Title: From groups to non-locality via categories

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Abstract: Symmetric monoidal categories provide a convenient and enlightening framework within which to compare and contrast physical theories on a common mathematical footing. In this talk we consider two theories: stabiliser qubit quantum mechanics and the toy bit theory proposed by Rob Spekkens. Expressed in the categorical framework the two theories look very similar mathematically, reflecting their common physical features. There are differences though: in particular a finite Abelian group emerges naturally in the categorical framework, and this group is different in each case (Z_4 for the stabiliser theory and $Z_2 \times Z_2$ for the toy bit theory). It turns out that this mathematical difference corresponds directly with a key physical difference between the theories: the stabiliser theory cannot be modelled by local hidden variables, while the toy bit theory can. This analysis can be extended to other Abelian groups yielding a group-theoretic criterion for determining the possibility of local hidden variable interpretations for other physical theories.

From groups to non-locality via categories

Bill Edwards

Includes joint work with Bob Coecke and Rob Spekkens

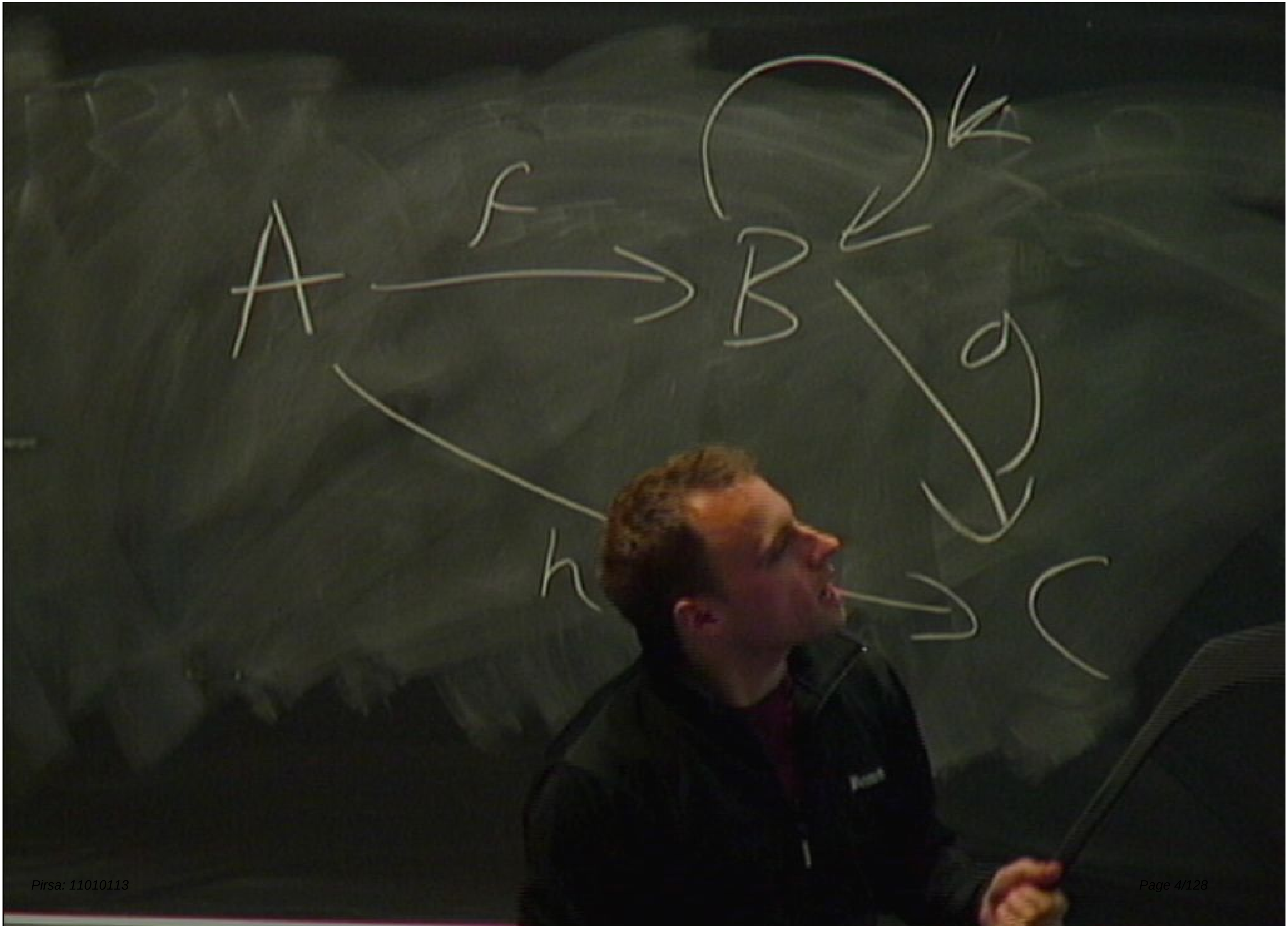
Quantum Foundations Seminar, PI, 25/01/11

Using the categorical approach to compare theories

- ☐ Use Abramsky and Coecke's categorical approach as a framework in which to compare different theories (inc. quantum mechanics).
- ☐ Every theory has an associated (symmetric monoidal) category - its *process* category.
- ☐ Allows comparison of theories which normally are not formulated in terms of the same mathematical structures.
- ☐ Comparing different physical theories allows us to pinpoint the categorical structures 'responsible' for different physical phenomena.

We will concentrate on a particular test case:

- ☐ Compare **qubit stabiliser theory** with **Spekkens's toy bit theory**.
- ☐ Concentrate on the the possibility of a **local hidden variable** interpretation.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Structure of the talk

1. Preliminaries:
   - □ local hidden variables
   - □ qubit stabiliser theory and Spekkens's toy bit theory
2. Review categorical framework, introduce process categories of stabiliser theory (**Stab**) and toy theory (**Spek**).
3. Identify key structures which arise in both of these categories, and note where they subtly differ.
4. Make a link between one of these structures, the *phase group*, and the issue of local hidden variables.

Hidden variables

Does quantum mechanics have an underlying hidden variable interpretation?

Quantum states give us the probabilities of different outcomes when measuring observables. Do these probabilities reflect an epistemic probability distribution over a set of *hidden states*, each of which has a definite value for each observable?

We all know that the answer is NO.

Mermin's no-go argument (1)

Consider three spatially separated qubits in a GHZ state: $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

$X_1 = ?$
$Y_1 = ?$

$X_2 = ?$
$Y_2 = ?$

$X_3 = ?$
$Y_3 = ?$

If there is a hidden variable interpretation then there are definite values for each of these six observables, either +1 or -1.

Mermin's no-go argument (2)

We can only measure one observable from each qubit in one go. Consider the following four combinations.

$$\begin{array}{cccr} X_1 & X_2 & X_3 & 1 \\ X_1 & Y_2 & Y_3 & -1 \\ Y_1 & X_2 & Y_3 & -1 \\ Y_1 & Y_2 & X_3 & -1 \end{array}$$

The GHZ state is such that only certain *parities* of outcomes are allowed. Finding a valid hidden state is equivalent to filling in the table such that the row parities are respected. This is impossible.

$\Longrightarrow$ there is no local hidden variable interpretation for quantum mechanics.

D. Mermin. Quantum mysteries revisited. *Am. J. Phys.*, 58:59-87, 1990

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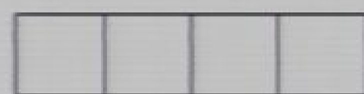
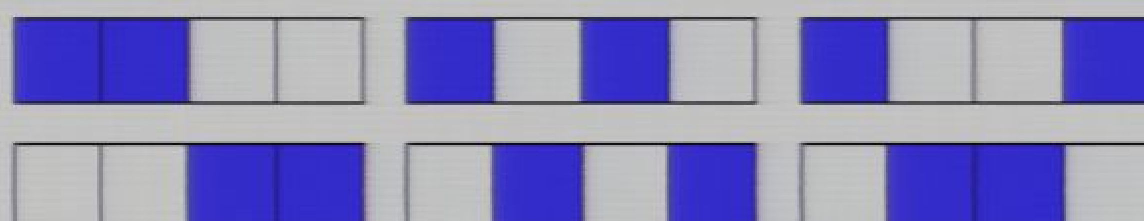
Spekkens's toy bit theory

R. Spekkens. Evidence for the epistemic view of quantum states: A toy theory. *Phys. Rev. A*, 75(032110), 2007.

There is one type of system in the theory, which can exist in one of four *ontic* states.



There are six possible states of maximal knowledge, consistent with the knowledge balance principle. These are called *epistemic* states.



Epistemic states described by *subsets*.

Measurements in the toy theory

A measurement consists of asking as many yes/no questions as is compatible with the knowledge balance principle.

A	A	B	B

A	B	A	B

A	B	B	A

A measurement induces an inevitable disturbance.

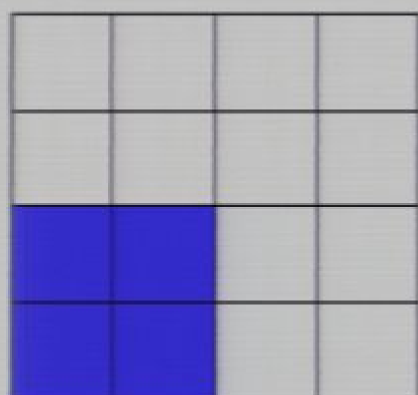
Such disturbances are described by *relations*.

Compound systems in the toy theory

The analysis can be extended to compound systems consisting of several elementary systems.

For example, epistemic states for systems with two components fall into two classes:

Uncorrelated - 'Separable'



Maximally correlated - 'Entangled'

Toy theory - displays some quantum features

The toy theory exhibits many characteristically quantum features:

- Incompatible observables
- No-cloning
- Protocols such as teleportation and dense coding

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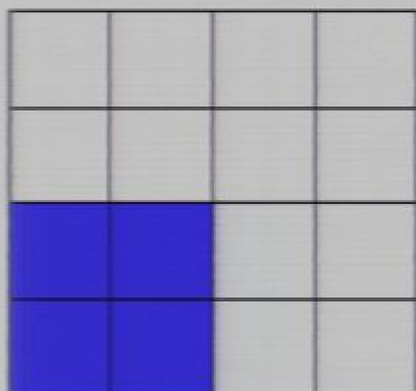

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Qubit stabiliser theory

Systems: *Qubits*
States: *Stabiliser states*
Processes: *Clifford operations*
Observables: *Pauli group*

1 qubit states:

- $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |i\rangle, |-i\rangle$

2 qubit states:

- 36 product states e.g. $|0\rangle \otimes |+\rangle$;
- 24 maximally entangled states e.g. $\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$

3 qubit states:

- Many more, including GHZ states e.g. $\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)$

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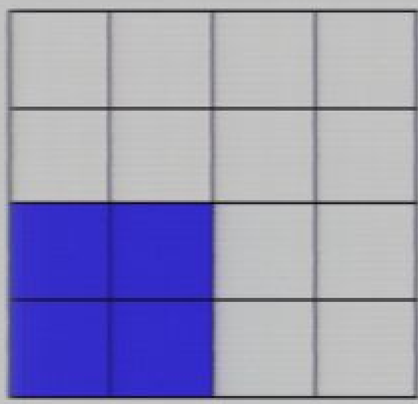

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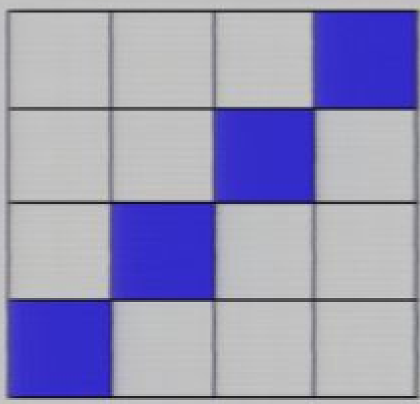

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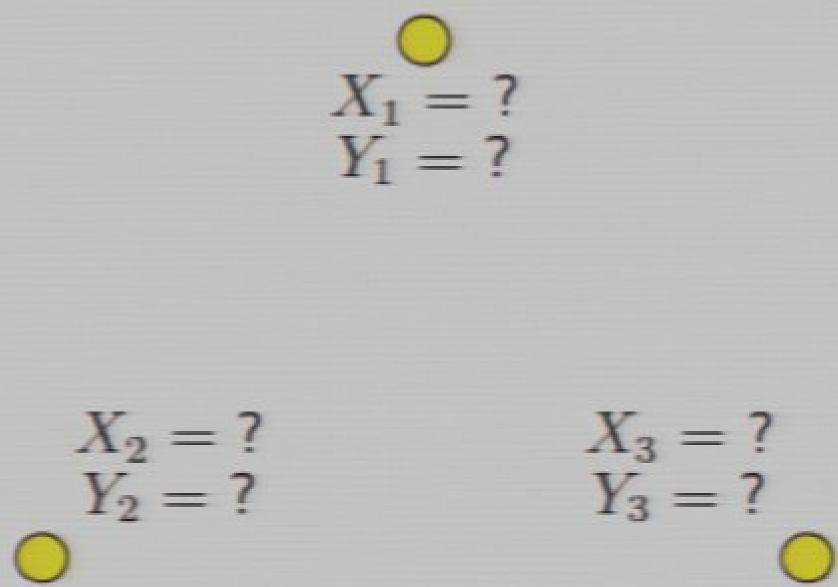

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Categorical approach - Algebra of processes

In quantum mechanics (and stabiliser theory) **states** are described by *vectors*, and **processes** by *linear maps*.
In the toy bit theory **states** are described by *subsets*, and **processes** by *relations*.

