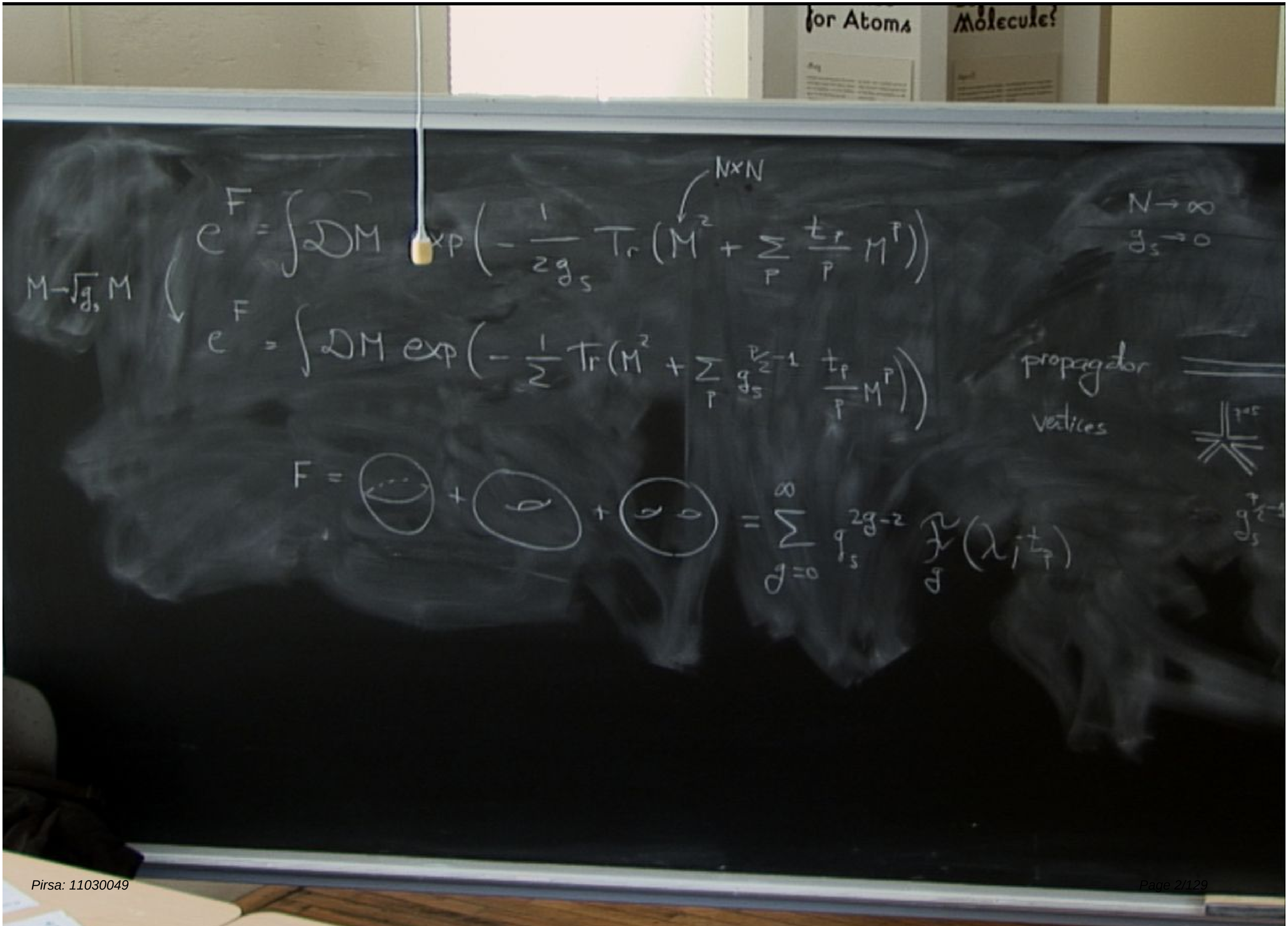
Title: Explorations in String Theory - Lecture 2

Date: Mar 15, 2011 11:30 AM

URL: http://pirsa.org/11030049

Abstract:



for Atoms
Molecule?
N×N
$e^F = \int \mathcal{D}M \exp\left(-\frac{1}{2g_s}\mathrm{Tr}\left(M^2+\sum_p \frac{t_p}{p}M^p\right)\right)$
$N\to\infty$
$g_s\to 0$
$M \to \sqrt{g_s}\, M$
$e^F = \int \mathcal{D}M \exp\left(-\frac{1}{2}\mathrm{Tr}\left(M^2+\sum_p g_s^{p/2-1}\frac{t_p}{p}M^p\right)\right)$
propagator
vertices
p=5
F = + + $= \sum_{g=0}^{\infty} g_s^{2g-2}\,\mathcal{F}_g(\lambda; t_p)$
$g_s^{p/2-1}$

$$e^F = \int \mathcal{D}M \exp\left(-\frac{1}{2g_s}\mathrm{Tr}\left(M^2 + \sum_p \frac{t_p}{p} M^p\right)\right)$$

$N\times N$

$N \to \infty$

$g_s \to 0$

$M \to \sqrt{g_s}\, M$

$$e^F = \int \mathcal{D}M \exp\left(-\frac{1}{2}\mathrm{Tr}\left(M^2 + \sum_p g_s^{\frac{p}{2}-1} \frac{t_p}{p} M^p\right)\right)$$

propagator

vertices

$p=5$

$g_s^{\frac{p}{2}-1}$

$$F = \;+\;+\; = \sum_{g=0}^{\infty} g_s^{2g-2}\, \mathcal{F}_g(\lambda, t_p)$$

$$\mathcal{F}_g = \sum_{\text{all planar diagrams}}$$

E

1 g_s

propagators

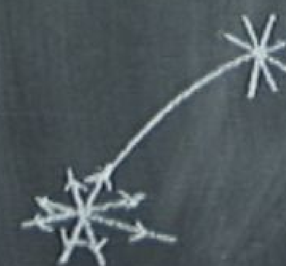
① $\underbrace{g_s^{E}}_{\text{propagators}} \quad \underbrace{g_s^{-V}}_{\text{vertices}} \quad \underbrace{N^{F}}_{\text{color loops}}$

② $\overbrace{1} \qquad \overbrace{\prod_p g_s^{(p/2-1)V_p}}$

① $\underbrace{g_s^{E}}_{\text{propagators}} \qquad \underbrace{g_s^{-V}}_{\text{vertices}} \qquad \underbrace{N^{F}}_{\text{color loops}}$

② $\overbrace{1} \qquad \overbrace{\prod_p g_s^{(p/2-1)V_p}} \qquad N^{F}$

$$\underbrace{g_s^{-\sum_p V_p}}_{g_s^{-V}} \quad g_s^{\sum_p \frac{pV_p}{2}}$$



V_8

$$\sum_p V_p\, p = 2E$$

$$\langle \underbrace{\mathrm{Tr}\, M}_{\text{gauge inv}}{}^4 \quad \mathrm{Tr}\, M^4 \rangle$$

$$M \to U M U^{-1}$$

$$\langle \underbrace{\mathrm{Tr}\, M^4}_{\text{gauge inv}} \quad \mathrm{Tr}\, M^4 \rangle_c$$

$M \to UMU^{-1}$

$$\langle \underbrace{\mathrm{Tr}\, M}_{\text{gauge inv}}{}^4 \quad \mathrm{Tr}\, M^4 \rangle_c$$

$M \to U M U^{-1}$

$$\langle \mathrm{Tr}\, M^4 \;\; \mathrm{Tr}\, M^4 \rangle_c$$

gauge inv

$M \to U M U^{-1}$

change

$\Downarrow$

$$\left\langle \underbrace{\frac{1}{N^2}\mathrm{Tr}\, M^4}_{\text{gauge inv}} \;\; \frac{1}{N^2}\mathrm{Tr}\, M^4 \right\rangle_c$$

$M \rightarrow U M U^{-1}$

$\sim N^4 \left(\frac{1}{N^2}\right)^2 \sim 1$

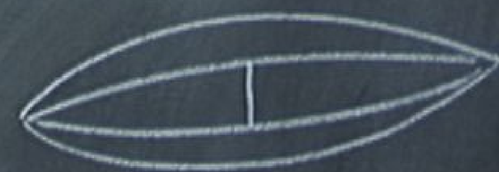
change

$\Downarrow$

$$\underbrace{\left\langle \frac{1}{N^2} \mathrm{Tr}\, M^4 \right\rangle_c^{\text{free}}}_{\mathcal{O}}$$

$\langle \mathcal{O}\mathcal{O} \rangle$

$\shortparallel$



MU^{-1}

$$\sim N^4 \left(\frac{1}{N^2}\right)^2 \sim 1$$

$\Downarrow$

$$\left\langle \overbrace{\frac{1}{N^2}\mathrm{Tr}\, M^4}^{\mathcal{O}} \right\rangle_c^{\text{free}}$$

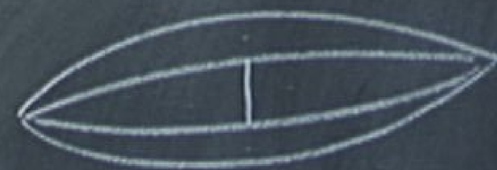
MU^{-1}

$$\sim N^4\left(\frac{1}{N^2}\right)^2 \sim 1$$

$$\Downarrow$$

$$\langle \mathcal{O}\mathcal{O} \rangle$$

$$\parallel$$

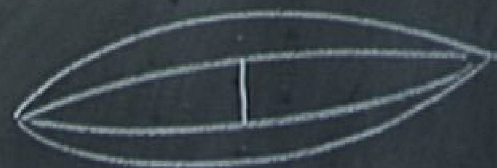


$$\sim \lambda$$

$$\langle \mathcal{O}\mathcal{O}\rangle = \underbrace{F(\lambda)}$$

$\lambda \longleftrightarrow$ string tension

$\parallel$

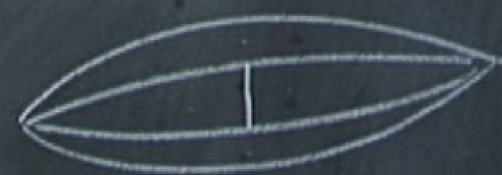
 $\sim \lambda$ CHECK

$$\sim N^4\left(\frac{1}{N^2}\right)^2 \sim 1$$

$$\langle \mathcal{O}\mathcal{O}\rangle = \underbrace{F(\lambda)}$$

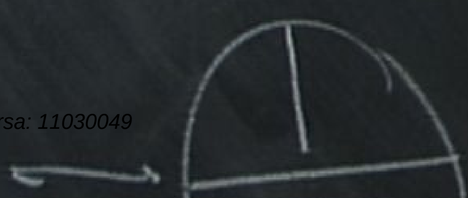
$\lambda \longleftrightarrow$ string tension

$$\parallel$$

$\sum$  $\sim \lambda$ CHECK

$$\sim N^4 \left(\frac{1}{N^2}\right)^2 \sim 1$$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$$



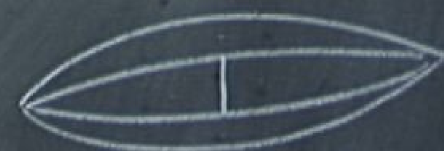
$$\Big\langle \underbrace{\frac{1}{N^2}\mathrm{Tr}\, M^4}_{\text{gauge inv}} \;\; \overbrace{\frac{1}{N^2}\mathrm{Tr}\, M^4}^{\mathcal{O}} \Big\rangle_c^{\text{free}}$$

gauge inv

$M \to U M U^{-1}$

$$\langle \mathcal{O}\,\mathcal{O}\rangle_c^{\text{int}} = \underbrace{F(\lambda)}_{\lambda \longleftrightarrow}$$

$\Vert$

$\sum$  $\sim \lambda$

1 string, 1 particle

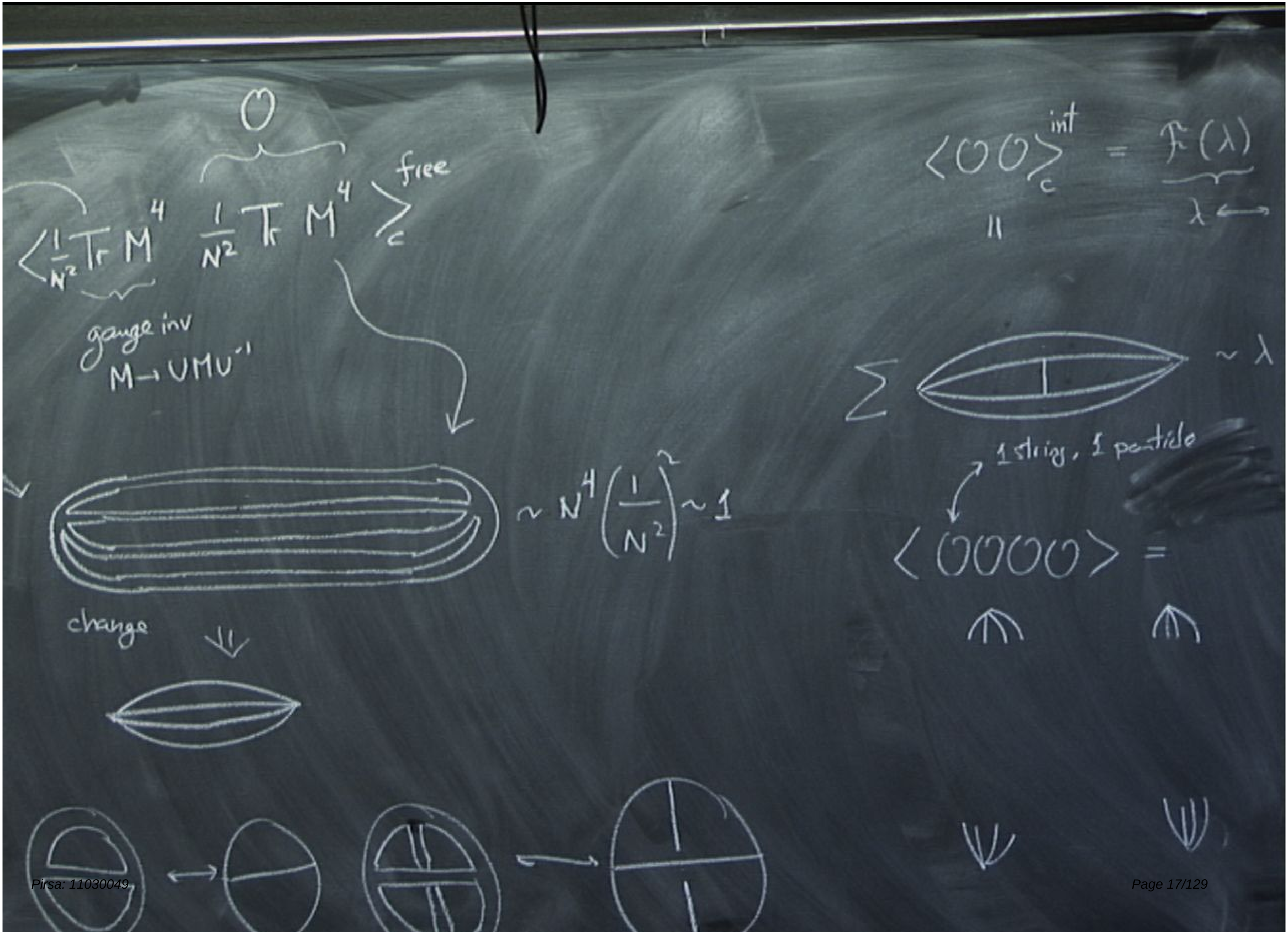
$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$

