Title: Coexistence of qubit observables

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Abstract: Quantum mechanics does not allow us to measure all possible combinations of observables on one system. Even in the simplest case of two observables we know, that measuring one of the observables changes the system in such way, that the other measurement will not give us desired precise information about the state of the system.

Prominent examples of such observables are measurement of position and momentum of a particle, or measuring spin along two orthogonal directions. However, once we accept the possibility of imprecise measurements, we can consider to perform such measurement within one experiment. This is the basis of the notion of coexistence. I will present the basics of coexistence by showing how to perform the spin measurement in two directions, while considering the imprecision of the measurement described by POVMs. We can also go a little further and consider coexistence of instruments, i.e. measurements, where on the output besides classical information we are also left out with quantum post-measurement state.

Coexistence of Qubit Observables

Daniel Reitzner

Research Center for Quantum Information
Institute of Physics
Slovak Academy of Sciences

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1. Introduction
   - Motivation
   - Generalized Measurements & POVMs

2. Coexistence and Joint Measurability of Simple Qubit Observables
   - Coexistence of Effects
   - Joint Measurability Criterions

3. Coexistence of Instruments
   - Instruments
   - Some Results

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Motivation I

- as a consequence of principles of QM, not every pair of observables can be jointly measurable:
  - position $\hat{x}$ and momentum $\hat{p}$
  - spin components
- *joint measurability (JM):* the ability to perform given measurements with one aparatus (while giving us "correct" probabilities for all states ρ)

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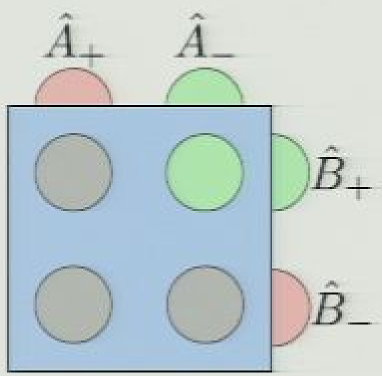

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Motivation I

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- *joint measurability (JM):* the ability to perform given measurements with one aparatus (while giving us "correct" probabilities for all states ρ)
- an imprecision in the measurement can be helpful in allowing us to perform joint measurement (Heisenberg's uncertainty relations)
- knowledge of JM of observables may be helpful in constructing complex measurements — subsequent measurements

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Motivation II

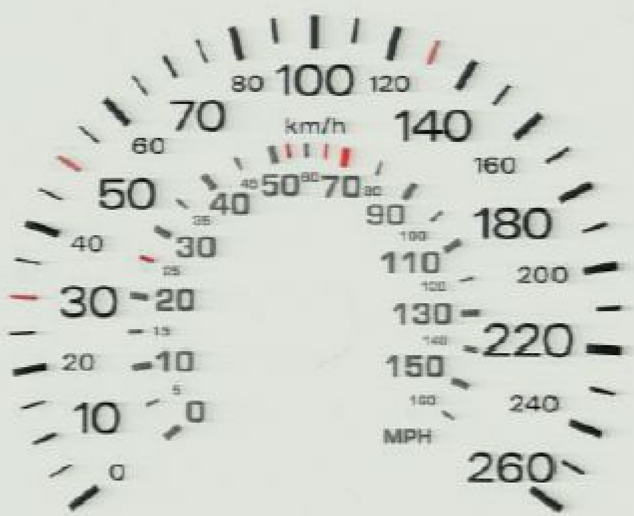
Other practical aspects might be consequences of joint measurability:

- M.M. Wolf *et al.*, Phys. Rev. Lett. **103,** 230402 (2009): every pair of 2-outcome observables that is not jointly measurable enables violation of CHSH Bell inequality
- this has consequences e.g. to security of BB84
- also if the Bell inequalities would be violated and corresponding effects would be coexistent it would imply possibility of superluminal communication
- if effects are not coexistent, it is interesting to study under what approximations this is possible:

P. Busch and T. Heinosaari, Quantum Inf Comp. ***8****, 797 (2008)*
T. Heinosaari, P. Stano and D. Reitzner, International Journal of Quantum Information ***6****, 975 (2008)*
M. Ozawa, Phys. Lett. A ***320****, 367 (2004)*

Standard Observables

- in QM measurements are identified with selfadjoint operators $\hat{A}^{\dagger} = \hat{A}$ — *observables*
- PVMs (projector valued measures) (in finite dimensions) are observables without the scale:
  - spectral decomposition $\hat{A} = \sum_{j\in\Omega} \lambda_j |\mu_j\rangle\langle\mu_j| \equiv \sum_j \lambda_j \hat{P}_j$
  - in measurement result λ_j is obtained with probability $p_j(\rho) = \mathrm{Tr}[\rho \hat{P}_j]$
  - average measured value in state ρ is $\langle \hat{A}\rangle_\rho = \sum_{j\in\Omega} \lambda_j p_j(\rho)$



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  - outcome with index $j \in \Omega$ is mapped to projector $\hat{P}_j$ and such result is obtained with probability $p_j(\rho) = \mathrm{Tr}[\rho \hat{P}_j]$
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Generalized Observables

- generally all probability measures are retrievable by considering the set of effects $\mathcal{E}(\mathcal{H})$ instead of $\mathcal{P}(\mathcal{H})$
- effect $\hat{E}$ is an operator satisfying $\hat{O} \leq \hat{E} \leq \hat{I}$, i.e. $\hat{I} - \hat{E} \geq \hat{0}$

Positive Operator Valued Measures

- POVM is a mapping $\mathsf{E} : 2^{\Omega} \mapsto \mathcal{E}(\mathcal{H})$
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- previous case of projectors is also POVM as $\mathcal{P}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H})$
- POVMs are observables with imprecisions

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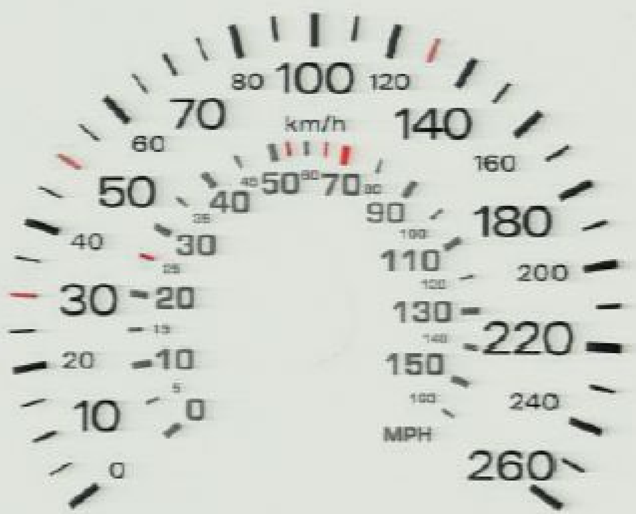

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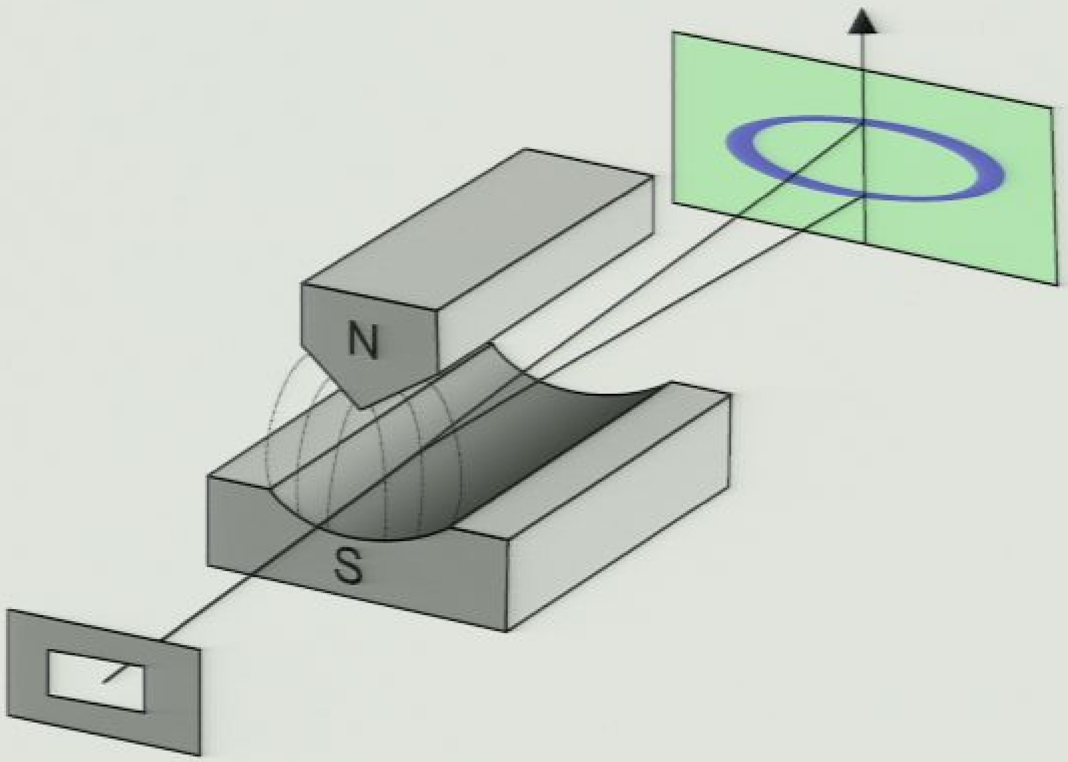
Example: Stern-Gerlach experiment

Setting

Ag — neutral with spin $1/2$ (1922)



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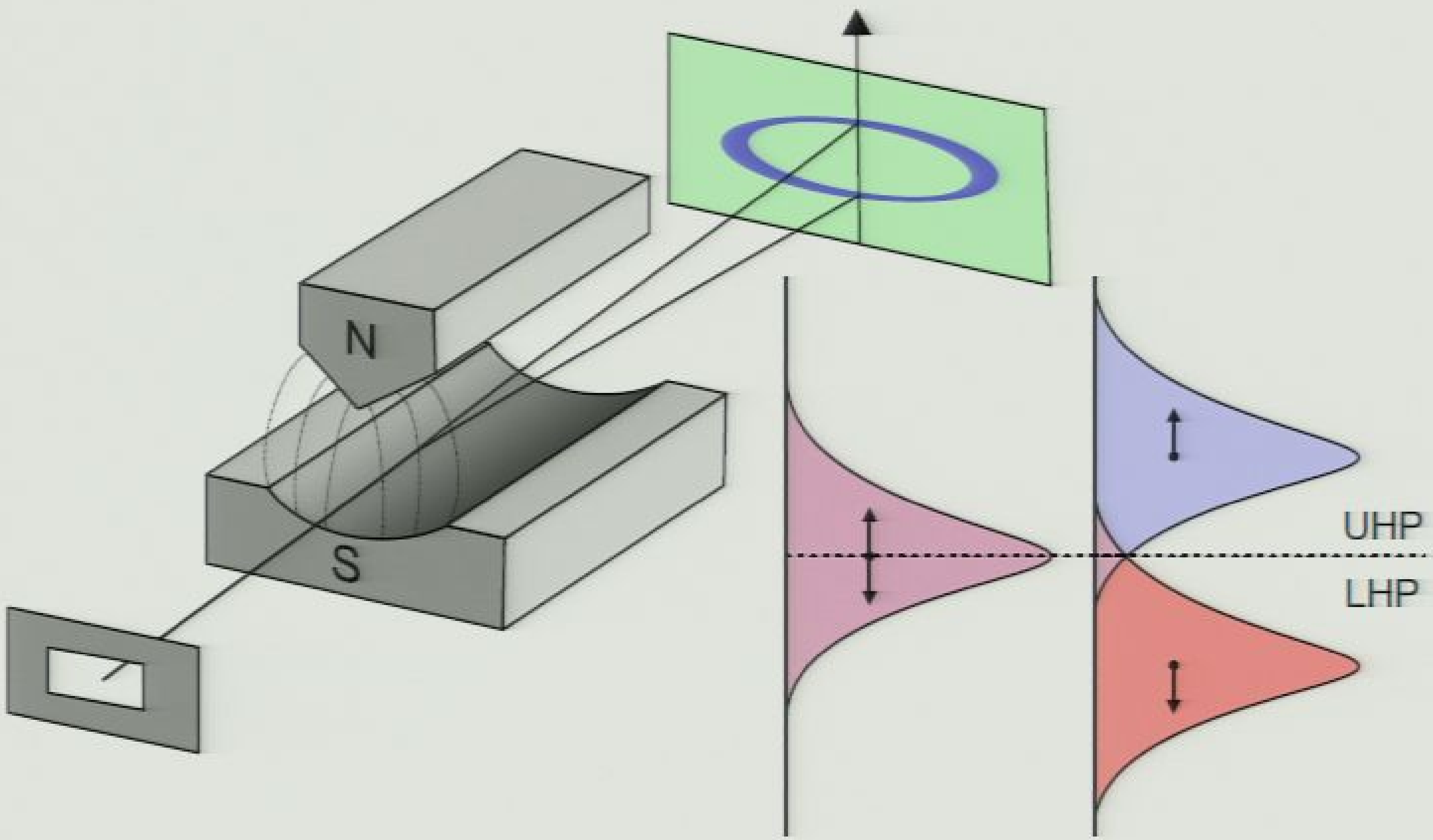

