Title: Cosmology and the Poisson summation formula

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Abstract: We show that, in a model of modified gravity based on the spectral action functional, there is a nontrivial coupling between cosmic topology and inflation, in the sense that the shape of the possible slow-roll inflation potentials obtained in the model from the nonperturbative form of the spectral action are sensitive not only to the geometry (flat or positively curved) of the universe, but also to the different possible non-simply connected topologies. We show this by explicitly computing the nonperturbative spectral action for some candidate cosmic topologies, spherical space forms and flat ones given by Bieberbach manifolds and showing that the resulting inflation potential differs from that of the sphere or flat torus by a multiplicative factor. We then show that, while the slow-roll parameters differ between the spherical and flat manifolds but do not distinguish different topologies within each class, the power spectra detect the different scalings of the slow-roll potential and therefore distinguish between the various topologies, both in the spherical and in the flat case. (Based on joint work with Elena Pierpaoli and Kevin Teh)

Cosmology and the Poisson summation formula

Matilde Marcolli

Perimeter Institute, 2011

This talk is based on:

- **MPT** M. Marcolli, E. Pierpaoli, K. Teh, *The spectral action and cosmic topology*, arXiv:1005.2256 (to appear in CMP)
- **MPT2** M. Marcolli, E. Pierpaoli, K. Teh, *The coupling of topology and inflation in Noncommutative Cosmology*, arXiv:1012.0780
- **CMT** B. Ćaćić, M. Marcolli, K. Teh, *Topological coupling of gravity to matter, spectral action and cosmic topology*, in preparation

The NCG standard model and cosmology

- **CCM** A. Chamseddine, A. Connes, M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007), no. 6, 991–1089.
- **MP** M. Marcolli, E. Pierpaoli, *Early universe models from noncommutative geometry*, arXiv:0908.3683
- **KM** D. Kolodrubetz, M. Marcolli, *Boundary conditions of the RGE flow in noncommutative cosmology*, arXiv:1006.4000

Two topics of current interest to cosmologists:

• Modified Gravity models in cosmology:
Einstein-Hilbert action (+cosmological term) replaced or extended with other gravity terms (conformal gravity, higher derivative terms) $\Rightarrow$ cosmological predictions

• The question of Cosmic Topology:
Nontrivial (non-simply-connected) spatial sections of spacetime, homogeneous spherical or flat spaces: how can this be detected from cosmological observations?

Our approach:

- NCG provides a modified gravity model through the spectral action
- The nonperturbative form of the spectral action determines a slow-roll inflation potential
- The underlying geometry (spherical/flat) affects the shape of the potential (possible models of inflation)
- Different inflation scenarios depending on geometry and topology of the cosmos
- More refined topological properties from coupling to matter

The noncommutative space $X \times F$ extra dimensions
product of 4-dim spacetime and finite NC space
The spectral action functional

$$\mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

$D_A = D + A + \varepsilon' J A J^{-1}$ Dirac operator with inner fluctuations
$A = A^* = \sum_k a_k[D, b_k]$

- Action functional for gravity on X (modified gravity)
- Gravity on $X \times F$ = gravity coupled to matter on X

Spectral triples $(\mathcal{A}, \mathcal{H}, D)$:

- involutive algebra $\mathcal{A}$
- representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator D on $\mathcal{H}$
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- even $\mathbb{Z}/2$-grading $[\gamma, a] = 0$ and $D\gamma = -\gamma D$
- real structure: antilinear isom $J : \mathcal{H} \to \mathcal{H}$ with $J^2 = \varepsilon$, $JD = \varepsilon' DJ$, and $J\gamma = \varepsilon'' \gamma J$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

- bimodule: $[a, b^0] = 0$ for $b^0 = Jb^* J^{-1}$
- order one condition: $[[D, a], b^0] = 0$

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Asymptotic formula for the spectral action (Chamseddine–Connes)

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{k \in \mathrm{DimSp}} f_k \Lambda^k \int\!\!\!\!\!- |D|^{-k} + f(0)\zeta_D(0) + o(1)$$

for large Λ with $f_k = \int_0^\infty f(v) v^{k-1} dv$ and integration given by residues of zeta function $\zeta_D(s) = \mathrm{Tr}(|D|^{-s})$; DimSp poles of zeta functions

Asymptotic expansion $\Rightarrow$ Effective Lagrangian
(modified gravity + matter)

At low energies: only nonperturbative form of the spectral action

$$\mathrm{Tr}(f(D_A/\Lambda))$$

Need explicit information on the Dirac spectrum!

Product geometry $(C^\infty(X), L^2(X,S), D_X) \cup (\mathcal{A}_F, \mathcal{H}_F, D_F)$

- $\mathcal{A} = C^\infty(X) \otimes \mathcal{A}_F = C^\infty(X, \mathcal{A}_F)$
- $\mathcal{H} = L^2(X,S) \otimes \mathcal{H}_F = L^2(X, S \otimes \mathcal{H}_F)$
- $D = D_X \otimes 1 + \gamma_5 \otimes D_F$

Inner fluctuations of the Dirac operator

$$D \to D_A = D + A + \varepsilon' J A J^{-1}$$

A self-adjoint operator

$$A = \sum a_j [D, b_j], \quad a_j, b_j \in \mathcal{A}$$

$\Rightarrow$ boson fields from inner fluctuations, fermions from $\mathcal{H}_F$

Get realistic particle physics models [CCM]
Need Ansatz for the NC space F

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

$\Rightarrow$ everything else follows by *computation*

- Representation: $\mathcal{M}_F$ sum of all inequiv irred odd $\mathcal{A}_{LR}$-bimodules (fix N generations) $\mathcal{H}_F = \oplus^N \mathcal{M}_F$ fermions
- Algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$: order one condition
- F zero dimensional but KO-dim 6
- $J_F =$ matter/antimatter, $\gamma_F =$ L/R chirality
- Classification of Dirac operators (moduli spaces)

Dirac operators and Majorana mass terms

$$D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}, \quad S = S_1 \oplus (S_3 \otimes 1_3), \quad T = Y_R : |\nu_R\rangle \to J_F\, |\nu_R\rangle$$

$$S_1 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix}$$

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Yukawa matrices: Dirac masses and mixing angles in $\mathrm{GL}_{N=3}(\mathbb{C})$

$Y_e = Y_{(\downarrow 1)}$ (charged leptons)

$Y_\nu = Y_{(\uparrow 1)}$ (neutrinos)