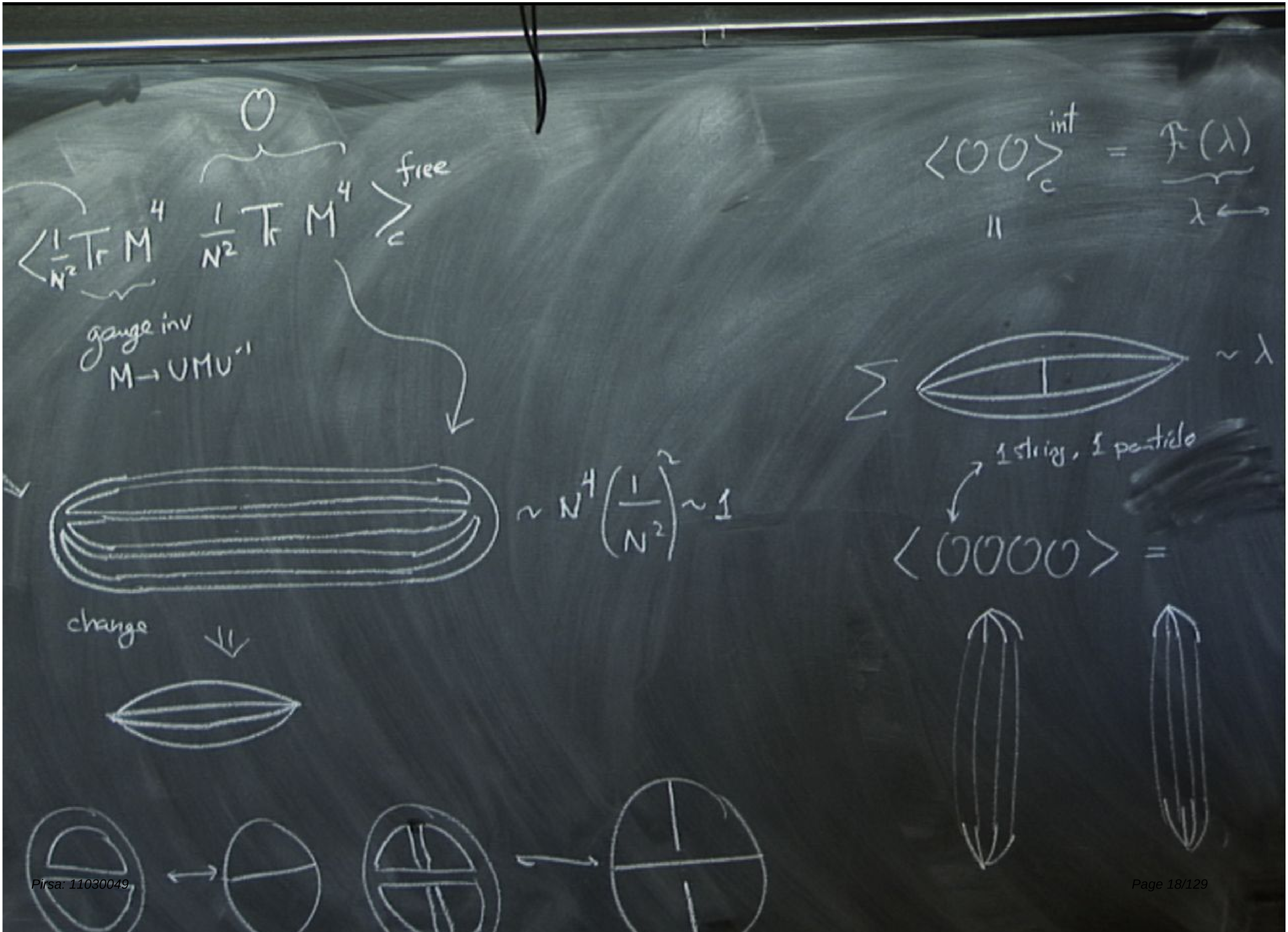
$\sim N^4\left(\frac{1}{N^2}\right)^2 \sim 1$

change

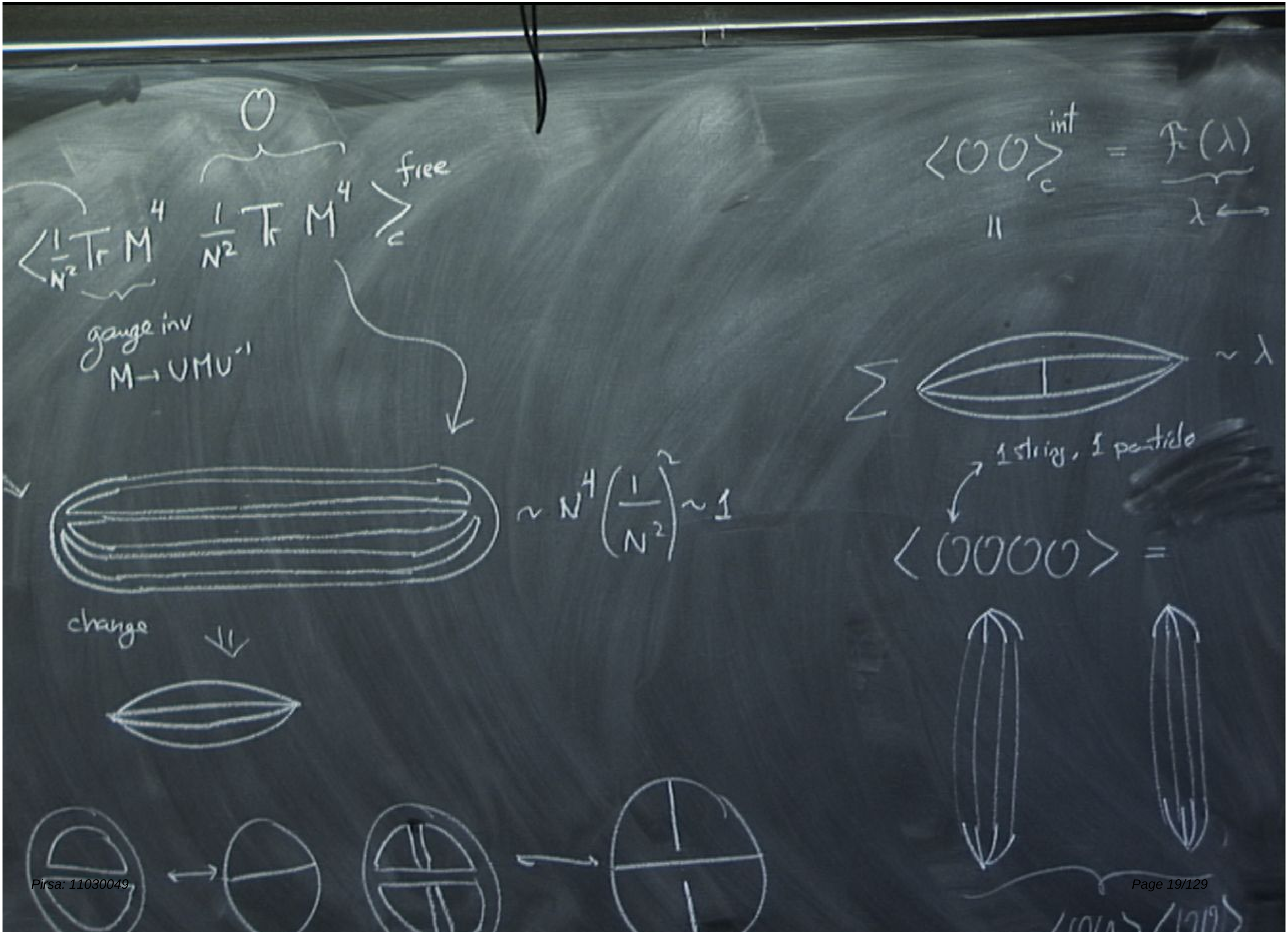
$\Downarrow$

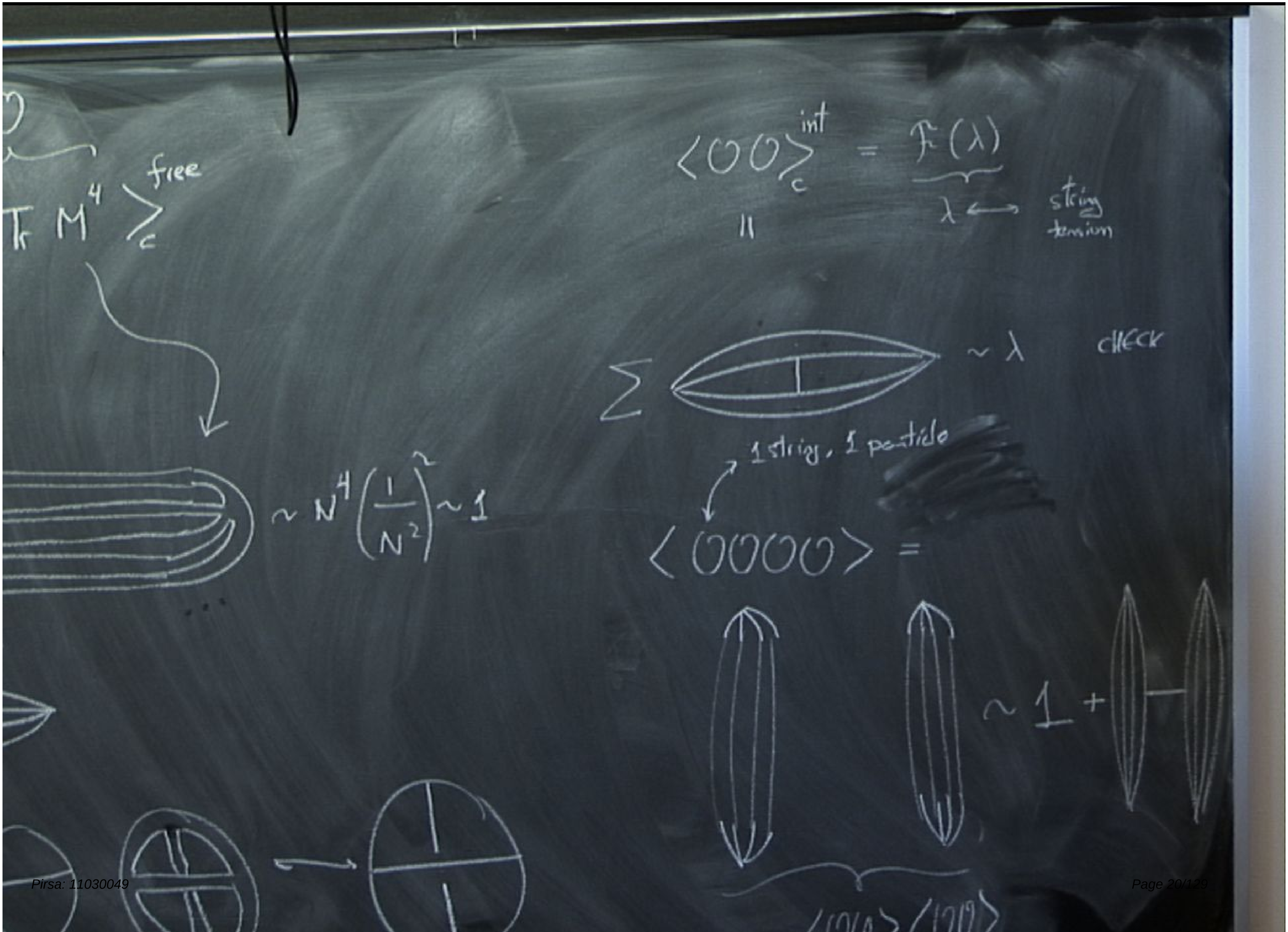
 $\longleftrightarrow$ 

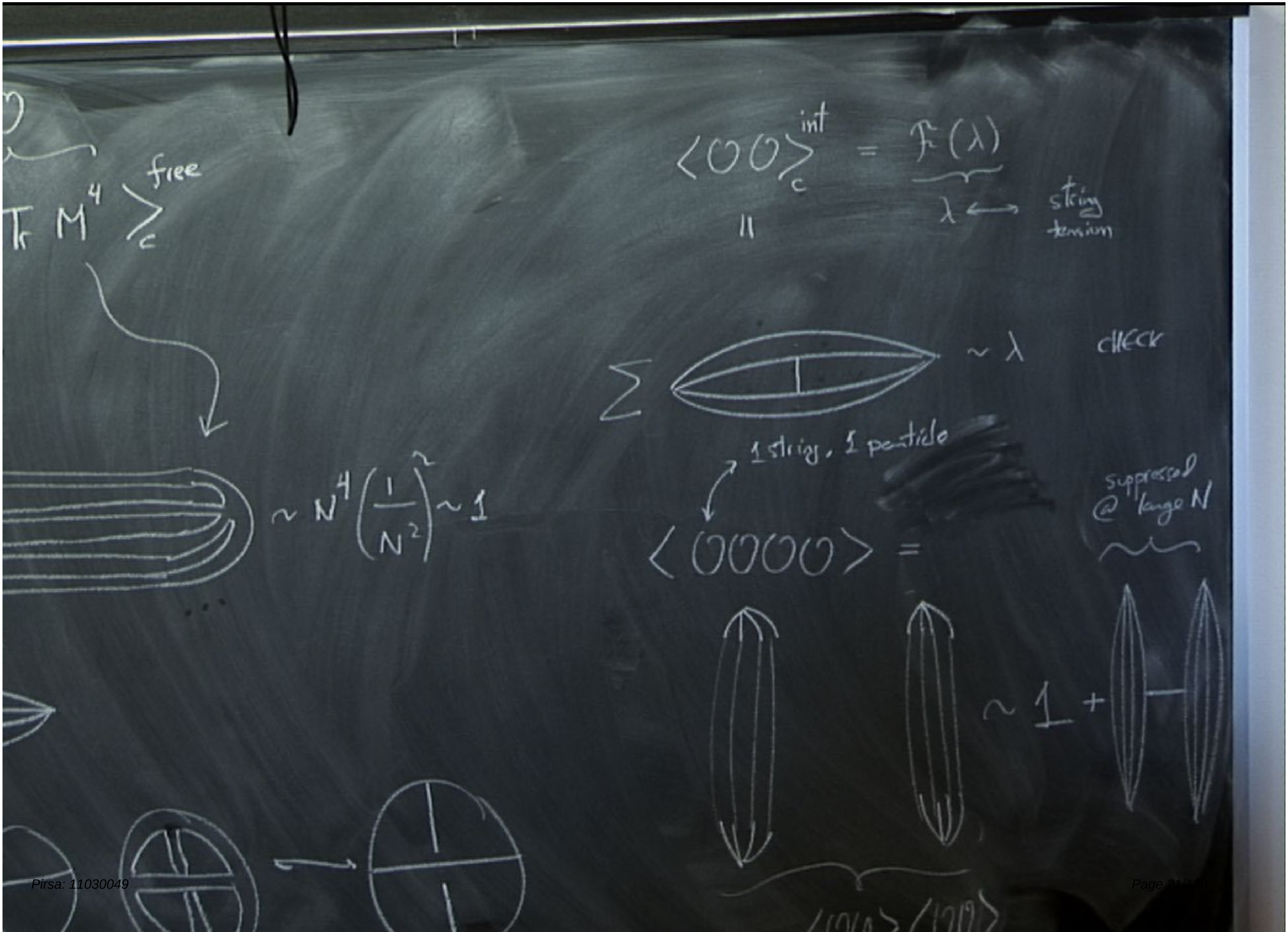

O
free
⟨ 1/N² Tr M⁴ 1/N² Tr M⁴ ⟩_c
gauge inv
M → UMU⁻¹
~ N⁴ (1/N²)² ~ 1
change
⟨OO⟩_c^int = F(λ)
λ ⟷
Σ
~ λ
1 string, 1 particle
⟨OOOO⟩ =

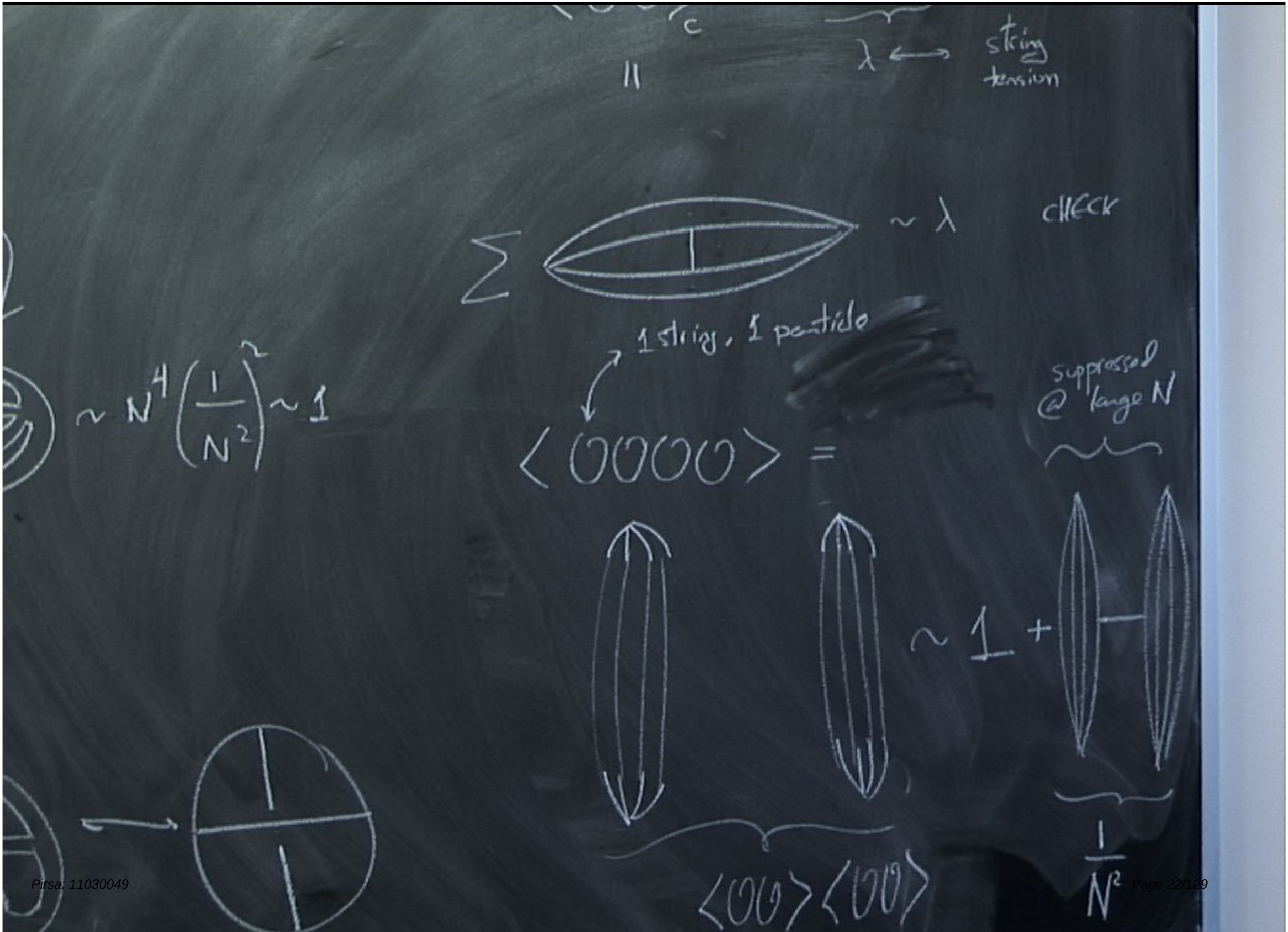


$\mathcal{O}$
free
$\langle \frac{1}{N^2} \mathrm{Tr}\, M^4 \ \frac{1}{N^2} \mathrm{Tr}\, M^4 \rangle_c$
gauge inv
$M \to U M U^{-1}$
$\sim N^4 \left(\frac{1}{N^2}\right)^2 \sim 1$
change
$\langle \mathcal{O} \mathcal{O} \rangle_c^{int} = F(\lambda)$
λ
Σ
$\sim \lambda$
1 string, 1 particle
$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle =$

