Title: Spin Foams and Noncommutative Geometry

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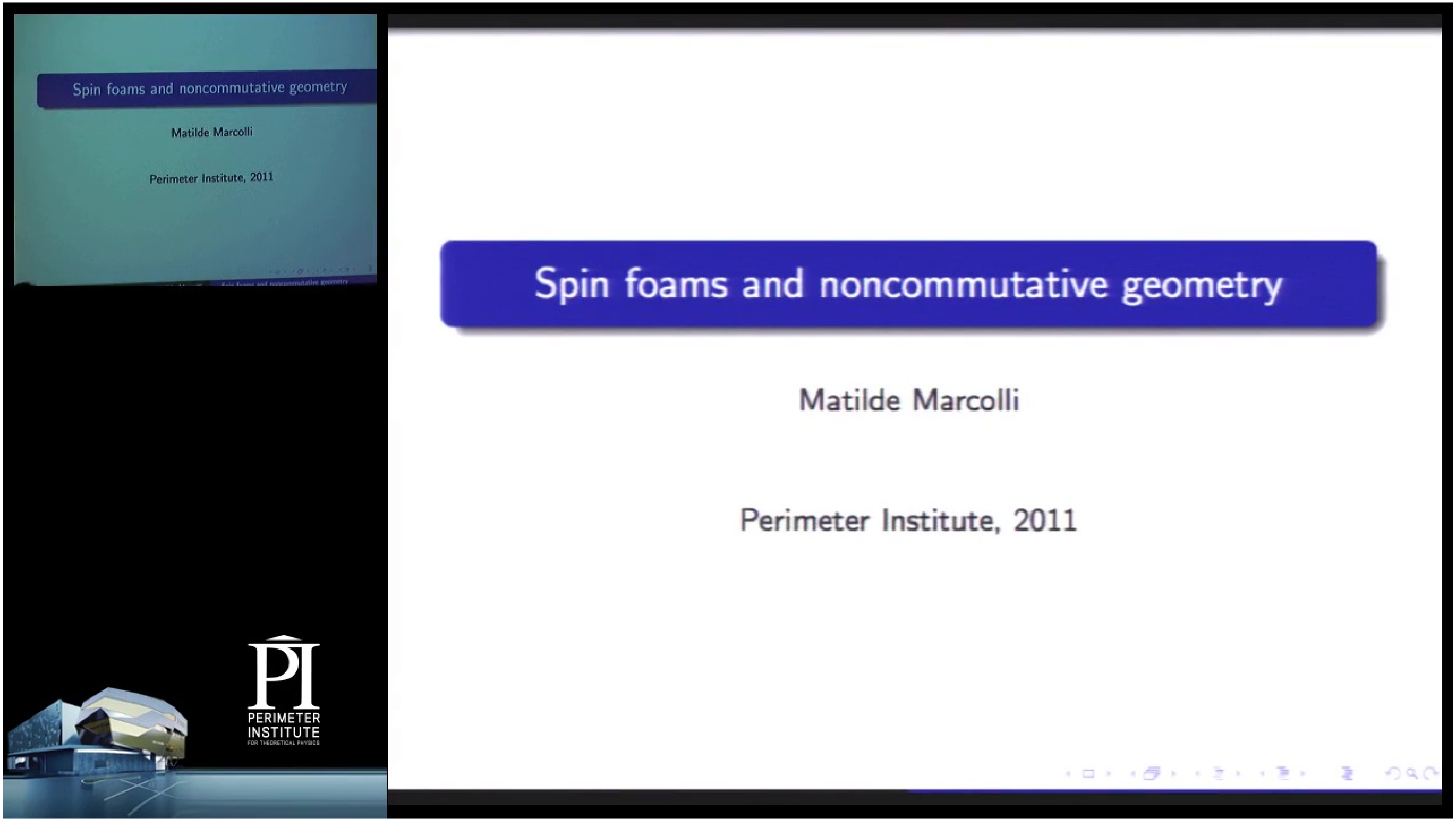
Abstract: We extend the formalism of embedded spin networks and spin foams to include topological data that encode the underlying three-manifold or
four-manifold as a branched cover. These data are expressed as monodromies, in a way similar to the encoding of the gravitational field via holonomies. We then describe convolution algebras of spin networks and spin foams, based on the different ways in which the same topology can be realized as a branched covering via covering moves, and on possible composition operations on spin foams. We illustrate the case of the groupoid algebra of the equivalence relation determined by covering moves and a 2-semigroupoid algebra arising from a 2-category of spin foams with composition operations corresponding to a fibered product of the branched coverings and the gluing of cobordisms. The spin foam amplitudes then give rise to dynamical flows on these algebras, and the existence of low temperature equilibrium states of Gibbs form is related to questions on the existence of topological invariants of embedded graphs and embedded two-complexes with given properties. We end by sketching a possible approach to combining the spin network and spin foam formalism with matter within the framework of spectral triples in noncommutative geometry.
(Based on joint work with Domenic Denicola and Ahmad Zainy al-Yasry)


