Title: Embedding DBI inflation in Scalar-tensor theory

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Abstract: The availability of high precision observational data in cosmology means that it is possible to go beyond simple descriptions of cosmic inflation in which the expansion is driven by a single scalar field. One set of models of particular interest involve the Dirac-Born-Infeld (DBI) action, arising in string cosmology, in which the dynamics of the field are affected by a speed limit in a manner akin to special relativity. In this talk, I will introduce a scalar-tensor theory in which the matter component is a field with a DBI action. Transforming to the Einstein frame, I will explore the effect of the resulting coupling on the background dynamics of the fields and the first-order perturbations. The coupling forces the scalar field into the minimum of its effective potential, so the dynamics are determined by the DBI field, which has the interesting effect of increasing the number of efolds of inflation and decreasing the boost factor of the DBI field. Focusing on this case, I will show that the power spectrum of the primordial perturbations is determined by the behaviour of the perturbations of the modified DBI field and calculate the effect of varying the model parameters on the inflationary observables.

# Outline

1. Introduction
   - Inflation with canonical scalar fields
   - Inflation with a DBI action
2. Coupled DBI inflation
   - The model
   - Minimal coupling
   - Non-minimal coupling
3. Cosmological perturbations
   - Perturbed equations
   - Numerical work
4. Summary

# Basic Equations

- The starting point of most cosmological models is the assumption that the universe is flat, isotropic and homogeneous on large scales.

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j .$$

- The matter content of the universe is assumed to be a perfect fluid

$$T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p)$$

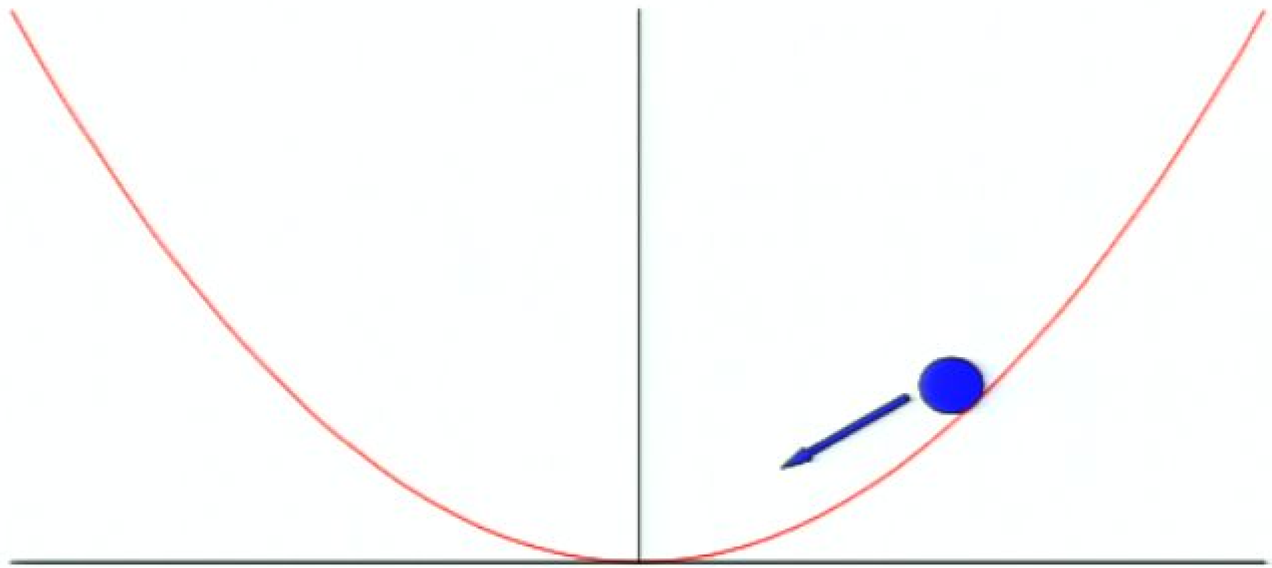
with energy density $\rho$, pressure $p$ and equation of state $p = w\rho$

The Einstein equations for this system are (where $M_{\rm Pl} = 1/\sqrt{8\pi G}$ and $H$ is the Hubble parameter)

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{\rho}{3M_{\rm Pl}^2} \qquad \frac{\ddot a}{a} = -\frac{1}{6M_{\rm Pl}^2}(1+3w)\rho$$

**The condition for an accelerating universe is $w < -\frac{1}{3}$.**

# Single-Field Inflation



- Simple models of inflation use a scalar field (the inflaton) and a power-law potential.
- The field slowly rolls down the potential and afterwards oscillates around the minimum, whereupon the inflaton decays.
- Fluctuations in the scalar field freeze-in as they cross the horizon, giving rise to an almost scale invariant spectrum of curvature perturbations.

# Single-Field Inflation

$$S_\phi = -\int d^4x\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + U(\phi)\right]$$

For a scalar field in an expanding background we have

$$\ddot{\phi} + 3H\dot{\phi} + U_{,\phi} = 0, \qquad w = \frac{\frac{1}{2}\dot{\phi}^2 - U}{\frac{1}{2}\dot{\phi}^2 + U}.$$

where $U_{,\phi} = \frac{dU(\phi)}{d\phi}$. If the field is slowly rolling ($\dot{\phi}^2 \ll U$), the first term can be neglected.

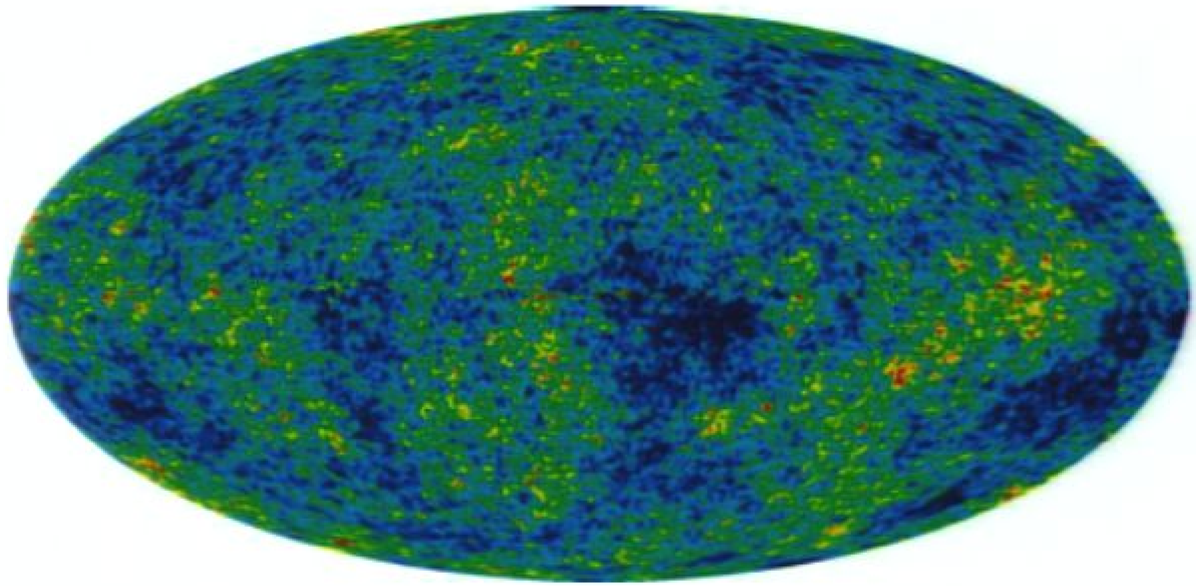
$$3H\dot{\phi} \approx -U_{,\phi}$$

## Slow-roll Parameters

$$\epsilon \approx \frac{1}{2}\left(\frac{U_{,\phi}}{U}\right)^2, \qquad \eta \equiv \frac{U_{,\phi\phi}}{U}.$$

These conditions can be written in terms of the slow roll parameters: $\epsilon$ and $\eta$. These satisfy $\epsilon, \eta \ll 1$ during inflation.

# Observational Handles



## Inflationary models can be constrained with observational data [1]

- Power spectrum amplitude: $P_{\mathrm{amp}} = 2.441^{+0.088}_{-0.092} \times 10^{-9}$
- Spectral index: $n_s = 0.963 \pm 0.012$
- Running of spectral index: $dn_S/d\ln k \simeq -0.03 \pm 0.02$
- Tensor-to-scalar ratio: $r \lesssim 0.35$
- Non-gaussianity: $|f_{NL}| \lesssim \mathcal{O}(100)$

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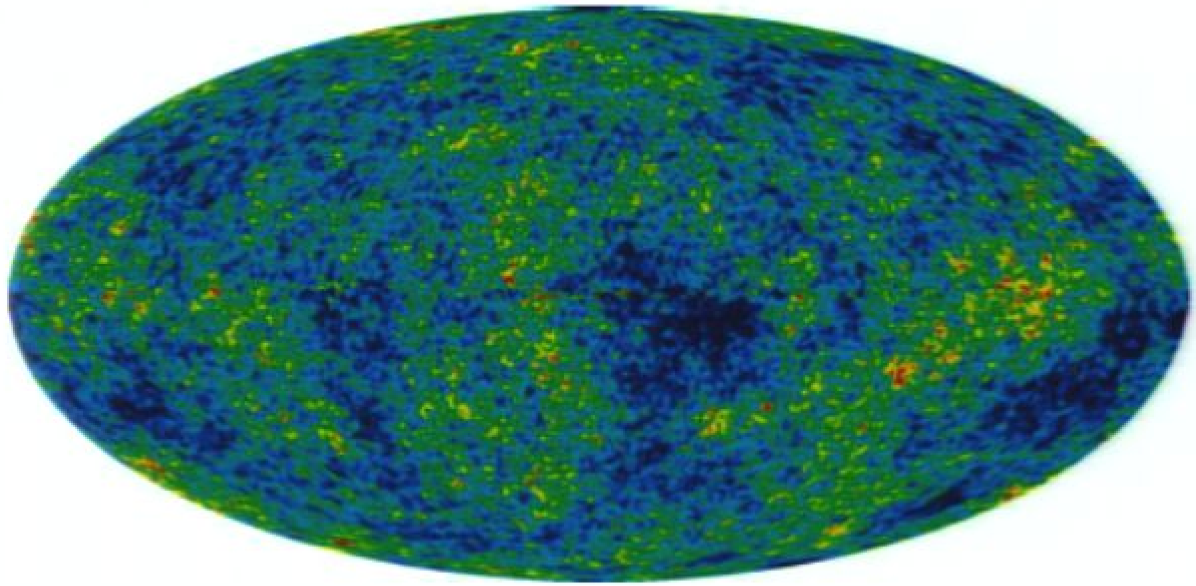

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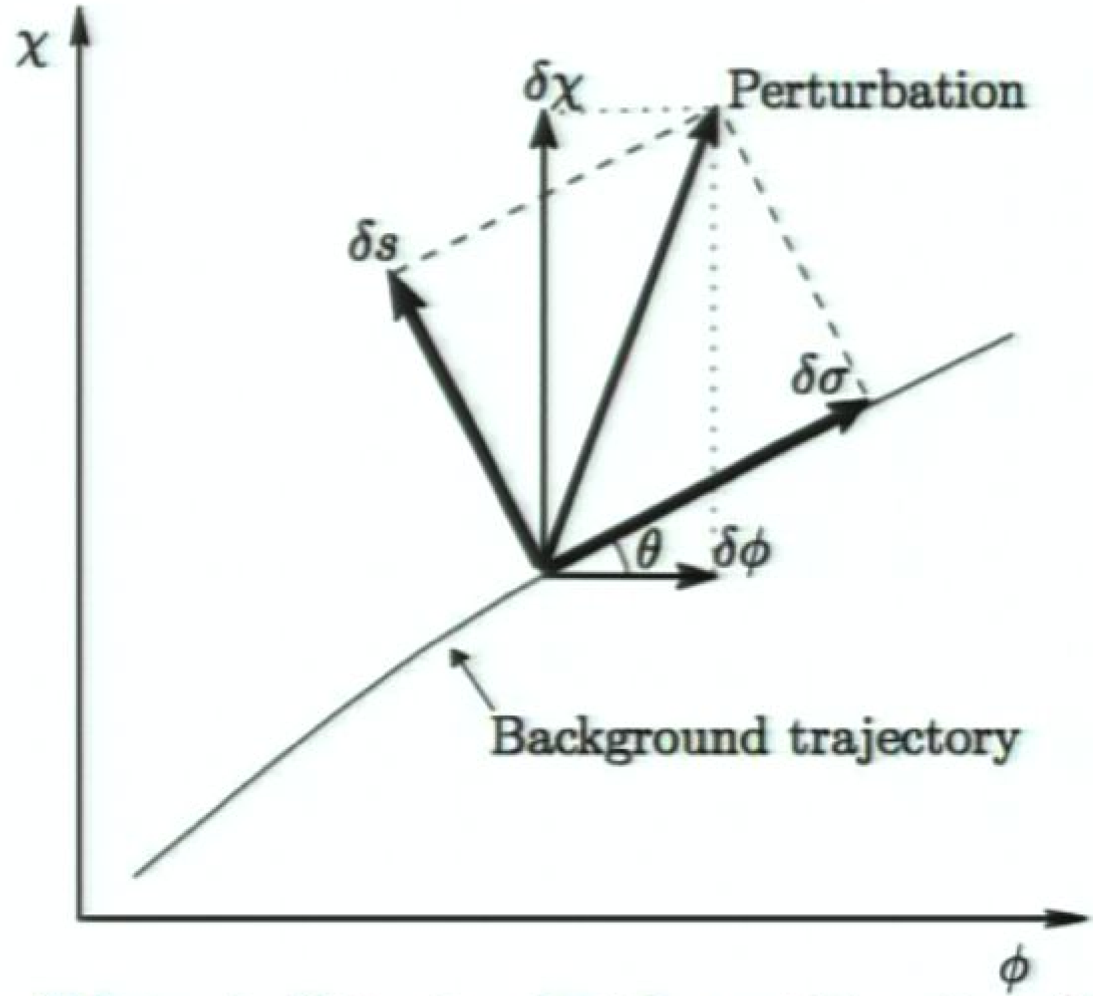
# Multiple Scalar Fields

- Many authors have studied inflation driven by multiple scalar fields.
- The perturbations of the fields are particularly interesting in multiple scalar field models as there are entropy (isocurvature) perturbations as well as adiabatic curvature perturbations to consider.
- The entropy modes act as an extra source for the curvature perturbation.
- The interaction between the fields can lead to deviations from the almost scale invariant spectrum.

# Multiple Scalar Fields




In the two-field case one can define a rotation to decompose the field perturbations into an adiabatic mode $\delta\sigma$ tangential to the background trajectory and an entropy mode $\delta s$ orthogonal to this.



picture credit: Gordon et al., 2000 [2]

# Adiabatic and Entropy Fields

Marco & Finelli, 2005 [3]

These ideas can be applied to coupled scalar field systems .

$$\mathcal{S} = \int d^4x \sqrt{-g}\left[\tfrac{1}{2}R - \tfrac{1}{2}(\nabla\phi)^2 - e^{2\beta\phi}(\nabla\chi)^2 - V(\phi,\chi)\right]$$

The fields have the following equations of motion.

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = \beta e^{2\beta\phi}\dot{\chi}^2, \qquad \ddot{\chi} + (3H + 2\beta\dot{\phi})\chi + e^{-2\beta\phi}V_{,\chi} = 0,$$

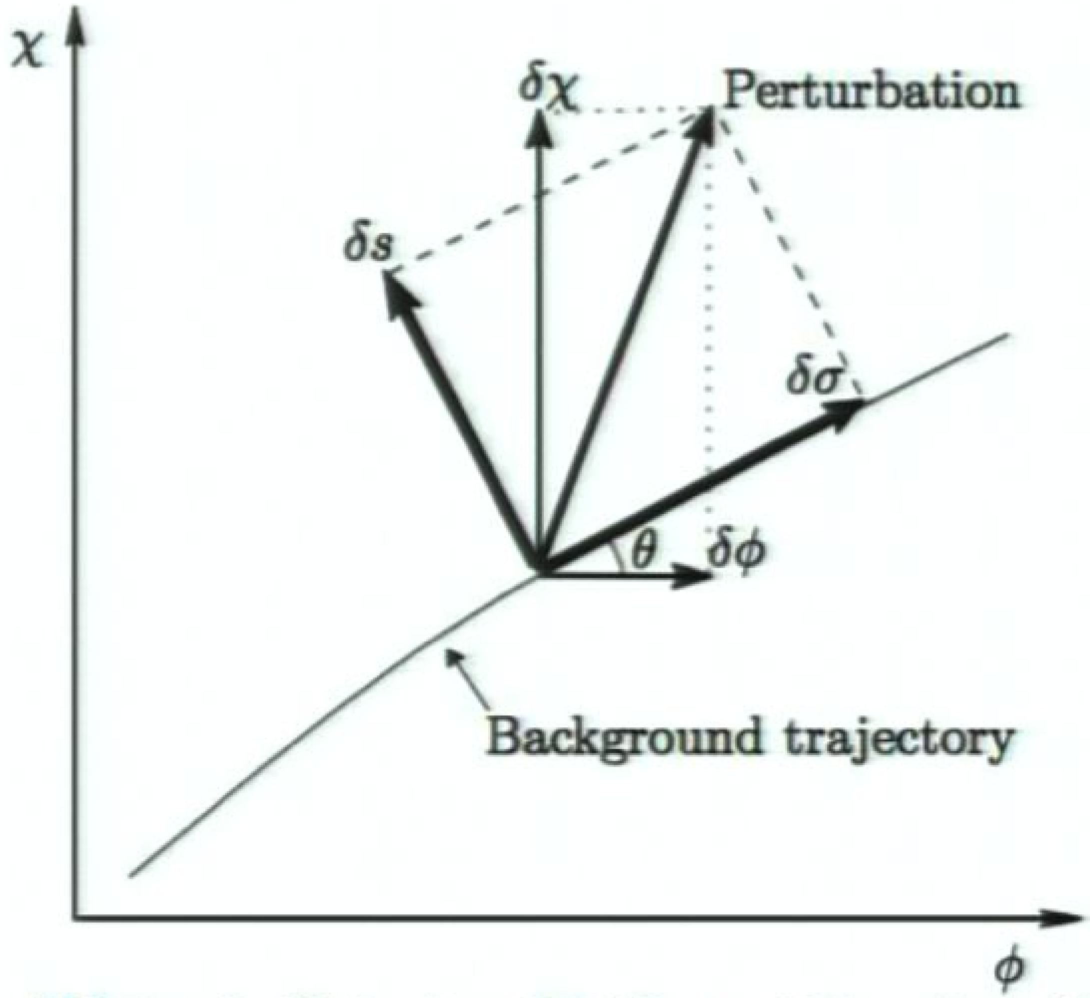
Along the background trajectory, $\dot{s} = 0$ and $\dot{\sigma} = \sqrt{-2\dot{H}}$.

Define the field rotation,

$$\delta\sigma = \cos\theta\delta\phi + \sin\theta e^{\beta\phi}\delta\chi, \qquad \delta s = e^{\beta\phi}\cos\theta\delta\chi - \sin\theta\delta\chi,$$

where $\cos\theta = \dot{\phi}/\dot{\sigma}$ and $\sin\theta = e^{\beta\phi}\dot{\chi}/\dot{\sigma}$.

# Multiple Scalar Fields

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# Adiabatic and Entropy Fields

- The evolution of the adiabatic and entropy field perturbations can now be calculated explicitly.

$$\ddot{\sigma} + 3H\dot{\sigma} + V_\sigma = 0, \qquad V_\sigma = V_{,\phi} \cos\theta + e^{-\beta\phi} V_{,\chi} \sin\theta.$$

- The (comoving) curvature perturbation $\mathcal{R}$ is related to the adiabatic perturbation by

$$\mathcal{R} = \frac{H}{\dot{\sigma}} Q_\sigma, \qquad \text{where} \qquad Q_\sigma = \delta\sigma + \frac{\dot{\sigma}}{H}\Psi$$

where $\Psi$ is the metric perturbation.

- At large scales, $\mathcal{R}$ is not constant, as in the single field case, but has a dependence on the entropy mode

$$\dot{\mathcal{R}} = -2H \frac{V_s}{\dot{\sigma}^2} \delta s$$

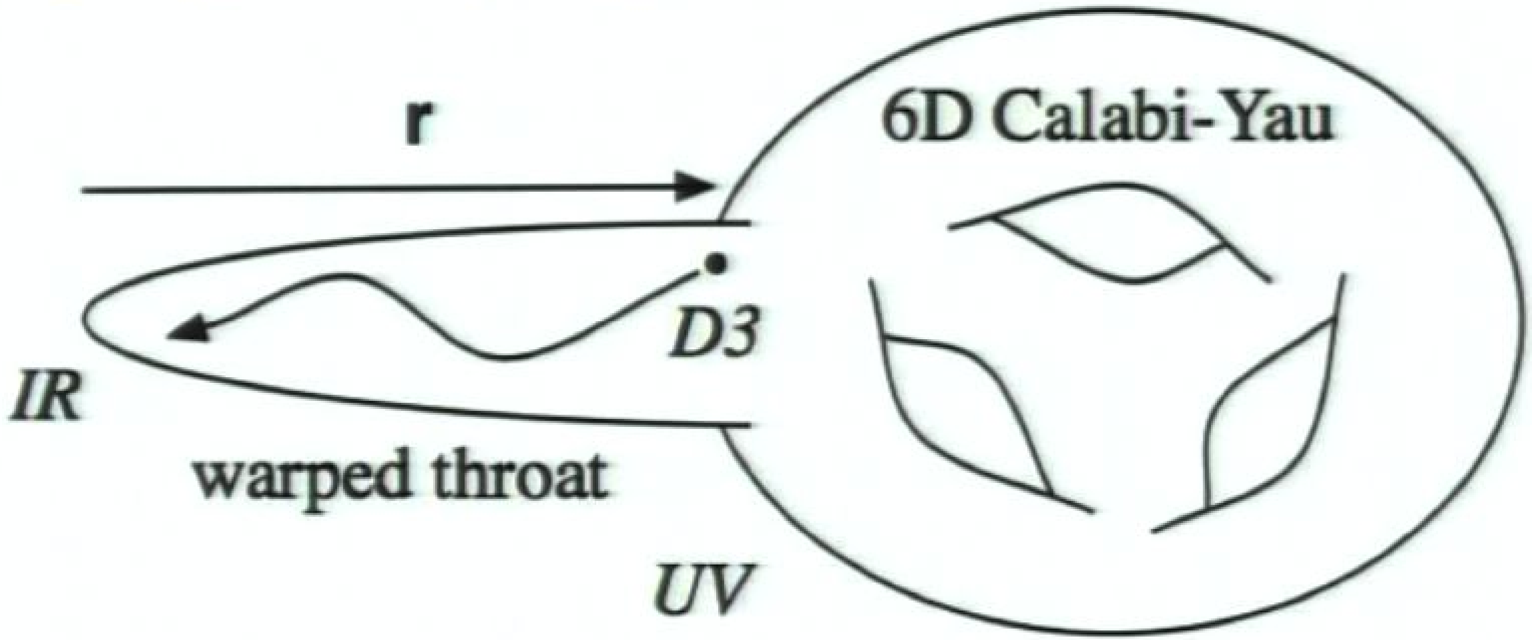
where $V_s = -V_{,\phi} \sin\theta + e^{-\beta\phi} V_{,\chi} \cos\theta$.