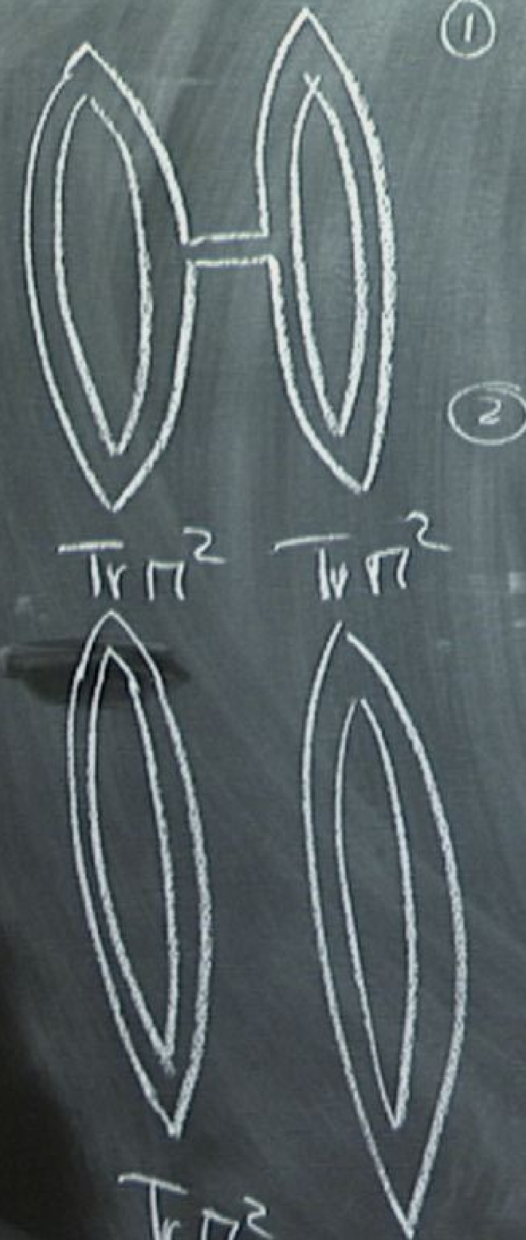

$\mathcal{O}$
free
$\left\langle \frac{1}{N^2} \mathrm{Tr}\, M^4 \; \frac{1}{N^2} \mathrm{Tr}\, M^4 \right\rangle_c$
gauge inv
$M \to U M U^{-1}$
$\sim N^4 \left(\frac{1}{N^2}\right)^2 \sim 1$
change
$\langle \mathcal{O}\mathcal{O} \rangle_c^{int} = F(\lambda)$
$\lambda \longleftrightarrow$
Σ
$\sim \lambda$
1 string, 1 particle
$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle =$


free
Tr M^4 Σ_c
⟨OO⟩_c^int = F(λ)
λ ⟷ string tension
~ λ
CHECK
1 string, 1 particle
~ N^4 (1/N^2)^2 ~ 1
⟨OOOO⟩ =
~ 1 +


free
Tr M^4 Σ_c
~ N^4 (1/N^2)^2 ~ 1
⟨OO⟩^int_c = F(λ)
λ ⟷ string tension
~ λ
CHECK
1 string, 1 particle
suppressed @ large N
⟨OOOO⟩ =
~ 1 +



c
λ ⟷ string tension
~ λ
CHECK
1 string, 1 particle
~ N^4 (1/N^2)^2 ~ 1
⟨OOOOO⟩ =
suppressed @ large N
~ 1 +
⟨OO⟩⟨OO⟩
1/N^2



$\mathrm{Tr}\,\Pi^2 \quad \mathrm{Tr}\,\Pi^2$

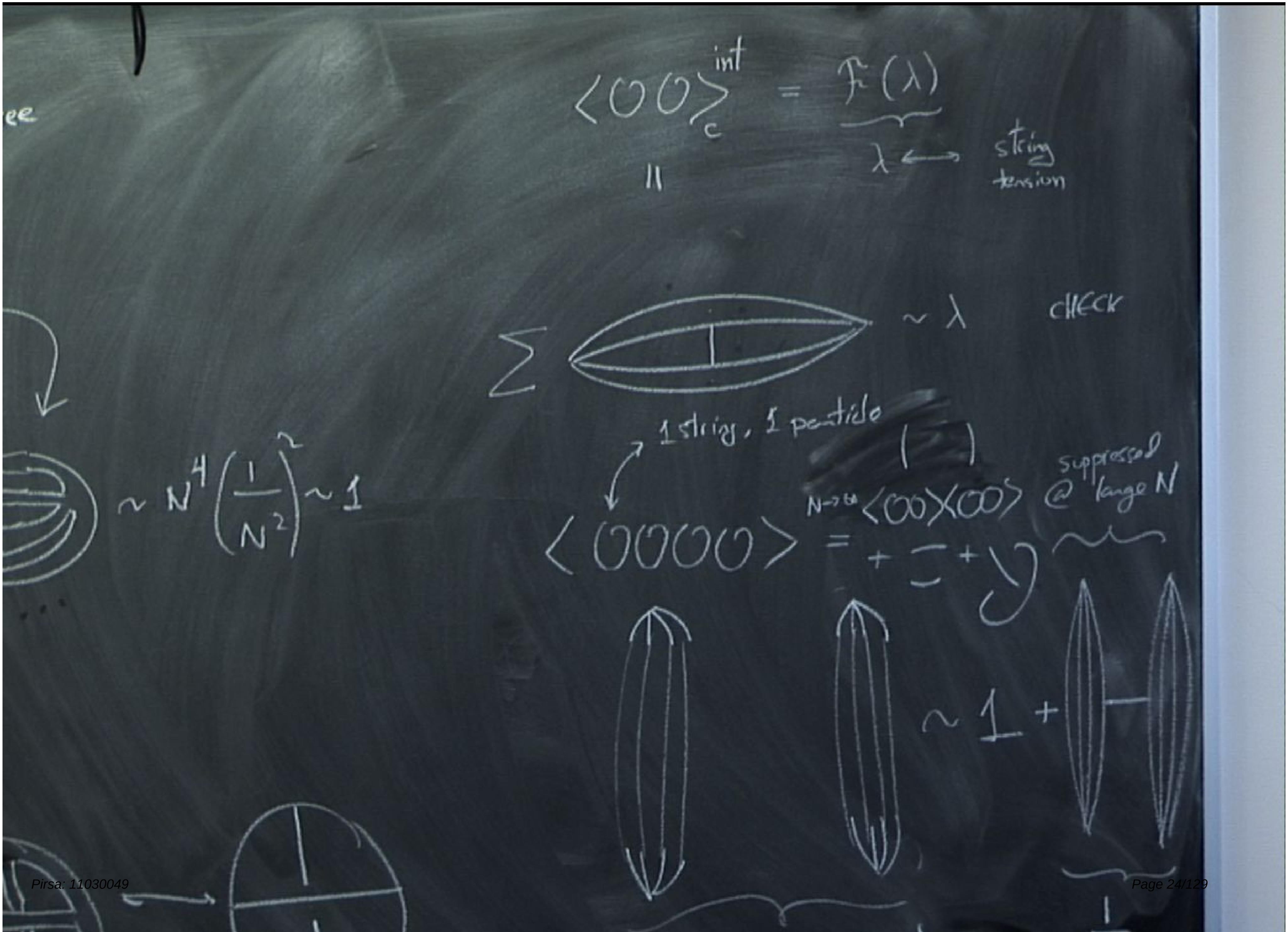
$\mathrm{Tr}\,\Pi^2 \quad \mathrm{Tr}\,\Pi^2$

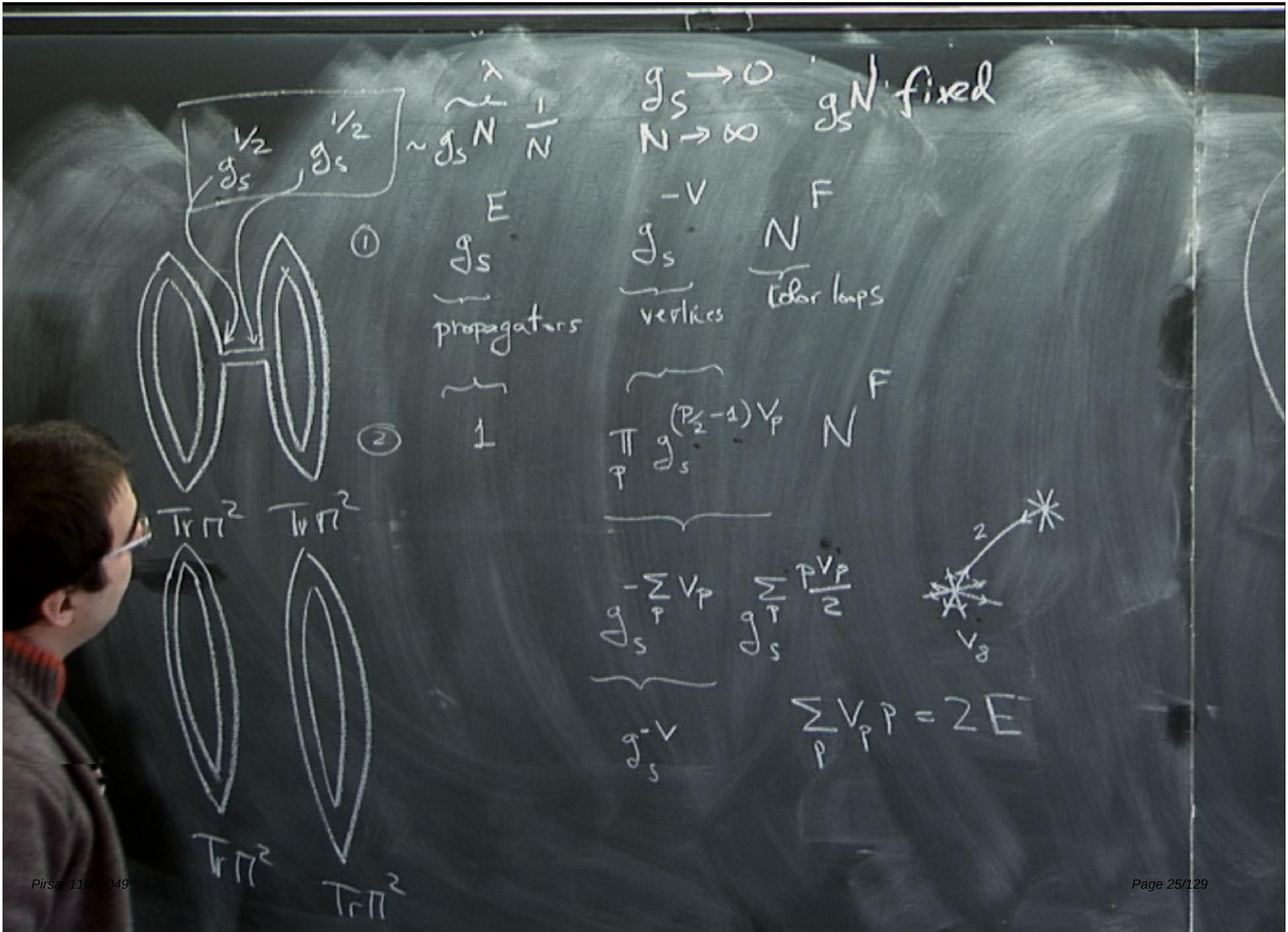
① $\underbrace{g_s}_{\text{propagators}} \quad \underbrace{g_s}_{\text{vertices}} \quad \underbrace{N}_{\text{col}}$

② $\underbrace{}_{1} \quad \underbrace{}_{\prod_p g_s^{(p/2-1)V_p}}$

$\underbrace{g_s^{-\sum_p V_p} \quad g_s^{\sum_p}}_{}$

$\underbrace{}_{g_s^{-V}}$


ee
⟨OO⟩_c^int = F(λ)
λ ⟷ string tension
~ λ
CHECK
1 string, 1 particle
~ N^4 (1/N^2)^2 ~ 1
N→∞ ⟨OO⟩⟨OO⟩
suppressed @ large N
⟨OOOO⟩ =
~ 1 +



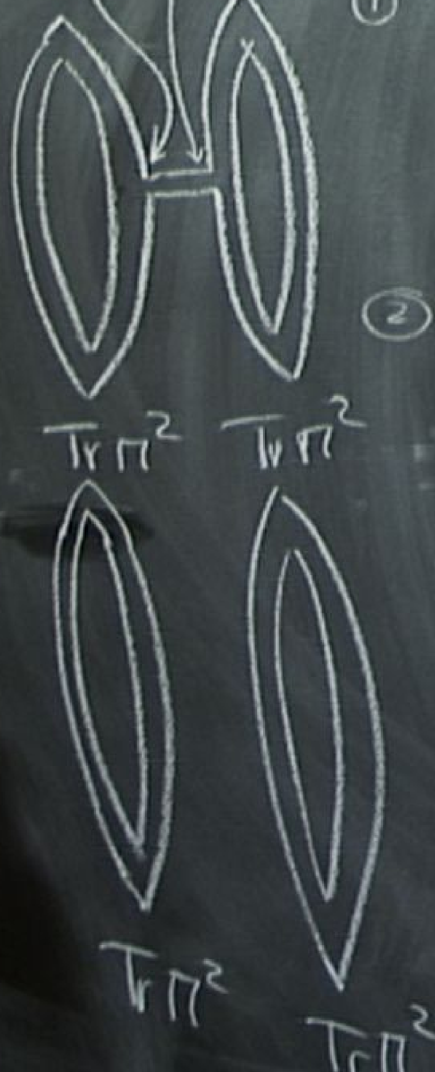
λ
$g_s \to 0$
$g_s N$ fixed
$N \to \infty$
$g_s^{1/2}$ $g_s^{1/2}$ ~ $g_s N$ $\frac{1}{N}$
① g_s^{E} g_s^{-V} N^{F}
propagators
vertices
color loops
② 1 $\prod_p g_s^{(p/2-1)V_p}$ N^{F}
$g_s^{-\sum_p V_p}$ $g_s^{\sum_p \frac{pV_p}{2}}$
g_s^{-V}
$\sum_p V_p p = 2E$
2
V_8
$\mathrm{Tr}\,\Pi^2$ $\mathrm{Tr}\,\Pi^2$
$\mathrm{Tr}\,\Pi^2$ $\mathrm{Tr}\,\Pi^2$

$$\boxed{g_s^{1/2} \quad g_s^{1/2}} \sim \overbrace{g_s N}^{\lambda} \frac{1}{N}$$

$g_s \to 0$

$N \to \infty$

$g_s N$ fixed



① $\underbrace{g_s^{E}}_{\text{propagators}} \quad \underbrace{g_s^{-V}}_{\text{vertices}} \quad \underbrace{N^{F}}_{\text{color loops}}$

② $1 \quad \prod_p g_s^{(p/2-1)V_p} \quad N^F$

$$g_s^{-\sum_p V_p} \quad g_s^{\sum_p \frac{pV_p}{2}}$$

$$g_s^{-V}$$

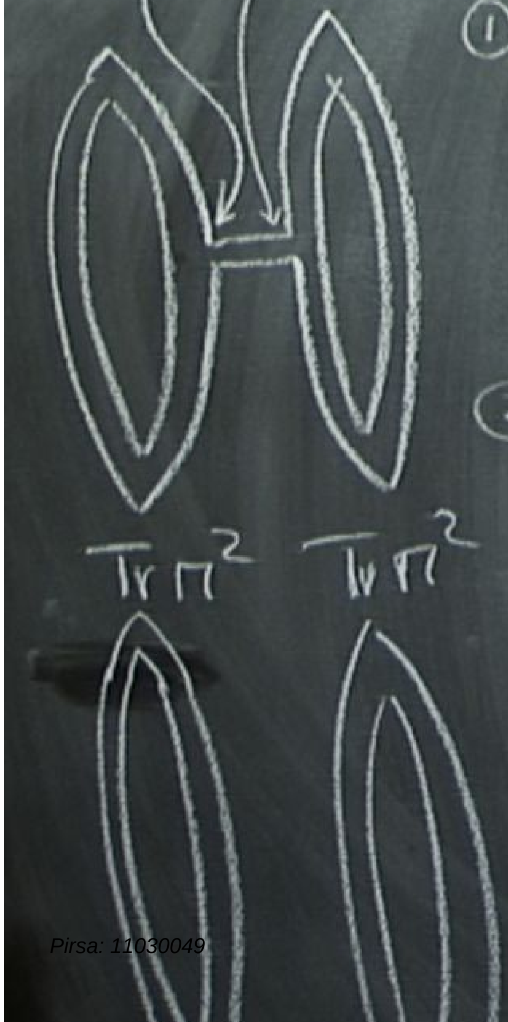
$$\sum_p V_p\, p = 2E$$

$$\boxed{g_s^{1/2} \quad g_s^{1/2}} \sim \underbrace{g_s N}_{\lambda} \frac{1}{N} \qquad g_s \to 0 \quad N \to \infty \qquad g_s N \text{ fixed}$$

$$E \qquad -V \qquad F$$

① $\underbrace{g_s}_{\text{propagators}} \qquad \underbrace{g_s}_{\text{vertices}} \qquad \underbrace{N}_{\text{color loops}}$

