Title: Quantum codes give counterexamples to the unique pre-image conjecture of the N-representability problem

Date: Feb 16, 2011 04:00 PM

URL: http://pirsa.org/11020102

Abstract: It is well known that the ground state energy of many-particle Hamiltonians involving only 2- body interactions can be obtained using constrained optimizations over density matrices which arise from reducing an N-particle state. While determining which 2-particle density matrices are 'N-representable' is a computationally hard problem, all known extreme N-representable 2-particle reduced density matrices arise from a unique N-particle pre-image, satisfying a conjecture established in 1972. We present explicit counterexamples to this conjecture through giving Hamiltonians with 2-body interactions which have degenerate ground states that cannot be distinguished by any 2-body operator. We relate the existence of such counterexamples to quantum error correction codes and topologically ordered spin systems.

Quantum codes give counterexamples to the unique pre-image conjecture of the N-representability problem

Sam Ocko

MIT

arXiv:1010.2717, PRL Accepted

Xie Chen(MIT), Bei Zeng(Guelph), Beni Yoshida(MIT),
Zhengfeng Ji(Perimeter Inst.), Mary Beth Ruskai(Tufts), Isaac Chuang(MIT)

February 16, 2011

Motivation

Idea: Represent an N-body fermionic state by it's 2-body reduced density matrix(RDM).

Motivation

Idea: Represent an N-body fermionic state by it's 2-body reduced density matrix(RDM).

- All physical Hamiltonians of fermions are comprised of 2-body interactions.
- Energy of a state in 2-body Hamiltonian is completely determined by it's 2-particle RDM.
- Find the lowest energy density matrix; solve Hamiltonians quickly and easily.

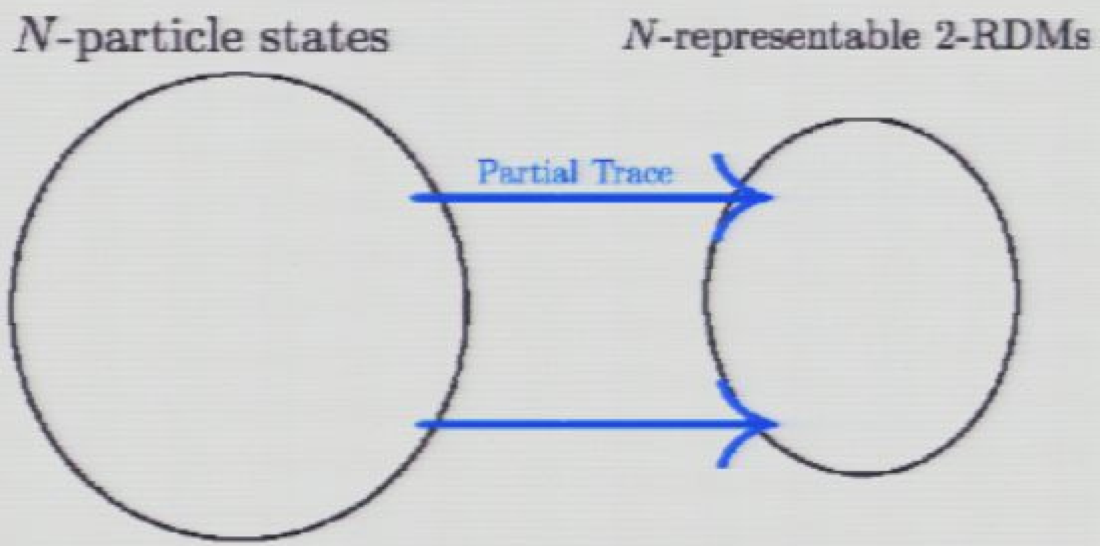
Motivation

Idea: Represent an N-body fermionic state by it's 2-body reduced density matrix(RDM).

- All physical Hamiltonians of fermions are comprised of 2-body interactions.
- Energy of a state in 2-body Hamiltonian is completely determined by it's 2-particle RDM.
- Find the lowest energy density matrix; solve Hamiltonians quickly and easily.

Problem: Not all density matrices are the reduced density matrices of some N-particle state!

N-Representability Problem:



What density matrices are the *reduced density matrices* of an N-particle state?
An N-particle state is called the pre-image of the state it maps to.

Importance and Connections to Quantum Information

If we could solve the N-representability problem, we could solve for the ground states of 2-body Hamiltonians!
Originally there was a hope the the N-representability problem would be solved.
The problem turned out to be far harder than expected; no progress was made for a long time.

