Title: Hidden symmetries and higher-dimensional rotating black holes

Date: Feb 23, 2011 11:00 AM

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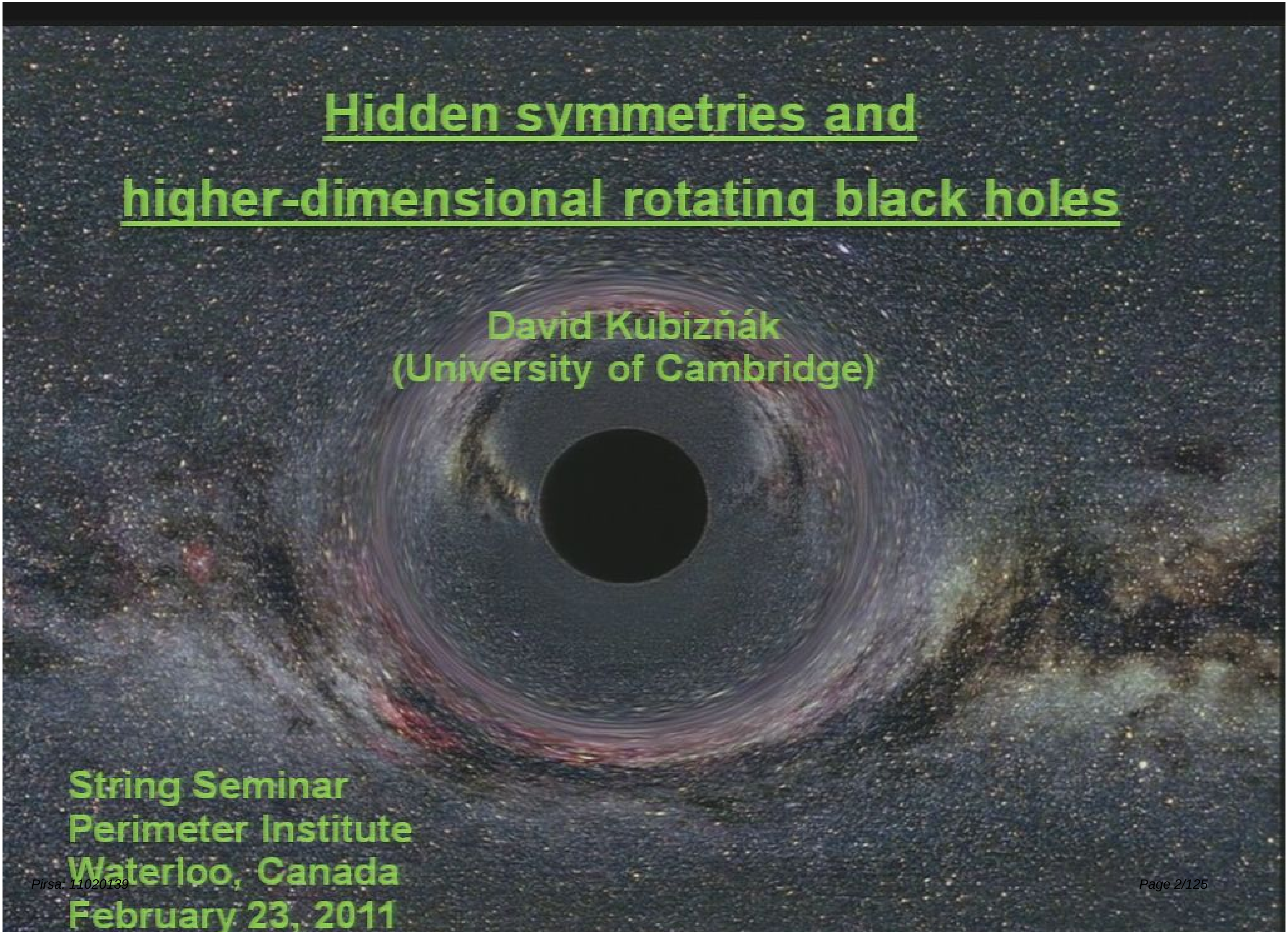
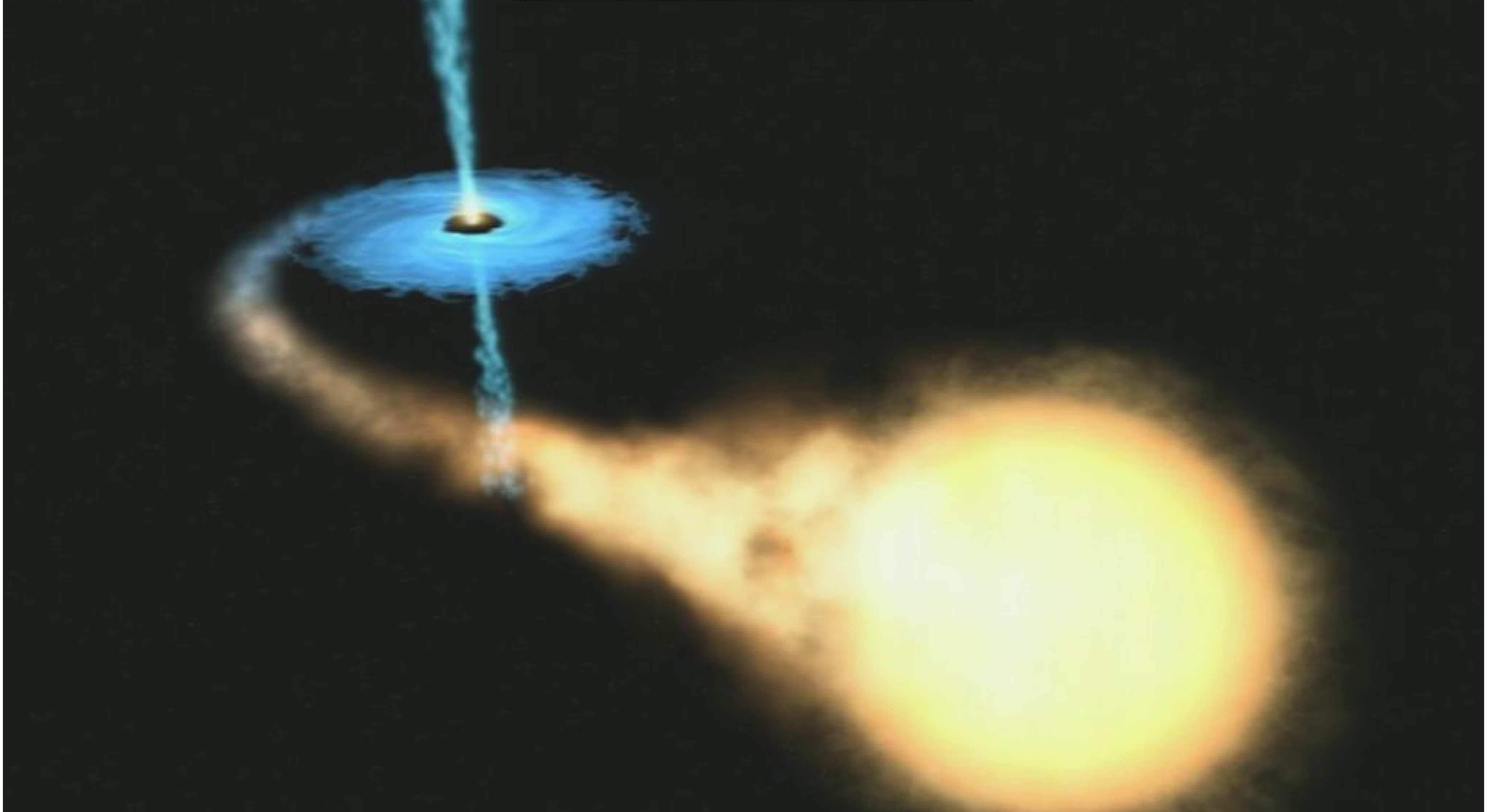
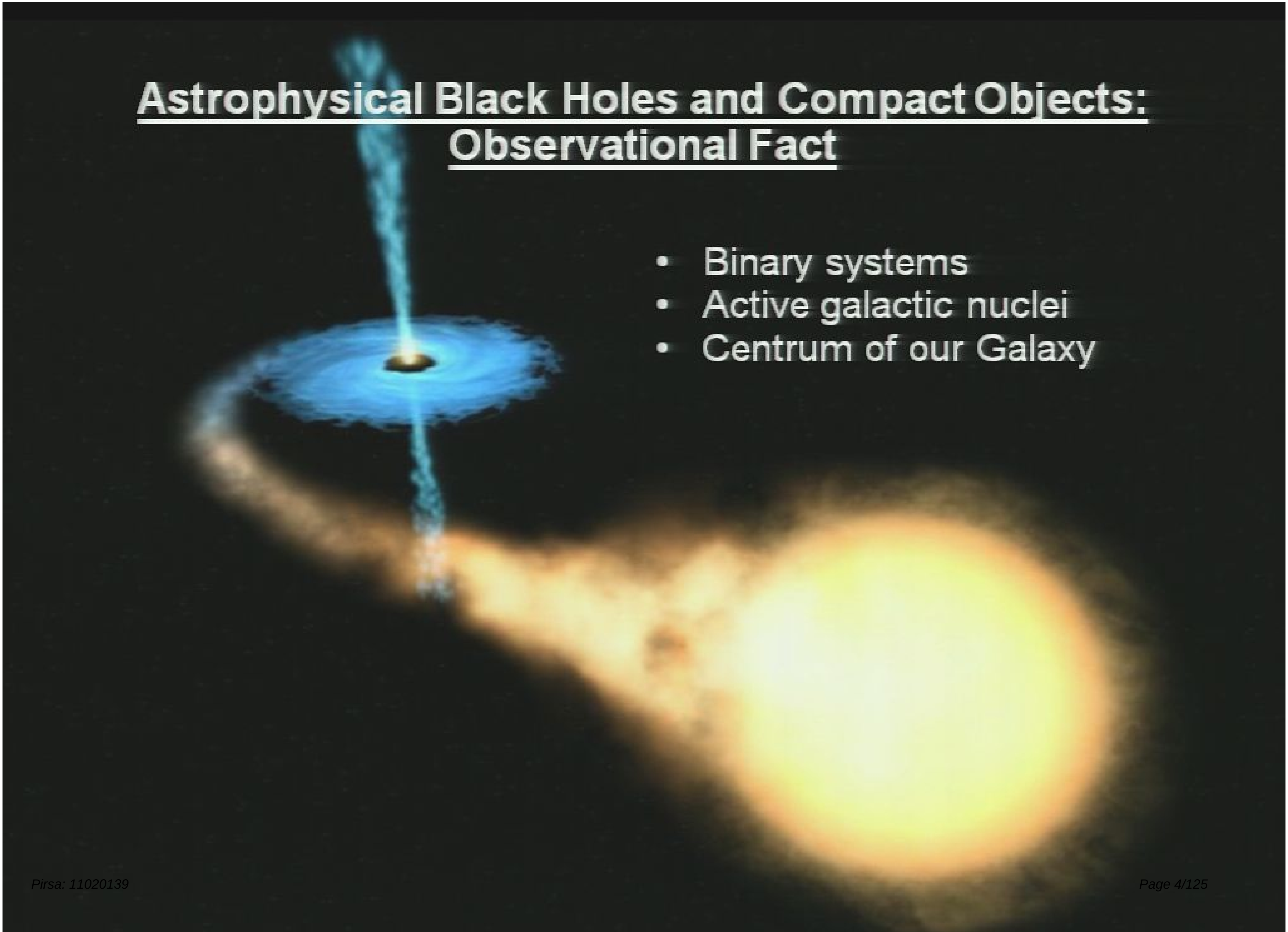
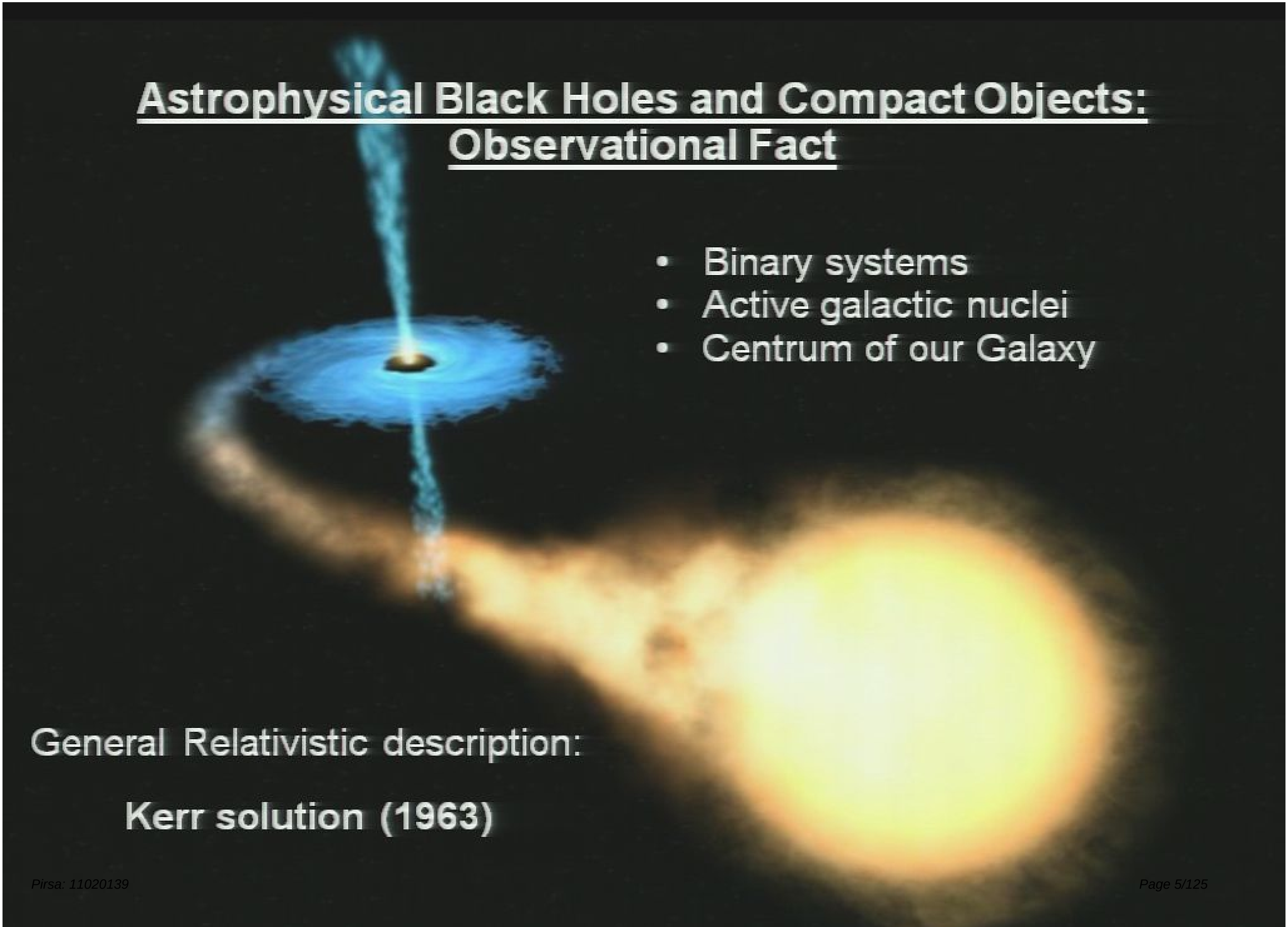
Abstract: The 4D rotating black hole described by the Kerr geometry possesses many of what was called by Chandrasekhar "miraculous" properties. Most of them are related to the existence of a fundamental hidden symmetry of a principal conformal Killing-Yano (PCKY) tensor. In my talk I shall demonstrate that hidden symmetry of the PCKY tensor plays exceptional role also in higher dimensions. Namely, I shall present the most general spacetime admitting the PCKY tensor and show that is possesses the following properties: 1) It is of the algebraic type D and admits the Kerr-Schild form 2) It allows a separation of variables for the Hamilton-Jacobi, Klein-Gordon, Dirac, and stationary string equations. 3) When the Einstein equations with the cosmological constant are imposed the metric describes the most general known (spherical) Kerr-NUT-AdS black hole spacetime. I will also discuss the generalization of Killing-Yano symmetries for spacetimes with natural "torsion 3-form", such as the black hole of D=5 minimal supergravity, or the Kerr-Sen solution of heterotic string theory, and comment on connection to special Riemannian manifolds admiting Killing spinors.