② $\underbrace{1} \qquad \underbrace{\prod_p g_s^{(p/2-1)V_p}} \qquad N^F$



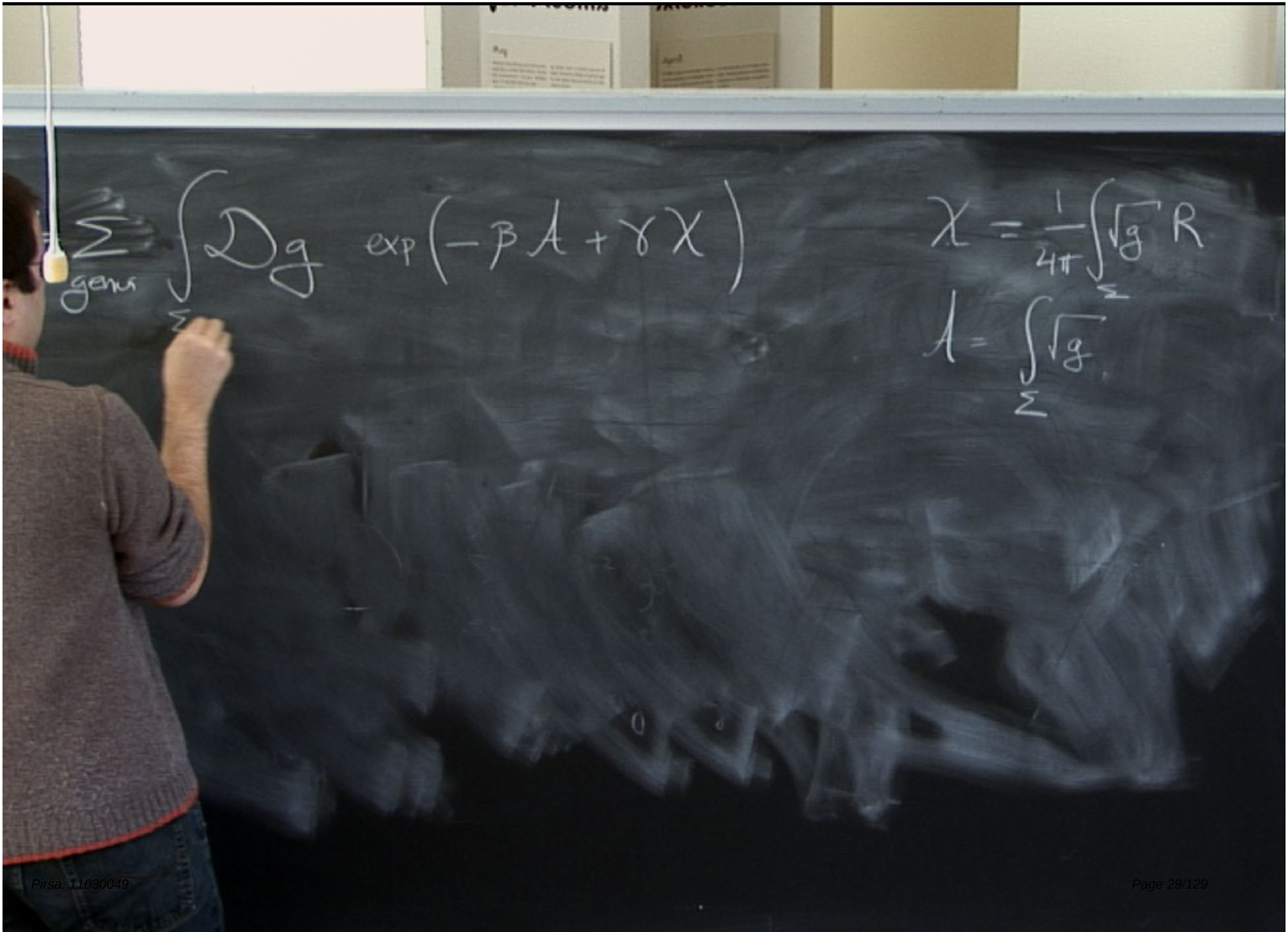
$$\underbrace{g_s^{-\sum_p V_p} \quad g_s^{\sum_p \frac{pV_p}{2}}}$$

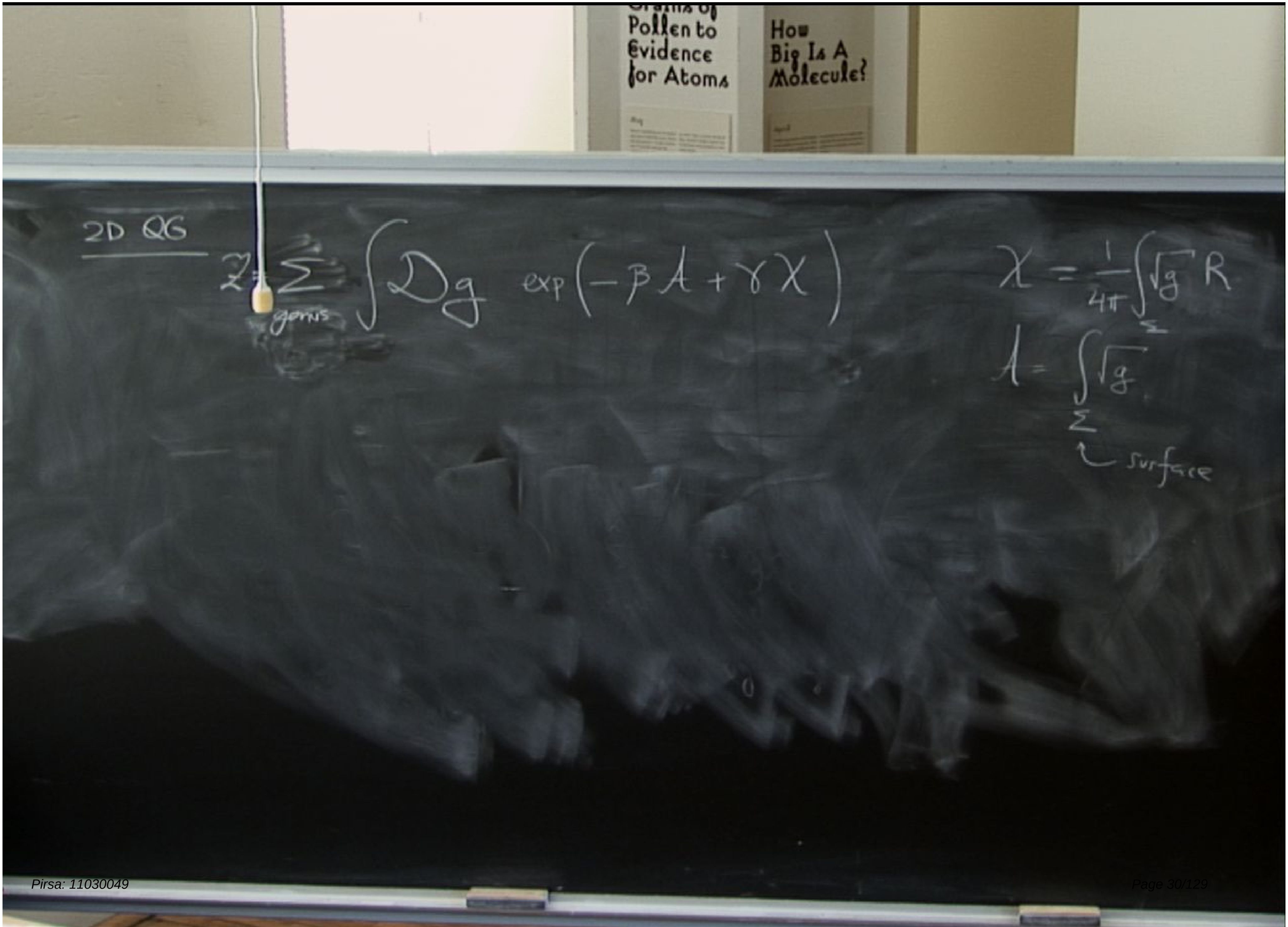
$$\underbrace{\qquad}$$

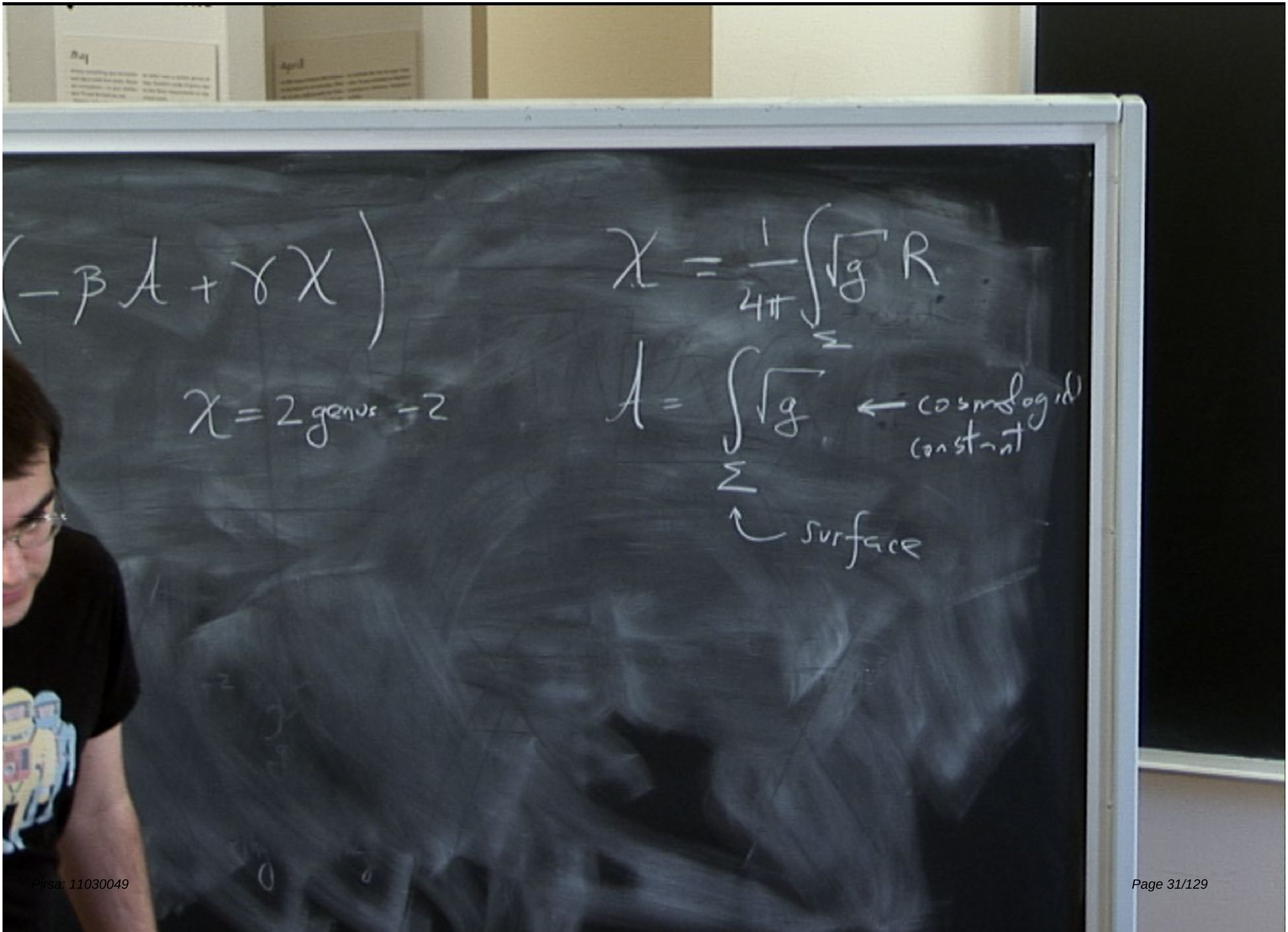
$$g^{-V} \qquad \sum_p V_p p = 2E$$

2D QG

$$Z = \sum_{\text{genus}} \int \mathcal{D}g \ \exp\left(-\beta A + \gamma \chi\right) \qquad \chi$$


$\sum_{\text{genus}} \int_{\Sigma} \mathcal{D}g \; \exp\left(-\beta A + \gamma \chi\right)$
$\chi = \frac{1}{4\pi} \int_{\Sigma} \sqrt{g}\, R$
$A = \int_{\Sigma} \sqrt{g}$


Pollen to
Evidence
for Atoms
How
Big Is A
Molecule?
2D QG
$Z = \sum_{\text{genus}} \int \mathcal{D}g \; \exp(-\beta A + \gamma \chi)$
$\chi = \frac{1}{4\pi} \int_\Sigma \sqrt{g}\, R$
$A = \int_\Sigma \sqrt{g}$
surface


$(-\beta A + \gamma \chi)$
$\chi = \frac{1}{4\pi}\int_\Sigma \sqrt{g}\, R$
$\chi = 2\,\text{genus} - 2$
$A = \int_\Sigma \sqrt{g}$ ← cosmological constant
surface

$$\left(-\beta A + \gamma \chi\right)$$

$$-\chi = 2\,\text{genus} - 2$$

$$\chi = \frac{1}{4\pi}\int_{\Sigma} \sqrt{g}\, R$$

$$A = \int_{\Sigma} \sqrt{g} \quad \leftarrow \text{cosmological constant}$$

Σ ← surface

the free parameters

$$g \quad \exp\left(-\beta A + \gamma \chi\right)$$

$$-\chi = 2\,\text{genus} - 2$$

$$\chi = \frac{1}{4\pi}\int_{\Sigma}\sqrt{g}\,R$$

$$A = \int_{\Sigma}\sqrt{g} \quad \leftarrow \text{cosmological constant}$$

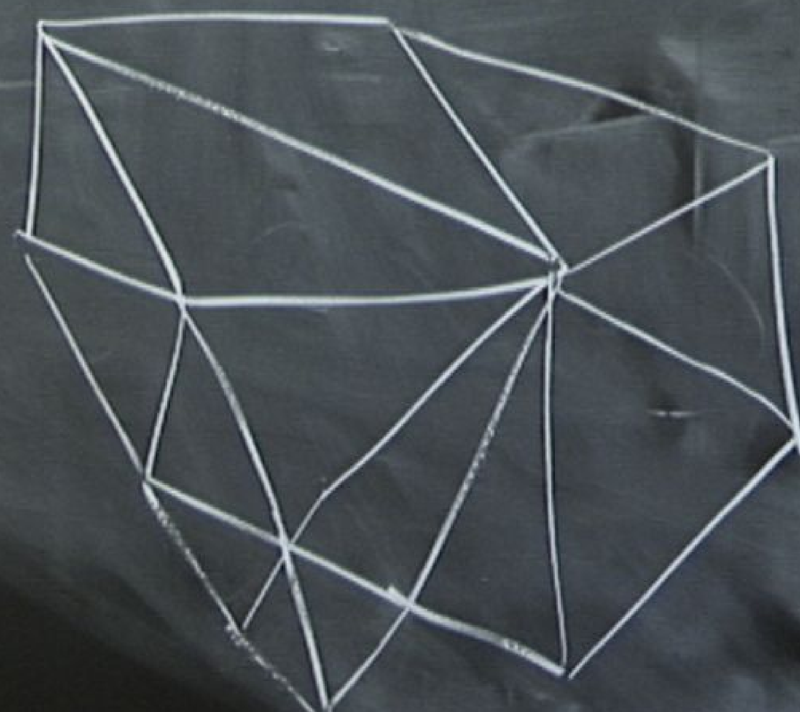
Σ surface

2D QG

$$Z[\beta,\gamma] = \sum_{\text{genus}} \int \mathcal{D}g \; \exp\left(-\beta A + \gamma \chi\right.$$

the free par

$$-\chi = 2\,\text{genu}$$



the free parameters

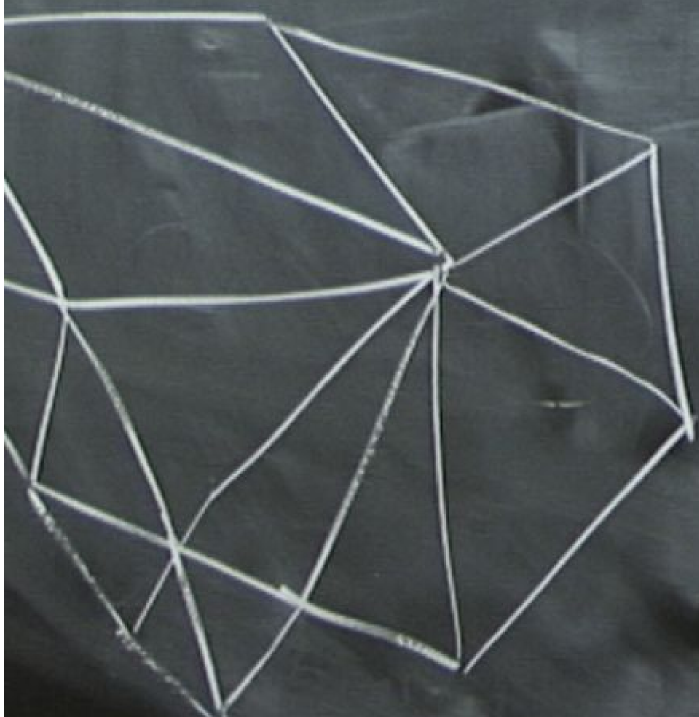
$$\sum_{\text{genus}} \int \mathcal{D}g \;\; \exp\left(-\beta A + \gamma \chi\right)$$

$$-\chi = 2\,\text{genus} - 2$$

$$\chi = \frac{1}{4\pi}\int$$

$$A = \int_{\Sigma} \sqrt{g}$$

sur



$$\int \sqrt{g}\, R \to \sum_{i=4}$$

the free parameters

$$g \quad \exp\left(-\beta A + \gamma \chi\right)$$

$$\chi = \frac{1}{4\pi}\int_\Sigma \sqrt{g}\, R$$

$$A = \int_\Sigma \sqrt{g}$$ ← cosmological constant

Σ ← surface

$$-\chi = 2\,\text{genus} - 2$$

$$\int \sqrt{g}\, R \rightarrow \sum_{i=1}^{V} 4\pi\left(1 - \frac{N_i}{6}\right)$$

