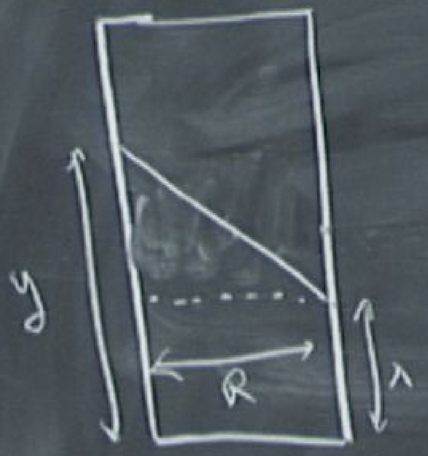


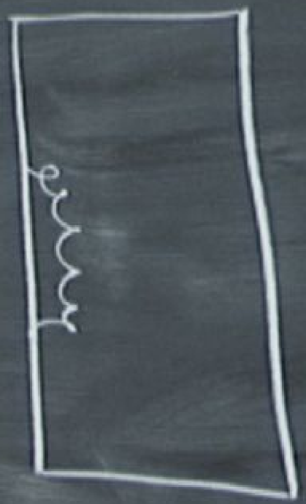
+ boring

etc

$R \ll T$

$$\int_0^T \int_0^T \frac{dx\,dy}{(x-y)^2+R^2}$$





+ boring

etc

$R \ll T$

$$\int_0^T\int_0^T \frac{dx\,dy}{(x-y)^2+R^2}$$

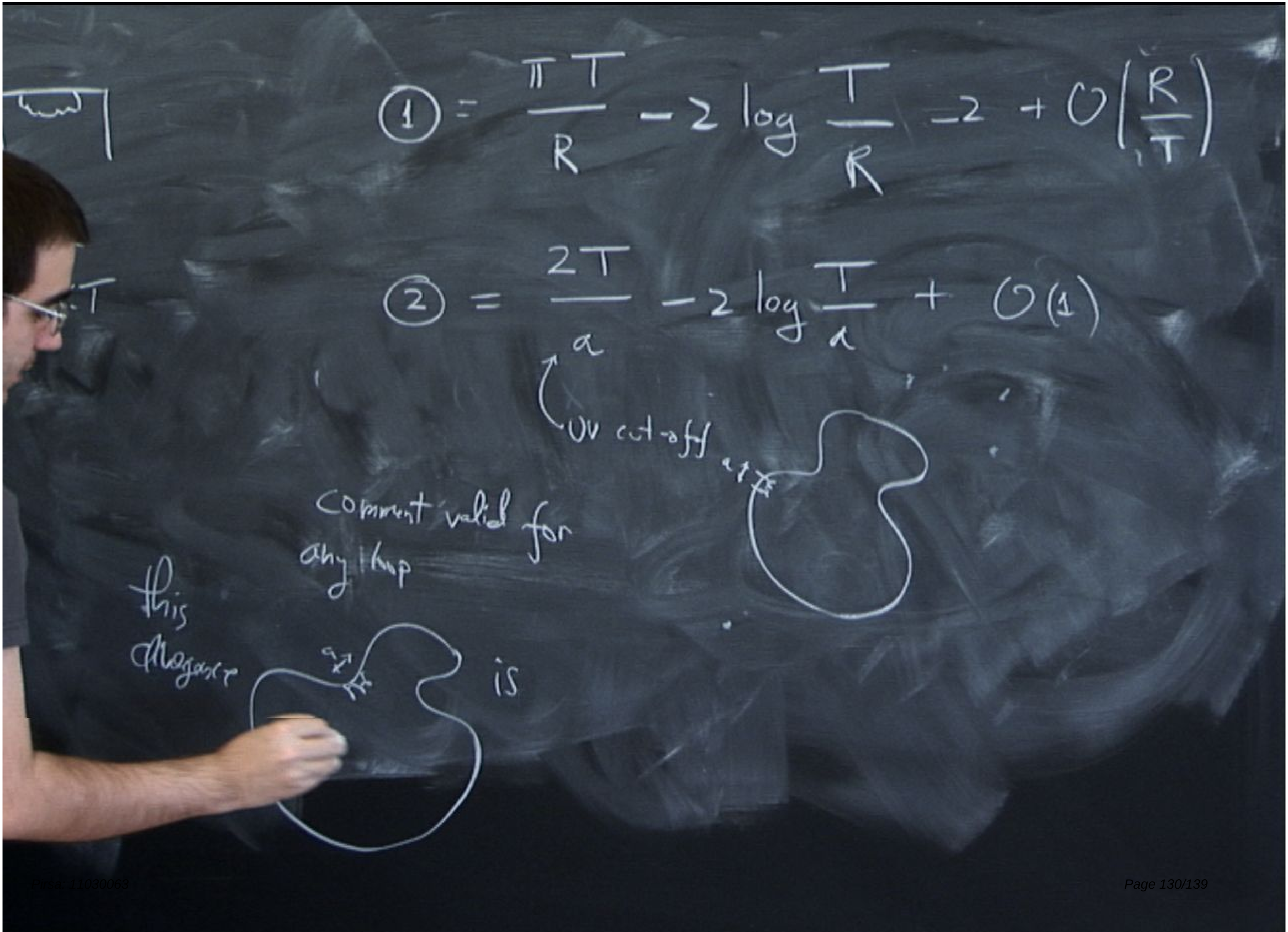
$$\int_0^T\int_0^T \frac{dx\,dy}{(x-y)^2}$$

