Title: Typicality in random matrix product states

Date: Feb 02, 2011 04:00 PM

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Abstract: Quantum Mechanics has been shown to provide a rigorous foundation for Statistical Mechanics. Concentration of measure, or typicality, is the main tool to construct a purely quantum derivation for the methods of Statistical Mechanics. From this point of view statistical ensembles are effective description for isolated quantum systems, since typically a random pure state of the system will have properties similar to those of the ensemble. Nevertheless, it is often argued that most of the states of the Hilbert space are not relevant for realistic systems. This talk will address this issue, presenting recent results on the emergence of typicality in the context of matrix product states, a set of physically and computationally relevant states associated to many-body Hamiltonians.

A new class of quantum random states with applications to statistical mechanics

Silvano Garnerone
University of Southern California

In collaboration with:
Thiago R. de Oliveira,
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SG et al, PRA 81,032336 (2010); SG et al, PRA 82, 052312 (2010)

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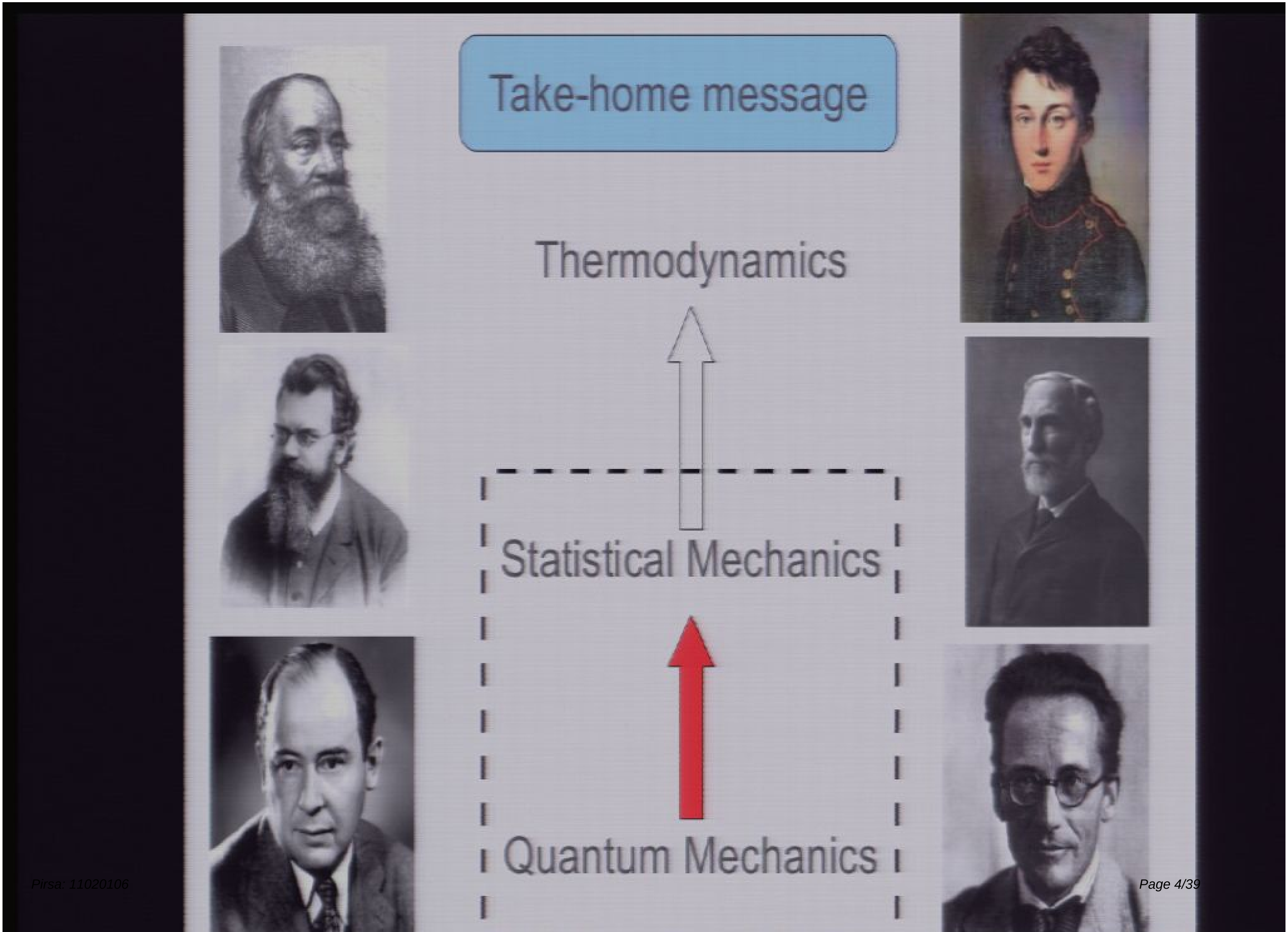
Take-home message
Thermodynamics
Statistical Mechanics
Quantum Mechanics