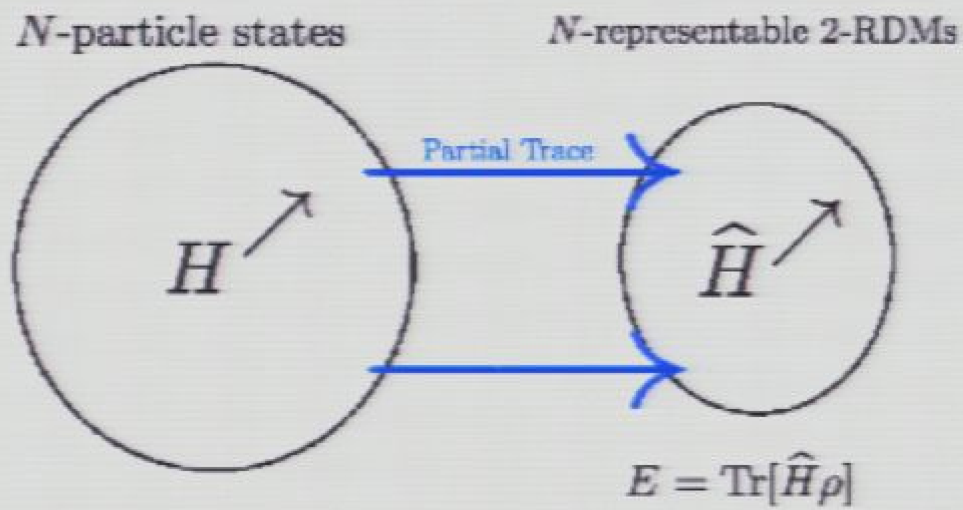
Importance and Connections to Quantum Information

If we could solve the N-representability problem, we could solve for the ground states of 2-body Hamiltonians!
Originally there was a hope the the N-representability problem would be solved.
The problem turned out to be far harder than expected; no progress was made for a long time.

(Liu et al., 2007): Deciding whether an arbitrary density matrix is N-representable is QMA-complete; hard even with a quantum computer, no exact solution possible.

Avenues of Attack

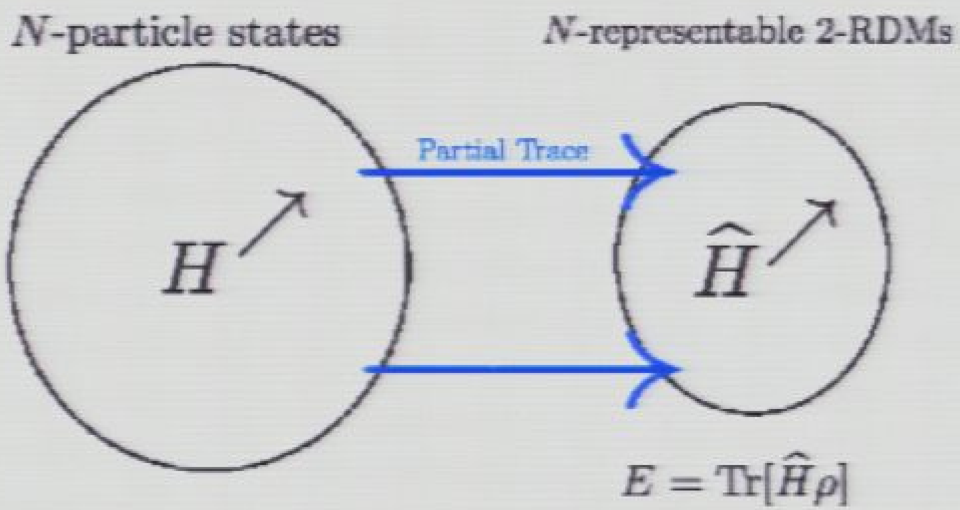
There are still many things we can say N-representable RDMs



- Form a convex set; the average of two is also N-representable.
- Energy($\mathrm{Tr}\,\widehat{H}\rho$) is a linear function of both RDM and Hamiltonian(like a dot product with proper choice of basis).

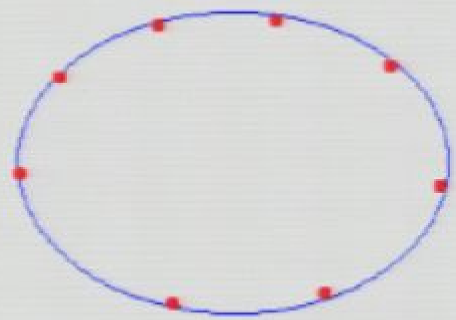
Avenues of Attack

There are still many things we can say N-representable RDMs



- Form a convex set; the average of two is also N-representable.
- Energy($\mathrm{Tr}\,\widehat{H}\rho$) is a linear function of both RDM and Hamiltonian(like a dot product with proper choice of basis).
- Linear optimization problem constrained to convex set.

Extreme Points



- Not average of any two points in set.
- Unique lowest energy point for some 2-body Hamiltonian.
- Completely characterize set; every other point is some weighted average of extreme points

Therefore important in linear optimization.

Unique Pre-Image Conjecture

Conjecture: (Erdahl, 1972) Every extreme N-representable fermionic RDM uniquely determines pre-image.