# Inflation from fundamental theory

- The general predictions are quite robust, however, for the details of inflation we need to understand the mechanism from the perspective of particle physics and/or fundamental theory.
- Since inflation takes place at high energies, there has been much interest in model building in the context of string theory, a framework in which there is no shortage of scalar fields.
- However, the abundance of light scalars (e.g. light moduli) not only complicates the dynamics of inflation, but also means that models must be tuned to prevent unwanted light moduli affecting the post-inflationary universe.
- As well as this, although heavy fields do not generally evolve during inflation they contribute to the potential that determines the evolution of the dynamical fields, further complicating the 'eta problem' in inflationary model building, in which one encounters large corrections to the flat potential required in slow-roll inflation.

# DBI inflation

(Silverstein & Tong, 2003 [5], Alishahiha et al., 2004 [6])




- Dirac-Born-Infeld (DBI) inflation is a specific example of 'stringy' inflation in which a D3 brane moves in a simplified compactified space, falling into a throat similar to a potential well.
- A speed limit is imposed upon the motion of the brane, which is dependent upon the throat geometry, which allows inflationary solutions with steep potentials.

# DBI inflation

## The action for the DBI field $\chi$ is

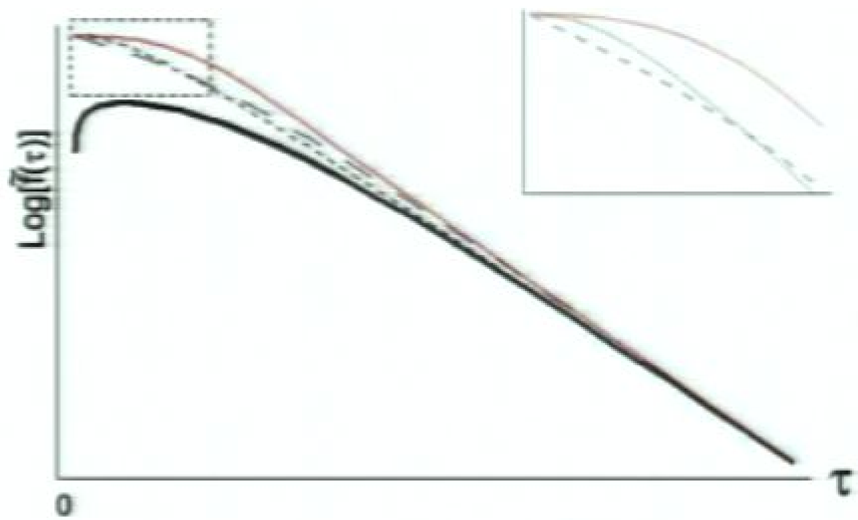
$$S = \int d^4x\sqrt{-g}\left[f^{-1}(\chi)(1-\gamma^{-1}) - V(\chi)\right]$$

**Warp factor** $f(\chi)$

- This is determined by the background geometry of the space, often taken to be a Klebanov-Strassler throat.
- This can be approximated phenomenologically by the mass-gap solution,

$$f(\chi) = \frac{\lambda}{(\chi^2+\mu^2)^2}.$$

which in the limit $\mu \to 0$, becomes the AdS solution. $f(\chi) = \lambda\chi^{-4}$.




Throat geometries as a function of a radial coordinate $\tau$ along the throat. Short dashed line is the full KS solution, red line is the mass gap solution and the long dashed line is the AdS solution. (Black is a log-corrected KS throat) (Picture credit: Kecskemeti et al., 2006 [7])

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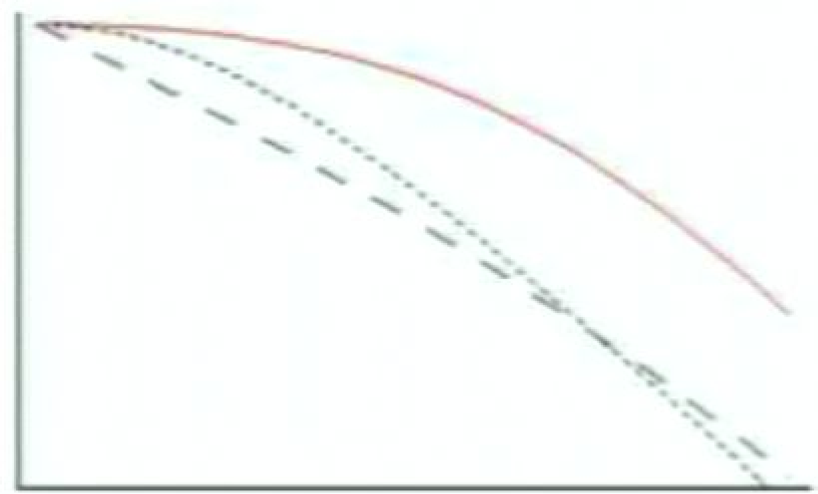

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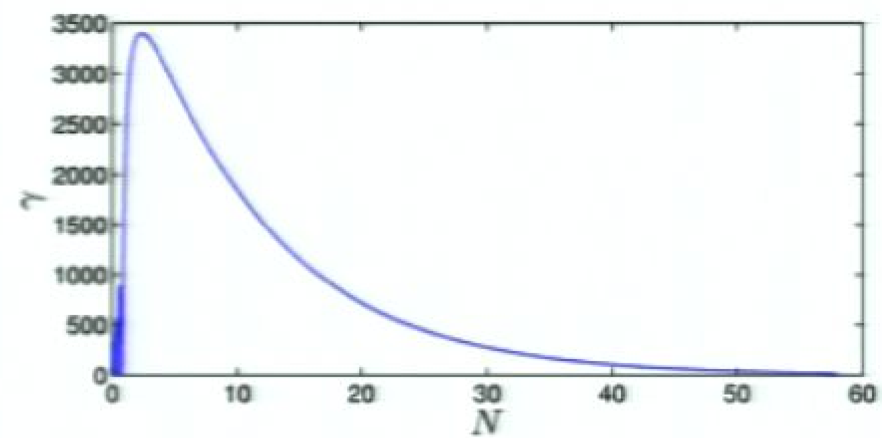
## Boost factor $\gamma$

- $\gamma$ is the boost factor, which takes a form similar to the Lorentz factor in special relativity,

$$\gamma = \frac{1}{\sqrt{1-f\dot{\chi}^2}}$$

- If $\gamma \simeq 1$ we recover the standard scalar field Lagrangian.




Boost factor $\gamma$ against efold number. Parameter Values: $\lambda = 10^{12}$, $m = 5\times10^{-5}$, $\mu = 0.1$.

# DBI inflation

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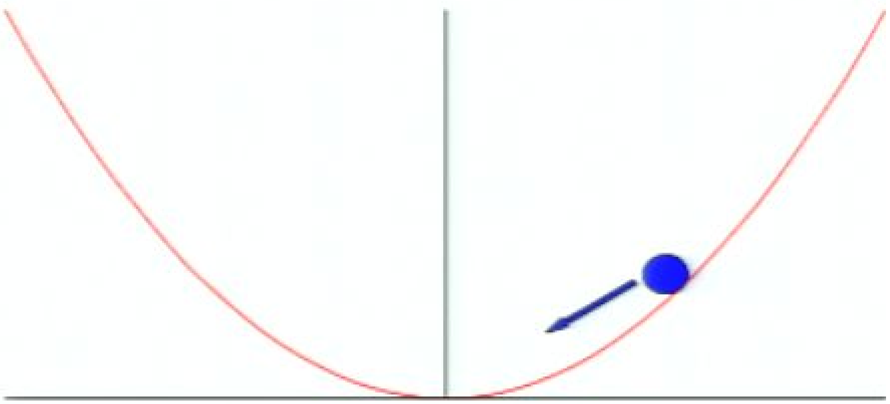
$$S = \int d^4x \sqrt{-g}\left[f^{-1}(\chi)(1-\gamma^{-1}) - V(\chi)\right]$$

**Potential** $V(\chi)$

- The DBI potential can be much steeper than than that in slow-roll inflation, as $\dot{\chi}$ is affected by the speed limit due to $\gamma$.
- The class of potentials that can drive DBI inflation is wide, but the quadratic potential

$$V(\chi) = \frac{1}{2}m^2\chi^2$$

is often used.



# DBI Inflation

We can study a flat FRW metric $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$ with scale factor $a(t)$, and get the Friedmann equations

$$3H^2 = f^{-1}(\gamma - 1) + V, \qquad -2\dot{H} = f^{-1}(\gamma - \gamma^{-1}) = \gamma\dot{\chi}^2.$$

The slow-roll parameter $\epsilon$ is given by,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}\frac{f^{-1}(\gamma - \gamma^{-1})}{f^{-1}(\gamma - 1) + V} \approx \frac{3}{2}\frac{\gamma}{\gamma + fV}.$$

The large value of $f$ required to satisfy observational constraints means that the potential term is dominant and $\epsilon \ll 1$. Thus, the equation of state $w$ is

$$w = \frac{p}{\rho} = \frac{f^{-1}(1 - \gamma^{-1}) - V}{f^{-1}(\gamma - 1) + V} \approx -\frac{fV}{\gamma + fV},$$

which is close to $-1$.

# DBI Inflation

**The DBI equation of motion is**

$$\ddot{\chi} + 3H\gamma^{-2}\dot{\chi} + \frac{1}{2}\frac{f_\chi}{f2}(1 - 3\gamma^{-2} + 2\gamma^{-3}) + \gamma^{-3}V_\chi = 0$$

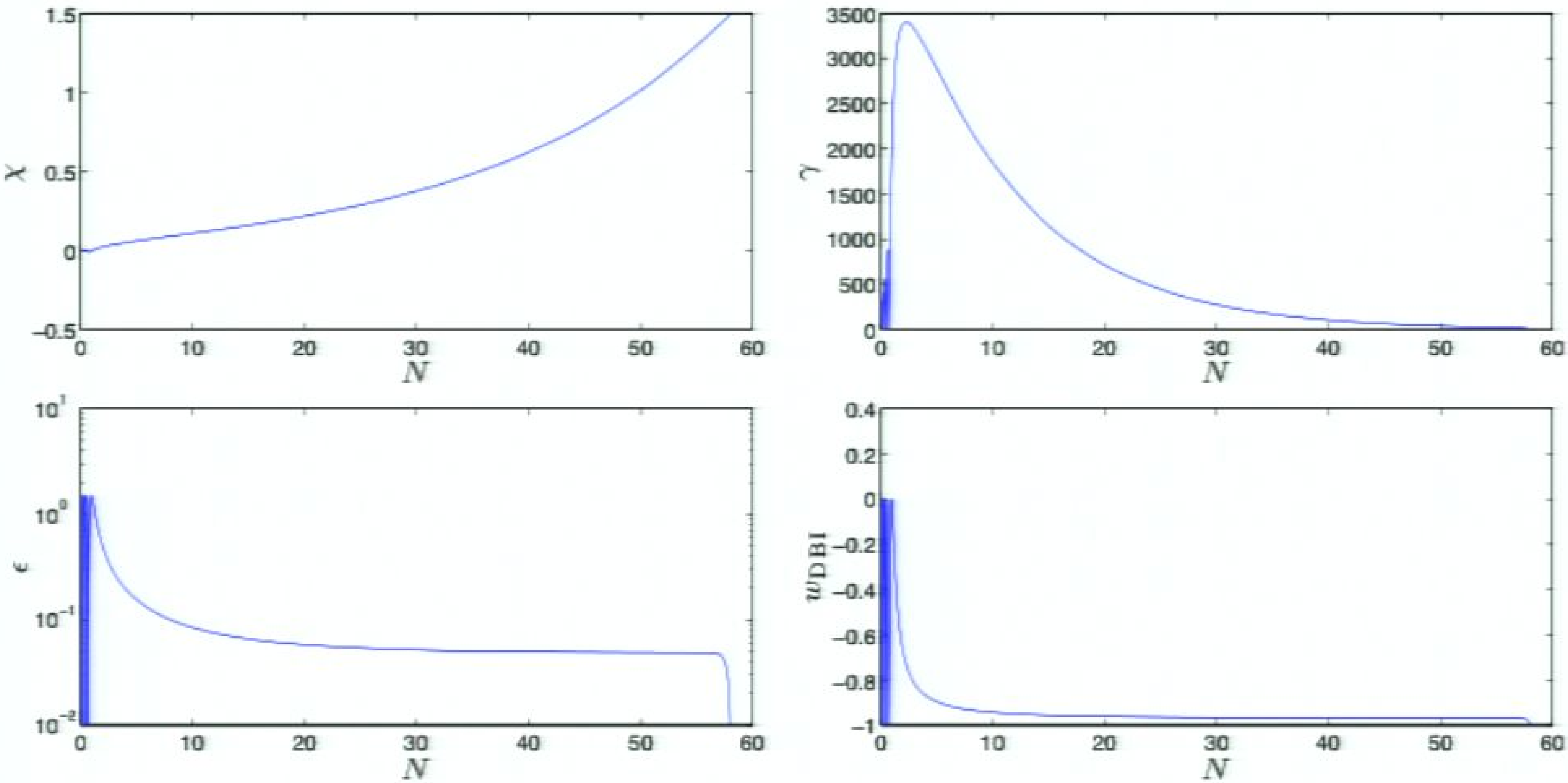
- As the DBI field starts to roll down its potential, the boost factor becomes large.
- Using $f = \lambda\chi^{-4}$, the late-time solution is

$$\ddot{\chi} - \frac{2}{\lambda}\chi^3 \approx 0 \qquad \Rightarrow \qquad \chi \to \sqrt{\lambda}/t$$

- This gives power law inflation $a = a_i t^{1/\epsilon}$ with

$$\epsilon \approx \sqrt{\frac{3}{\lambda}\frac{1}{m}}, \quad \gamma \to \sqrt{\frac{4}{3\lambda}}mt^2, \quad H \to \frac{1}{\epsilon t}.$$

# DBI inflation



# K-inflation

Armendariz-Picon et al., 1999 [8]; Garriga & Mukhanov, 1999 [9]

DBI inflation is an example of a k-inflation model, for which the action can be written

$$S = \int d^4x \sqrt{-g} P(\chi, X), \qquad \text{with} \qquad X = -\tfrac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi,$$

which gives the background equation

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \frac{\partial P}{\partial X} g^{\mu\nu} \partial_\nu \chi \right) + \frac{\partial P}{\partial \chi} = 0.$$

## cf. canonical scalar field

$$P = X - V(\phi) \qquad \Rightarrow \qquad P_{,X} = 1.$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) - \frac{\partial V}{\partial \phi} = 0.$$

# K-inflation

To simplify the perturbations we can define an auxiliary variable $\nu = z\mathcal{R}$ where $\mathcal{R}$ is the comoving curvature perturbation and

$$z \equiv \frac{a(\rho + p)^{1/2}}{c_s H} \qquad \text{with} \qquad c_s^2 = \frac{P_{,X}}{\rho_{,X}} = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}},$$

so the perturbation equation (in terms of conformal time $\tau$) is

$$\frac{d^2\nu_k}{d\tau^2} + \left(c_s^2 k^2 - \frac{1}{z}\frac{d^2 z}{d\tau^2}\right)\nu_k = 0.$$

so the perturbations do not travel at the speed of light.

**On small scales ($kc_s \gg aH$)**

$$\nu_k = \frac{1}{\sqrt{2kc_s}} e^{-ikc_s\tau}.$$

**On large scales ($kc_s \ll aH$)**

$$\nu_k \sim z.$$

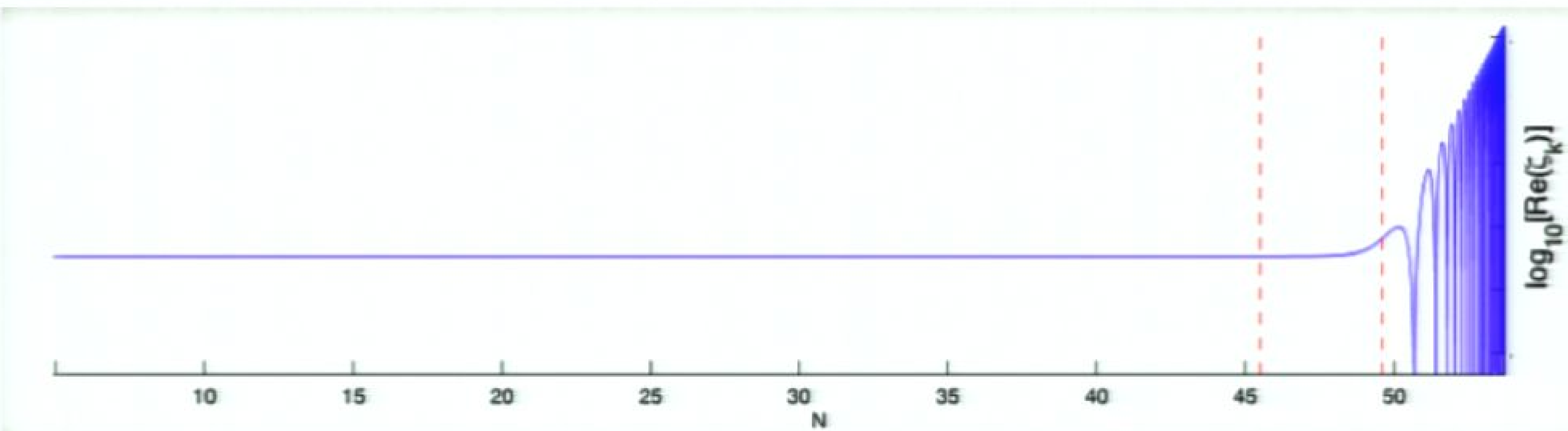
# Perturbations in DBI inflation

In the DBI case, we have $P_{,X} = \gamma$, so

$$c_s^2 = \frac{\gamma}{\gamma + 2X\gamma_{,X}} = \gamma^{-2} \qquad \Rightarrow \qquad z = \frac{a\gamma^{3/2}\dot{\chi}}{H}$$

and the perturbations satisfy

$$\frac{d^2 v_k}{d\tau^2} + \left( \frac{k^2}{\gamma^2} - \frac{1}{z}\frac{d^2 z}{d\tau^2} \right) v_k = 0.$$




**The sound speed of the perturbations is $c_s^2 = \gamma^{-2}$**

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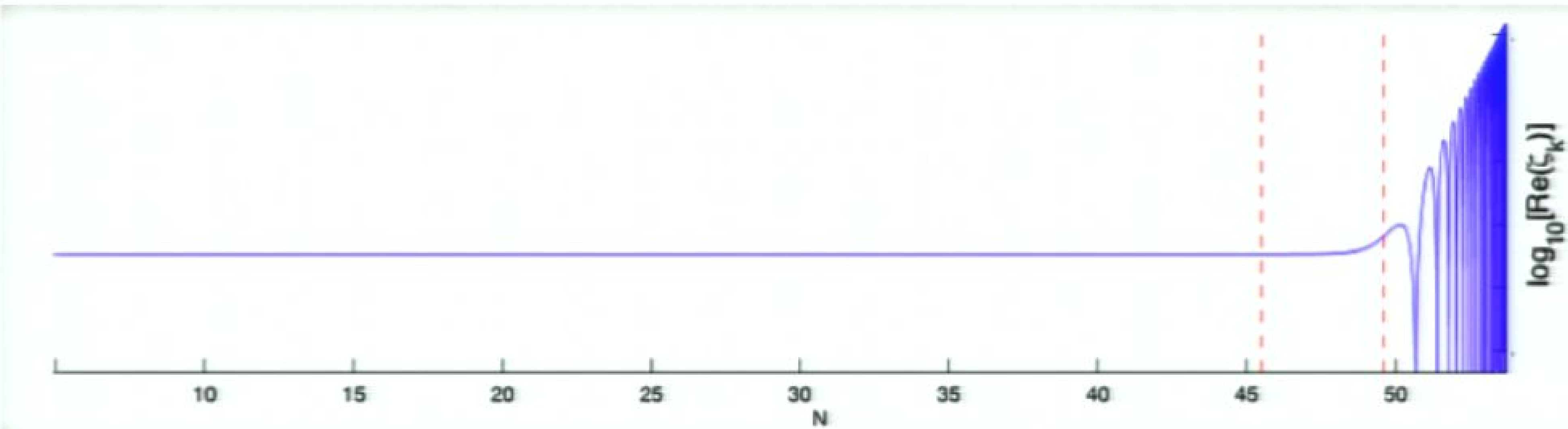


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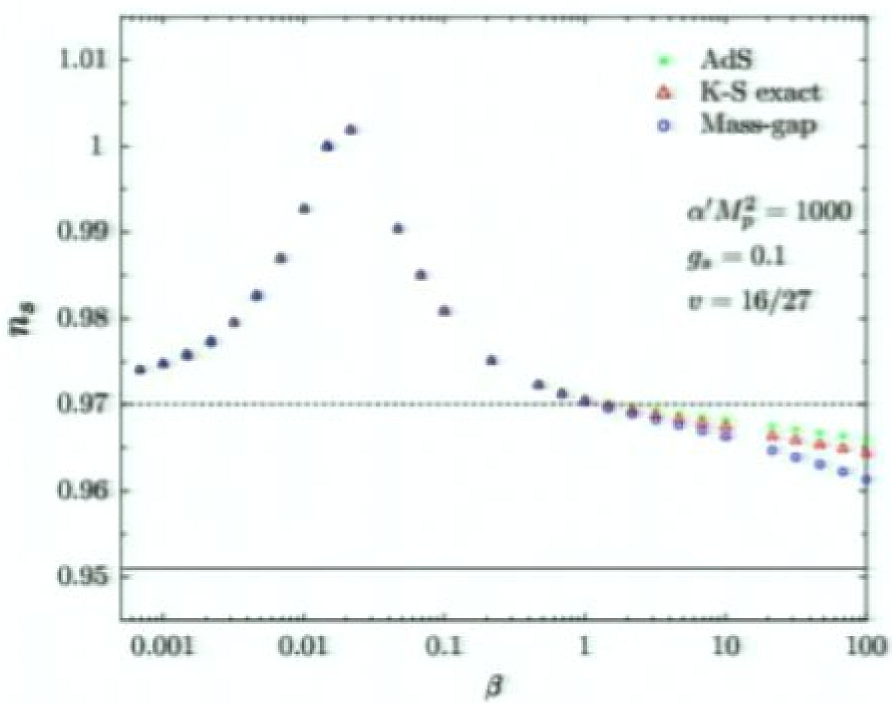
# Perturbations in DBI inflation

Using the approximate solution for the AdS throat we have

$$\mathcal{P}_{\mathcal{R}} = \frac{1}{8\pi^2}\left(\frac{H^2}{c_s\epsilon}\right)_{|c_s k = aH} \approx \frac{1}{36\pi^2} m^4 \lambda$$

- One can use the power spectrum amplitude to fix the parameters, but this can lead to a relatively small number of efolds of inflation.
- Modes freeze-in at smaller scales as inflation progresses, cancelling the red-tilt due to the evolution of $H$.
- The spectral index is dependent on the warped geometry and the background dynamics.