N_i = lines emanating from vertex i

$$\sum_{i=1}^{V} 4\pi = 4\pi V$$

$$- \sum_{i=1}^{V} \frac{4\pi}{6} N_i$$

now

$$2E = \sum_i N_i$$

$$\sum_{i=1}^{V} 4\pi = 4\pi V$$

$$- \sum_{i=1}^{V} \frac{4\pi}{6} N_i$$

now

$$2E = \sum_i N_i \ , \quad 2E = 3F$$

$$\sum_{i=1}^{V} 4\pi = 4\pi V$$

$$-\sum_{i=1}^{V} \frac{4\pi}{6} N_i = -4\pi\left(\frac{F}{2}\right)$$

now

$$2E = \sum_i N_i \quad , \quad 2E = 3F$$

each face has $\underline{3}$ edges and each edge is shared by $\underline{2}$ faces

$$\sum_{i=1}^{V} 4\pi = 4\pi V$$

$$-\sum_{i=1}^{V} \frac{4\pi}{6} N_i = -4\pi\left(\frac{F}{2}\right)$$

now

$$2E = \sum_i N_i \quad , \quad 2E = 3F$$

each face has $\underline{\underline{3}}$ edges and each edge is shared by $\underline{\underline{2}}$ faces

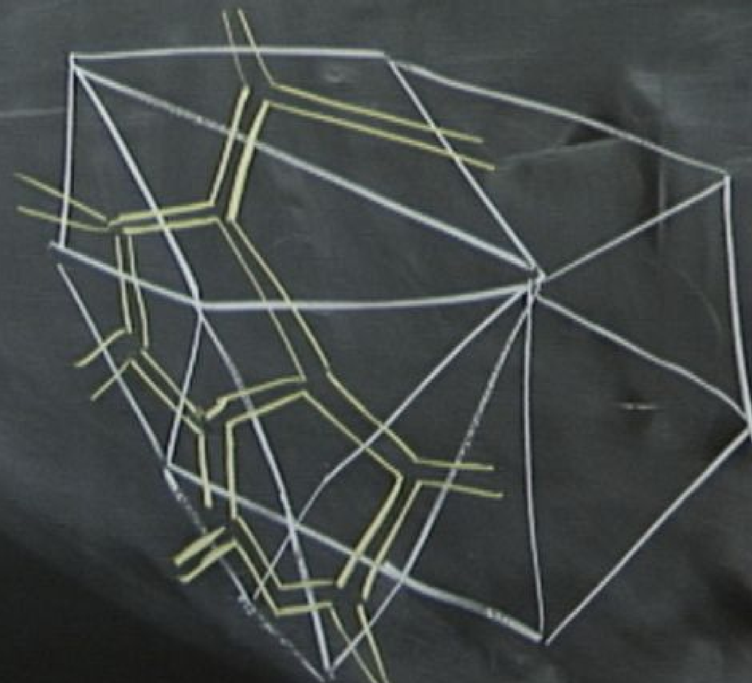
$$\# = 4\pi\left(V - E + F\right) = 4\pi \chi$$

2D QG

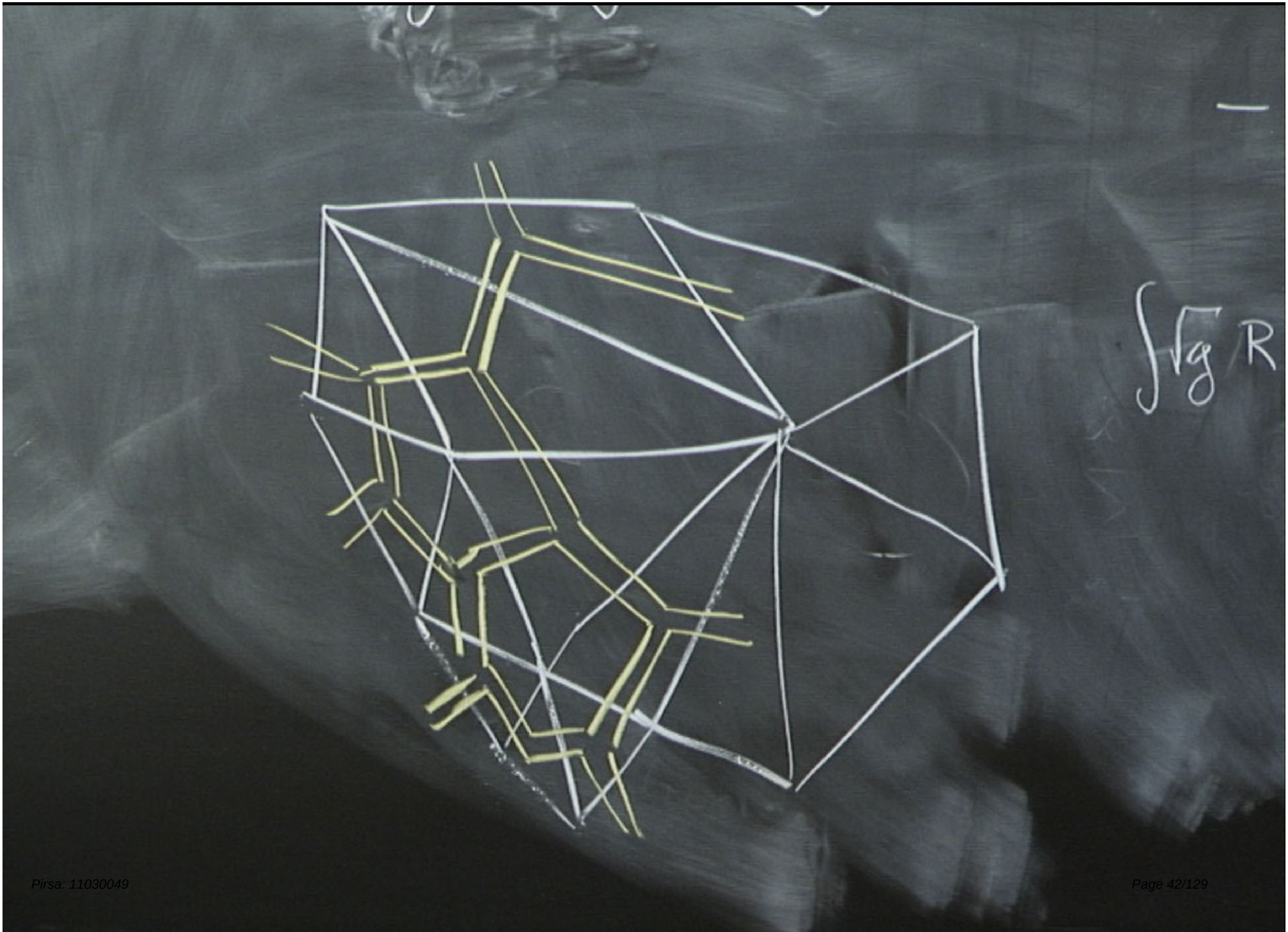
the free parameter

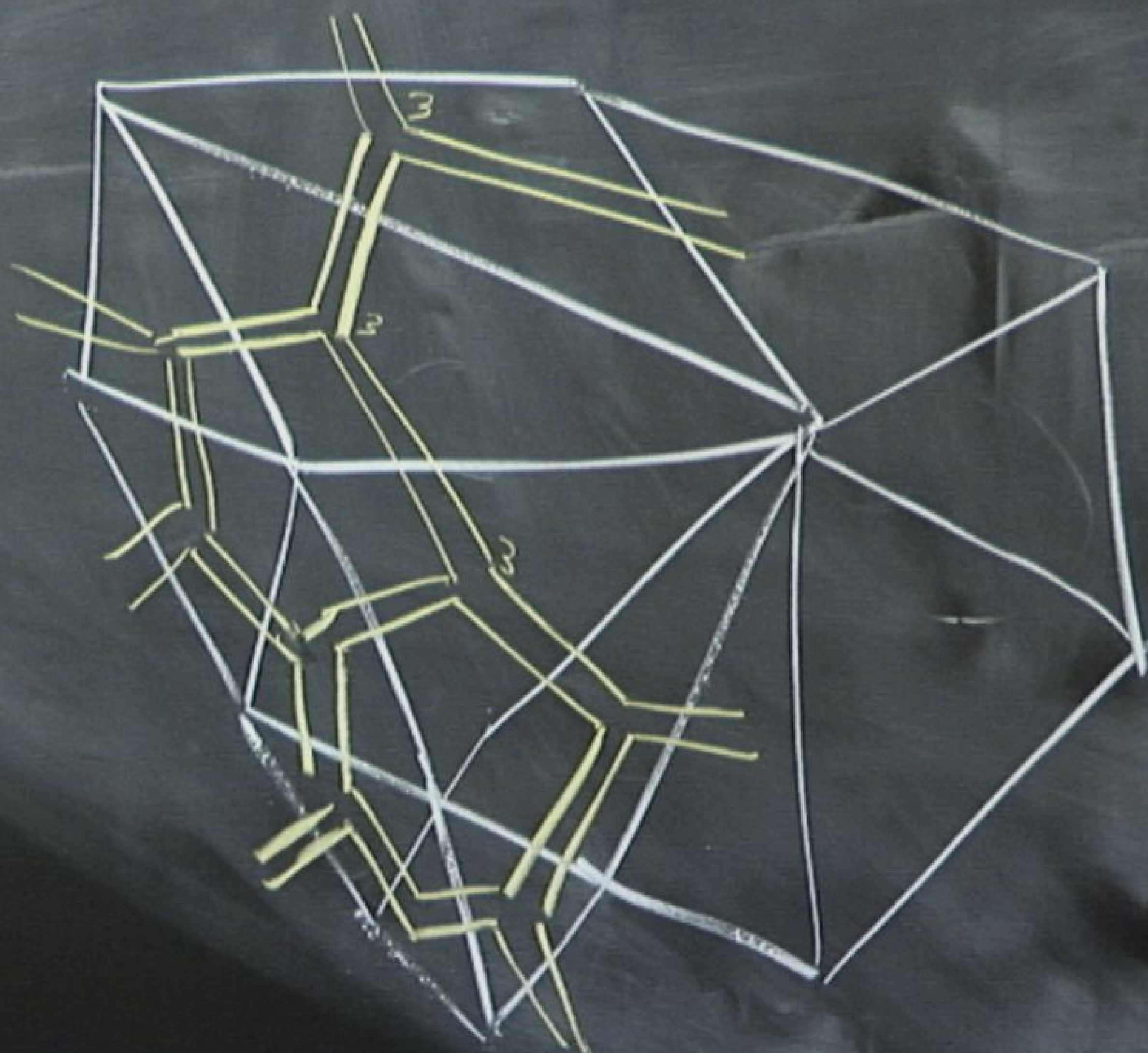
$$Z[\beta,\gamma] = \sum_{\text{genus}} \int \mathcal{D}g \; \exp\left(-\beta A + \gamma \chi\right)$$

$$-\chi = 2\,\text{genus} - 2$$



$$\int \sqrt{g}\, R \to \sum_{i=1} 4\pi \left(1\right.$$


$\int \sqrt{g}\, R$



$\int \sqrt{g}\, R$

$$e^{Z} = \int \mathcal{D}M \, e^{-\frac{1}{2}\mathrm{Tr}\,M^2 + \frac{g}{\sqrt{N}}\mathrm{tr}\,M^3}$$

$\langle OO\rangle$

$\parallel$

$N \equiv e^{\gamma}$

$g \equiv e^{-\beta}$

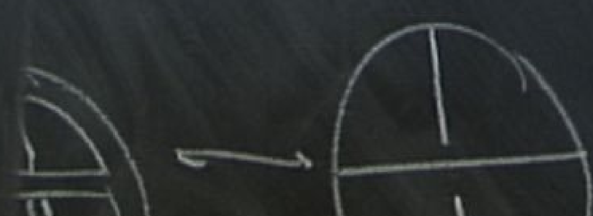
EXERCISE:
CHECK THE
DETAILS

$\sum$ 

1 stri

$(N^2)^{\sim 1}$

$\langle OOOO$



se

$$e^{Z} = \int \mathcal{D}M \, e^{-\frac{1}{2}\mathrm{Tr}\,M^{2} + \frac{g}{\sqrt{N}}\mathrm{tr}\,M^{3}}$$

$\langle OO \rangle$

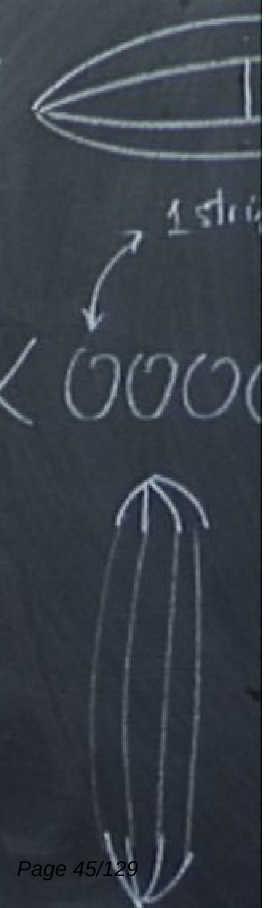
$=$

$$N \equiv e^{\gamma}$$
$$g \equiv e^{-\beta}$$

EXERCISE: CHECK THE DETAILS

$$Z = N^{2} Z_{0}(g) + Z_{1}(g) + \chi_{2}(g)/N^{2} + \ldots$$

$\sum$



1 stri

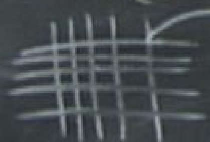
$\langle OOOO$

2D Ising Model on a random surface

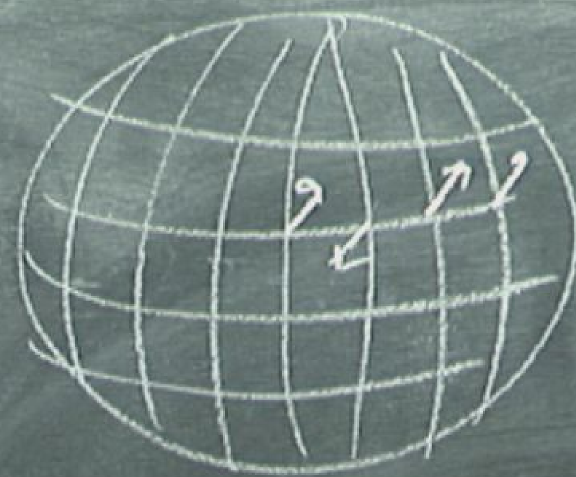
$$\mathcal{Z}_{\text{ising model in a fixed lattice}} = \sum_{\{\sigma_a = \pm 1\}} e^{+\beta \frac{J}{2} \sum_{\langle ab \rangle} (\sigma_a \sigma_b - 1) - \beta H \sum_a \sigma_a}$$