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Moduli space of Dirac operators on finite NC space F

$$\mathcal{C}_3 \times \mathcal{C}_1$$

- $\mathcal{C}_3 =$ pairs $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo W_j unitary matrices:

$$Y'_{(\downarrow 3)} = W_1\, Y_{(\downarrow 3)}\, W_3^*\,,\; Y'_{(\uparrow 3)} = W_2\, Y_{(\uparrow 3)}\, W_3^*$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$: $\quad \mathcal{C}_3 = (K \times K) \backslash (G \times G)/K$
$\dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4$ (eigenval, coset 3 angles 1 phase)

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$\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$ (dim fiber 12-1=11)

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Parameters of νMSM

- three coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters
- QCD vacuum angle

Moduli space of Dirac operators on F $\Rightarrow$ geometric form of all the Yukawa and Majorana parameters

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Gauge group $SU(\mathcal{A}_F) = U(1) \times SU(2) \times SU(3)$
(up to fin abelian group)

- Hypercharges: adjoint action of $\mathrm{U}(1)$ (in powers of $\lambda \in \mathrm{U}(1)$)

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Explicit computation gives part of SM Larangian with

- $\mathcal{L}_{Hf}$ = coupling of Higgs to fermions
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The asymptotic expansion of the spectral action from [CCM]

$$
\begin{aligned}
S = \; & \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, \mathfrak{c} + \frac{f_0}{4}\mathfrak{d}) \int \sqrt{g}\, d^4x \\
+ \; & \frac{96\, f_2\, \Lambda^2 - f_0\, \mathfrak{c}}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
+ \; & \frac{f_0}{10\, \pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x \\
+ \; & \frac{(-2\, \mathfrak{a}\, f_2\, \Lambda^2 + \mathfrak{e}\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \; & \frac{f_0 \mathfrak{a}}{2\, \pi^2} \int |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
- \; & \frac{f_0 \mathfrak{a}}{12\, \pi^2} \int R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
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Parameters:

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$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ functions of Yukawa parameters of SM+r.h.ν

$$\begin{aligned}
\mathfrak{a} &= \mathrm{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)) \\
\mathfrak{b} &= \mathrm{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2) \\
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$\Rightarrow$ Correct hypercharges to the fermions

Action functional

$$\mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

Fermion part: antisymmetric bilinear form $\mathfrak{A}(\tilde{\xi})$ on

$$\mathcal{H}^+ = \{\xi \in \mathcal{H} \,|\, \gamma\xi = \xi\}$$

$\Rightarrow$ nonzero on Grassmann variables
Euclidean functional integral $\Rightarrow$ Pfaffian

$$Pf(\mathfrak{A}) = \int e^{-\frac{1}{2}\mathfrak{A}(\tilde{\xi})} D[\tilde{\xi}]$$

(avoids Fermion doubling problem of previous models based on symmetric $\langle \xi, D_A\xi\rangle$ for NC space with KO-dim=0)

Explicit computation gives part of SM Larangian with

- $\mathcal{L}_{Hf}$ = coupling of Higgs to fermions
- $\mathcal{L}_{gf}$ = coupling of gauge bosons to fermions
- $\mathcal{L}_f$ = fermion terms

Moduli space of Dirac operators on finite NC space F

$$\mathcal{C}_3 \times \mathcal{C}_1$$

- $\mathcal{C}_3 =$ pairs $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo W_j unitary matrices:

$$Y'_{(\downarrow 3)} = W_1\, Y_{(\downarrow 3)}\, W_3^*\,,\; Y'_{(\uparrow 3)} = W_2\, Y_{(\uparrow 3)}\, W_3^*$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$: $\quad \mathcal{C}_3 = (K \times K)\backslash(G \times G)/K$
$\dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4$ (eigenval, coset 3 angles 1 phase)

- $\mathcal{C}_1 =$ triplets $(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R)$ with Y_R symmetric modulo

$$Y'_{(\downarrow 1)} = V_1\, Y_{(\downarrow 1)} V_3^*\,,\; Y'_{(\uparrow 1)} = V_2\, Y_{(\uparrow 1)}\, V_3^*\,,$$

$$Y'_R = V_2\, Y_R\, \bar{V}_2^*$$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection forgets Y_R fiber symm matrices mod $Y_R \mapsto \lambda^2 Y_R$
$\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$ (dim fiber 12-1=11)

Parameters of νMSM

- three coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters
- QCD vacuum angle

Moduli space of Dirac operators on F $\Rightarrow$ geometric form of all the Yukawa and Majorana parameters

Fields content of the model

- Bosons: inner fluctuations $A = \sum_j a_j[D, b_j]$

- In M direction: $U(1)$, $SU(2)$, and $SU(3)$ gauge bosons
- In F direction: Higgs field $H = \varphi_1 + \varphi_2 j$

- Fermions: basis of $\mathcal{H}_F$

$$|\uparrow\rangle \otimes \mathbf{3}^0, \ |\downarrow\rangle \otimes \mathbf{3}^0, \ |\uparrow\rangle \otimes \mathbf{1}^0, \ |\downarrow\rangle \otimes \mathbf{1}^0$$

Gauge group $SU(\mathcal{A}_F) = U(1) \times SU(2) \times SU(3)$
(up to fin abelian group)

- Hypercharges: adjoint action of $\mathrm{U}(1)$ (in powers of $\lambda \in \mathrm{U}(1)$)

	$\uparrow \otimes \mathbf{1}^0$	$\downarrow \otimes \mathbf{1}^0$	$\uparrow \otimes \mathbf{3}^0$	$\downarrow \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

$\Rightarrow$ Correct hypercharges to the fermions

The asymptotic expansion of the spectral action from [CCM]

$$
\begin{aligned}
S = \; & \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, \mathfrak{c} + \frac{f_0}{4}\mathfrak{d}) \int \sqrt{g}\, d^4x \\
+ \; & \frac{96\, f_2\, \Lambda^2 - f_0\, \mathfrak{c}}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
+ \; & \frac{f_0}{10\,\pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x \\
+ \; & \frac{(-2\, \mathfrak{a}\, f_2\, \Lambda^2 + \mathfrak{e}\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \; & \frac{f_0 \mathfrak{a}}{2\,\pi^2} \int |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
- \; & \frac{f_0 \mathfrak{a}}{12\,\pi^2} \int R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \; & \frac{f_0 \mathfrak{b}}{2\,\pi^2} \int |\varphi|^4\, \sqrt{g}\, d^4x \\
+ \; & \frac{f_0}{2\,\pi^2} \int (g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x,
\end{aligned}
$$