Unique Pre-Image Conjecture

Conjecture: (Erdahl, 1972) Every extreme N-representable fermionic RDM uniquely determines pre-image.

- Holds for all known extreme points so far.

Unique Pre-Image Conjecture

Conjecture: (Erdahl, 1972) Every extreme N-representable fermionic RDM uniquely determines pre-image.

- Holds for all known extreme points so far.
- Seems natural for systems in quantum chemistry, where $H = -\nabla^2 + V$.

Unique Pre-Image Conjecture

Conjecture: (Erdahl, 1972) Every extreme N-representable fermionic RDM uniquely determines pre-image.

- Holds for all known extreme points so far.
- Seems natural for systems in quantum chemistry, where $H = -\nabla^2 + V$.
- Similar observations have been made for translationally invariant spin systems.

Unique Pre-Image Conjecture

Conjecture: (Erdahl, 1972) Every extreme N-representable fermionic RDM uniquely determines pre-image.

- Holds for all known extreme points so far.
- Seems natural for systems in quantum chemistry, where $H = -\nabla^2 + V$.
- Similar observations have been made for translationally invariant spin systems.
- Proven for m-body density matrices where $m \geq N/2$.

Main Result

In this talk: We disprove Erdahl's conjecture using recent advances in quantum coding theory.

Main Result

In this talk: We disprove Erdahl's conjecture using recent advances in quantum coding theory.

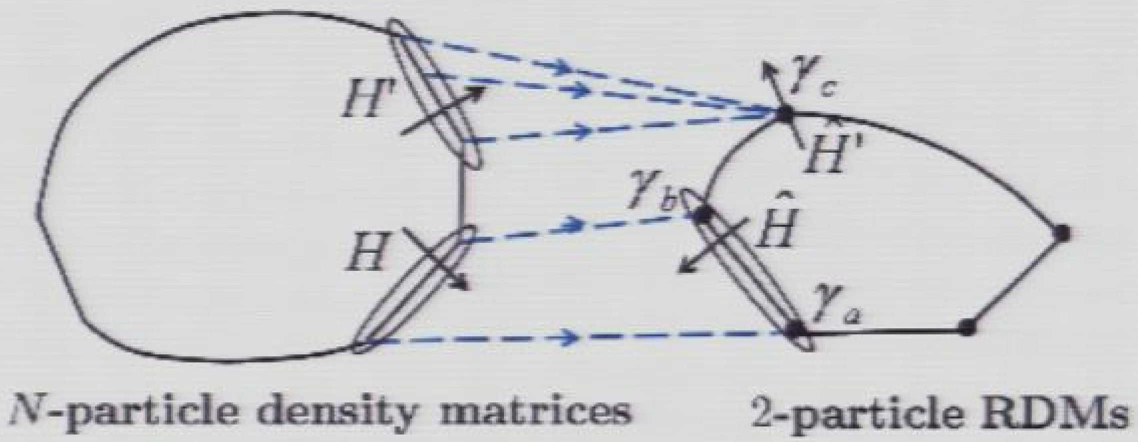
-Set of extreme points is much more complex than previously believed.

Roadmap

1. Translate Erdahl's conjecture into the language of Hamiltonians.
2. Present spin lattice Hamiltonian.
3. Map spin lattice case to fermionic case-disprove conjecture.
4. Connect to quantum coding theory.

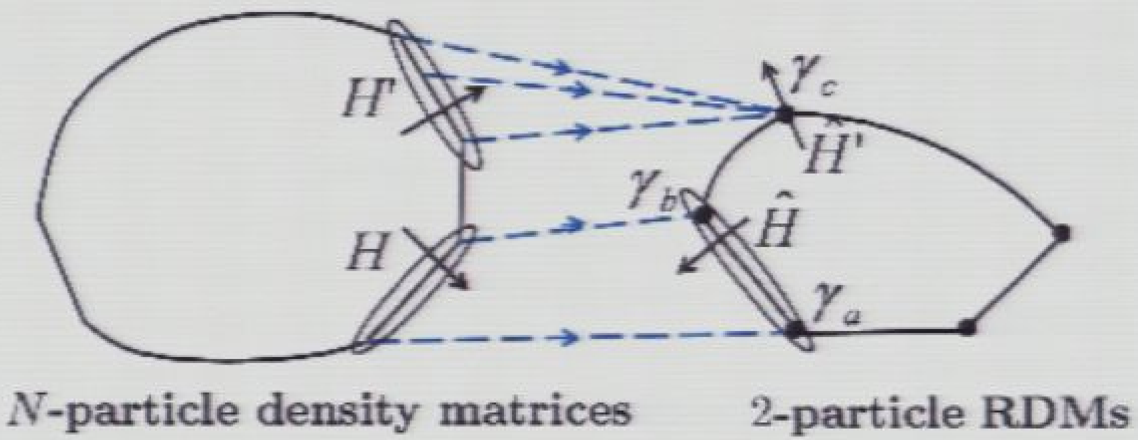
Translating into Hamiltonian Language

Every extreme point γ corresponds to the ground space of some 2-body Hamiltonian.