Spectral index as a function of $\beta$, which parameterizes the contribution of the quadratic term to the energy density $m^2 = \beta H^2$. (Picture Credit: Bean et al., 2007 [10])

# Perturbations in DBI inflation

- In standard single field inflation, the perturbations are Gaussian in the sense that all higher order correlation functions are given in terms of the two-point function.
- In DBI inflation, there can be non-Gaussian corrections to the power spectrum Fluctuations can be correlated as the modes freeze-in at different length scales. The non-linearity parameter is a typical measure of the level of non-Gaussianity in the perturbation.

$$\zeta = \zeta_L - \frac{3}{5} f_{NL} \zeta_L^2$$

- Perturbations in DBI inflation are characterised by high levels of non-Gaussianities

$$f_{NL} = \frac{35}{108}\left(\frac{1}{c_s^2} - 1\right) \quad \Rightarrow \quad \gamma \lesssim 30,$$

which could be used to distinguish these types of models from single field inflation.

# DBI inflation in scalar-tensor theory

- Non-minimally coupled scalar fields arise in a number of scenarios in high-energy physics, such as low-energy effective actions from higher dimensional theories, quantum field theory in curved space and $f(R)$ models of gravity.
- We can consider the DBI inflationary scenario is embedded into a scalar-tensor theory, with DBI field $\chi$ and canonical field $\varphi$.
- The additional field could describe the degrees of freedom associated with additional moduli fields in the higher-dimensional theory (although this scenario is treated as a phenomenological model).

cf. *Embedding DBI inflation in scalar-tensor theory*, van de Bruck, Mota and Weller, 2010 (arXiv:1012.1567 [11])

# The model in the Jordan Frame

## Jordan Frame Action

$$S = \int d^4x\sqrt{-g}\left[F(\varphi)\frac{R}{2} - \frac{1}{2}(\nabla\varphi)^2 - U(\varphi) + f^{-1}(\chi)\left[1-\gamma^{-1}\right] - V(\chi)\right]$$

The function $F(\varphi)$ complicates the $\varphi$ equation of motion

$$2\varpi\Box\varphi = F_\varphi T - \varpi_\varphi(\nabla\varphi)^2 - 4UF_\varphi + 2U_\varphi F,$$

where $\varpi \equiv F + \frac{3}{2}F_\varphi^2$, but the DBI equation is unchanged

$$\nabla_\mu\left[\gamma g^{\mu\nu}\nabla_\nu\chi\right] = \frac{f_\chi}{f^2}(1 - \tfrac{1}{2}\gamma - \tfrac{1}{2}\gamma^{-1}) + V_\chi.$$

However, the Einstein equations are difficult to work with in this frame

$$FG_{\mu\nu} = T^{\mathrm{DBI}}_{\mu\nu} + [\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}g_{\mu\nu}(\nabla\varphi)^2] - g_{\mu\nu}U + \nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F.$$

# A conformal transformation

## Jordan Frame Action

$$S = \int d^4x\sqrt{-g}\left[F(\varphi)\frac{R}{2} - \frac{1}{2}(\nabla\varphi)^2 - U(\varphi) + f^{-1}(\chi)\left[1-\gamma^{-1}\right] - V(\chi)\right]$$

We can perform a conformal transformation

$$\tilde{g}_{\mu\nu} = F(\varphi)g_{\mu\nu} \qquad \text{with} \qquad A(\tilde{\varphi}) = F^{-1/2}(\varphi)$$

into the Einstein frame, and redefine the field so its action takes the canonical form

$$\frac{d\tilde{\varphi}}{d\varphi} = \sqrt{\frac{3}{2}\left(\frac{F_\varphi}{F}\right)^2 + \frac{1}{F}}, \qquad \tilde{U}(\tilde{\varphi}) = U(\varphi)F^{-2},$$

## Einstein Frame Action

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{2} - \frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\varphi}_{,\mu}\tilde{\varphi}_{,\mu} - \tilde{U}(\tilde{\varphi})\right] + S_{\rm DBI}\left[A^2(\tilde{\varphi})\tilde{g}_{\mu\nu}\right]$$

# The model in the Einstein Frame

## Einstein Frame Action

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\varphi}_{,\mu} \tilde{\varphi}_{,\mu} - \tilde{U}(\tilde{\varphi}) \right] + S_{\rm DBI} \left[ A^2(\tilde{\varphi}) \tilde{g}_{\mu\nu} \right]$$

In the Einstein Frame, the DBI action is modified:

$$S_{\rm DBI} \left[ A^2 \tilde{g}_{\mu\nu} \right] = \int d^4x \sqrt{-\tilde{g}} A^4 \left\{ f^{-1}(\chi) \left( 1 - \tilde{\gamma}^{-1} \right) - V(\chi) \right\}.$$

Not only is there is an overall factor of the coupling $A(\tilde{\varphi})$ but the boost factor is also modified

$$\tilde{\gamma} = \frac{1}{\sqrt{1 + A^{-2} f(\chi) \tilde{g}^{\mu\nu} \chi_{,\mu} \chi_{,\mu}}} \qquad \Rightarrow \qquad \tilde{\gamma} = \frac{1}{\sqrt{1 - A^{-2} f \dot{\chi}^2}},$$

using a flat FRW metric in the Einstein frame.

# The model in the Einstein Frame

## The equations of motion are

$$\begin{aligned}\ddot{\chi} + 3H\gamma^{-2}\dot{\chi} + \tfrac{1}{2}A^2\frac{f_\chi}{f^2}(1 - 3\gamma^{-2} + 2\gamma^{-3}) + A^2\gamma^{-3}V_\chi &= -\beta\dot{\chi}\dot{\varphi}(3\gamma^{-2} - 1), \\ \ddot{\varphi} + 3H\dot{\varphi} + U_\varphi &= \beta T_{\rm DBI}\end{aligned}$$

where $\beta = d\ln A/d\varphi$. Letting $\beta =$ constant means the coupling takes the form

$$A(\varphi) = \exp(\beta\varphi).$$

$T_{\rm DBI}$ is the trace of the DBI stress-energy tensor, which is

$$T_{\rm DBI} = A^4\left[f^{-1}(4 - 3\gamma^{-1} - \gamma) - 4V\right] \simeq -4A^4V$$

and the Friedmann equations are

$$3H^2 = \tfrac{1}{2}\dot{\varphi}^2 + U + A^4\left[f^{-1}(\gamma - 1) + V\right], \qquad -2\dot{H} = \dot{\varphi}^2 + \gamma A^2\dot{\chi}^2.$$

# Minimal coupling

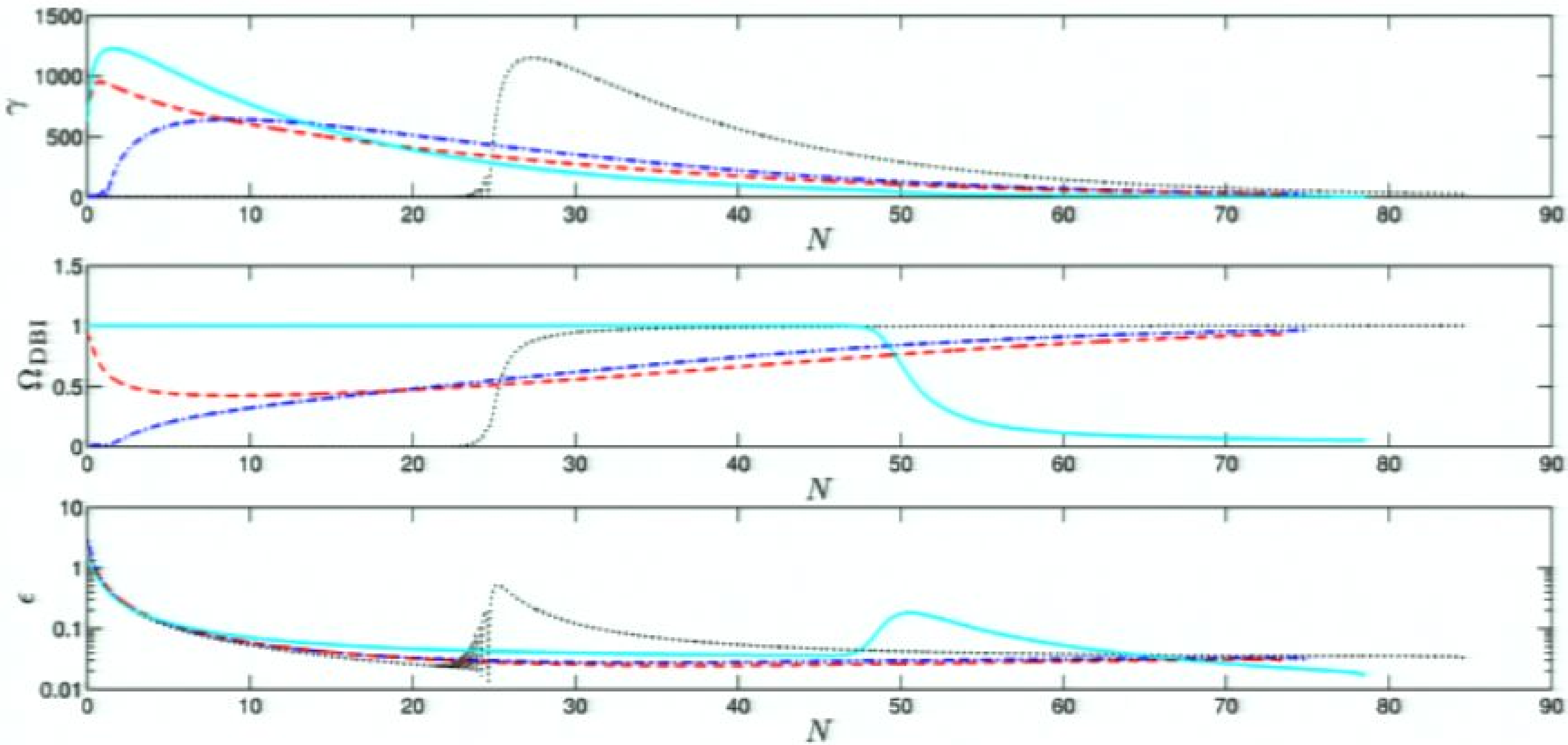
If there is no coupling between the fields, they evolve separately, each affected by its own potential.

- The DBI field needs a relatively large mass to drive inflation cf. $\epsilon \approx \sqrt{3/\lambda m^2}$ for the single field case.
- However, the canonical field needs to be light.

**Without fine-tuning the masses, this means that one field is generally dominant.**

- If the DBI field is initially dominant, $\varphi$ is almost frozen and there is a period of DBI inflation followed by slow-roll inflation.
- If $\varphi$ is initially dominant, it drives slow-roll inflation with no DBI effects until it reaches its minimum, whereupon the DBI field drives inflation.

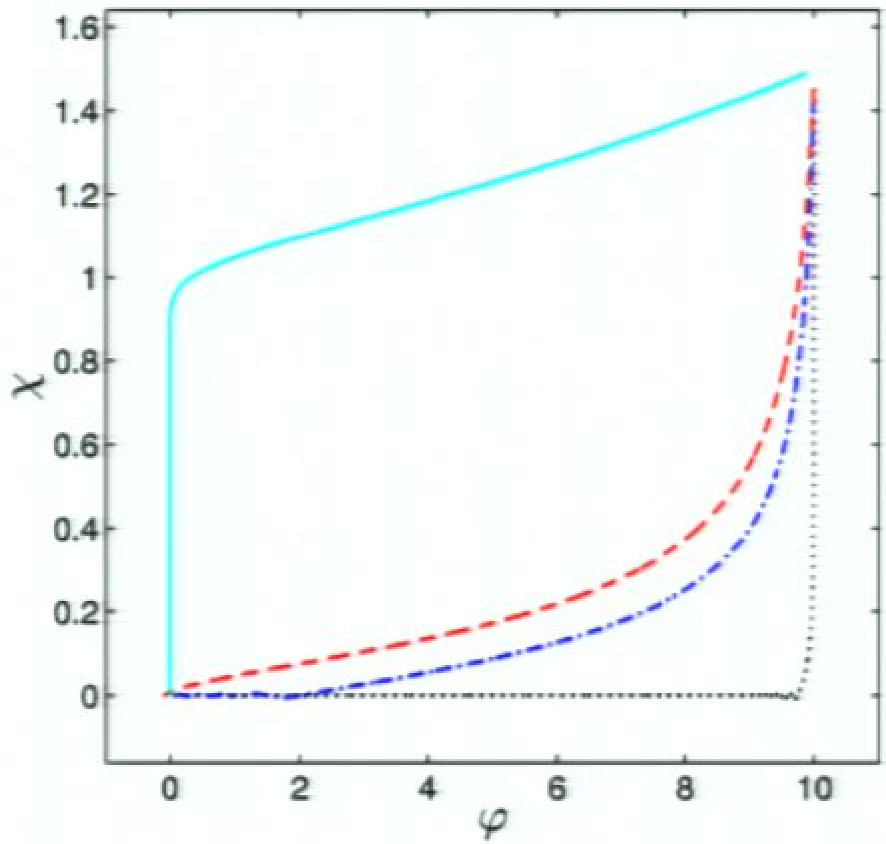
# Minimal coupling




Minimal coupling with a quadratic potential $U(\varphi) = U_0\varphi^2$ with $U_0 = 10^{-14}$ (black, dotted), $10^{-12}$ (blue, dot-dashed), $2 \times 10^{-12}$ (red, dashed) and $5 \times 10^{-10}$ (cyan, solid). Other parameter values: $\lambda = 2 \times 10^{12}$, $m = 5 \times 10^{-5}$ and $\mu = 0.2$.

# Minimal coupling

- These trajectories exhibit a sharp turn in field space.
- If one fine-tunes the mass scales so that both fields contribute roughly equally to the total energy density for a considerable number of efolds, the dynamics of the DBI field can be significantly affected by the additional contribution to the Hubble damping.
- However, this is sensitive to the masses of both fields and initial conditions.




**As both fields are evolving in this case, isocurvature fluctuations could play an important role.**

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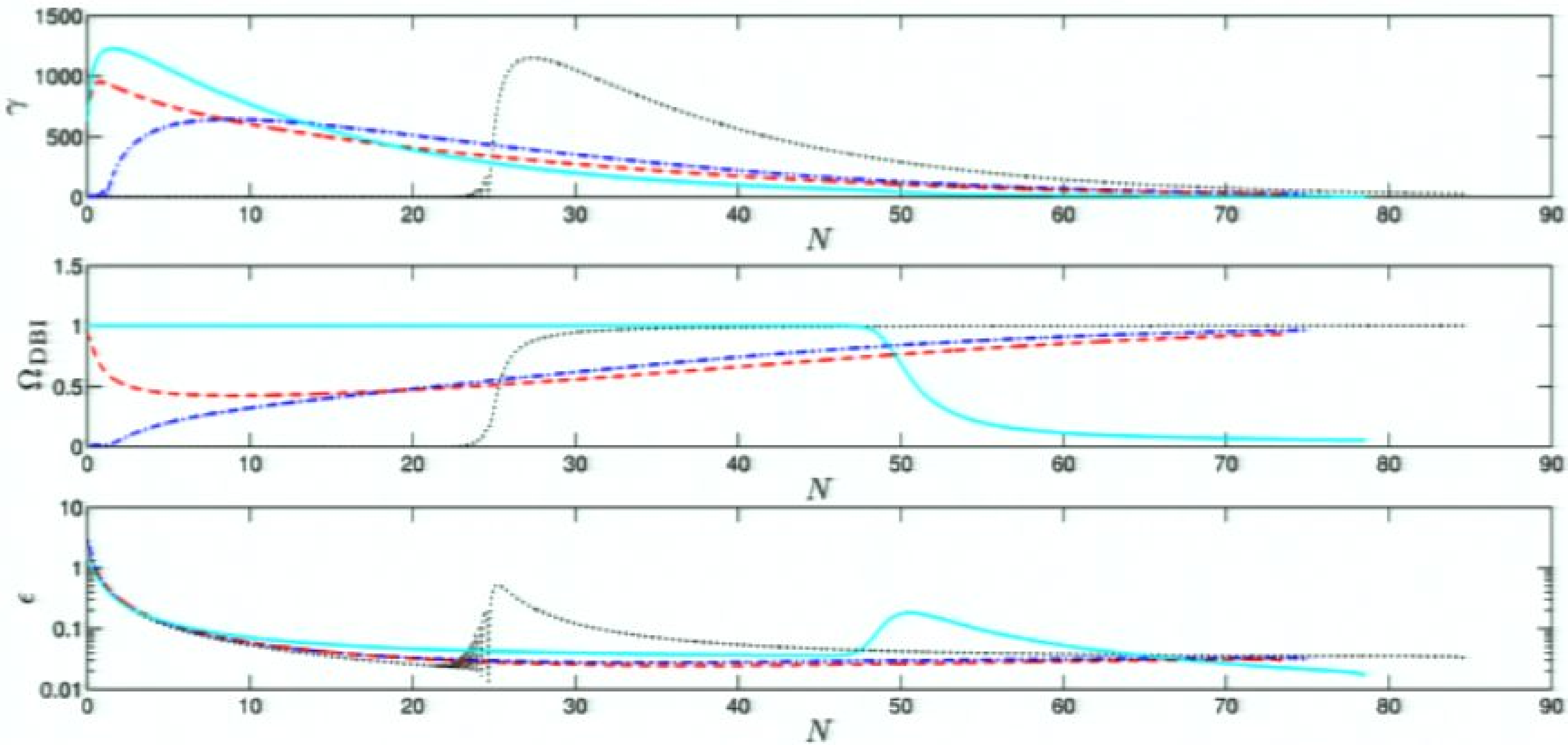


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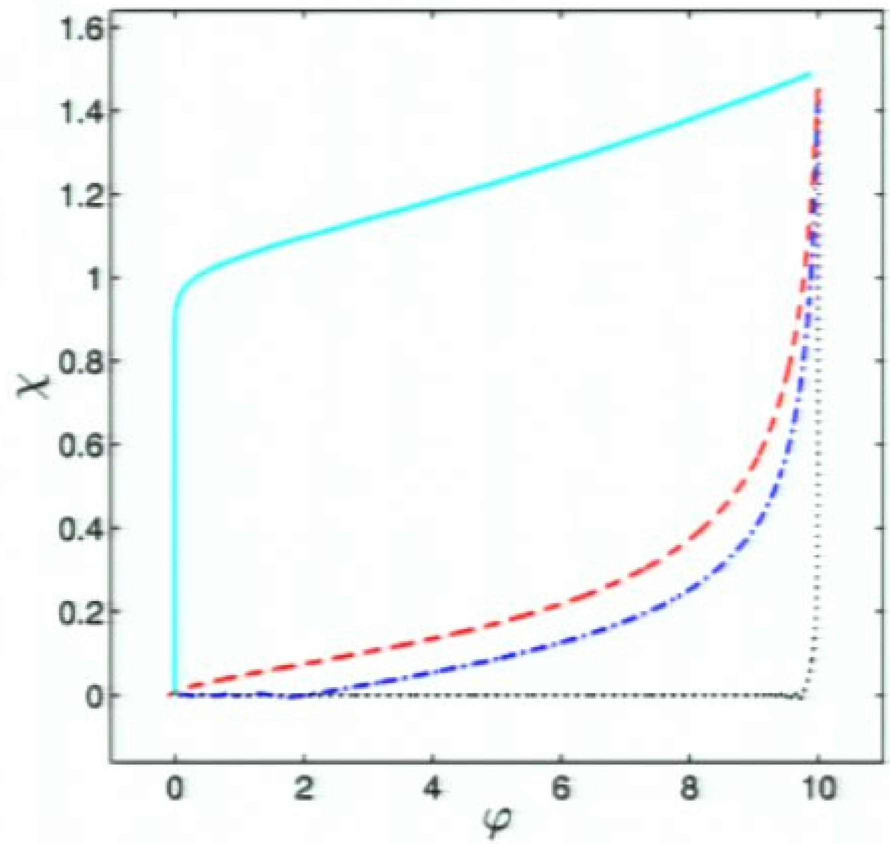