Parameters:

- f_0, f_2, f_4 free parameters, $f_0 = f(0)$ and, for $k > 0$,

$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ functions of Yukawa parameters of SM+r.h.ν

$$\begin{aligned}
\mathfrak{a} = & \quad \mathrm{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)) \\
\mathfrak{b} = & \quad \mathrm{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2) \\
\mathfrak{c} = & \quad \mathrm{Tr}(MM^\dagger) \\
\mathfrak{d} = & \quad \mathrm{Tr}((MM^\dagger)^2) \\
\mathfrak{e} = & \quad \mathrm{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu).
\end{aligned}$$

Normalization and coefficients

$$
\begin{aligned}
S = \; & \frac{1}{2\kappa_0^2}\int R\sqrt{g}\, d^4x + \gamma_0 \int \sqrt{g}\, d^4x \\
+ \; & \alpha_0 \int C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma}\sqrt{g}\, d^4x + \tau_0 \int R^* R^* \sqrt{g}\, d^4x \\
+ \; & \frac{1}{2}\int |DH|^2 \sqrt{g}\, d^4x - \mu_0^2 \int |H|^2 \sqrt{g}\, d^4x \\
- \; & \xi_0 \int R\,|H|^2 \sqrt{g}\, d^4x + \lambda_0 \int |H|^4 \sqrt{g}\, d^4x \\
+ \; & \frac{1}{4}\int (G^i_{\mu\nu}\, G^{\mu\nu i} + F^{\alpha}_{\mu\nu}\, F^{\mu\nu\alpha} + B_{\mu\nu}\, B^{\mu\nu})\sqrt{g}\, d^4x,
\end{aligned}
$$

Energy scale: Unification ($10^{15} - 10^{17}$ GeV)

$$
\frac{g^2 f_0}{2\pi^2} = \frac{1}{4}
$$

Preferred energy scale, unification of coupling constants

Coefficients

$$\frac{1}{2\kappa_0^2} = \frac{96 f_2 \Lambda^2 - f_0 \mathfrak{c}}{24\pi^2} \qquad \gamma_0 = \frac{1}{\pi^2}\left(48 f_4 \Lambda^4 - f_2 \Lambda^2 \mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right)$$

$$\alpha_0 = -\frac{3 f_0}{10\pi^2} \qquad \tau_0 = \frac{11 f_0}{60\pi^2}$$

$$\mu_0^2 = 2\frac{f_2 \Lambda^2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \qquad \xi_0 = \frac{1}{12}$$

$$\lambda_0 = \frac{\pi^2 \mathfrak{b}}{2 f_0 \mathfrak{a}^2}$$

In [MP] [KM]: running coefficients with RGE flow of particle physics content from unification energy down to electroweak.
$\Rightarrow$ Very early universe models! $(10^{-36}s < t < 10^{-12}s)$

Effective gravitational constant

$$G_{\text{eff}} = \frac{\kappa_0^2}{8\pi} = \frac{3\pi}{192 f_2 \Lambda^2 - 2 f_0 \mathfrak{c}(\Lambda)}$$

Effective cosmological constant

$$\gamma_0 = \frac{1}{4\pi^2}(192 f_4 \Lambda^4 - 4 f_2 \Lambda^2 \mathfrak{c}(\Lambda) + f_0 \mathfrak{d}(\Lambda))$$

Conformal non-minimal coupling of Higgs and gravity

$$\frac{1}{16\pi G_{\text{eff}}} \int R \sqrt{g} d^4x - \frac{1}{12} \int R |H|^2 \sqrt{g} d^4x$$

Conformal gravity

$$\frac{-3 f_0}{10\pi^2} \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g} d^4x$$

$C^{\mu\nu\rho\sigma}$ = Weyl curvature tensor (trace free part of Riemann tensor)

Cosmological implications of the NCG SM [MP]

- Linde's hypothesis (antigravity in the early universe)
- Primordial black holes and gravitational memory
- Gravitational waves in modified gravity
- Gravity balls
- Varying effective cosmological constant
- Higgs based slow-roll inflation
- Spontaneously arising Hoyle-Narlikar in EH backgrounds

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- Remark: Cannot extrapolate to modern universe, nonperturbative effects in the spectral action: requires nonperturbative spectral action

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Cosmological models for the not-so-early-universe?

Need to work with non-perturbative form of the spectral action

Can to for specially symmetric geometries!

Concentrate on pure gravity part: X instead of $X \times F$

The spectral action and the question of cosmic topology

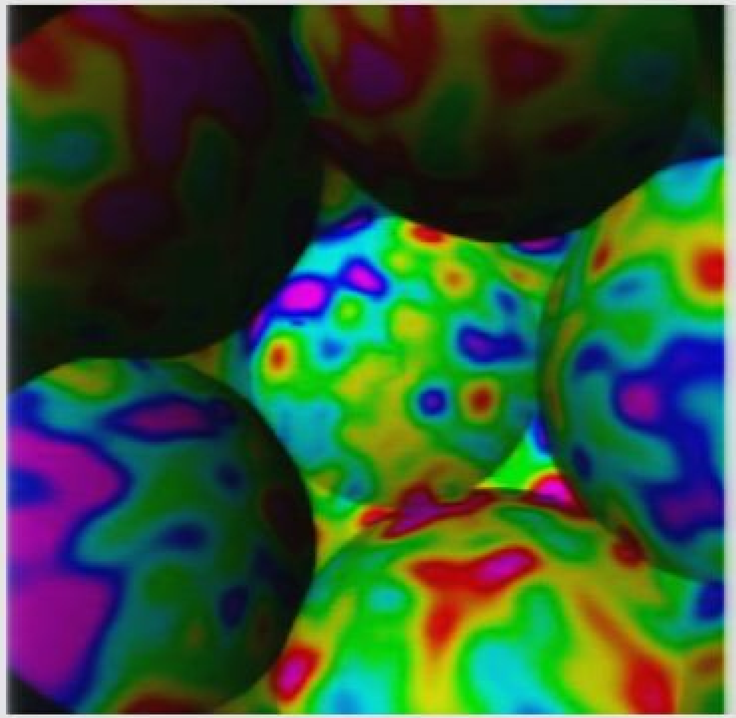
(with E. Pierpaoli and K. Teh)

Spatial sections of spacetime closed 3-manifolds $\neq S^3$?

- Cosmologists search for signatures of topology in the CMB
- Model based on NCG distinguishes cosmic topologies?