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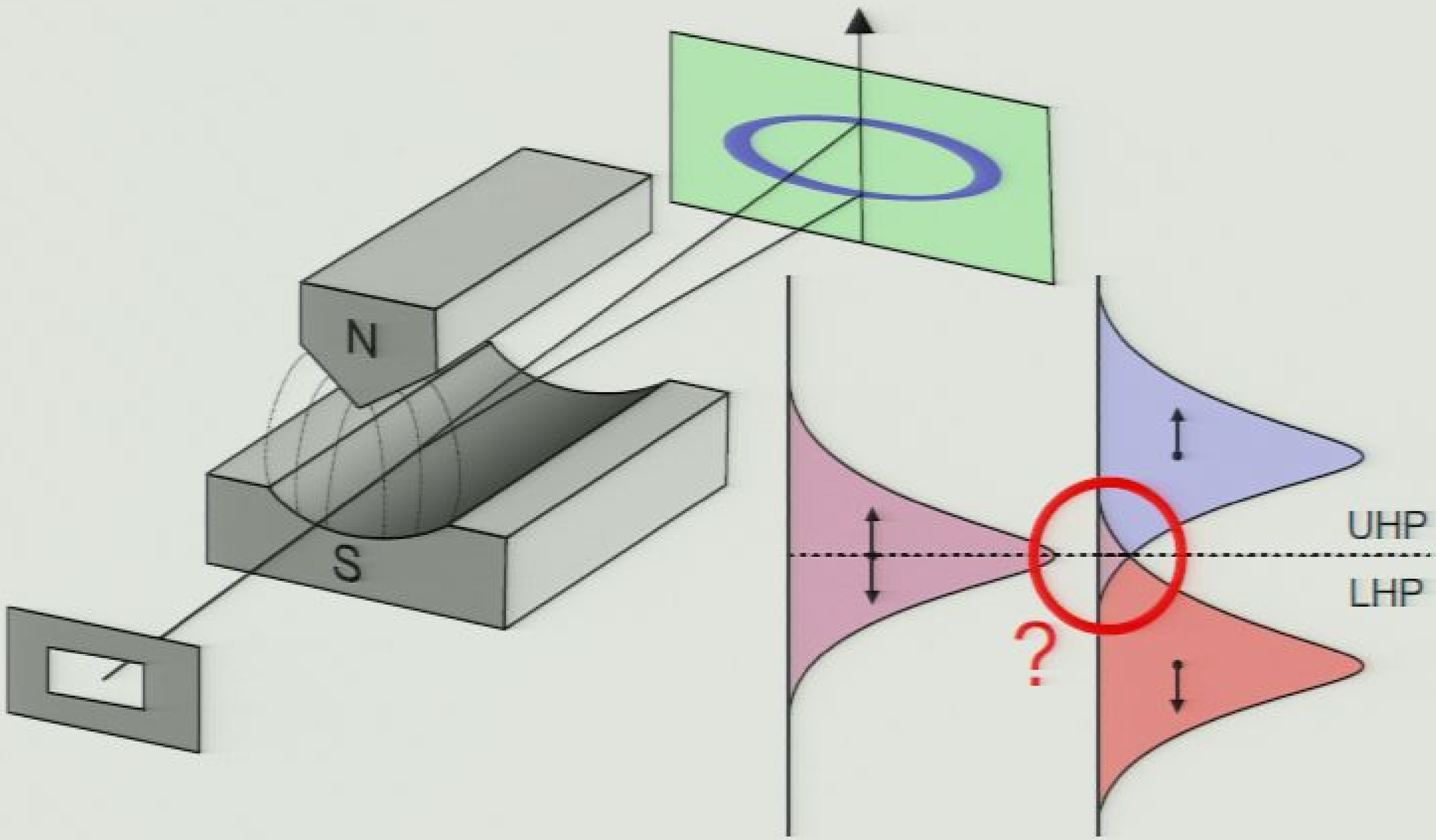
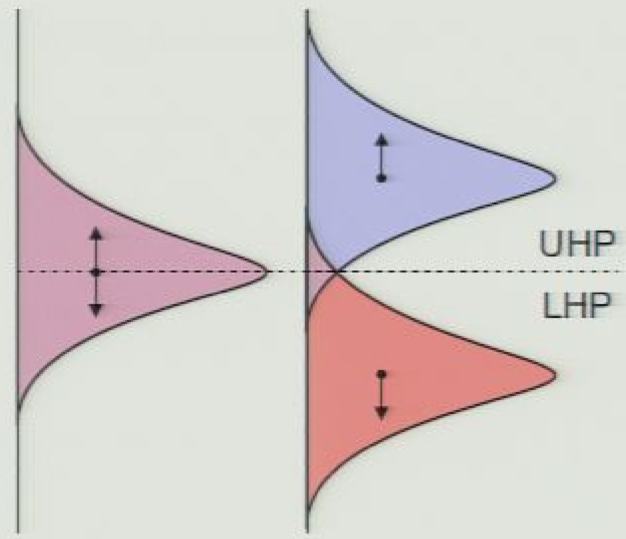


Example: Stern-Gerlach Experiment

Imprecisions I

- indirect measurement of spin through additional spatial "pointer system":

$$|\psi_\uparrow\rangle \otimes |\phi\rangle \mapsto |\psi_\uparrow\rangle \otimes |\phi_\uparrow\rangle$$
$$|\psi_\downarrow\rangle \otimes |\phi\rangle \mapsto |\psi_\downarrow\rangle \otimes |\phi_\downarrow\rangle$$



- initial state is

$$\rho = \frac{1}{2}\left[|\psi_\uparrow\rangle\langle\psi_\uparrow| + |\psi_\downarrow\rangle\langle\psi_\downarrow|\right] \otimes |\phi\rangle\langle\phi|$$

- state after coupling is

$$\rho' = \frac{1}{2}|\psi_\uparrow\rangle\langle\psi_\uparrow| \otimes |\phi_\uparrow\rangle\langle\phi_\uparrow| + |\psi_\downarrow\rangle\langle\psi_\downarrow| \otimes |\phi_\downarrow\rangle\langle\phi_\downarrow|$$

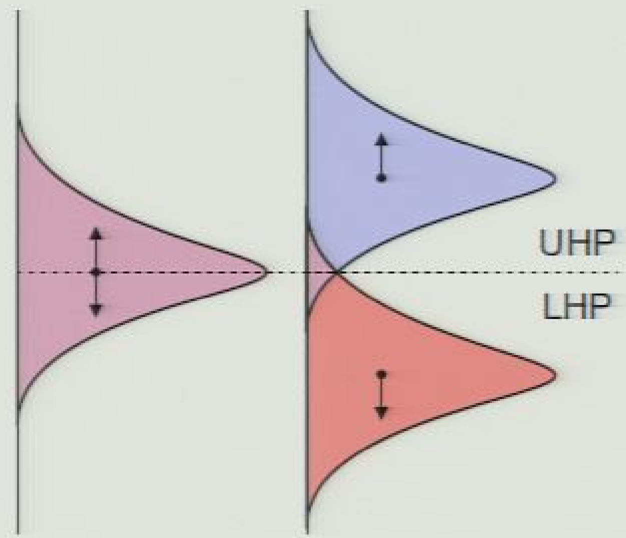
Example: Stern-Gerlach Experiment

Imprecisions II

- state after coupling is

$$\rho' = \frac{1}{2}|\psi_\uparrow\rangle\langle\psi_\uparrow| \otimes |\phi_\uparrow\rangle\langle\phi_\uparrow| + |\psi_\downarrow\rangle\langle\psi_\downarrow| \otimes |\phi_\downarrow\rangle\langle\phi_\downarrow|$$

- corresponding measurement probabilities ($p_? = \mathrm{Tr}[\rho'(\hat{I}\otimes\hat{P}_\pm)]$) are



$$\begin{aligned} p_\uparrow &= \frac{1}{2}\langle\phi_\uparrow|\hat{P}_\uparrow|\phi_\uparrow\rangle + \frac{1}{2}\langle\phi_\downarrow|\hat{P}_\uparrow|\phi_\downarrow\rangle \\ p_\downarrow &= \frac{1}{2}\langle\phi_\uparrow|\hat{P}_\downarrow|\phi_\uparrow\rangle + \frac{1}{2}\langle\phi_\downarrow|\hat{P}_\downarrow|\phi_\downarrow\rangle \end{aligned}$$

$$\hat{E}_? = \langle\phi_\uparrow|\hat{P}_?|\phi_\uparrow\rangle|\psi_\uparrow\rangle\langle\psi_\uparrow| + \langle\phi_\downarrow|\hat{P}_?|\phi_\downarrow\rangle|\psi_\downarrow\rangle\langle\psi_\downarrow|$$

Example: Qubit Effects

- qubit selfadjoint operators can be parametrized by four real parameters α and $\mathbf{a}$:

$$\hat{E} = \frac{1}{2}(\alpha\hat{I} + \mathbf{a}\cdot\boldsymbol{\sigma})$$

- for $\hat{E}$ to be an effect ($\hat{0} \leq \hat{E} \leq \hat{I}$) following has to hold:

$$0 \leq a \leq \alpha \leq 2 - a,$$

where $a \equiv \|\mathbf{a}\|$

- imprecisions are caused by coefficients:
  - *sharpness* a: as for $\alpha = 1$, $p_{\hat{E}}(\rho_{\mathbf{a}}) = \mathrm{Tr}[\rho_{\mathbf{a}}\hat{E}] = \frac{1+a}{2}$
  - *bias* α, as $p_{\hat{E}}(\frac{1}{2}\hat{I}) = \mathrm{Tr}[\frac{1}{2}\hat{I}\hat{E}] = \frac{\alpha}{2}$