- Thermodynamics is a highly corroborated theory, though phenomenological.
- Stat Mech explains the effectiveness of thermodynamics.
- How can we explain the effectiveness of Stat Mech?...ergodicity, principle of equal a priori probability...but these have to be assumed most of the time.
- **Quantum Mechanics** can provide a justification for the effectiveness of SM ensembles.

Stat Mech from Quantum Mechanics

Universe

$$\mathcal{H}_R \subseteq \mathcal{H}_S \otimes \mathcal{H}_E$$

Microcanonical state

$$\mathcal{E}_R = \frac{\mathbb{1}_R}{d_R}$$

$$\rho_S = \mathrm{Tr}_E(|\phi\rangle\langle\phi|)$$

State of the subsystem given a random pure state of the system

$$\Omega_S = \mathrm{Tr}_E(\mathcal{E}_R)$$

General canonical state

4

Popescu et al. Nature Physics 2 754 (2006)

Principle of apparently equal a priori probability

Given a sufficiently small subsystem of the universe, almost every pure state of the universe is such that the subsystem is approximately in the canonical state.

$\rho_S = \mathrm{Tr}_E(|\phi\rangle\langle\phi|)$

$\rho_S \approx \Omega_S$

$\Omega_S = \mathrm{Tr}_E(\mathcal{E}_R)$

$$\Pr\left\{\mathcal{D}(\rho^S, \rho^S_{\mathrm{mc}}) \geq 2\epsilon + 2\sqrt{\frac{d_S}{d_B^{\mathrm{eff}}}}\right\} \leq 2\, e^{-C\, d_R \epsilon^2}$$

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From subsystems to particular states

- Sampling the entire Hilbert space could not be necessary...
- What if the set of realizable states are just a small portion of the Hilbert space?
- Would, in this case, a typicality argument still hold true?

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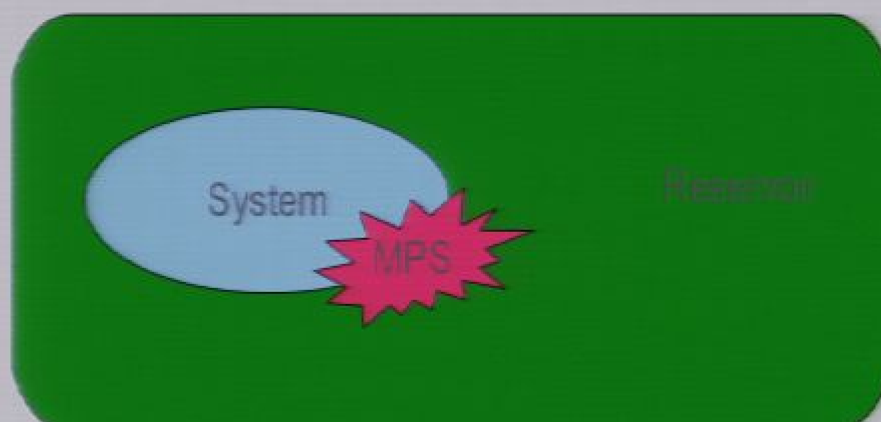
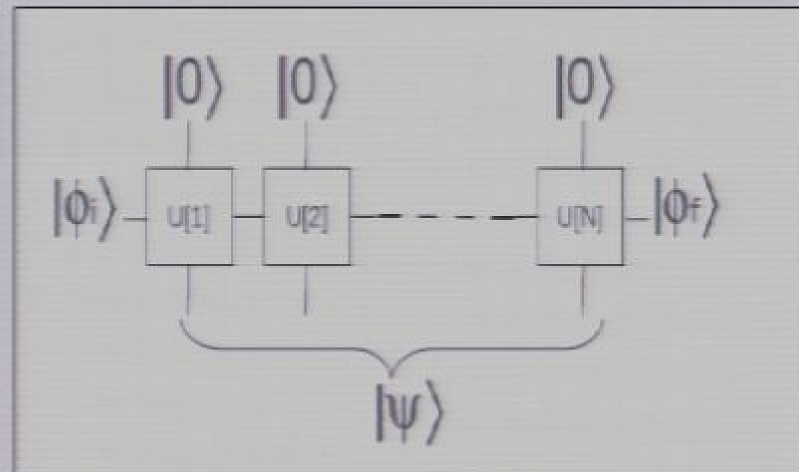
Random Matrix Product States