Hidden symmetries and
higher-dimensional rotating black holes
David Kubizňák
(University of Cambridge)
String Seminar
Perimeter Institute
Waterloo, Canada
February 23, 2011

Astrophysical Black Holes and Compact Objects: Observational Fact




Astrophysical Black Holes and Compact Objects:
Observational Fact
• Binary systems
• Active galactic nuclei
• Centrum of our Galaxy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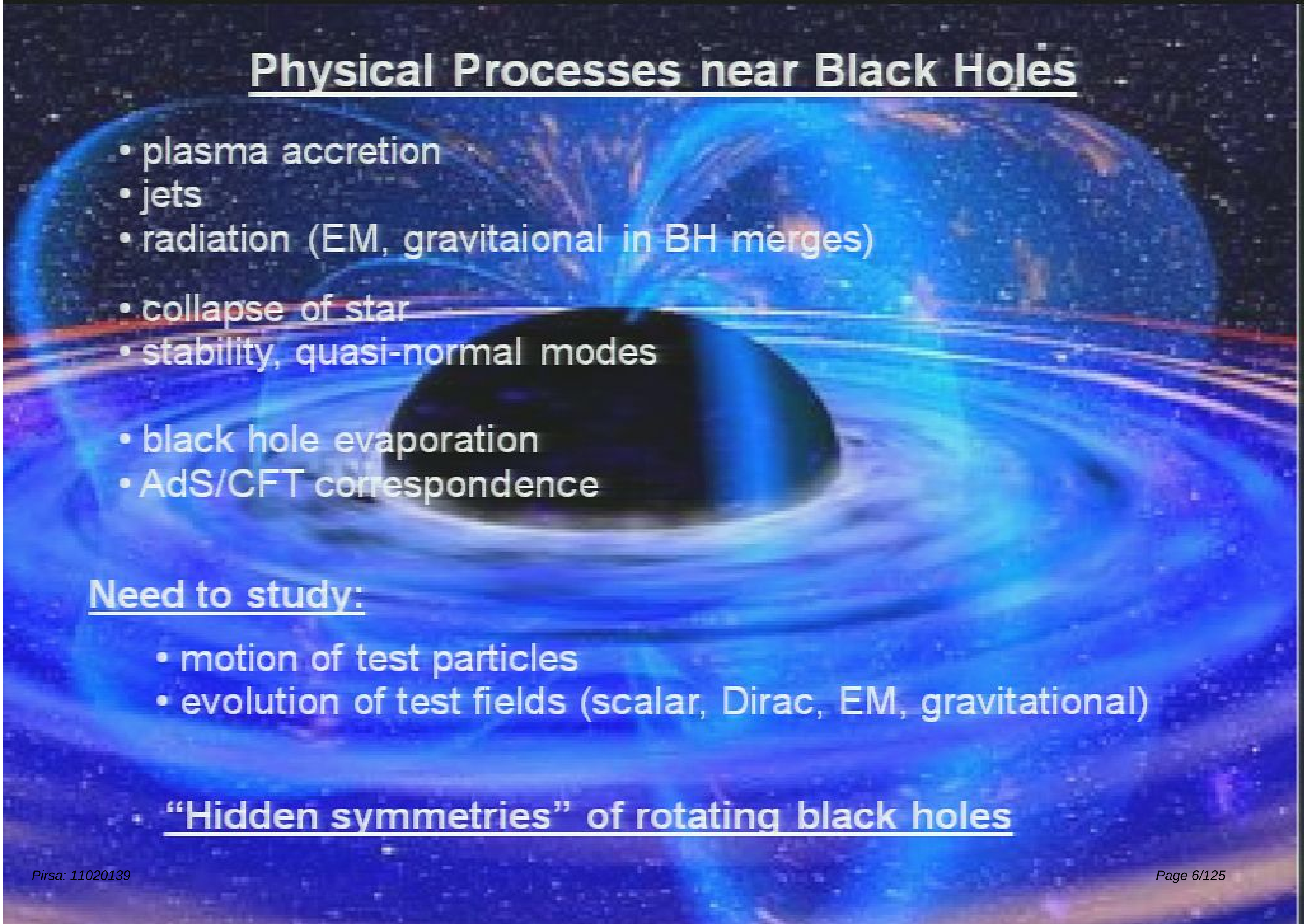
General Relativistic description:
Kerr solution (1963)


Physical Processes near Black Holes
• plasma accretion
• jets
• radiation (EM, gravitaional in BH merges)
• collapse of star
• stability, quasi-normal modes
• black hole evaporation
• AdS/CFT correspondence
Need to study:
• motion of test particles
• evolution of test fields (scalar, Dirac, EM, gravitational)
• "Hidden symmetries" of rotating black holes

Plan of the talk

I. Miraculous properties of 4D black holes

II. Hidden symmetries in all dimensions
- Killing-Yano tensors and PCKY 2-form
- Canonical metric and Kerr-NUT-AdS spacetimes
- Parallel transport
- Separability of test field equations
- Algebraic type and Kerr-Schild form
- Generalized Killing-Yano symmetries
- Killing spinors & special Riemannian manifolds

III. Future directions

Collaborators:

M. Cariglia, V.P. Frolov, T. Houri, P. Krtous, H.K. Kunduri, D.N. Page, M. Vasudevan, C.M. Warnick, Y. Yasui

I) Miraculous properties of 4D black holes

Symmmetries in GR

- Killing vectors - isometries of spacetime

Symmmetries in GR

- Killing vectors - isometries of spacetime

$\Phi_\xi : M \to M$ diffeomorphism

ISOMETRY of spacetime $\Longleftrightarrow \Phi^*_\xi g_{\mu\nu} = g_{\mu\nu}$

$$\pounds_\xi g_{\mu\nu} = 0 \Longleftrightarrow \xi_{(\mu;\nu)} = 0.$$

Symmmetries in GR

- Killing vectors - isometries of spacetime

$$\xi_{(\mu;\nu)} = 0.$$

Noether theorem: conserved quantities

particle: integrals of motion linear in momenta $C_1 = \xi_\mu u^\mu$

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- Stackel-Killing tensors

M. Walker and R. Penrose, Comm. Math. Phys. 18 , 265 (1970).

$$K_{\mu\nu} = K_{(\mu\nu)}, \qquad K_{(\mu\nu;\lambda)} = 0.$$

particle: integrals of motion of higher order in momenta $C_2 = K_{\mu\nu} u^\mu u^\nu$

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R. Penrose, Ann. N.Y. Acad. Sci. , 125 (1973).
R. Floyd, The dynamics of Kerr fields, PhD Thesis, London (1973).

$$f_{\mu\nu} = f_{[\mu\nu]}\,, \qquad f_{\mu(\nu;\lambda)} = 0\,.$$

particle: parallel transported $w^\mu = f^{\mu\nu} u_\nu$

History of Remarkable properties of Kerr geometry

Kerr-Schild
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Kerr-Schild form

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H l_\mu l_\nu$$

$\eta_{\mu\nu}$ is a flat metric

l_μ is a null vector

Einstein equations
become linear

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Kerr-Schild
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Isometries
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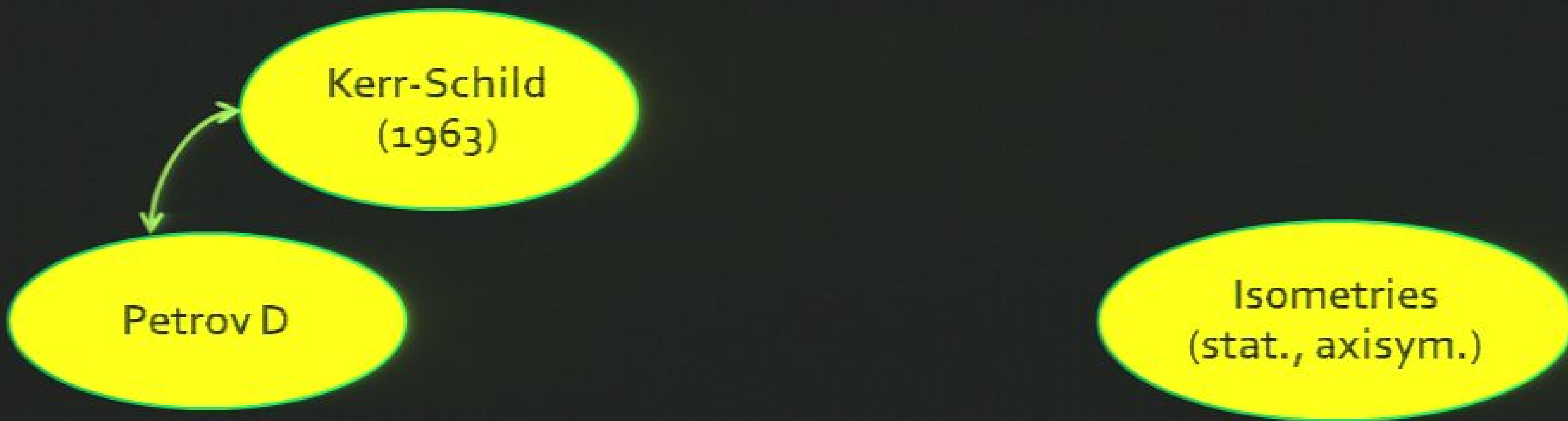

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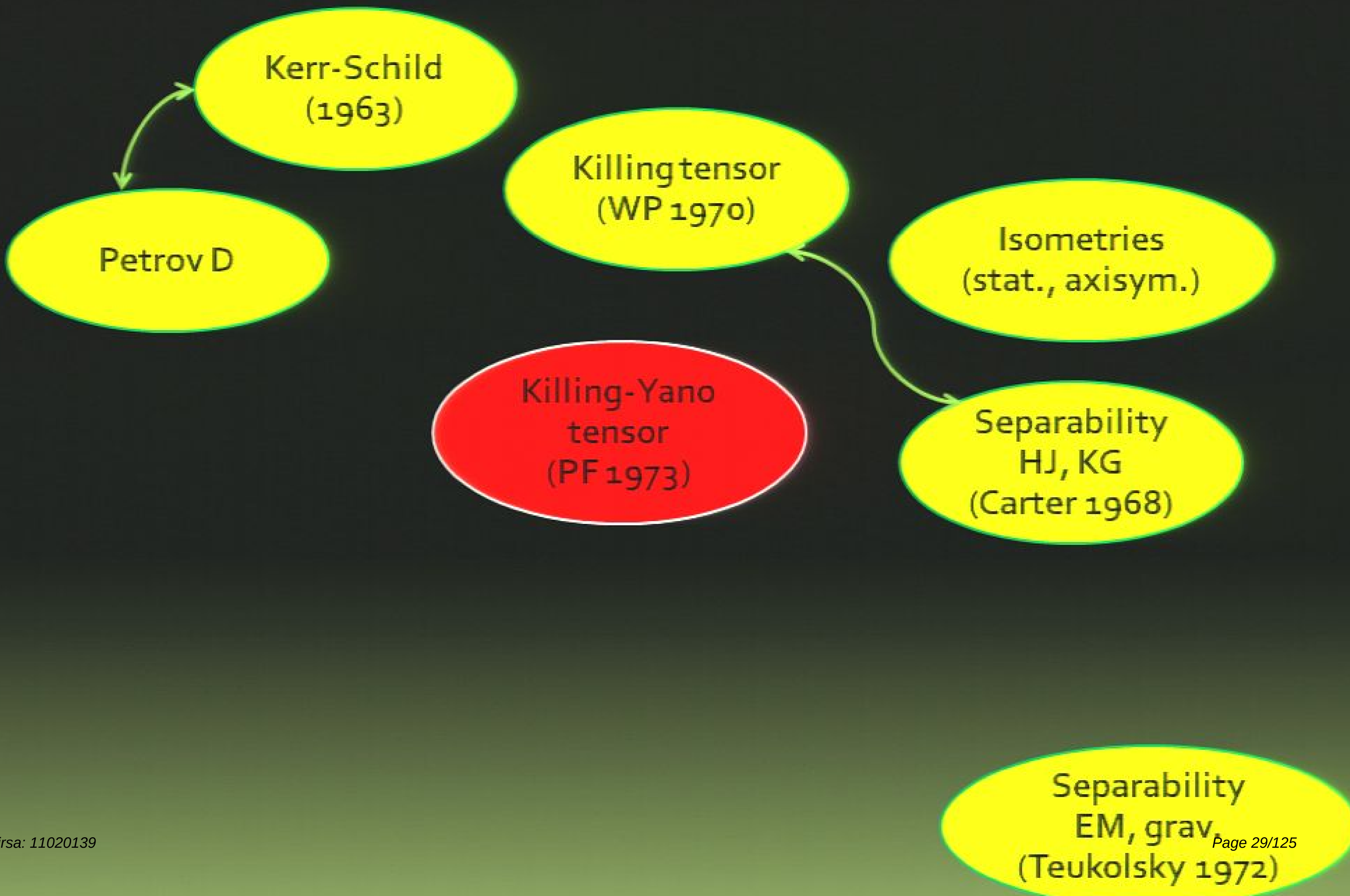

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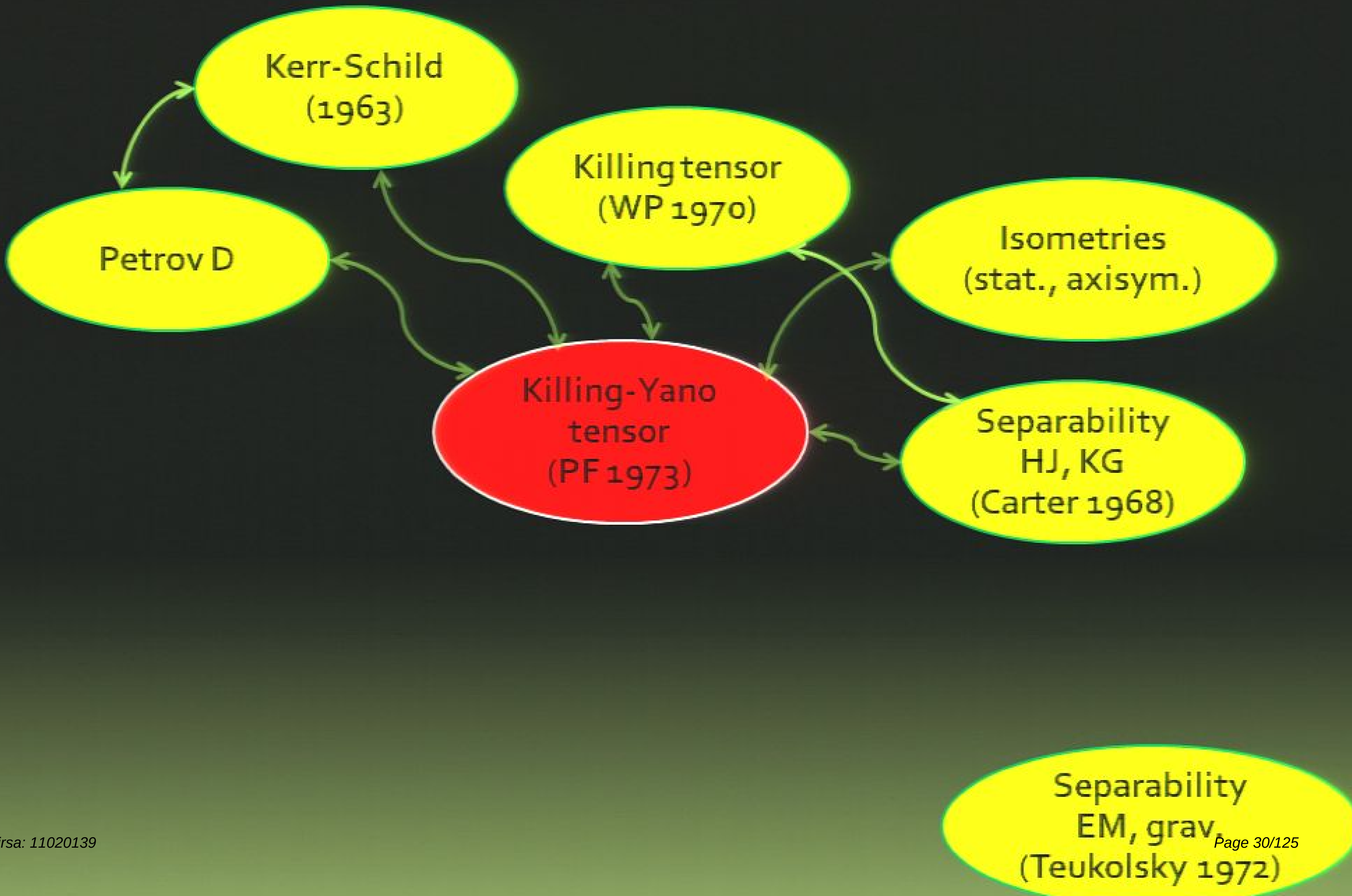