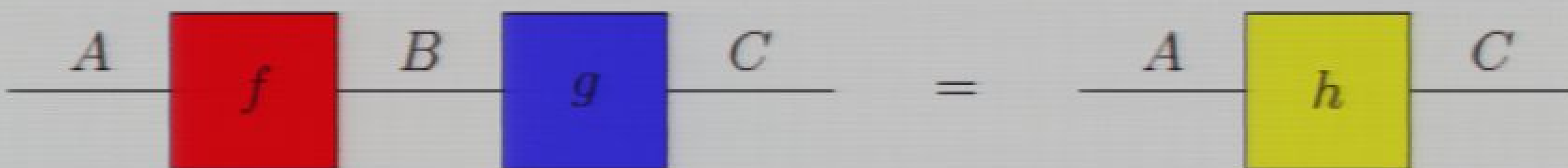
We are uninterested in these details: we confine our attention to the *algebra of how the processes combine*.

Graphical depiction of processes

We use diagrams to represent processes and their composition.

A process transforming a system of type A into a system of type B:

The composition of two processes is equal to a third:

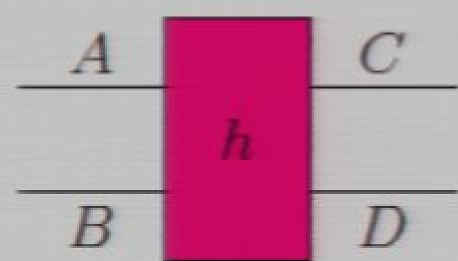


Composite systems and parallel processes

Parallel processes:

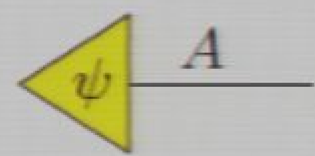
In QM, *tensor product* $-\otimes-$. In toy theory, *Cartesian product* $-\times-$.

A process involving interaction:



Preparation of states, scalars

The following diagram depicts the preparation of a new state of system A:



In reality would occur as part of a combination:

We term this kind of combination a *scalar*:

In QM scalars $\in \mathbb{C}$, in toy theory scalars $\in \mathbb{B}_2$.

Our categories

Quantum mechanics: **FHilb** (already well known to mathematicians).

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Toy bit theory: **Spek**

Will not describe structure in detail.

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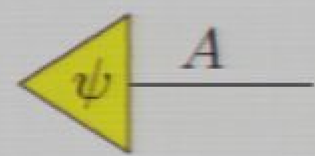

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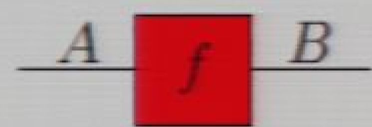
The dagger operation

Bijection between processes of this type:



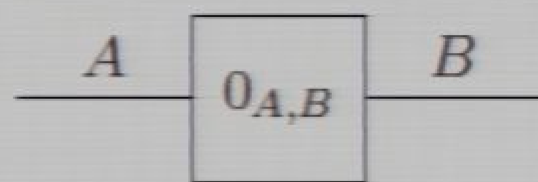
and this type:

In QM (and stabiliser theory) corresponds to the *adjoint*. In toy theory corresponds to *relational converse*.

Impossible processes

We need something to represent an *impossible* process. This could represent the result of composing two operations which can never occur one after the other.

For every pair of systems A and B we have a *zero* process.



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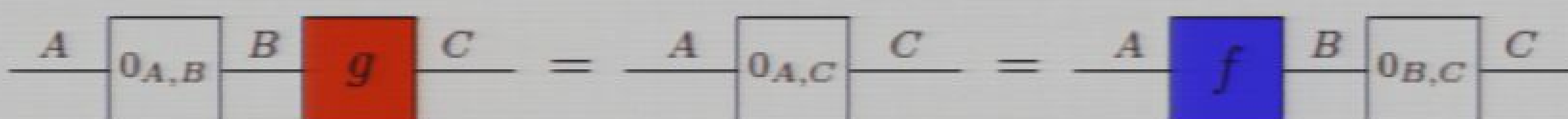

In particular we also have *zero scalars*.

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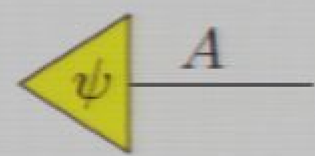

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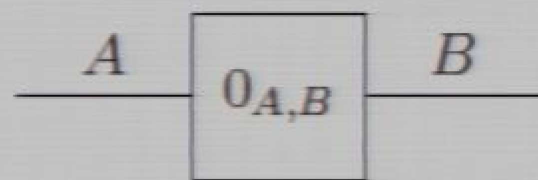

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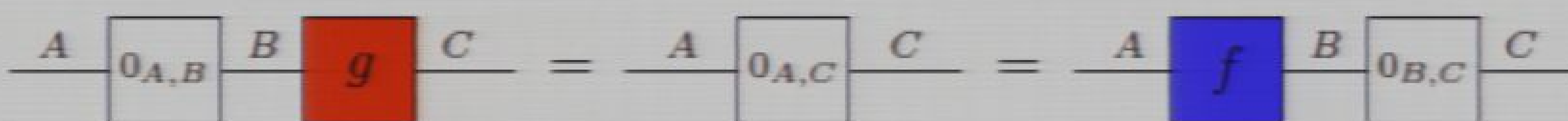

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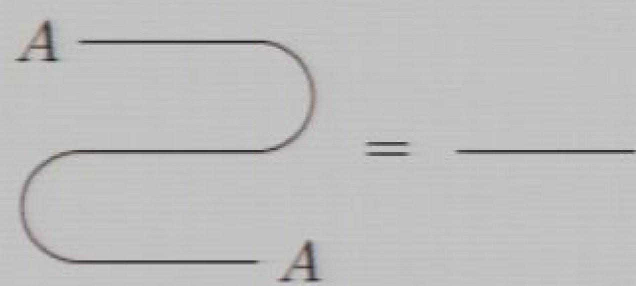
Compact structures

A system A has a *compact structure* if there exist a state and co-state:

Which satisfy the following property:



In QM, every system has such a state and co-state. The state is the Bell state.

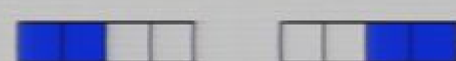
Basis structures in the toy theory

An example of a basis structure in the two theory. We will label the four ontic states of a single elementary system simply as $1, 2, 3$ and 4. Then the following two relations constitute a basis structure:

$$\begin{aligned} \delta ::\quad & 1 \sim \{(1,1),(2,2)\} \\ & 2 \sim \{(1,2),(2,1)\} \\ & 3 \sim \{(3,3),(4,4)\} \\ & 4 \sim \{(3,4),(4,3)\} \end{aligned}$$

$$\begin{aligned} \epsilon ::\quad & 1 \sim \{*\} \\ & 3 \sim \{*\} \end{aligned}$$

It's not obvious, but δ copies the following two states:



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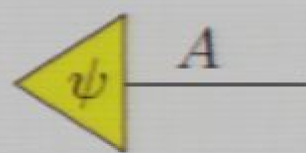

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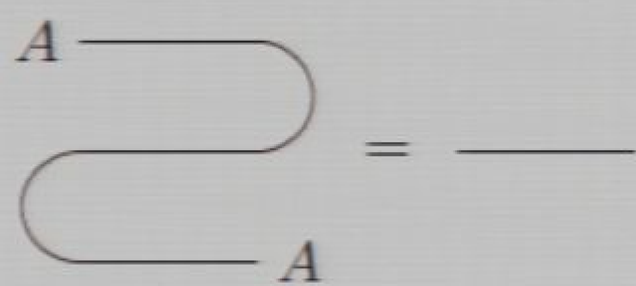

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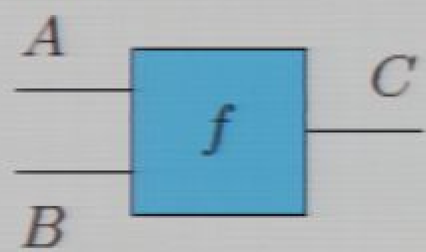
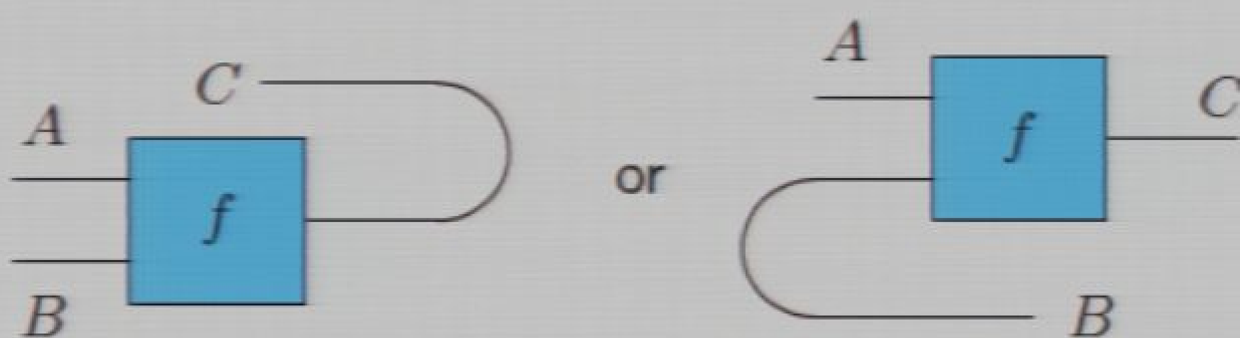
Bijections between different types of process

One of the key consequences of compact structure is that it generates a whole series of bijections between processes of different types:
If we start with a process of this type:



We can get other processes of different types, for example:



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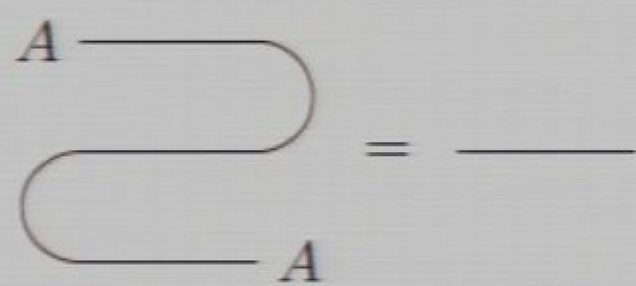

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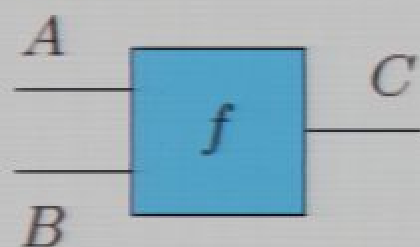

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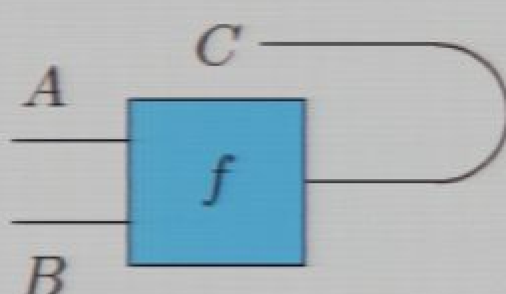