- We want to construct a “small” set of physically relevant states and check if typicality still apply.
- We focus on <u>MPS</u>.
- We need to define an ensemble of <u>random MPS</u>.



Random MPS

<u>Sequential generation of MPS:</u>



<u>U is random</u> (Haar-distributed of size χD)

$$A^{i}_{\alpha,\beta}[k] \equiv \langle i,\alpha|U[k]|\beta,0\rangle$$

Latin letters label the physical space, greek letters label the ancilla space

OBC →
$$|\psi\rangle = \sum_{i_1,\dots,i_N} \langle\phi_I|A^{i_1}[1]\cdots A^{i_N}[N]|\phi_F\rangle|i_1\cdots i_N\rangle$$

PBC →
$$\sum_{i_1,\dots,i_N} \mathrm{Tr}(A^{i_1}[1]\cdots A^{i_N}[N])|i_1\cdots i_N\rangle$$

Schoen et al PRL 95 110503 (2005); SG et al PRA 81,032336 (2010)

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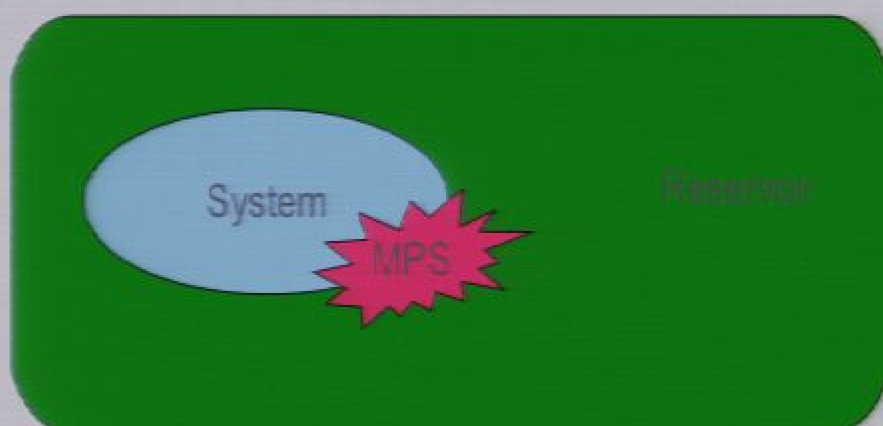

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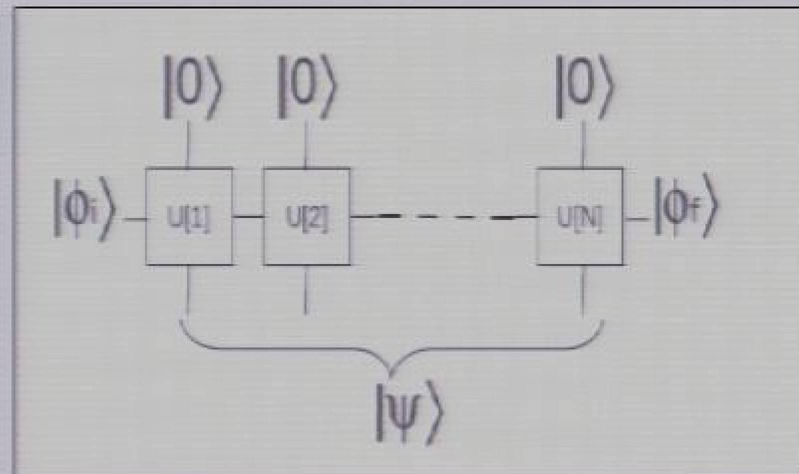