- Consider a 2-body Hamiltonian H' whose ground states can't be distinguished by 2-body operators.
- All ground states have the same 2-particle RDM; otherwise 2-body operators would distinguish them.
- We define such a Hamiltonian to be 2-blind.

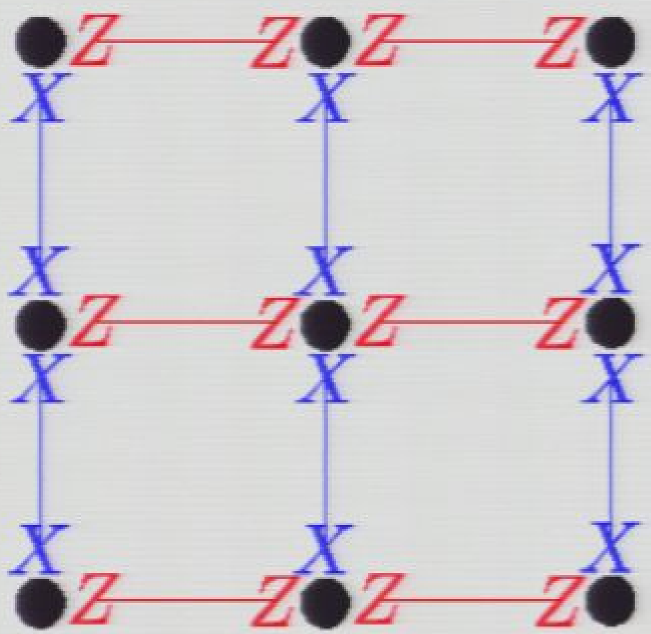
Translation cont.



- 2-blind H' has a unique lowest-energy reduced density matrix γ, which is therefore extreme.
- Since multiple ground states of H' map to γ, γ is an extreme point with multiple pre-images.
- Erdahl's conjecture, translated, is that there's no 2-blind fermionic Hamiltonian.

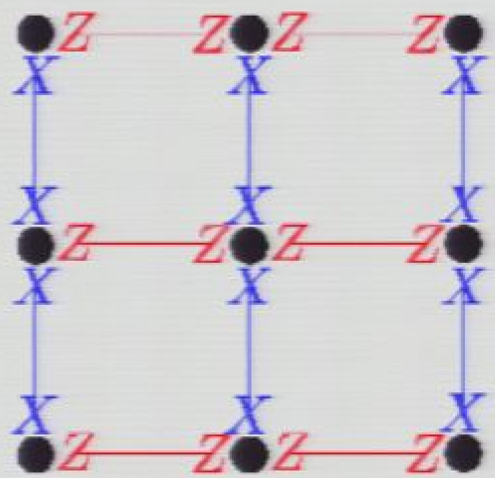
Bacon-Shor Code: A Simple 2-Blind Spin Hamiltonian

Subsystem code with gauge generators:



$$H_{BS} \equiv \sum_{jk} \left(- X_{j,k} X_{j+1,k} - Z_{j,k} Z_{j,k+1} \right)$$

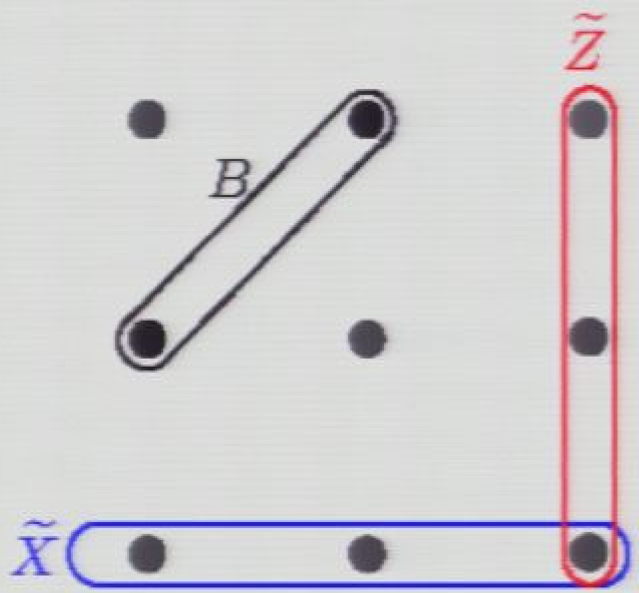
Bacon-Shor Code: Ground States(Codewords)



- H_{BS} has 2 ground states, each stabilized by stabilizer group (all columns have same parity).
- Ground states have even parity($|C_0\rangle$) or odd parity($|C_1\rangle$).