Yes! the non-perturbative spectral action predicts different models of slow-roll inflation

Cosmic topology



(Luminet, Lehoucq, Riazuelo, Weeks, et al.: simulated CMB sky)
Best candidates: Poincaré homology 3-sphere and other spherical forms (quaternionic space), flat tori
Testable Cosmological predictions? (in various gravity models)

What to look for? (in the background radiation)
Friedmann metric (expanding universe)

$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

Separate tensor and scalar perturbation h_{ij} of metric $\Rightarrow$ Fourier modes: power spectra for scalar and tensor fluctuations, $\mathcal{P}_s(k)$ and $\mathcal{P}_t(k)$ satisfy power law

$$\mathcal{P}_s(k) \sim \mathcal{P}_s(k_0) \left(\frac{k}{k_0}\right)^{1-n_s+\frac{\alpha_s}{2}\log(k/k_0)}$$

$$\mathcal{P}_t(k) \sim \mathcal{P}_t(k_0) \left(\frac{k}{k_0}\right)^{n_t+\frac{\alpha_t}{2}\log(k/k_0)}$$

Amplitudes and exponents: constrained by observational parameters and predicted by models of *slow roll inflation* (slow roll potential)
Main Question: Can get predictions of power spectra from slow roll inflation via NCG model, so that distinguish topologies?

Slow roll parameters Minkowskian Friedmann metric on $Y \times \mathbb{R}$

$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

accelerated expansion $\frac{\ddot{a}}{a} = H^2(1-\epsilon)$ Hubble parameter

$$H^2(\phi)\left(1 - \frac{1}{3}\epsilon(\phi)\right) = \frac{8\pi}{3m_{Pl}^2} V(\phi)$$

m_{Pl} Planck mass, inflation phase $\epsilon(\phi) < 1$

$$\epsilon(\phi) = \frac{m_{Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

$$\eta(\phi) = \frac{m_{Pl}^2}{8\pi}\frac{V''(\phi)}{V(\phi)}$$

$$\xi(\phi) = \frac{m_{Pl}^4}{64\pi^2}\frac{V'(\phi)V'''(\phi)}{V^2(\phi)}$$

$\Rightarrow$ measurable quantities

$$n_s \simeq 1 - 6\epsilon + 2\eta, \quad n_t \simeq -2\epsilon, \quad r = 16\epsilon,$$

$$\alpha_s \simeq 16\epsilon\eta - 24\epsilon^2 - 2\xi, \quad \alpha_t \simeq 4\epsilon\eta - 8\epsilon^2$$

Spectral action and Poisson summation formula

$$\sum_{n\in\mathbb{Z}} h(x+\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \exp\left(\frac{2\pi i n x}{\lambda}\right)\, \widehat{h}(\frac{n}{\lambda})$$

$\lambda \in \mathbb{R}_+^*$ and $x \in \mathbb{R}$ with

$$\widehat{h}(x) = \int_{\mathbb{R}} h(u)\, e^{-2\pi i u x}\, du$$

Idea: write $\mathrm{Tr}(f(D/\Lambda))$ as sums over lattices

- Need explicit spectrum of D with multiplicities
- Need to write as a union of arithmetic progressions $\lambda_{n,i}$, $n \in \mathbb{Z}$
- Multiplicities polynomial functions $m_{\lambda_{n,i}} = P_i(\lambda_{n,i})$

$$\mathrm{Tr}(f(D/\Lambda)) = \sum_i \sum_{n\in\mathbb{Z}} P_i(\lambda_{n,i}) f(\lambda_{n,i}/\Lambda)$$

The standard topology S^3 (Chamseddine–Connes)

Dirac spectrum $\pm a^{-1}(\frac{1}{2}+n)$ for $n \in \mathbb{Z}$, with multiplicity $n(n+1)$

$$\mathrm{Tr}(f(D/\Lambda)) = (\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{4}(\Lambda a)\widehat{f}(0) + O((\Lambda a)^{-k})$$

with $\widehat{f}^{(2)}$ Fourier transform of $v^2 f(v)$ 4-dimensional Euclidean $S^3 \times S^1$

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \pi \Lambda^4 a^3 \beta \int_0^\infty u\, h(u)\, du - \frac{1}{2}\pi \Lambda a \beta \int_0^\infty h(u)\, du + O(\Lambda^{-k})$$

$$g(u,v) = 2P(u)\, h(u^2(\Lambda a)^{-2} + v^2(\Lambda\beta)^{-2})$$

$$\widehat{g}(n,m) = \int_{\mathbb{R}^2} g(u,v) e^{-2\pi i(xu+yv)}\, du\, dv$$

A slow roll potential from non-perturbative effects
perturbation $D^2 \mapsto D^2 + \phi^2$ gives potential $V(\phi)$ scalar field coupled to gravity

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2))) = \pi\Lambda^4\beta a^3 \int_0^\infty u h(u) du - \frac{\pi}{2}\Lambda^2 \beta a \int_0^\infty h(u) du$$

$$+\pi\Lambda^4\beta a^3\, \mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\, \mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))du, \qquad \mathcal{W}(x) = \int_0^x h(u) du$$

Slow-roll parameters from spectral action $S = S^3$

$$\epsilon(x) = \frac{m_{Pl}^2}{16\pi}\left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}\right)^2$$

$$\eta(x) = \frac{m_{Pl}^2}{8\pi}\frac{h'(x) + 2\pi(\Lambda a)^2 h(x)}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}$$

In Minkowskian Friedmann metric $\Lambda(t) \sim 1/a(t)$

Also independent of β (artificial Euclidean compactification)

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The quaternionic space $SU(2)/Q8$ (quaternion units $\pm 1, \pm\sigma_k$)

Dirac spectrum (Ginoux)

$$\frac{3}{2} + 4k \quad \text{with multiplicity} \quad 2(k+1)(2k+1)$$

$$\frac{3}{2} + 4k + 2 \quad \text{with multiplicity} \quad 4k(k+1)$$

Polynomial interpolation of multiplicities

$$P_1(u) = \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16}$$

$$P_2(u) = \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}$$

Spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + O(\Lambda^{-k})$$

(1/8 of action for S^3) with $g_i(u) = P_i(u) f(u/\Lambda)$:

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from Poisson summation $\Rightarrow$ Same slow-roll parameters

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from Poisson summation $\Rightarrow$ Same slow-roll parameters