Coexistence of Effects I

Definition

Effects $\hat{A}, \hat{B} \in \mathcal{E}(\mathcal{H})$ are called coexistent if there exists an observable G : $2^{\Omega} \to \mathcal{E}(\mathcal{H})$ and events $X, Y \in 2^{\Omega}$ such that $\hat{A} = \hat{G}(X)$ and $\hat{B} = \hat{G}(Y)$

Helpful is following proposition

Theorem

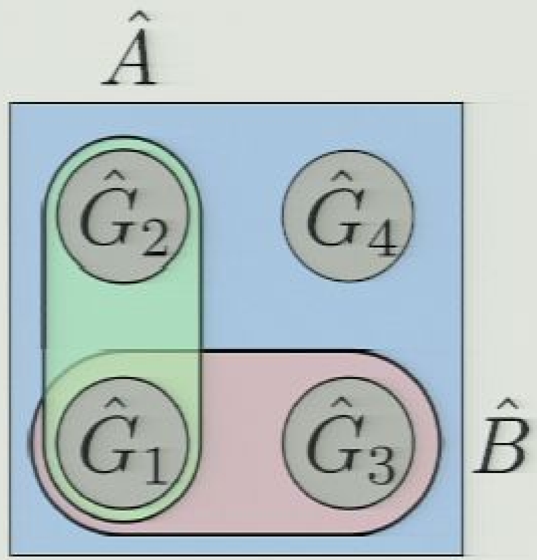
Effects $\hat{A}$ and $\hat{B}$ are coexistent if and only if there exists an observable G *with four outcomes $\{1, 2, 3, 4\}$ such that*

$$\hat{A} = \hat{G}_1 + \hat{G}_2, \qquad \hat{B} = \hat{G}_1 + \hat{G}_3$$

Coexistence of Effects II

The proposition tells us, that if we find an effect $\hat{0} \leq \hat{G} \equiv \hat{G}_1$, following has to hold:

- $0 \leq G_2 = \hat{A} - \hat{G}_1$ implying $\hat{G} \leq \hat{A}$
- $0 \leq G_3 = \hat{B} - \hat{G}_1$ implying $\hat{G} \leq \hat{B}$
- $0 \leq G_4 = \hat{I} - \hat{G}_1 - \hat{G}_2 - \hat{G}_3$ implying $\hat{I} + \hat{G} \geq \hat{A} + \hat{B}$



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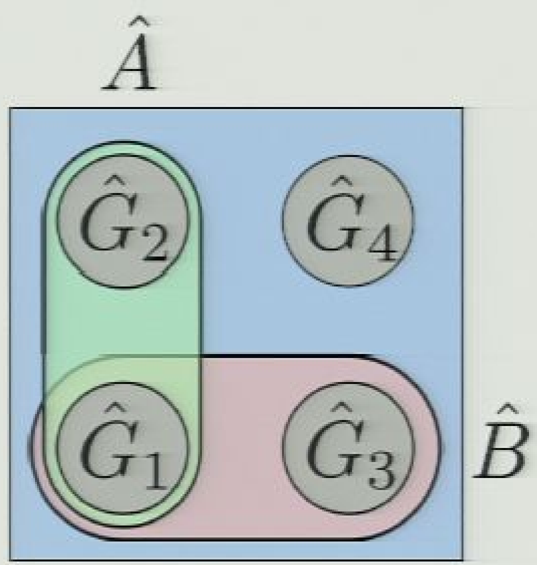

Example: Unambiguous Discrimination

- unambiguous discrimination of two pure states $\Phi_1 = |\phi_1\rangle\langle\phi_1|$ and $\Phi_2 = |\phi_2\rangle\langle\phi_2|$ is always possible, but perfectly only when ϕ_1 and ϕ_2 are orthogonal
- unambiguous discrimination of Φ_1 is determined by effect $\hat{E}_1 = \hat{I} - \Phi_2$ (also $\hat{E}_1^{\perp} = \Phi_2$) as

$$\begin{aligned} \mathrm{Tr}[\Phi_1 \hat{E}_1] &= 1 - |\langle\phi_1|\phi_2\rangle|^2 \\ \mathrm{Tr}[\Phi_2 \hat{E}_1] &= 0 \end{aligned}$$

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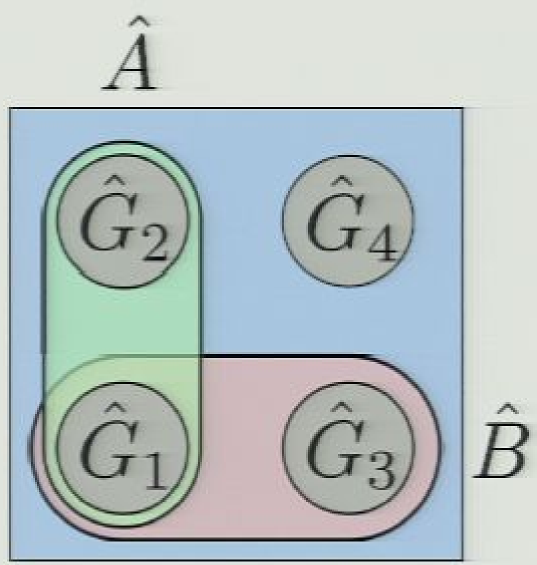

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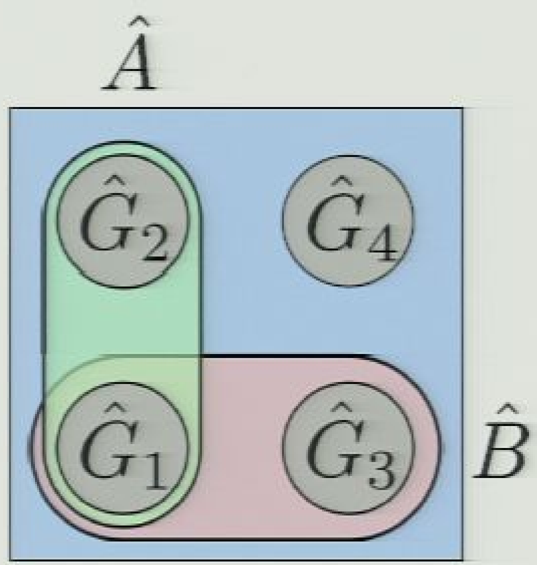

Example: Unambiguous Discrimination

- unambiguous discrimination of two pure states $\Phi_1 = |\phi_1\rangle\langle\phi_1|$ and $\Phi_2 = |\phi_2\rangle\langle\phi_2|$ is always possible, but perfectly only when ϕ_1 and ϕ_2 are orthogonal
- unambiguous discrimination of Φ_1 is determined by effect $\hat{E}_1 = \hat{I} - \Phi_2$ (also $\hat{E}_1^{\perp} = \Phi_2$) as

$$\begin{aligned} \mathrm{Tr}[\Phi_1 \hat{E}_1] &= 1 - |\langle\phi_1|\phi_2\rangle|^2 \\ \mathrm{Tr}[\Phi_2 \hat{E}_1] &= 0 \end{aligned}$$

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- unambiguous discrimination of both Φ_1 and Φ_2 is determined by effects $\hat{F}_1 = \hat{E}_1/2$ and $\hat{F}_2 = \hat{E}_2/2$ with complementary (indecisive) effect $\hat{F}_? = \hat{I} - \hat{F}_1 - \hat{F}_2 \geq \hat{0}$ — increased uncertainty makes it possible to perform both measurements within one, where $\hat{G} = \hat{0}$ as $\hat{F}_1 + \hat{F}_2 \leq \hat{I}$

Example: Commuting Effects

If effects $\hat{A}$ and $\hat{B}$ commute we can choose

- $\hat{G}_1 = \hat{A}\hat{B}$
- $\hat{G}_2 = \hat{A} - \hat{G}_1 = \hat{A}(\hat{I} - \hat{B})$
- $\hat{G}_3 = \hat{B} - \hat{G}_1 = \hat{B}(\hat{I} - \hat{A})$
- $\hat{G}_4 = \hat{I} - \hat{G}_1 - \hat{G}_2 - \hat{G}_3 = (\hat{I} - \hat{A})(\hat{I} - \hat{B})$

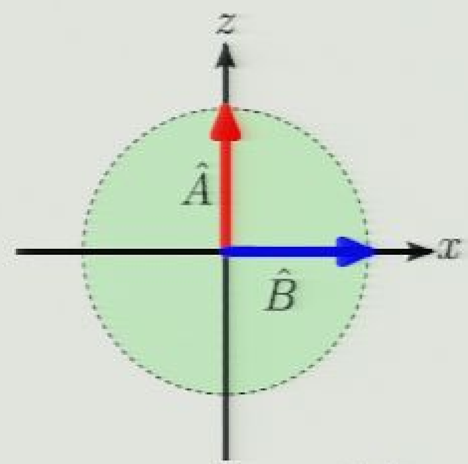
Indeed,

$$\hat{G}_1 + \hat{G}_2 + \hat{G}_3 + \hat{G}_4 = \hat{I} + [\hat{A}, \hat{B}] = \hat{I}$$

xample: Measuring two orthogonal spin components

Coexistence of effects

$$\hat{A} = \frac{1}{2}(\hat{I} + \mu\sigma_z) \quad \text{and} \quad \hat{B} = \frac{1}{2}(\hat{I} + \mu\sigma_x)$$



is conditioned by μ:

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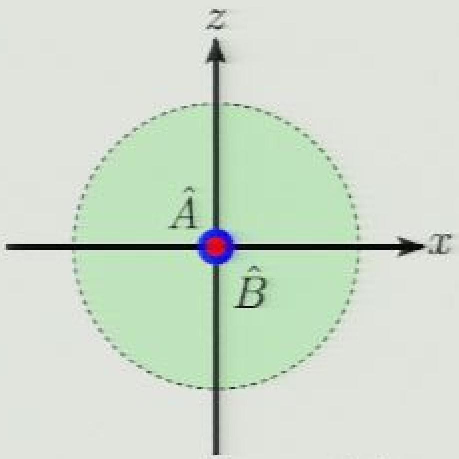

is conditioned by μ:

- if $\mu = 1$ effects correspond to projective measurement in directions x and z — clearly not coexistent
- if $\mu = 0$ we get $\hat{A} = \hat{B} = \frac{1}{2}\hat{I}$ — trivially coexistent

xample: Measuring two orthogonal spin components

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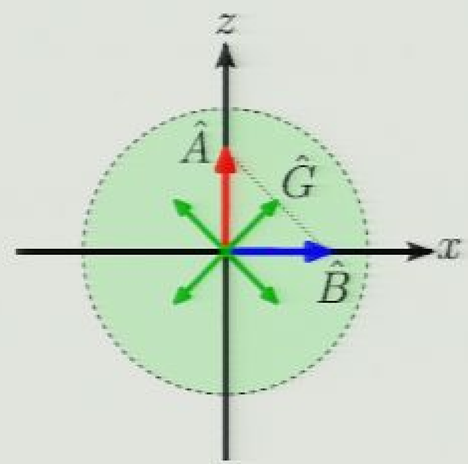

is conditioned by μ:

- in general case choose

$$\hat{G} = \frac{1}{4}\left[\hat{I} + \mu(\sigma_x + \sigma_z)\right]$$

  in this case $\alpha = 1/2$ and $a = \mu/\sqrt{2}$ — conditions for $\hat{G}$ to be effect are $0 \leq a \leq \alpha \leq 2 - a$; they are fulfilled if $\mu \leq 1/\sqrt{2}$
- it can be easily shown, that if $\hat{G}$ is effect, then all the conditions for coexistence are fulfilled