Spin foams and noncommutative geometry
Matilde Marcolli
Perimeter Institute, 2011
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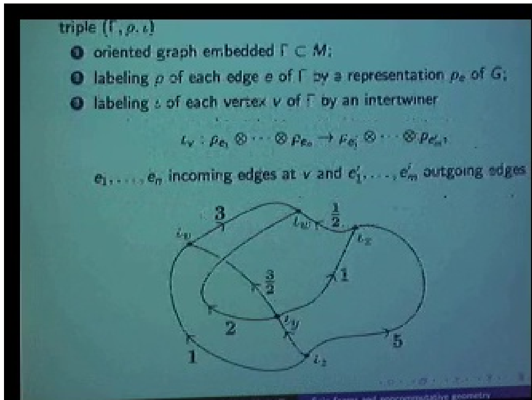

Based on

• Domenic Denicola, Matilde Marcolli, Ahmed Zainy al-Yasry, *Spin foams and noncommutative geometry*, Classical and Quantum Gravity, 27 (2010) 205025 [53 pages]

• M.Marcolli, A. Zainy al-Yasry, *Coverings, correspondences and noncommutative geometry*, Journal of Geometry and Physics, Vol.58 (2008) N.12, 1639–1661.

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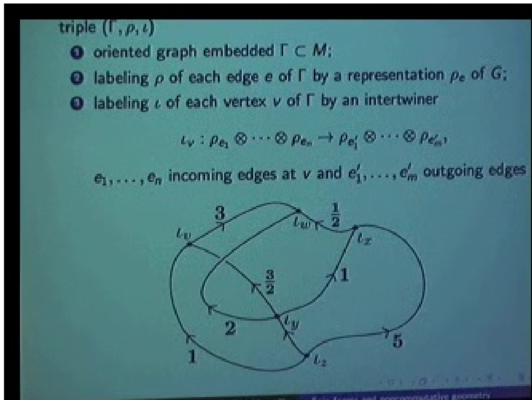
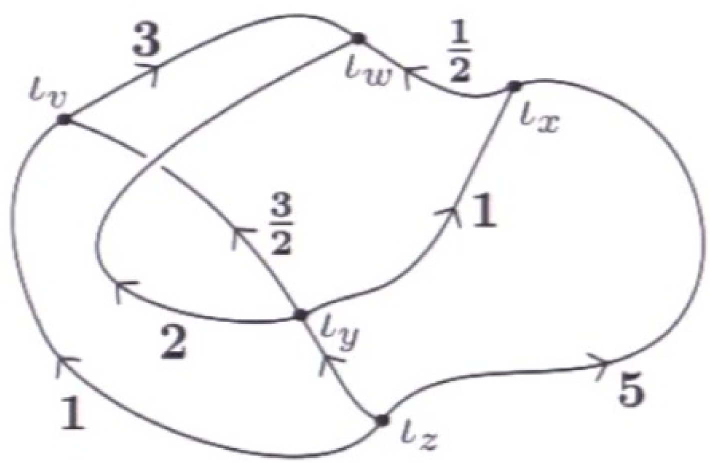


Spin networks (comp Lie group G) in a 3-manifold M:
triple (Γ, ρ, ι)

1. oriented graph embedded $\Gamma \subset M$;
2. labeling ρ of each edge e of Γ by a representation ρ_e of G;
3. labeling ι of each vertex v of Γ by an intertwiner

$$\iota_v : \rho_{e_1} \otimes \cdots \otimes \rho_{e_n} \to \rho_{e'_1} \otimes \cdots \otimes \rho_{e'_m},$$

$e_1, \ldots, e_n$ incoming edges at v and $e'_1, \ldots, e'_m$ outgoing edges



Idea: a "quantum three-geometry"
• vertices ⇒ quanta of volume
• egdes ⇒ quanta of area separating them
• representation data encode *holonomies* ⇒ gravitational field
• ambient topology M is fixed (eg Turaev-Viro invariants)
Idea of additional topological data: topspin networks
• ambient topology variable encoded in spin network data
• M encoded as a branched covering of S^3
• *monodromies* in addition to *holonomies*