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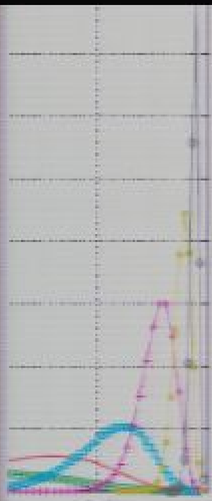
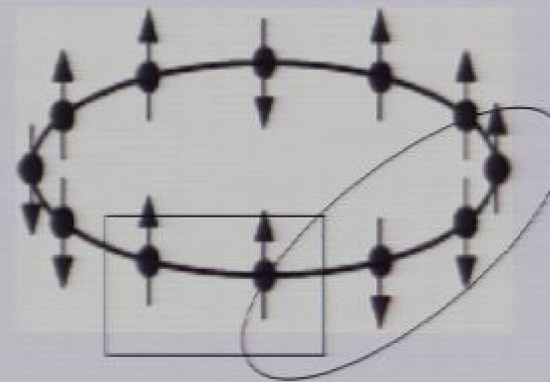


Typicality for RMPS

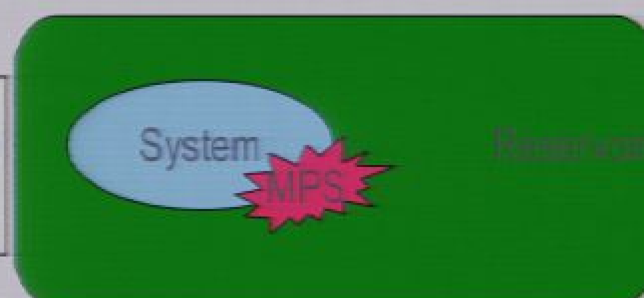
- MPS are specified by and exponentially smaller number of parameters, with respect to a general state.
- Can we have typicality for A-matrices of constant size?
- If not how does the **size χ** of the A-matrix has to scale?

1)Why χ cannot be constant?
2)Why χ cannot scale as exp(N)?



1)The size of the matrix A has to scale with the system size, because of the cut-off on the correlations.
2)If χ scales like exp(N) then one recovers the general result.

$$|\psi\rangle = \sum_{i_1,\ldots,i_N} \langle\phi_I|A^{i_1}[1]\cdots A^{i_N}[N]|\phi_F\rangle|i_1\cdots i_N\rangle$$



Numerical evidence for local observables

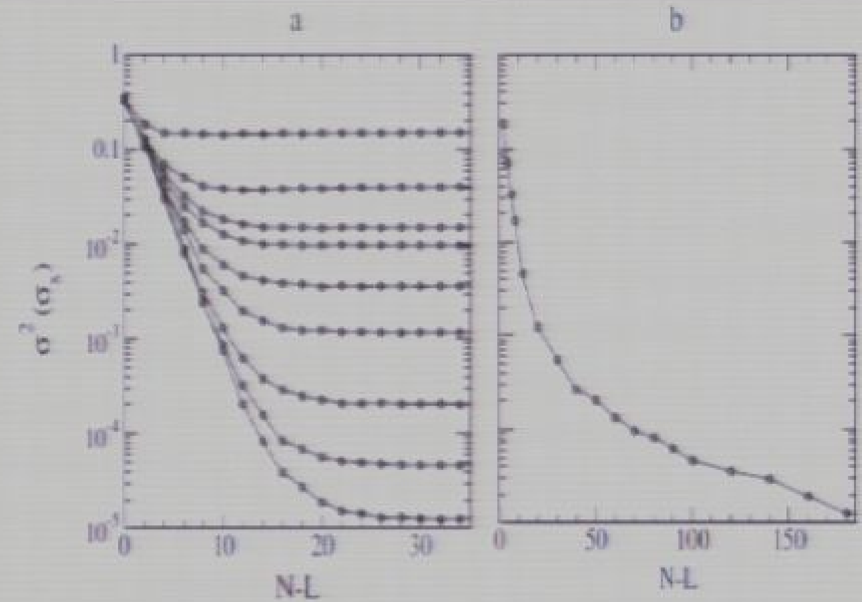


FIG. 5. (a) The variance of the expectation value of σ_x ($L = 1$) increasing the size of the system and for fixed but different values of $\chi = 2, 4, 6, 8, 12, 20, 50, 100,$ and 180 (from top to bottom). (b) The variance of the expectation value of σ_x ($L = 1$) for increasing system size when the MPS dimension increases linearly with the number of particles in the bath: $\chi = N - L$.