The dodecahedral space Poincaré homology sphere S^3/Γ

binary icosahedral group 120 elements

Dirac spectrum: eigenvalues of S^3 different multiplicities $\Rightarrow$ generating function (Bär)

$$F_+(z) = \sum_{k=0}^{\infty} m(\frac{3}{2}+k, D)z^k \quad F_-(z) = \sum_{k=0}^{\infty} m(-(\frac{3}{2}+k), D)z^k$$

$$F_+(z) = -\frac{16(710647 + 317811\sqrt{5})G^+(z)}{(7+3\sqrt{5})^3(2207+987\sqrt{5})H^+(z)}$$

$$G^+(z) = 6z^{11}+18z^{13}+24z^{15}+12z^{17}-2z^{19}-6z^{21}-2z^{23}+2z^{25}+4z^{27}+3z^{29}+z^{31}$$

$$H^+(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
$$-9z^{22}-6z^{24}-2z^{26}+2z^{28}+4z^{30}+3z^{32}+z^{34}$$

$$F_-(z) = -\frac{1024(5374978561 + 2403763488\sqrt{5})G^-(z)}{(7+3\sqrt{5})^8(2207+987\sqrt{5})H^-(z)}$$

$$G^-(z) = 1+3z^2+4z^4+2z^6-2z^8-6z^{10}-2z^{12}+12z^{14}+24z^{16}+18z^{18}+6z^{20}$$

$$H^-(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
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Polynomial interpolation of multiplicities: 60 polynomials $P_i(u)$

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}$$

Spectral action: functions $g_j(u) = P_j(u) f(u/\Lambda)$

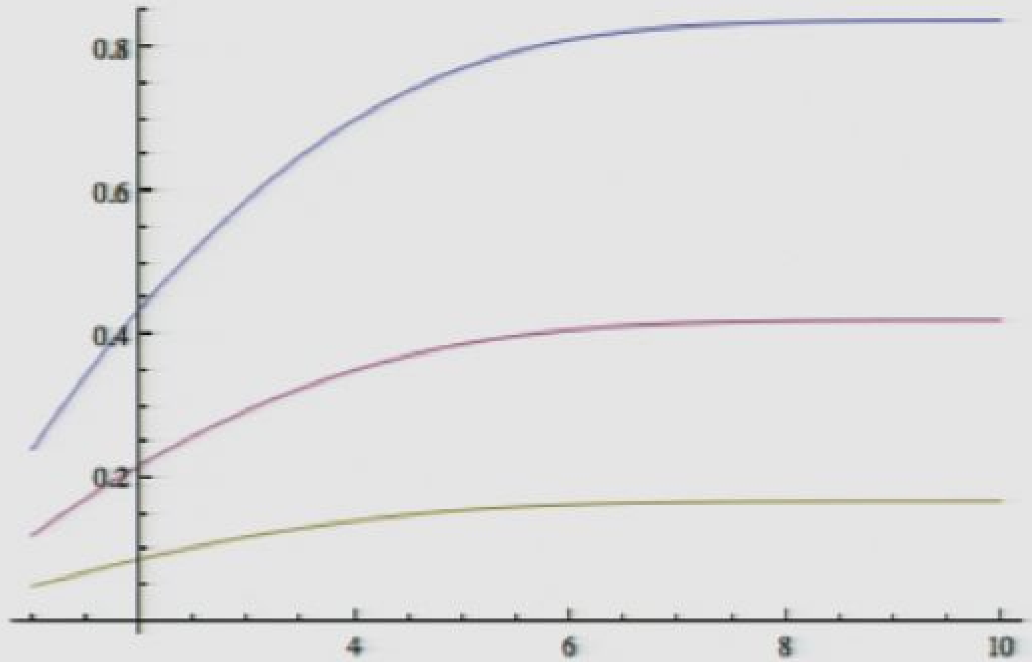
$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60} \sum_{j=0}^{59} \widehat{g}_j(0) + O(\Lambda^{-k})$$

$$= \frac{1}{60} \int_{\mathbb{R}} \sum_j P_j(u) f(u/\Lambda) du + O(\Lambda^{-k})$$

by Poisson summation $\Rightarrow$ 1/120 of action for S^3

Same slow-roll parameters

But ... different amplitudes of power spectra:
multiplicative factor of potential $V(\phi)$



$$\mathcal{P}_s(k) \sim \frac{V^3}{(V')^2}, \quad \mathcal{P}_t(k) \sim V$$

$$V \mapsto \lambda V \quad \Rightarrow \mathcal{P}_s(k_0) \mapsto \lambda \mathcal{P}_s(k_0), \quad \mathcal{P}_t(k_0) \mapsto \lambda \mathcal{P}_t(k_0)$$

$\Rightarrow$ distinguish different spherical topologies

Topological factors (spherical cases):

Theorem (K.Teh): spherical forms $Y = S^3/\Gamma$ spectral action

$$\mathrm{Tr}(f(D_Y/\Lambda)) = \frac{1}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) = \frac{1}{\#\Gamma}\mathrm{Tr}(f(D_{S^3}/\Lambda))$$

up to order $O(\Lambda^{-\infty})$ with

Y spherical	λ_Y
sphere	1
lens N	$1/N$
binary dihedral $4N$	$1/(4N)$
binary tetrahedral	$1/24$
binary octahedral	$1/48$
binary icosahedral	$1/120$

Note: λ_Y does not distinguish all of them

The flat tori

Dirac spectrum (Bär)

$$\pm 2\pi \parallel (m,n,p)+(m_0,n_0,p_0) \parallel , \qquad (1)$$

$(m,n,p)\in\mathbb{Z}^3$ multiplicity 1 and constant vector (m_0,n_0,p_0) depending on spin structure

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \sum_{(m,n,p)\in\mathbb{Z}^3} 2f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Poisson summation

$$\sum_{\mathbb{Z}^3} g(m,n,p) = \sum_{\mathbb{Z}^3} \widehat{g}(m,n,p)$$

$$\widehat{g}(m,n,p) = \int_{\mathbb{R}^3} g(u,v,w)e^{-2\pi i(mu+nv+pw)}dudvdw$$

$$g(m,n,p) = f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Spectral action for the flat tori

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)\,du\,dv\,dw + O(\Lambda^{-k})$$