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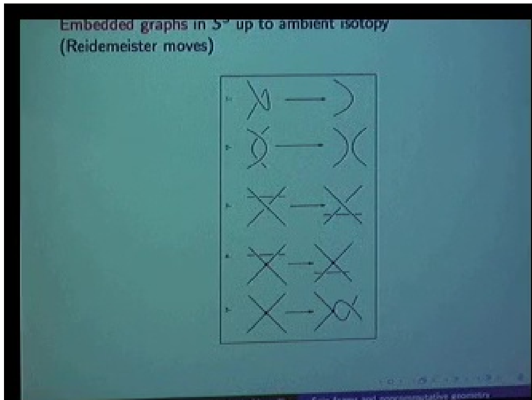
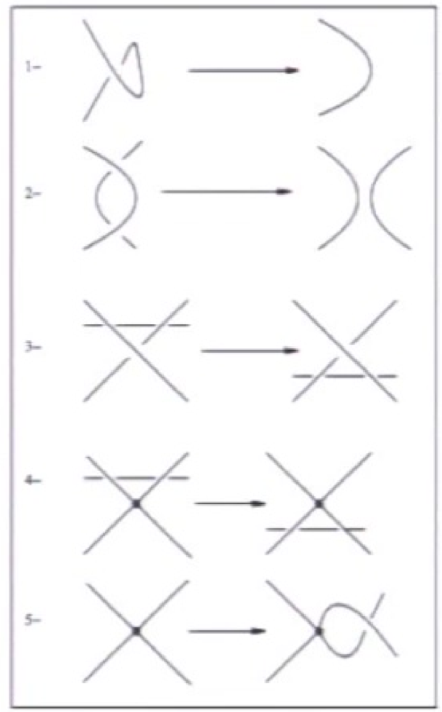


Embedded graphs in S^3 up to ambient isotopy
(Reidemeister moves)



3-manifolds as branched covers

$p : M \to S^3$ with restriction $p_| : M \setminus p^{-1}(\Gamma) \to S^3 \setminus \Gamma$ to complement of an embedded graph $\Gamma \subseteq S^3$ an ordinary covering of some degree n

Non-unique: Poincaré homology sphere fivefold covering of S^3 branched along the trefoil knot $K_{2,3}$ or threefold covering branched along the $(2,5)$ torus knot $K_{2,5}$

PL 4-manifolds: branched coverings of the four-sphere S^4, branched along an embedded simplicial two-complex (Piergallini)

Branched cover cobordism: 3-manifolds M_0 and M_1 branched coverings $p_i : M_i \to S^3$ along embedded graphs $\Gamma_i \subset S^3$, 4-dim cobordism W with $\partial W = M_0 \cup \bar{M}_1$ branched cover $q : W \to S^3 \times [0,1]$, branched along $\Sigma \subset S^3 \times [0,1]$ with $\partial \Sigma = \Gamma_0 \cup \bar{\Gamma}_1$ and $q|_{t=0} = p_0$, $q|_{t=1} = p_1$

Fundamental group representations
Branched covering $p : M \to S^3$ determined by representation

$$\sigma : \pi_1(S^3 \smallsetminus \Gamma) \to S_n$$

Wirtinger presentation: $D(\Gamma)$ planar diagram
permutations $\sigma_i \in S_n$ assigned to arcs of $D(\Gamma)$

$$\sigma_j = \sigma_k \sigma_i \sigma_k^{-1}$$

at crossings

$$\prod_i \sigma_i \prod_j \sigma_j^{-1} = 1$$

at vertices (σ_i incoming, σ_j outgoing arcs)
presentation by monodromies around edges of the embedded graph

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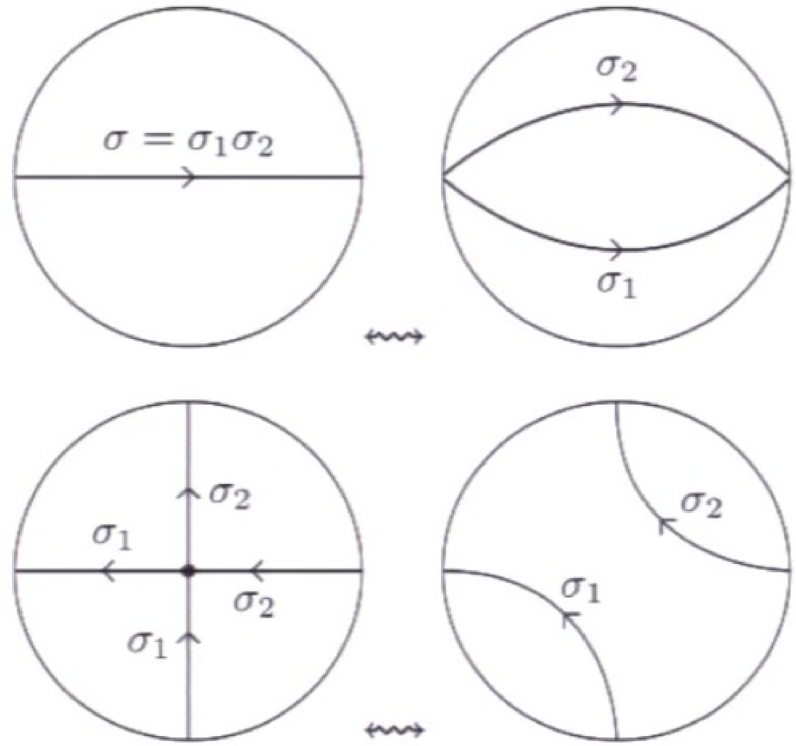
Covering moves: different coverings with same 3-manifold (Bobtcheva–Piergallini)

At vertices



Topspin networks (topologically enriched spin networks)

1. a spin network (Γ, ρ, ι) with $\Gamma \subset S^3$,
2. a representation $\sigma : \pi_1(S^3 \setminus \Gamma) \to S_n$.