$$① = \frac{\pi T}{R} - 2 \log \frac{T}{R} - 2 + O\left(\frac{R}{T}\right)$$

$$① = \frac{\pi T}{R} - 2\log\frac{T}{R} - 2 + O\left(\frac{R}{T}\right)$$

$$① = \frac{\pi T}{R} - 2\log\frac{T}{R} - 2 + O\left(\frac{R}{T}\right)$$

$$② = \frac{2T}{a} - 2\log\frac{T}{a} + O(1)$$



$$① = \frac{\pi T}{R} - 2\log\frac{T}{R} - 2 + O\left(\frac{R}{T}\right)$$

$$② = \frac{2T}{a} - 2\log\frac{T}{a} + O(1)$$

UV cut-off

comment valid for any loop

this diagram is

etc

$R \ll T$

$$① = \frac{\pi T}{R} - 2\log\frac{T}{R} - 2 + O\left(\frac{R}{T}\right)$$

$$② = \frac{2T}{a} - 2\log\frac{T}{a} + O(1)$$

$\uparrow$ UV cut-off

Comment valid for any loop

this diagram

is anyway there

etc

$R \ll T$

$$① = \frac{\pi T}{R} - 2\log\frac{T}{R} - 2 + O\left(\frac{R}{T}\right)$$

$$② = \frac{2T}{a} - 2\log\frac{T}{a} + O(1)$$

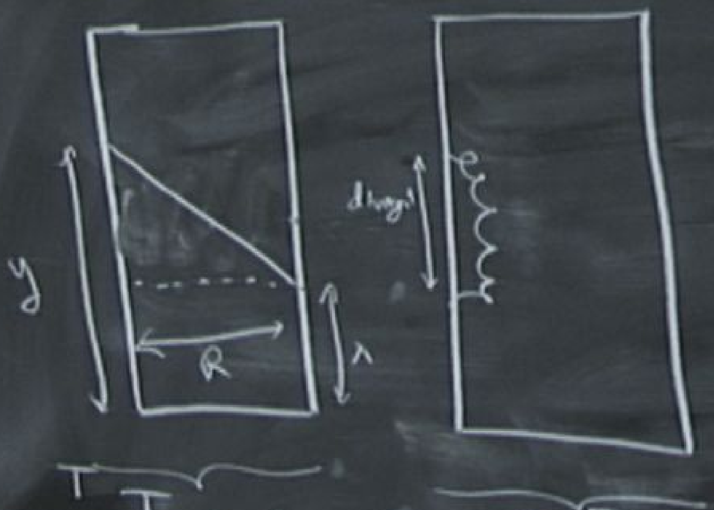
a ← UV cut-off

comment valid for any loop

this diagram

is always there

$$\sim L\,\frac{1}{a}\,\#$$ ← number

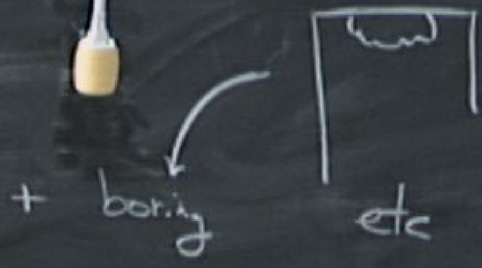


$$\int_0^T\int_0^T \frac{dx\,dy}{(x-y)^2+R^2}$$

①

$$\int_0^T\int_0^T \frac{dx\,dy}{(x-y)^2}$$

②



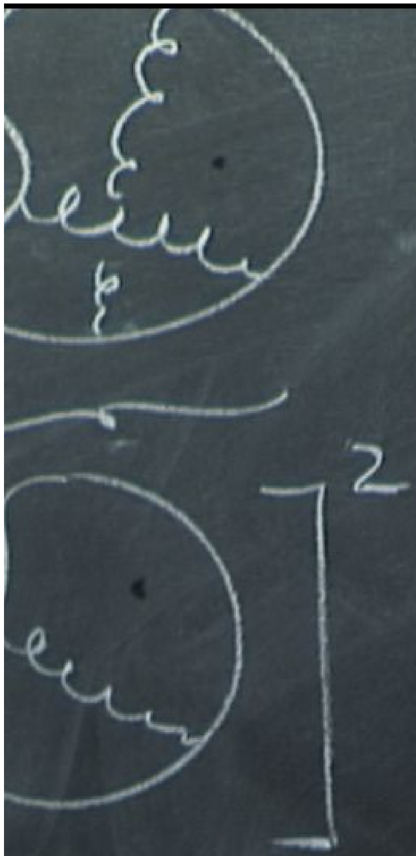
$R \ll T$

① =

② =

this diagram

comment valid
any loop

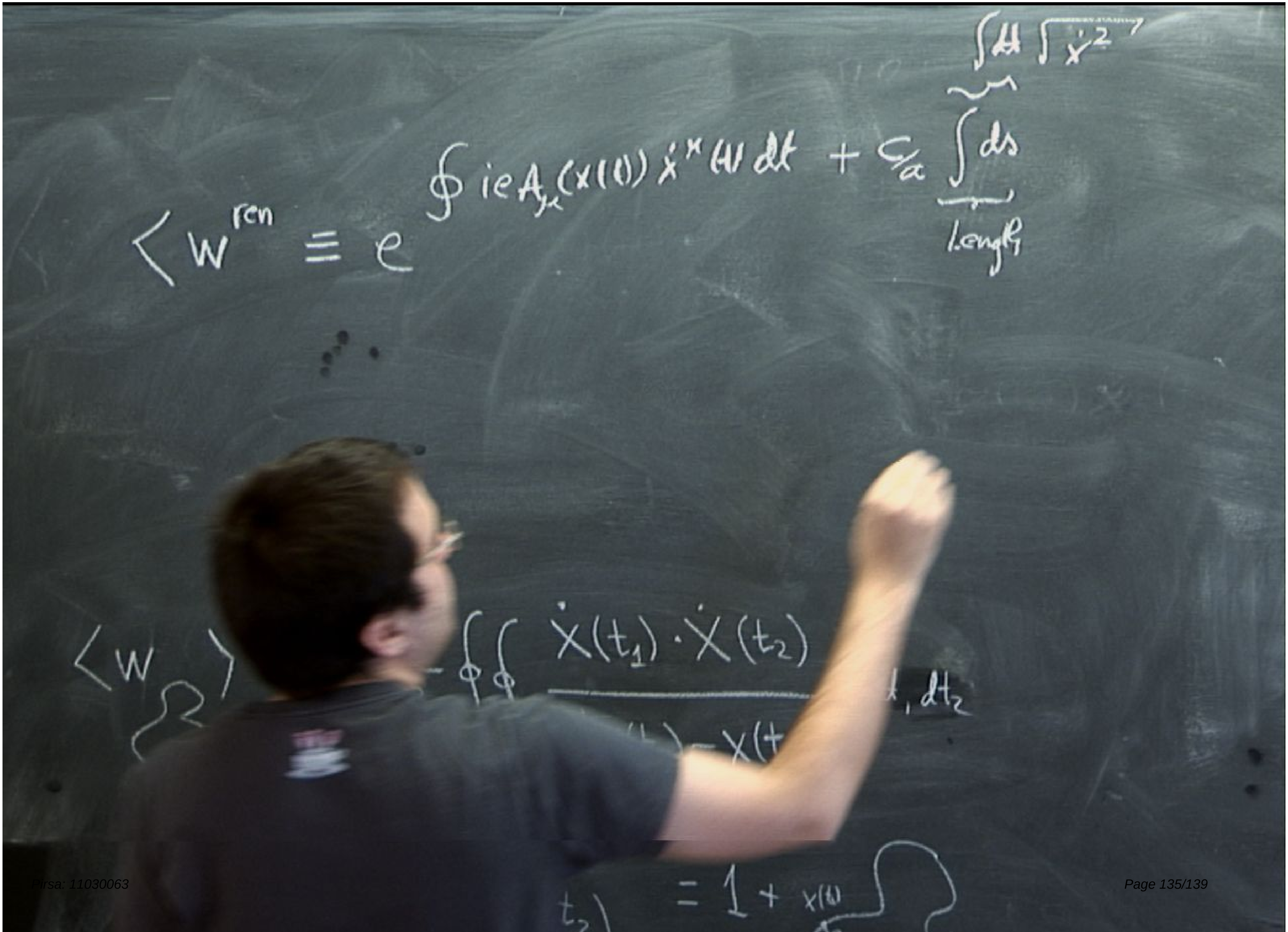


$\dot{x} = (1,0,0,0)$

$$dt_1\, dt_2 \; \frac{\dot{x}(t_1)\cdot\dot{x}(t_2)}{|x(t_1)-x(t_2)|^2} \Bigg)$$