**As both fields are evolving in this case, isocurvature fluctuations could play an important role.**

# Non-minimal coupling

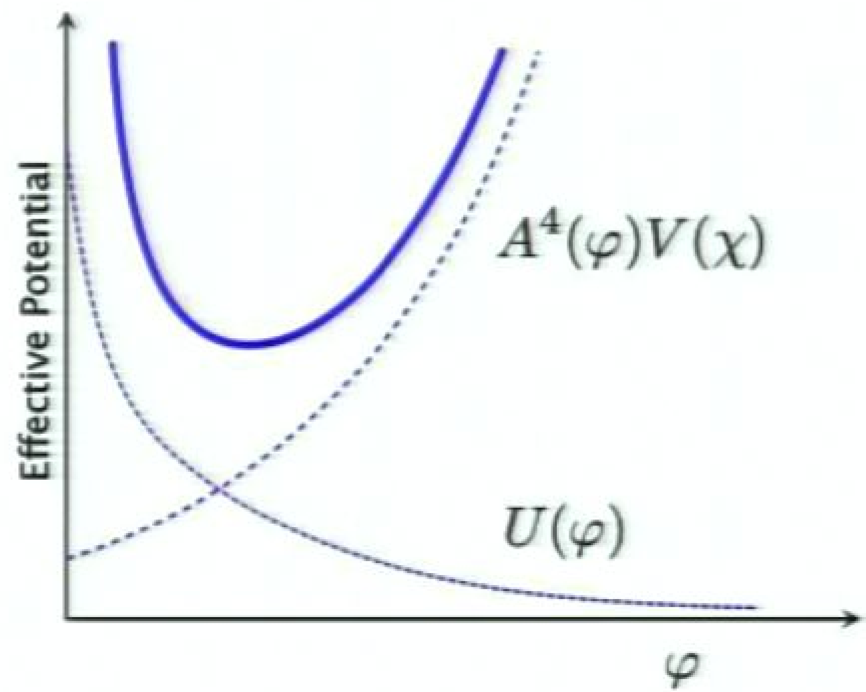
$$\ddot{\chi} + 3H\gamma^{-2}\dot{\chi} + \tfrac{1}{2}A^2\frac{f_\chi}{f^2}(1 - 3\gamma^{-2} + 2\gamma^{-3}) + A^2\gamma^{-3}V_\chi = -\beta\dot{\chi}\dot{\varphi}(3\gamma^{-2} - 1),$$
$$\ddot{\varphi} + 3H\dot{\varphi} + U_\varphi = \beta T_{\mathrm{DBI}}$$

- When $\beta \neq 0$ the scalar field moves in an effective potential $U_{\mathrm{eff}} = U - \frac{1}{4}T_{\mathrm{DBI}}$, that can have a minimum at

$$\left.\frac{dU}{d\varphi}\right|_{\varphi=\varphi_{\min}} - \beta e^{4\beta\varphi_{min}} T^b_{\mathrm{DBI}} = 0,$$

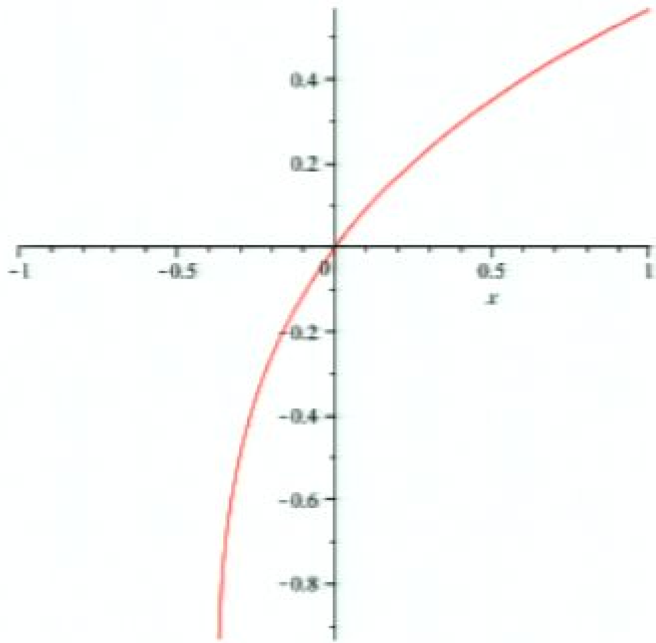
  where $T^b_{\mathrm{DBI}} \approx -4V$.

- $\beta$ is positive by definition, so we need a potential with $dU/d\varphi < 0$.



# Potentials

I will consider two types of potential, one with a minimum and one without.




## Offset quadratic potential

$$U(\varphi) = U_0(\varphi - \eta)^2$$

$$\therefore \varphi_{min} \approx \eta - \frac{1}{4\beta} W\left(\frac{8\beta^2 V e^{4\beta\eta}}{U_0}\right)$$

## Exponential potential

$$U(\varphi) = U_0 \exp(-\eta\varphi)$$

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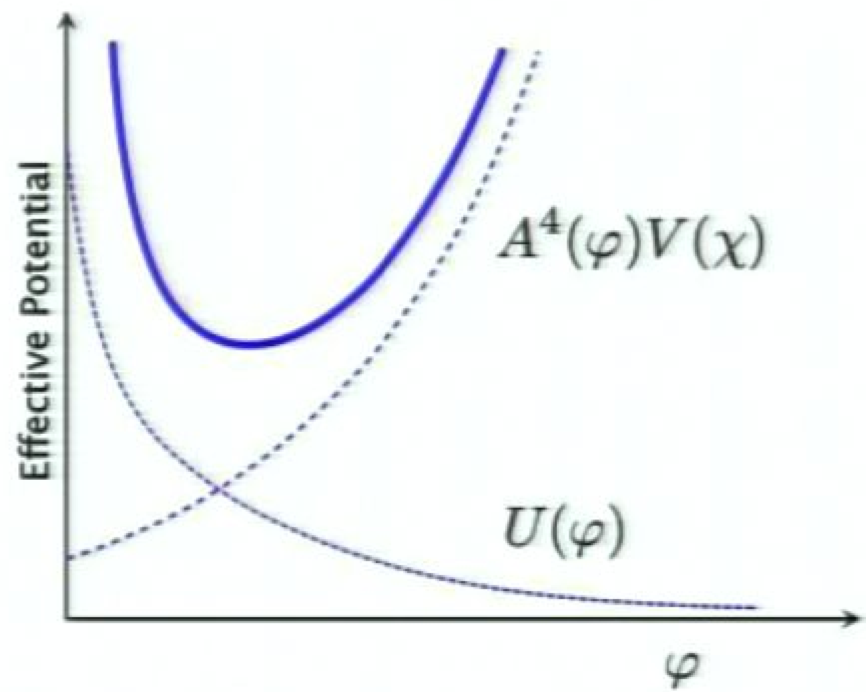


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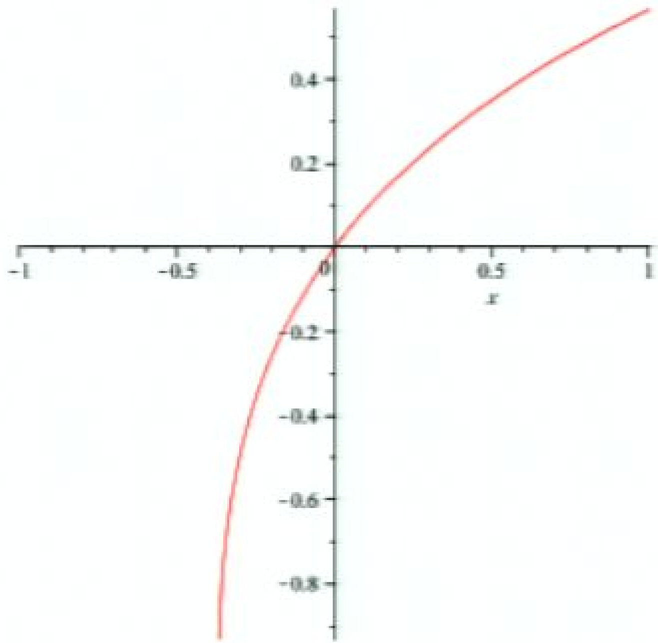


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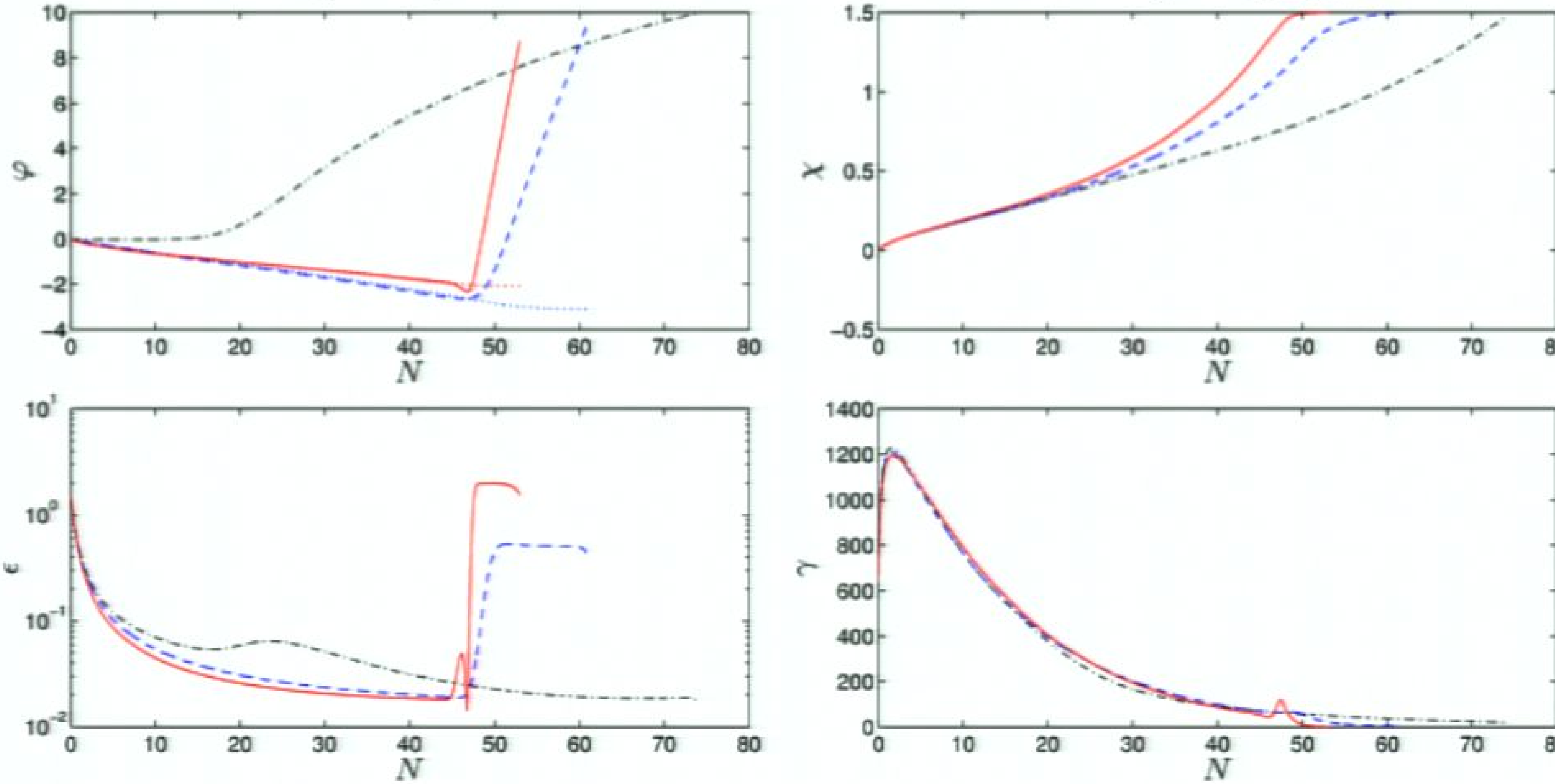
# Background dynamics

The scalar field quickly finds its minimum so we can write $\varphi = \varphi_{\rm min}$.




Quadratic potential with $\beta = 0, 0.25, 0.5$. $\lambda = 2 \times 10^{12}$, $m = 5 \times 10^{-5}$, $U0 = 2 \times 10^{-11}$, $\mu = 0.2$, $\eta = 0$

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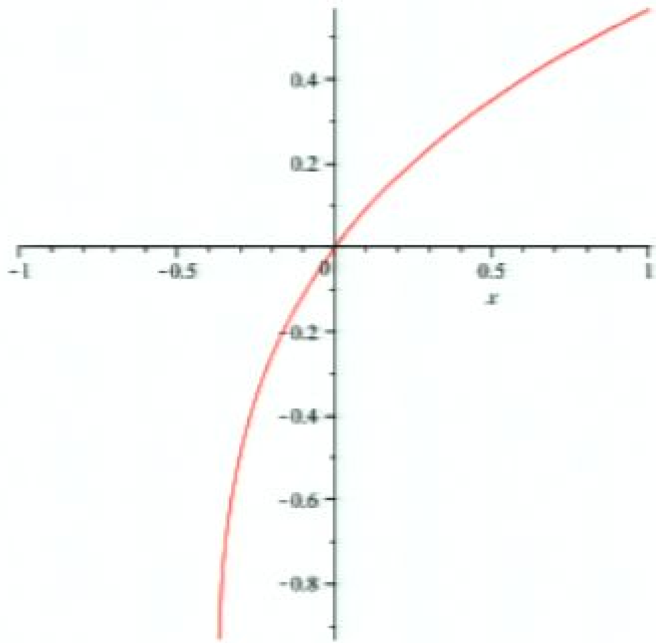


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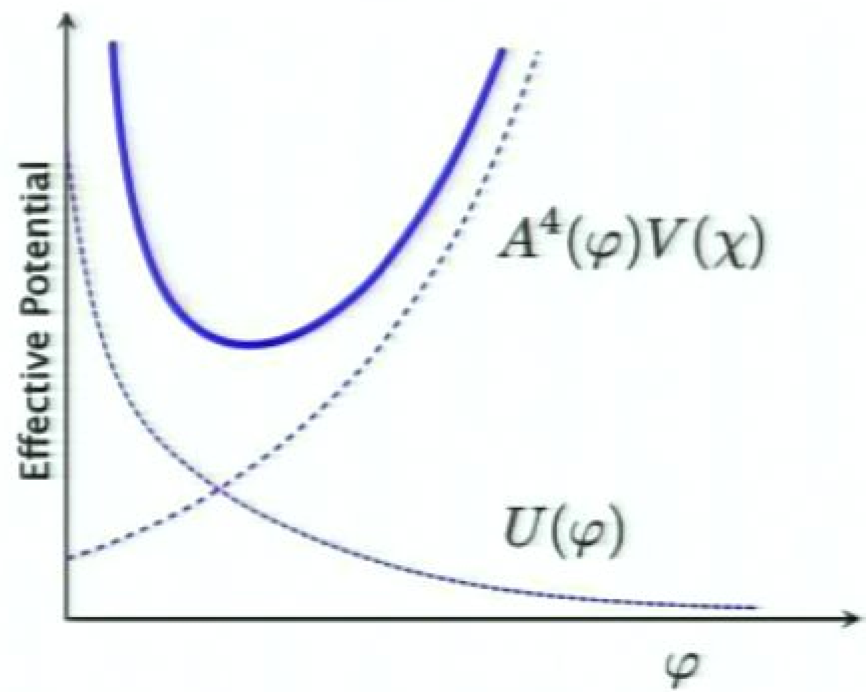

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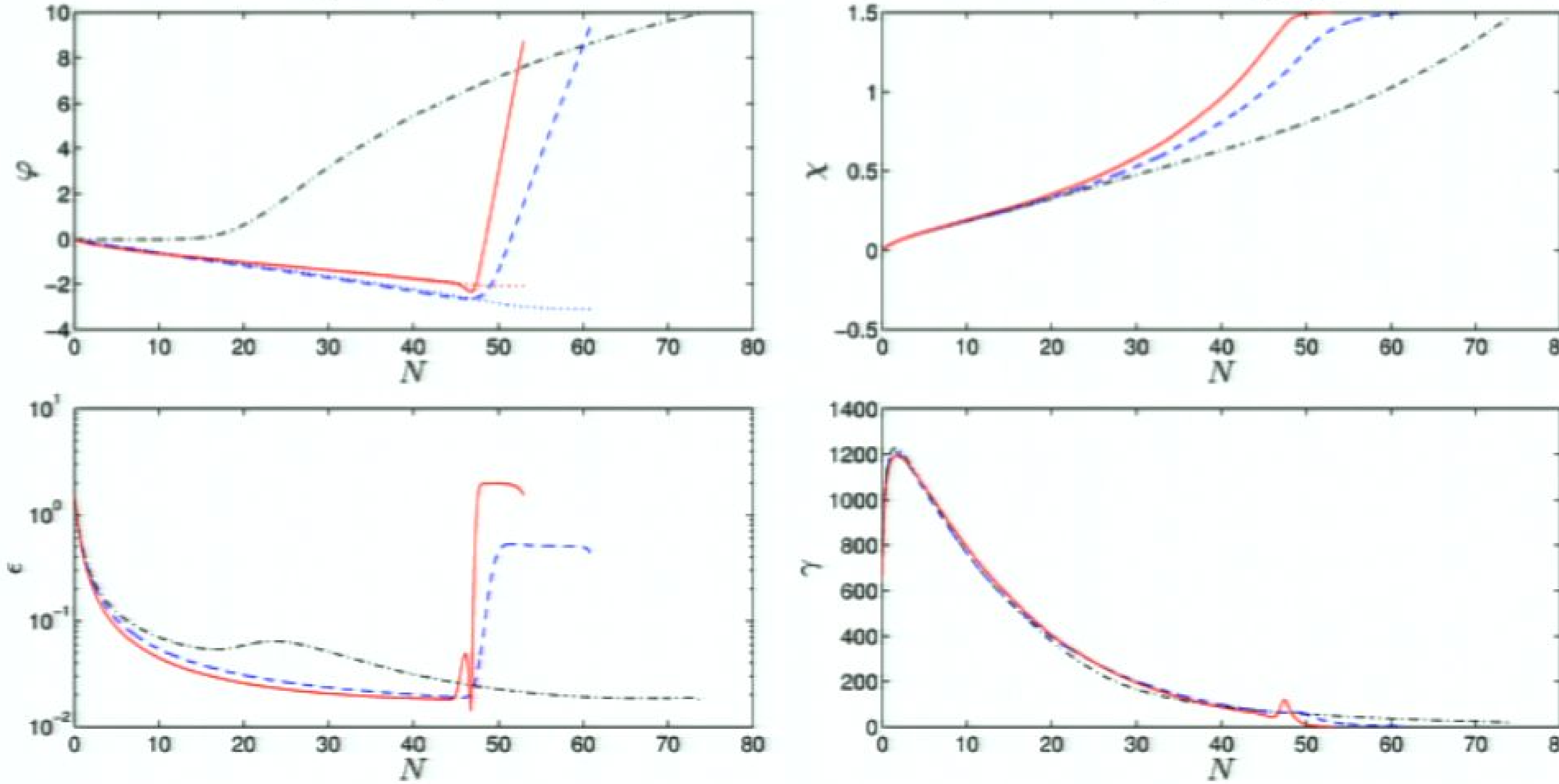


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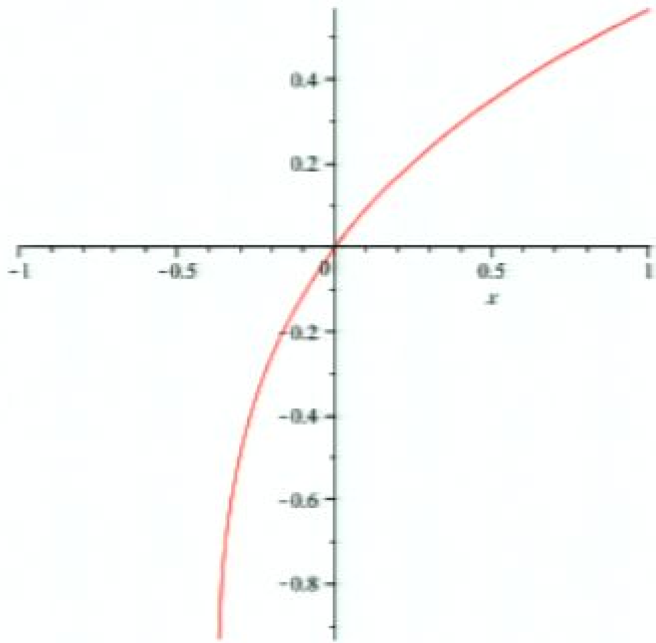