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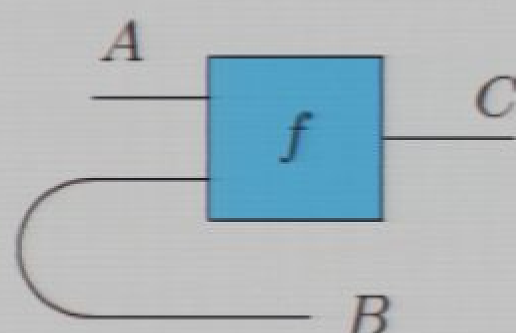
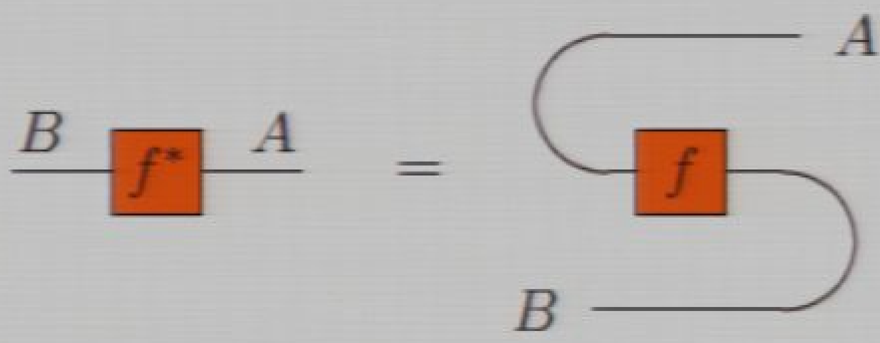
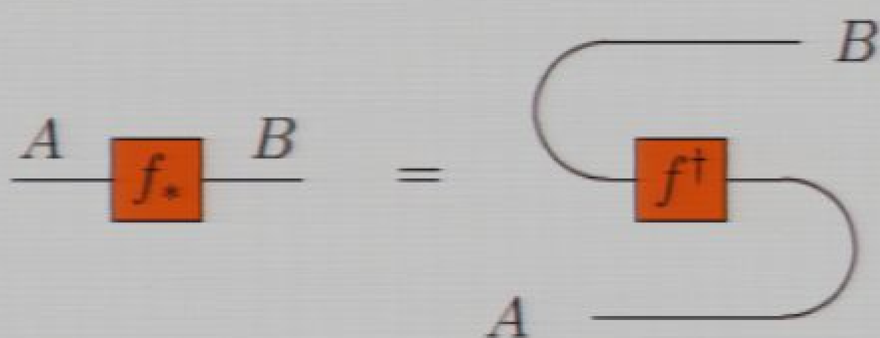


Upper and lower star bijections



In QM, *transposition*. In the toy theory, *relational converse*.

And we can define $f_* = (f^\dagger)^*$:



In QM, *complex conjugation*. In the toy theory, *identity*.

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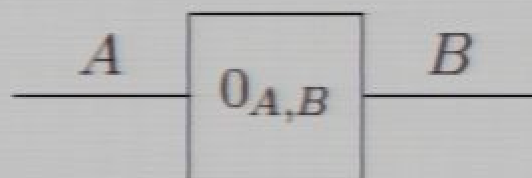

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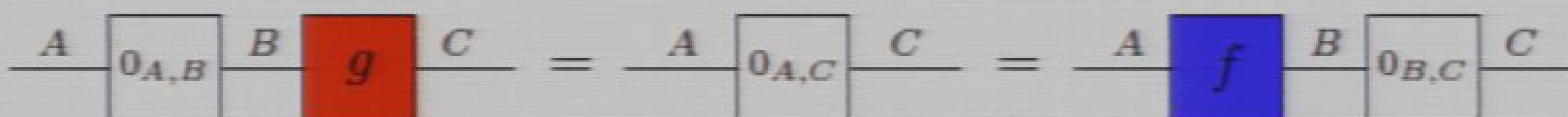

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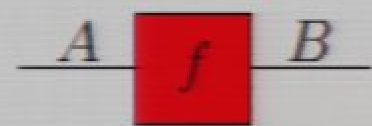

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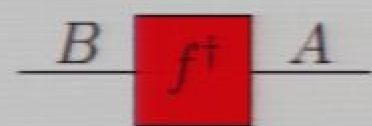

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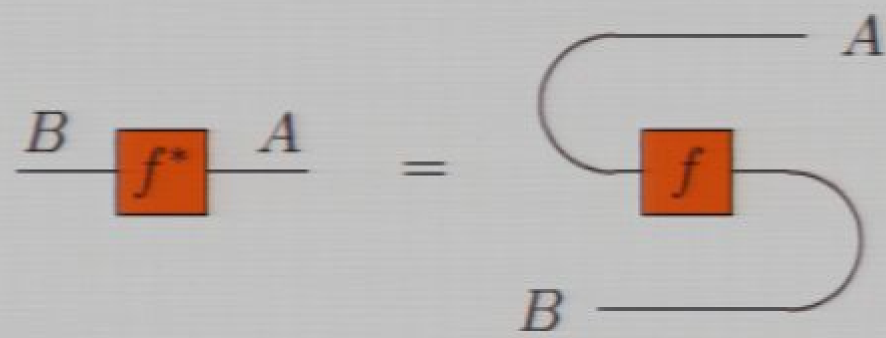

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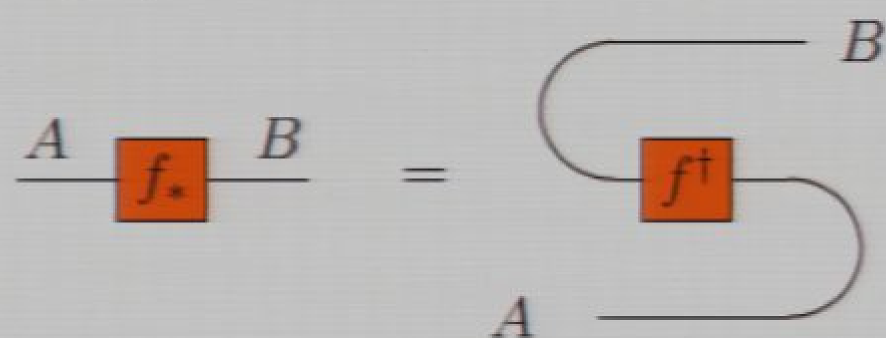

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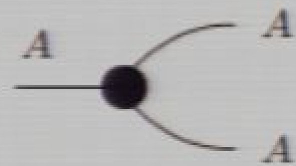
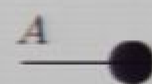
Basis structures - definition

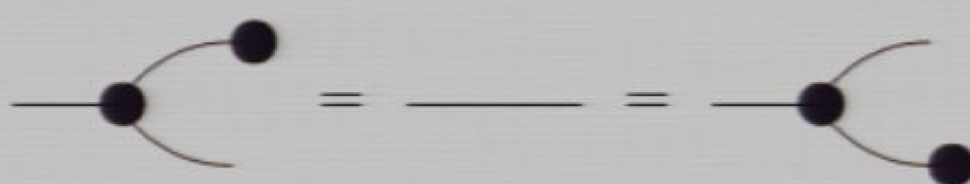
A basis structure on an object A consists of a pair of operations

δ 

ϵ 

satisfying the following five conditions:

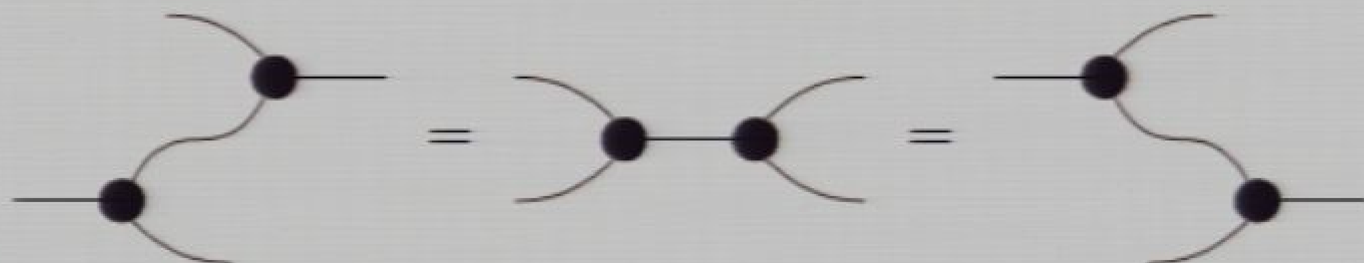
(Coassociativity) (1)



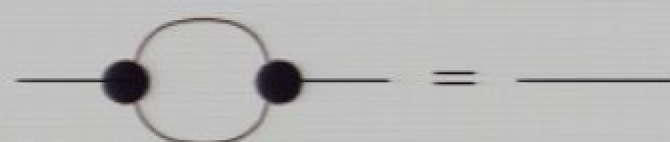
(Counit) (2)

Basis structures - definition (continued)

(Cocommutativity) (3)



(Frobenius) (4)



(Speciality) (5)

where

$\delta^\dagger$  $\epsilon^\dagger$ 

Basis structures in QM

In quantum mechanics there is a bijective correspondence between basis structures and orthonormal bases. Explicitly:

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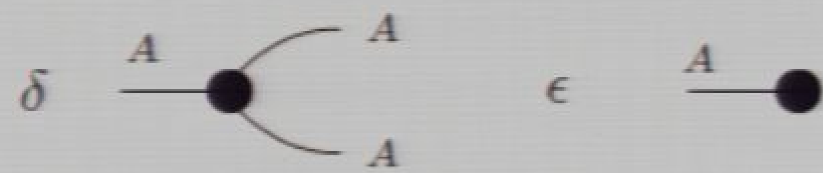

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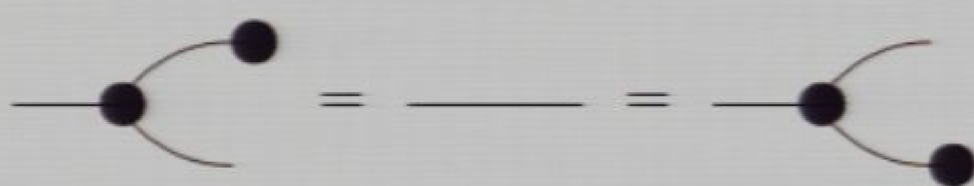

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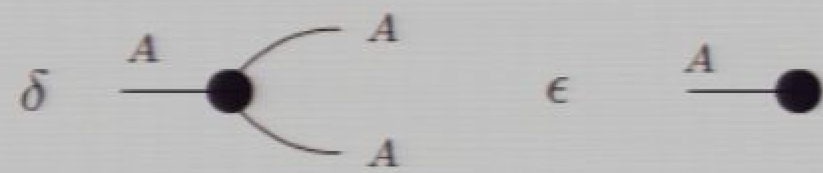

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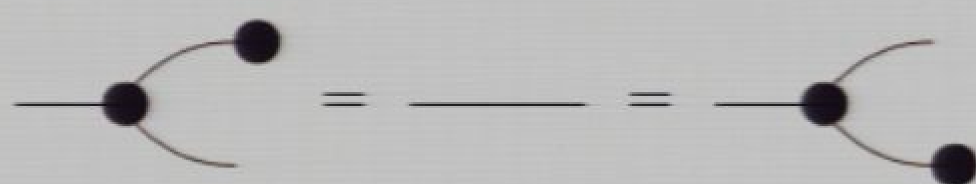

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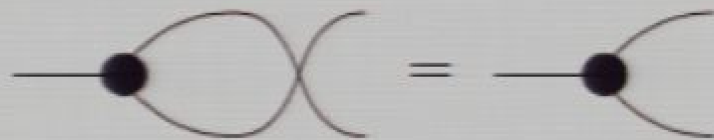