$\Rightarrow$ gives a spin network in M branched covering of S^3 (topology of M in spin network data through monodromies)

Spin network data and covering moves compatibility: way to extend holonomy data ρ, ι compatibly with covering moves

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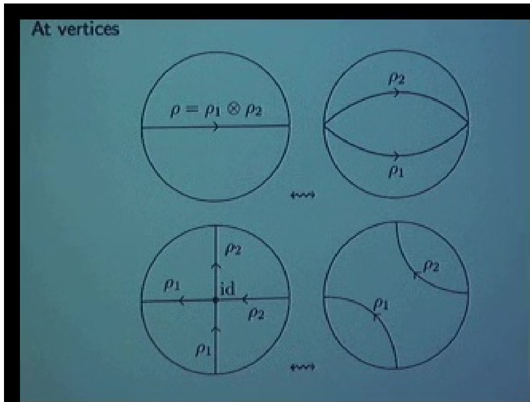
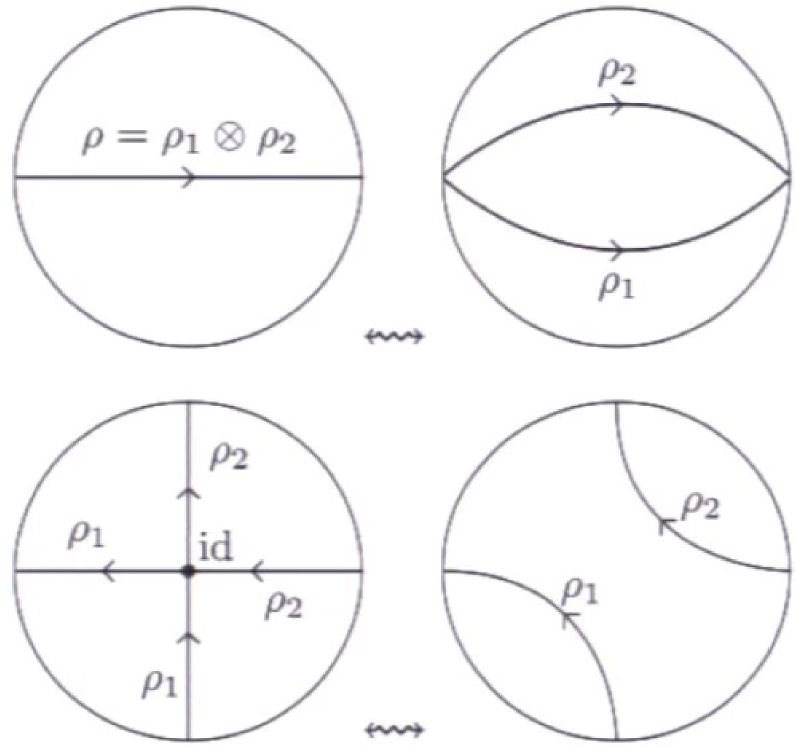


At vertices



$\rho = \rho_1 \otimes \rho_2$
ρ_2
ρ_1
ρ_2
ρ_1
id
ρ_2
ρ_1
ρ_2
ρ_1

Spin foams (spin network cobordisms)
$\psi = (\Gamma, \rho, \iota)$ and $\psi' = (\Gamma', \rho', \iota')$ spin networks, graphs Γ and Γ' embedded in M and M'.
Spin foam $\Psi : \psi \to \psi'$ in a cobordism W with $\partial W = M \cup \bar{M}'$ is a triple $\Psi = (\Sigma, \tilde{\rho}, \tilde{\iota})$:
1. an oriented two-complex $\Sigma \subseteq W$, with $\partial\Sigma = \Gamma \cup \bar{\Gamma}'$
2. a labeling $\tilde{\rho}$ of each face f of Σ by a representation $\tilde{\rho}_f$ of G;
3. a labeling $\tilde{\iota}$ of each edge e of Σ that does not lie in Γ or Γ' by an intertwiner
$$\tilde{\iota}_e : \bigotimes_{f: e\in\partial(f)} \tilde{\rho}_f \to \bigotimes_{f': \bar{e}\in\partial(f')} \tilde{\rho}_{f'}$$
additional consistency conditions:
1. edge e in Γ and f_e face bordered by e then $\tilde{\rho}_{f_e} = \rho_e$ (or dual depending on orientation)
2. vertex v of Γ and e_v edge adjacent to v in Σ then $\tilde{\iota}_{e_v} = \iota_v$ (or dual depending on orientation)
3. similar for Γ'