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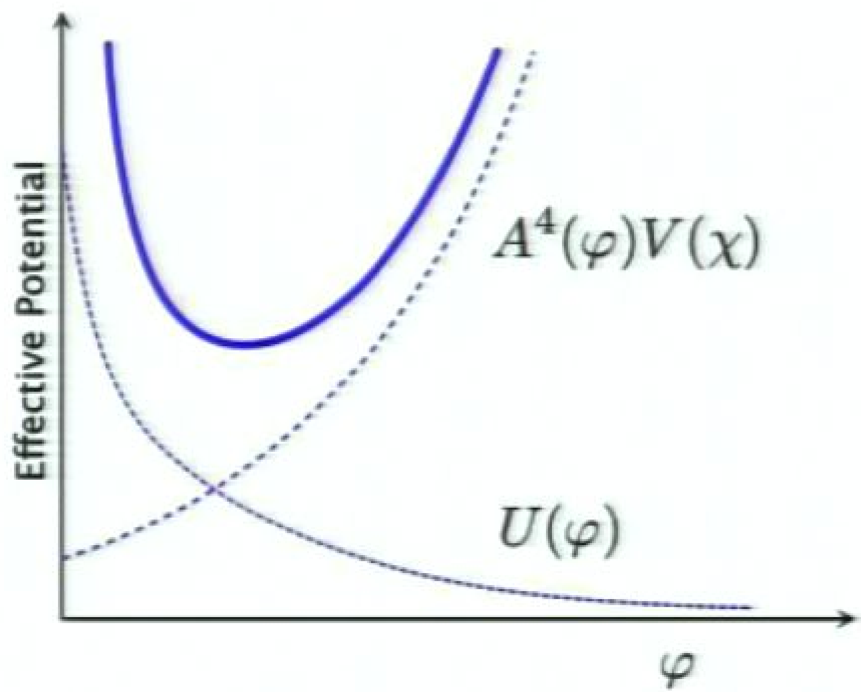

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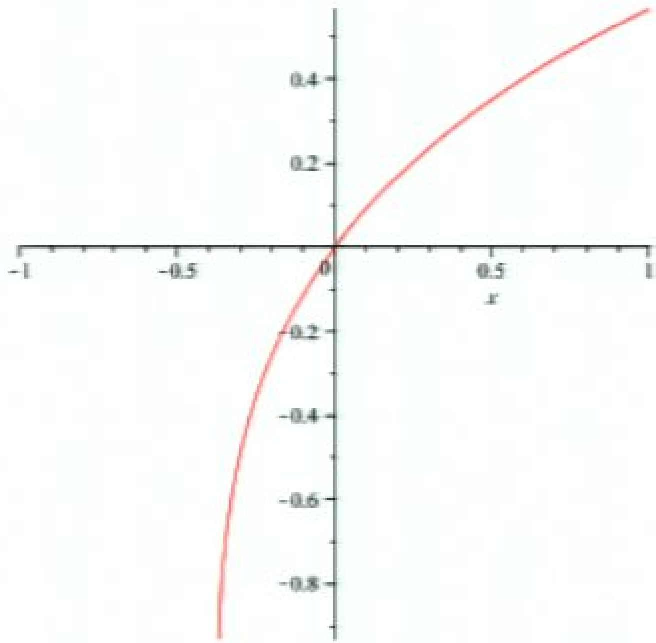


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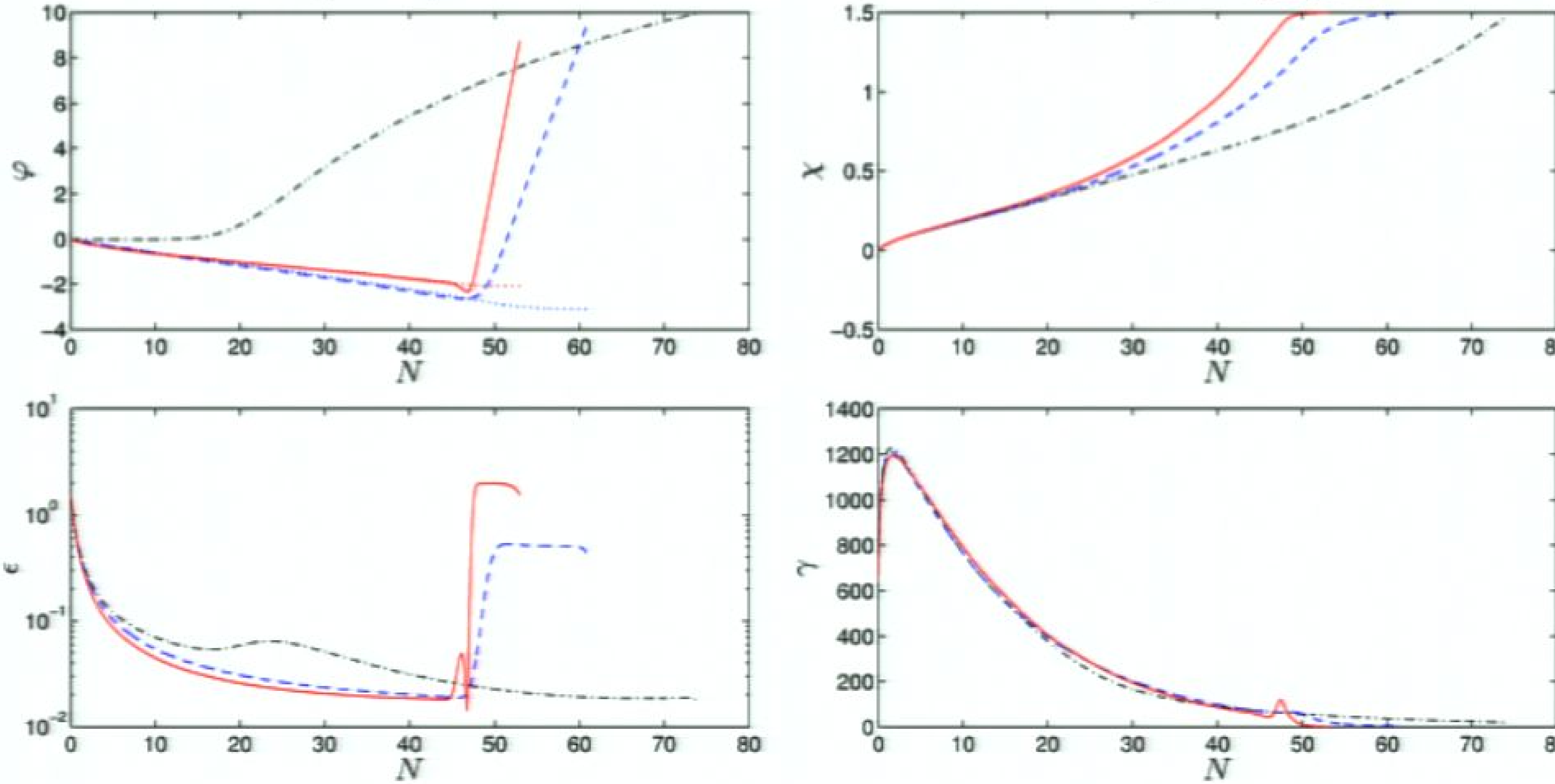



Quadratic potential with $\beta = 0, 0.25, 0.5$. $\lambda = 2 \times 10^{12}$, $m = 5 \times 10^{-5}$, $U0 = 2 \times 10^{-11}$, $\mu = 0.2$, $\eta = 0$

# Background dynamics: exponential potential

For the exponential potential $U(\varphi) = U_0 \exp(-\eta\varphi)$ the condition for the minimum is

$$\varphi_{\rm min} = \frac{1}{4\beta+\eta}\log\left(\frac{\eta U_0}{4\beta V}\right) \quad \Rightarrow \quad \dot\varphi_{\rm min} \simeq -\frac{1}{\eta(1+4\beta\eta)}\frac{\dot\chi}{\chi}$$

so the condition $\dot\varphi^2 \ll A^2\gamma\dot\chi^2$ implies that

$$\gamma \gg \left[\frac{(1+4\beta/\eta)}{6\eta(1+4\beta\eta)}\right]\left(\frac{A^2m^2}{H^2}\right) \gtrsim \mathcal{O}(1),$$

must be satisfied, which is true when $\chi$ is affected by the DBI terms in the action. We can rewrite the Friedmann equation as

$$3H^2 \simeq (1+4\beta/\eta)A^4V,$$

so the slow-roll parameter is **smaller** than the standard DBI case when $A \gtrsim 1$.

$$\epsilon = \frac{\dot\varphi^2 + A^2\gamma\dot\chi^2}{2H^2} \simeq \frac{3A^2\gamma\dot\chi^2}{2(1+4\beta/\eta)A^4V} \simeq \frac{1}{A^2(1+4\beta/\eta)}\left(\frac{3\gamma\dot\chi^2}{2V}\right) < \epsilon_{\rm DBI}$$

# Background dynamics: exponential potential

$$\epsilon \simeq \frac{1}{A^2(1+4\beta/\eta)}\left(\frac{3\gamma\dot{\chi}^2}{2V}\right) < \epsilon_{\rm DBI}$$

- Since $\epsilon$ is inversely related to the number of efolds of inflation $N_{\rm max} = \ln(a_f/a_i)$ by

$$N_{\rm max} = \int_{t_i}^{t_f} H dt = \int_{t_i}^{t_f} \frac{1}{\sqrt{\epsilon}}\sqrt{-\dot{H}} dt \simeq \int_0^{\chi_{\rm ini}} A\sqrt{\frac{\gamma}{2\epsilon}} d\chi,$$

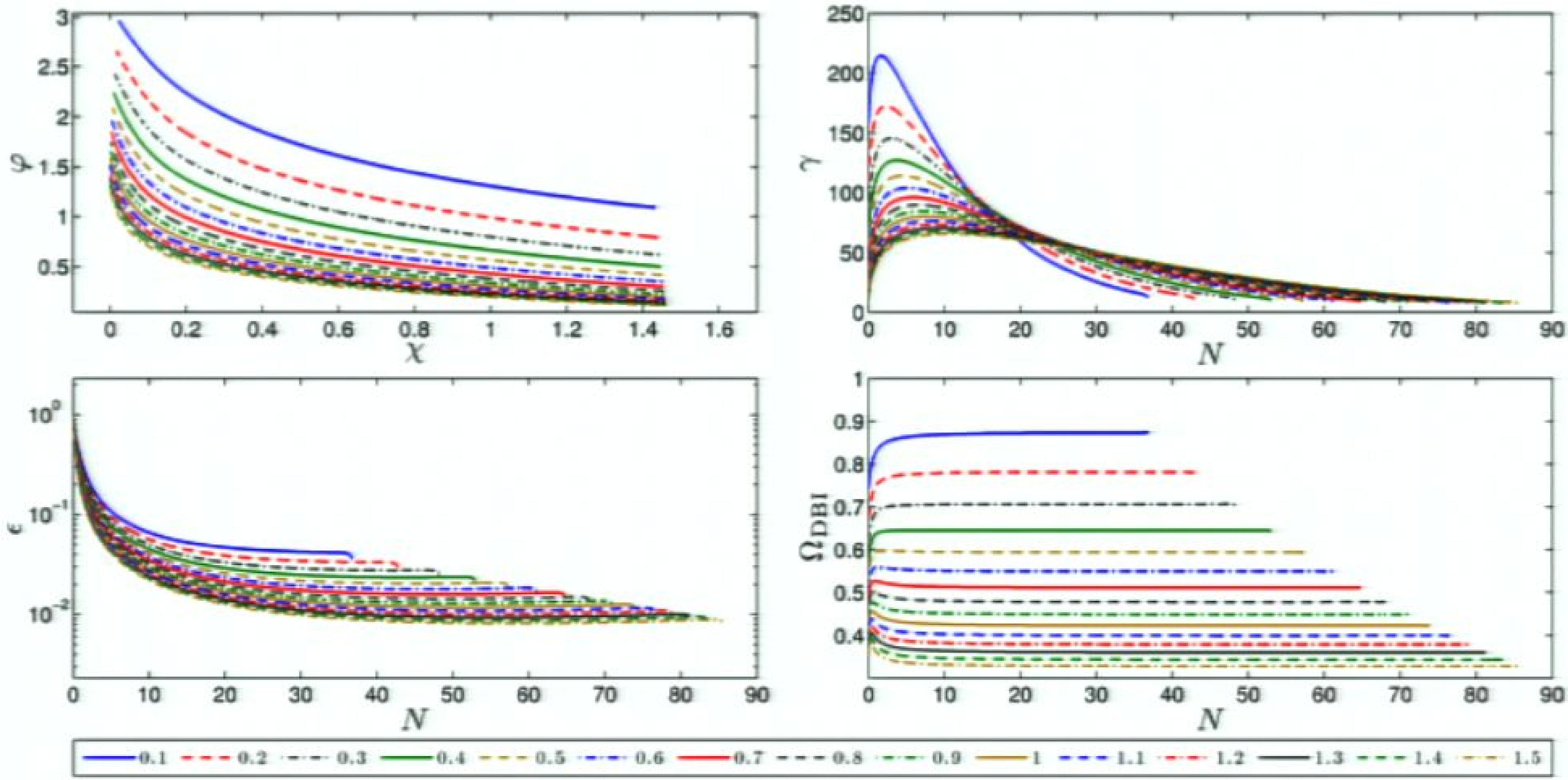
  the duration of inflation is increased by the coupling when $A > 1$ ($\varphi_{\rm min} > 0$)
- Also, the boost factor,

$$\gamma = \frac{1}{\sqrt{1 - A^{-2} f \dot{\chi}^2}}$$



  contains a factor of $A^{-2}$ so will reach smaller values.

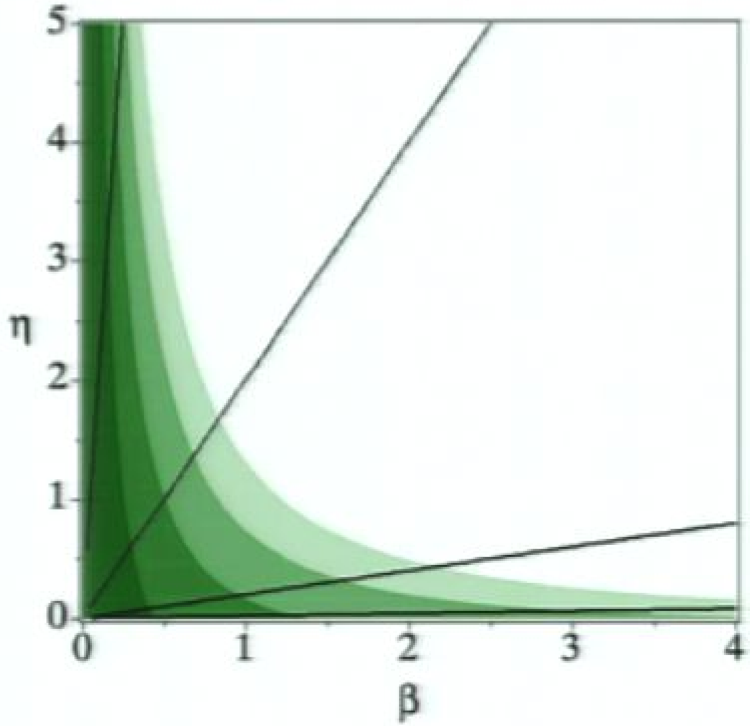
# Background dynamics: exponential potential



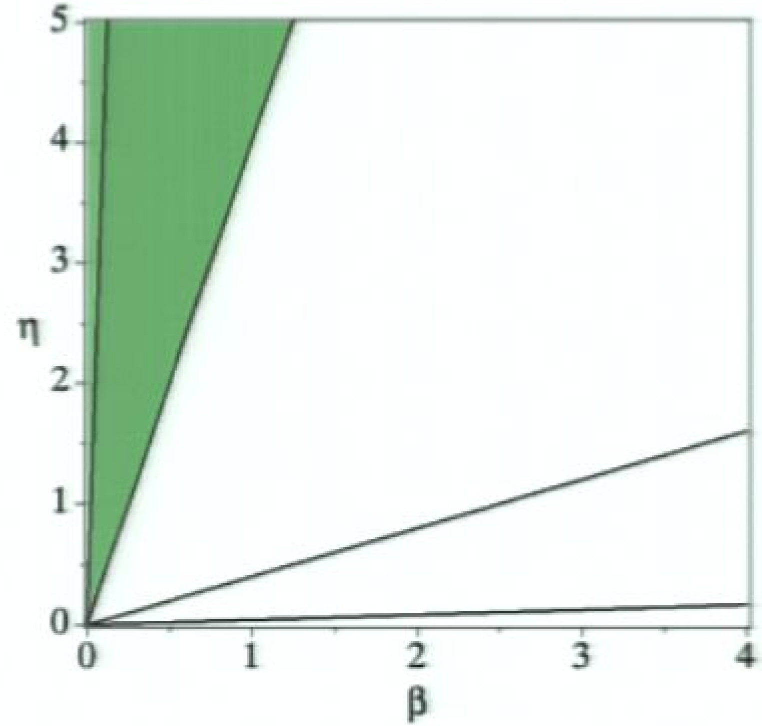

Exponential potential with $\lambda = 2 \times 10^{13}$, $m = 10^{-7}$, $U0 = 5.5^{-10}$, $\mu = 0.32$, $\eta = 2.9$

# What is the condition for $A > 1$?

Offset quadratic potential




Exponential potential




$$U(\varphi) = U_0(\varphi - \eta)^2$$

- $\varphi_{\rm min} > 0 \Rightarrow W(x) < 4\beta\eta$
- $\Omega_{\rm DBI} > \Omega_\varphi \Rightarrow W(x) < 2$

$$U(\varphi) = U_0 \exp(-\eta\varphi)$$

- $\varphi_{\rm min} > 0 \Rightarrow \eta > 4\beta(V/U_0)$
- $\Omega_{\rm DBI} > \Omega_\varphi \Rightarrow \eta > 4\beta$



where $x \approx 8\beta^2 V e^{4\beta\eta}/U_0$.

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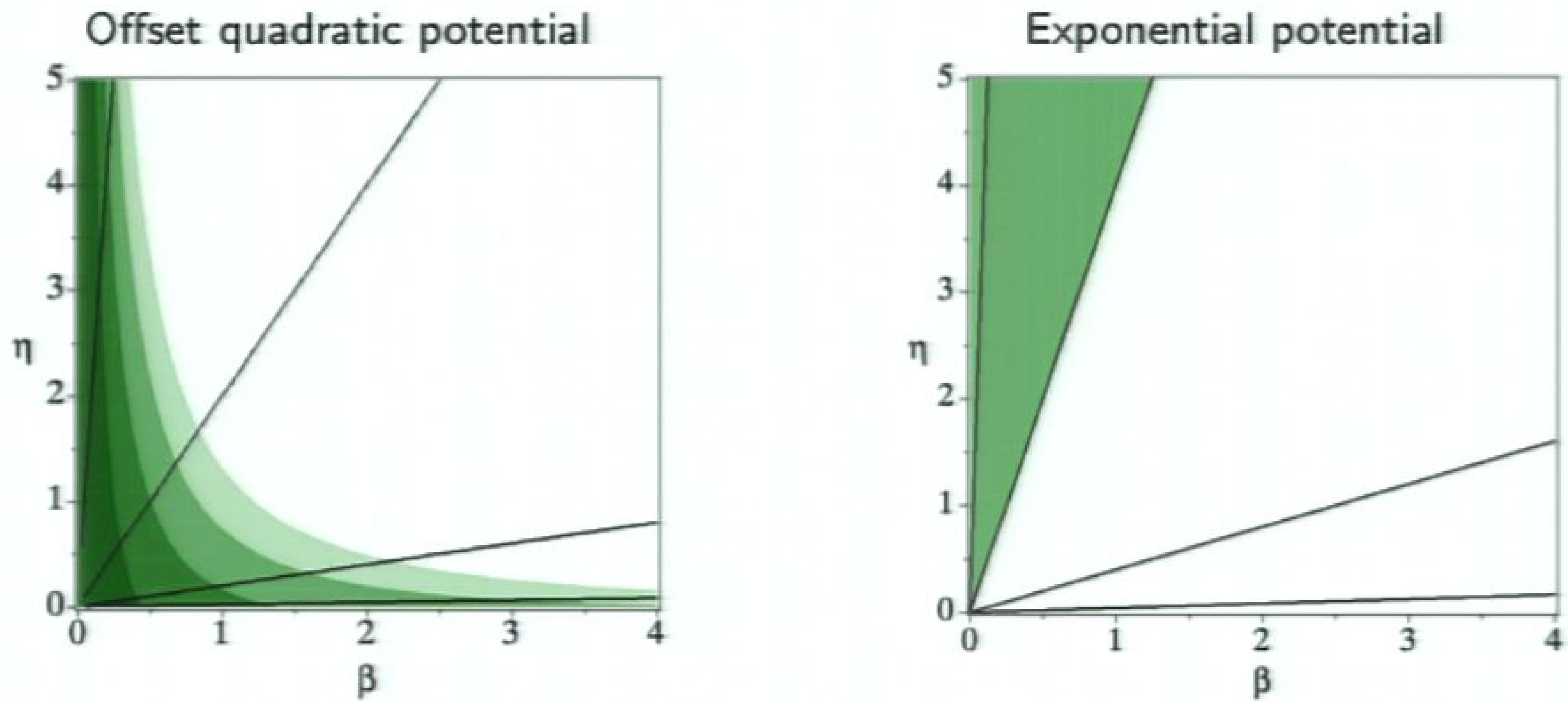
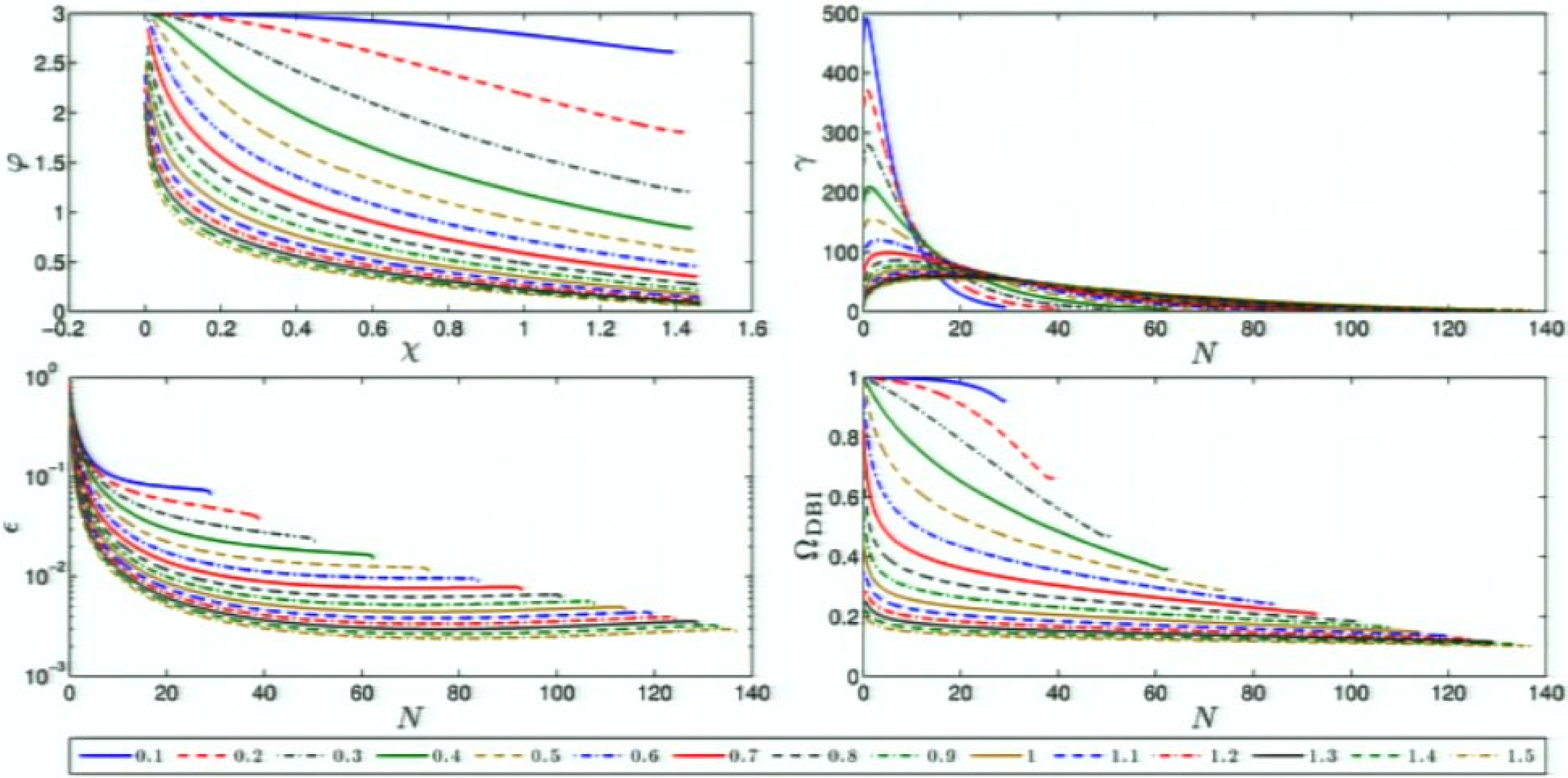