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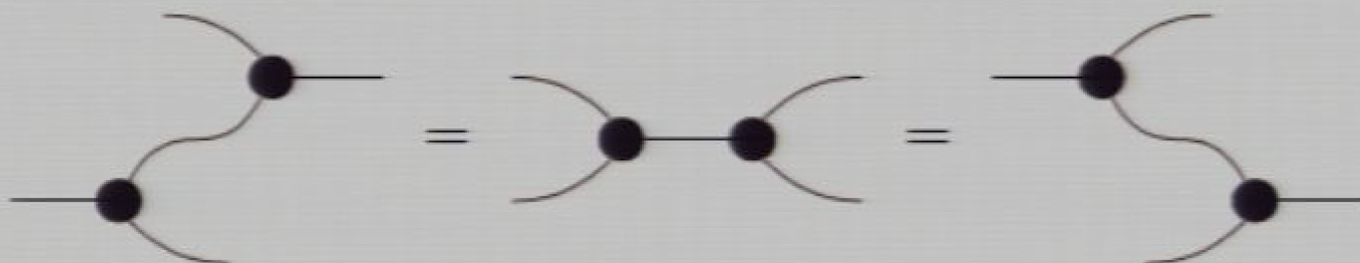

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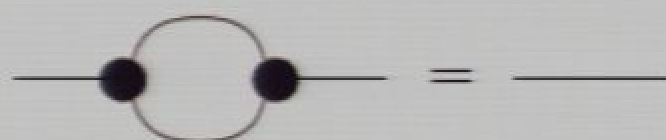

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An example of a basis structure in the two theory. We will label the four ontic states of a single elementary system simply as $1, 2, 3$ and 4. Then the following two relations constitute a basis structure:

$$\begin{aligned} \delta :: \quad & 1 \sim \{(1,1),(2,2)\} \\ & 2 \sim \{(1,2),(2,1)\} \\ & 3 \sim \{(3,3),(4,4)\} \\ & 4 \sim \{(3,4),(4,3)\} \end{aligned}$$

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It's not obvious, but δ copies the following two states:

Qubit stabiliser theory

Systems: *Qubits*
States: *Stabiliser states*
Processes: *Clifford operations*
Observables: *Pauli group*

1 qubit states:

- $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |i\rangle, |-i\rangle$

2 qubit states:

- 36 product states e.g. $|0\rangle \otimes |+\rangle$;
- 24 maximally entangled states e.g. $\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$

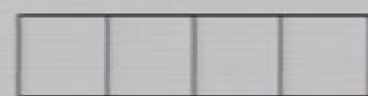
3 qubit states:

- Many more, including GHZ states e.g. $\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)$

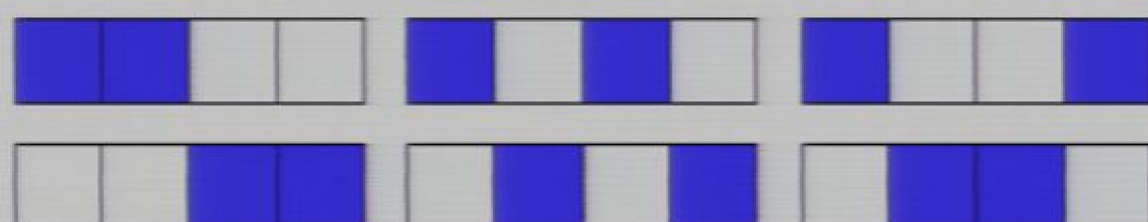
Spekkens's toy bit theory

R. Spekkens. Evidence for the epistemic view of quantum states: A toy theory. *Phys. Rev. A*, 75(032110), 2007.

There is one type of system in the theory, which can exist in one of four *ontic* states.



There are six possible states of maximal knowledge, consistent with the knowledge balance principle. These are called *epistemic* states.

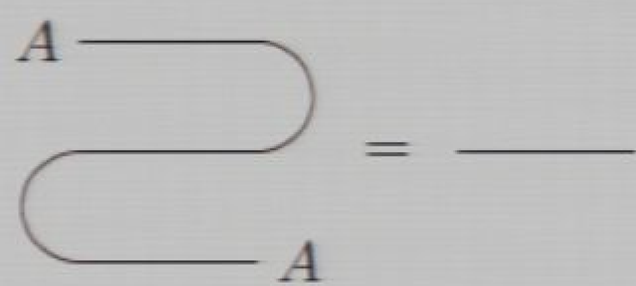


Epistemic states described by *subsets*.

Compact structures

A system A has a *compact structure* if there exist a state and co-state:

Which satisfy the following property:



In QM, every system has such a state and co-state. The state is the Bell state.

Basis structures in QM

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We term the states which are copied by δ *eigenstates.*

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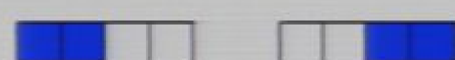

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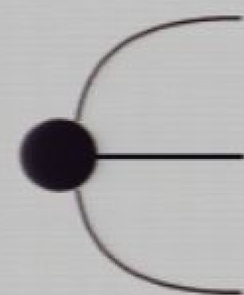
Abstract GHZ states

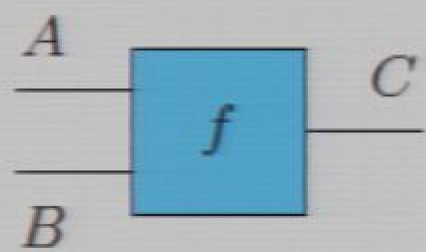
We can bend around the input line to get a diagram with three outputs, describing preparation of a tripartite state:



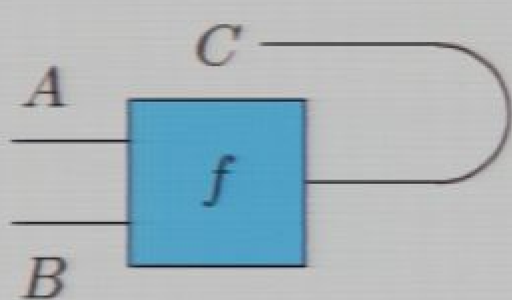
In quantum mechanics, if we began with the δ which copies $|0\rangle$ and $|1\rangle$ then this state is the (un-normalised) GHZ state $|000\rangle + |111\rangle$.

Bijections between different types of process

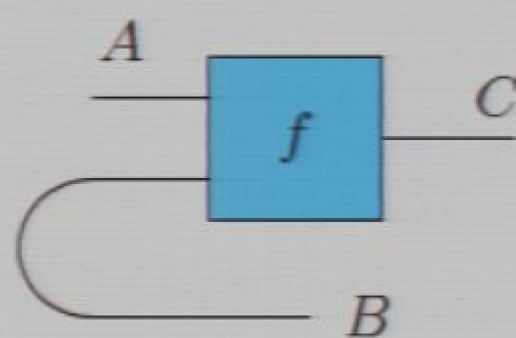
One of the key consequences of compact structure is that it generates a whole series of bijections between processes of different types:
If we start with a process of this type:



We can get other processes of different types, for example:



or



The dagger operation

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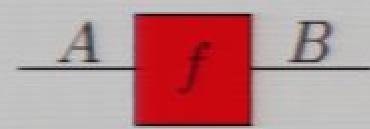

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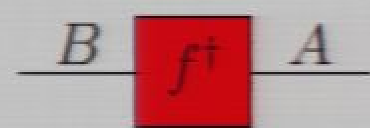

In QM (and stabiliser theory) corresponds to the *adjoint*. In toy theory corresponds to *relational converse*.

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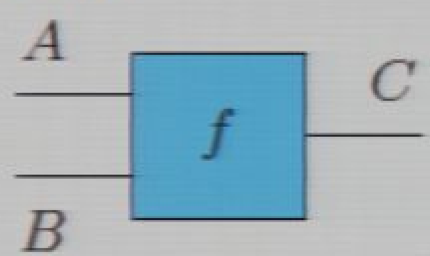

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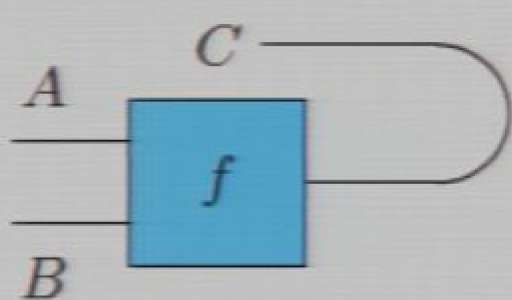

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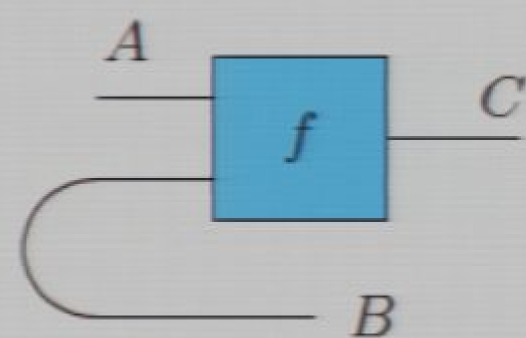

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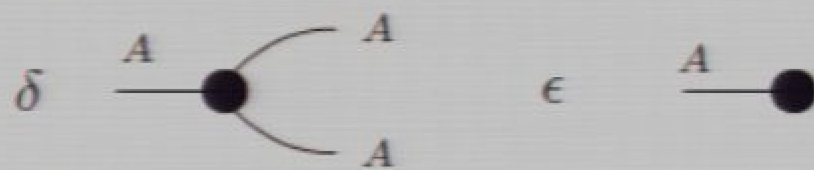

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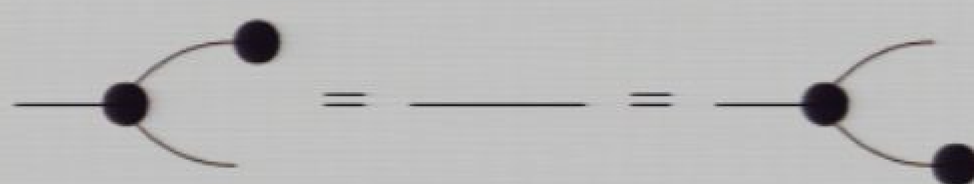

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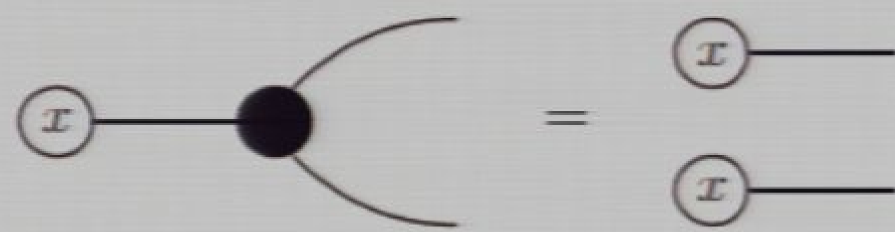

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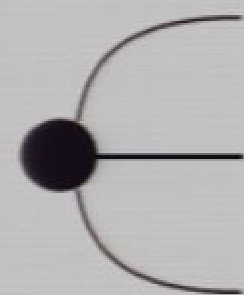

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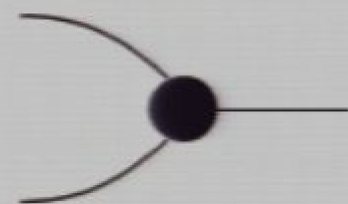

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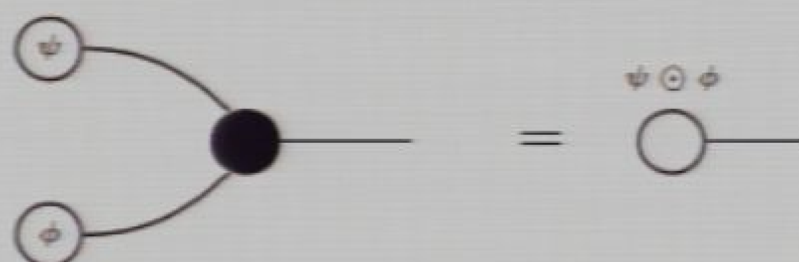

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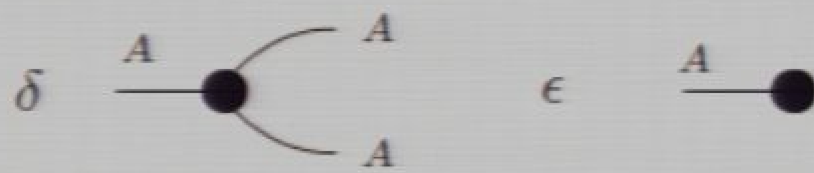