SG et al PRA 81,032336 (2010)

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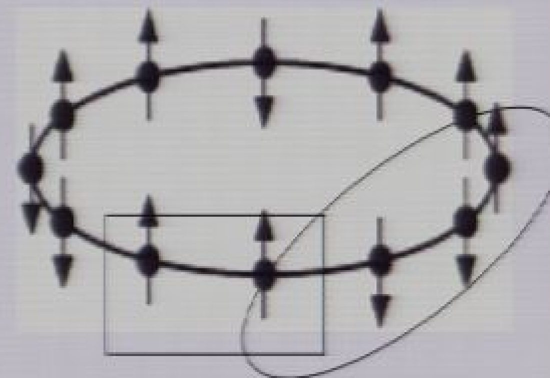

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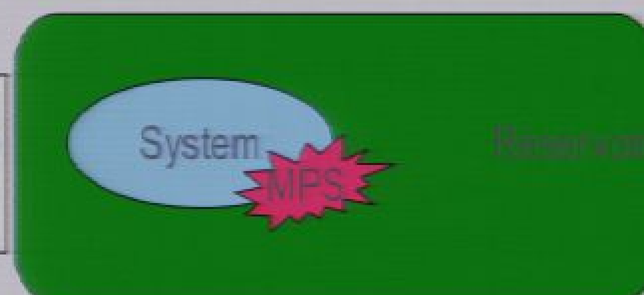

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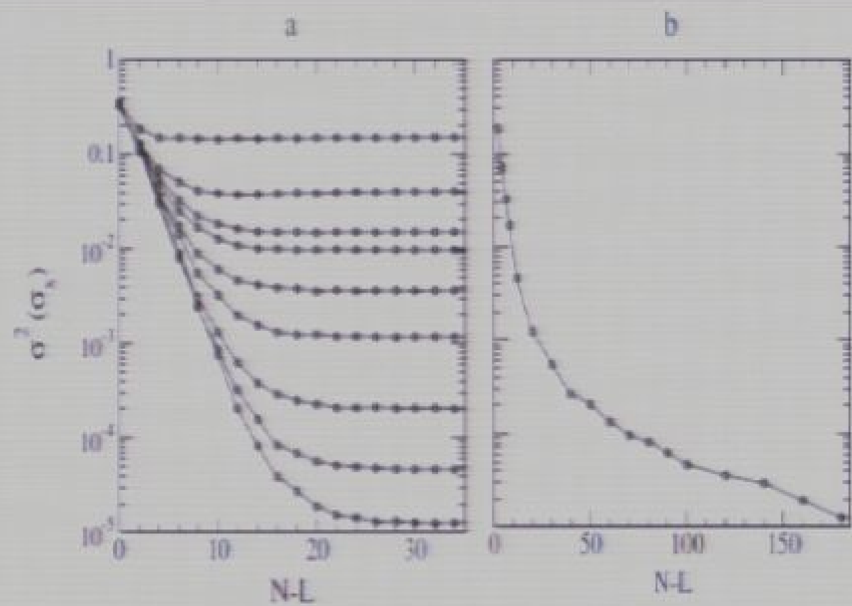

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Methods

- Use concentration of measure for the unitary group.
- Provide an upper bound for the fluctuations of the random variable: $\eta \leqslant 4D^{2L+2}N\|O\|_{\infty}^{L}$

Typicality for RMPS:

$$\mathrm{Prob}\Big[\,|f(\phi)-\langle f\rangle| \geq \epsilon\Big] \leq 2\exp\left(\frac{-2C(d+1)\epsilon^2}{\eta^2}\right)$$