Coexistence of Effects II

The proposition tells us, that if we find an effect $\hat{0} \leq \hat{G} \equiv \hat{G}_1$, following has to hold:

- $0 \leq G_2 = \hat{A} - \hat{G}_1$ implying $\hat{G} \leq \hat{A}$
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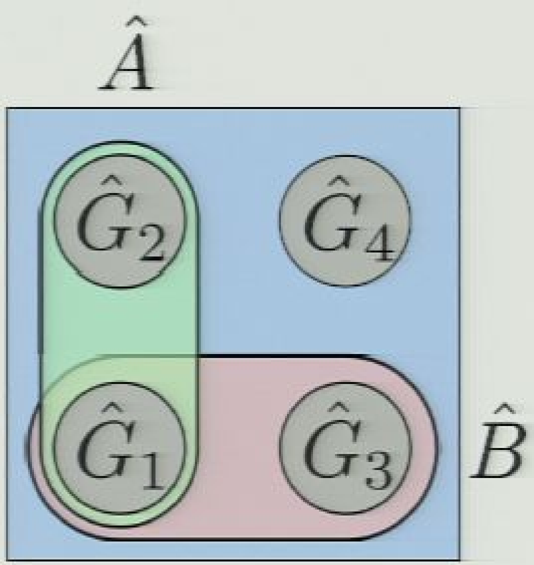

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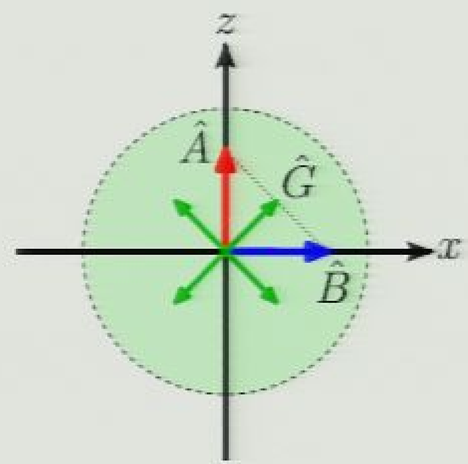

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Obvious Coexistence Criterions

Criterion 1

If effects $\hat{A}$ and $\hat{B}$ fulfill condition $\hat{A} + \hat{B} \leq \hat{I}$ they are coexistent.

Criterion 2

If effects $\hat{A}$ and $\hat{B}$ commute they are coexistent.

And one not so obvious:

Criterion 3

If effects $\hat{A}$ and $\hat{B}$ are "unsharp" enough, they are coexistent.

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oint Measurability Criterions for Simple Observables

Definition

An observable composed only of two non-trivial outcomes ($\Omega = \{-1, 1\}$) is called simple.

Simple observable A is determined by only one effect $\hat{A} \equiv \hat{A}_+$ (corresponding to outcome $+1$) as for the outcome -1 corresponding effect is $\hat{A}_- = \hat{I} - \hat{A}$.

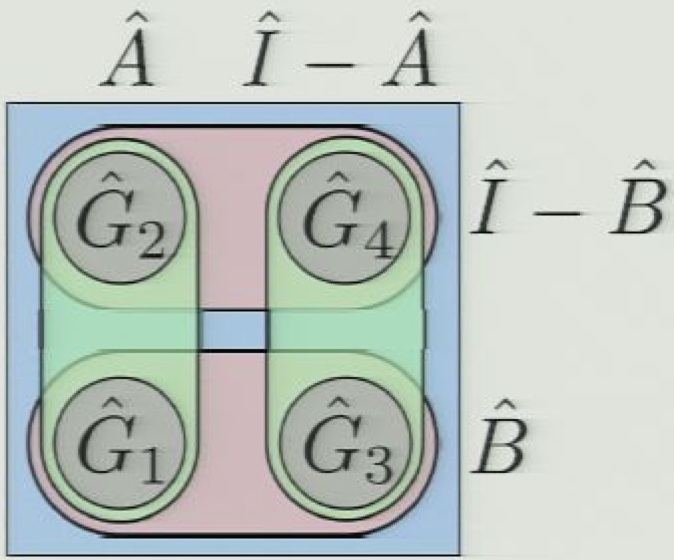
Theorem

If effects $\hat{A}$ and $\hat{B}$ are coexistent, then also effects $\hat{A}$ and $\hat{I} - \hat{B}$, resp. $\hat{I} - \hat{A}$ and $\hat{B}$, resp. $\hat{I} - \hat{A}$ and $\hat{I} - \hat{B}$, are coexistent.

oint Measurability Criterions for Simple Observables

Definition

Simple observables A and B are jointly measurable, if corresponding defining effects $\hat{A}$ and $\hat{B}$ are coexistent.



Joint Measurability of Simple Qubit Observables

Task

Having simple qubit observables A and B given by effects

$$\hat{A} = \frac{1}{2}(\alpha\hat{I} + \mathbf{a}\cdot\boldsymbol{\sigma}) \qquad \text{and} \qquad \hat{B} = \frac{1}{2}(\beta\hat{I} + \mathbf{b}\cdot\boldsymbol{\sigma})$$

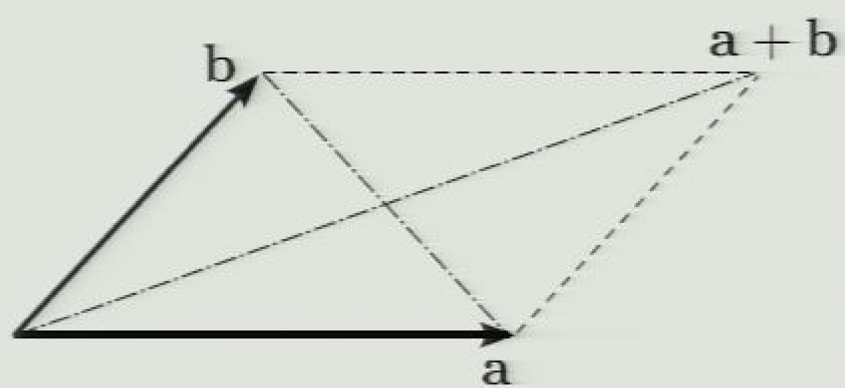
determine for which parameters α, β, $\mathbf{a}$ and $\mathbf{b}$ these are jointly measurable.

- it is basically the simplest problem of joint measurability
- from symmetry and previous theorems all we need to consider are parameters $\alpha \in [0;1]$, $\beta \in [0;1]$, $\mathbf{a} \parallel x$-axis, i.e. relevant is only a, and two parameters of $\mathbf{b}$, $b_{\parallel}$ and $b_{\perp}$ (we can restrict ourselves to a plane given by $\mathbf{a}$ and $\mathbf{b}$)
- we will characterize all $\mathbf{b}$ for given parameters α, a and β

Geometrical Conditions

Taking $\hat{G} = \frac{1}{2}(\gamma \hat{I} + \mathbf{g} \cdot \boldsymbol{\sigma})$ we have following conditions:

$$\hat{0} \leq \hat{G} \qquad \hat{G} \leq \hat{A}$$
$$\hat{A} + \hat{B} \leq \hat{I} + \hat{G} \qquad \hat{G} \leq \hat{B}$$



Geometrical Conditions

Taking $\hat{G} = \frac{1}{2}(\gamma\hat{I} + \mathbf{g}\cdot\boldsymbol{\sigma})$ we have following conditions:

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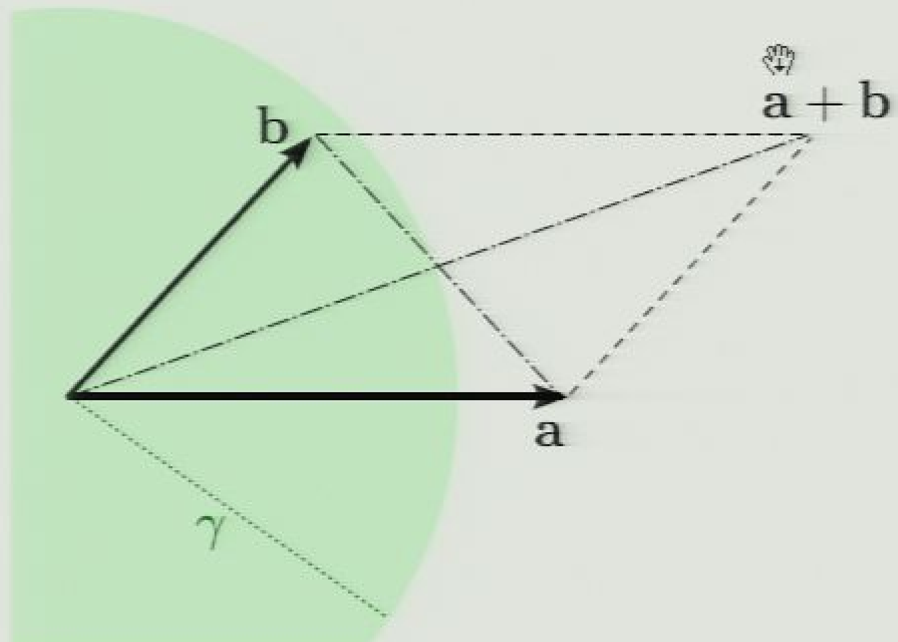

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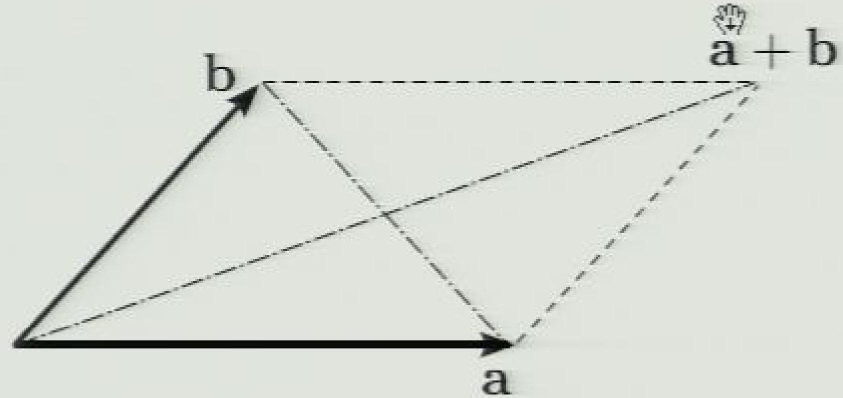

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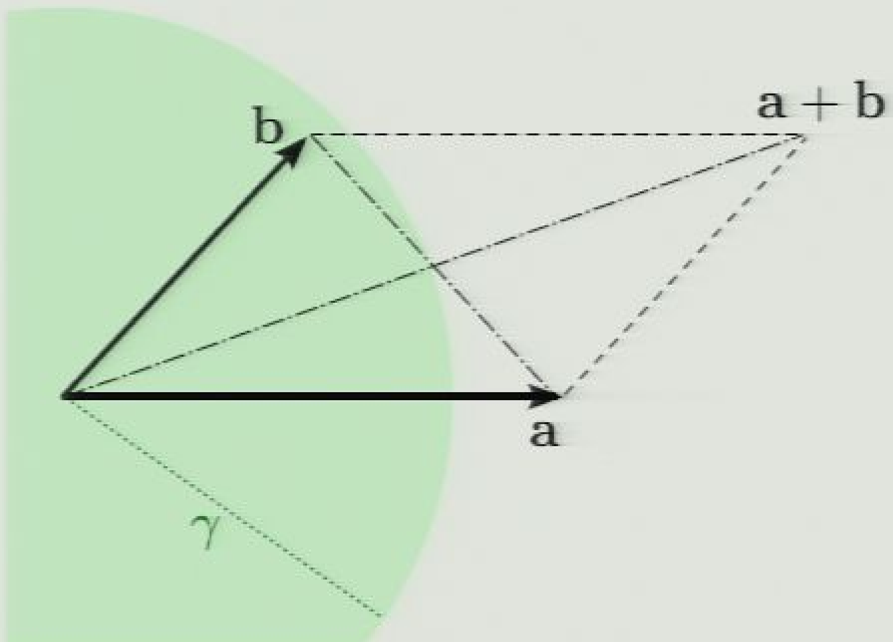