- regions above the solid lines satisfy the condition for $\varphi_{\rm min} > 0$ for ratio $V/U_0 = 0.01, 0.1, 1, 10$ (starting from the bottom).
- Shaded regions indicate where the DBI energy is dominant (with the lightest region corresponding to $V/U_0 = 0.01$).
- **The DBI energy density is subdominant for a wide range of**

# Background dynamics: offset quadratic potential




Offset quadratic potential with $\lambda = 8 \times 10^{12}$, $m = 8 \times 10^{-6}$, $U0 = 10^{-10}$, $\mu = 0.15$, $\eta = 3$

# Perturbed equations

We decompose the fields $\varphi$ and $\chi$ into a homogeneous and perturbed part

$$\varphi(t, \mathbf{x}) = \varphi(t) + \delta\varphi(t, \mathbf{x}), \qquad \chi(t, \mathbf{x}) = \chi(t) + \delta\chi(t, \mathbf{x}).$$

In the longitudinal gauge and in the absence of anisotropic stress, the scalar perturbations of the FRW metric can be expressed as

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 - 2\Psi)\delta_{ij}dx^i dx^j.$$

The perturbed Einstein equations and the equations of motion for $\delta\varphi$ and $\delta\chi$ can be used to form a closed system of equations in terms of the gauge-invariant Mukhanov-Sasaki variables.

$$Q_\varphi \equiv (\delta\varphi) + \frac{\dot{\varphi}}{H}\Psi, \qquad Q_\chi \equiv (\delta\chi) + \frac{\dot{\chi}}{H}\Psi.$$

# Perturbed equations

Using $c_s^2 = \gamma^{-2}$, this gives

$$\ddot{Q}\varphi \quad + \quad 3H\dot{Q}\varphi + B_\varphi \dot{Q}_\chi + \left(\frac{k^2}{a^2} + C_{\varphi\varphi}\right) Q_\varphi + C_{\varphi\chi} Q_\chi = 0,$$

$$\ddot{Q}\chi \quad + \quad \left[3H + 2\beta\dot{\varphi} - 3\frac{\dot{c}_s}{c_s}\right] \dot{Q}\chi + B_\chi \dot{Q}_\varphi + \left(\frac{k^2}{a^2}c_s^2 + C_{\chi\chi}\right) Q_\chi + C_{\chi\varphi} Q_\varphi = 0$$

The coefficients are made up of terms:

- proportional to the non-minimal coupling parameter $\beta$
- with factors of $c_s$ due to the non-canonical kinetic term of the DBI field.
- involving the potentials of both fields.

For example consider the cross-kinetic terms

$$B_\chi = \left[\beta(3c_s^2 - 1) - \frac{\dot{\varphi}}{2H}(1 - c_s^2)\right]\dot{\chi}, \qquad B_\varphi = -A^2 c_s^{-3} B_\varphi.$$

These are not present with two uncoupled canonical fields and $B_{\varphi,\chi} \to 0$ as $\beta \to 0$, $c_s \to 1$.

# Perturbed equations

In terms of the total potential $V_T(\varphi,\chi) = U(\varphi) + A^4(\varphi)V(\chi)$, the other coefficients are

$$C_{\varphi\varphi} = \beta\left(\frac{\dot\varphi}{H}\right)A^4 f^{-1}(3c_s+1)(1-c_s^{-1})^3 - \beta^2 A^4 f^{-1}(16 - 8c_s^{-1} - 9c_s + c_s^{-3}) + +3\dot\varphi^2 - c_s^{-3}(1+c_s^2)A^2\frac{\dot\varphi^2\dot\chi^2}{4H^2} - \frac{\dot\varphi^4}{2H^2} + \frac{2\dot\varphi V_{T,\varphi}}{H} + V_{T,\varphi\varphi}, \quad (1)$$

$$C_{\varphi\chi} = \frac{A^4\dot\varphi}{4H}\frac{f_\chi}{f^2}c_s(1-c_s^{-1})^2(c_s^{-2}+2c_s^{-1}-1) + 3c_s^{-1}A^2\dot\chi\dot\varphi - c_s^{-4}(1+c_s^2)A^4\frac{\dot\varphi\dot\chi^3}{4H^2} - -\tfrac{1}{2}\beta A^4 f^{-1}(3c_s+1)(1-c_s^{-1})^3\left[\frac{f_\chi}{f} - \frac{A^2\dot\chi}{Hc_s}\right] - c_s^{-1}A^2\frac{\dot\varphi^3\dot\chi}{2H^2} + \frac{V_{T,\chi}\dot\varphi}{H} + +\frac{c_s^{-1}A^2\dot\chi}{H}V_{T,\varphi} + V_{T,\varphi\chi}, \quad (2)$$

$$C_{\chi\chi} = \frac{A^4\dot\chi}{H}\frac{f_\chi}{f^2}(1-c_s)^2 - \left[\frac{f_\chi}{f} - \frac{A^2\dot\chi}{Hc_s}\right]\frac{\dot c_s}{c_s}\dot\chi - \tfrac{1}{2}c_s f_\chi A^{-4}\dot\chi^2 V_{T,\chi} + +\tfrac{1}{2}A^2(1-c_s)^2\left[c_s\left(\frac{f_\chi}{f^2}\right)_{,\chi} + (1+c_s)f^{-1}\left(\frac{f_\chi}{f}\right)_{,\chi}\right] + \tfrac{3}{2}A^2\dot\chi^2 c_s^{-1}(1+c_s^2) - -A^4 c_s^{-2}\frac{\dot\chi^4}{2H^2} - A^2 c_s^{-1}(1+c_s^2)\frac{\dot\chi^2\dot\varphi^2}{4H^2} + \frac{\dot\chi V_{T,\chi}}{H}(1+c_s^2) + c_s^3 A^{-2}V_{T,\chi\chi}, \quad (3)$$

$$C_{\chi\varphi} = \left(2\beta + \frac{\dot\varphi}{H}\right)\left[\tfrac{1}{2}A^2\frac{f_\chi}{f^2}c_s(1-c_s)^2 + \frac{\dot c_s}{c_s}\dot\chi\right] - 2\beta\frac{A^4\dot\chi}{H}f^{-1}(1-c_s)^2 - c_s^{-1}\frac{A^2\dot\varphi\dot\chi^3}{2H^2} + +\tfrac{1}{2}(1+c_s^2)\left[3\dot\varphi\dot\chi - \frac{\dot\varphi^3\dot\chi}{2H^2} + 2\beta c_s A^{-2}V_{T,\chi} + c_s\frac{\dot\varphi A^{-2}V_{T,\chi}}{H} + \frac{V_{T,\varphi}\dot\chi}{H}\right]. \quad (4)$$

# Initial conditions

To get the initial conditions, we can define auxiliary variables similar to the single field case

$$\nu_\varphi = r_\varphi Q_\varphi, \qquad \nu_\chi = r_\chi Q_\chi,$$

where $r_\varphi = a$ and $r_\chi = aA\gamma^{3/2}$. Differentiating wrt conformal time ($' = d/d\tau$) gives

$$\begin{aligned}
\nu_\varphi'' - B\nu_\chi' + \left[k^2 + a^2 C_{\varphi\varphi} - \frac{r_\varphi''}{r_\varphi}\right]\nu_\varphi + \left[\left(\frac{r_\varphi}{r_\chi}\right)a^2 C_{\varphi\chi} + B\frac{r_\chi'}{r_\chi}\right]\nu_\chi &= 0, \\
\nu_\chi'' + B\nu_\varphi' + \left[k^2 c_s^2 + a^2 C_{\chi\chi} - \frac{r_\chi''}{r_\chi}\right]\nu_\chi + \left[\left(\frac{r_\chi}{r_\varphi}\right)a^2 C_{\chi\varphi} - B\frac{r_\varphi'}{r_\varphi}\right]\nu_\varphi &= 0,
\end{aligned}$$

with $B = r_\chi B_\chi$. We can neglect correlations between the perturbations for modes well within the horizon, so the short wavelength solutions for the $\nu_\varphi$ and $\nu_\chi$ are

$$\nu_\varphi = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \qquad \nu_\chi = \frac{1}{\sqrt{2kc_s}} e^{-ikc_s\tau},$$

and the relative normalisation of the perturbations is dependent on the sound speed of the DBI field.

# Perturbed equations

In terms of the total potential $V_T(\varphi,\chi) = U(\varphi) + A^4(\varphi)V(\chi)$, the other coefficients are

$$\begin{aligned}
C_{\varphi\varphi} &= \beta\left(\frac{\dot\varphi}{H}\right)A^4 f^{-1}(3c_s+1)(1-c_s^{-1})^3 - \beta^2 A^4 f^{-1}(16 - 8c_s^{-1} - 9c_s + c_s^{-3}) + \\
&+3\dot\varphi^2 - c_s^{-3}(1+c_s^2)A^2\frac{\dot\varphi^2\dot\chi^2}{4H^2} - \frac{\dot\varphi^4}{2H^2} + \frac{2\dot\varphi V_{T,\varphi}}{H} + V_{T,\varphi\varphi}, \qquad (1)\\
C_{\varphi\chi} &= \frac{A^4\dot\varphi}{4H}\frac{f_\chi}{f^2}c_s(1-c_s^{-1})^2(c_s^{-2}+2c_s^{-1}-1) + 3c_s^{-1}A^2\dot\chi\dot\varphi - c_s^{-4}(1+c_s^2)A^4\frac{\dot\varphi\dot\chi^3}{4H^2} - \\
&-\tfrac{1}{2}\beta A^4 f^{-1}(3c_s+1)(1-c_s^{-1})^3\left[\frac{f_\chi}{f} - \frac{A^2\dot\chi}{Hc_s}\right] - c_s^{-1}A^2\frac{\dot\varphi^3\dot\chi}{2H^2} + \frac{V_{T,\chi}\dot\varphi}{H} + \\
&+\frac{c_s^{-1}A^2\dot\chi}{H}V_{T,\varphi} + V_{T,\varphi\chi}, \qquad (2)\\
C_{\chi\chi} &= \frac{A^4\dot\chi}{H}\frac{f_\chi}{f^2}(1-c_s)^2 - \left[\frac{f_\chi}{f} - \frac{A^2\dot\chi}{Hc_s}\right]\frac{\dot c_s}{c_s}\dot\chi - \tfrac{1}{2}c_s f_\chi A^{-4}\dot\chi^2 V_{T,\chi} + \\
&+\tfrac{1}{2}A^2(1-c_s)^2\left[c_s\left(\frac{f_\chi}{f^2}\right)_{,\chi} + (1+c_s)f^{-1}\left(\frac{f_\chi}{f}\right)_{,\chi}\right] + \tfrac{3}{2}A^2\dot\chi^2 c_s^{-1}(1+c_s^2) - \\
&-A^4 c_s^{-2}\frac{\dot\chi^4}{2H^2} - A^2 c_s^{-1}(1+c_s^2)\frac{\dot\chi^2\dot\varphi^2}{4H^2} + \frac{\dot\chi V_{T,\chi}}{H}(1+c_s^2) + c_s^3 A^{-2}V_{T,\chi\chi}, \qquad (3)\\
C_{\chi\varphi} &= \left(2\beta + \frac{\dot\varphi}{H}\right)\left[\tfrac{1}{2}A^2\frac{f_\chi}{f^2}c_s(1-c_s)^2 + \frac{\dot c_s}{c_s}\dot\chi\right] - 2\beta\frac{A^4\dot\chi}{H}f^{-1}(1-c_s)^2 - c_s^{-1}\frac{A^2\dot\varphi\dot\chi^3}{2H^2} + \\
&+\tfrac{1}{2}(1+c_s^2)\left[3\dot\varphi\dot\chi - \frac{\dot\varphi^3\dot\chi}{2H^2} + 2\beta c_s A^{-2}V_{T,\chi} + c_s\frac{\dot\varphi A^{-2}V_{T,\chi}}{H} + \frac{V_{T,\varphi}\dot\chi}{H}\right]. \qquad (4)
\end{aligned}$$

# Perturbed equations

Using $c_s^2 = \gamma^{-2}$, this gives

$$\ddot{Q}\varphi \quad + \quad 3H\dot{Q}\varphi + B_\varphi \dot{Q}_\chi + \left(\frac{k^2}{a^2} + C_{\varphi\varphi}\right) Q_\varphi + C_{\varphi\chi} Q_\chi = 0,$$

$$\ddot{Q}\chi \quad + \quad \left[3H + 2\beta\dot{\varphi} - 3\frac{\dot{c}_s}{c_s}\right] \dot{Q}\chi + B_\chi \dot{Q}_\varphi + \left(\frac{k^2}{a^2}c_s^2 + C_{\chi\chi}\right) Q_\chi + C_{\chi\varphi} Q_\varphi = 0$$

The coefficients are made up of terms:

- proportional to the non-minimal coupling parameter $\beta$
- with factors of $c_s$ due to the non-canonical kinetic term of the DBI field.
- involving the potentials of both fields.

For example consider the cross-kinetic terms

$$B_\chi = \left[\beta(3c_s^2 - 1) - \frac{\dot{\varphi}}{2H}(1 - c_s^2)\right] \dot{\chi}, \qquad B_\varphi = -A^2 c_s^{-3} B_\varphi.$$

These are not present with two uncoupled canonical fields and $B_{\varphi,\chi} \to 0$ as $\beta \to 0$, $c_s \to 1$.

# Perturbed equations

In terms of the total potential $V_T(\varphi,\chi) = U(\varphi) + A^4(\varphi)V(\chi)$, the other coefficients are

$$
\begin{aligned}
C_{\varphi\varphi} &= \beta\left(\frac{\dot\varphi}{H}\right)A^4 f^{-1}(3c_s+1)(1-c_s^{-1})^3 - \beta^2 A^4 f^{-1}(16-8c_s^{-1}-9c_s+c_s^{-3}) + \\
&\quad + 3\dot\varphi^2 - c_s^{-3}(1+c_s^2)A^2\frac{\dot\varphi^2\dot\chi^2}{4H^2} - \frac{\dot\varphi^4}{2H^2} + \frac{2\dot\varphi V_{T,\varphi}}{H} + V_{T,\varphi\varphi}, \qquad (1)\\
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&\quad - \tfrac{1}{2}\beta A^4 f^{-1}(3c_s+1)(1-c_s^{-1})^3\left[\frac{f_\chi}{f} - \frac{A^2\dot\chi}{Hc_s}\right] - c_s^{-1}A^2\frac{\dot\varphi^3\dot\chi}{2H^2} + \frac{V_{T,\chi}\dot\varphi}{H} + \\
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&\quad + \tfrac{1}{2}A^2(1-c_s)^2\left[c_s\left(\frac{f_\chi}{f^2}\right)_{,\chi} + (1+c_s)f^{-1}\left(\frac{f_\chi}{f}\right)_{,\chi}\right] + \tfrac{3}{2}A^2\dot\chi^2 c_s^{-1}(1+c_s^2) - \\
&\quad - A^4 c_s^{-2}\frac{\dot\chi^4}{2H^2} - A^2 c_s^{-1}(1+c_s^2)\frac{\dot\chi^2\dot\varphi^2}{4H^2} + \frac{\dot\chi V_{T,\chi}}{H}(1+c_s^2) + c_s^3 A^{-2}V_{T,\chi\chi}, \qquad (3)\\
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&\quad + \tfrac{1}{2}(1+c_s^2)\left[3\dot\varphi\dot\chi - \frac{\dot\varphi^3\dot\chi}{2H^2} + 2\beta c_s A^{-2}V_{T,\chi} + c_s\frac{\dot\varphi A^{-2}V_{T,\chi}}{H} + \frac{V_{T,\varphi}\dot\chi}{H}\right]. \qquad (4)
\end{aligned}
$$

# Initial conditions

To get the initial conditions, we can define auxiliary variables similar to the single field case

$$\nu_\varphi = r_\varphi Q_\varphi, \qquad \nu_\chi = r_\chi Q_\chi,$$

where $r_\varphi = a$ and $r_\chi = aA\gamma^{3/2}$. Differentiating wrt conformal time ($' = d/d\tau$) gives

$$\begin{aligned}
\nu''_\varphi - B\nu'_\chi + \left[k^2 + a^2 C_{\varphi\varphi} - \frac{r''_\varphi}{r_\varphi}\right]\nu_\varphi + \left[\left(\frac{r_\varphi}{r_\chi}\right) a^2 C_{\varphi\chi} + B\frac{r'_\chi}{r_\chi}\right]\nu_\chi &= 0,\\
\nu''_\chi + B\nu'_\varphi + \left[k^2 c_s^2 + a^2 C_{\chi\chi} - \frac{r''_\chi}{r_\chi}\right]\nu_\chi + \left[\left(\frac{r_\chi}{r_\varphi}\right) a^2 C_{\chi\varphi} - B\frac{r'_\varphi}{r_\varphi}\right]\nu_\varphi &= 0,
\end{aligned}$$

with $B = r_\chi B_\chi$. We can neglect correlations between the perturbations for modes well within the horizon, so the short wavelength solutions for the $\nu_\varphi$ and $\nu_\chi$ are

$$\nu_\varphi = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \qquad \nu_\chi = \frac{1}{\sqrt{2kc_s}} e^{-ikc_s\tau},$$

and the relative normalisation of the perturbations is dependent on the sound speed of the DBI field.

# Initial conditions

The modes should be uncorrelated deep within the horizon. As in the standard two-field case, to implement this numerically one should perform two runs (cf. Tsujikawa et al., 2003 [12])

1. One with $\nu_\varphi$ in the Bunch-Davies vacuum and $\nu_\chi = 0$.
2. One with $\nu_\varphi = 0$ and $\nu_\chi$ in the (quasi) Bunch-Davies vacuum.

One can then calculate the (comoving) curvature perturbation

$$\mathcal{R} = \frac{H}{(-2\dot{H})}\left[\dot{\varphi}Q_\varphi + A^2\gamma\dot{\chi}Q_\chi\right]$$

for each run and combine the results to get the power spectrum

$$\mathcal{P}_\mathcal{R} = \frac{k^3}{2\pi^2}\left(|\mathcal{R}_1|^2 + |\mathcal{R}_2|^2\right)$$

# Adiabatic and entropic fields

The rate of change of the curvature perturbation can be written,

$$\dot{\mathcal{R}} = \frac{H}{(-2\dot{H})}\left[(\delta p_{nad}) + \frac{\dot{p}}{\dot{\rho}}(\delta\rho_m)\right]$$

where non-adiabatic pressure is defined by,

$$(\delta p_{nad}) \equiv (\delta p) - \frac{\dot{p}}{\dot{\rho}}(\delta\rho).$$

The quantity $\delta\rho_m \equiv \delta\rho - 3H\delta q$ is the gauge invariant comoving density perturbation. This appears on the RHS of the Poisson equation,

$$\frac{k^2}{a^2}\Psi = -\tfrac{1}{2}\delta\rho_m,$$

and thus decays in the long-wavelength/late-time limit.