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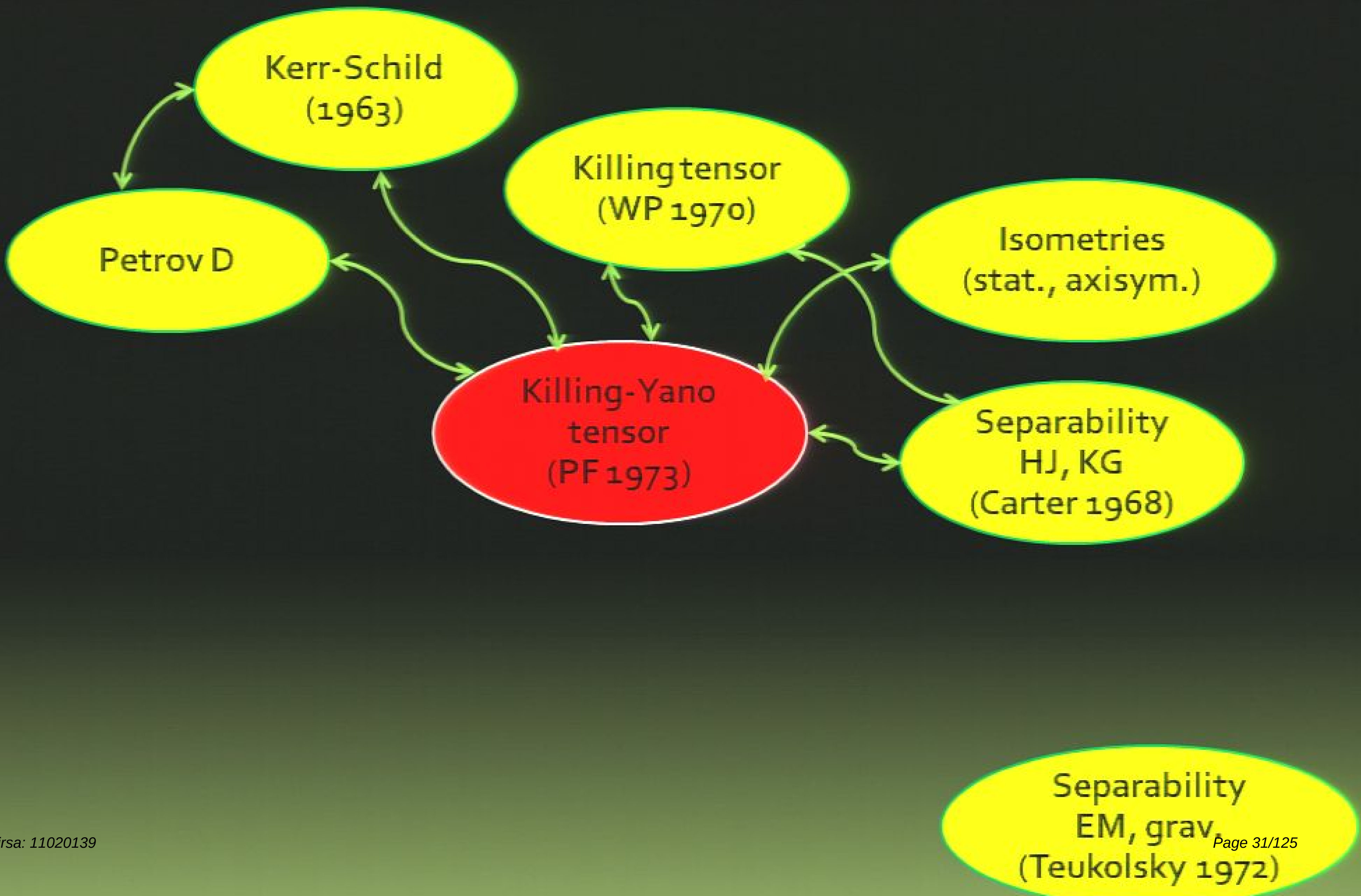

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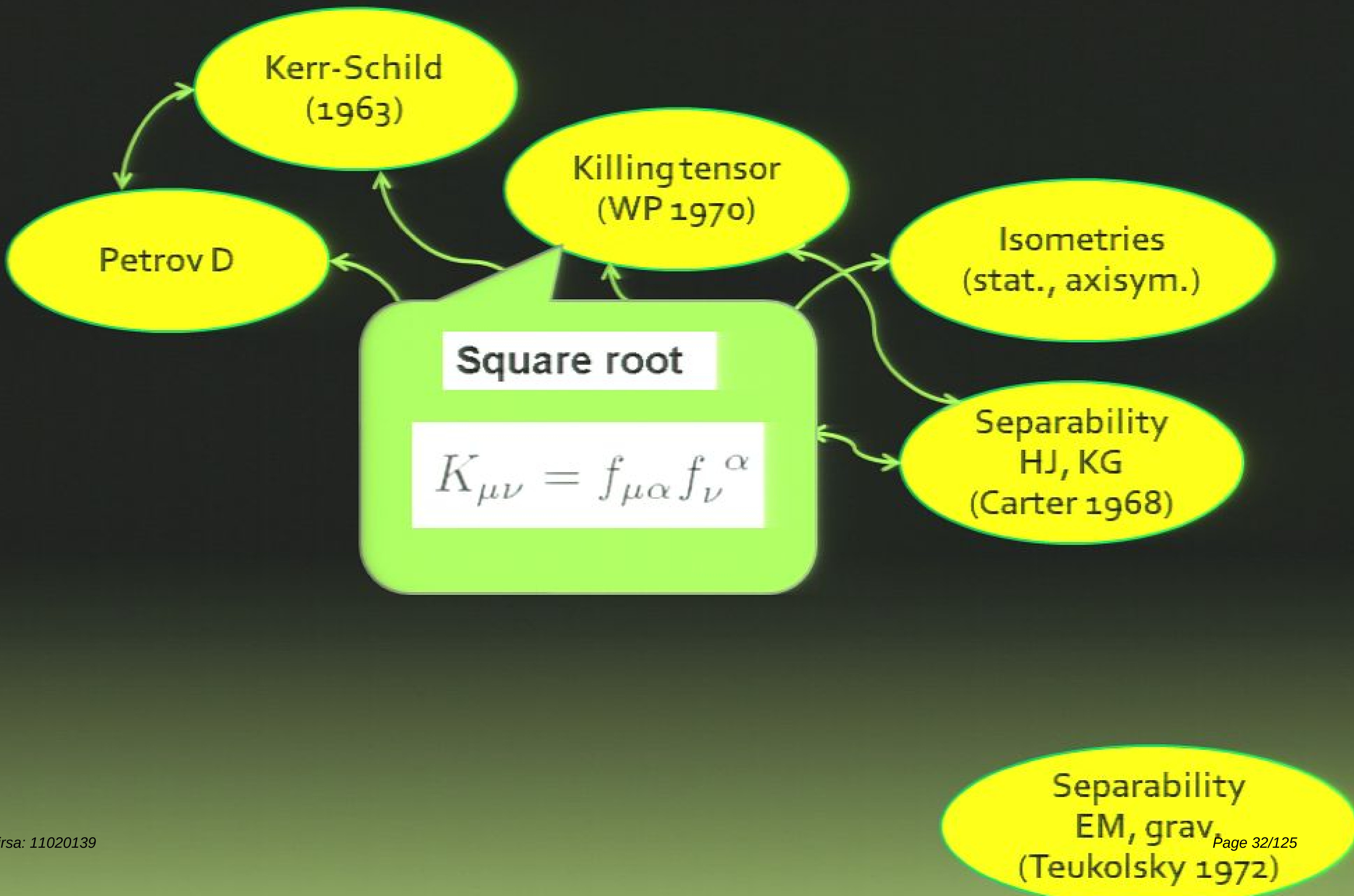

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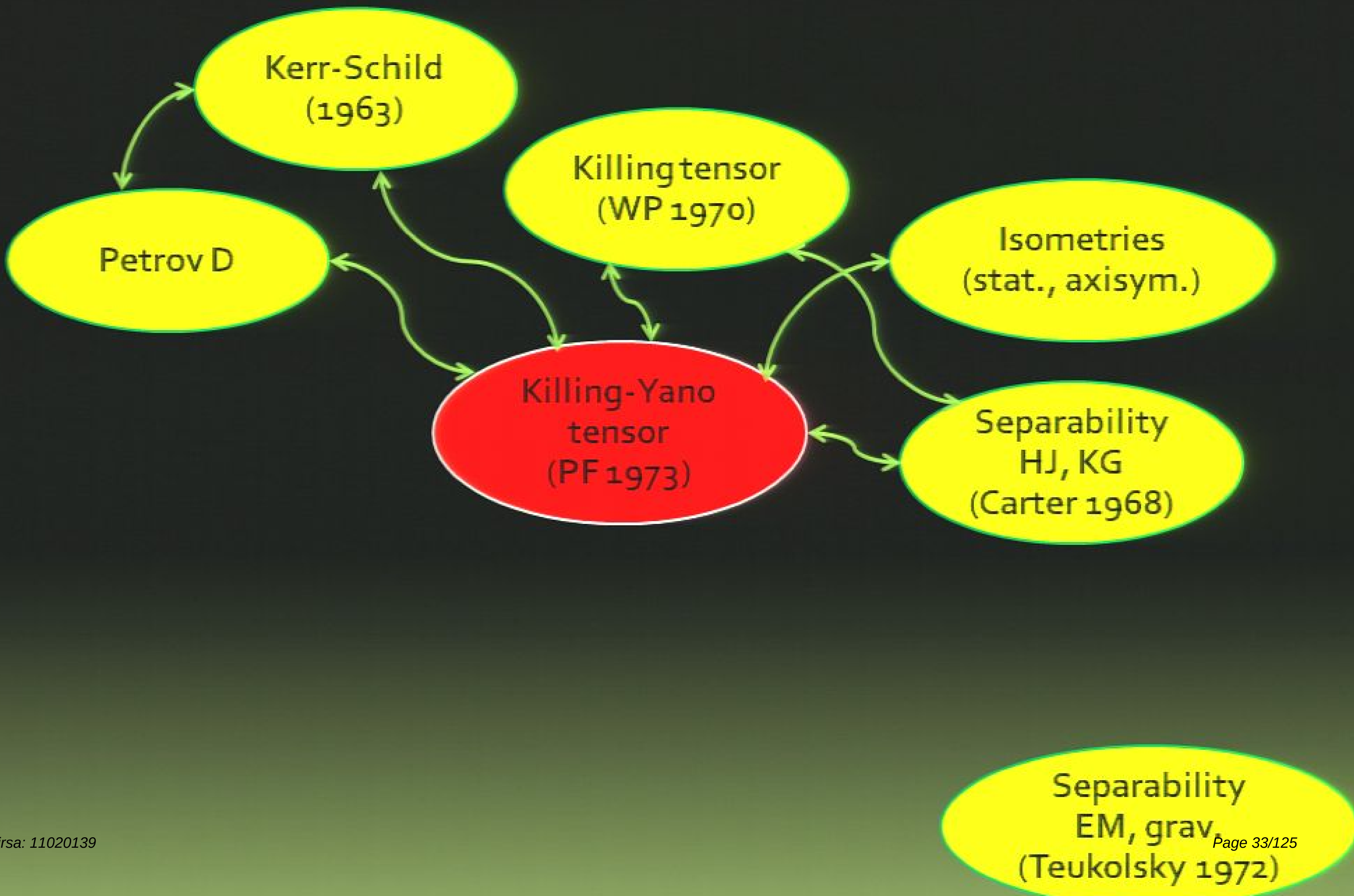

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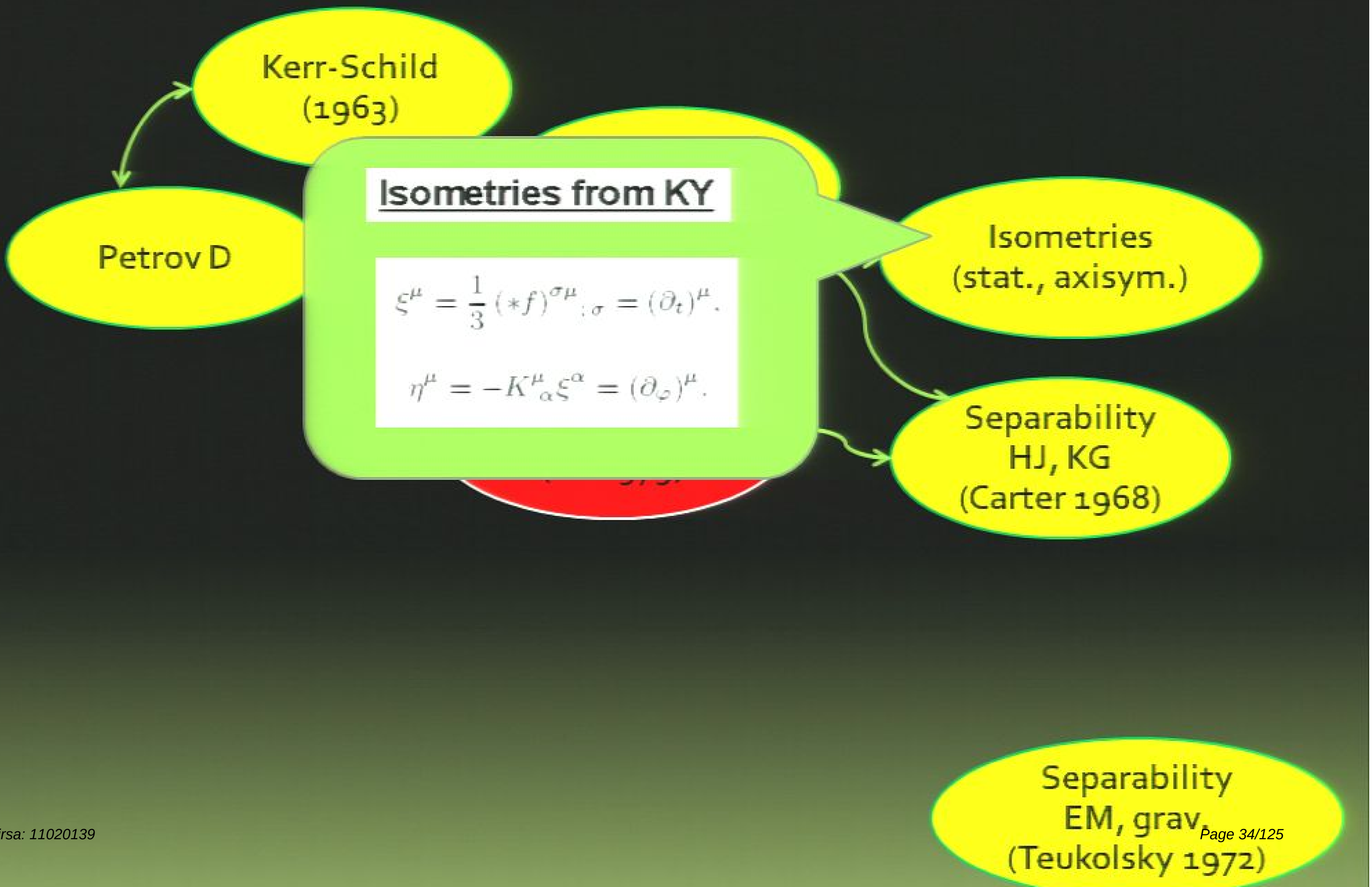