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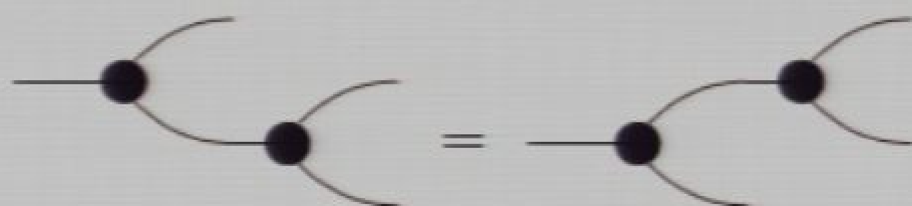

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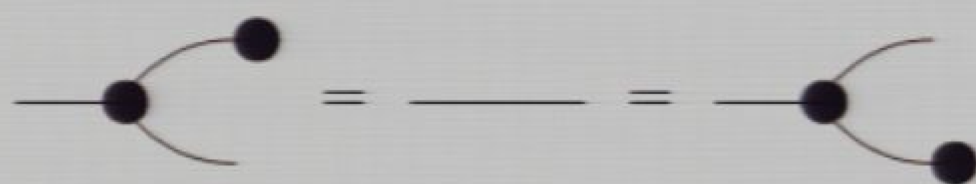

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Basis structures - definition (continued)

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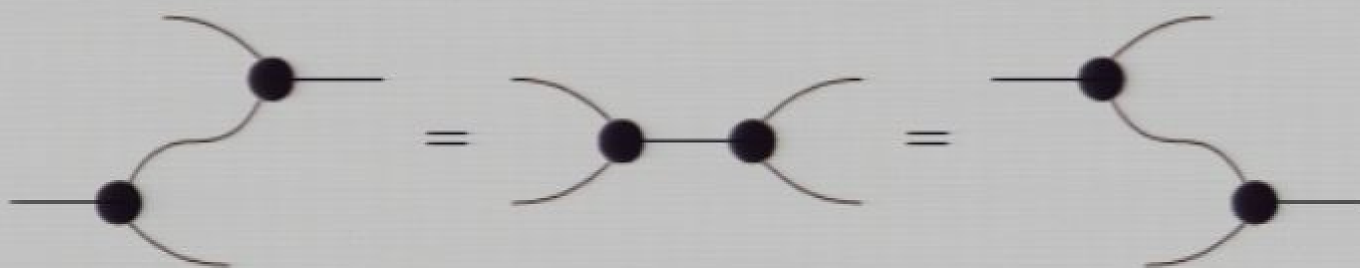

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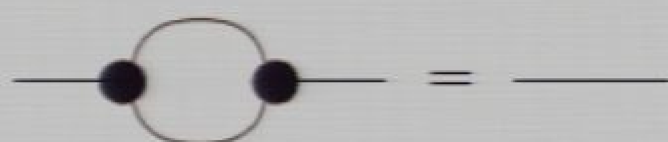

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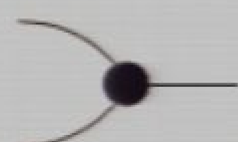
$\delta^\dagger$  $\epsilon^\dagger$ 

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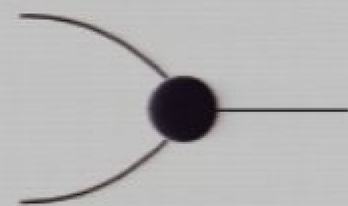

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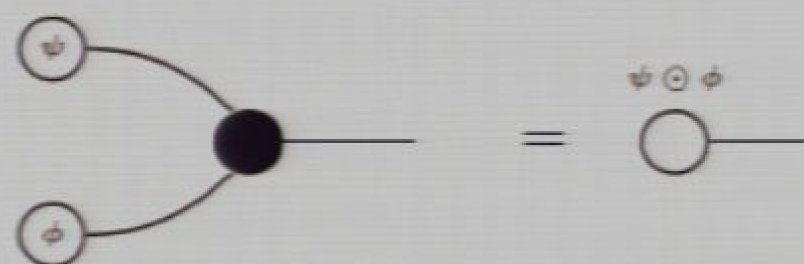

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If we express $|\psi\rangle$ and $|\phi\rangle$ in terms of the basis *which is copied by δ*:

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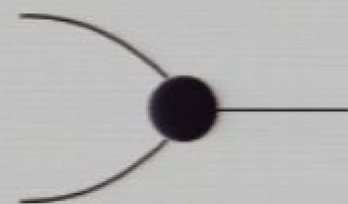

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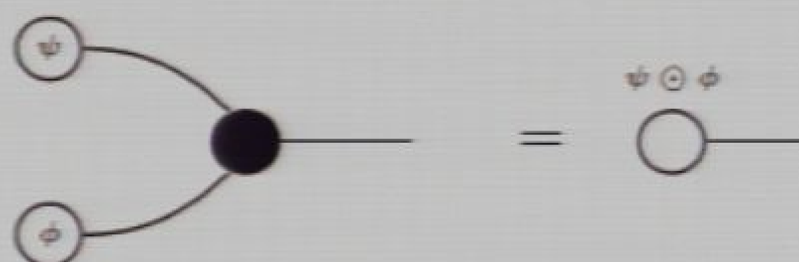

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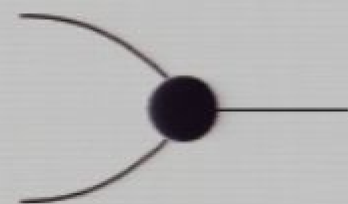

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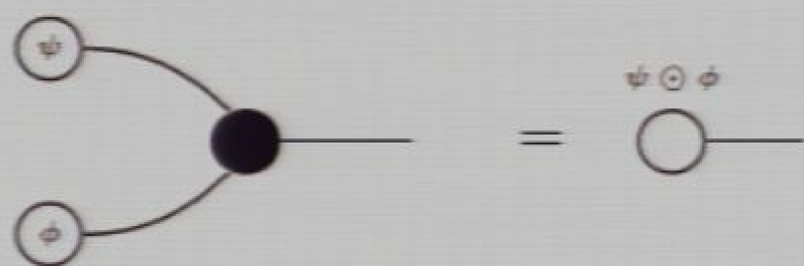

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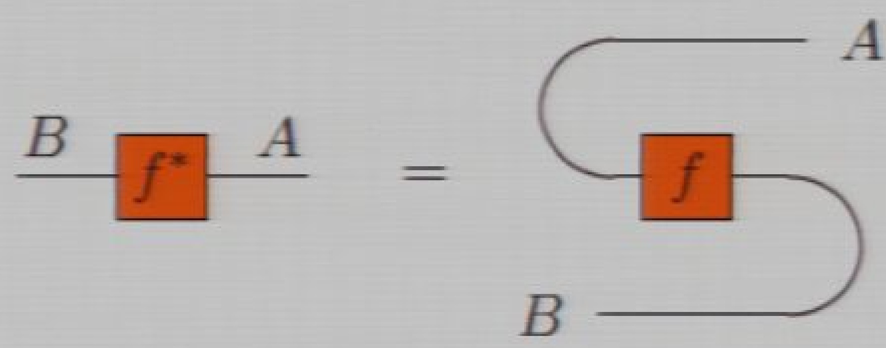
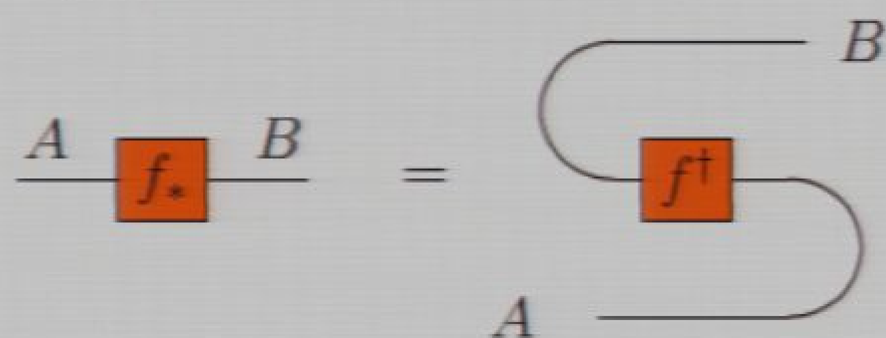
Upper and lower star bijections



In QM, *transposition*. In the toy theory, *relational converse*.

And we can define $f_* = (f^\dagger)^*$:



In QM, *complex conjugation*. In the toy theory, *identity*.

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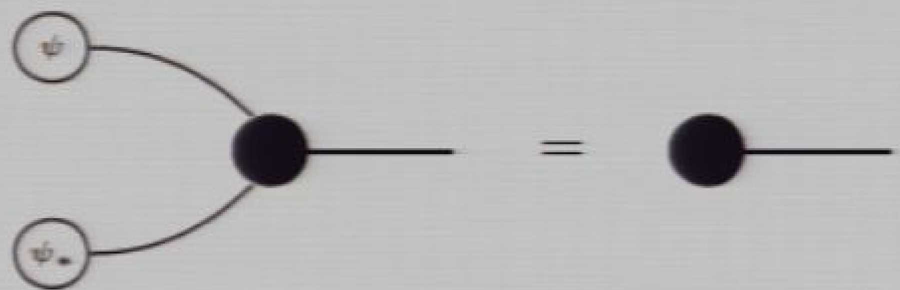
Unbiased states

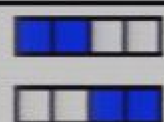
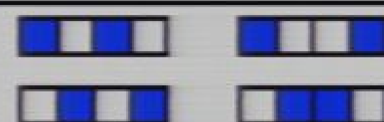
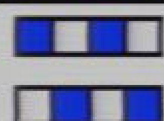
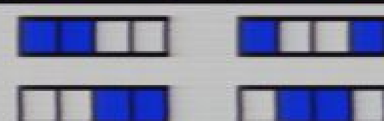
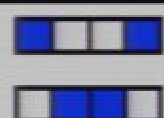
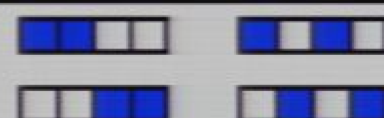
Inspired by this we make the following definition in the general categorical setting. A state ψ is *unbiased* with respect to a basis structure if:



We can show that under the action of the basis structure monoid, these states are closed. Thus they form an Abelian sub-group of the monoid, which we refer to as the *phase group*.

Basis structures in Stab and Spek

	Eigenstates	Unbiased	Phase group
δ_Z	$\lvert 0\rangle, \lvert 1\rangle$	$\lvert +\rangle, \lvert -\rangle, \lvert i\rangle, \lvert -i\rangle$	
δ_X	$\lvert +\rangle, \lvert -\rangle$	$\lvert 0\rangle, \lvert 1\rangle, \lvert i\rangle, \lvert -i\rangle$	Z_4
δ_Y	$\lvert i\rangle, \lvert -i\rangle$	$\lvert +\rangle, \lvert -\rangle, \lvert 0\rangle, \lvert 1\rangle$	

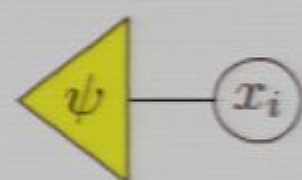
δ_Z			
δ_X			$Z_2 \times Z_2$
δ_Y			

Description of observables in this approach

In both quantum mechanics and the toy theory, we observe the following correspondence:

- Observable — Basis structure
- Outcome of measurement — Eigenstate

Furthermore, if we prepare a system in state ψ, the probability of obtaining a measurement outcome corresponding to an eigenstate x_i is some function of this scalar:



Specifically, if this scalar is equal to the zero scalar, then the probability is zero i.e. this outcome is impossible.