# Adiabatic and entropic fields

If we introduce adiabatic and entropic fields in a similar manner to the coupled two-field system, so that $\mathcal{R} = (H/\sqrt{-2\dot{H}})Q_\sigma$ i.e.

$$Q_\sigma = \frac{1}{\sqrt{-2\dot{H}}}\left[\dot{\varphi}Q_\varphi + A^2\gamma\dot{\chi}Q_\chi\right], \qquad \delta s = \frac{A\gamma^{1/2}}{\sqrt{-2\dot{H}}}\left[\dot{\varphi}Q_\chi - \dot{\chi}Q_\varphi\right],$$

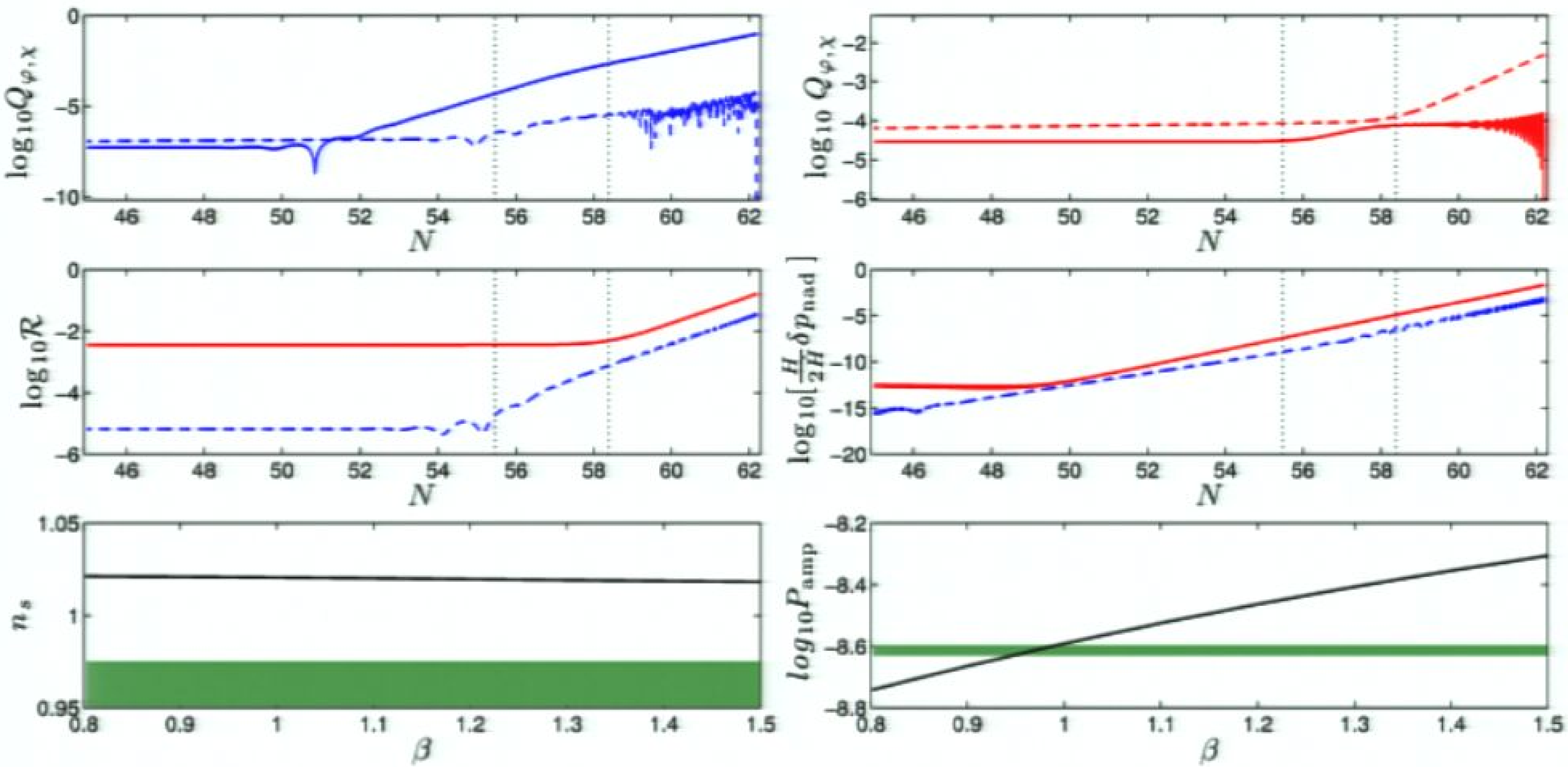
the non-adiabatic pressure can be written

$$\delta p_{\rm nad} = \left[1 - \frac{\dot{p}}{\dot{\rho}}\right]\delta\rho_m + \frac{2}{\sqrt{-2\dot{H}}}\left[\sqrt{-2\dot{H}}P_s + \beta A^3\gamma^{3/2}\dot{\chi}^3\right](\delta s) + \left[\frac{\dot{\gamma}}{\sqrt{-2\dot{H}}}Q_\sigma - Q_\gamma\right]A^2\dot{\chi}^2.$$

where $P_s$ is the partial derivative of the pressure wrt the entropy field and $Q_\gamma = \delta\gamma + \frac{\dot{\gamma}}{H}\Psi$ is a gauge invariant quantity dependent on the sound speed.

**Unlike the coupled two-field case, the non-adiabatic pressure cannot be written only in terms of the entropy field.**

# Perturbations: exponential potential (large $\gamma$)




Perturbation quantities for pivot scale with $\beta = 1$. The lower panels show the variation in the resulting value of the spectral index $n_s$ (bottom-left) and power spectrum amplitude $P_{\mathrm{amp}}$ (bottom-right) with the coupling $\beta$. The shaded regions shows observational values of these quantities (WMAP+ BAO+H0) at 68% c.l [1].

# Observational quantities

There are six parameters in the model that can affect the behaviour of the perturbations:

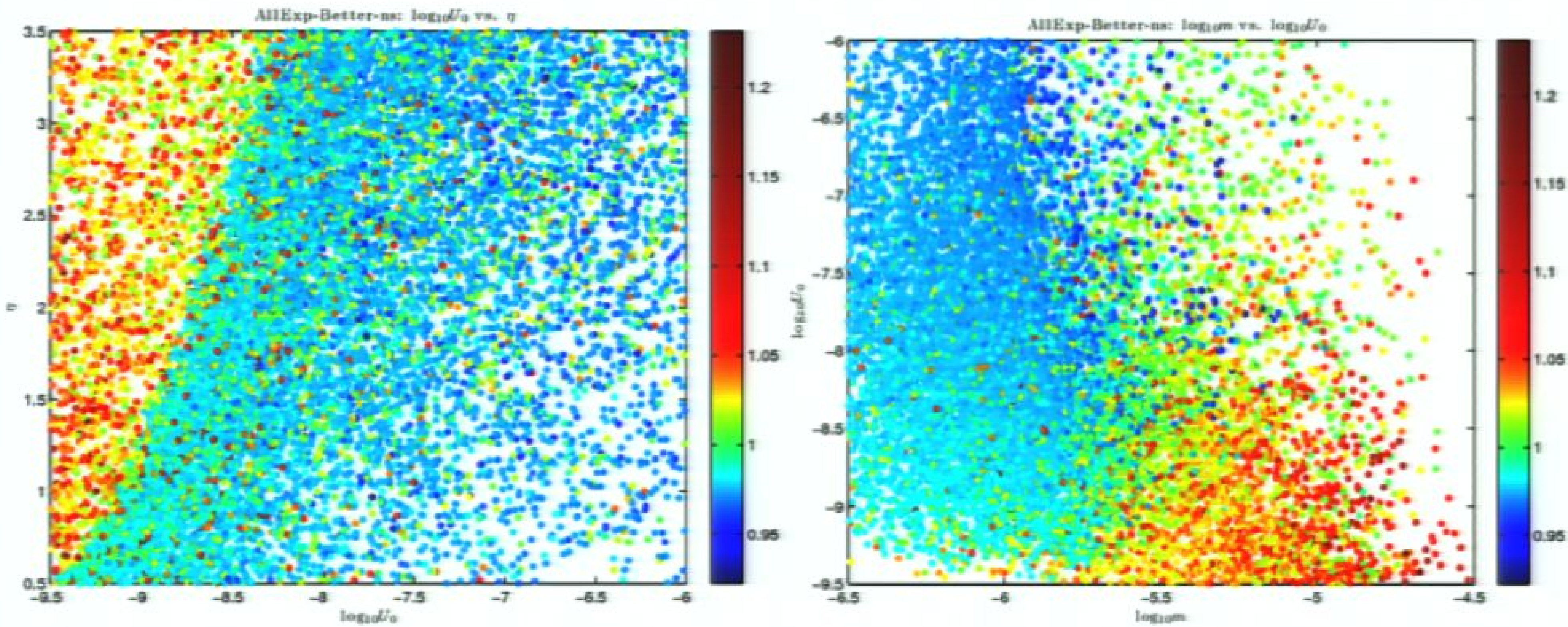
- The coupling $\beta$
- DBI warp factor parameters $\lambda$ and $\mu$
- Parameters in the field potentials $U_0$, $m$, $\eta$

We can constrain the parameter space by considering the effect on:

- background quantities e.g. no. of efolds, boost factor
- perturbation quantities e.g. the power spectrum amplitude, spectral index

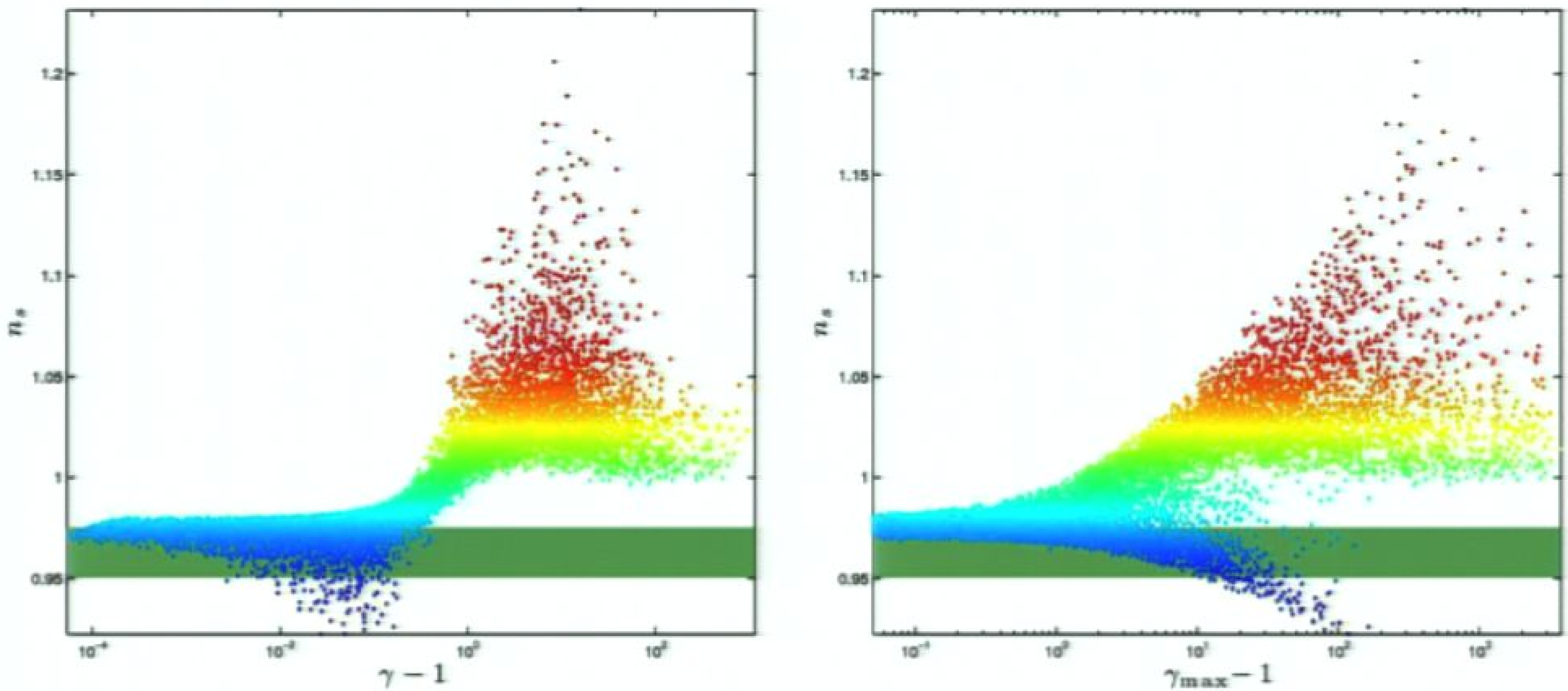
**Integrate the perturbations $5 \times 10^6$ times with random permutations of parameters. Calculate $n_s$ for those with $P_{\mathrm{amp}}$ in the observational range.**

# Numerical results: spectral index $n_s$




Looking at the runs with $P_{\mathrm{amp}} = 2.441^{+0.088}_{-0.092} \times 10^{-9}$, we can identify two overlapping regions in the parameter space based on the value of the spectral index.

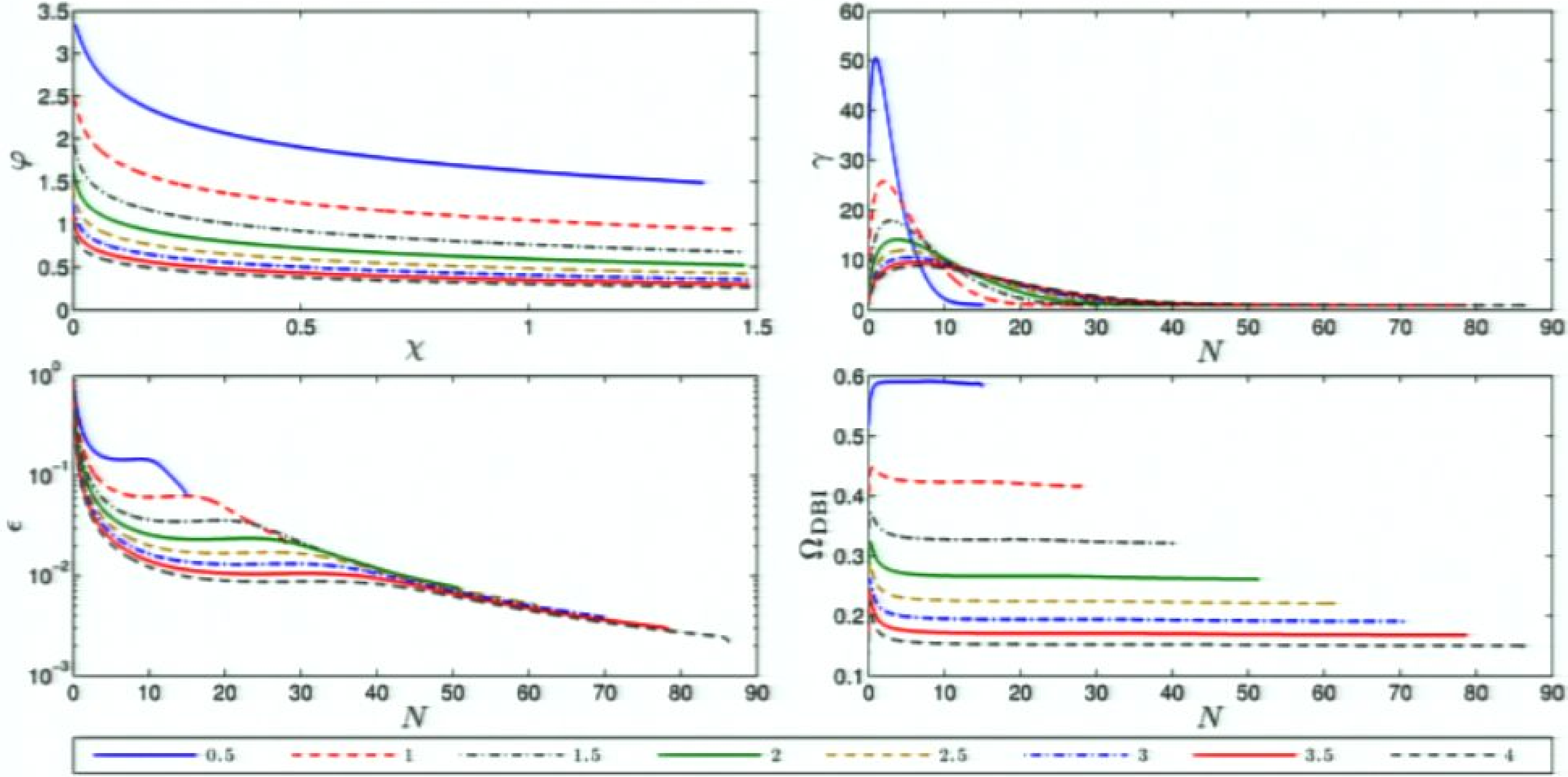
# Numerical results: spectral index $n_s$




- Yellow/red points with $n_s \gtrsim 1$ exhibit significant DBI characteristics at horizon crossing and afterwards.
- Points with $n_s \lesssim 0.96$ have DBI characteristics suppressed $[(\gamma - 1) \ll 1]$ at horizon crossing but have $\gamma_{\max} \sim 10 - 100$.

# Background dynamics: exponential potential (small $\gamma$)

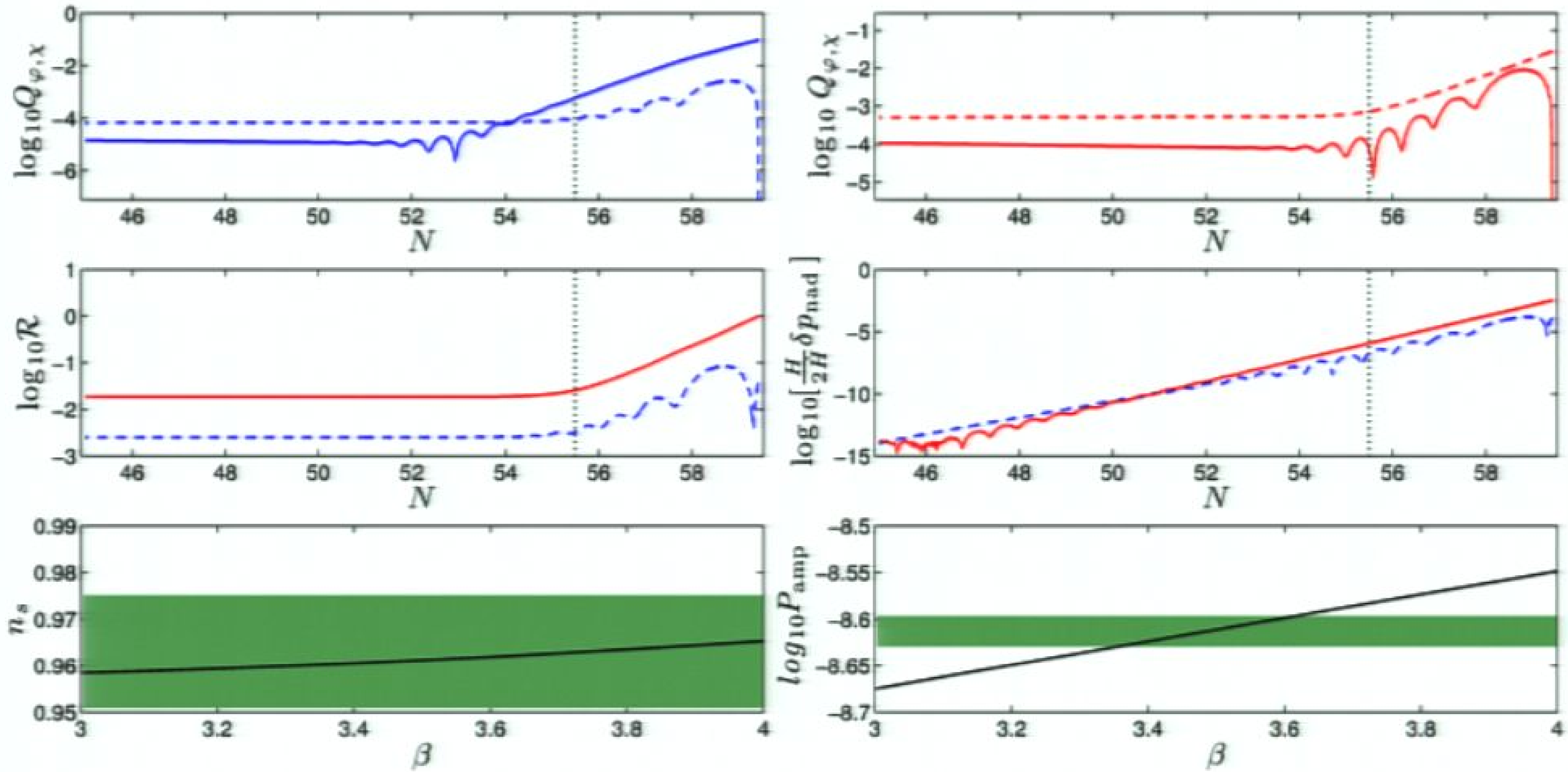
In some cases with relatively large coupling, the effective warp factor $A^{-2}f$ is small enough that $\chi$ does not exhibit DBI behaviour until it has rolled to sufficiently small values.