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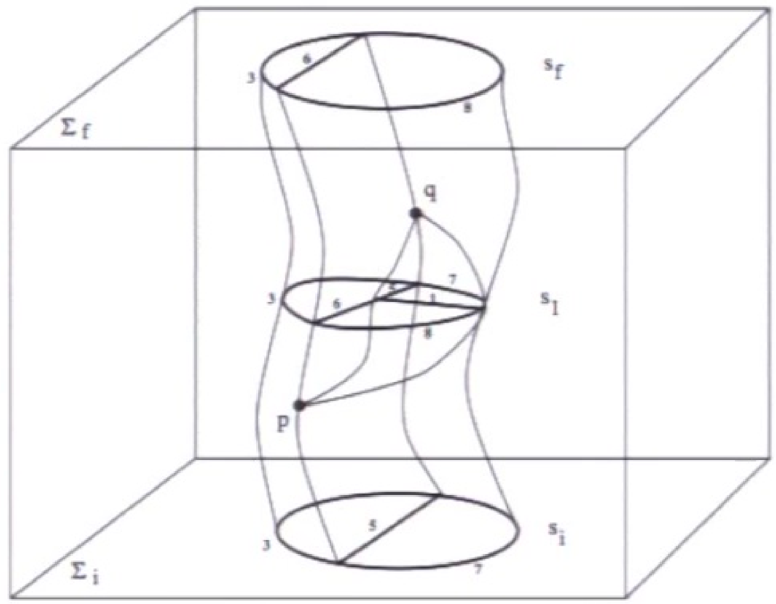
Spin foams as a cobordism of spin networks



Topspin foams (topologically enriched)
$\psi = (\Gamma, \rho, \iota, \sigma)$ and $\psi' = (\Gamma', \rho', \iota', \sigma')$ are topspin networks with monodromy reps in same S_n (and $\Gamma, \Gamma' \subset S^3$)
A topspin foam $\Psi : \psi \to \psi'$ is $\Psi = (\Sigma, \tilde{\rho}, \tilde{\iota}, \tilde{\sigma})$ with

1. a spin foam $(\Sigma, \tilde{\rho}, \tilde{\iota})$ between ψ and ψ' with $\Sigma \subset S^3 \times [0,1]$,
2. a representation $\tilde{\sigma} : \pi_1((S^3 \times [0,1]) \smallsetminus \Sigma) \to S_n$, defining branched cover cobordism W between M and M' (branched coverings defined by (Γ, σ) and (Γ', σ'))

PL (smooth) 4-manifold W cobordism encoded in the spin foam data, like M and M' with spin networks

Note In a path integral formulation, the sum over geometries is now also a sum over topologies, through the monodromy data

Diagrams version

3-dimensional projection diagram $D(\Sigma)$

1. assigning to each one-dimensional strand e_i of $D(\Sigma)$ the same intertwiner $\tilde{\iota}_e$ assigned to the edge e;
2. assigning to each two-dimensional strand f_α of $D(\Sigma)$ the same representation $\tilde{\rho}_f$ of G assigned to the face f;
3. assigning to each two-dimensional strand f_α of $D(\Sigma)$ a topological label $\tilde{\sigma}_\alpha \in S_n$ such that taken in total such assignments satisfy the Wirtinger relations $\tilde{\sigma}_\alpha = \tilde{\sigma}_\beta \tilde{\sigma}_{\alpha'} \tilde{\sigma}_\beta^{-1}$ at crossings of faces and along edges

$$\prod_{\alpha : e \in \partial(f_\alpha)} \tilde{\sigma}_\alpha \prod_{\alpha' : \bar{e} \in \partial(f_{\alpha'})} \tilde{\sigma}_{\alpha'}^{-1} = 1$$

Categories of branched cover 3-manifolds and 4-dim cobordisms
3-manifolds (realized in different ways as branched covers) as correspondences between embedded graphs

$$\mathcal{C}(\Gamma,\Gamma') = \{\Gamma \subset E \subset S^3 \xleftarrow{\pi} M \xrightarrow{\pi'} S^3 \supset E' \supset \Gamma'\}$$

Composition: fibered product (motivated by KK-theory)

$$\mathcal{C}(\Gamma,\Gamma') \times \mathcal{C}(\Gamma',\Gamma'') \to \mathcal{C}(\Gamma,\Gamma'')$$

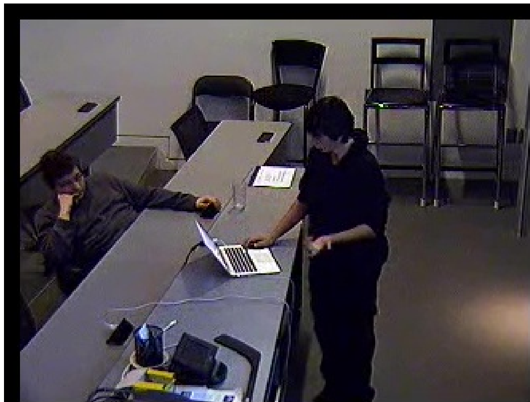
$$\Gamma \subset E \cup \pi\pi_1^{-1}(E_2) \subset S^3 \leftarrow M \times_{S^3} M' \to S^3 \supset E'' \cup \pi''\pi_2^{-1}(E_1) \supset \Gamma''$$