Proof of Indistinguishability by 2-Body Operators:

Given any operator B acting on two qubits, we can define logical operators $\widetilde{X}, \widetilde{Z}$ which don't touch the same qubits;



Therefore, Bacon-Shor Hamiltonian is 2-blind.

Mapping to Fermions

Replace system of N spins with system of N fermions and N sites.
Map each direct product state to a Slater determinant:

$$V : |s_1\rangle \otimes \ldots \otimes |s_N\rangle \mapsto a^\dagger_{1,s_1} \ldots a^\dagger_{N,s_N} |\Omega\rangle$$

Mapping to Fermions

Replace system of N spins with system of N fermions and N sites. Map each direct product state to a Slater determinant:

$$V : |s_1\rangle \otimes \ldots \otimes |s_N\rangle \mapsto a^\dagger_{1,s_1} \ldots a^\dagger_{N,s_N} |\Omega\rangle$$

May "fermionize" one-body and two-body spin operators:

$$Z_j \mapsto Z_j^{\mathrm{ferm}} = a^\dagger_{j,\uparrow} a_{j,\uparrow} - a^\dagger_{j,\downarrow} a_{j,\downarrow}, \qquad X_j \mapsto X_j^{\mathrm{ferm}} = a^\dagger_{j,\uparrow} a_{j,\downarrow} + a^\dagger_{j,\downarrow} a_{j,\uparrow}$$

Mapping to Fermions

Replace system of N spins with system of N fermions and N sites. Map each direct product state to a Slater determinant:

$$V : |s_1\rangle \otimes \ldots \otimes |s_N\rangle \mapsto a^\dagger_{1,s_1} \ldots a^\dagger_{N,s_N} |\Omega\rangle$$

May "fermionize" one-body and two-body spin operators:

$$Z_j \mapsto Z_j^{\text{ferm}} = a^\dagger_{j,\uparrow} a_{j,\uparrow} - a^\dagger_{j,\downarrow} a_{j,\downarrow}, \qquad X_j \mapsto X_j^{\text{ferm}} = a^\dagger_{j,\uparrow} a_{j,\downarrow} + a^\dagger_{j,\downarrow} a_{j,\uparrow}$$

Fermionized spin operators have the same algebra and expectation values on fermionized states that spin operators have on spin states:

$$Z_j^{\text{ferm}} a^\dagger_{j,\uparrow} |\Omega\rangle = a^\dagger_{j,\uparrow} |\Omega\rangle, \qquad Z_j^{\text{ferm}} a^\dagger_{j,\downarrow} |\Omega\rangle = -a^\dagger_{j,\downarrow} |\Omega\rangle, \text{ etc.}$$

Fermionizing the Hamiltonian

- Fermionize $H_{BS} \mapsto H_{BS}^{\text{ferm}}$
- Fermionize ground states $|C_0\rangle, |C_1\rangle \mapsto V|C_0\rangle, V|C_1\rangle$
- Since ground states of H_{BS} can't be distinguished by 2-body spin operators, ground states of H_{BS}^{ferm} can't be distinguished by 2-body fermionic operators; map to same 2-RDM.
- H_{BS}^{ferm} is 2-blind and gives an extreme point with multiple pre-images (it's ground space), disproving Erdahl's conjecture.

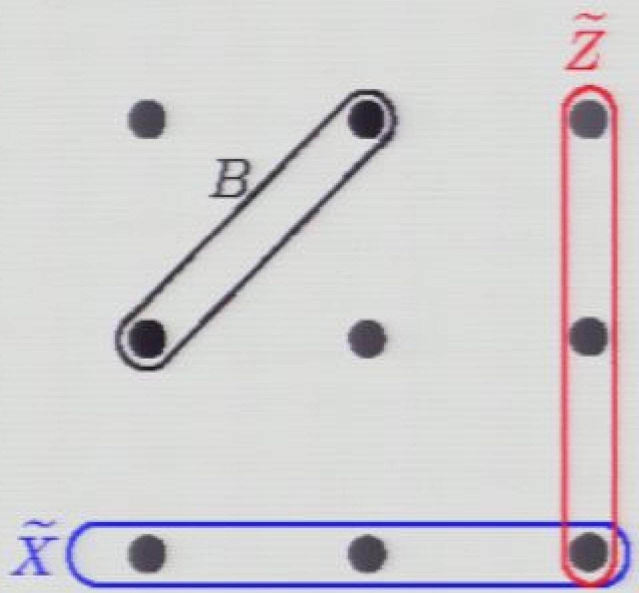
Mapping to Fermions

Replace system of N spins with system of N fermions and N sites.
Map each direct product state to a Slater determinant:

$$V : |s_1\rangle \otimes \ldots \otimes |s_N\rangle \mapsto a^\dagger_{1,s_1} \ldots a^\dagger_{N,s_N}|\Omega\rangle$$

Proof of Indistinguishability by 2-Body Operators:

Given any operator B acting on two qubits, we can define logical operators $\widetilde{X}, \widetilde{Z}$ which don't touch the same qubits;

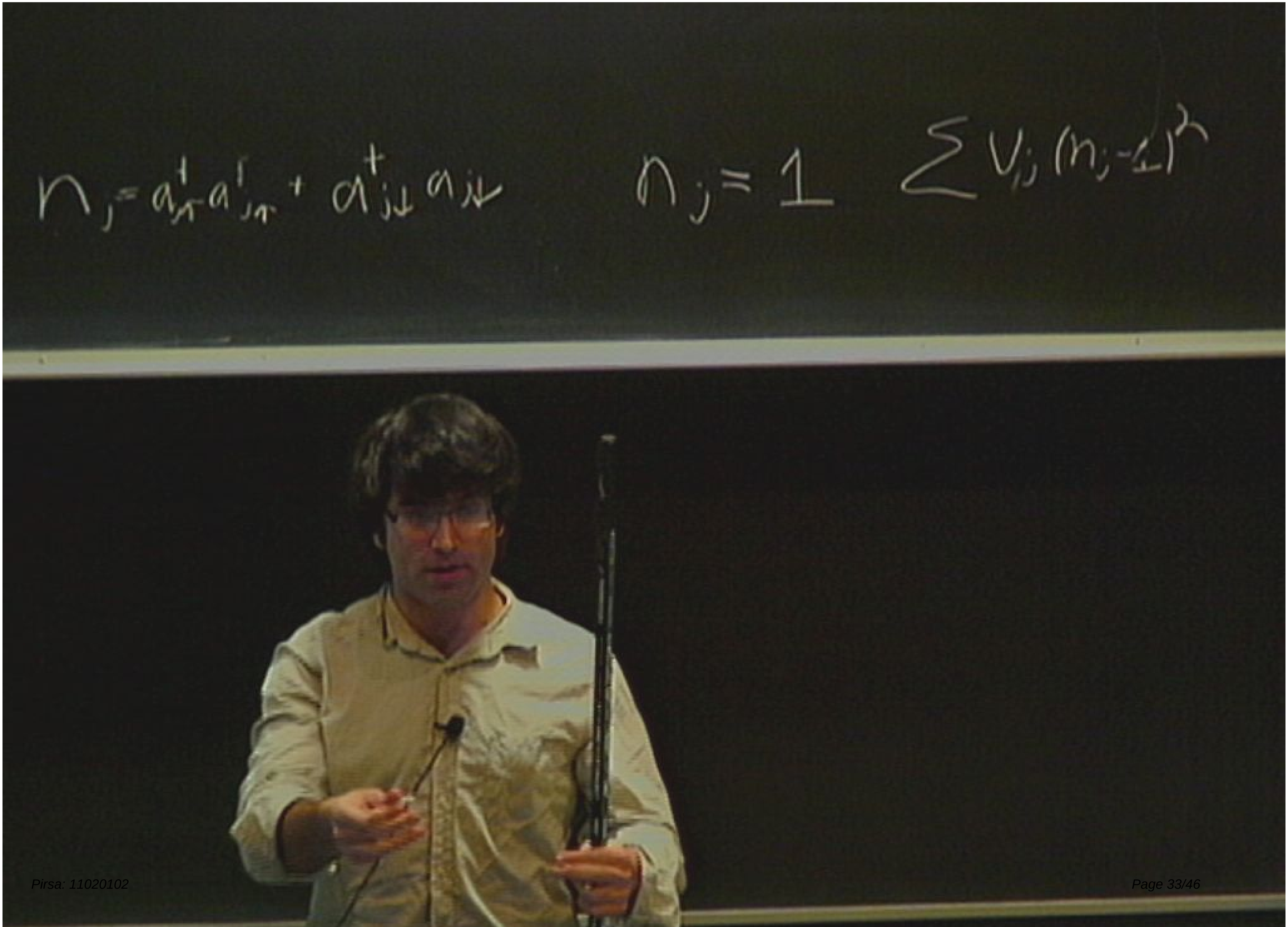


Therefore, Bacon-Shor Hamiltonian is 2-blind.

Mapping to Fermions

Replace system of N spins with system of N fermions and N sites. Map each direct product state to a Slater determinant:

$$V : |s_1\rangle \otimes \ldots \otimes |s_N\rangle \mapsto a^\dagger_{1,s_1} \ldots a^\dagger_{N,s_N}|\Omega\rangle$$


$n_j = a^\dagger_{j\uparrow} a_{j\uparrow} + a^\dagger_{j\downarrow} a_{j\downarrow}$
$n_j = 1$
$\sum V_j (n_j - 1)^2$

Mapping to Fermions

Replace system of N spins with system of N fermions and N sites.
Map each direct product state to a Slater determinant:

$$V : |s_1\rangle \otimes \ldots \otimes |s_N\rangle \mapsto a^\dagger_{1,s_1} \ldots a^\dagger_{N,s_N} |\Omega\rangle$$

Fermionizing the Hamiltonian

- Fermionize $H_{BS} \mapsto H_{BS}^{\mathrm{ferm}}$
- Fermionize ground states $|C_0\rangle, |C_1\rangle \mapsto V|C_0\rangle, V|C_1\rangle$
- Since ground states of H_{BS} can't be distinguished by 2-body spin operators, ground states of H_{BS}^{ferm} can't be distinguished by 2-body fermionic operators; map to same 2-RDM.
- H_{BS}^{ferm} is 2-blind and gives an extreme point with multiple pre-images (it's ground space), disproving Erdahl's conjecture.