$$\sim e^{-c\,\frac{\text{Length}}{a}}$$

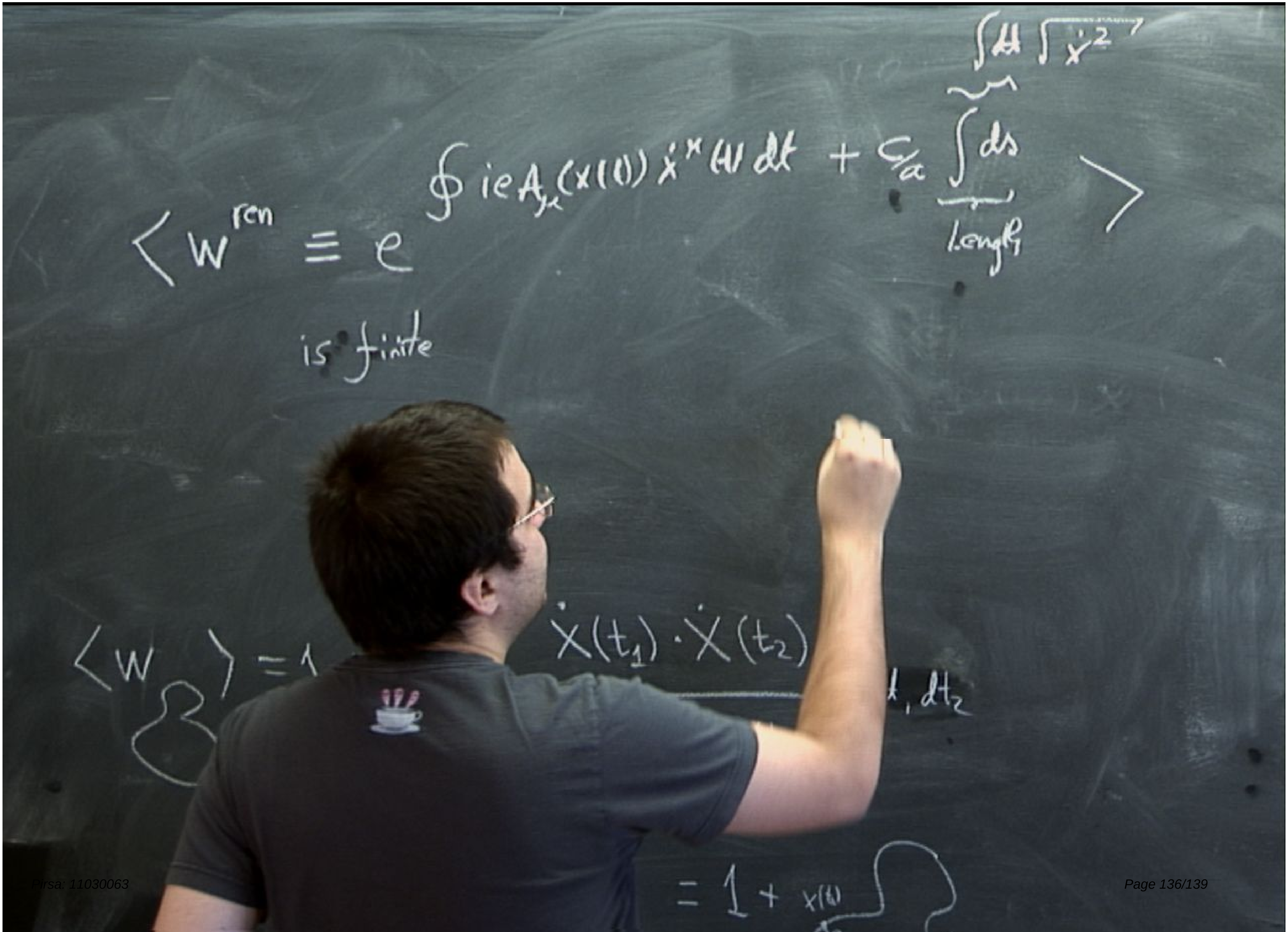
↑ contains a divergence



$$\langle W^{ren} \equiv e^{\oint ie A_\mu(x(t))\, \dot{x}^\mu(t)\, dt \;+\; c/a \underbrace{\overbrace{\int ds}^{\int dt \sqrt{\dot{x}^2}}}_{Length}}$$

$$\langle W \rangle \;\; - \oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{\quad (t_1) - X(t \quad} dt_1\, dt_2$$

$$t_2) \quad = 1 + x(t)$$



$$\left\langle W^{ren} \equiv e^{\oint ieA_\mu(x(t))\,\dot x^\mu(t)\,dt \;+\; \frac{c}{a}\underbrace{\overbrace{\int ds}^{\int dt\sqrt{\dot x^2}}}_{Length}}\right\rangle$$