2-morphisms: branched cover cobordisms

$$\Sigma \subset S \subset S^3 \times I \leftarrow W \to S^3 \times I \supset S' \supset \Sigma'$$

$$\partial W = M_1 \cup \bar{M}_2, \quad \partial\Sigma = \Gamma_1 \cup \bar{\Gamma}_2, \quad \partial\Sigma' = \Gamma_1' \cup \bar{\Gamma}_2'$$

Σ, Σ', S, S' embedded 2-complexes



2-category:

- Objects $Obj(\mathcal{C}) \ni X$
- 1-morphisms $\mathcal{C}(X,Y) \ni \varphi$, composition

$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$

- 2-morphisms $\mathcal{C}^{(2)}(\varphi,\psi) \ni \Phi$
- Vertical composition $\mathcal{C}^{(2)}(\varphi,\psi) \times \mathcal{C}^{(2)}(\varphi,\psi) \to \mathcal{C}^{(2)}(\varphi,\psi)$
- Horizontal composition $\mathcal{C}^{(2)}(\varphi,\psi) \times \mathcal{C}^{(2)}(\psi,\eta) \to \mathcal{C}^{(2)}(\varphi,\eta)$

Vertical and horizontal composition of 2-morphisms:

Vertical composition: gluing cobordisms along a common boundary

$$W_1 \bullet W_2 = W_1 \cup_M W_2$$

Horizontal composition: fibered product along branched covering maps

$$W_1 \circ W_2 = W_1 \times_{S^3 \times [0,1]} W_2$$

Horizontal composition as in KK-product in D-brane geometry (see Connes–Skandalis and Mathai–Rosenberg)

Algebras from categories
– Group algebra $C^*(G)$: discrete group G group ring $\mathbb{C}[G]$, finitely supported functions with convolution
$$(f_1 \star f_2)(g) = \sum_{g=g_1 g_2} f_1(g_1) f_2(g_2)$$
involution $f^*(g) \equiv \overline{f(g^{-1})}$, norm closure
– Semigroup algebra $f : S \to \mathbb{C}$ with convolution
$$(f_1 \star f_2)(s) = \sum_{s=s_1 s_2} f_1(s_1) f_2(s_2)$$
no longer necessarily involutive: represent on $\ell^2(S)$ by isometries $\delta_s^* \delta_s = 1$ but $\delta_s \delta_s^* = e_s$ idempotent
– Groupoid algebra $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, s, t)$ functions $f : \mathcal{G}^{(1)} \to \mathbb{C}$ with convolution
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– Semigroupoid (small category) algebra: functions of morphisms with convolution

$$(f_1 \star f_2)(\phi) = \sum_{\phi = \phi_1 \circ \phi_2} f_1(\phi_1) f_2(\phi_2)$$

– 2-semigroupoid algebra has two associative multiplications ($\circ$ and $\bullet$) with

$$(a_1 \circ b_1) \bullet (a_2 \circ b_2) = (a_1 \bullet a_2) \circ (b_1 \bullet b_2)$$

– small 2-category, functions on 2-morphisms $f : \mathcal{C}^{(2)} \to \mathbb{C}$

$$(f_1 \bullet f_2)(\Phi) = \sum_{\Phi = \Phi_1 \bullet \Phi_2} f_1(\Phi_1) f_2(\Phi_2)$$

$$(f_1 \circ f_2)(\Phi) = \sum_{\Phi = \Psi \circ \Upsilon} f_1(\Psi) f_2(\Upsilon)$$

with compatibility

Algebras and NC spaces

Associative convolution algebra = NC space of "quotient"

Equivalence relation $\mathcal{R}$ on X: quotient $Y = X/\mathcal{R}$ (often not good: too few functions) classical functions on the quotient

$$\mathcal{A}(Y) := \{ f \in \mathcal{A}(X) \mid f \text{ is } \mathcal{R} - \text{invariant} \}$$

NCG: $\mathcal{A}(Y)$ noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation with convolution

$$(f_1 * f_2)(x,y) = \sum_{x \sim u \sim y} f_1(x,u) f_2(u,y)$$

and involution $f^*(x,y) = \overline{f(y,x)}$.