Connection to Quantum Coding Theory

- Want to encode qubit using codewords $|C_p\rangle$.
- If an error from set $\{E_m\}$ is applied, we want to identify and correct the error without disturbing the encoded state:

$$\langle C_p|E_\ell^\dagger E_m|C_q\rangle = \delta_{pq} Q_{\ell m}.$$

Connection to Quantum Coding Theory

- Want to encode qubit using codewords $|C_p\rangle$.
- If an error from set $\{E_m\}$ is applied, we want to identify and correct the error without disturbing the encoded state:

$$\langle C_p|E_\ell^\dagger E_m|C_q\rangle = \delta_{pq}Q_{\ell m}.$$

- When $\{E_m\}$ consists of all single-bit errors, $E_\ell^\dagger E_m$ is basis for all 2-body operators.

Connection to Quantum Coding Theory

- Want to encode qubit using codewords $|C_p\rangle$.
- If an error from set $\{E_m\}$ is applied, we want to identify and correct the error without disturbing the encoded state:

$$\langle C_p|E_\ell^\dagger E_m|C_q\rangle = \delta_{pq} Q_{\ell m}.$$

- When $\{E_m\}$ consists of all single-bit errors, $E_\ell^\dagger E_m$ is basis for all 2-body operators.
- QECC criteria = indistinguishability criteria;
  Code corrects 1-bit errors $\Longleftrightarrow$ all codespace has same 2-RDMs (Bravyi et al. 2010).

Only a few codes form the ground space of 2-blind Ham (Bacon 2006, Bombin et al. 2010, etc.)

Generalization to m-Body RDMs

Original conjecture was that all extreme N-representable m-particle RDMs have unique pre-images.

- m-blind Hamiltonians give extreme m-body RDM with multiple pre-images(toric code, topological stabilizer codes, etc.)
- Ground space gives a code which can correct $\lfloor \frac{m}{2} \rfloor$ errors.

Future Work: Tightening the Rules of the Game

- Intuition behind conjecture came from Hamiltonians with Laplacians: $H = -\nabla^2 + V$.
- Our 2-blind Hamiltonian has no Laplacian term, each site has exactly one fermion at all times.

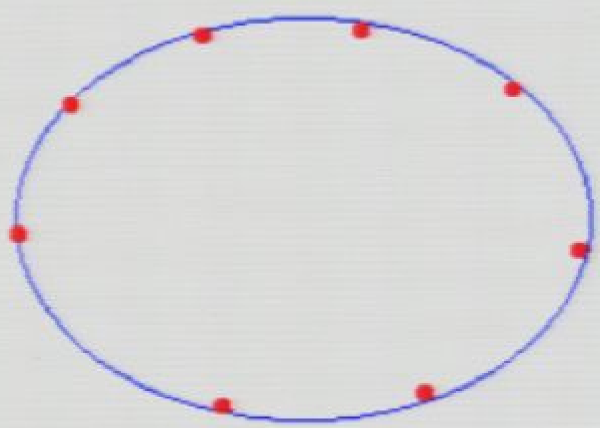
Future Work: Tightening the Rules of the Game

- Intuition behind conjecture came from Hamiltonians with Laplacians: $H = -\nabla^2 + V$.
- Our 2-blind Hamiltonian has no Laplacian term, each site has exactly one fermion at all times.

What classes of fermionic Hamiltonians can be 2-blind?

Quantum Chemistry

- Extreme N-representable RDMs much different than previously believed.
- What are the implications for N-representability?
- Can we find a better set of extreme points for variational calculations?



Holonomic Quantum Computation

Curved boundaries represent states which can be transformed adiabatically as ground states of 2-body Hamiltonians.