Exponential potential with $\lambda = 7.8 \times 10^{11}$, $m = 2.8 \times 10^{-6}$, $U0 = 7 \times 10^{-9}$, $\mu = 0.07$, $n = 2.8$

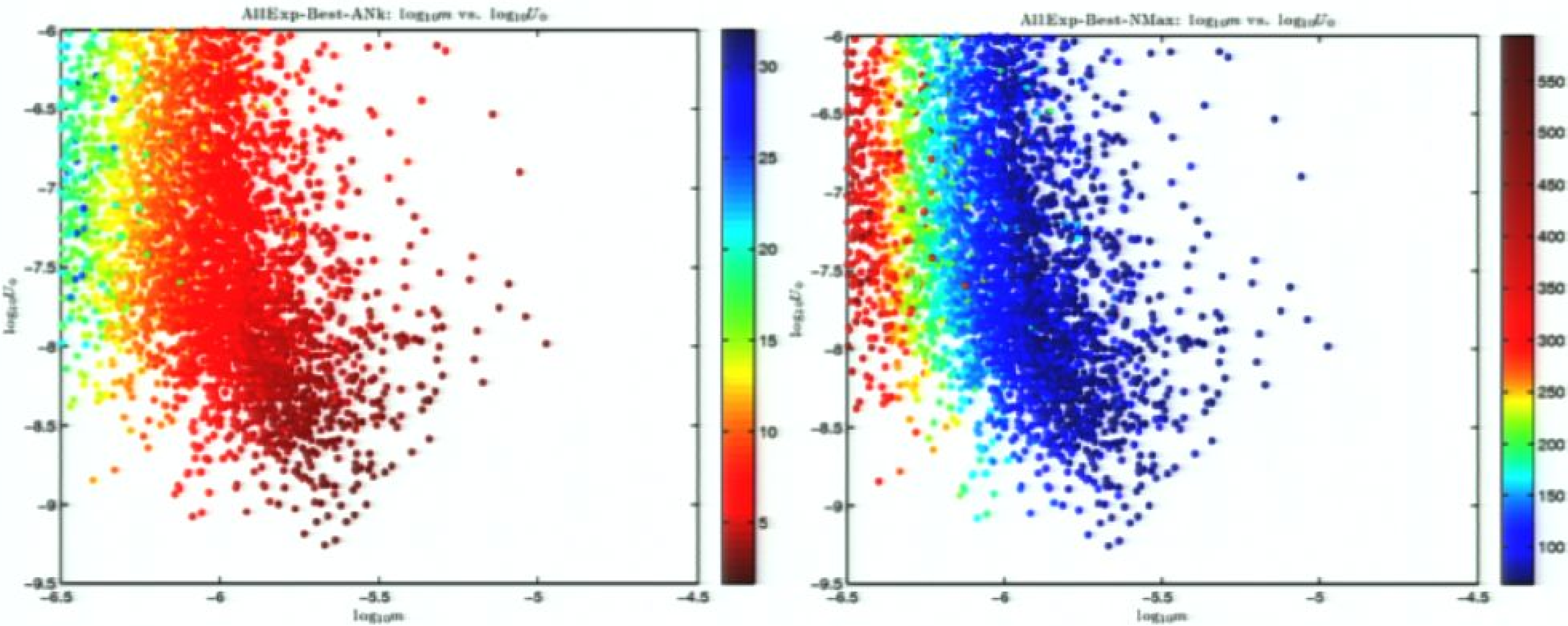
# Perturbations: exponential potential (small $\gamma$)




Perturbation quantities for pivot scale with $\beta = 3.5$.

# Background quantities: potential terms $m$ vs. $U_0$

The region of the parameter space with both $n_s = 0.963 \pm 0.012$ and $P_{\rm amp} = 2.441^{+0.088}_{-0.092} \times 10^{-9}$ is more tightly constrained.

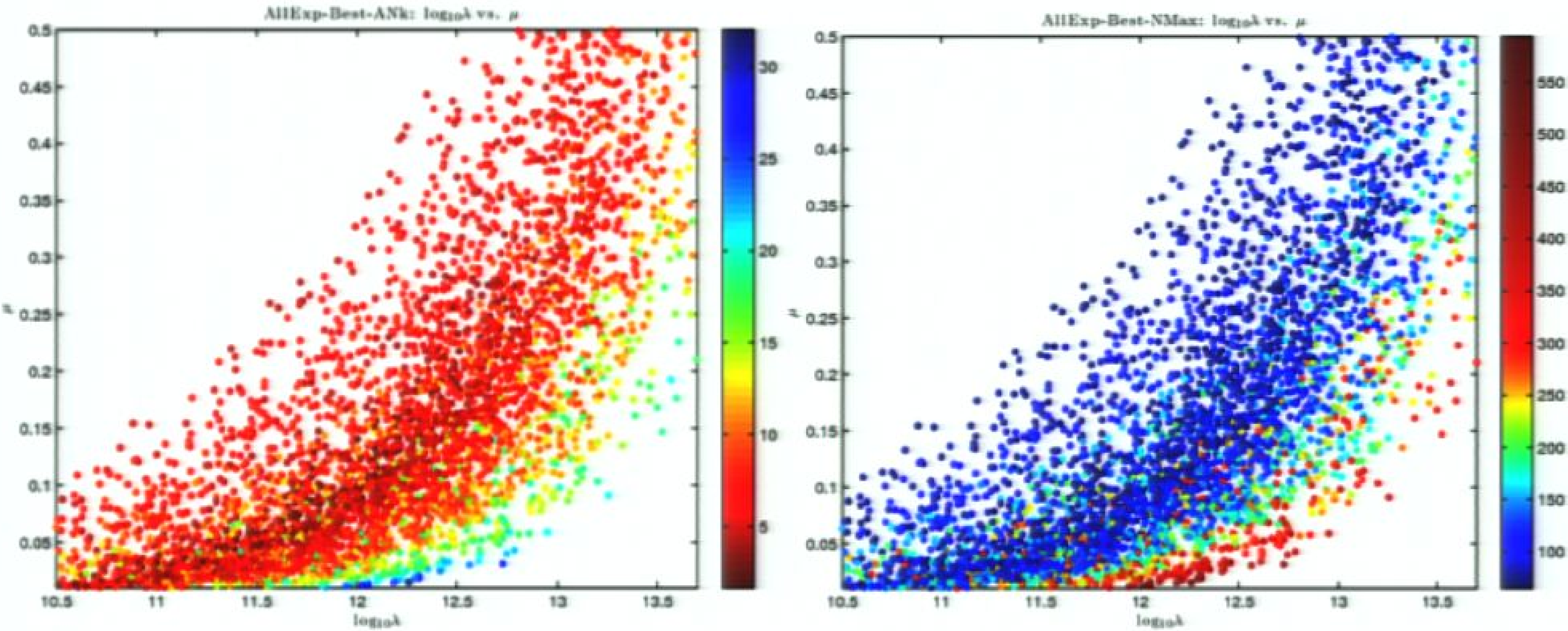



Left plot colour-coded by $A(\varphi)|_{k=aH}$; right plot colour-coded by $N_{\rm max}$.

**As we saw before, large coupling is correlated with a long period of inflation.**

# Background quantities: warp factor terms $\lambda$ vs. $\mu$

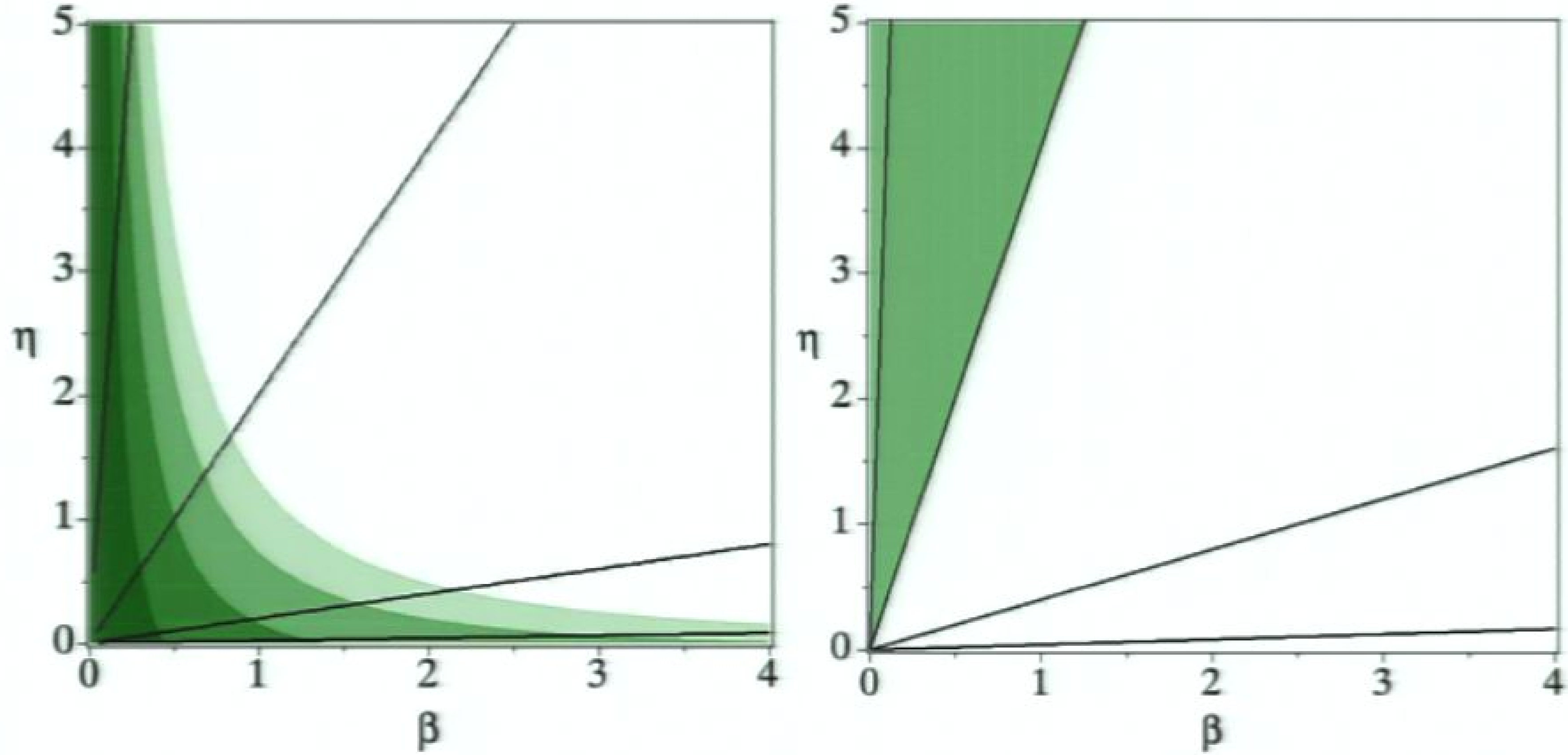
The region of the parameter space with both $n_s = 0.963 \pm 0.012$ and $P_{\mathrm{amp}} = 2.441^{+0.088}_{-0.092} \times 10^{-9}$ is more tightly constrained.




Left plot colour-coded by $A(\varphi)|_{k=aH}$; right plot colour-coded by $N_{\max}$.

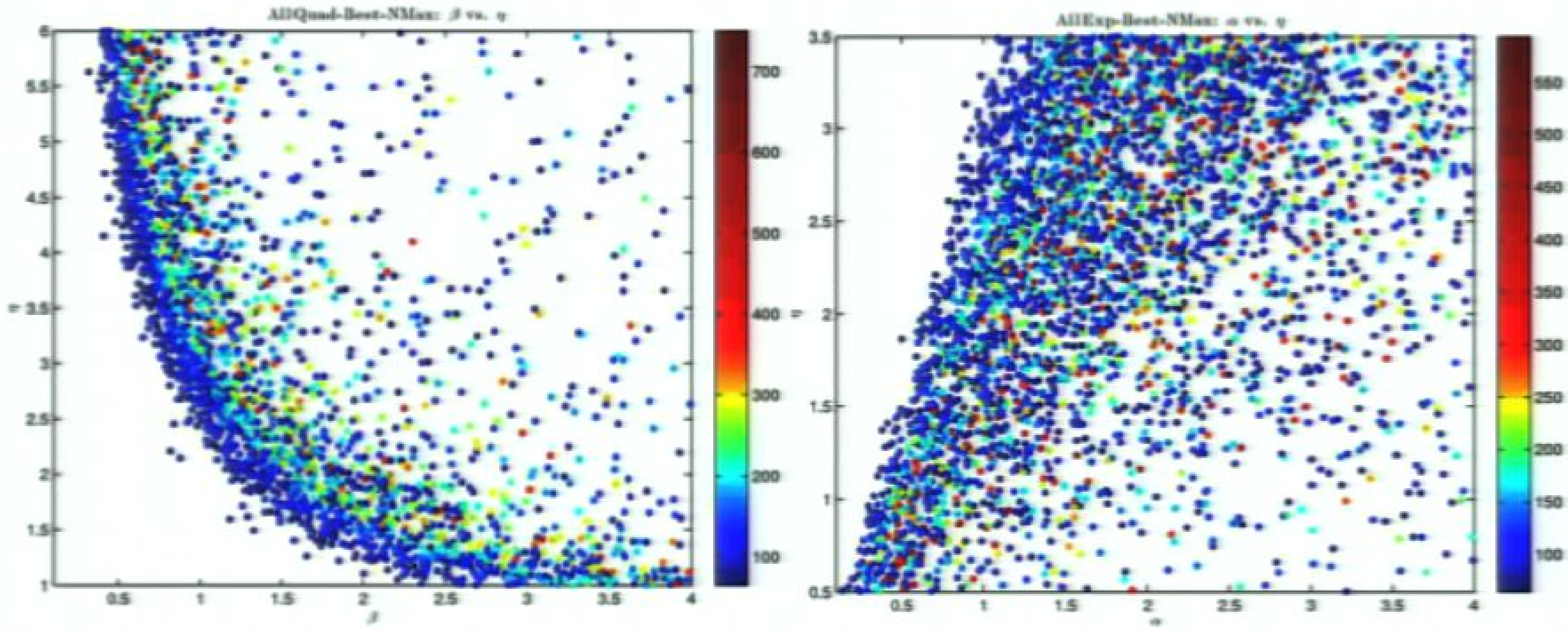
**As we saw before, large coupling is correlated with a long period of inflation.**

# Solutions with $\Omega_\varphi \gtrsim \Omega_{\rm DBI}$



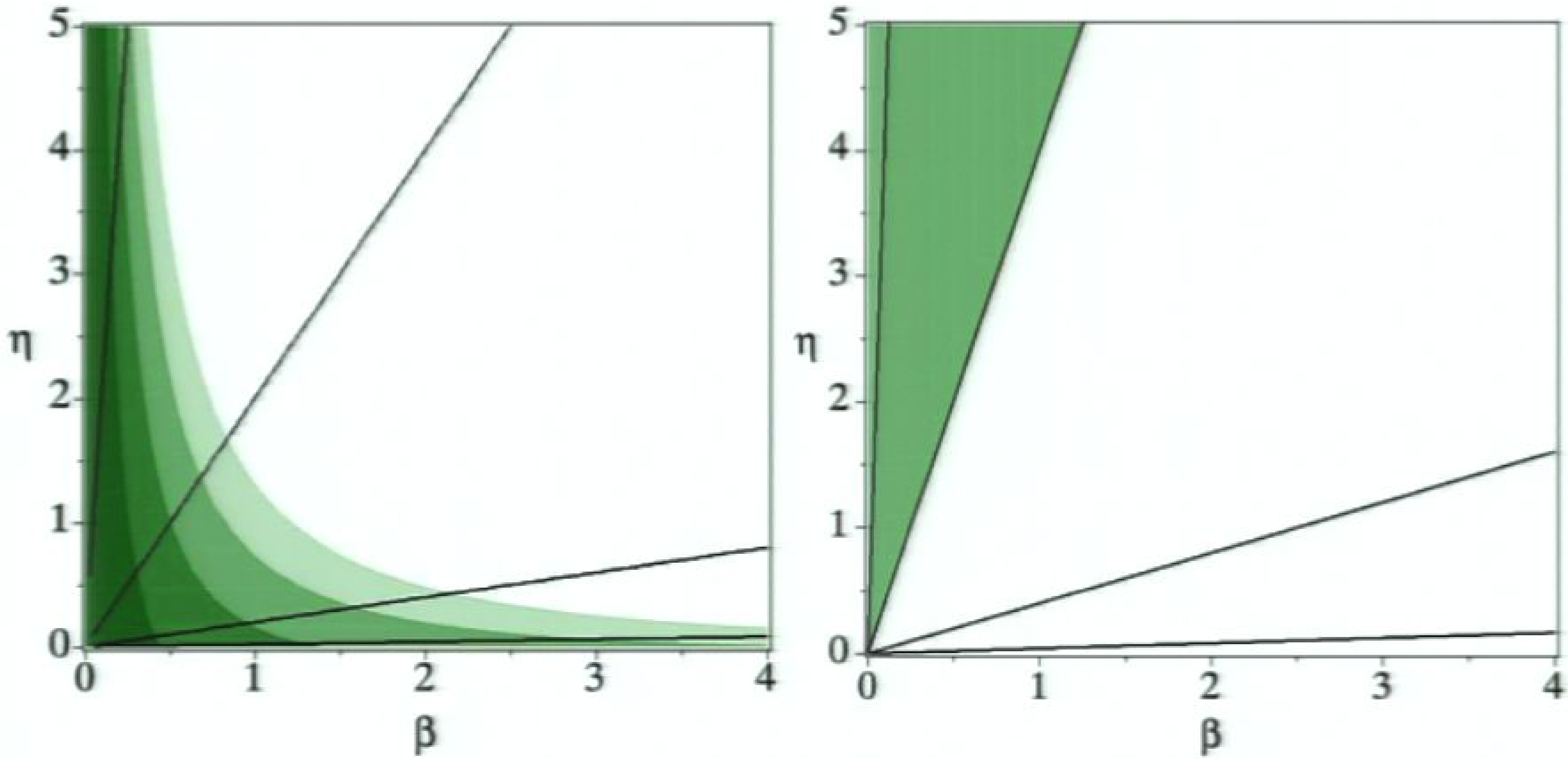

Earlier I showed these plots, in which the shaded regions mark out regions with $\Omega_{\rm DBI} > \Omega_\varphi$ (for different values of $V/U_0$).

# Solutions with $\Omega_\varphi \gtrsim \Omega_{\mathrm{DBI}}$



Comparing to the numerical results, we see that DBI dominant solutions are excluded.

# Solutions with $\Omega_\varphi \gtrsim \Omega_{\rm DBI}$

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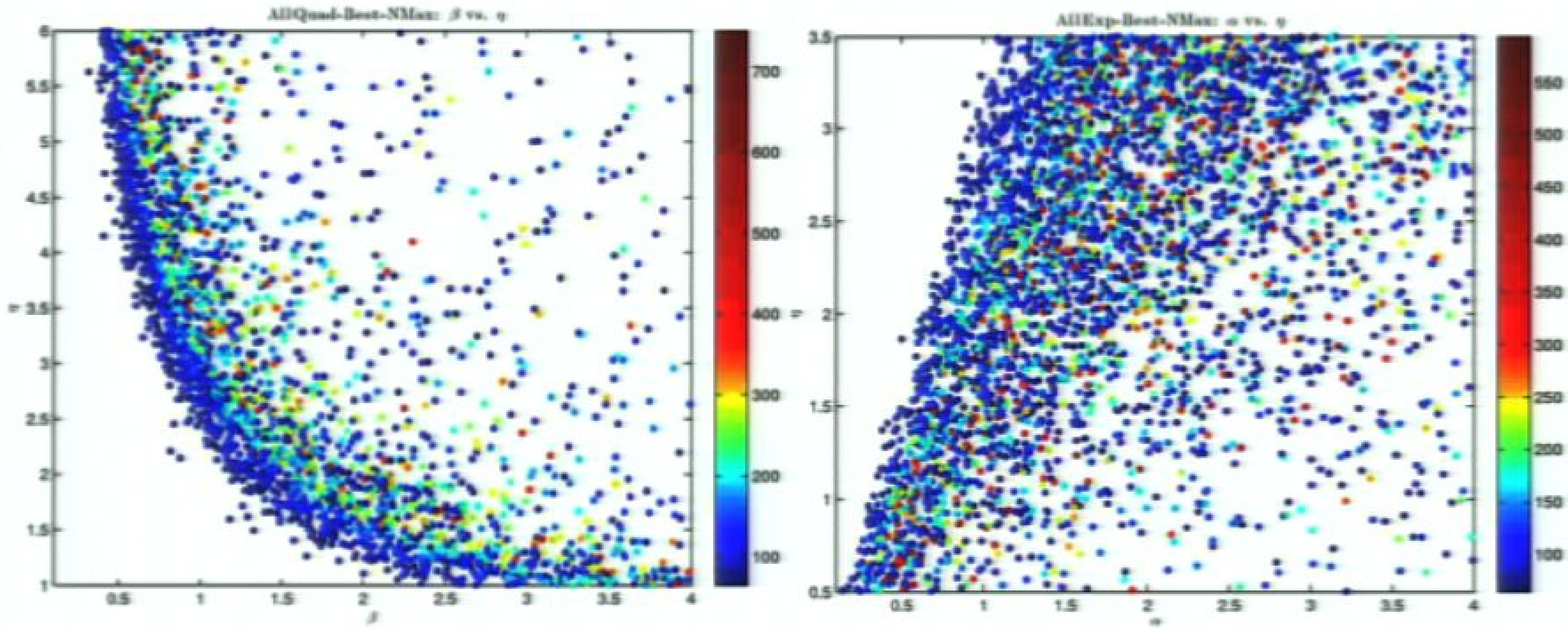
# Solutions with $\Omega_\varphi \gtrsim \Omega_{\mathrm{DBI}}$



Comparing to the numerical results, we see that DBI dominant solutions are excluded.

# Extensions and future work

- So far, only the spectral index and power spectrum amplitude have been considered. To go further, one can other quantities such the tensor-to-scalar ratio $r$ and and the running of the spectral index $\alpha$ to constrain the parameter space.
- Another important extension is to investigate the minimally coupled case, in which both fields are dynamically important.
- Finally, it would be of interest to investigate the properties of the non-Gaussian signature produced in this scenario.

# Summary

- DBI provides an interesting example of k-inflation with 'stringy' motivations. However, a large boost factor and short duration of inflation can be a problem.
- Realistic inflationary models are likely to involve multiple scalar fields and couplings. Coupled DBI inflation combines these elements in a two-field model.
- The coupling forces the scalar field into the minimum of its effective potential, extending the number of efolds of DBI inflation and decreasing the boost factor. The $\varphi$ perturbations are negligible when the field is in the minimum so the level of non-Gaussianity (normally $\propto \gamma^2$) would be much smaller than the standard DBI case.
- A range of parameters affect the prediction for the spectral index (for both the exponential and the quadratic potential) including a considerable set compatible with current observational limits.