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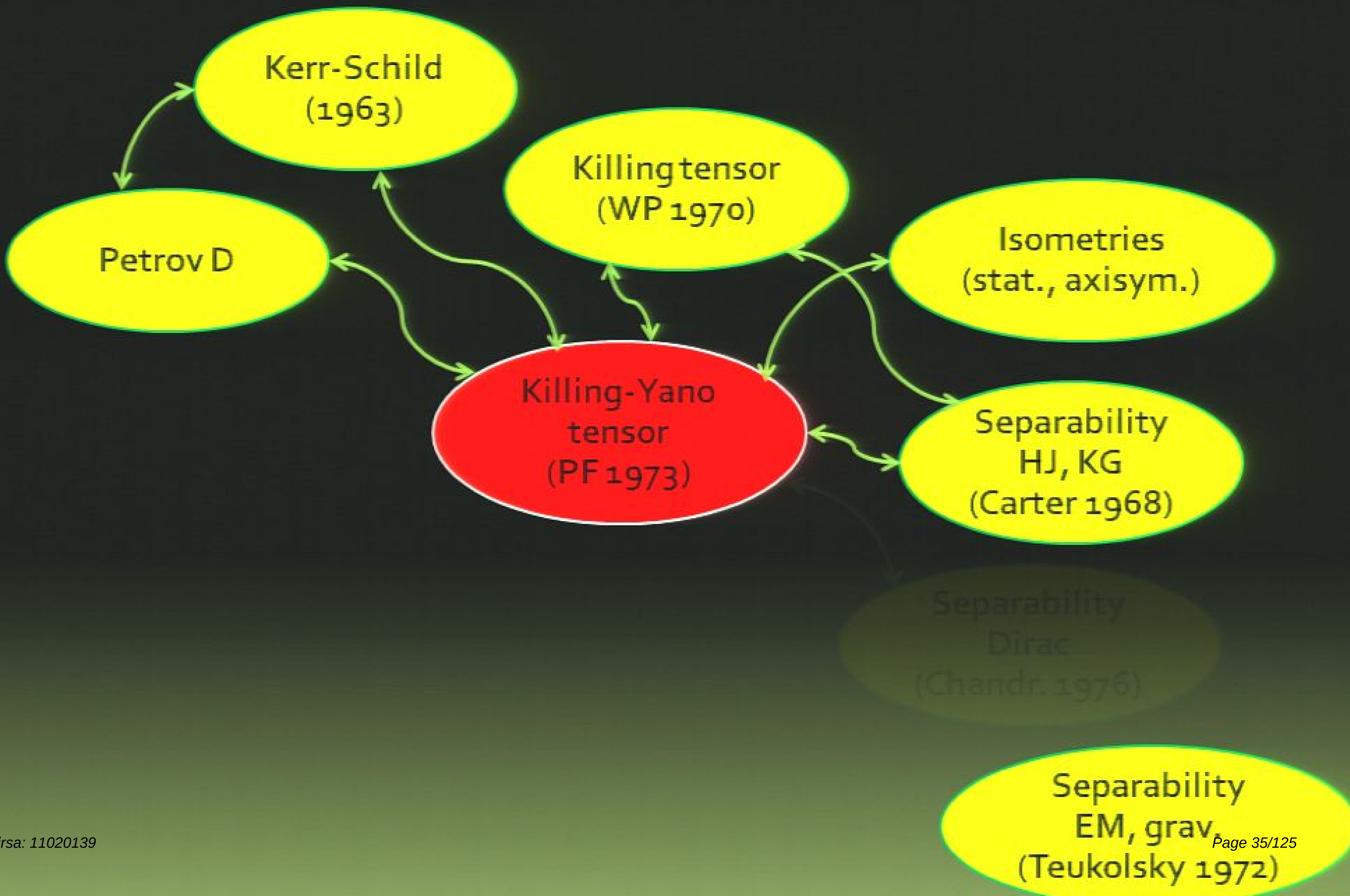

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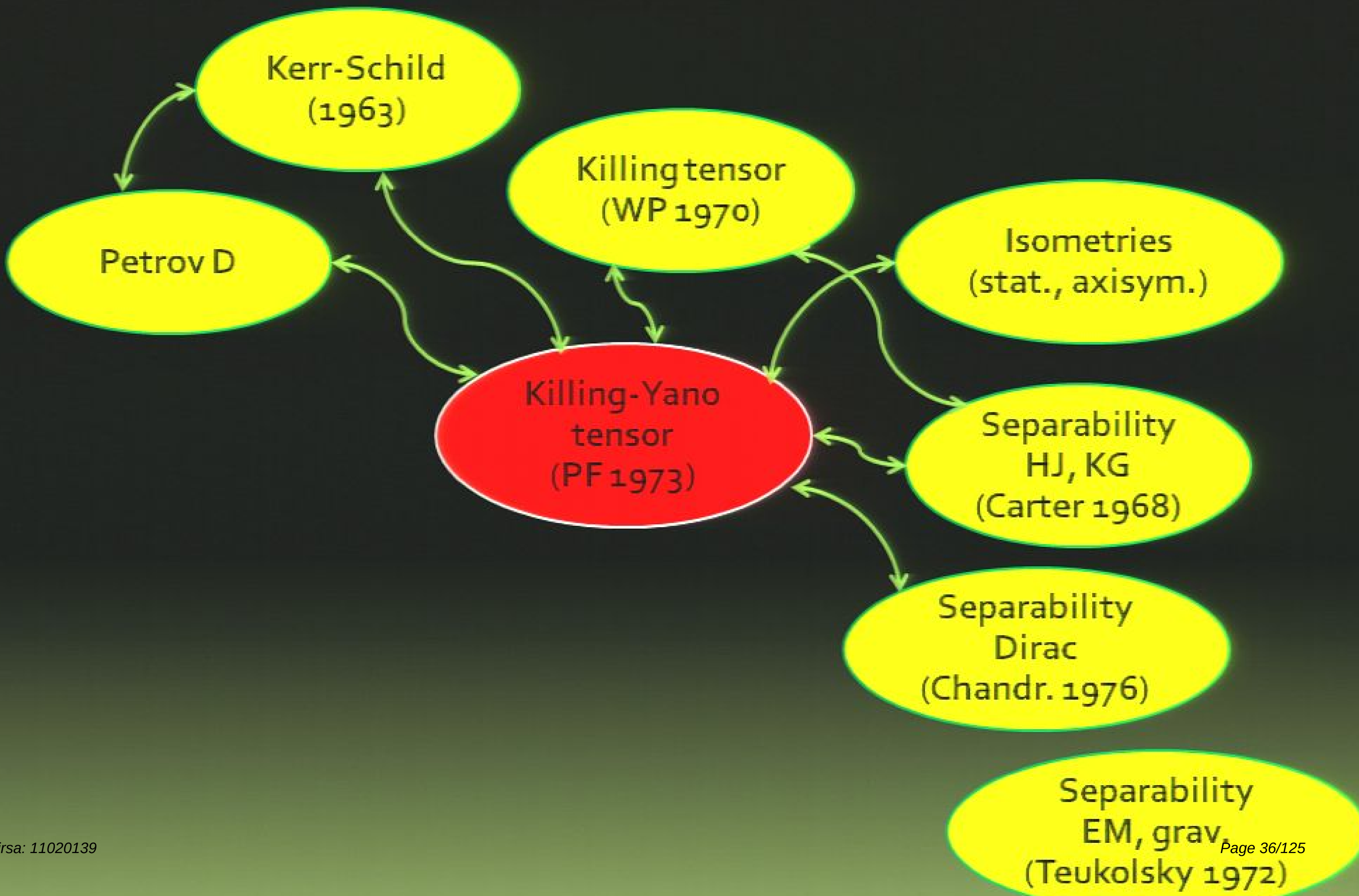

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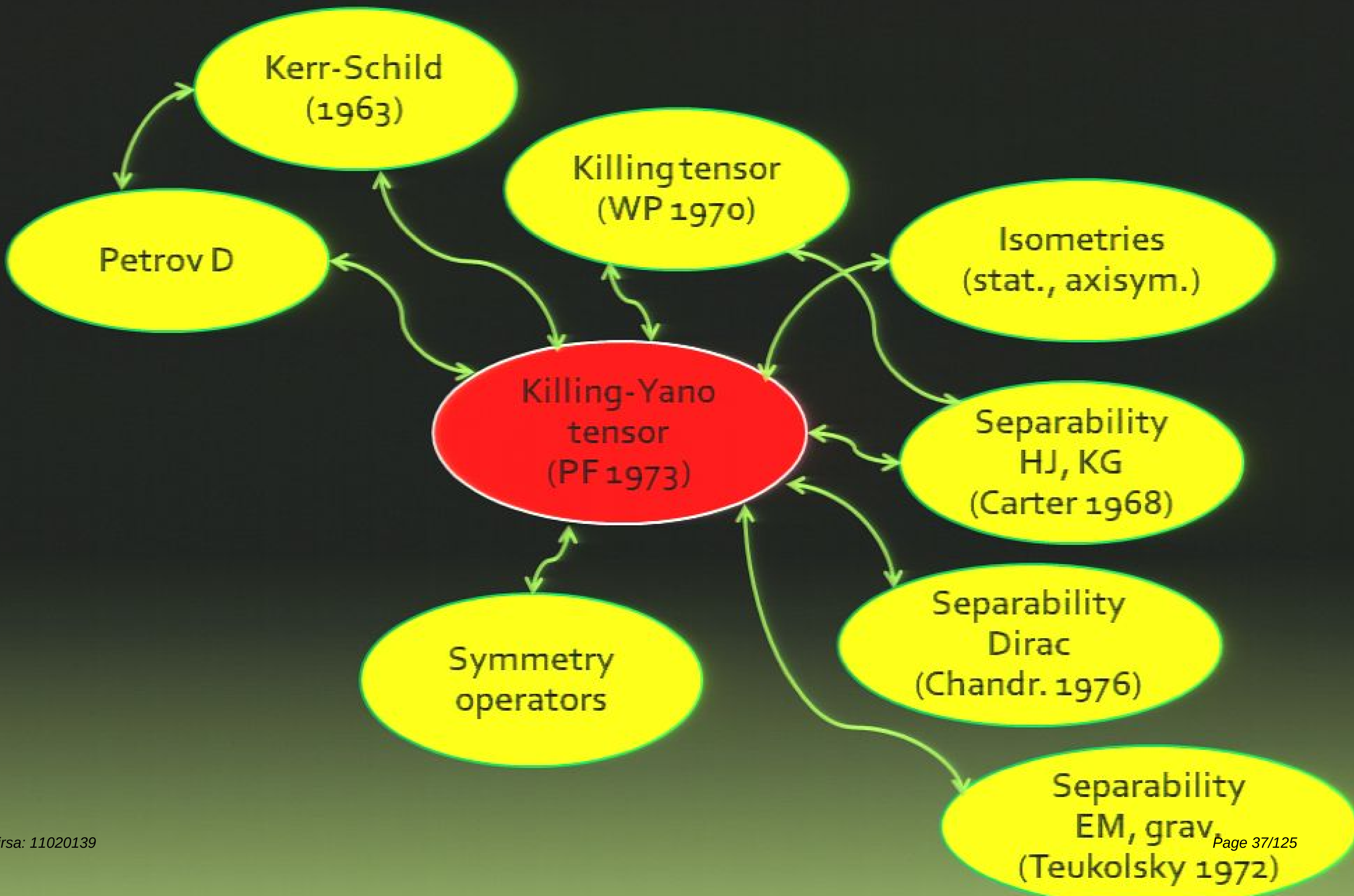


History of Remarkable properties of Kerr geometry

Symmetry Operators

- Scalar field (1977 Carter)

$$\Box = \nabla_a g^{ab} \nabla_b .$$

$$\hat{\xi} = \frac{i}{2} \left(\xi^a \nabla_a + \nabla_a \xi^a \right) \qquad \hat{K} = \nabla_a K^{ab} \nabla_b .$$

$$[\Box . \hat{K}\,] = 0 = [\Box . \hat{\xi}\,] .$$

- Dirac field (1979 Carter & McLenaghan)

$$\hat{f} = i\gamma_5 \gamma^\mu \left(f_\mu{}^\nu D_\nu - \frac{1}{6} \gamma^\nu \gamma^\rho f_{\mu\nu;\rho} \right)$$

Isometries (stat., axisym.)

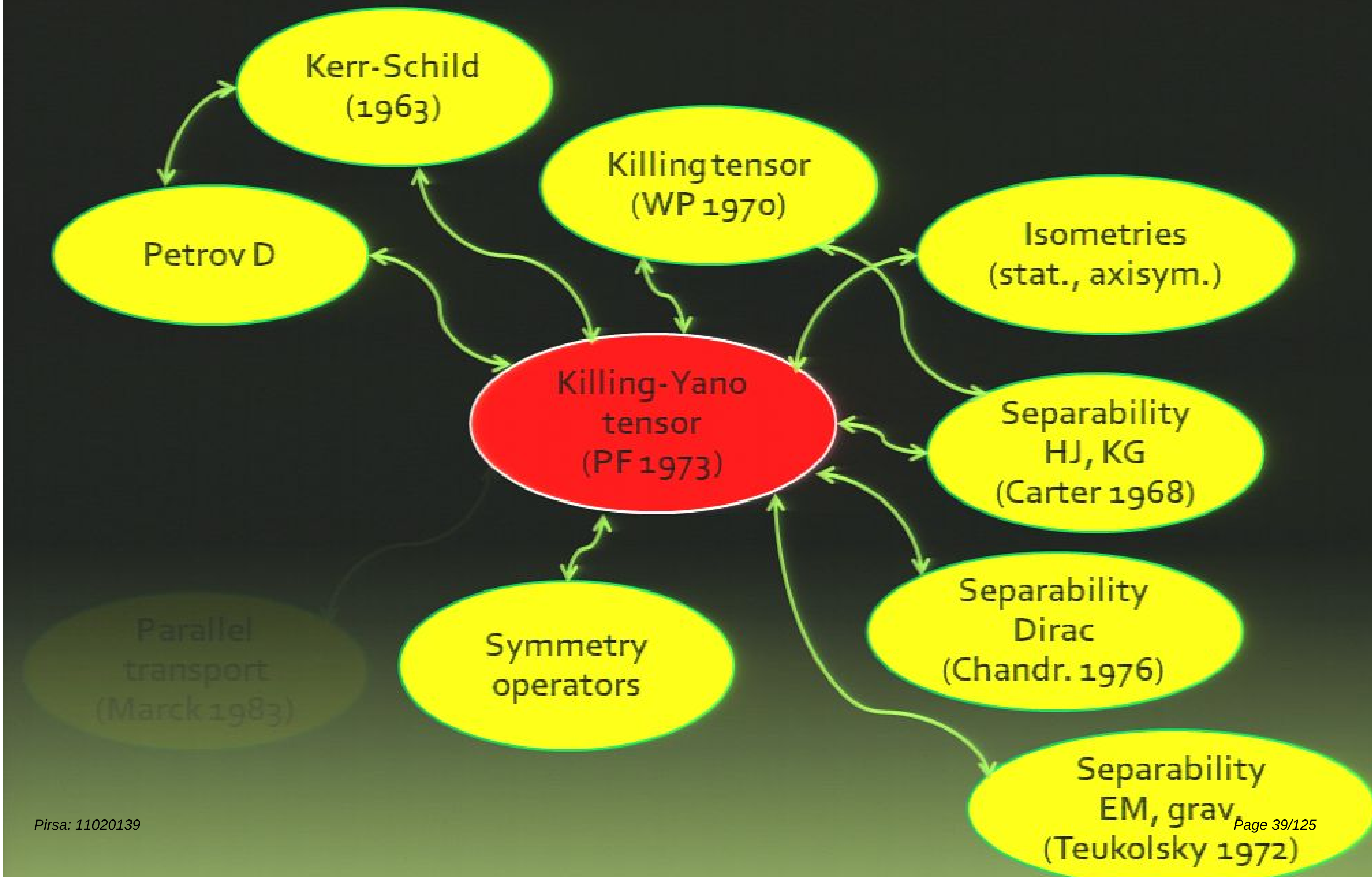
Separability HJ, KG (Carter 1968)