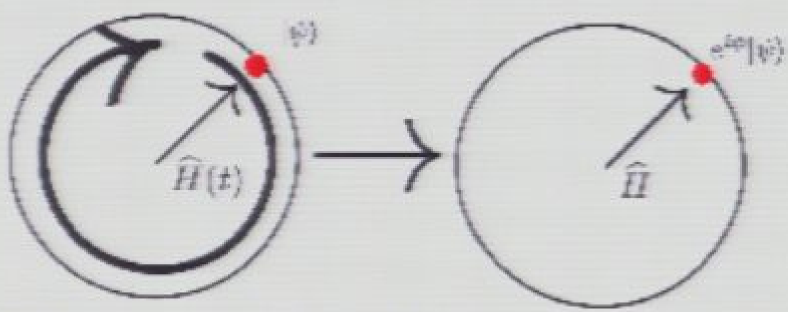


Figure: Extreme points have unique pre-images

Holonomic Quantum Computation

Curved boundaries represent states which can be transformed adiabatically as ground states of 2-body Hamiltonians.

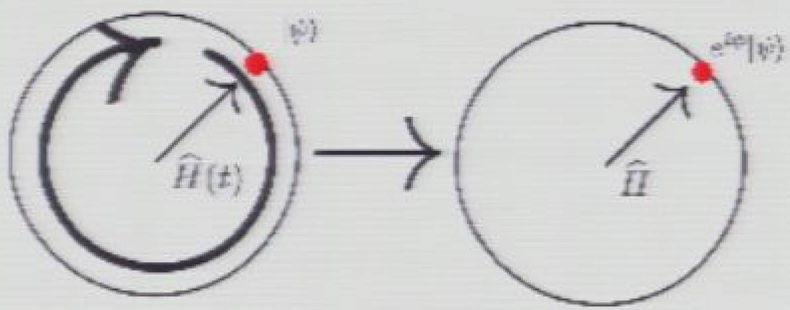


Figure: Extreme points have unique pre-images

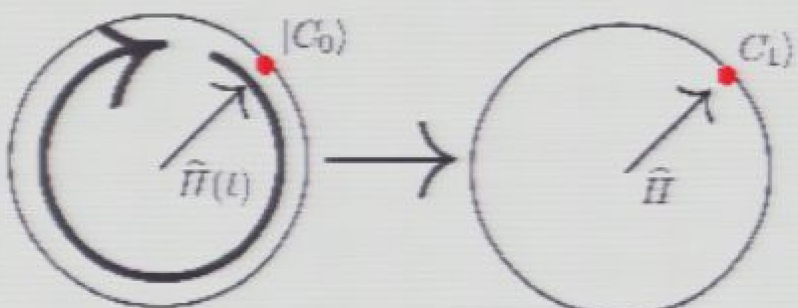


Figure: Extreme points have multiple pre-images

Holonomic Quantum Computation

Curved boundaries represent states which can be transformed adiabatically as ground states of 2-body Hamiltonians.

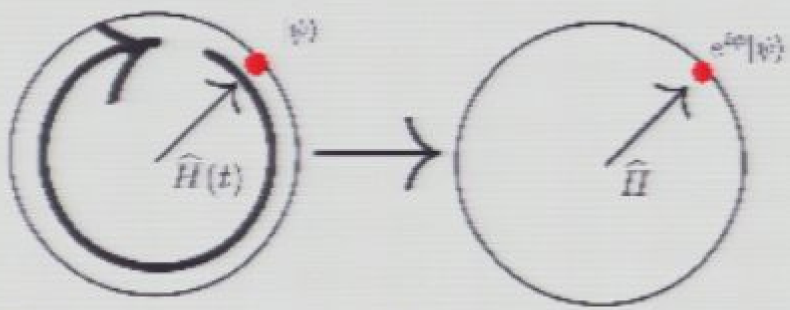


Figure: Extreme points have unique pre-images

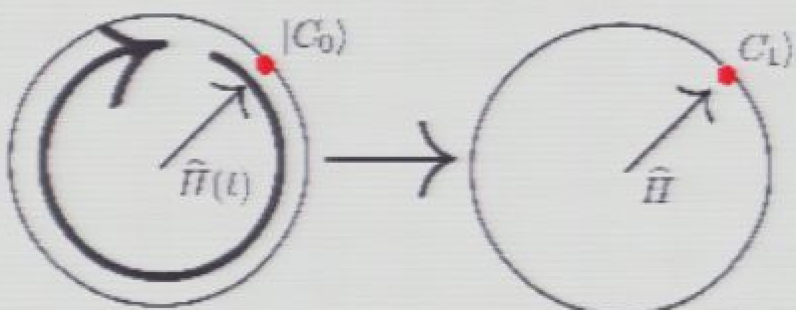


Figure: Extreme points have multiple pre-images

Universal Holonomic quantum computation from time-evolving 2-blind Hamiltonian? (Oreshkov et al. 2009)

Topological Quantum Codes

- Bacon-Shor code, other CSS subsystem codes give 2-blind Hamiltonians. Not believed to be gapped in the thermodynamic limit.
- How can we get topological quantum codes from 2-body interactions? (Kitaev, Bombin et al., Brell et al. . . .)
- 2-body topologically ordered Hamiltonians with exactly solvable ground states and low-energy excitations? (Ocko and Yoshida, in preparation)