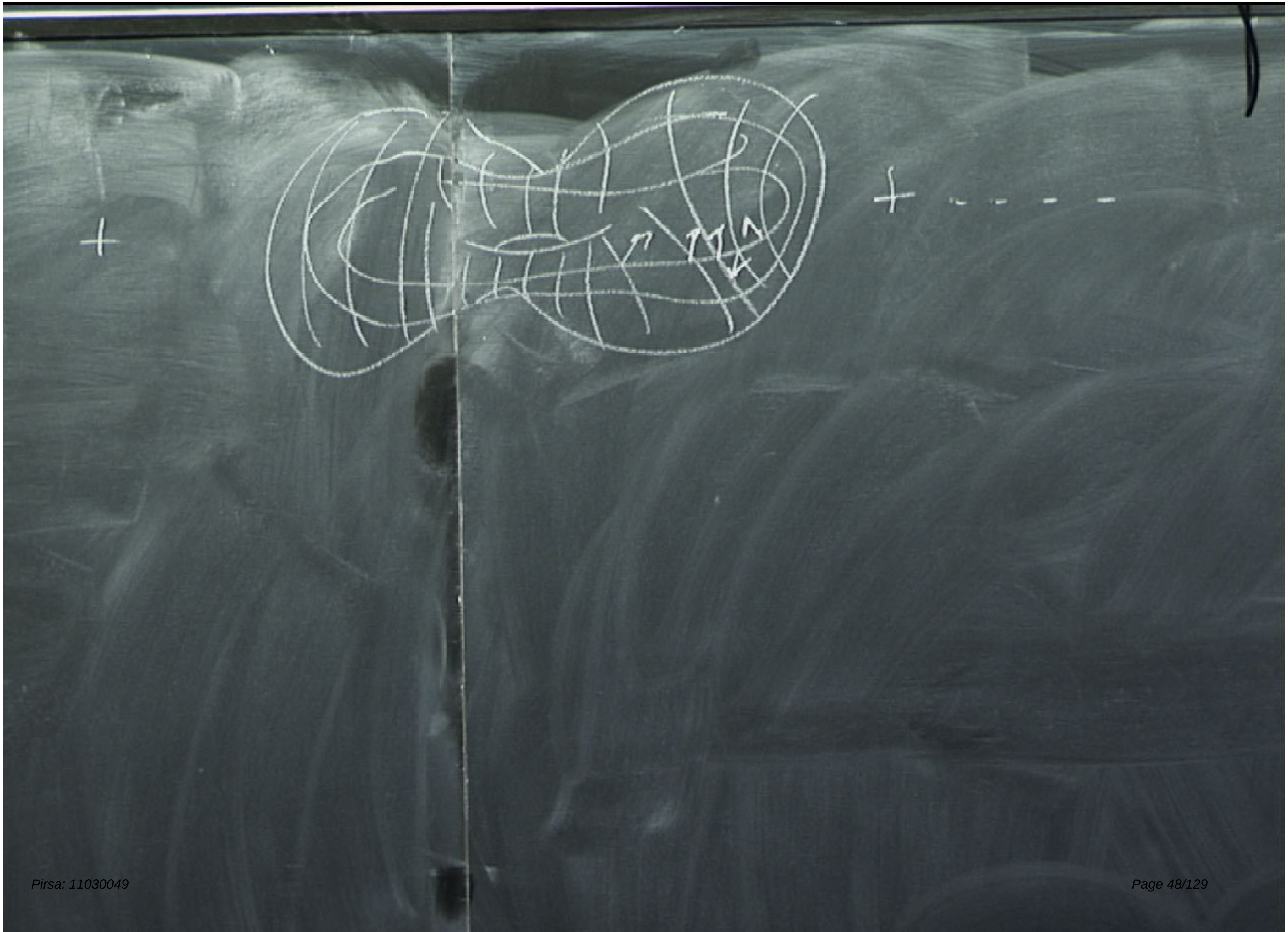
spin coupling

magnetic field



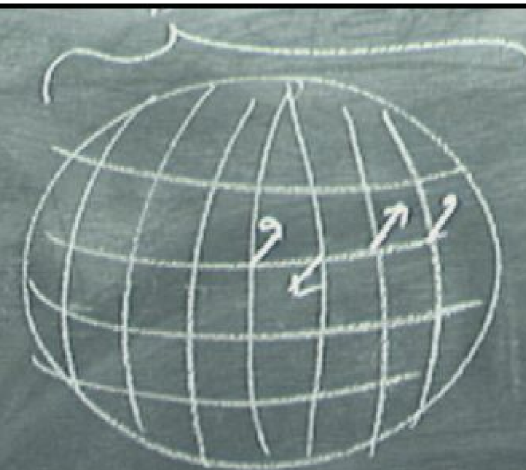
$\sigma = \pm 1$

$Z_{\text{ising model on a random lattice}} = \sum_{\uparrow\downarrow}$  $+$

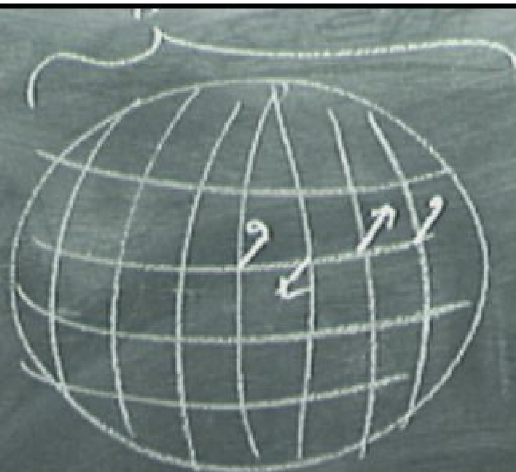


$Z_{\text{ising model on a random lattice}} = \sum_{\uparrow\downarrow}$  $+$ 

$$= \sum_{g} N^{2-2g}$$

Z ising model on a random lattice $= \sum_{\uparrow\downarrow}$  $+$ 

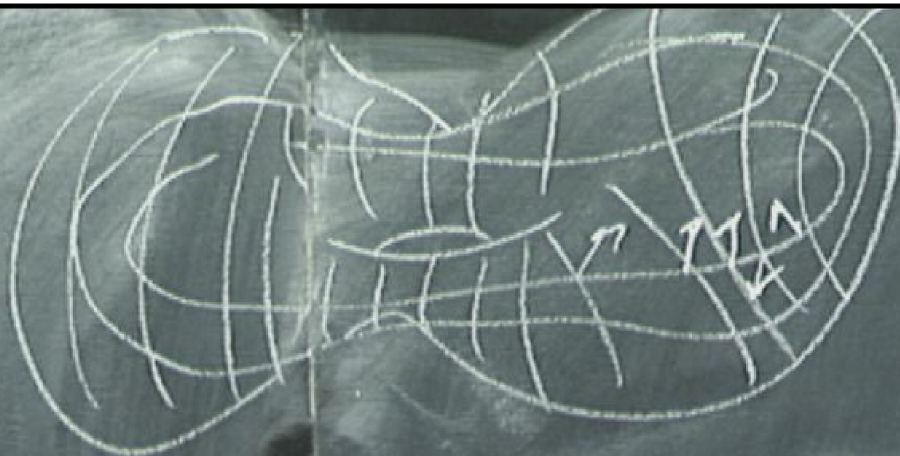
$$= \sum_{g} N^{2-2g} \sum_{n=1}^{\infty} e^{-\mu n}$$

$$= \sum_{\uparrow\downarrow}$$  $$+$$ 

$$N^{2-2g} \sum_{n=1}^{\infty} e^{-\mu n \beta} \sum_{\Sigma_{n,g}}$$

surfaces

$+$



$+ \ldots\ldots$

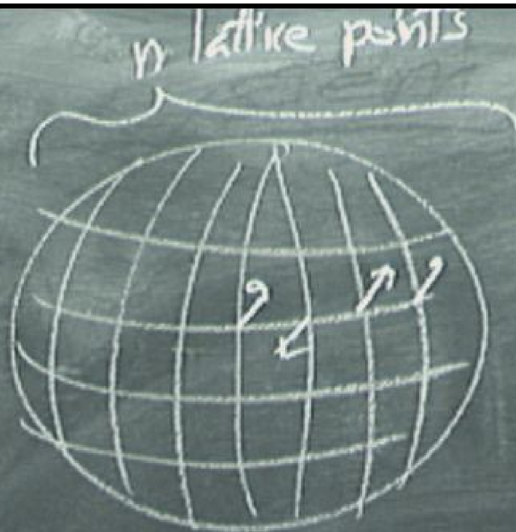
$$\sum_{\Sigma_{n,g}} \sum_{\{\sigma_a = \pm 1\}} e^{-\frac{\beta}{2} J \sum_{\langle ab \rangle} (1 - \sigma_a \sigma_b) - \beta H \sum_a}$$

surfaces

$+ \ldots$

$$e^{-\frac{\beta}{2} J \sum_{\langle ab \rangle} (1 - \sigma_a \sigma_b) - \beta H \sum_a \sigma_a}$$

Z ising model on a random lattice $= \sum_{\uparrow\downarrow}$

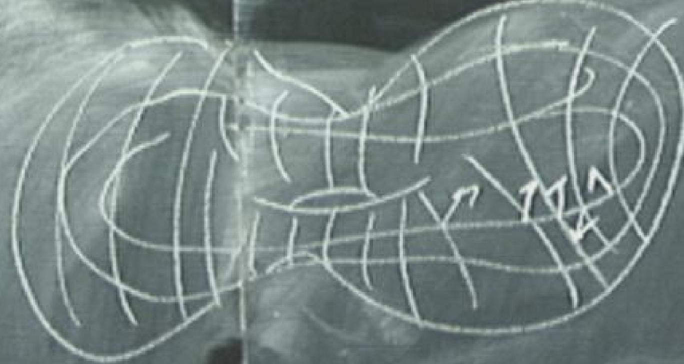


$+$

$$Z \equiv \sum_{g} N^{2-2g} \sum_{n=1}^{\infty} e^{-\mu n \beta} \sum_{\substack{\Sigma_{n,g} \\ \text{surfaces}}} \sum_{\{\sigma_a = \pm 1\}}$$

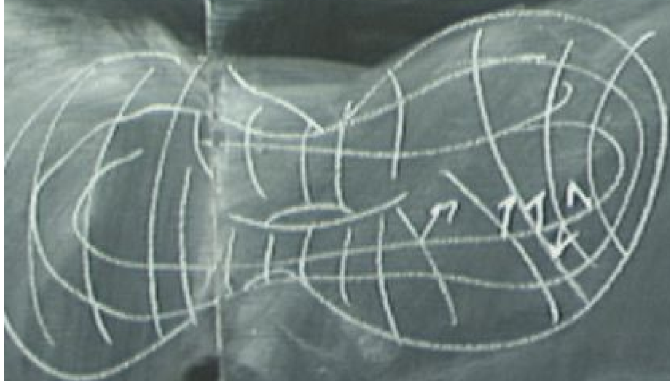
<u>claim</u>

$$e^{Z} = \int \mathcal{D}A\,\mathcal{D}B \exp\left(-\mathrm{Tr}\left[\frac{A^2 + B^2}{\quad}\right.\right.$$

$+$  $+ \cdots$

$$n\beta \quad \sum_{\substack{\Sigma_{n,g} \\ \text{(surfaces)}}} \ \sum_{\{\sigma_a = \pm 1\}} e^{-\frac{\beta}{2}J\sum_{\langle ab\rangle}(1-\sigma_a\sigma_b) - \beta H \sum_a \sigma_a}$$

$$\Big(-\mathrm{Tr}\Big[\frac{A^2 + B^2 - 2e^{-\beta J}AB}{1-e^{-2\beta J}} - \frac{e^{-\beta\mu}}{N}\mathrm{Tr}\Big(e^{-\beta H}A^4 + e^{\beta H}$$



$+ \cdots$

$$\sum_{\{\sigma_a = \pm 1\}} e^{-\frac{\beta}{2} J \sum_{\langle ab \rangle} (1 - \sigma_a \sigma_b) - \beta H \sum_a \sigma_a}$$

$$\frac{\cdots + B^2 - 2e^{-\beta J} AB}{1 - e^{-2\beta J}} - \frac{e^{-\beta\mu}}{N} \mathrm{Tr}\left(e^{-\beta H} A^4 + e^{\beta H} B^4 \right) \Big]$$

$+ \cdots\cdots$

$$e^{-\frac{\beta}{2} J \sum_{\langle ab \rangle} (1-\sigma_a \sigma_b) - \beta H \sum_a \sigma_a}$$

$$\frac{-2e^{-\beta J} AB}{1-e^{-2\beta J}} - \frac{e^{-\beta \mu}}{N} \mathrm{Tr}\left(e^{-\beta H} A^4 + e^{\beta H} B^4\right)\Big]\Big)$$

n lattice points

Z ising model on a random lattice with coordination number 4 $= \sum_{\uparrow\downarrow}$  $+$ 

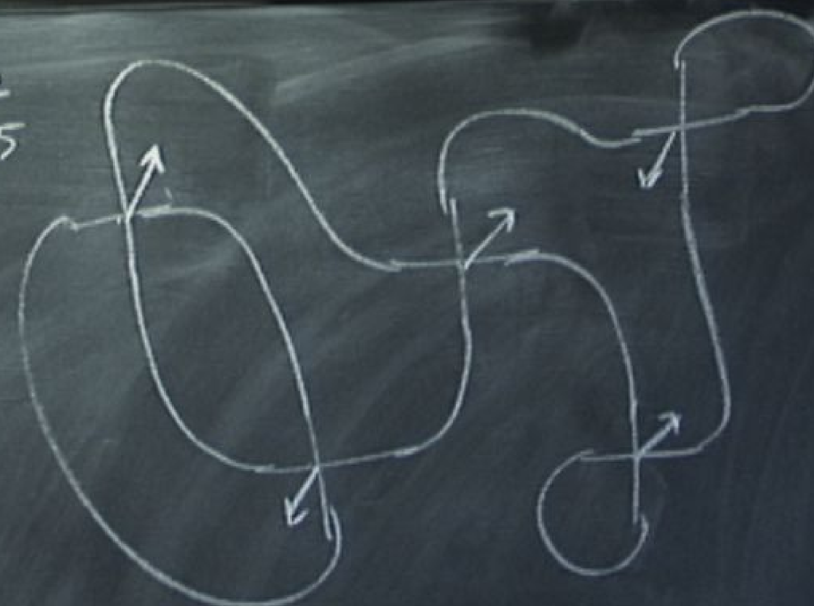
$$Z \equiv \sum_{g=0}^{\infty} N^{2-2g} \sum_{n=1}^{\infty} e^{-\mu n \beta} \sum_{\substack{\Sigma_{n,g} \\ \text{surfaces}}} \sum_{\{\sigma_a = \pm 1\}} e^{-\ldots}$$

claim

$$e^{Z} = \int \mathcal{D}A\,\mathcal{D}B \exp\Big(-\mathrm{Tr}\Big[\frac{A^2+B^2 - 2e\ldots}{1-e^{-\ldots}}$$

$+ \cdots$

example for n=5



$$e^{-\frac{\beta}{2}J\sum_{\langle ab\rangle}(1-\sigma_a\sigma_b) - \beta H\sum_a \sigma_a}$$

$$\frac{-2e^{-\beta J}AB}{-e^{-2\beta J}} - \frac{e^{-\beta\mu}}{N}\mathrm{Tr}\left(e^{-\beta H}A^4 + e^{\beta H}B^4\right)\Big]\Big)$$