$X = T^3 \times S^1_\beta$:

$$\mathrm{Tr}(h(D_X^2/\Lambda^2)) = \frac{\Lambda^4\beta\ell^3}{4\pi}\int_0^\infty u h(u)du + O(\Lambda^{-k})$$

using

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} 2\,h\left(\frac{4\pi^2}{(\Lambda\ell)^2}((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)+\frac{1}{(\Lambda\beta)^2}(r+\frac{1}{2})^2\right)$$

$$g(u,v,w,y) = 2\,h\left(\frac{4\pi^2}{\Lambda^2}(u^2+v^2+w^2)+\frac{y^2}{(\Lambda\beta)^2}\right)$$

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} g(m+m_0, n+n_0, p+p_0, r+\frac{1}{2}) = \sum_{(m,n,p,r)\in\mathbb{Z}^4} (-1)^r\,\widehat{g}(m,n,p,r)$$

Different slow-roll potential and parameters Introducing the perturbation $D^2 \mapsto D^2 + \phi^2$:

$$\mathrm{Tr}(h((D_X^2 + \phi^2)/\Lambda^2)) = \mathrm{Tr}(h(D_X^2/\Lambda^2)) + \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

slow-roll potential

$$V(\phi) = \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u\,(h(u+x) - h(u))\,du$$

Slow-roll parameters (different from spherical cases)

$$\epsilon = \frac{m_{Pl}^2}{16\pi} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta = \frac{m_{Pl}^2}{8\pi} \left(\frac{h(x)}{\int_0^\infty u(h(u+x) - h(u))du} \right)$$

Bieberbach manifolds

Quotients of T^3 by group actions: $G2$, $G3$, $G4$, $G5$, $G6$

spin structures

	δ_1	δ_2	δ_3
(a)	± 1	1	1
(b)	± 1	-1	1
(c)	± 1	1	-1
(d)	± 1	-1	-1

$G2(a)$, $G2(b)$, $G2(c)$, $G2(d)$, etc.

Dirac spectra known (Pfäffle):

spectra often different for different spin structures

but spectral action same!

Bieberbach cosmic topologies (t_i = translations by a_i)

- $G2$ = half turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (T,S,0)$, with $H, L, S \in \mathbb{R}_+^*$ and $T \in \mathbb{R}$

$$\alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

- $G3$ = third turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (-\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, for H and L in $\mathbb{R}_+^*$

$$\alpha^3 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$$

- $G4$ = quarter turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (0,L,0)$, with $H, L > 0$

$$\alpha^4 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1}$$

- $G5 =$ sixth turn space

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$ and $a_3 = (\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, $H, L > 0$

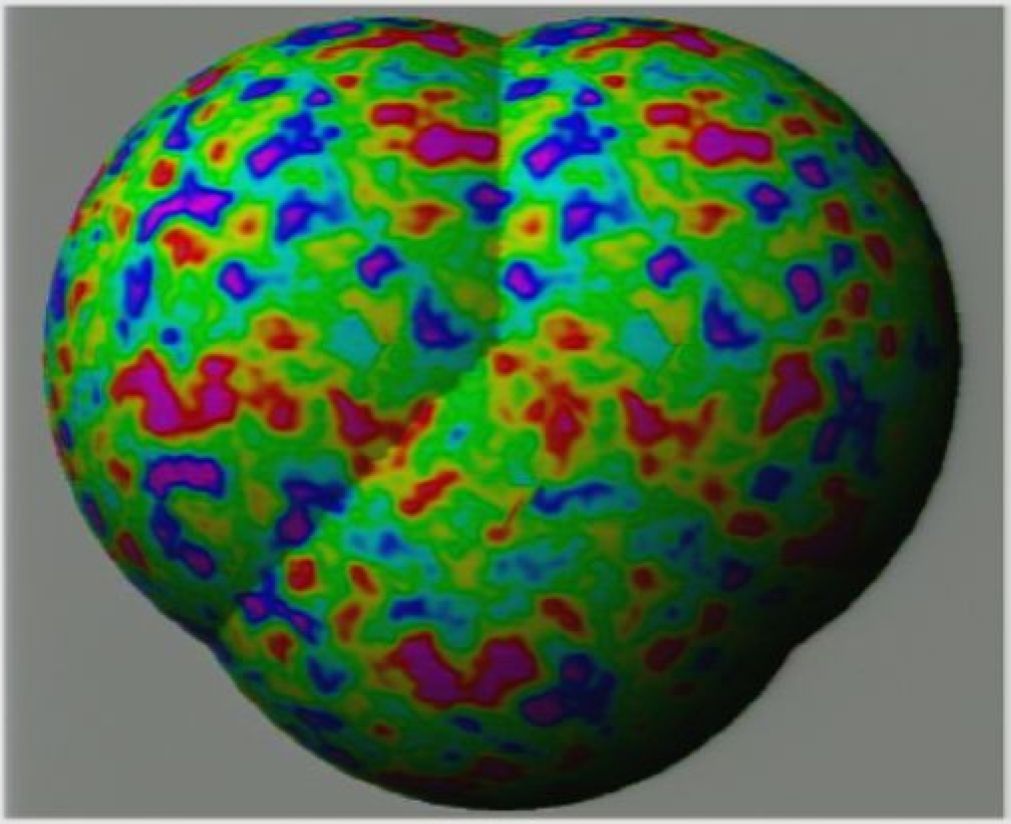
$$\alpha^6 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3$$

- $G6 =$ Hantzsche–Wendt space (π-twist along each coordinate axis)

lattice $a_1 = (0, 0, H)$, $a_2 = (L, 0, 0)$, and $a_3 = (0, S, 0)$, with $H, L, S > 0$

$$\begin{gathered}\alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta^2 = t_2, \quad \beta t_1 \beta^{-1} = t_1^{-1}, \quad \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma^2 = t_3, \quad \gamma t_1 \gamma^{-1} = t_1^{-1}, \quad \gamma t_2 \gamma^{-1} = t_2^{-1}, \\ \gamma\beta\alpha = t_1 t_3.\end{gathered}$$

Simulated CMB sky for a Bieberbach G6-cosmology



(from Riazuelo, Weeks, Uzan, Lehoucq, Luminet, 2003)

Topological factors (flat cases):