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Quantum Statistical Mechanics and NCG
$\mathcal{A}=$ algebra of observables (C^*-algebra)
State: $\varphi:\mathcal{A}\to\mathbb{C}$ linear

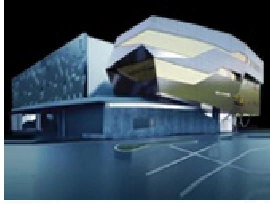
$$\varphi(1)=1,\quad \varphi(a^*a)\geq 0$$

Time evolution $\sigma_t\in \mathrm{Aut}(\mathcal{A})$
Rep π on Hilbert space $\mathcal{H}\Rightarrow$ Hamiltonian $H=\frac{d}{dt}\sigma_t|_{t=0}$

$$\pi(\sigma_t(a))=e^{itH}\pi(a)e^{-itH}$$

Equilibrium state (inverse temperature $\beta=1/kT$)

$$\frac{1}{Z(\beta)}\mathrm{Tr}\left(a\,e^{-\beta H}\right)\qquad Z(\beta)=\mathrm{Tr}\left(e^{-\beta H}\right)$$



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KMS states $\varphi \in \mathrm{KMS}_\beta$ $(0 < \beta < \infty)$
$\forall a, b \in \mathcal{A}$ $\exists$ holom function $F_{a,b}(z)$ on strip: $\forall t \in \mathbb{R}$

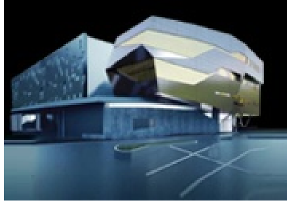
$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$$

Ground states $\quad (\beta = \infty,\ T = 0)$
At $T > 0$ simplex KMS_β $\rightsquigarrow$ extremal $\mathcal{E}_\beta$
(Points on NC space $\mathcal{A}$)
At $T = 0$: KMS_∞ = weak limits of KMS_β

$$\varphi_\infty(a) = \lim_{\beta \to \infty} \varphi_\beta(a)$$

Idea: extremal KMS_β states are classical points of a noncommutative space

Motivation N.1: NCG and arithmetic, $\mathbb{Q}$-lattices
(Λ, ϕ) $\mathbb{Q}$-lattice in $\mathbb{R}^n$
lattice $\Lambda \subset \mathbb{R}^n$ + labels of torsion points
$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$
group homomorphism (invertible $\mathbb{Q}$-lat is isom)
Commensurability $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2$ mod $\Lambda_1 + \Lambda_2$
$\mathbb{Q}$-lattices / Commensurability $\Rightarrow$ NC space
Groupoid algebra of equivalence relation



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$\mathbb{Q}$-lattices / Commensurability $\Rightarrow$ NC space

Groupoid algebra of equivalence relation

Main properties:

- Partition function $\zeta(\beta)$ Riemann zeta function
- Low temperature KMS states = invertible $\mathbb{Q}$-lattices = $\hat{\mathbb{Z}}^*$
- Galois group action $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$
- Dual system with scaling action (spectral realization of $\zeta(s)$)

Generalizations: $GL(2)$, $\mathbb{Q}(\sqrt{-D})$, Shimura varieties, number fields, function fields (Connes and M.M. and Ramachandran, Ha and Paugam, Jacob, Consani and M.M., Laca and Larsen and Neshveyev, Cornelissen and M.M.)

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Motivation N.2: Standard Model in NCG

- Almost commutative geometry $M \times F$
- Moduli spaces of Dirac operators (Yukawa parameters)
- Spectral action recovers gravity coupled to matter
- Planck scale ? Quantum gravity ?

Would like to have:

- Algebra of "spectral correspondences" (cobordisms) with "degenerate" Dirac operators.
- Dynamics such that equilibrium states al low temperature recover "good" (nondegenerate) geometries (emergent geometry)

Dictionary of analogies between these two settings

Chapter 4, §8 of Connes–Marcolli book (2008)

2-semigroupoid algebras and time evolutions: $f = \sum_\Phi c_\Phi \delta_\Phi$

Two associative products: vertical and horizontal

$$(f_1 \bullet f_2)(\Phi) = \sum_{\Phi = \Phi_1 \bullet \Phi_2} f_1(\Phi_1) f_2(\Phi_2)$$

$$(f_2 \circ f_2)(\Phi) = \sum_{\Phi = \Phi_1 \circ \Phi_2} f_1(\Phi_1) f_2(\Phi_2)$$

Time evolutions: vertical and horizontal

$$\sigma_t(f_1 \bullet f_2) = \sigma_t(f_1) \bullet \sigma_t(f_2)$$

$$\sigma_t(f_2 \circ f_2) = \sigma_t(f_1) \circ \sigma_t(f_2)$$

Topological factor

$$\omega(\Sigma_1 \cup_\Gamma \Sigma_2) = \omega(\Sigma_1)\omega(\Sigma_2)$$

(eg exp of additive invariant) needed for time evolution
Hamiltonian (infinitesimal generator of time evolution)

$$\mathbb{H}\xi(\Psi') = \log \mathbb{A}(\Psi')\,\xi(\Psi')$$

on space of $\Psi' \sim \Psi$ under covering moves
To have Gibbs states

$$\varphi_\beta(f) = \frac{\mathrm{Tr}(\pi_\Psi(f)e^{-\beta\mathbb{H}})}{\mathrm{Tr}(e^{-\beta\mathbb{H}})}$$

condition $\mathrm{Tr}(e^{-\beta\mathbb{H}}) < \infty \Rightarrow$ problem of multiplicities in the spectrum