Separability Dirac (Chandr. 1976)

Symmetry operators

Separability EM, grav. (Teukolsky 1972)

History of Remarkable properties of Kerr geometry



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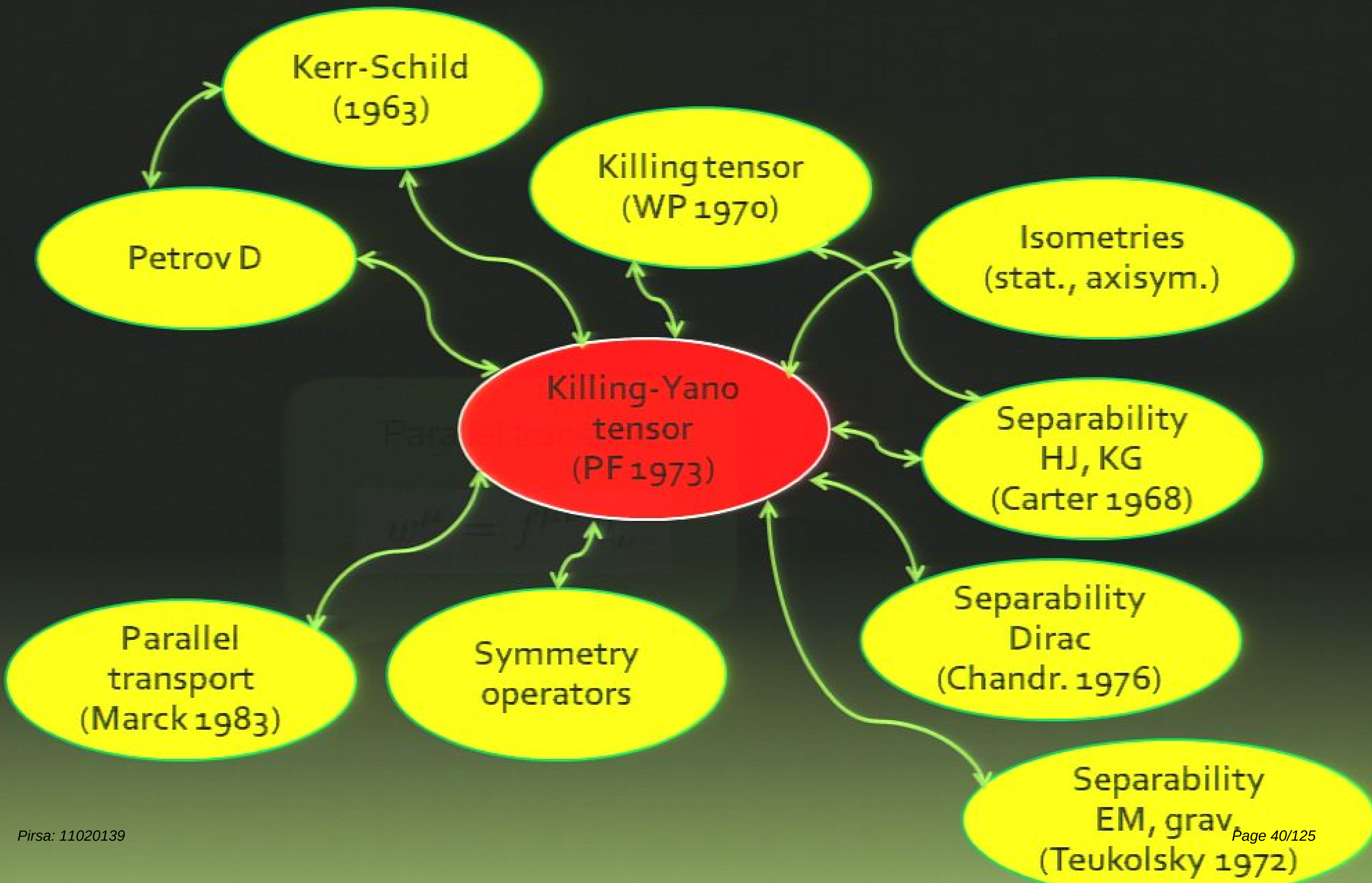

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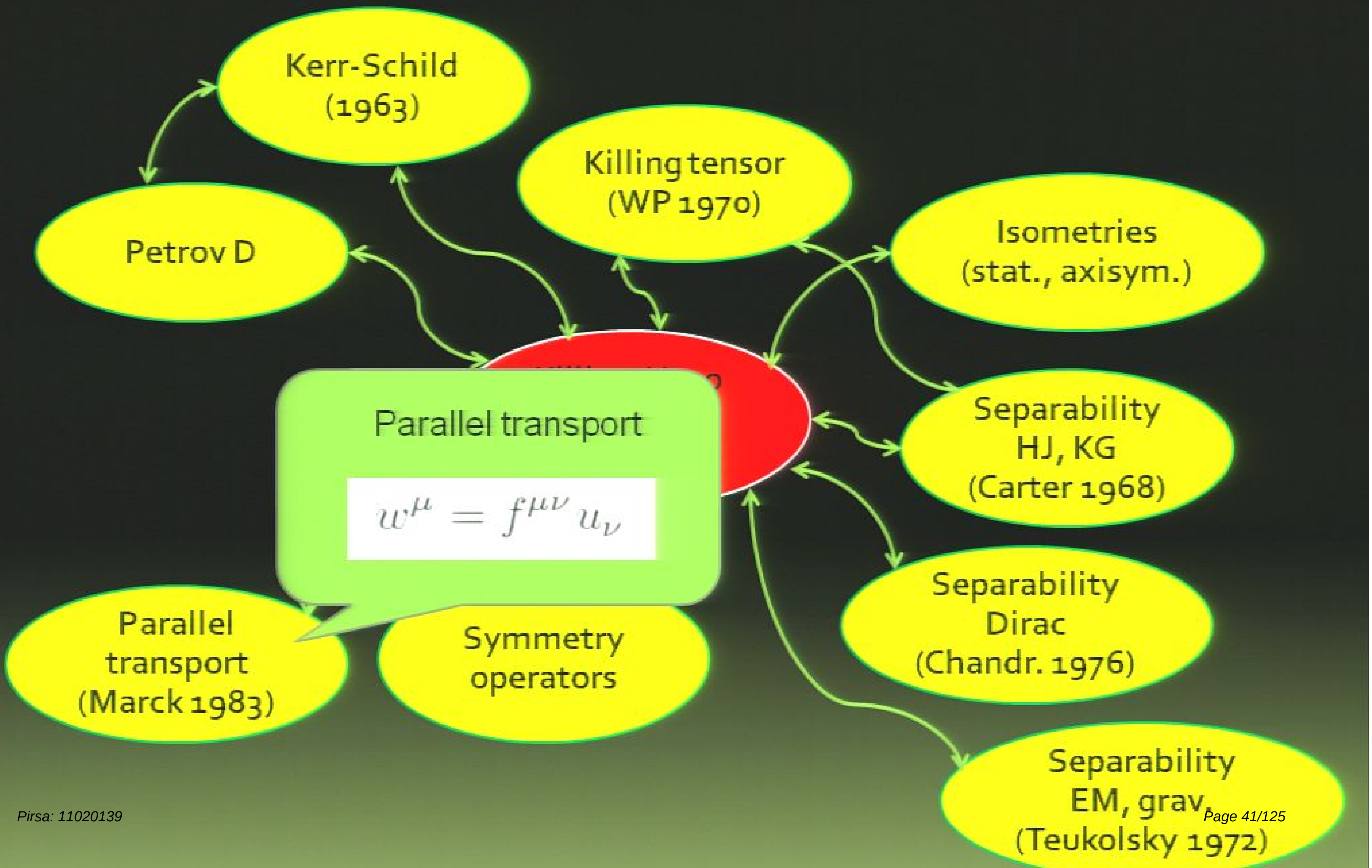

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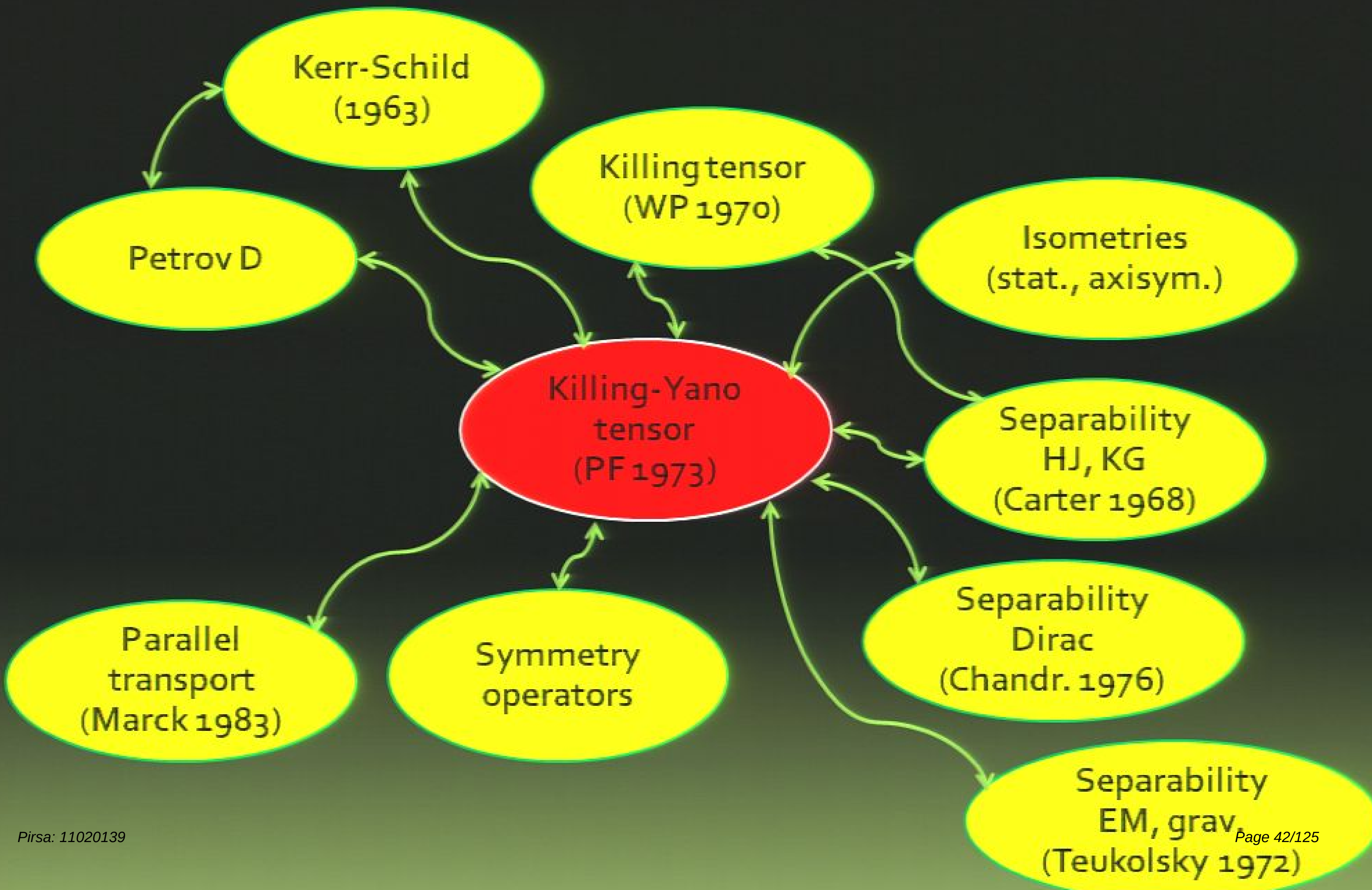

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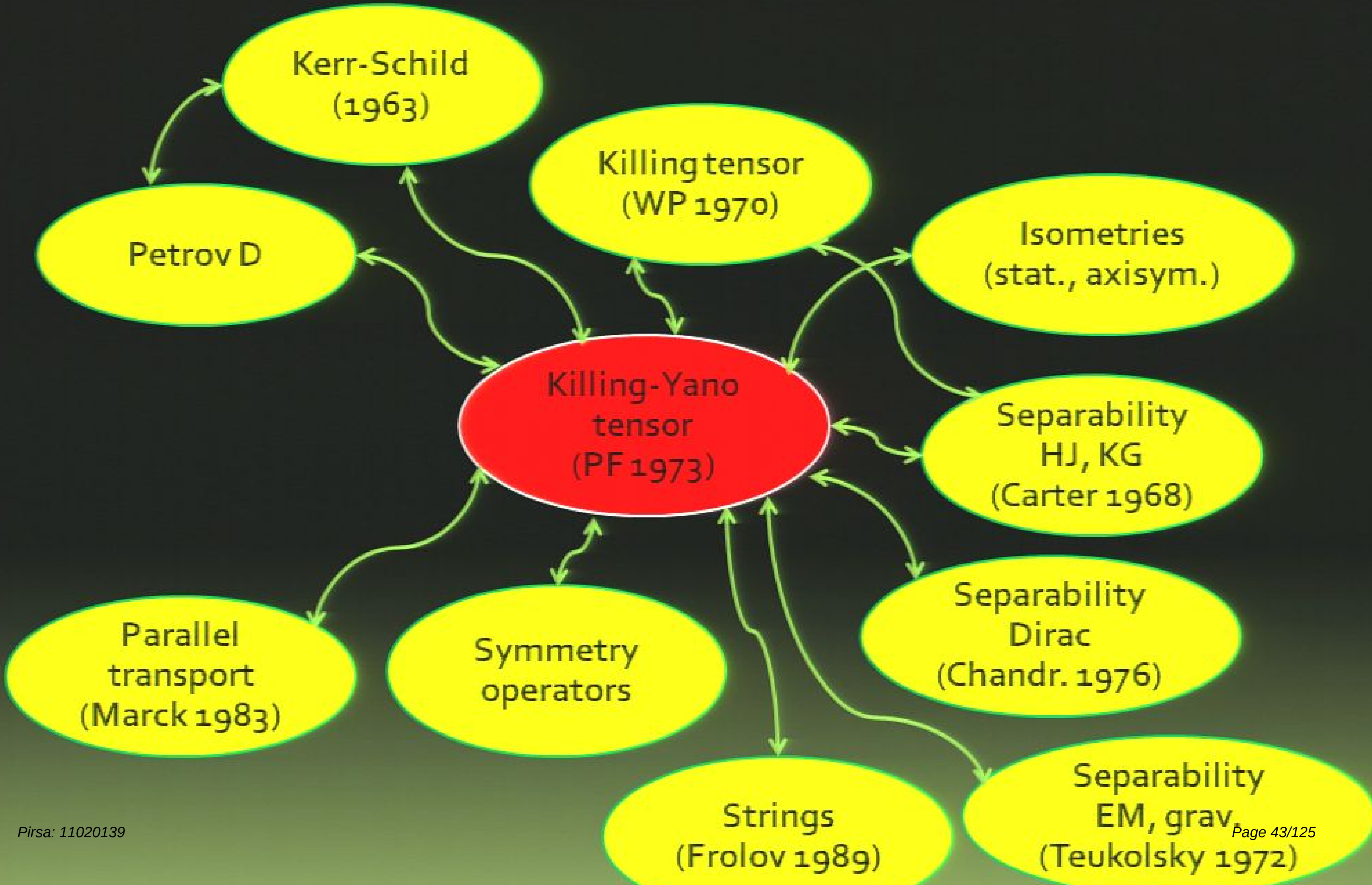