Basis structures in Stab and Spek

	Eigenstates	Unbiased	Phase group
δ_Z	$\|0\rangle, \|1\rangle$	$\|+\rangle, \|-\rangle, \|i\rangle, \|-i\rangle$	Z_4
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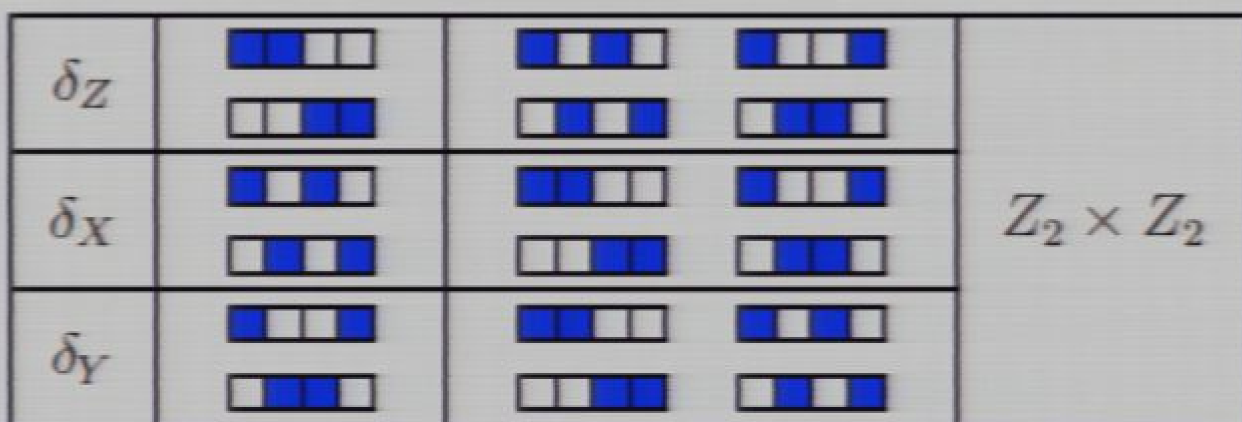
δ_Y	$\|i\rangle, \|-i\rangle$	$\|+\rangle, \|-\rangle, \|0\rangle, \|1\rangle$	



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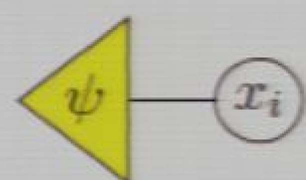
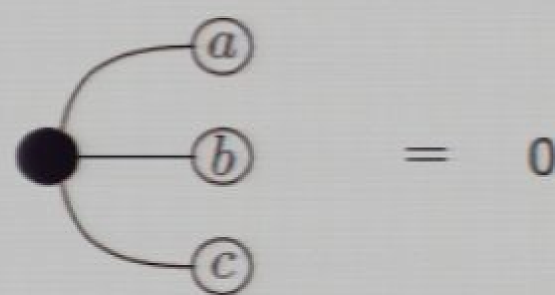


Specifically, if this scalar is equal to the zero scalar, then the probability is zero i.e. this outcome is impossible.

Forbidden triples

We will be particularly interested in triples of outcomes of measurements applied to abstract GHZ states for which the corresponding scalar is the zero scalar.



where a, b and c are each an eigenstate of some basis structure (not necessarily the same one).

(a, b, c) is a *forbidden triple.*

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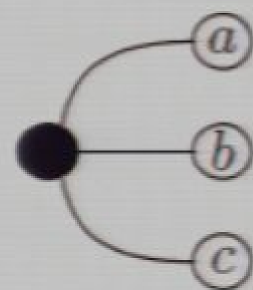

$$= \quad 0$$

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Basis structures in Stab and Spek

	Eigenstates	Unbiased	Phase group
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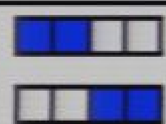
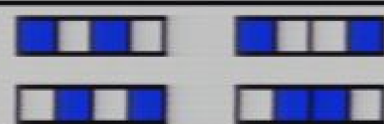
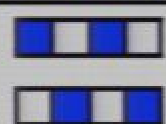
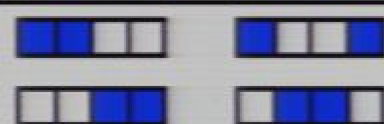
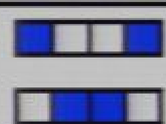
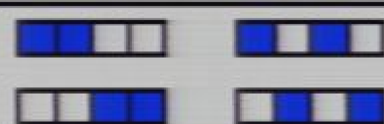
δ_Z			$Z_2 \times Z_2$
δ_X			
δ_Y			

Forbidden triples

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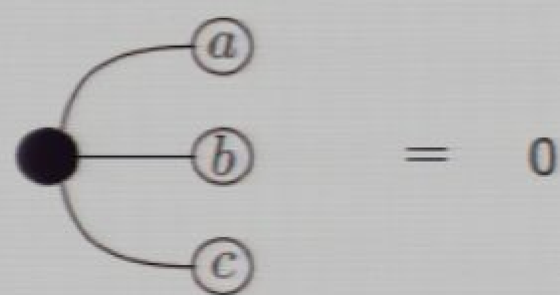
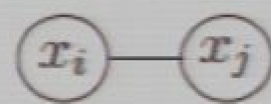


where a, b and c are each an eigenstate of some basis structure (not necessarily the same one).

(a, b, c) is a *forbidden triple*.

Some key results (without proof)

1. For two eigenstates x_i and x_j of the same basis structure, the following state-outcome scalar is an idempotent.

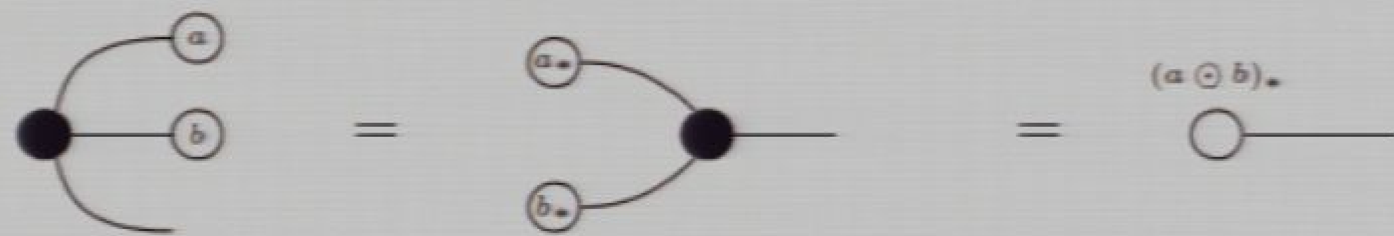


2. If this scalar is the identity scalar, then $x_i = x_j$.
3. Under certain reasonable assumptions we can show that the only idempotent scalars in the category of a quantum-like theory are the identity and the zero scalar.

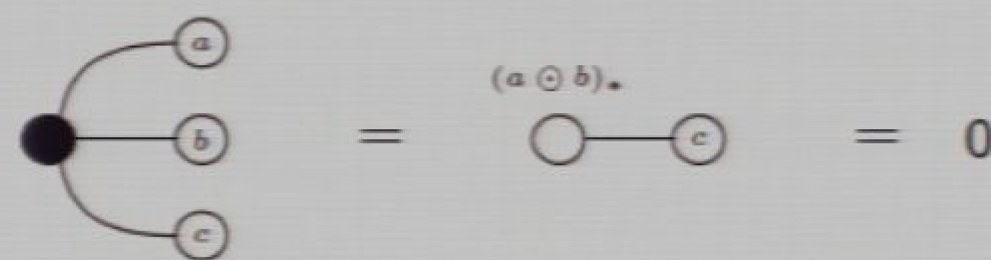
Thus we conclude that if $x_i \neq x_j$, then:

$= 0$

Monoid determines allowed and forbidden triples

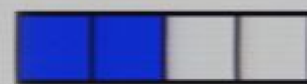
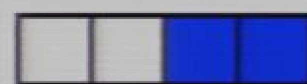
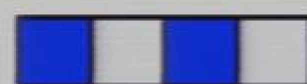
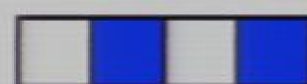
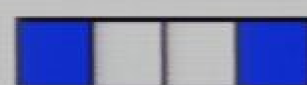


Now suppose c and $(a \odot b)_*$ are both eigenstates of some (possibly different) basis structure.



$(a, b, (a \odot b)_*)$ is an allowed triple. (a, b, c) is a forbidden triple.

Labelling of states

	Stab	Spek
z_0	$\|0\rangle$	
z_1	$\|1\rangle$	
x_0	$\|+\rangle$	
x_1	$\|-\rangle$	
y_0	$\|i\rangle$	
y_1	$\|-i\rangle$	

Phase group tables

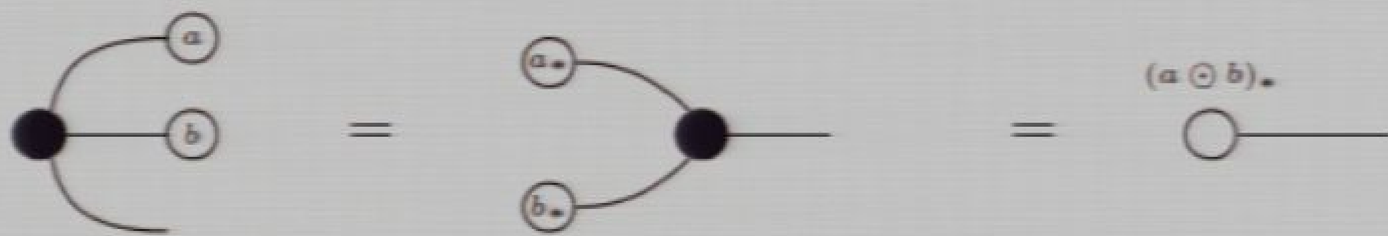
Spek

$Z_2 \times Z_2$	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

Stab

Z_4	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_1	x_0
y_1	y_1	y_0	x_0	x_1

Monoid determines allowed and forbidden triples



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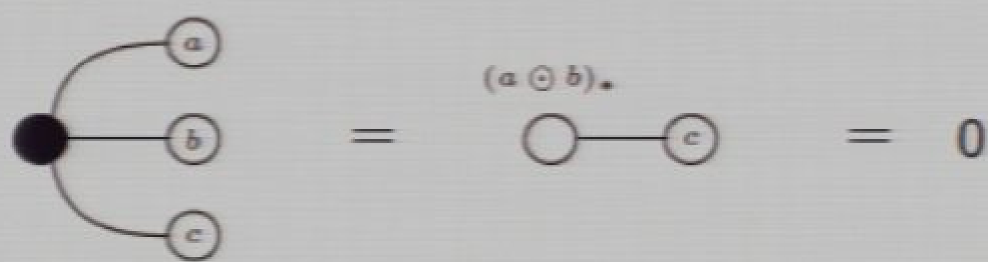

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y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

Stab

Z_4	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_1	x_0
y_1	y_1	y_0	x_0	x_1

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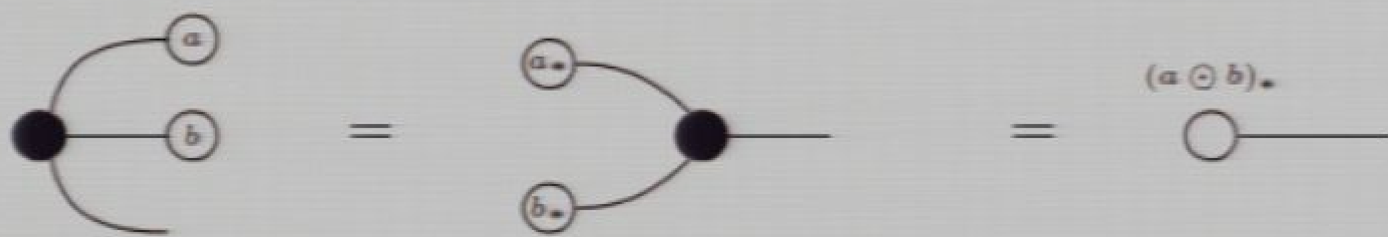

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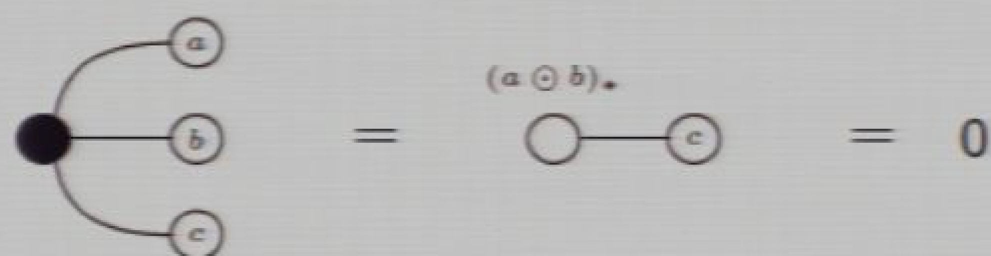

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Phase group tables

Spek

$Z_2 \times Z_2$	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

Stab

Z_4	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_1	x_0
y_1	y_1	y_0	x_0	x_1

Allowed triple tables

Spek allowed triples

'$Z_2 \times Z_2$'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

Stab allowed triples

'Z_4'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_1	y_0
x_1	x_1	x_0	y_0	y_1
y_0	y_1	y_0	x_1	x_0
y_1	y_0	y_1	x_0	x_1

Since allowed triples are of form $(a, b, (a \odot b)_*)$.