Theorem [MPT2]: Bieberbach manifolds spectral action

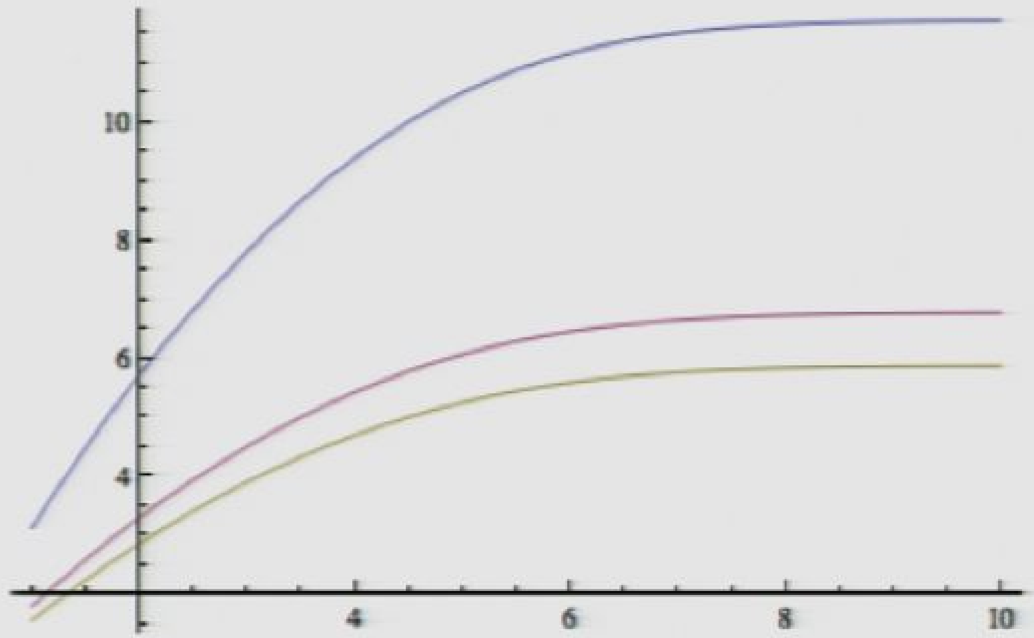
$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{\lambda_Y \Lambda^3}{4\pi^3} \int_{\mathbb{R}^3} f(u^2+v^2+w^2)\,dudvdw$$

up to oder $O(\Lambda^{-\infty})$ with factors

$$\lambda_Y = \begin{cases} \frac{HSL}{2} & G2 \\ \frac{HL^2}{2\sqrt{3}} & G3 \\ \frac{HL^2}{4} & G4 \\ \frac{HLS}{4} & G6 \end{cases}$$

Note lattice summation technique not immediately suitable for $G5$, but expect like $G3$ up to factor of 2

Topological factors and inflation slow-roll potential



⇒ Multiplicative factor in amplitude of power spectra

Adding the coupling to matter $Y \times F$

Not only product but nontrivial fibration

Vector bundle V over 3-manifold Y, fiber $\mathcal{H}_F$ (fermion content)

Dirac operator D_Y twisted with connection on V (bosons)

Spectra of twisted Dirac operators on spherical manifolds (Cisneros–Molina)

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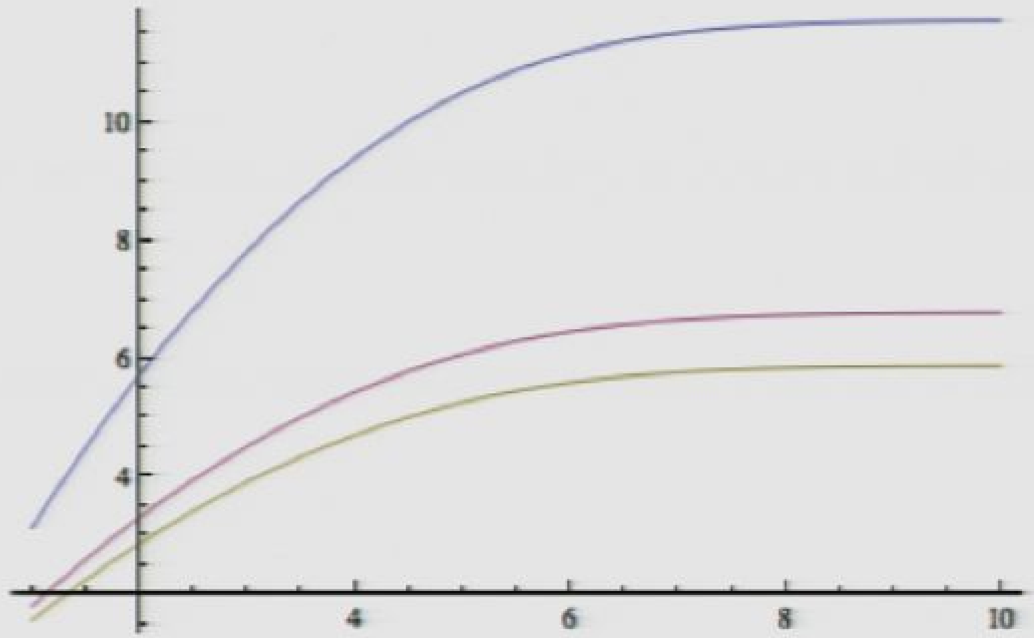
up to order $O(\Lambda^{-\infty})$

representation V dimension N; spherical form $Y = S^3/\Gamma$

$\Rightarrow$ topological factor $\lambda_Y \mapsto N\lambda_Y$

Topological factors and inflation slow-roll potential



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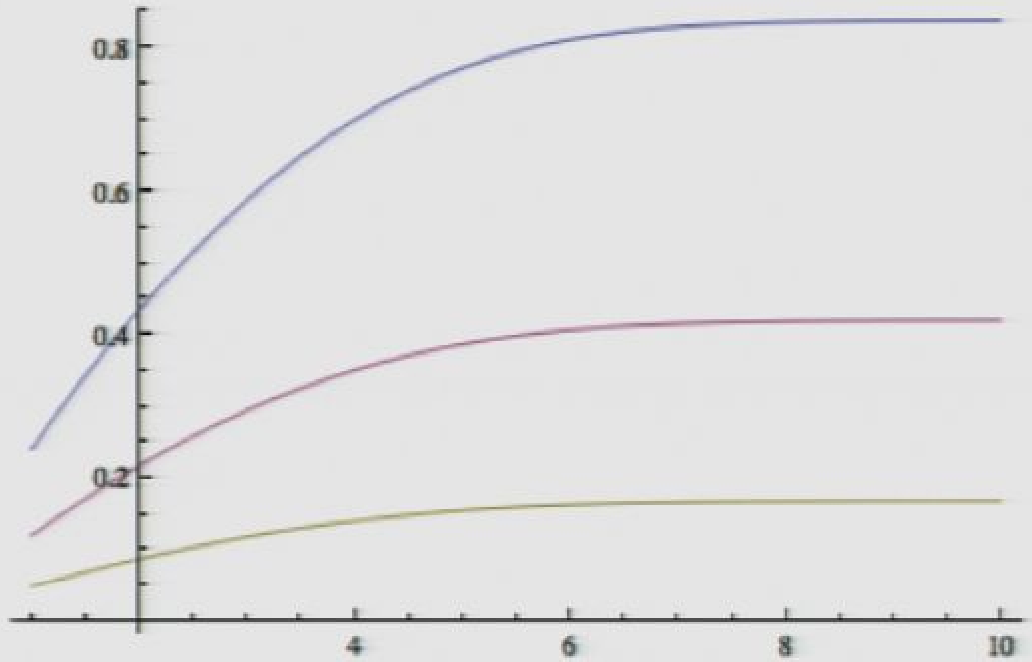