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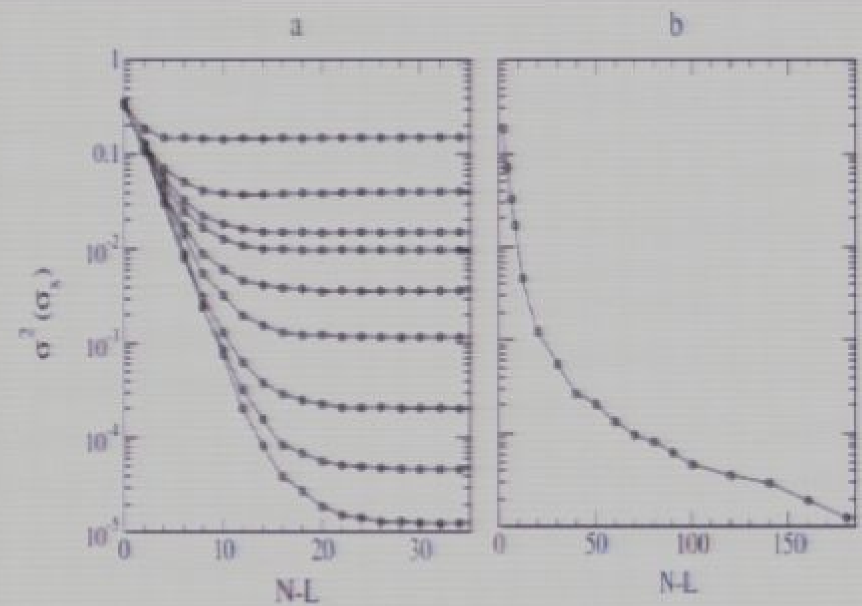

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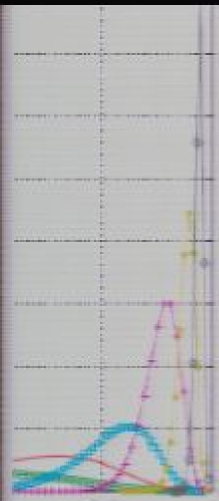

Results

1) $$\Pr[|f - \bar{f}| \geqslant \epsilon] \leqslant c_1 \exp(-c_2' \epsilon^2 \chi(N)/N^2)$$

expectation values of local observables

2) $$\Pr[\|\rho_s - \overline{\rho_s}\|_1 \geqslant 4^{3L/2}\epsilon] \leqslant 4^L c_1 \exp\left(-c_2 \epsilon^2 D \frac{\chi}{N^2}\right)$$

Typically the RDM will be closed to the canonical state

3) $$\overline{\psi}^{\text{RMPS}} = \overline{\psi}^{\text{CUE}}$$

The average state is the microcanonical state (for non-homogeneous RMPS)

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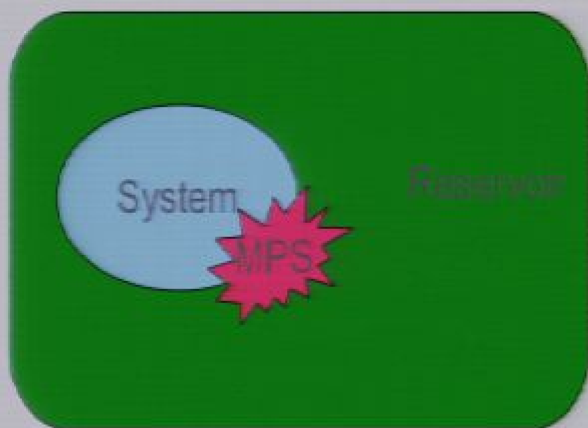
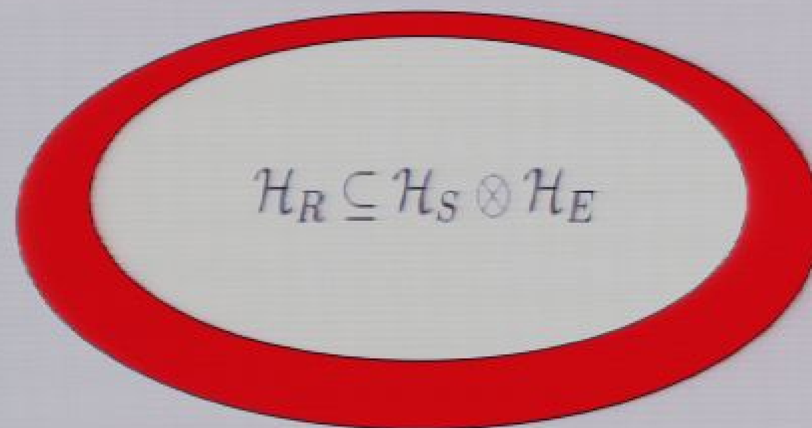
SG et al PRA 81 032336 (2010); SG et al PRA 82 052312 (2010)