Subgroup and cosets

Spek allowed triples

'$Z_2 \times Z_2$'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

XXX XYY

YXY YYX

Stab allowed triples

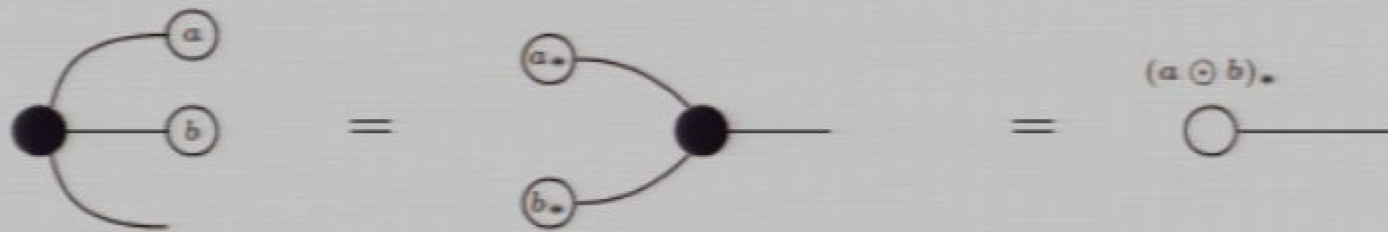
'Z_4'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_1	y_0
x_1	x_1	x_0	y_0	y_1
y_0	y_1	y_0	x_1	x_0
y_1	y_0	y_1	x_0	x_1

XXX XYY

YXY YYX

For the four triples of observables thus singled out, any outcome triple which doesn't appear in the table is forbidden $\implies$ phase group gives complete information on their allowed/forbidden triples.

Monoid determines allowed and forbidden triples



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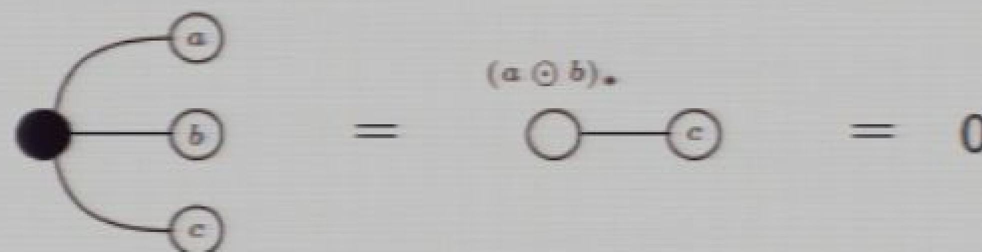

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Subgroup and cosets

Spek allowed triples

'$Z_2 \times Z_2$'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

Stab allowed triples

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Mermin table

Write out these special four triples of observables in rows:

$$\begin{array}{ccc} X_1 & X_2 & X_3 \\ X_1 & Y_2 & Y_3 \\ Y_1 & X_2 & Y_3 \\ Y_1 & Y_2 & X_3 \end{array}$$

A possible assignment of values to the observables?

$$\begin{array}{ccc} X_1 = x_0 & X_2 = x_0 & X_3 = x_0 \\ X_1 = x_0 & Y_2 = y_0 & Y_3 = y_1 \\ Y_1 = y_1 & X_2 = x_0 & Y_3 = y_1 \\ Y_1 = y_1 & Y_2 = y_0 & X_3 = x_0 \end{array}$$

Subgroup and cosets

Spek allowed triples

'Z_2 × Z_2'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

Stab allowed triples

'Z_4'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_1	y_0
x_1	x_1	x_0	y_0	y_1
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Allowed triple parities

Spek parities

$`Z_2 \times Z_2`$	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

	0	0
	0	0

Stab parities

$`Z_4`$	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_1	y_0
x_1	x_1	x_0	y_0	y_1
y_0	y_1	y_0	x_1	x_0
y_1	y_0	y_1	x_0	x_1

	0	1
	1	1

No-go proof for Stab

Calculate the parity of the whole table either by
(i) Calculating row parities (ii) Calculating column parities

Row parities can be read off from allowed triple table.

$$\begin{array}{cccc} X_1 & X_2 & X_3 & 0 \\ X_1 & Y_2 & Y_3 & 1 \\ Y_1 & X_2 & Y_3 & 1 \\ Y_1 & Y_2 & X_3 & 1 \\ 0 & 0 & 0 & \end{array}$$

No-go proof for Spek?

In the case of **Spek**, we can no longer derive a contradiction: the parities for the rows don't allow it.

$$\begin{array}{cccc} X_1 & X_2 & X_3 & 0 \\ X_1 & Y_2 & Y_3 & 0 \\ Y_1 & X_2 & Y_3 & 0 \\ Y_1 & Y_2 & X_3 & 0 \\ 0 & 0 & 0 & \end{array}$$

Should expect this, since the toy theory is a local hidden variable theory.

Note that the origin in this difference re. locality lies in the difference between the phase group tables.

General phase groups?

- If observable coset condition applies, we can generalise the Mermin table.
- We can generalise the notion of parity in such a way that every row of the generalised table has a well-defined generalised parity.
- Argument about column parities can be extended in *some* cases.

Whether or not a phase group passes or fails the Mermin table test seems to be related to the *group extension problem*.

"Given G_1, G_2 find G such that G_1 is a normal subgroup of G, and $G/G_1 \cong G_2$."

If $G_1, G_2 = Z_2$, then both Z_4 and $Z_2 \times Z_2$ are valid group extensions.

Subgroup and cosets

Spek allowed triples

'$Z_2 \times Z_2$'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_0	y_1
x_1	x_1	x_0	y_1	y_0
y_0	y_0	y_1	x_0	x_1
y_1	y_1	y_0	x_1	x_0

Stab allowed triples

'Z_4'	x_0	x_1	y_0	y_1
x_0	x_0	x_1	y_1	y_0
x_1	x_1	x_0	y_0	y_1
y_0	y_1	y_0	x_1	x_0
y_1	y_0	y_1	x_0	x_1

For the four triples of observables thus singled out, any outcome triple which doesn't appear in the table is forbidden $\Longrightarrow$ phase group gives complete information on their allowed/forbidden triples.

General phase groups?

- If observable coset condition applies, we can generalise the Mermin table.
- We can generalise the notion of parity in such a way that every row of the generalised table has a well-defined generalised parity.
- Argument about column parities can be extended in *some* cases.

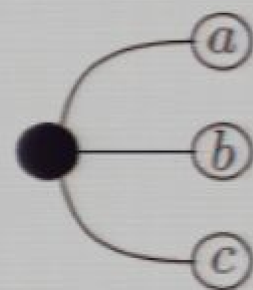
Whether or not a phase group passes or fails the Mermin table test seems to be related to the *group extension problem*.

“Given G_1, G_2 find G such that G_1 is a normal subgroup of G, and $G/G_1 \cong G_2$.”

If $G_1, G_2 = Z_2$, then both Z_4 and $Z_2 \times Z_2$ are valid group extensions.

Forbidden triples

We will be particularly interested in triples of outcomes of measurements applied to abstract GHZ states for which the corresponding scalar is the zero scalar.



$$= \quad 0$$

where a, b and c are each an eigenstate of some basis structure (not necessarily the same one).

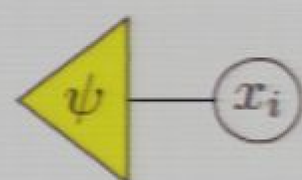
(a, b, c) is a *forbidden triple.*

Description of observables in this approach

In both quantum mechanics and the toy theory, we observe the following correspondence:

- Observable — Basis structure
- Outcome of measurement — Eigenstate

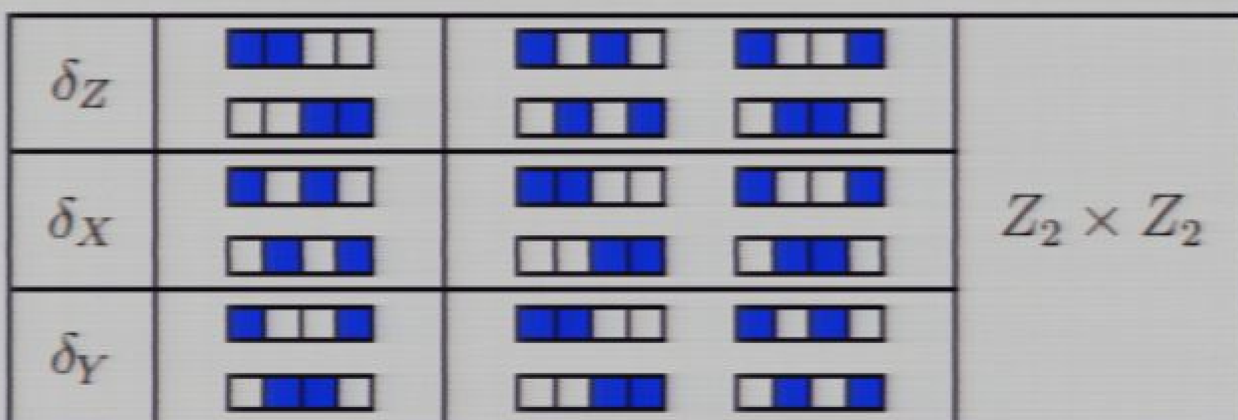
Furthermore, if we prepare a system in state ψ, the probability of obtaining a measurement outcome corresponding to an eigenstate x_i is some function of this scalar:



Specifically, if this scalar is equal to the zero scalar, then the probability is zero i.e. this outcome is impossible.

Basis structures in Stab and Spek

	Eigenstates	Unbiased	Phase group
δ_Z	$\|0\rangle, \|1\rangle$	$\|+\rangle, \|-\rangle, \|i\rangle, \|-i\rangle$	Z_4
δ_X	$\|+\rangle, \|-\rangle$	$\|0\rangle, \|1\rangle, \|i\rangle, \|-i\rangle$	
δ_Y	$\|i\rangle, \|-i\rangle$	$\|+\rangle, \|-\rangle, \|0\rangle, \|1\rangle$	



Basis structures in QM

In quantum mechanics there is a bijective correspondence between basis structures and orthonormal bases. Explicitly:

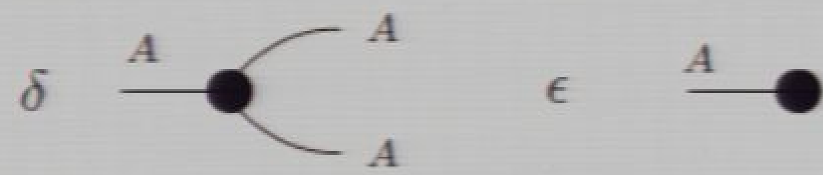
$$\delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |i\rangle \otimes |i\rangle \qquad \epsilon : \mathcal{H} \to \mathbb{C} :: |i\rangle \mapsto 1$$

For example, the qubit in stabiliser theory has three basis structures:

- ☐ δ_Z copies $|0\rangle$ and $|1\rangle$.
- ☐ δ_X copies $|+\rangle$ and $|-\rangle$.
- ☐ δ_Y copies $|i\rangle$ and $|-i\rangle$.