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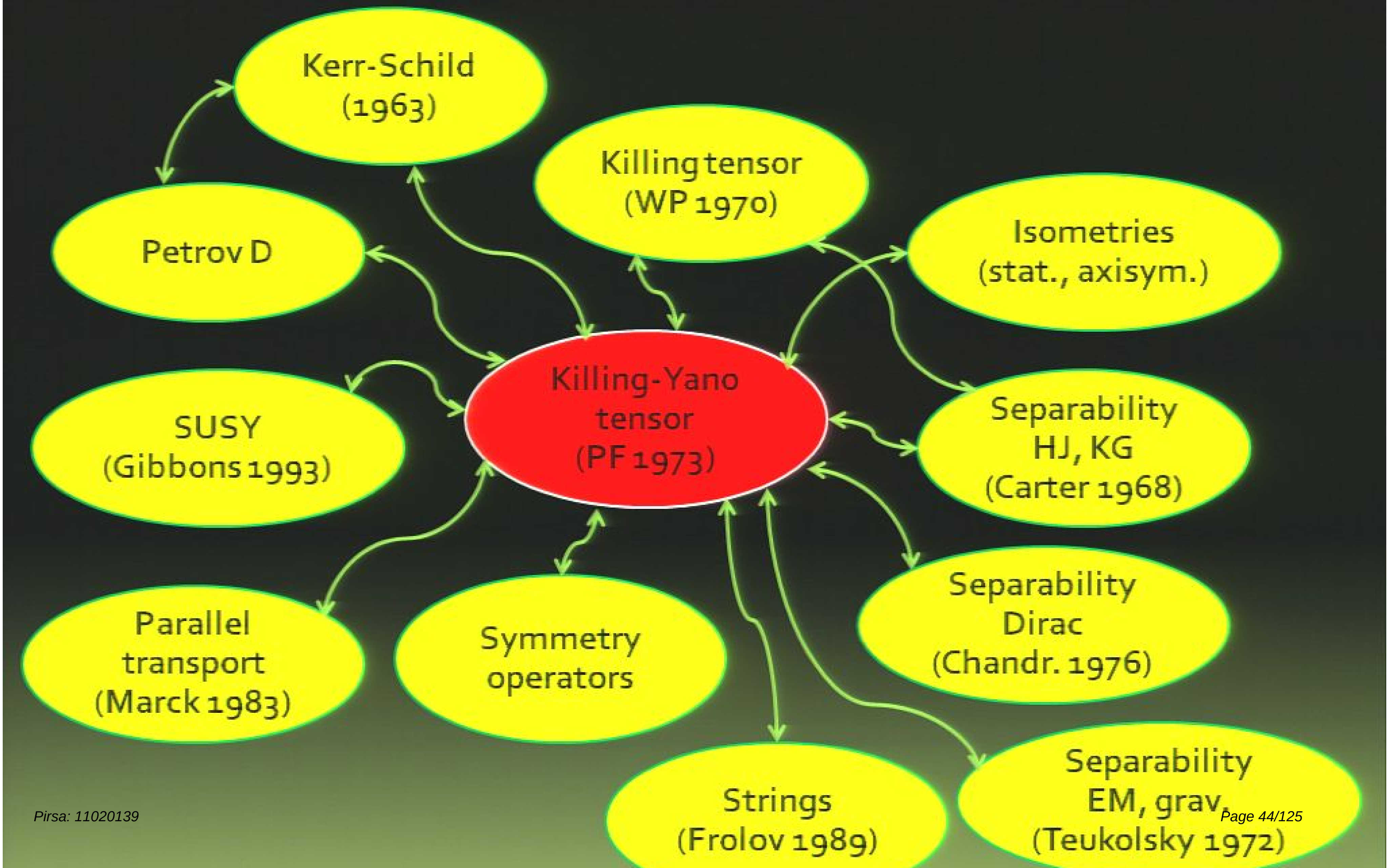

Why to study higher dimensions?

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String theory, AdS/CFT

General Relativity

Braneworlds

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String theory, AdS/CFT

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Do the remarkable properties of black holes extend to higher dimensions?

II) Hidden symmetries in all dimensions

1) Killing-Yano tensors

for a general differential form

$$\nabla\omega \propto \text{exterior} + \text{divergence} + \text{symmetric parts}$$

- Conformal Killing-Yano (CKY) tensor

$$\nabla_X \boldsymbol{k} = \frac{1}{p+1} X \lrcorner\, d\boldsymbol{k} - \frac{1}{D-p+1} X^\flat \wedge \delta \boldsymbol{k}\,.$$

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- Killing-Yano (KY) tensor: divergence part is missing
- closed CKY tensor exterior part is missing

Under Hodge duality divergence part transforms into exterior part and vice versa.

$$*(\text{closed CKY}) \propto \text{KY}\,.$$

2) Principal conformal Killing-Yano (PCKY) tensor

= (non-degenerate) closed CKY 2-form

$$\nabla_X \boldsymbol{h} = \boldsymbol{X}^{\flat} \wedge \boldsymbol{\xi}\,.$$

$$\nabla_X h_{ab} = 2X_{[a}\xi_{b]}\,, \quad \xi_b = \frac{1}{D-1}\nabla_d h^d{}_b\,.$$

Let us postulate the existence of this 2-form and find the consequences (less restrictive than Kahler 2-form)

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Let us postulate the existence of this 2-form and find the consequences (less restrictive than Kahler 2-form)

Eigenvalues of $-\boldsymbol{h}^2$:

$$\{\underbrace{x_1^2 \ldots\ldots x_1^2}_{2l_1} \ldots\ldots \underbrace{x_n^2 \ldots\ldots x_n^2}_{2l_n} . \underbrace{\xi_1^2 \ldots\ldots \xi_1^2}_{2m_1} \ldots\ldots \underbrace{\xi_N^2 \ldots\ldots \xi_N^2}_{2m_N} . \underbrace{0 \ldots\ldots 0}_{K}\} \,.$$

Canonical metric element:

- T. Houri, T. Oota, Y. Yasui, Phys. Lett. B666 (2008) 391, [arXiv:0805.0838].
- T. Houri, T. Oota, Y. Yasui, Class. Quantum Grav. 26 (2009) 045015, [arXiv: 0805.3877].

3) Canonical metric element

a) Darboux basis:

$$g = \delta_{ab}\boldsymbol{\omega}^{\hat{a}}\boldsymbol{\omega}^{\hat{b}} = \sum_{\mu=1}^{n}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon\boldsymbol{\omega}^{\hat{0}}\boldsymbol{\omega}^{\hat{0}} \,.$$

$$h = \sum_{\mu=1}^{n} x_\mu \boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}} \,.$$

$$\boxed{D = 2n + \varepsilon \,.}$$

(PCKY is non-degenerate)

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b) Towers of symmetries:

construction based on the following Lemma:

Lemma ([Krtouš *et al.*, 2007b]). *Let $\boldsymbol{k}^{(1)}$ and $\boldsymbol{k}^{(2)}$ be two closed CKY tensors. Then their exterior product $\boldsymbol{k} \equiv \boldsymbol{k}^{(1)} \wedge \boldsymbol{k}^{(2)}$ is also a closed CKY tensor.*

P. Krtouš, DK, D.N. Page, V.P. Frolov, Killing-Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions, JHEP 0702 (2007) 004.

Towers of hidden symmetries:

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Killing tensors:

$$K^{(j)}_{ab} \equiv \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}{}_{ac_1 \ldots c_{D-2j-1}} f^{(j)}{}_{b}{}^{c_1 \ldots c_{D-2j-1}} .$$

$$K^{(j)} = \sum_{\mu=1}^{n} A^{(j)}_{\mu} (\omega^{\hat{\mu}} \omega^{\hat{\mu}} + \tilde{\omega}^{\hat{\mu}} \tilde{\omega}^{\hat{\mu}}) + \varepsilon A^{(j)} \omega^{\hat{0}} \omega^{\hat{0}} .$$

$$\nabla_{(a} K^{(j)}_{bc)} = 0$$

where

$$A^{(j)} = \sum_{\nu_1 < \cdots < \nu_j} x^2_{\nu_1} \ldots x^2_{\nu_j} , \quad A^{(j)}_{\mu} = \sum_{\substack{\nu_1 < \cdots < \nu_j \\ \nu_i \neq \mu}} x^2_{\nu_1} \ldots x^2_{\nu_j} ,$$

Tower of explicit symmetries:

Primary Killing vector:

$$\boldsymbol{\xi}^{(0)} \equiv \boldsymbol{\xi}$$

$$\xi_b^{(0)} \equiv \xi_b = \frac{1}{D-1} \nabla_d h^d{}_b \,.$$

Secondary Killing vectors:

$$\boldsymbol{\xi}^{(j)} \equiv \boldsymbol{\eta}^{(j)} \; (j = 1, \ldots, n-1)$$

$$\xi^{(j)\,a} \equiv \eta^{(j)\,a} \equiv K^{(j)a}{}_b \xi^b \,.$$

Last Killing vector (odd dimensions):

$$\xi_{(\mu;\nu)} = 0.$$

$$\boldsymbol{\xi}^{(n)} \equiv \boldsymbol{f}^{(n)}$$

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$$[\boldsymbol{\xi}^{(i)}, \boldsymbol{K}^{(j)}] = 0\,, \quad [\boldsymbol{\xi}^{(i)}, \boldsymbol{\xi}^{(j)}] = 0\,.$$

$$\left[K^{(j)}, K^{(l)}\right]_{abc} \equiv K^{(j)}_{e(a}\nabla^e K^{(l)}_{bc)} - K^{(l)}_{e(a}\nabla^e K^{(j)}_{bc)} = 0$$

c) Canonical coordinates:

- the 'eigenvalues' x_μ are 'natural' coordinates.

- these n coordinates can be 'upgraded' by adding $n+\varepsilon$ new coordinates ψ_i

$$\boldsymbol{\xi}^{(0)} = \boldsymbol{\partial}_{\psi_0}\,, \quad \boldsymbol{\xi}^{(i)} = \boldsymbol{\partial}_{\psi_i}\,, \quad \boldsymbol{\xi}^{(n)} = \boldsymbol{\partial}_{\psi_n}\,.$$

d) Canonical metric:

$$\boldsymbol{g} = \delta_{ab}\boldsymbol{\omega}^{\hat{a}}\boldsymbol{\omega}^{\hat{b}} = \sum_{\mu=1}^{n}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon\boldsymbol{\omega}^{\hat{0}}\boldsymbol{\omega}^{\hat{0}}\,,$$

$$\boldsymbol{\omega}^{\hat{\mu}} = \frac{\boldsymbol{d}x_\mu}{\sqrt{Q_\mu}}\,, \quad \tilde{\boldsymbol{\omega}}^{\hat{\mu}} = \sqrt{Q_\mu}\sum_{j=0}^{n-1}A_\mu^{(j)}\boldsymbol{d}\psi_j\,, \quad \boldsymbol{\omega}^{\hat{0}} = \sqrt{\frac{-c}{A^{(n)}}}\sum_{j=0}^{n}A^{(j)}\boldsymbol{d}\psi_j\,.$$

$$Q_\mu = \frac{X_\mu}{U_\mu}\,, \quad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2)\,.$$

$$X_\mu = X_\mu(x_\mu)\,.$$

4) Kerr-NUT-(A)dS spacetime

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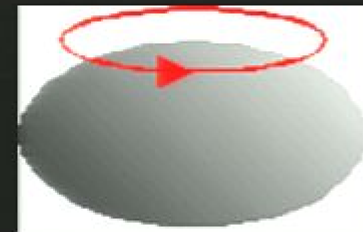


$$X_\mu = X_\mu(x_\mu).$$

Einstein space condition:

$$R_{ab} = (-1)^n (D-1) c_n \, g_{ab} \, ,$$

implies the specific form of metric functions:

$$X_\mu = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2} \, .$$

W. Chen, H. Lü and C. N. Pope, Class. Quant. Grav. 23 , 5323 (2006).

Constants are related to mass, NUT parameters, rotations, and cosmological constant

. T. Houri, T. Oota, Y. Yasui, Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness, Phys.Lett.B656 (2007) 214.

. P. Krtouš, V. P. Frolov, DK, Hidden Symmetries of Higher Dimensional Black Holes and Uniqueness of the Kerr-NUT-(A)dS spacetime, Phys. Rev. D 78 (2008) 064022.

5) Parallel transport

- Let $\gamma(\tau)$ be a timelike geodesic
- τ be its affine parameter
- u^a be its normalized tangent vector

$$u^a = \frac{dx^a}{d\tau}\,.$$

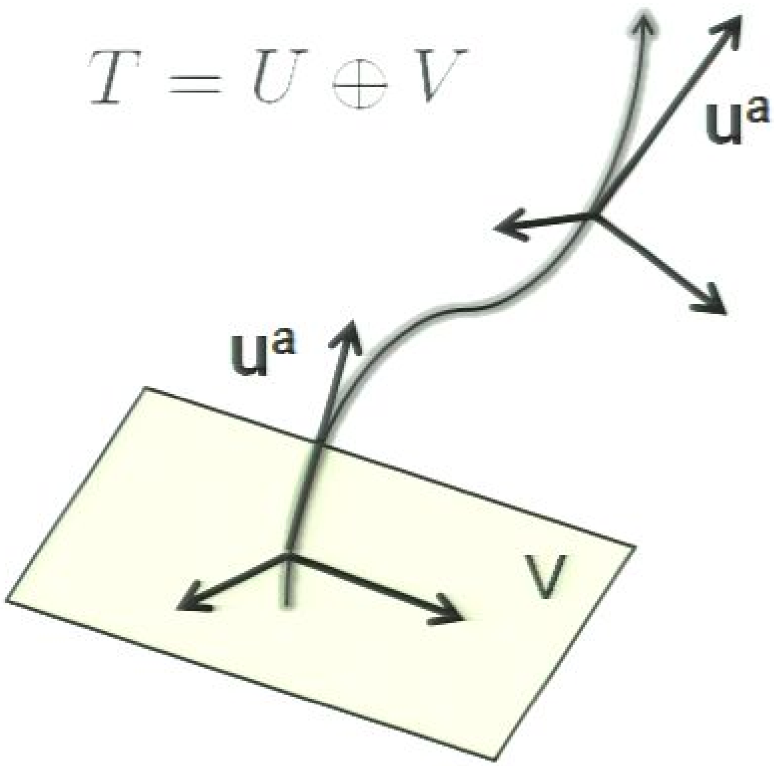
- We denote covariant derivative of tensor T along γ by

$$\dot{\boldsymbol{T}} = \nabla_u \boldsymbol{T} = u^a \nabla_a \boldsymbol{T}\,.$$

- In particular $\dot{u}^a = 0\,.$

a) Parallel-transported frame

$$T = U \oplus V$$



Projector along geodesic:

$$P : T \to V$$

given by

$$P_a^b = \delta_a^b + u^b u_a$$

b) 2-form F

$$F_{ab} = P_a^c P_b^d h_{cd} \,.$$

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It is parallel-transported !

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where we have used

$$P_a^c u_c = 0 \,, \quad \dot{P}_a^c = 0 \,.$$

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Hence, any object constructed from F and metric g is parallel transported. In particular, this is true for the invariants constructed from F, such as its eigenvalues.

c) Complete integrability of geodesic motion

Definition. A motion in M^D is *completely integrable* if there exist D functionally independent integrals of motion which are in *involution*, that is, they mutually Poisson commute of one another [Arnol'd, 1989], [Kozlov, V. V., 1983].