Comparison with general states: local observables





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MPS

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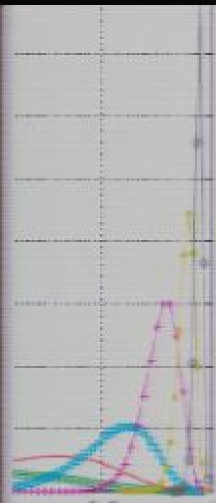

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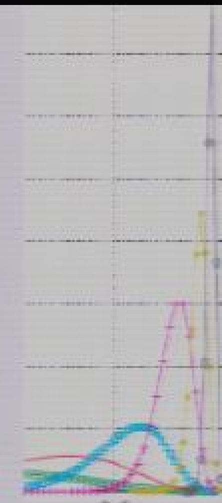

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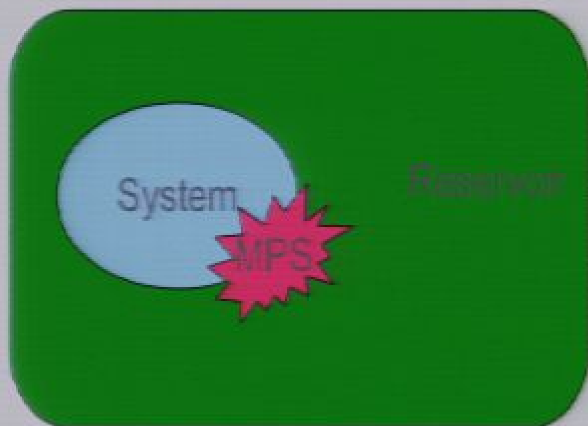

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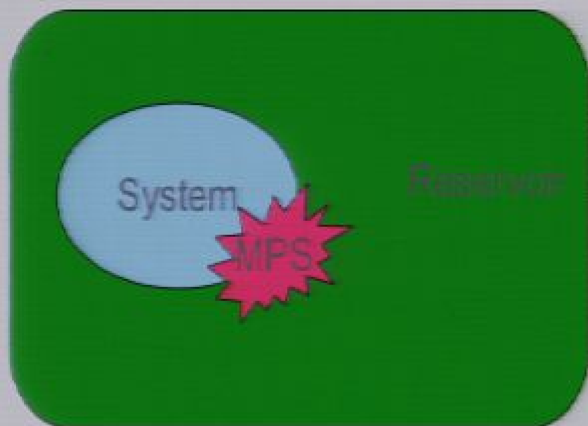

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Future directions

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- Connections with random Hamiltonians.
- Possible algorithmic applications.
- Stronger bounds.
- Higher dimensions.

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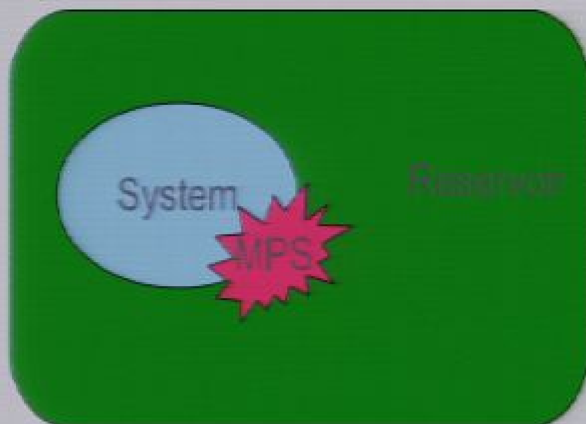