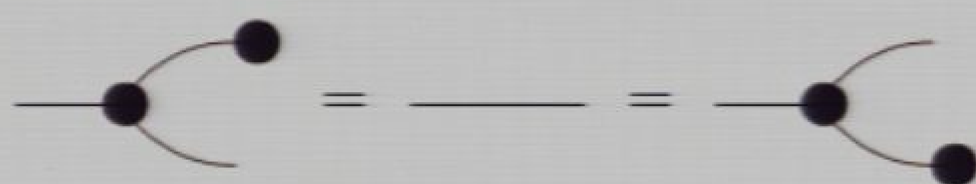
Basis structures - definition

A basis structure on an object A consists of a pair of operations



satisfying the following five conditions:

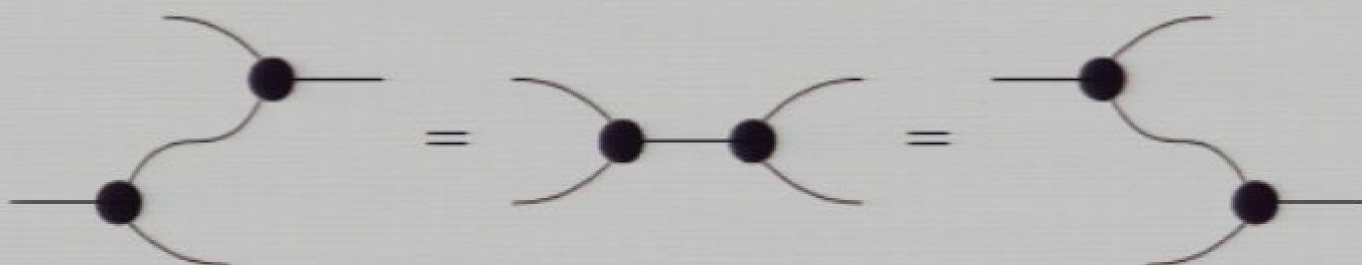
(Coassociativity) (1)



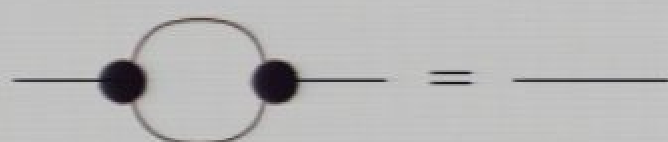
(Counit) (2)

Basis structures - definition (continued)

(Cocommutativity) (3)



(Frobenius) (4)



(Speciality) (5)

where

$\delta^\dagger$  $\epsilon^\dagger$ 

Basis structure monoid in quantum mechanics

If we express $|\psi\rangle$ and $|\phi\rangle$ in terms of the basis *which is copied by* δ:

$$|\psi\rangle = (\psi_1, \psi_2, \ldots, \psi_n), \quad |\phi\rangle = (\phi_1, \phi_2, \ldots, \phi_n)$$

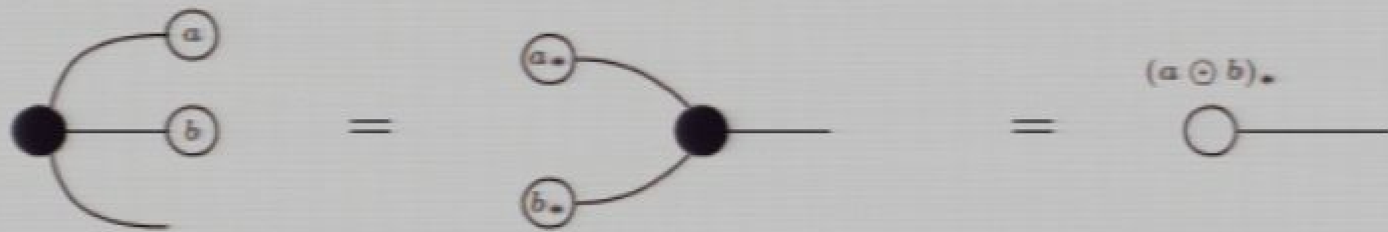
then $|\psi \odot \phi\rangle$ is written as:

$$|\psi \odot \phi\rangle = (\psi_1.\phi_1, \psi_2.\phi_2, \ldots, \psi_n.\phi_n)$$

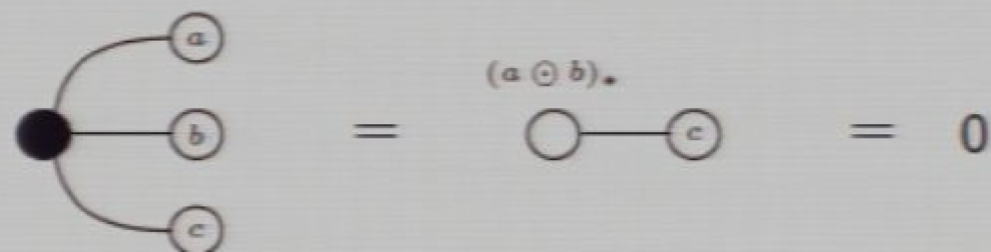
Recalling that $|\psi_*\rangle = (\bar{\psi_1}, \bar{\psi_2}, \ldots, \bar{\psi_n})$, and neglecting normalisation, we note that if $|\psi\rangle$ is *unbiased* with respect to this basis then:

$$|\psi \odot \psi_*\rangle = (|\psi_1|^2, |\psi_2|^2, \ldots, |\psi_n|^2) = (1, 1, \ldots, 1) = \epsilon^\dagger$$

Monoid determines allowed and forbidden triples



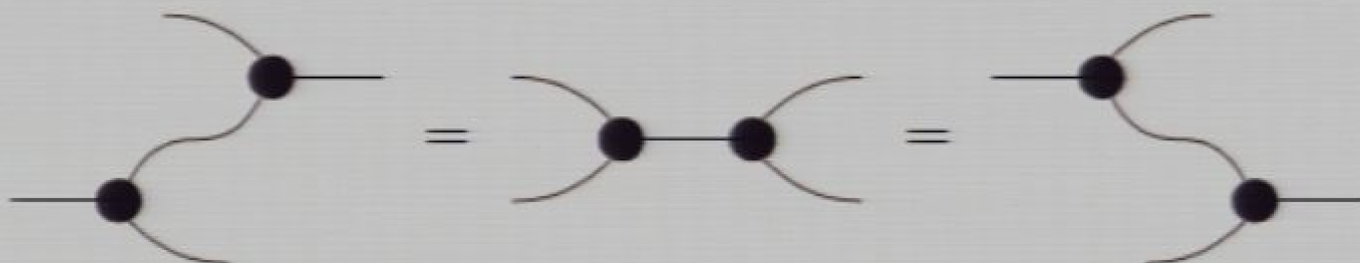
Now suppose c and $(a \odot b)_*$ are both eigenstates of some (possibly different) basis structure.



$(a, b, (a \odot b)_*)$ is an allowed triple. (a, b, c) is a forbidden triple.

Basis structures - definition (continued)

(Cocommutativity) (3)



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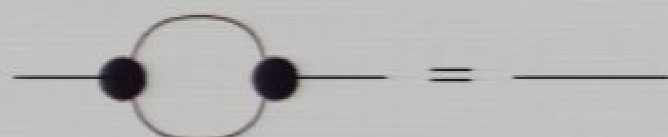

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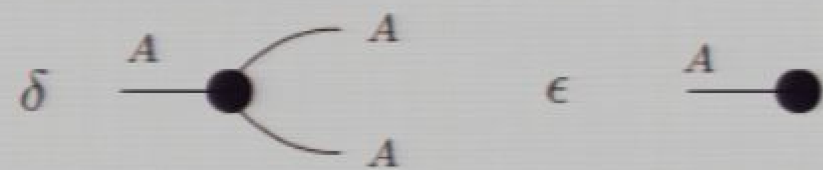


satisfying the following five conditions:

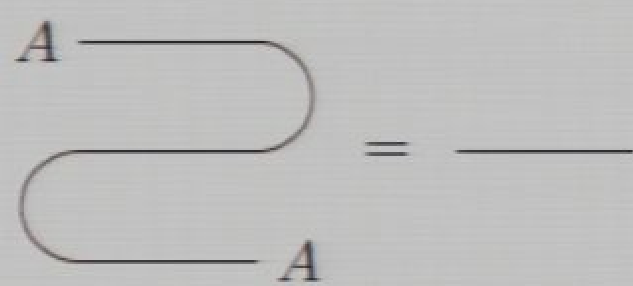
(Coassociativity) (1)

(Counit) (2)

Compact structures

A system A has a *compact structure* if there exist a state and co-state:

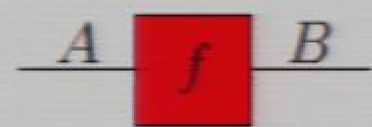
Which satisfy the following property:



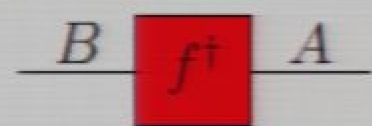
In QM, every system has such a state and co-state. The state is the Bell state.

The dagger operation

Bijection between processes of this type:



and this type:



In QM (and stabiliser theory) corresponds to the *adjoint*. In toy theory corresponds to *relational converse*.

Our categories

Quantum mechanics: **FHilb** (already well known to mathematicians).

Stabiliser theory: **Stab**

Toy bit theory: **Spek**

Will not describe structure in detail.

Qubit stabiliser theory

Systems: *Qubits*
States: *Stabiliser states*
Processes: *Clifford operations*
Observables: *Pauli group*

1 qubit states:

- $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |i\rangle, |-i\rangle$

2 qubit states:

- 36 product states e.g. $|0\rangle \otimes |+\rangle$;
- 24 maximally entangled states e.g. $\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$

3 qubit states:

- Many more, including GHZ states e.g. $\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle)$

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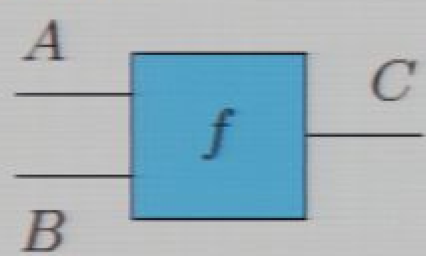
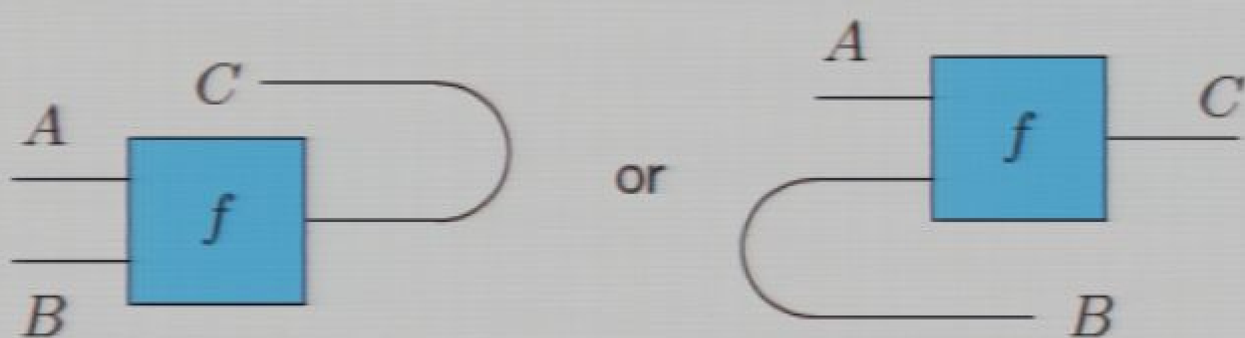
Bijections between different types of process

One of the key consequences of compact structure is that it generates a whole series of bijections between processes of different types:
If we start with a process of this type:



We can get other processes of different types, for example:



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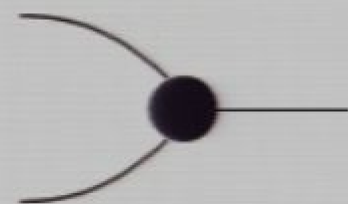
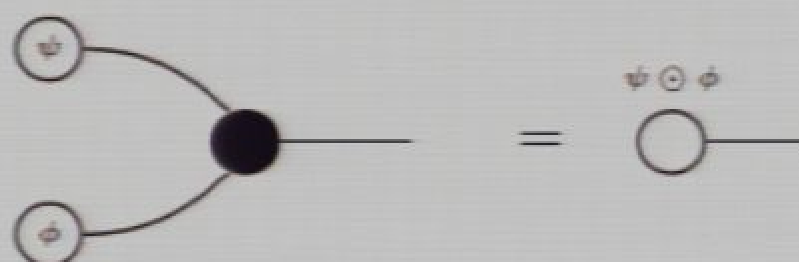
Basis structure monoid

Now consider the action of $\delta^\dagger$:



Now consider plugging states into the inputs



From the axioms defining a basis structure this turns out to be a commutative monoid.

Basis structure monoid in quantum mechanics

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