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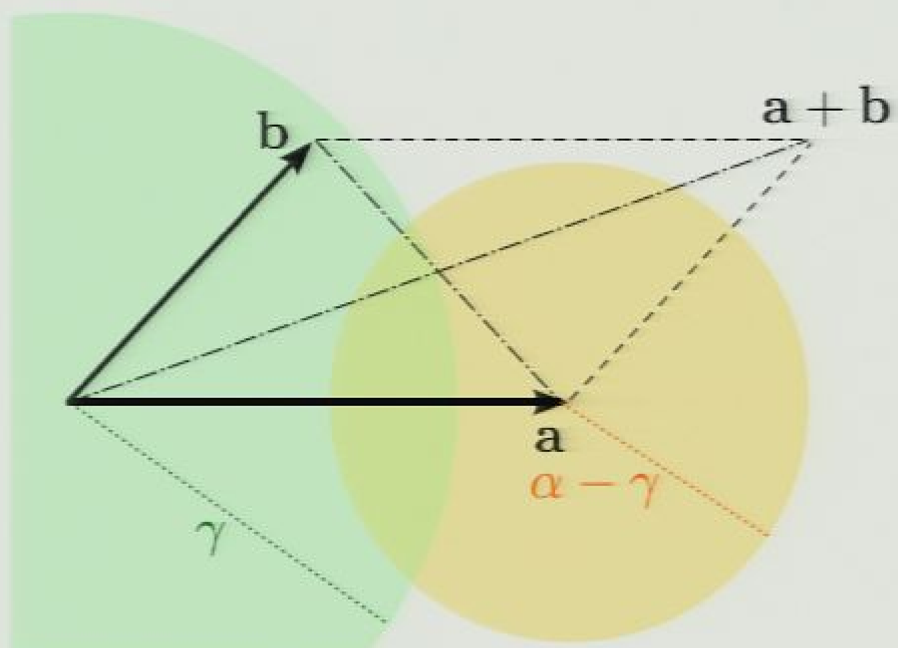

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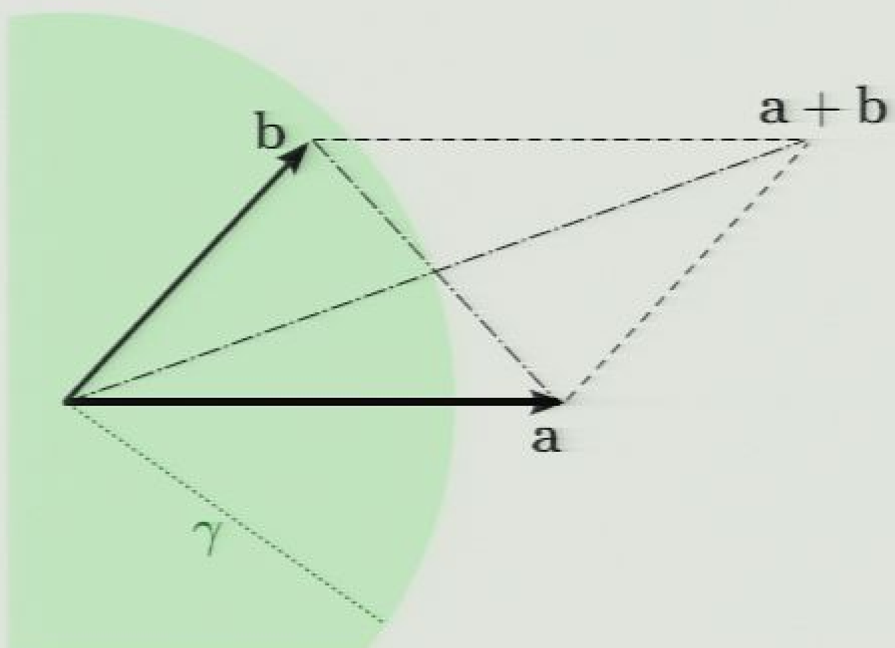

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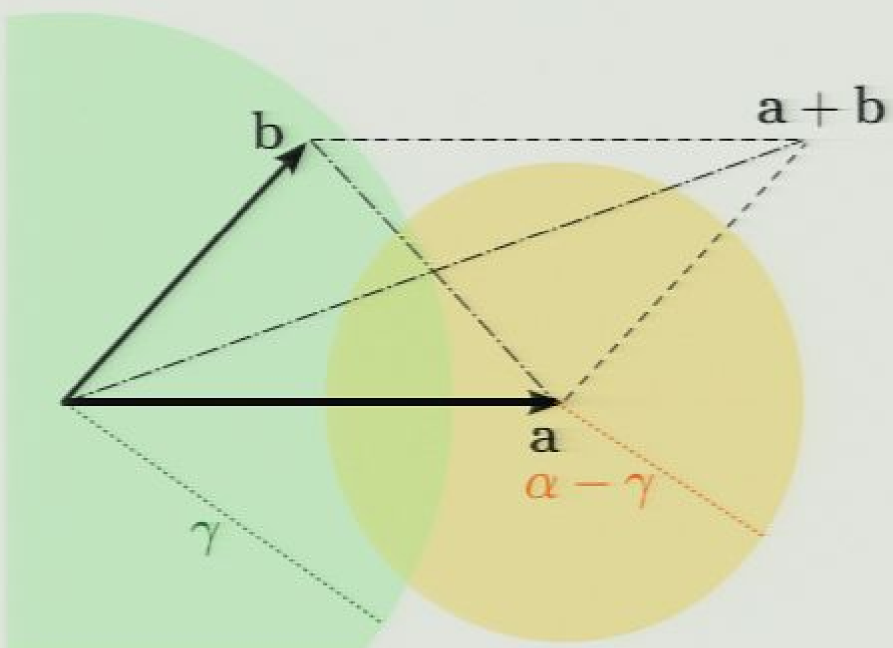

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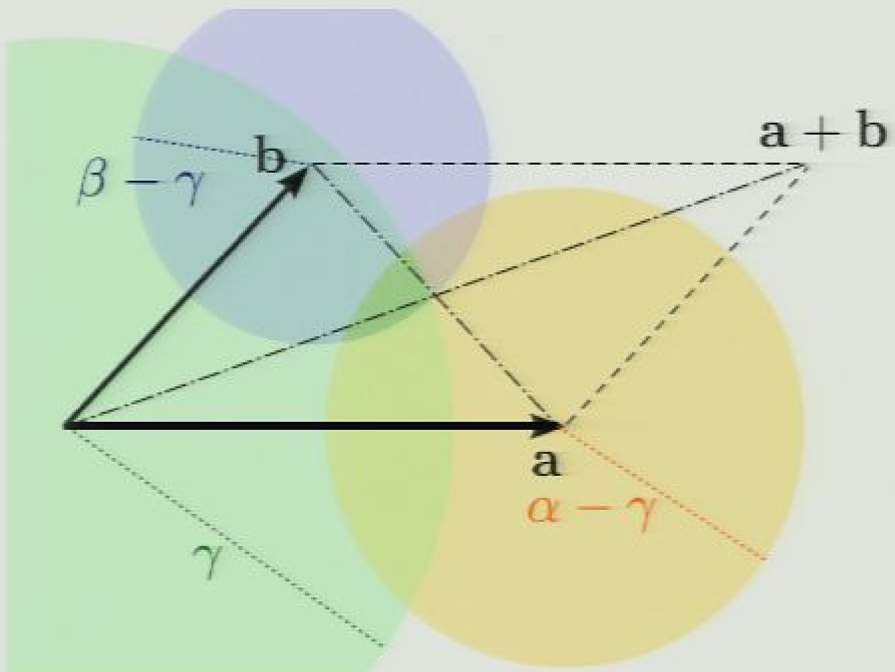

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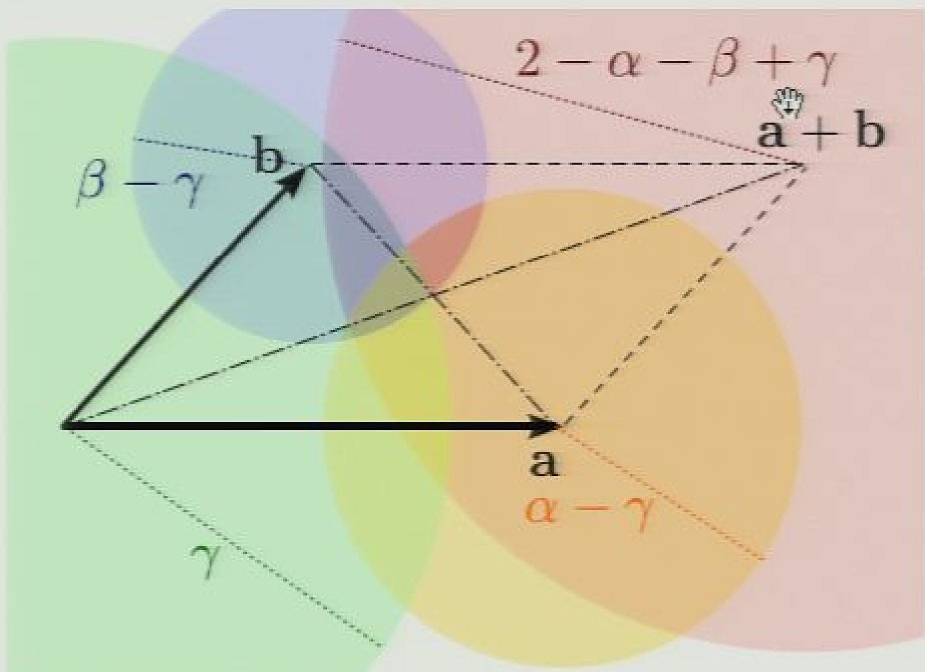