2D Ising Model on a random surface

A ═══ A

B ═══ B

A ═══ B

2D Ising Model on a random surface

$$A \;=\!=\; A \;= 1$$

$$B \;=\!=\; B \;= 1$$

$$A \;=\!=\; B \;= e^{-\beta J}$$

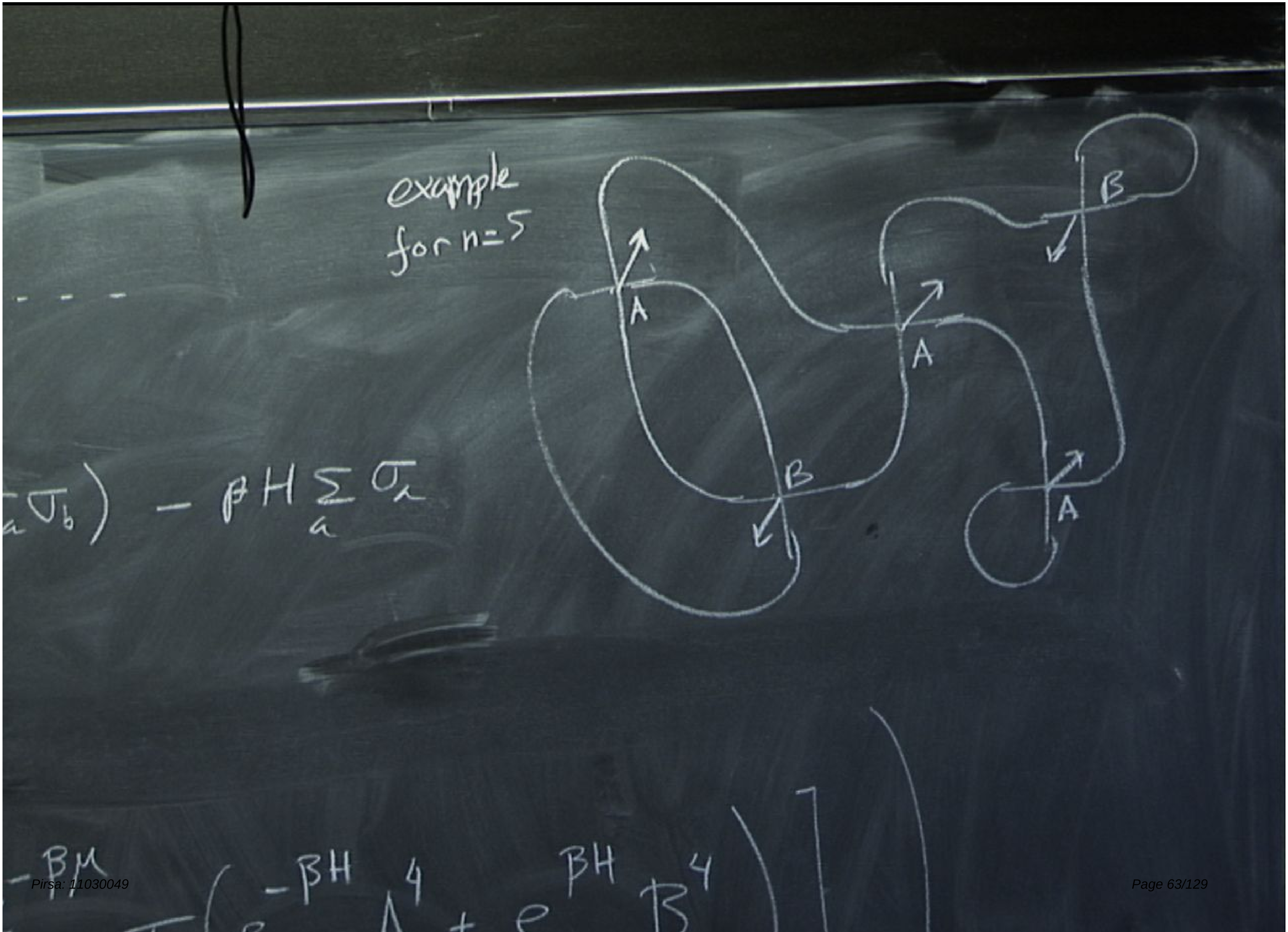
2D Ising Model on a random surface

$$A = \dot{A} = 1$$

$$B = B = 1$$

$$A = B = e^{-\beta J}$$

this is why we chose that kin. term


example
for n=5
A
B
A
B
A
$-\beta H \sum_a \sigma_a$

Ising Model on a random surface

idea $A \longleftrightarrow \uparrow$

$B \longleftrightarrow \downarrow$

$$A \;=\!=\!=\; A = 1$$

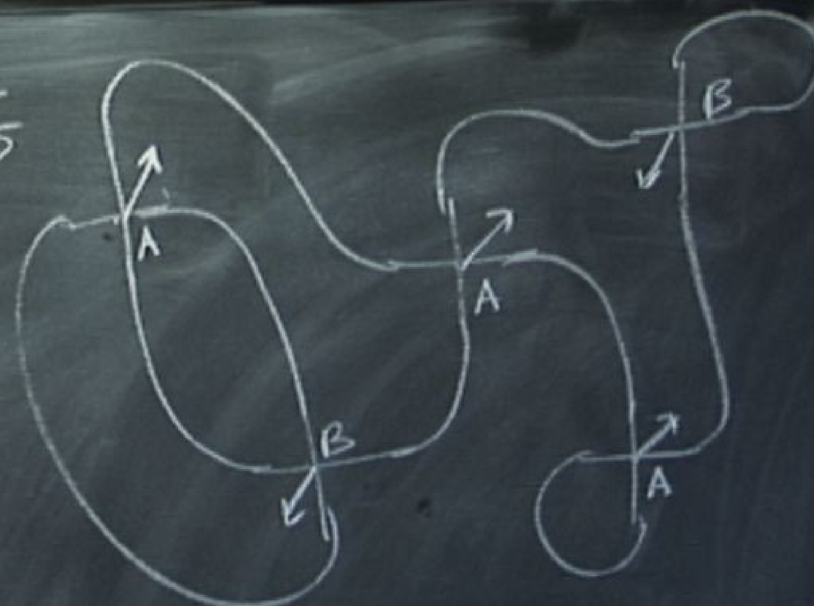
$$B \;=\!=\!=\; B = 1$$

$$A \;=\!=\!=\; B = e^{-\beta J}$$

this is why we chose that kin. term

$+ \cdots$

example for n=5



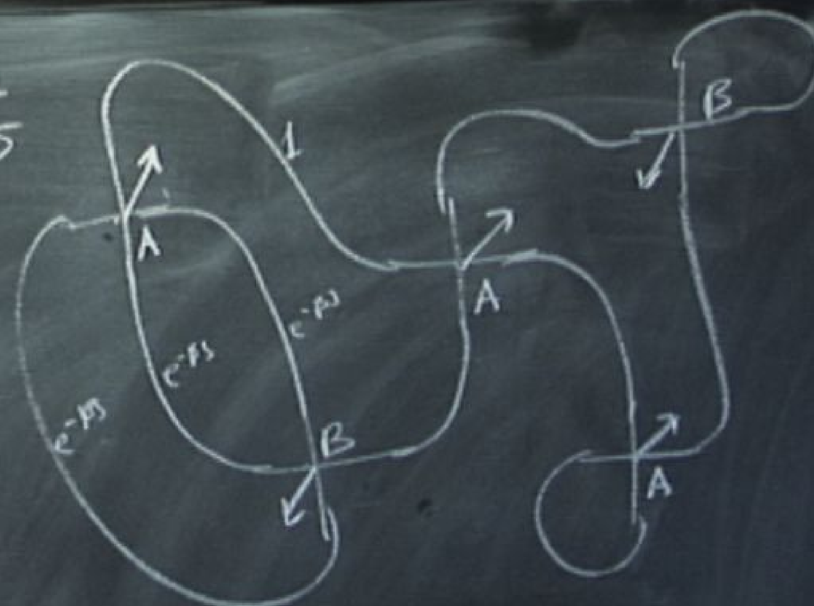
$$\underbrace{e^{-\frac{\beta}{2} J \sum_{\langle ab\rangle}(1-\sigma_a\sigma_b) - \beta H \sum_a \sigma_a}}$$

$e^{-\beta J}$ for each ↙ connected to ↗

$$- \frac{2e^{-\beta J} AB}{1-e^{-2\beta J}} - \frac{e^{-\beta\mu}}{N} \mathrm{Tr}\left(e^{-\beta H} A^4 + e^{\beta H} B^4\right)\Big)\Big]\Big)$$

$+ \cdots$

example for $n=5$



$$e^{\underbrace{-\frac{\beta}{2}J\sum_{\langle ab\rangle}(1-\sigma_a\sigma_b)}_{e^{-\beta J}\text{ for each }\swarrow\text{ connected to }\nearrow} - \beta H\sum_a \sigma_a}$$

$$-\frac{2e^{-\beta J}AB}{1-e^{-2\beta J}} - \frac{e^{-\beta\mu}}{N}\mathrm{Tr}\left(e^{-\beta H}A^4 + e^{\beta H}B^4\right)\Big)\Big]\Big)$$

on a random surface

idea $A \longleftrightarrow$

$B \longleftrightarrow$

$= \dot{A} = 1$

$= B = 1$

$= B = e^{-\beta J}$

this is why we chose that kin. term

$+ \; e^{-\beta\mu} \; e^{-\beta H}$

random surface

$= 1$

$B \quad = 1$

$B \quad = e^{-\beta J}$

this is why we chose that kin. term

idea $A \longleftrightarrow \nearrow$

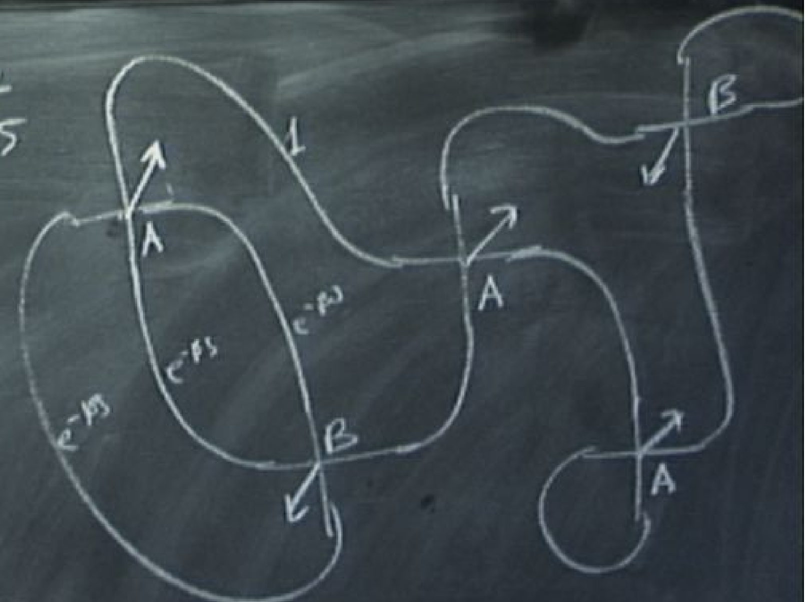
$B \longleftrightarrow \swarrow$

$A + \quad e^{-\beta\mu} \quad e^{-\beta H}$

$B + \quad e^{-\beta\mu} \quad e^{+\beta H}$

 $+ \cdots\cdots$

example for $n=5$

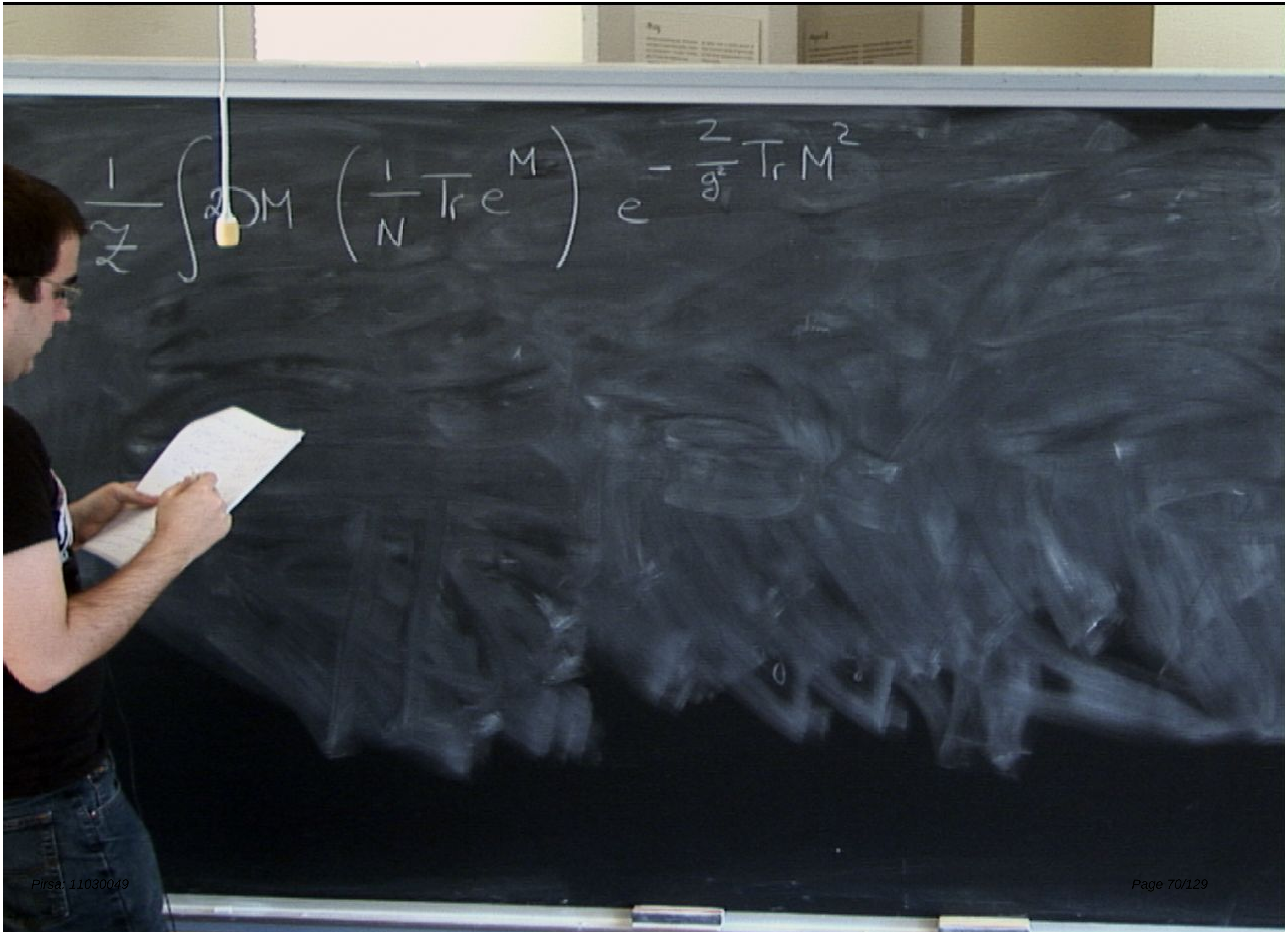


$\pm 1\}$

$$e^{\underbrace{-\frac{\beta}{2}J\sum_{\langle ab\rangle}\left(1-\sigma_a\sigma_b\right)}_{e^{-\beta J}\text{ for each }\swarrow\text{ connected to }\nearrow}\;\underbrace{-\;\beta H\sum_a \sigma_a}_{\nearrow\, e^{-\beta H},\ \swarrow\, e^{+\beta H}}}$$

$$\cdots B^2 - 2e^{-\beta J}AB \over 1-e^{-2\beta J}$$

$$-\;\frac{e^{-\beta\mu}}{N}\,\mathrm{Tr}\left(e^{-\beta H}A^4 + e^{\beta H}B^4\right)\Big]\Big)$$


$\frac{1}{Z}\int \mathcal{D}M \left(\frac{1}{N}\mathrm{Tr}\, e^{M}\right) e^{-\frac{2}{g^2}\mathrm{Tr}\, M^2}$

$$\frac{1}{Z}\int \mathcal{D}M \left(\frac{1}{N}\operatorname{Tr} e^{M}\right) e^{-\frac{2}{g^2}\operatorname{Tr} M^2}$$

N×N matrices

$$\int \mathcal{D}M \left(\frac{1}{N}\mathrm{Tr}\, e^{M}\right) e^{-\frac{2}{g^2}\mathrm{Tr}\, M^2} = \frac{1}{N}\underbrace{L^{1}_{N-1}}_{\text{Laguerre Polynomial}}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$$

$N \times N$ matrices

$$\lambda = g^2 N$$

$$\int \mathcal{D}M \left(\frac{1}{N}\mathrm{Tr}\, e^{M}\right) e^{-\frac{2}{g^2}\mathrm{Tr}\, M^2} = \frac{1}{N}\underbrace{L^{1}_{N-1}}_{\text{Laguerre Polynomial}}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$$

$N \times N$ matrices

$\lambda = g^2 N$

$$= \underbrace{\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})}_{\text{all planar diagrams!}} + \frac{\lambda}{8\pi N^2}\overbrace{I_2(\sqrt{\lambda})}^{\text{all torus diags!}} + \cdots$$

Bessel functions

$$\int \mathcal{D}M \left(\frac{1}{N}\mathrm{Tr}\, e^{M}\right) e^{-\frac{2}{g^2}\mathrm{Tr}\, M^2} = \frac{1}{N}\underbrace{L^{1}_{N-1}}_{\text{Laguerre Polynomial}}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$$

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Bessel functions

$$L^{m}_{n} = \frac{e^{x}}{n!} x^{-m} \frac{d^{n}}{dx^{n}}\left(e^{-x} x^{n+m}\right)$$

DERIVATION

$M \to U M U^{-1}$ huge symmetry

DERIVATION

$M \rightarrow U M U^{-1}$ huge symmetry

eigenvalues of M not transformed

DERIVATION

$M \longrightarrow UMU^{-1}$ huge symmetry

eigenvalues of M not transformed

$$\underbrace{M}_{N^2\text{ variables}} \longrightarrow \underbrace{\{\lambda_i\}}_{N\text{ variables}}$$

$$S[M] = e^{-(\mathrm{Tr}\, M^2)\frac{1}{2g^2}} = e^{-\frac{1}{2g^2}\sum_i \lambda_i^2}$$

DERIVATION

$M \to U M U^{-1}$ huge symmetry

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$$\underbrace{M}_{N^2 \text{ variables}} \longrightarrow \underbrace{\{\lambda_i\}}_{N \text{ variables}}$$

$$S[M] = e^{-(\mathrm{Tr}\, M^2)\frac{1}{2g^2}} = e^{-\frac{1}{2g^2}\sum_i \lambda_i^2}$$

$$U\begin{pmatrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{pmatrix}U^{-1}U\begin{pmatrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{pmatrix}U^{-1}$$

in #

$$\mathcal{D}M \longrightarrow d\lambda_1 \ldots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

in #

$$\mathcal{D}M \to d\lambda_1 \dots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

$$\int \mathcal{D}M \, f(M)$$

in #

$$\mathcal{D}M \longrightarrow d\lambda_1 \ldots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

$$\int \mathcal{D}M \, f(M)$$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}(UMU^+ - \Lambda)$$

in #

$$\mathcal{D}M \longrightarrow d\lambda_1 \cdots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

$$\int \mathcal{D}M\, f(M)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N \lambda_i\, dU\, \delta^{(N^2)}\left(UMU^{+} - \Lambda\right) \Delta^2(\{\lambda\})$$

in #

$$\mathcal{D}M \longrightarrow d\lambda_1 \cdots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$\int \mathcal{D}M \; f(M) \; 1$$

$$1 \equiv \int d^N \lambda_i \; dU \; \delta^{(N^2)}\left(UMU^{+} - \Lambda\right) \Delta^2(\{\lambda\})$$

def of Δ^2

in #

$$\mathcal{D}M \longrightarrow d\lambda_1 \cdots d\lambda_N \ \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

$$\int \mathcal{D}M \, f(M)\, 1 = \int dU \int \mathcal{D}\Pi \int d^N\lambda \, f(\lambda)\, \Delta^2(\lambda)\, \delta^{(N^2)}(U\Pi U^{-1} -$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N\lambda_i \, dU \ \delta^{(N^2)}(UMU^{+} - \Lambda)\, \Delta^2(\{\lambda\})$$

def of Δ^2

in #

$$\mathcal{D}M \to d\lambda_1 \dots d\lambda_N \; \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

$$\int \mathcal{D}M \; f(M)\; 1 = \int dU \int \mathcal{D}M \int d^N\lambda \; f(\lambda)\, \Delta^2(\lambda)\, \delta^{(N^2)}\big(\overbrace{UMU^{-1}}^{\tilde{M}} - \Lambda\big)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N \lambda_i \; dU \;\; \delta^{(N^2)}\left(UMU^{+} - \Lambda\right) \Delta^2(\{\lambda\})$$

def of Δ^2

in #

$$\mathcal{D}M \to \underset{\text{Jacobian}}{d\lambda_1 \dots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}}$$

(no measure!) $\mathcal{D}M \to \mathcal{D}\tilde{M}$

$$\int \mathcal{D}M\, f(M)\, 1 = \int dU \int \mathcal{D}M \int d^N\lambda\, f(\lambda)\, \Delta^2(\lambda)\, \delta^{(N^2)}\big(\overbrace{UMU^{-1}}^{\tilde{M}} - \Lambda\big)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N\lambda_i\, dU\, \delta^{(N^2)}(UMU^\dagger - \Lambda)\, \Delta^2(\{\lambda\})$$

def of Δ^2

in #

$$\mathcal{D}M \longrightarrow d\lambda_1 \dots d\lambda_N \; \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

(no measure!) $\mathcal{D}M \to \mathcal{D}\tilde{M}$

$$\int \mathcal{D}M \, f(M) \, 1 = \int dU \int \mathcal{D}M \int d^N\lambda \, f(\lambda) \, \Delta^2(\lambda) \, \delta^{(N^2)}\Big(\overbrace{UMU^{-1}}^{\tilde{M}} - \Lambda\Big)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N\lambda_i \, dU \; \delta^{(N^2)}\left(UMU^{\dagger} - \Lambda\right) \Delta^2(\{\lambda\})$$

def of Δ^2

$$\text{Tr} = \underbrace{\left(\int dU\right)}_{1} \int d^N\lambda \, f(\lambda) \, \Delta^2(\{\lambda\})$$

in #

$\mathcal{D}M \longrightarrow d\lambda_1 \ldots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$

(no measure!)