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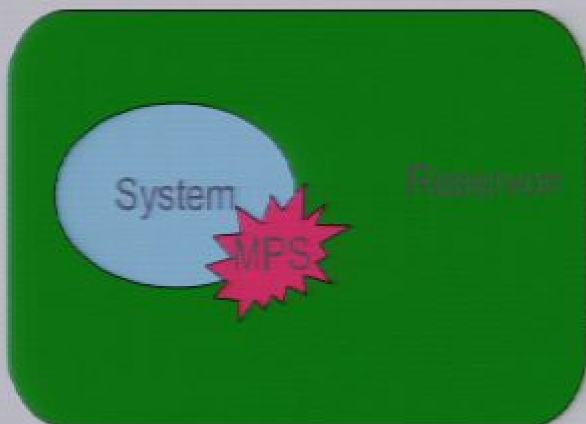
Conclusions

- RMPS show measure concentration.
- General canonical principle holds true for MPS, though in a weaker sense wrt to general states.
- RMPS provides a new "interesting" ensemble of random states.

Thanks!

Comparison with general states:RDM



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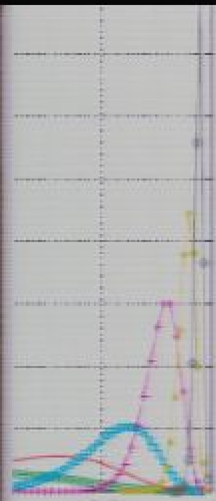

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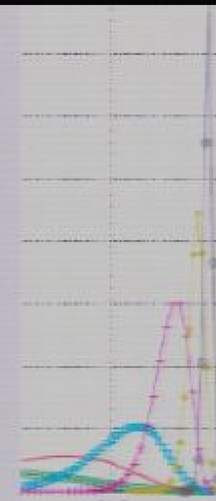

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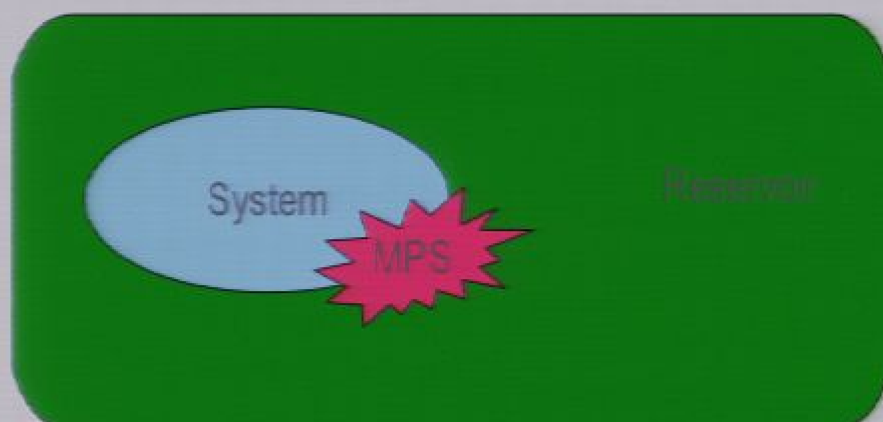

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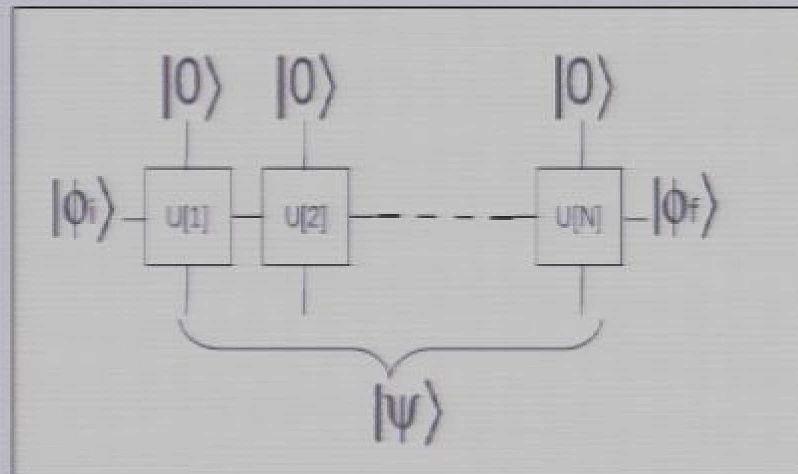

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