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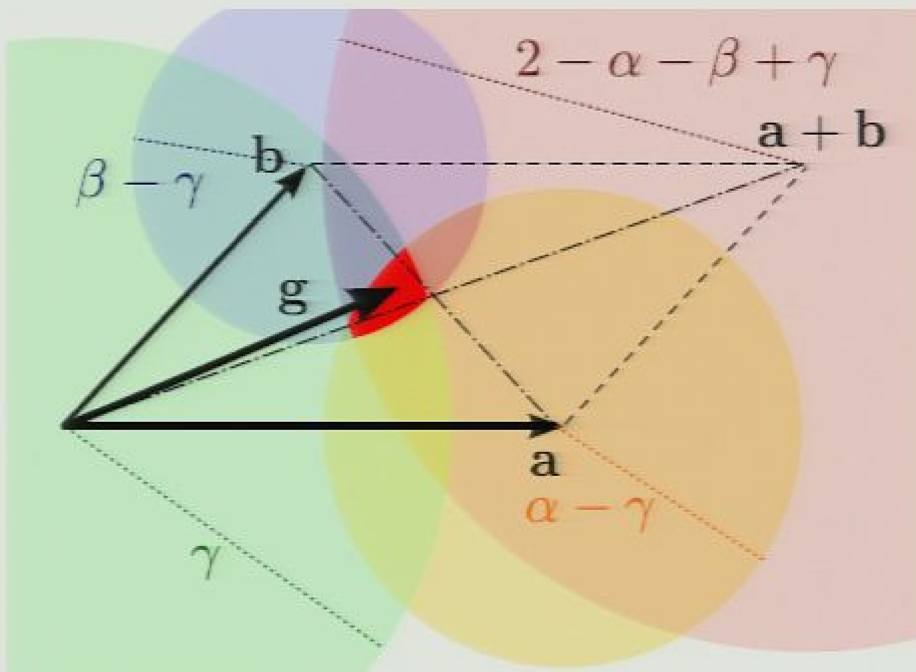
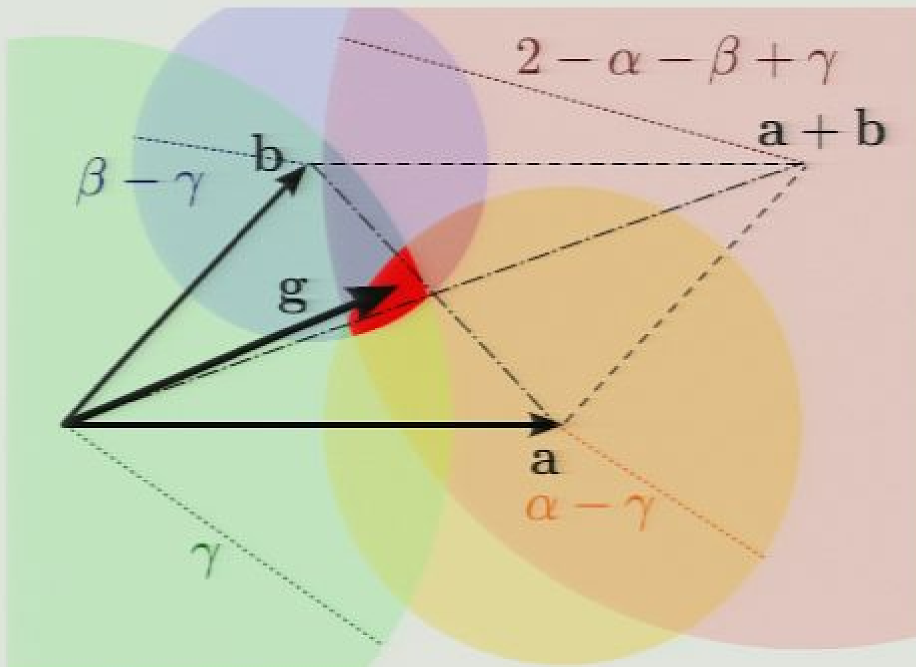


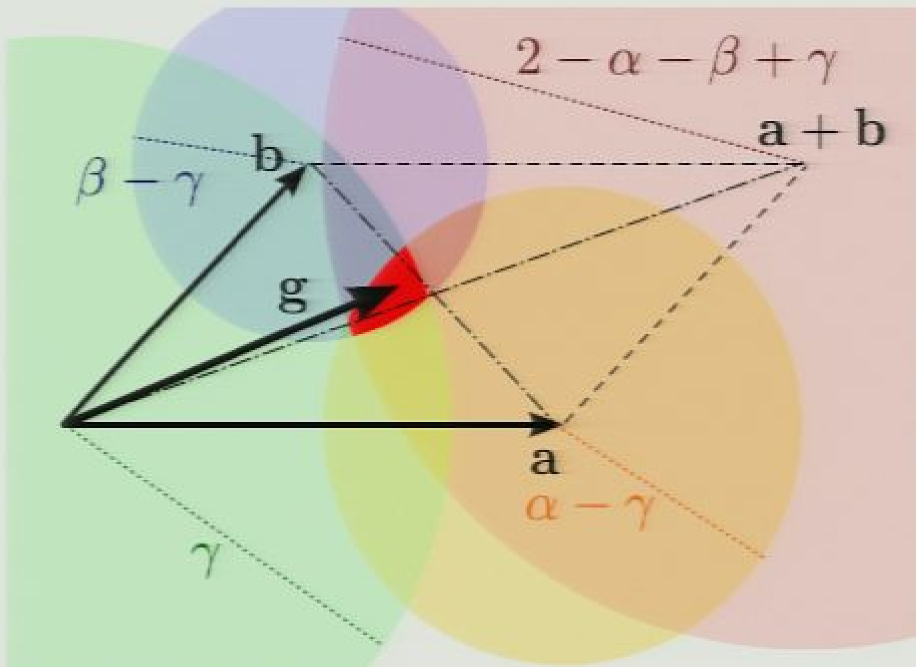
Necessary Condition



we see that

$$\begin{aligned} \|\mathbf{a}+\mathbf{b}\| &= \|\mathbf{g}-[\mathbf{g}-(\mathbf{a}+\mathbf{b})]\| \le 2\gamma+2-\alpha-\beta \\ \|\mathbf{a}-\mathbf{b}\| &= \|(\mathbf{g}-\mathbf{b})-(\mathbf{g}-\mathbf{a})\| \le \alpha+\beta-2\gamma \end{aligned}$$

Necessary Condition



we see that

$$\|\mathbf{a}+\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\| \leq 2$$

Necessary Condition

Theorem (P. Busch, Phys. Rev. D 33, 2253 (1986))

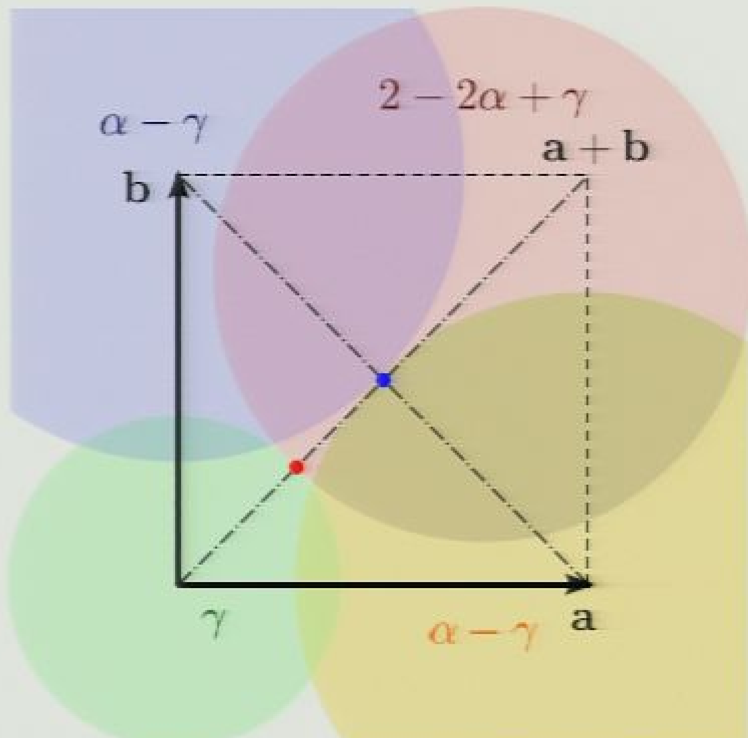
If $\hat{\mathrm{A}}$ and $\hat{\mathrm{B}}$ are jointly measurable, then

$$||\mathbf{a}+\mathbf{b}|| + ||\mathbf{a}-\mathbf{b}|| \leq 2$$

- if $\alpha = \beta = 1$, then this is also sufficient condition ($\gamma = 1/2$) and $\mathbf{g} = (\mathbf{a}+\mathbf{b})/2$
- moreover if $\mathbf{a} \perp \mathbf{b}$, $||\mathbf{a}+\mathbf{b}|| = ||\mathbf{a}-\mathbf{b}|| \leq 1$
- and moreover if $a = b$, $||\mathbf{a}|| \leq 1/\sqrt{2}$ — previous example
- it is not a sufficient condition in general!

Not a sufficient condition!

Suppose $\alpha = \beta < 1$, $a = b = 1/\sqrt{2}$ and $\mathbf{a} \perp \mathbf{b}$



If we choose γ such that $2\alpha - 2\gamma = 1$, one common point of opposite circles lies in the "centre", while at the same time the other circles touch, but they do not have the same radiuses, so

Sufficient condition

Definition

Function $\mathfrak{S}$ from $\mathcal{E}(\mathcal{H}_2)$ to $[0;1]$,

$$\mathfrak{S}(\hat{A}) = \frac{1}{2}\left\{a^2 + \alpha(2-\alpha) - \sqrt{(\alpha^2 - a^2)\left[(2-\alpha)^2 - a^2\right]}\right\}$$

will be called sharpness.

- $\mathfrak{S}(\hat{I} - \hat{A}) = \mathfrak{S}(\hat{A})$ and it does not depend on the choice of the basis
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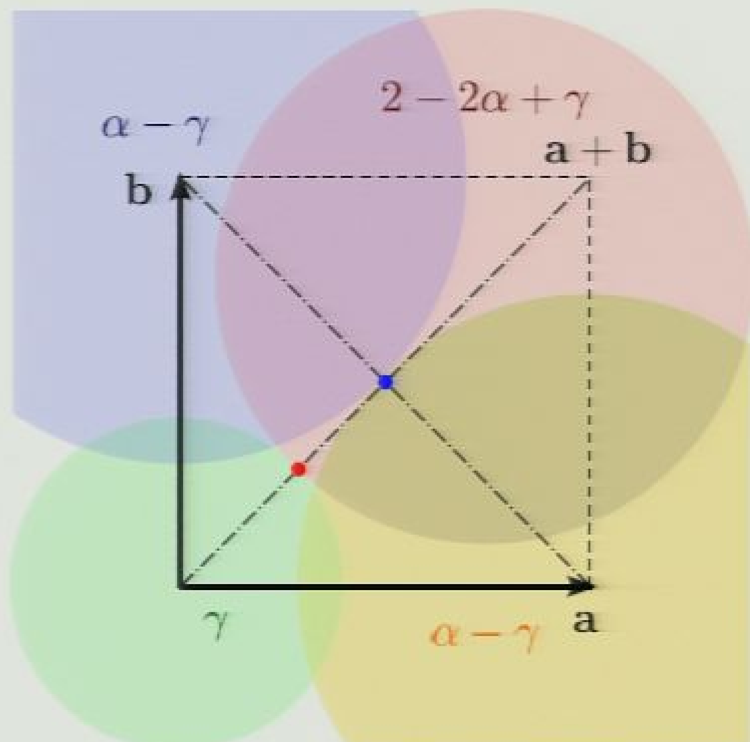

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Theorem (P. Stano, D. Reitzner and T. Heinosaari, PRA **78**, 012315 (2008))

An effect $\hat{B}$ is coexistent with $\hat{A}$ if and only if it falls into one of the following three disjoint cases:

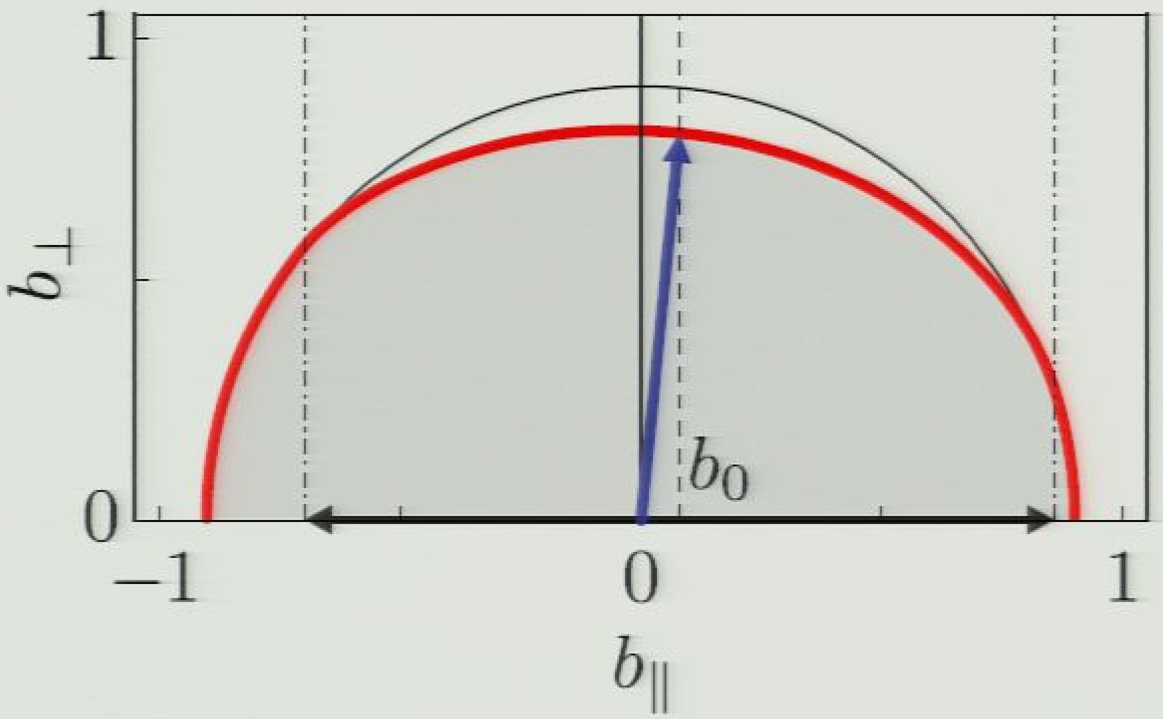
1. *if* $\beta \leq 1 - \mathfrak{S}(\hat{A})$, *irrespectively of* $\mathbf{b}$,
2. *if* $\beta > 1 - \mathfrak{S}(\hat{A})$ *and* $|b_{\|} - b_0| \geq w$,
3. *if* $\beta > 1 - \mathfrak{S}(\hat{A})$, $|b_{\|} - b_0| < w$ *and if and only if* $b_{\perp} \leq b_{\perp}^{\max}$,

where $b_0 = \frac{1}{a}(1-\alpha)(1-\beta)$,

$$w = \frac{1}{a}\sqrt{(1-\alpha)^2 - \beta[(1-\alpha)^2 + 1 - a^2] + \beta^2}$$

$$b_{\perp}^{\max} = \frac{1}{2a}\sqrt{[(2-\alpha)^2 - a^2]\{a^2 - [a(b_{\|} - b_0) + (1-\beta)^2\}} + \frac{1}{2a}\sqrt{[\alpha^2 - a^2]\{a^2 - [a(b_{\|} - b_0) - (1-\beta)^2\}}$$

Examples



$\alpha = 0.6$, $a = 0.5$ and $\beta = 0.9$; note: $b_0 + w \leq \beta$ (if applicable)

Sufficient condition

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$$\mathfrak{S}(\hat{A}) = \frac{1}{2}\left\{a^2 + \alpha(2-\alpha) - \sqrt{(\alpha^2 - a^2)\left[(2-\alpha)^2 - a^2\right]}\right\}$$

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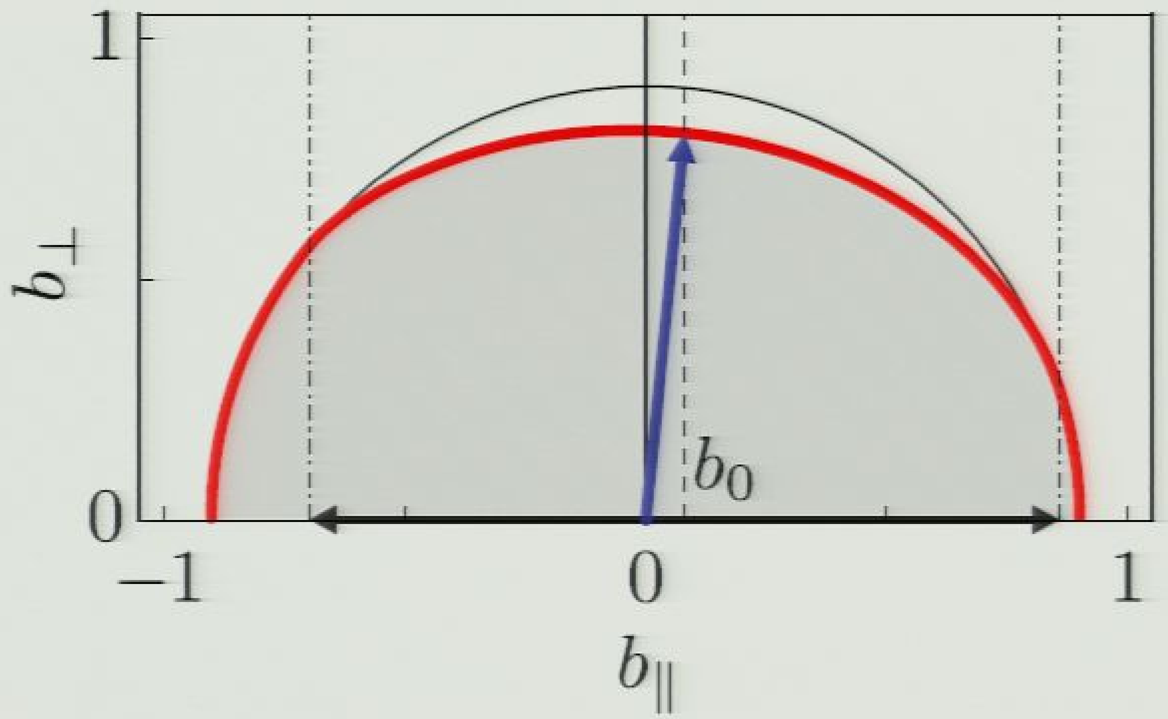
3. *if* $\beta > 1 - \mathfrak{S}(\hat{A})$, $|b_{\|} - b_0| < w$ *and if and only if* $b_{\perp} \leq b_{\perp}^{\max}$,

where $b_0 = \frac{1}{a}(1-\alpha)(1-\beta)$,

$$w = \frac{1}{a}\sqrt{(1-\alpha)^2 - \beta[(1-\alpha)^2 + 1 - a^2] + \beta^2}$$

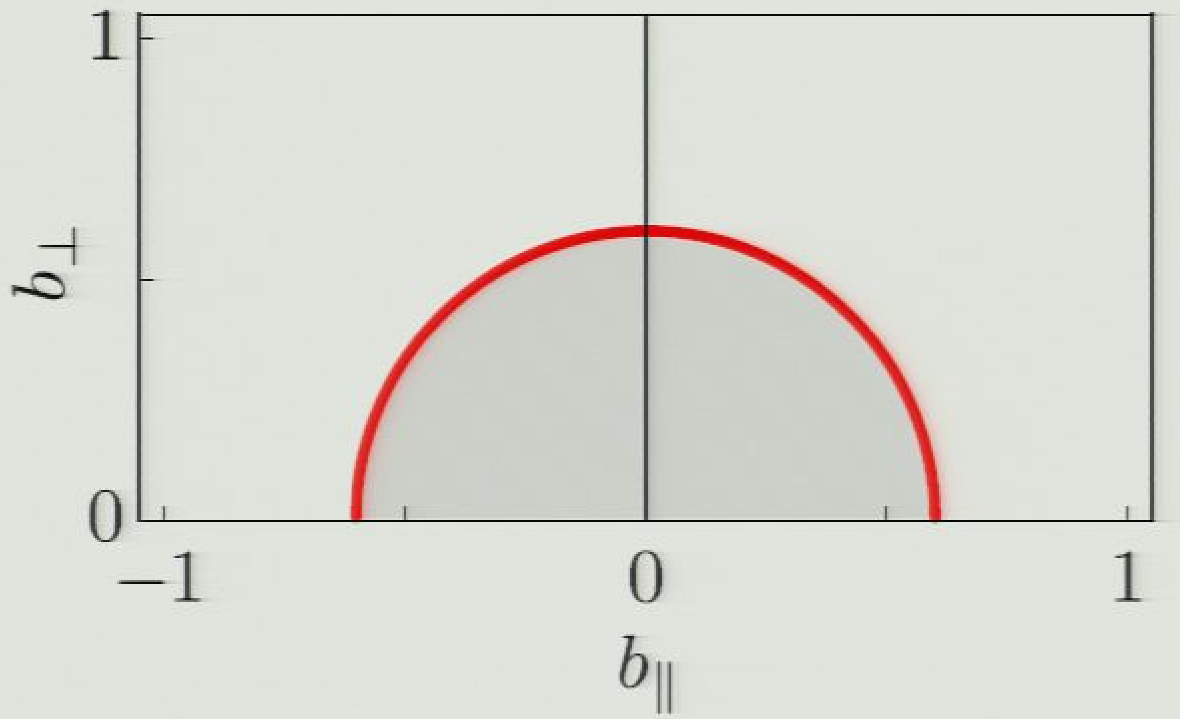
$$b_{\perp}^{\max} = \frac{1}{2a}\sqrt{[(2-\alpha)^2 - a^2]\{a^2 - [a(b_{\|} - b_0) + (1-\beta)^2\}} + \frac{1}{2a}\sqrt{[\alpha^2 - a^2]\{a^2 - [a(b_{\|} - b_0) - (1-\beta)^2\}}$$