$\mathcal{D}M \to \mathcal{D}\tilde{M}$

$$\int \mathcal{D}M\, f(M)\, 1 = \int dU \int \mathcal{D}M \int d^4\lambda\, f(\lambda)\, \Delta^2(\lambda)\, \delta^{(N^2)}\big(\overbrace{UMU^{-1}}^{\tilde{M}} - \Lambda\big)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N\lambda_i\, dU\ \delta^{(N^2)}(UMU^+ - \Lambda)\, \Delta^2(\{\lambda\})$$

def of Δ^2

$$= \underbrace{\left(\int dU\right)}_{1} \int d^4\lambda\, f(\lambda)\, \Delta^2(\{\lambda\})$$

in #

$\mathcal{D}M \to d\lambda_1 \dots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$

(no measure!)

$\mathcal{D}M \to \mathcal{D}\tilde{M}$

$$\int \mathcal{D}M\, f(M)\, 1 = \int dU \int \mathcal{D}M \int d^4\lambda\, f(\lambda)\, \Delta^2(\lambda)\, \delta^{(N^2)}(\underbrace{UMU^{-1}}_{\tilde{M}} - \Lambda)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N \lambda_i\, dU\, \delta^{(N^2)}(UMU^+ - \Lambda)\, \Delta^2(\{\lambda\})$$

def of Δ^2

$$T_1 = \underbrace{\left(\int dU\right)}_{1} \int d^4\lambda\, f(\lambda)\, \Delta^2(\{\lambda\})$$

next use to compute Δ

in #

$$\mathcal{D}M \longrightarrow d\lambda_1 \ldots d\lambda_N \underbrace{\Delta^2(\{\lambda\})}_{\text{Jacobian}}$$

(no measure!)

$$\mathcal{D}M \to \mathcal{D}\tilde{M}$$

$$\int \mathcal{D}M\, f(M)\, 1 = \int dU \int \mathcal{D}M \int d^N\lambda\, f(\lambda)\, \Delta^2(\lambda)\, \delta^{(N^2)}\big(\overbrace{UMU^{-1}}^{\tilde{M}} - \Lambda\big)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

$$1 \equiv \int d^N\lambda_i\, dU\, \delta^{(N^2)}\left(UMU^{+} - \Lambda\right) \Delta^2(\{\lambda\})$$

def of Δ^2

$$= \underbrace{\left(\int dU\right)}_{1} \int d^N\lambda\, f(\lambda)\, \Delta^2(\{\lambda\})$$

next. use to compute Δ

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}\left(UMU^{-1} - \Lambda\right)\Delta^2(\lambda)$$

for a generic $M = U_0^{-1}\tilde{\Lambda}U_0$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}\left(UMU^{-1} - \Lambda \right) \Delta^2(\lambda)$$

for a generic $M = U_0^{-1} \underbrace{\tilde{\Lambda}}_{\begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_N' \end{pmatrix}} U_0$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}\left(UMU^{-1} - \Lambda\right)\Delta^2(\lambda)$$

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U will <u>not</u>

give

$$UMU^{+} - \Lambda = 0 \; !$$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}\left(UMU^{-1} - \Lambda\right) \Delta^2(\lambda)$$

for a generic $M = U_0^{-1} \underbrace{\Lambda}_{\begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_N' \end{pmatrix}} U_0$

U will not give $UMU^{+} - \Lambda = 0$!

only if $U \simeq U_0$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}(UMU^{-1} - \Lambda)\, \Delta^2(\lambda)$$

for a generic $M = U_0^{-1} \underbrace{\tilde{\Lambda}}_{\begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_N' \end{pmatrix}} U_0$

U will not

give $UMU^+ - \Lambda = 0$!

only if $U \simeq U_0$

$U = (1 + T)U_0$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}(UMU^{-1} - \Lambda)\, \Delta^2(\lambda)$$

for a generic $M = U_0^{-1} \underbrace{\tilde{\Lambda}}_{\begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_N' \end{pmatrix}} U_0$

U will <u>not</u>

give $UMU^{+} - \Lambda = 0$!

only if $U \simeq U_0$

only $U = (1+T)U_0$ matter

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}\left(UMU^{-1} - \Lambda\right)\Delta^2(\lambda)$$

for a generic $M = U_0^{-1} \underbrace{\Lambda}_{\begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_N' \end{pmatrix}} U_0$

U will <u>not</u> give $UMU^{+} - \Lambda = 0$! $U^{+} = U^{-1}$

only if $U \simeq U_0$

only $U = (1 + T)U_0$ matter

$UMU^{-1} = \Lambda' + [T, \Lambda]$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}(UMU^{-1} - \Lambda)\, \Delta^2(\lambda)$$

for a generic $M = U_0^{-1} \underbrace{\tilde{\Lambda}}\, U_0$

$$\begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_N' \end{pmatrix}$$

U will <u>not</u>

give $UMU^{+} - \Lambda = 0$! $U^{+} = U^{-1}$

only if $U \simeq U_0$

only $U = (1 + T)U_0$ matter

$UMU^{-1} = \tilde{\Lambda} + [T, \tilde{\Lambda}]$

$$1 = \int d^N\lambda_i \, dU \, \delta^{(N^2)}\left(UMU^{-1} - \Lambda\right) \Delta^2(\lambda)$$

for a generic $M = U_0^{-1} \underbrace{\tilde{\Lambda}}_{\begin{pmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_N' \end{pmatrix}} U_0$

U will <u>not</u> give $UMU^{+} - \Lambda = 0$! $U^{+} = U^{-1}$

only if $U \simeq U_0$

only $U = (1+T)U_0$ matter

$UMU^{-1} = \tilde{\Lambda} + [T, \tilde{\Lambda}]$

$$\delta^{(\lambda^2)}(\ldots) = \delta(\Lambda - \Lambda')\,\delta([T, \bar{\Lambda}'])$$

$$1 = \int d^N\lambda \prod_{i<j} dT_{ij} \left[\delta^{(2)}(\ldots) = \delta(\Lambda - \Lambda') \, \delta([T, \Lambda']) \right] \Delta^2(\lambda)$$

$\delta(\mathrm{Re}(\ldots))\,\delta(\mathrm{Im}(\ldots))$

$$1 = \Delta^2(\lambda') \int \underbrace{\prod_{i<j} dT_{ij}}_{d\mathrm{Re}T_{ij}\, d\mathrm{Im}T_{ij}} \prod_{i<j} \delta^{(2)}\left(T_{ij}(\lambda_j - \lambda_i)\right), \quad 1 = \frac{\Delta^2(\lambda')}{\prod_{i<j}(\lambda'_i - \lambda'_j)^2}$$

$$\boxed{\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)}$$

Vandermonde Det

$$\int \mathcal{D}M \, f(M) = \int d^N\lambda \, \Delta^2(\lambda) \, f(\lambda)$$

$$N^2 \longrightarrow N \quad !$$

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$$= \det \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & & \lambda_2^{N-1} \\ 1 & \lambda_3 & \lambda_3^2 & \cdots & \lambda_3^{N-1} \\ \vdots & \vdots & \vdots & & \\ 1 & \lambda_N & \lambda_N^2 & & \lambda_N^{N-1} \end{vmatrix}$$

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in

$\mathcal{D}$

$\int$

1

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Ex: $N=3$

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Ex: $N=3$

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PROOF: LHS and RHS are both Polynomials with the same degree, zeros and large λ_i behavior. Hence they are the same

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$\lambda_1 \to \infty$ LHS

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$\lambda_1 \to \infty$ LHS $\lambda_1^{N-1} \underbrace{\prod_{\substack{i<j \\ i>1}} (\lambda_i - \lambda_j)}_{\Delta_{N-1}(\lambda_2 \cdots \lambda_N)}$

$$\prod_{i<j}(\lambda_i - \lambda_j) = \det \lambda_i^{j-1}$$

PROOF: LHS and RHS are both Polynomials with the same degree, zeros and large λ_i behavior. Hence they are the same

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$$\lambda_1 \to \infty \qquad \text{LHS} \quad \lambda_1^{N-1} \underbrace{\prod_{\substack{i<j \\ i>1}} (\lambda_i - \lambda_j)}_{\Delta_{N-1}(\lambda_2 \ldots \lambda_N)} \qquad \text{QED}$$

$$\text{RHS} \quad \lambda_1^{N-1} \ldots = \lambda_1^{N-1} \Delta_{N-1}(\lambda_2 \ldots \lambda_N)$$

$$\left\langle \frac{1}{N} \mathrm{Tr}\, e^{M} \right\rangle = \frac{1}{Z} \int \prod_{i=1}^{N} d\lambda_i \left(\det_{i,j} \lambda_i^{j-1} \right)^2 \underbrace{e^{\lambda_1}}_{\frac{1}{N} \sum_i e^{\lambda_i}} e^{-\frac{2}{g^2} \sum_{i=1}^{N} \lambda_i^2}$$

GOAL : $$\left\langle \frac{1}{N}\mathrm{tr}\left(e^{M}\right) \right\rangle = \frac{1}{Z}\int \prod_{i=1}^{N} d\lambda_i \left(\det_{i,j} \lambda_i^{\,j-1}\right)^2 \underbrace{e^{\lambda_1}}_{\frac{1}{N}\sum_i e^{\lambda_i}} e^{-\frac{2}{g^2}\sum_{i=1}^{N}\lambda_i^2}$$

$$\rangle = \frac{1}{Z}\int \prod_{i=1}^{N} d\lambda_i \left(\det_{i,j} \lambda_i^{j-1}\right)^2 \underbrace{e^{\lambda_1}}_{\frac{1}{N}\sum_i e^{\lambda_i}} \; e^{-\frac{2}{g^2}\sum_{i=1}^{N}\lambda_i^2}$$

$$= \frac{1}{Z}\int \prod^{N} d\lambda_i \left(\det \lambda_i^{j-1}\right)^2 e^{\frac{g}{\sqrt{2}}\lambda_1} \; e^{-\sum_{i=1}^{N}\lambda_i^2}$$

$$\int \mathcal{D}M \; f(M) = \int d^N\lambda \; \Delta^2(\lambda) \, f(\lambda)$$

$$N^2 \longrightarrow N \; !$$

we can replace

$\lambda_i^{j-1} \to P_{j-1}(\lambda_i)$

$= \lambda_i^{j-1} + \# \lambda_i^{j-2} + \cdots$

$$\Delta(\lambda) = \det_{i,j} \lambda_i^{j-1}$$

$$= \prod_{i<j} (\lambda_i - \lambda_j)$$

$$= \det \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{j-1} & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \vdots & & \lambda_2^{N-1} \\ 1 & \lambda_3 & \lambda_3^2 & & \vdots & & \lambda_3^{N-1} \\ \vdots & \vdots & \vdots & & \vdots & & \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{j-1} & \cdots & \lambda_N^{N-1} \end{vmatrix}$$

Ex: $N=3$

$\prod_{i<j} (\lambda$

PROOF

deg

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$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 + \alpha\lambda_1 + \beta \\ 1 & \lambda_2 & \lambda_2^2 + \alpha\lambda_2 + \beta \\ 1 & \lambda_3 & \lambda_3^2 + \alpha\lambda_3 + \beta \end{vmatrix}$$

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$$\lambda_i^{j-1} \to P_{j-1}(\lambda_i) = \lambda_i^{j-1} + \# \lambda_i^{j-2} + \cdots$$

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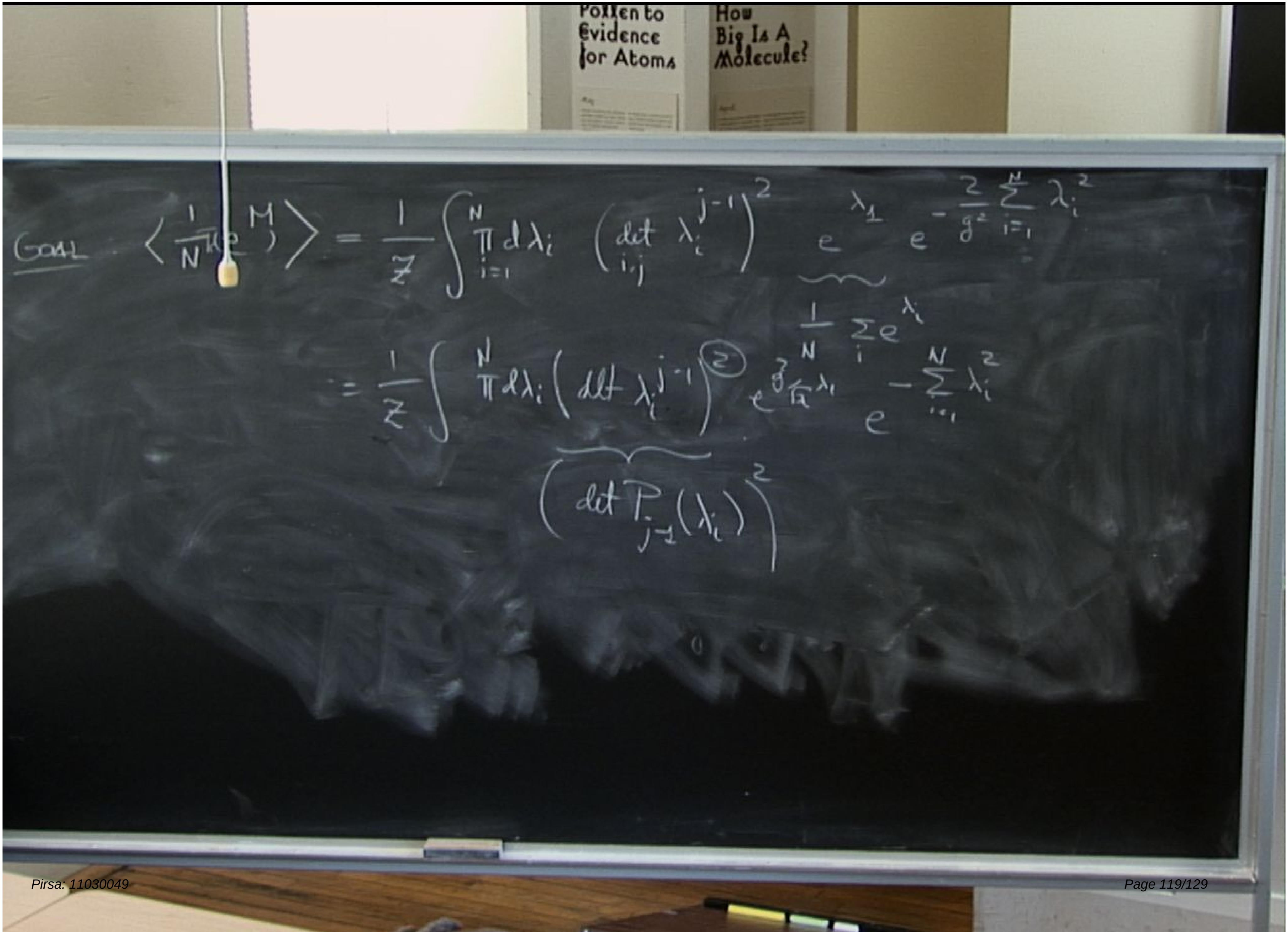
Pollen to Evidence for Atoms
How Big Is A Molecule?
Goal: $\left\langle \frac{1}{N} \mathrm{Tr}\, e^{M} \right\rangle = \frac{1}{Z} \int \prod_{i=1}^{N} d\lambda_i \left(\det_{i,j} \lambda_i^{j-1} \right)^2 \underbrace{e^{\lambda_1}}_{\frac{1}{N} \sum_i e^{\lambda_i}} e^{-\frac{2}{g^2} \sum_{i=1}^{N} \lambda_i^2}$
$= \frac{1}{Z} \int \prod^{N} d\lambda_i \underbrace{\left(\det \lambda_i^{j-1} \right)^2}_{\left(\det P_{j-1}(\lambda_i) \right)^2} e^{\frac{g}{\sqrt{2}} \lambda_1} e^{-\sum_{i=1}^{N} \lambda_i^2}$

$$\frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \underbrace{H_n(x)}_{x^n + \cdots\cdots}$$

$$P_n(x) \equiv \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \underbrace{H_n(x)}_{x^n + \cdots}$$

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2} P_n(x) P_m(x) = \delta_{nm}$$

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$$\left(\det_{i,j} P_j(\lambda_i) \right) \left(\det_{k,l} P_k(\lambda_l) \right)$$

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$$= \sum_{j=0}^{N-1} \int d\lambda_1 P_j(\lambda_1)^2 e^{-\lambda_1^2 + \frac{\lambda_1}{\sqrt{2}} g^2} \quad \nabla_0 \quad 1 \int \nabla_0$$

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$$\int_{-\infty}^{+\infty} dx\, e^{-x^2} P_n(x) P_m(x) = \delta_{nm}$$

$$= \sum_{j=0} \frac{1}{N} e^{\frac{\lambda}{8N}} L_j^0\left(-\frac{\lambda}{4N}\right)$$

$$= \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$$

$$\int d\lambda_1 \cdots d\lambda_N \left(\det_{i,j} P_j(\lambda_i)\right)\left(\det_{k,l} P_k(\lambda_l)\right) \underbrace{e^{-\lambda_2^2}\, e^{-\lambda_3^2} \cdots e^{-\lambda_N^2}}\; e^{-\lambda_1^2 + \frac{\lambda_1}{\sqrt{2}} g^2}$$

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$$= \frac{1}{N} L^1_{N-1}\left(-\frac{\lambda}{4N}\right) e^{\frac{\lambda}{8N}}$$

$$= \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{1}{N^2} \frac{\lambda}{48^2} I_2(\sqrt{\lambda}) + \ldots$$

$$\int d\lambda_1 \ldots d\lambda_N \left(\det_{i,j} P_j(\lambda_i)\right)\left(\det_{k,l} P_k(\lambda_l)\right) \underbrace{e^{-\lambda_2^2} e^{-\lambda_3^2} \cdots e^{-\lambda_N^2}}\, e^{-\lambda_1^2 + \frac{\lambda_1}{\sqrt{2}} g^2}$$

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