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$$\kappa_j = K^{(j)}_{\ \ ab} u^a u^b = \boldsymbol{u} \cdot \boldsymbol{K}^{(j)} \cdot \boldsymbol{u} \,. \qquad \nabla_{(a} K^{(j)}_{bc)} = 0$$

Geodesic velocity:

$$\dot{x}_\mu = \frac{\sigma_\mu}{|U_\mu|}\left(X_\mu V_\mu - W_\mu^2\right)^{1/2},$$

$$\dot{\psi}_k = \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu X_\mu} W_\mu - \varepsilon \frac{\Psi_n}{cA^{(n)}} \delta_{kn}.$$

where:

Constants $\sigma_\mu = \pm 1$ *are independent of one another, and*

$$V_\mu \equiv \sum_{j=0}^{m} (-x_\mu^2)^{n-1-j} \kappa_j, \quad W_\mu \equiv \sum_{k=0}^{m} (-x_\mu^2)^{n-1-k} \Psi_k, \quad \kappa_n \equiv -\frac{\Psi_n^2}{c}.$$

- D.N. Page, DK, M. Vasudevan, P. Krtouš, Complete Integrability of Geodesic Motion in General Higher-Dimensional Rotating Black-Hole Spacetimes, Phys. Rev. Lett. 98 (2007) 061102.
- P. Krtouš, DK, D.N. Page, M. Vasudevan, Constants of Geodesic Motion in Higher-Dimensional Black-Hole Spacetimes, Phys. Rev. D 76 (2007) 084034.
- T. Houri, T. Oota, Y.Yasui, Closed conformal Killing-Yano tensor and geodesic integrability, J.Phys.A41 (2008) 025204.

d) Darboux spaces of F

Eigenvalue problem:

$$\boldsymbol{F}^2\boldsymbol{v} = -\lambda_\mu^2\boldsymbol{v}\,, \quad \boldsymbol{v} \in V_\mu\,.$$

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_p\,.$$

then we have

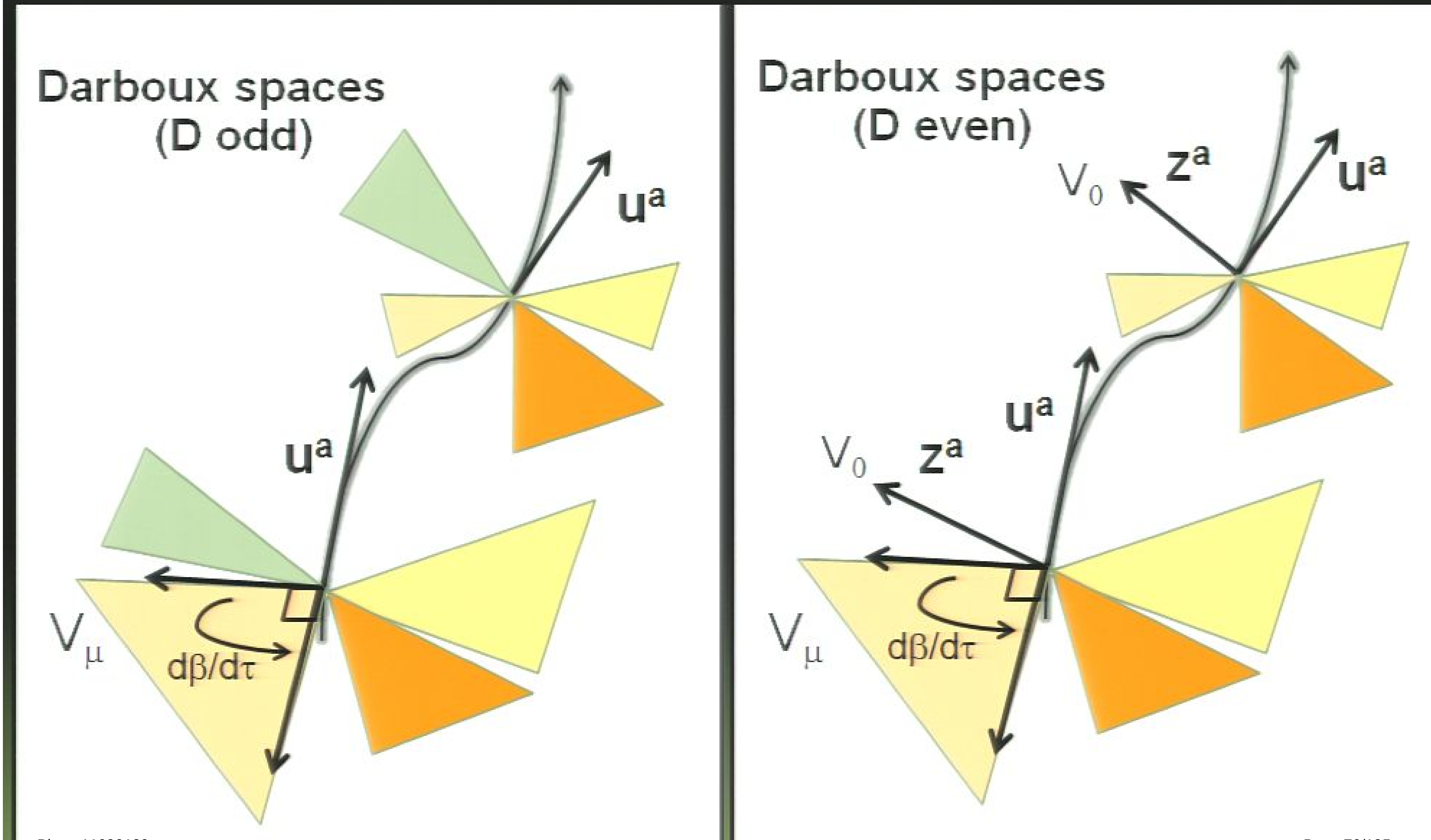
$$\boldsymbol{F}^2\dot{\boldsymbol{v}} = \nabla_u(\boldsymbol{F}^2\boldsymbol{v}) = \nabla_u(-\lambda_\mu^2\boldsymbol{v}) = -\lambda_\mu^2\dot{\boldsymbol{v}}$$

$$\dot{\boldsymbol{v}} \in V_\mu \quad \text{for}\ \forall\, \boldsymbol{v} \in V_\mu$$

Darboux spaces of F (eigenspaces of F^2) are independently parallel-transported !!!

P. Connell, V.P. Frolov, DK, Solving parallel transport equations in the higher-dimensional Kerr-NUT-(A)dS spacetimes, Phys. Rev. D 78 (2008) 024042.

e) Generic picture of parallel transport



6) Separability of test field equations

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a) Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \lambda} + g^{ab}\,\partial_a S\,\partial_b S = 0\ .$$

additive separation

$$S = -w\lambda + \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_{k=0}^{m} \Psi_k c_k\ .$$

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Wonderful Identities:

$$U \equiv \prod_{\mu,\nu=1}^{n} (x_\mu^2 - x_\nu^2)\ , \quad \Omega_\mu \equiv \frac{U}{U_\mu}\ .$$

$$\sum_{\mu=1}^{n} x_\mu^{2(n-1)} \Omega_\mu = (-1)^{n-1} U\ .$$

$$\sum_{\mu=1}^{n} x_\mu^{2k}\, \Omega_\mu = 0 \quad \text{for} \quad k = 0, \ldots, n-2\ .$$

$$\sum_{\mu=1}^{n} \frac{1}{x_\mu^2} \Omega_\mu = \frac{U}{A^{(n)}}\ .$$

$$\sum_{\mu=1}^{n} \frac{A_\mu^{(k)}}{x_\mu^2} \Omega_\mu = \frac{A^{(k)}}{A^{(n)}} U \quad \text{for} \quad k = 0, \ldots, n-1\ .$$

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b) Klein-Gordon equation

$$\Box\Phi = \frac{1}{\sqrt{|g|}}\partial_a(\sqrt{|g|}g^{ab}\partial_b\Phi) = \mu^2\Phi.$$

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$$\Phi = \prod_{\mu=1}^{n} R_\mu(x_\mu)\prod_{j=0}^{m} e^{i\Psi_j\psi_j}.$$

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determinant:

$$\sqrt{|g|} \propto U P^{\varepsilon}, \quad P \equiv \prod_{\mu=1}^{n} x_\mu\,.$$

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$$(X_\mu R_\mu')' + \varepsilon\frac{X_\mu}{x_\mu}R_\mu' + \left(V_\mu - \frac{W_\mu^2}{X_\mu}\right)R_\mu = 0\ .$$

V.P. Frolov, P. Krtouš, DK, Separability of Hamilton-Jacobi and Klein-Gordon Equations in General Kerr-NUT-AdS Spacetimes, JHEP 0702 (2007) 005.

c) Intrinsic characterization: Separability Structures

= classes of separable charts for the Hamilton-Jacobi equation

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Theorem. A manifold $(V_D, \boldsymbol{g})$ admits a δ_m-separability structure iff it admits m commuting Killing vectors ψ_k $(k = 1, \ldots, m)$ and $D - m$ Killing tensors $\boldsymbol{K}^{(\alpha)}$ $(\alpha = 0, \ldots, D - m - 1)$, all of them independent, which satisfy:
(i) In the Lie algebra of Killing tensors with Schouten–Nijenhuis brackets the commutation relations

$$K^{(\alpha)}_{e(a} \nabla^e K^{(\beta)}_{bc)} - K^{(\beta)}_{e(a} \nabla^e K^{(\alpha)}_{bc)} = 0 \, .$$

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- S. Benenti and M. Francaviglia, Gen. Rel. Grav. 10 , 79 (1979).
- M. Demianski and M. Francaviglia, J. Theor. Phys. 19, 675 (1980).

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Symmetries generated from the PCKY tensor provide a textbook example of this theory !

Separability structure and Klein-Gordon equation

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"Quantum anomaly" disappears:

$$[H, \nabla_a K^{ab} \nabla_b] = \frac{4}{3} \nabla_a \left(K_c{}^{[a} R^{b]c} \right) \nabla_b$$

1) In Einstein space

Corollary. If the manifols is an Einstein space, the Hamilton–Jacobi equation is separable iff the same holds for the Klein–Gordon equation.

2) In Killing-Yano case

d) Dirac equation

Explicit separation: T. Oota, Y. Yasui, Phys. Lett. B659, 688-693, 2008.

Theory of separability is not well established!

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$$L = 2X^a \lrcorner\, \omega \nabla_a + \frac{\pi - 1}{\pi} d\omega - \frac{n - \pi - 1}{n - \pi}\delta\omega \tag{1.1}$$

and obeys

$$DL - (\eta L)D = \frac{\eta}{n - \pi}\delta\omega D\,. \tag{1.2}$$

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For separability we need a complete set of mutually commuting operators.

- By appropriate choice of α we can prove the following:

Corollary (Commuting operator). *The most general first-order operator $\tilde{L}$ which commutes with the Dirac operator D, $[\tilde{L}, D] = 0$, splits into the even and odd parts*

$$\tilde{L} = \tilde{L}_e + \tilde{L}_o\,. \tag{2.15}$$

where

$$\tilde{L}_e \equiv K_{\omega_o} = X^a \lrcorner\, \omega_o \nabla_a + \frac{\pi - 1}{2\pi} d\omega_o\,. \quad \omega_o \text{ is odd KY form}\,. \tag{2.16}$$

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- The requirement $[L_1, L_2] = L_3$ imposes algebraic conditions on ω.

- for separability we need to be able to find a complete subset of mutually commuting operators

• Algebra of symmetry operators

Lemma 1. *Let k and m be two Killing 1-forms and K_k and K_m the corresponding operators (3.2). Then the following statement holds:*

$$[K_k, K_m] = K_{L_k m} \, . \tag{2.25}$$

Proof:

$$[K_k, K_m] = k^a m^b \nabla_{(a}\nabla_{b)} - m^a k^b \nabla_{(a}\nabla_{b)} + \left[k^a(\nabla_a m^b) - m^a(\nabla_a k^b)\right]\nabla_b + \text{ zeroth order terms.}$$

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Similarly one can show that under certain (strong) algebraic conditions we have:

$$[K_\kappa, K_\lambda] = K_{[\kappa,\lambda]_{\rm KY}}\,,\quad [K_\mu, M_\omega] = M_{[\mu,\omega]_{\rm KY}}\,,\quad [M_\alpha, M_\beta] = K_{[\alpha,\beta]_{\rm KY}}$$

where new tensors are given in terms of Killing-Yano brackets. Some of them related to Schouten-Nijenhuis brackets:

$$\frac{p!q!}{(p+q-1)!}[\alpha, \beta]_{NS} = (X^a \lrcorner\, \alpha) \wedge \nabla_a \beta + (-1)^{pq}(X^a \lrcorner\, \beta) \wedge \nabla_a \alpha\,.$$

Complete set of commuting operators

Proposition 3.2 (Complete set of commuting operators). *The most general spacetime admitting the PCKY tensor admits the following complete set of commuting operators:*

$$\{D, K_{\xi^{(0)}}, \dots, K_{\xi^{(N-1+\varepsilon)}}, M_{h^{(1)}}, \dots, M_{h^{(N-1)}}\}. \tag{3.23}$$

In odd dimensions another complete set is

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Proof:
- D commutes with all these operators
- Algebraic conditions are satisfied

$$[K_\kappa, K_\lambda] = K_{[\kappa,\lambda]_{\mathrm{KY}}}\,,\quad [K_\mu, M_\omega] = M_{[\mu,\omega]_{\mathrm{KY}}}\,,\quad [M_\alpha, M_\beta] = K_{[\alpha,\beta]_{\mathrm{KY}}}$$

however

$$\mathcal{L}_{\xi^{(k)}\sharp}\xi^{(l)} = 0\,,\quad \mathcal{L}_{\xi^{(k)}\sharp}h^{(l)} = 0\,,\qquad [h^{(k)}, h^{(l)}]_{\mathrm{KY}} = 0\,.$$

all operators mutually commute & independent → **complete set**

M. Cariglia, P. Krtous, DK, Commuting symmetry operators of the Dirac equation, Killing-Yano and Schouten-Nijenhuis brackets, today

7) Algebraic type and Kerr-Schild form

Special algebraic type:

Integrability condition for PCKY

$$C_{AB}{}^{C} h_{C} = 0$$

$$\left(C_{bcd}{}^{e}\delta_a^f + C_{adc}{}^{e}\delta_b^f + C_{dab}{}^{e}\delta_c^f + C_{cba}{}^{e}\delta_d^f\right) h_{fe} = 0\,.$$

using the canonical form ⇒ Many components of the Weyl tensor vanish, **type D**

(L. Mason & A. Taghavi-Chabert, J.Geom. Phys. 60, 907-923, 2010.)

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Multi Kerr-Schild form:

$$\boldsymbol{g} = \boldsymbol{g}_0 + \sum_{\mu=1}^{n} \frac{H_\mu(x_\mu)}{U_\mu}\,\boldsymbol{l}_\mu^2\,, \qquad \boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{d}x_\mu \wedge \boldsymbol{l}_\mu\,.$$

(W. Chen and H. Lu, Phys. Lett. B658, 158, 2008.)

8) Generalized hidden symmetries

a) Motivation

• All the "miraculous" results connected with CKY tensors limited to canonical spacetimes (vacuum, type D)

• One would like to extend to wider class of non-vacuum spacetimes, for example to BHs of various supergravities

• It was known that some of these solutions possess Killing tensor and allow separability of HJ and KG equations (D.D.K. Chow, 0811.1264)

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What about KY tensors?
(in the presence of matter fields)

b) Systematic derivation:

By studying symmetry operators of "flux-modified" Dirac operator

$$\mathcal{D} = \gamma^a \nabla_a + \sum_p \frac{1}{p!} B_{a_1 \dots a_p} \gamma^{a_1} \dots \gamma^{a_p}$$

This includes the case of a massive Dirac operator, the Dirac operator minimally coupled to a Maxwell field, the Dirac operator in the presence of torsion, as well as more general operators. (In the backgrounds considered for superstring or supergravity theories, the metric is often supplemented by other fields or fluxes which couple to the spinor field and modify the Dirac equation.)

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Generalized conformal Killing-Yano system

$$K_a \omega + \{B, \omega_a\}_\perp = 0\,.$$

(In general, a coupled system of linear first order partial differential equations for homogeneous parts of inhomogeneous form ω. This system decouples if B is a combination of a function, 1-form and 3-form.)

c) CKY tensors in the presence of torsion

$$\nabla^T_X k - \frac{1}{p+1} X \lrcorner\, d^T k + \frac{1}{D-p+1} X^\flat \wedge \delta^T k = 0\,.$$

- DK, H.K. Kunduri, Y. Yasui, Phys. Lett. B678 (2009) 240.
- Yano & Bochner, Curvature and Betti numbers, 1952.