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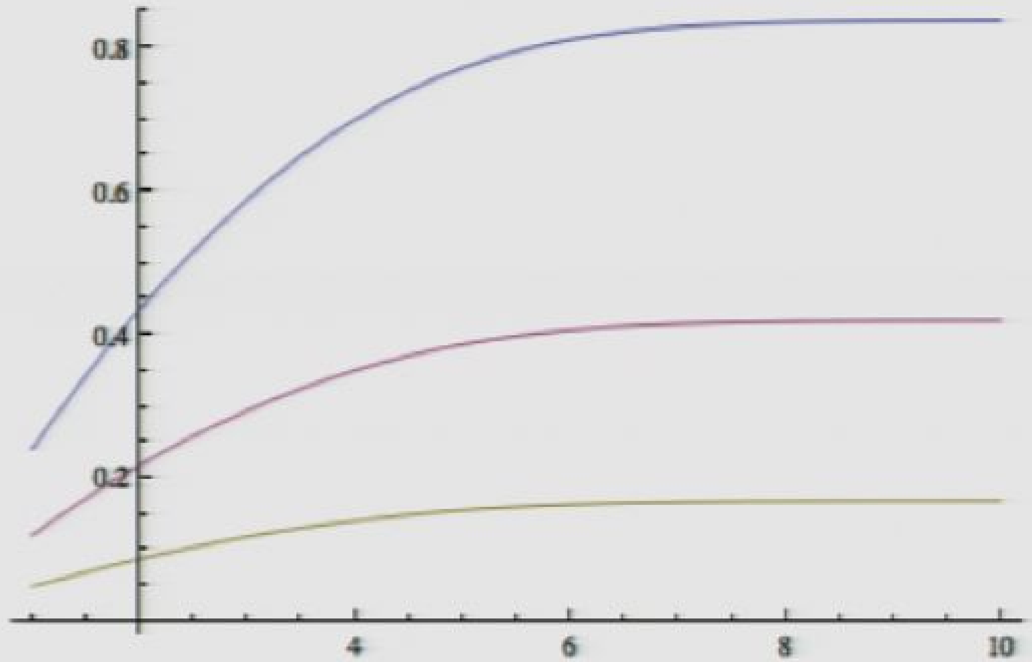

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by Poisson summation $\Rightarrow$ $1/120$ of action for S^3
Same slow-roll parameters

The quaternionic space $SU(2)/Q8$ (quaternion units $\pm 1, \pm\sigma_k$)

Dirac spectrum (Ginoux)

$$\frac{3}{2} + 4k \quad \text{with multiplicity} \quad 2(k+1)(2k+1)$$

$$\frac{3}{2} + 4k + 2 \quad \text{with multiplicity} \quad 4k(k+1)$$

Polynomial interpolation of multiplicities

$$P_1(u) = \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16}$$

$$P_2(u) = \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}$$

Spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f^{(2)}}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + O(\Lambda^{-k})$$

(1/8 of action for S^3) with $g_i(u) = P_i(u) f(u/\Lambda)$:

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{4}(\widehat{g}_1(0) + \widehat{g}_2(0)) + O(\Lambda^{-k})$$

from Poisson summation $\Rightarrow$ Same slow-roll parameters

Slow-roll parameters from spectral action $S = S^3$

$$\epsilon(x) = \frac{m_{Pl}^2}{16\pi} \left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta(x) = \frac{m_{Pl}^2}{8\pi} \frac{h'(x) + 2\pi(\Lambda a)^2 h(x)}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}$$

In Minkowskian Friedmann metric $\Lambda(t) \sim 1/a(t)$
Also independent of β (artificial Euclidean compactification)

A slow roll potential from non-perturbative effects

perturbation $D^2 \mapsto D^2 + \phi^2$ gives potential $V(\phi)$ scalar field coupled to gravity

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2))) = \pi\Lambda^4\beta a^3 \int_0^\infty uh(u)du - \frac{\pi}{2}\Lambda^2\beta a \int_0^\infty h(u)du$$

$$+\pi\Lambda^4\beta a^3\,\mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\,\mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))du, \qquad \mathcal{W}(x) = \int_0^x h(u)du$$

The standard topology S^3 (Chamseddine–Connes)

Dirac spectrum $\pm a^{-1}(\frac{1}{2}+n)$ for $n \in \mathbb{Z}$, with multiplicity $n(n+1)$

$$\mathrm{Tr}(f(D/\Lambda)) = (\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{4}(\Lambda a)\widehat{f}(0) + O((\Lambda a)^{-k})$$

with $\widehat{f}^{(2)}$ Fourier transform of $v^2 f(v)$ 4-dimensional Euclidean $S^3 \times S^1$

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \pi \Lambda^4 a^3 \beta \int_0^\infty u\, h(u)\, du - \frac{1}{2}\pi \Lambda a \beta \int_0^\infty h(u)\, du + O(\Lambda^{-k})$$

$$g(u,v) = 2P(u)\, h(u^2(\Lambda a)^{-2} + v^2(\Lambda\beta)^{-2})$$

$$\widehat{g}(n,m) = \int_{\mathbb{R}^2} g(u,v) e^{-2\pi i(xu+yv)}\, du\, dv$$

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$$+\pi\Lambda^4\beta a^3 \mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a \mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))du, \qquad \mathcal{W}(x) = \int_0^x h(u)du$$

Spectral action for the flat tori

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$$g(u,v,w,y) = 2\,h\left(\frac{4\pi^2}{\Lambda^2}(u^2+v^2+w^2)+\frac{y^2}{(\Lambda\beta)^2}\right)$$

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$\Rightarrow$ topological factor $\lambda_Y \mapsto N\lambda_Y$