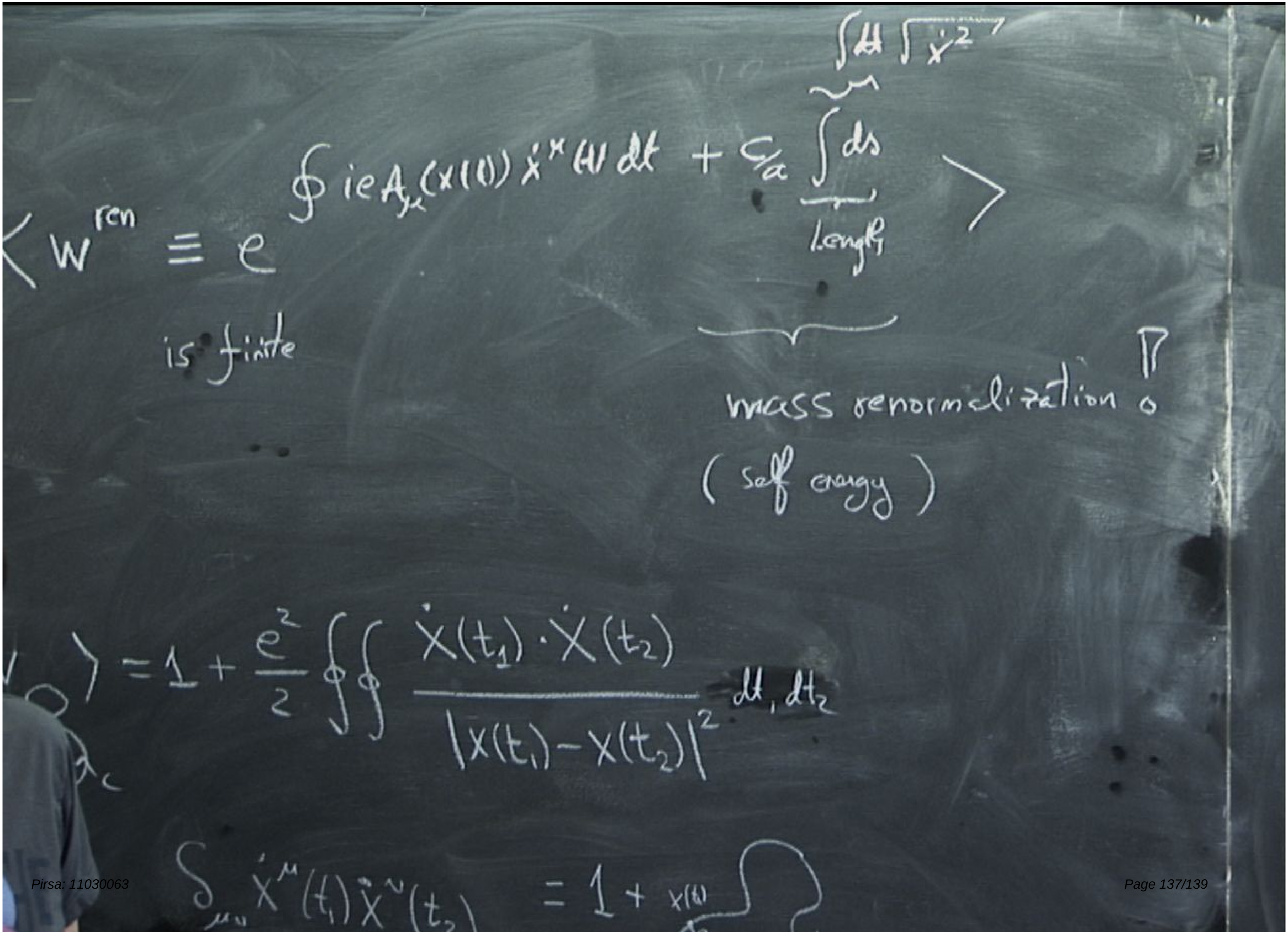
is finite

$\langle W \rangle = 1$

$\dot X(t_1)\cdot\dot X(t_2)$

$dt_1\,dt_2$

$= 1 + x(t)$



$$\langle W^{ren} \equiv e^{\oint ie A_\mu(x(t))\,\dot{x}^\mu(t)\,dt \;+\; c/a \underbrace{\overbrace{\int ds}^{\int dt \sqrt{\dot{x}^2}}}_{\text{Length}}} \rangle$$

is finite

mass renormalization !

(self energy)

$$_{\bigcirc}\rangle = 1 + \frac{e^2}{2}\oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|x(t_1)-x(t_2)|^2}\,dt_1\,dt_2$$

x_c

$$\delta_{\mu\nu}\,\dot{x}^\mu(t_1)\,\dot{x}^\nu(t_2) \quad = 1 + x(t)$$

$$\langle W^{ren} \equiv e^{\oint ieA_\mu(x(t))\,\dot{x}^\mu(t)\,dt \;+\; \underbrace{c/a\,\underbrace{\overbrace{\int ds}^{\int dt\sqrt{\dot{x}^2}}}_{\text{Length}}}_{\text{mass renormalization !}}} \rangle$$

(self energy)

is finite

$$\rangle = 1 + \frac{e^2}{2}\oint\oint \frac{\dot{X}(t_1)\cdot\dot{X}(t_2)}{|x(t_1)-x(t_2)|^2}\,dt_1\,dt_2$$

$$_{\mu\nu}\,\dot{x}^\mu(t_1)\,\dot{x}^\nu(t_2) \quad = 1 + \; x(t)$$

$$\langle W^{ren} \equiv e^{\oint ie A_\mu(x(t))\, \dot{x}^\mu(t)\, dt \;+\; c/a \underbrace{\overbrace{\int ds}^{\int dt \sqrt{\dot{x}^2}}}_{\text{Length}}} \rangle$$

is finite

$$\underbrace{\qquad\qquad}_{\text{mass renormalization !}}$$

(self energy)

$$\langle W_{\Box} \rangle = e^{-T\left(\underbrace{\frac{e^2}{8\pi^2 a}}_{\text{self energy}} - \underbrace{\frac{e^2}{4\pi^2 R^2}}_{\text{Coulomb}}\right)}$$

(does $\underline{\underline{\text{not}}}$ depend on R)