Dynamics on *groupoid* algebra of topspin netoworks (or foams)

$$\sigma_t(f)(\Psi,\Psi') = \left(\frac{\mathbb{A}(\Psi)}{\mathbb{A}(\Psi')}\right)^{it} f(\Psi,\Psi')$$

include topological data through character $\chi : S_\infty \to \mathrm{U}(1)$

$$\mathfrak{W}(\psi) \equiv \left(\prod_{v\in V(\Gamma)} \prod_{e:v\in\partial(e)} \sigma_e \prod_{e:v\in\bar{\partial}(e)} \sigma_e^{-1} \right)$$

$\mathfrak{W}(\psi) \in S_n$ product of Wirtinger relations at vertices, is $=1$ for actual (nondegenerate) geometries, and nontrivial otherwise

$$\sigma_t(f)(\Psi,\Psi') = \left(\frac{\mathbb{A}(\Psi)}{\mathbb{A}(\Psi')}\right)^{it} \chi(\mathfrak{W}(\Psi)\mathfrak{W}(\Psi')^{-1})^t f(\Psi,\Psi')$$

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Multiplicities: question on existence of an invariant of embedded graphs $\omega(\Gamma)$ with

1. $\omega(\Gamma)$ depends on the ambient isotopy class
2. values of $\omega(\Gamma)$ form discrete set of positive real numbers growing at least exponentially $\sim e^{cn}$ for large n
3. number of embedded graph Γ combinatorially equivalent to a given Γ_0 with fixed $\omega(\Gamma)$ is finite and grows at most like $e^{\kappa n}$ some $\kappa > 0$

Same question for invariant $\omega(\Sigma)$ of embedded two-complexes $\Sigma \subset S^3 \times [0, 1]$

$\Rightarrow$ ensures the existence of low temperature Gibbs states

Quantized area of spin networks

$S \subset S^3$ be a closed embedded smooth (or PL) surface, generically intersects Γ transversely finite number of points

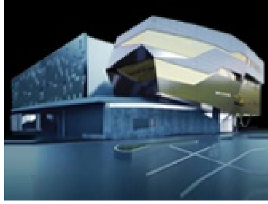
$$A_S\, f({}_\psi M_{\psi'}) = \hbar \left(\sum_{x \in S \cap \Gamma} (j_x(j_x+1))^{1/2} \right) f({}_\psi M_{\psi'})$$

for $f({}_\psi M_{\psi'})$ in convolution algebra of topspin networks with fibered product, $j_x = j_e$ spin of $SU(2)$ rep ρ_e of edge containing x

More generally $N : \bigcup_\Gamma E(\Gamma) \to \mathbb{Z}$

$$A\, f({}_\psi M_{\psi'}) = \hbar \left(\sum_{e \in E(\Gamma)} N(e)\, (j_e(j_e+1))^{1/2} \right) f({}_\psi M_{\psi'})$$

generalizes multiplicity of inters with S



PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Amplitude and time evolution for 2-semigroupoid algebra of topspin foams

$$A_s\, f(W) = \hbar \left(\sum_{\mathfrak{f}\in F(\hat{\Sigma})} \chi_\Sigma(\mathfrak{f})\, (j_{\mathfrak{f}}(j_{\mathfrak{f}}+1))^{1/2} \right) f(W)$$

time evolution (up to topological factor $e^{it\chi(\Sigma,W)} = \omega(\Sigma)$)

$$\sigma_t(f) = e^{it(A_s - A_t)} f$$

Topological condition: Question invariant $\chi(\Sigma, W)$ of embedded two-complexes Σ and branched cover data $q : W \to S^3 \times [0,1]$

1. values of $\chi(\Gamma, W)$ discrete set in $\mathbb{R}_+^*$ growing at least linearly $c_1 n + c_0$ for large n, $c_i > 0$
2. for fixed branched cover number of embedded Σ with $\chi(\Sigma, W)$ fixed grows at most like $e^{\kappa n}$ some $\kappa > 0$ (indep of W)
3. on fibered product $\tilde{W} = W \times_{S^3\times I} W' \Rightarrow$ $\chi(\Sigma \cup q q_1^{-1}(\Sigma_2), \tilde{W}) = \chi(\Sigma, W) + \chi(\Sigma_2, W')$

$\Sigma \subset S^3 \times I \xleftarrow{q} W \xrightarrow{q_1} S^3 \times I \supset \Sigma_1$ and $\Sigma_2 \subset S^3 \times I \xleftarrow{q_2} W' \xrightarrow{q'} S^3 \times I \supset \Sigma'$

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Spin foams and almost commutative geometries
A possible approach to coupling with matter
• coupling gravity to matter via almost-commutative geometries (NCG models of particle physics and cosmology) $X \times F$
• when discretize spacetime replace 4-dim X with spin foam and 3-dim with spin networks
• keep the finite NC geometry F describing matter
• product geometry $X \times F$ spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$
• replace Dirac operator on X with analog spectral triples on spin foams (Aastrup–Grimstrup–Nest)
• similar dynamics but also involving spectral action

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