Similar properties as standard CKY tensors:

- 1-form is a conformal Killing 1-form
- Hodge duality
- Conformal rescaling, Lie derivative
- GCCKY tensors form a graded algebra w.r.t. wedge product
- GCCKY 2-form produces the tower of GCCKY tensors
- These give rise to standard Killing tensors
- Relation to symmetry operators (anomalies)

PCKY tensor in the presence of torsion

$$\nabla^T_X h = X^\flat \wedge \xi\,, \quad \xi = -\frac{1}{4}\,\delta^T h\,.$$

- S-Q. Wu, Phys. Rev. D80 (2009) 069902; Phys. Rev. D80 (2009) 084009.
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Minimal gauged D=5 supergravity

$$\pounds = *(R+\Lambda) - \frac{1}{2} F \wedge *F + \frac{1}{3\sqrt{3}} F \wedge F \wedge A\,.$$

Identify $T = \frac{1}{\sqrt{3}} *F\,.$ → $\delta^T T = 0\,, \quad d^T T = 0\,.$

(Torsion is "harmonic")

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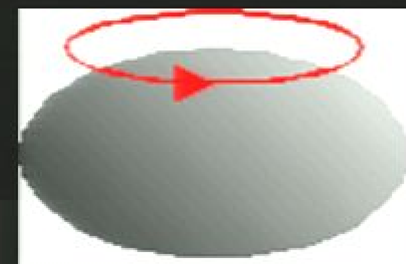
Chong-Cvetic-Lu-Pope black hole

(Phys. Rev. Lett 95 (2005) 161301)

Another example: Kerr-Sen black hole

[A. Sen, Phys. Rev. Lett 69 (1992) 1006]



$$S = -\int d^4x \sqrt{-g_s} e^{-\Phi} \Big(-R_s + \frac{1}{12} H_{abc} H^{abc} - g_s^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{8} F_{ab} F^{ab} \Big) \, .$$

identify $$\boldsymbol{T = H}$$

(T. Houri, DK, C. Warnick, Y. Yasui, JHEP 1007:055,2010)

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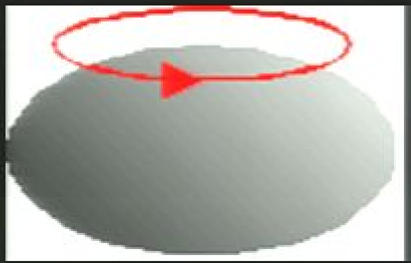

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Possesses GPCKY tensor with all the nice properties !!!

The same remains true for HD generalizations:

- M. Cvetic and D. Youm, Nucl. Phys. B477 (1996) 449.
- D.D.K. Chow, Class. Quant. Grav. 27 (2010) 205009.

9) Relation to Killing spinors and special Riemannian manifolds

- Conformal (twistor) spinor

$$\nabla^T_X \psi - \frac{1}{n} X \cdot D^T \psi = 0 \,.$$

$\Rightarrow$

$$\omega = (\psi_1 , e^{a_1 \ldots a_p} \psi_2) e_{a_1 \ldots a_p}$$

CKY forms (with torsion)

9) Relation to Killing spinors and special Riemannian manifolds

- **Conformal (twistor) spinor**

$$\nabla^T_X \psi - \frac{1}{n} X^\flat D^T \psi = 0\,.$$ $\Rightarrow$ $$\omega = (\psi_1 .\, e^{a_1 \dots a_p} \psi_2) e_{a_1 \dots a_p}$$

CKY forms (with torsion)

- **Special Riemannian manifolds:**

Wick rotate the Lorentzian manifold admitting (non-degenerate) hidden symmetry. Perform the "degenerating" (BPS) limit. Impose regularity.

example: Start with even-dimensional canonical metric admitting PCKY tensor

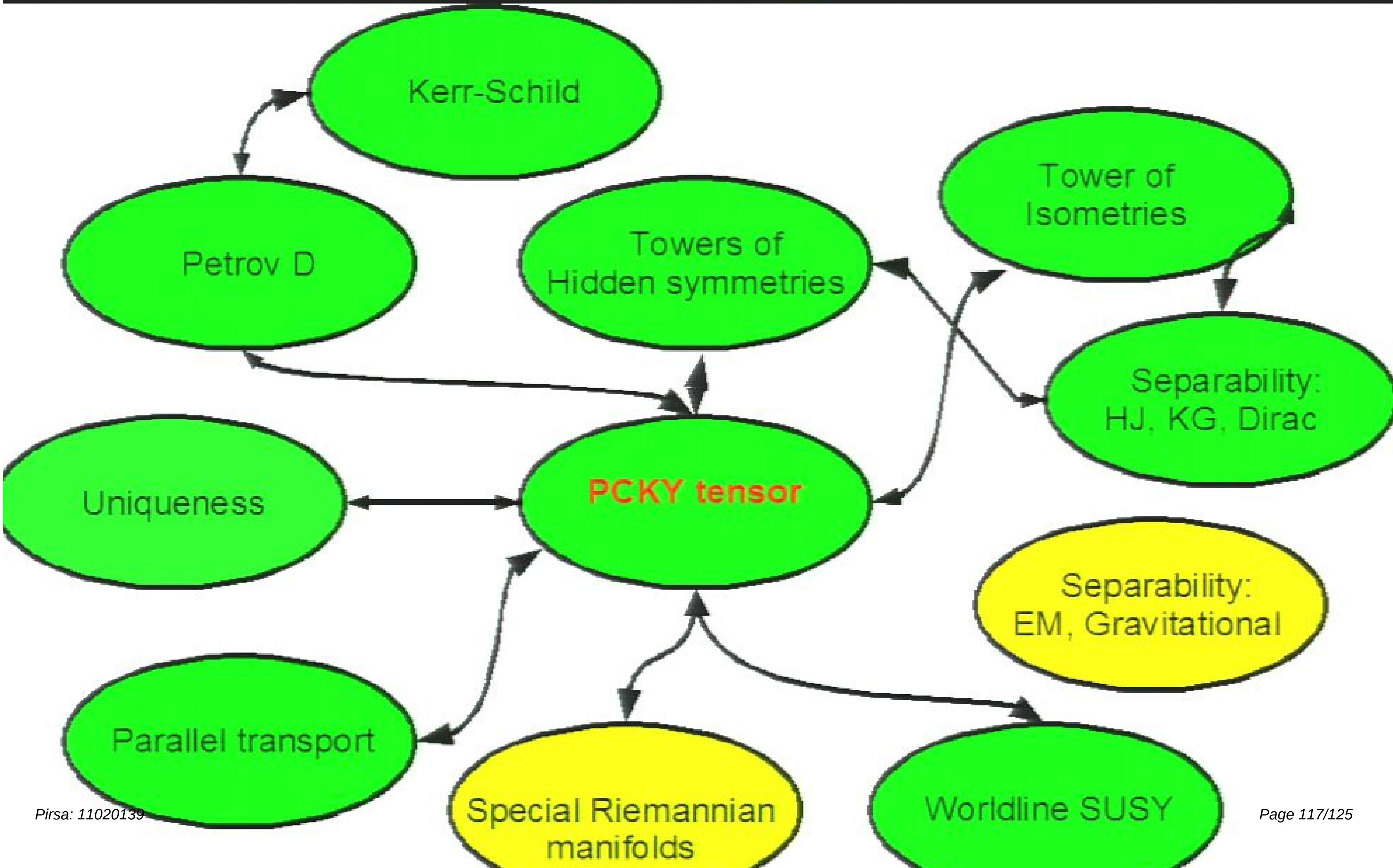
$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat{\mu}}\,.$$

perform a "degenerating" limit

$$x_\mu \to 1 - \epsilon x_\mu \qquad \varepsilon \to 0.$$

(In this way one can recover the most general explicitly known Einstein-Kahler manifold in all dimensions)

10) Hidden symmetry in higher-dimensions: summary



IV) Future directions

1) Is there a connection between KY tensors and separability of higher spin equations?

2) Can we find other examples of BH spacetimes where this symmetry exists? Is it possible to exploit generalized Killing-Yano tensors for the construction of new exact solutions?

3) Killing-Yano tensors: Are there further unknown connections? Is there hidden an "additional purpose"?

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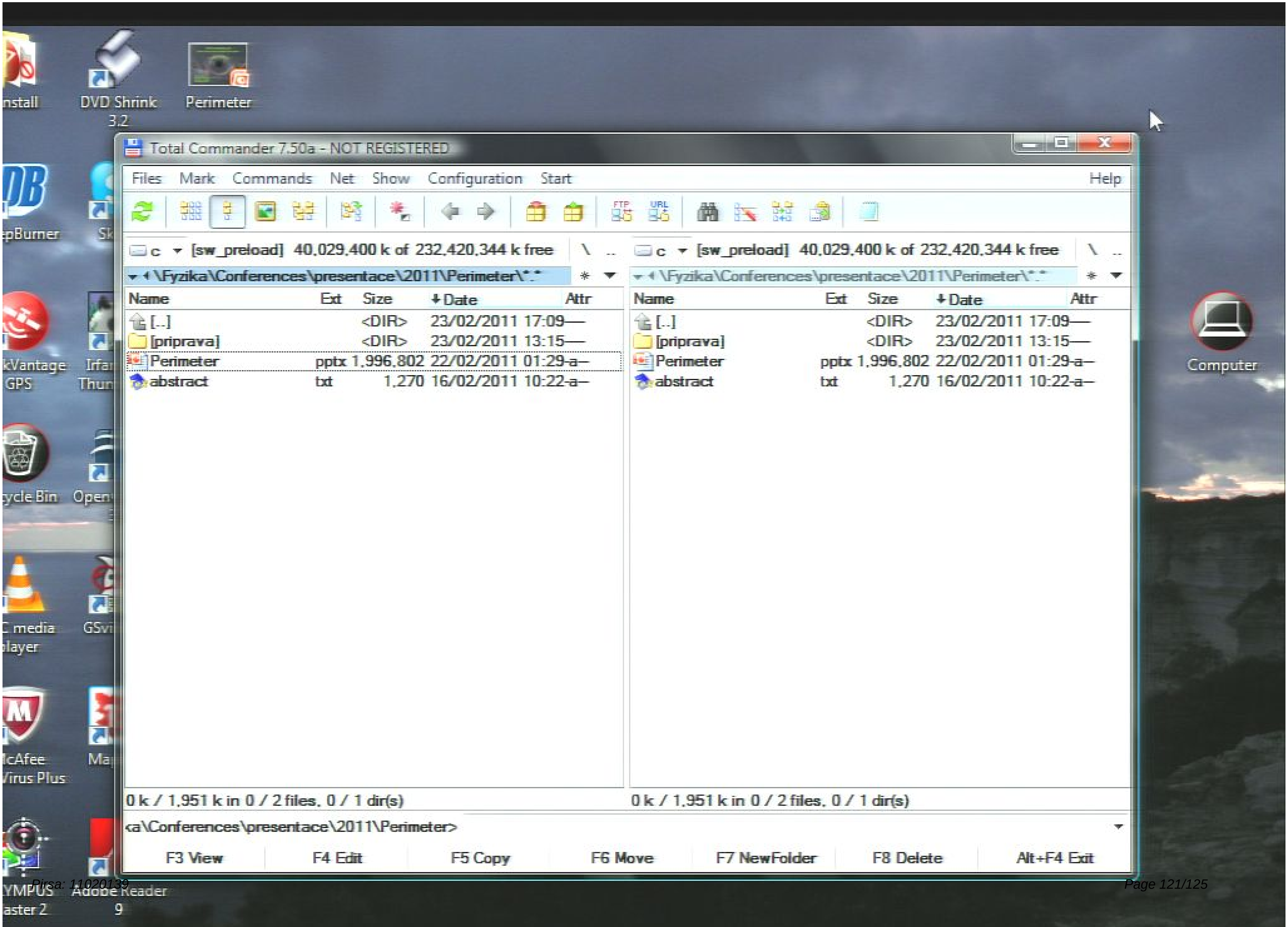
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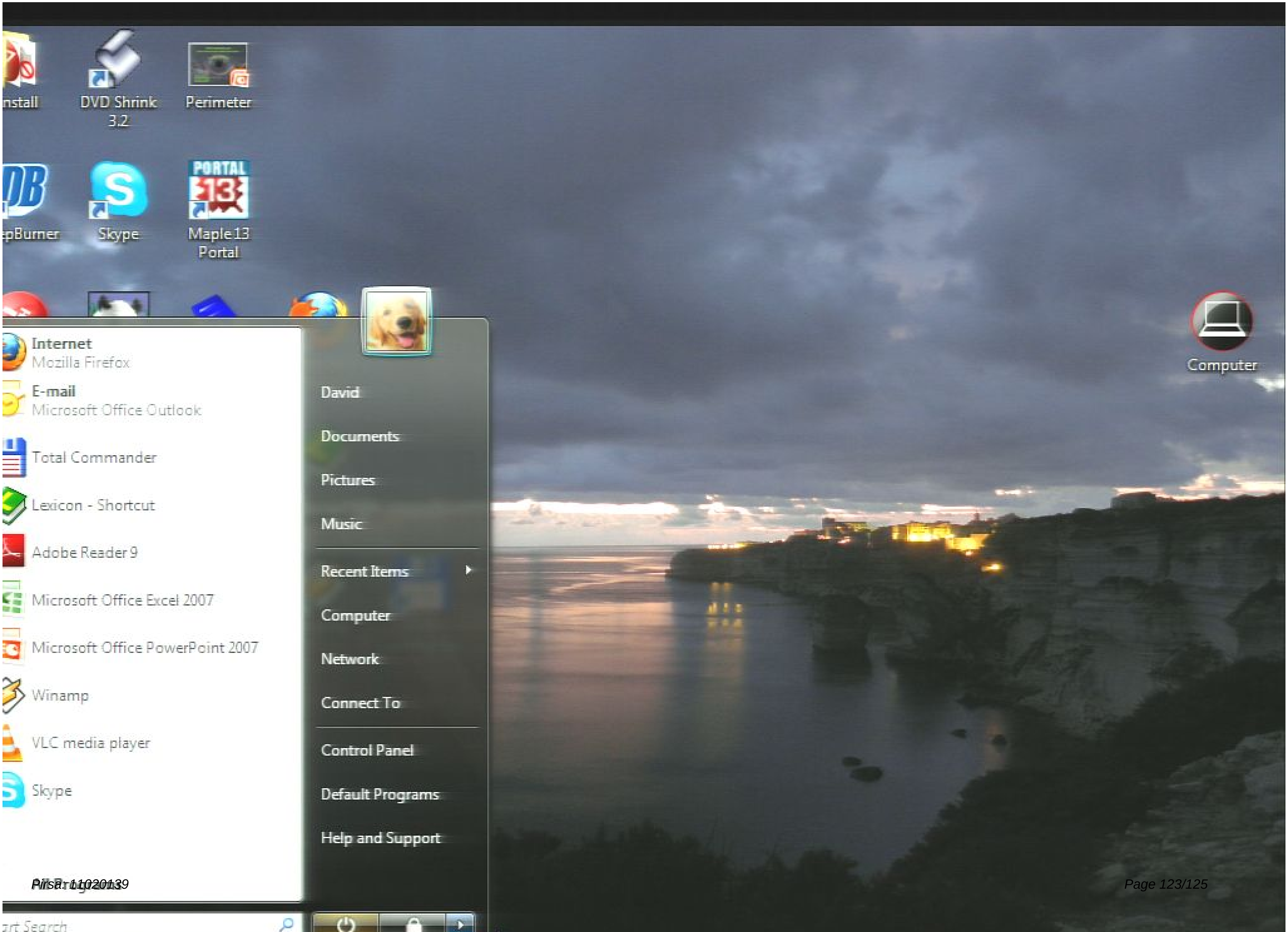
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