Examples



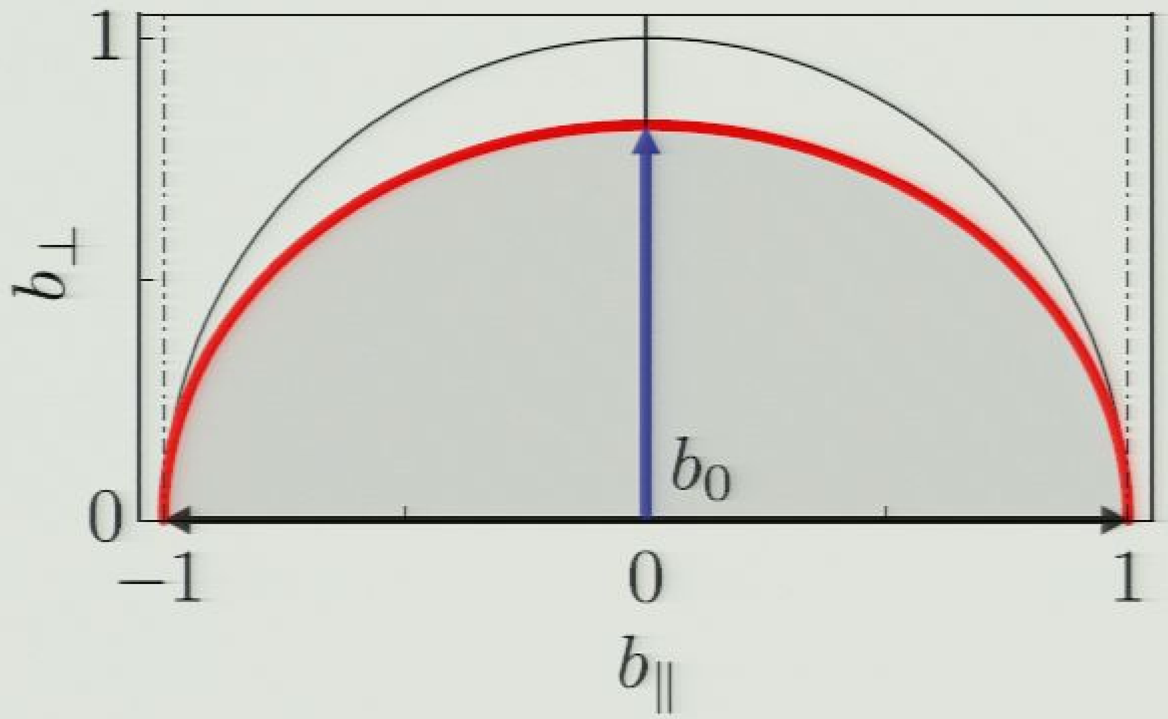
$\alpha = 0.6$, $a = 0.5$ and $\beta = 0.9$; note: $b_0 + w \leq \beta$ (if applicable)

Examples



$\alpha = 0.6$, $a = 0.5$ and $\beta = 0.6$ — both $\hat{A}$ and $\hat{B}$ are unsharp
$\mathfrak{S}(\hat{A}) + \mathfrak{S}(\hat{B}) \leq 1$

Examples



$\alpha = 0.6$, $a = 0.5$ and $\beta = 1$, when $w = 1$ and $b_0 = 0$

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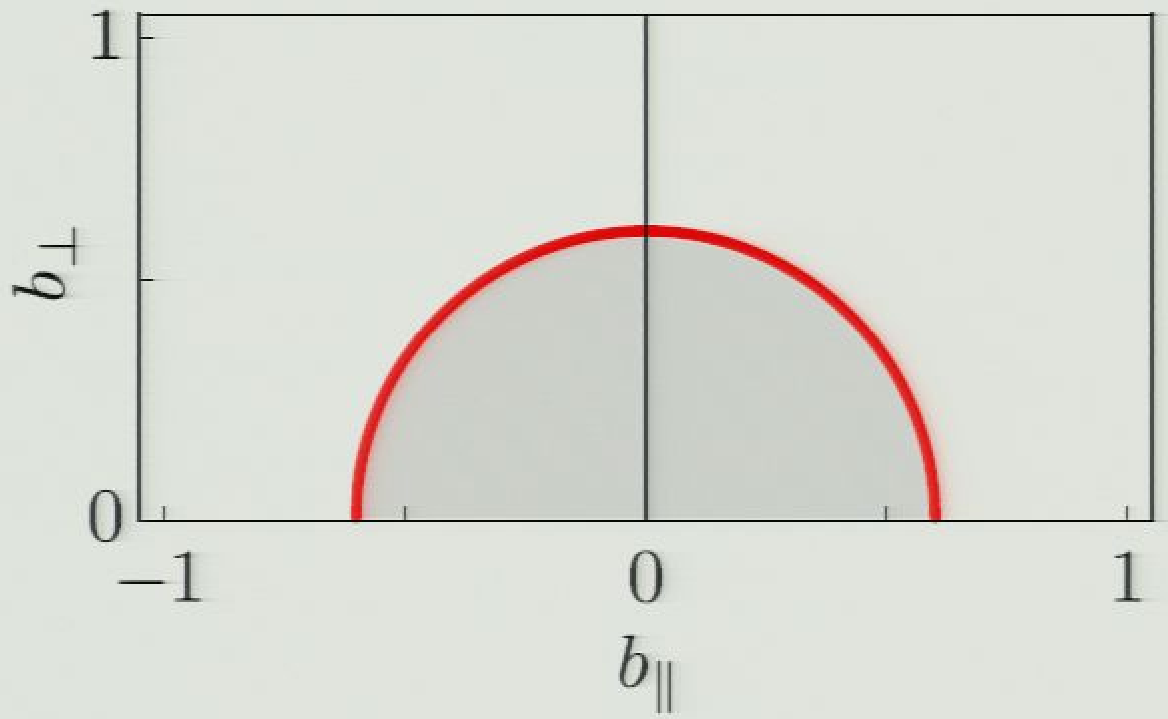

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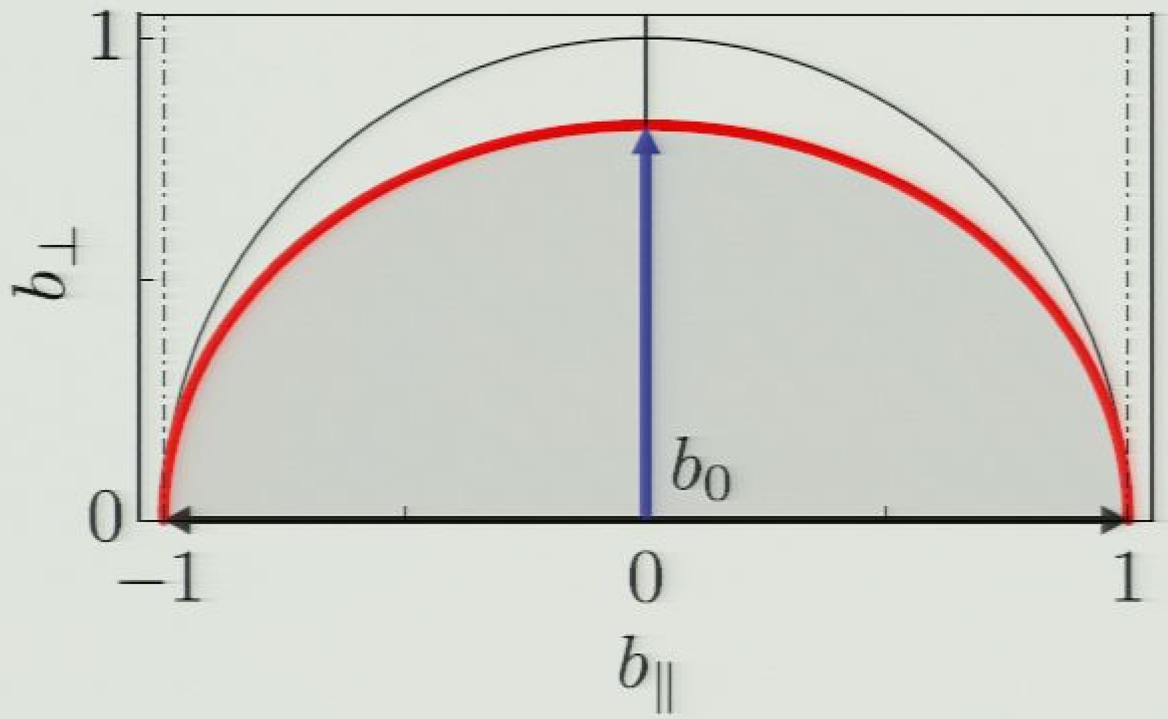

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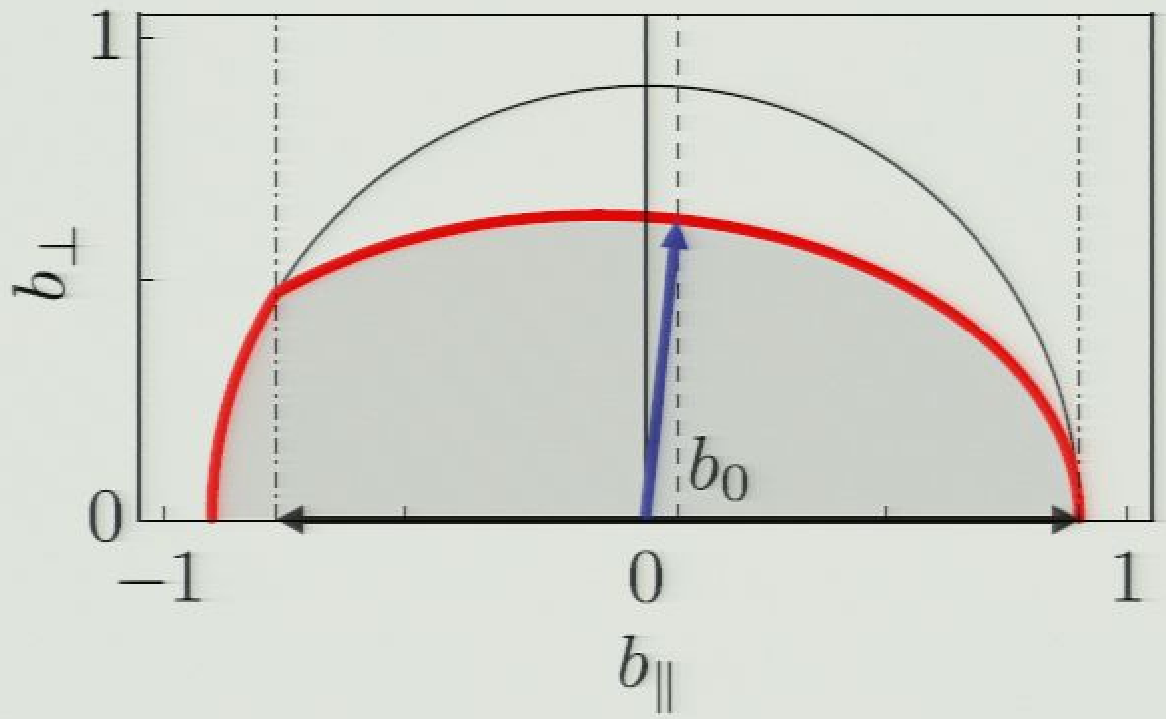


$\alpha = 0.6 = a$ and $\beta = 0.9$, when $b_0 + w = \beta$

Other results

Quantum Inf Process (2010) 9:143–169
DOI 10.1007/s11128-009-0109-x

Coexistence of qubit effects

Paul Busch · Heinz-Jürgen Schmidt

Published online: 13 March 2009

Abstract Two quantum events, represented by positive operators (effects), are *coexistent* if they can occur as possible outcomes in a single measurement scheme. Equivalently, the corresponding effects are coexistent if and only if they are contained in the ranges of a single (joint) observable. Here we give several equivalent characterizations of coexistent pairs of qubit effects. We also establish the equivalence between our results and those obtained independently by other authors. Our approach makes explicit use of the Minkowski space geometry inherent in the four-dimensional real vector space of selfadjoint operators in a two-dimensional complex Hilbert space.

Other results

Joint measurement of two unsharp observables of a qubit

Sixia Yu,[1,2] Nai-le Liu,[1] Li Li,[1] and C. H. Oh[2]

[1]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China*
[2]*Centre for Quantum Technologies and Physics Department, National University of Singapore, 2 Science Drive 3, Singapore 117542*

We present a single inequality as the necessary and sufficient condition for two unsharp observables of a two-level system to be jointly measurable in a single apparatus and construct explicitly the joint observables. A complementarity inequality arising from the condition of joint measurement, which generalizes Englert's duality inequality, is derived as the trade-off between the unsharpnesses of two jointly measurable observables.

PACS numbers: 03.65.Ta, 03.67.-a

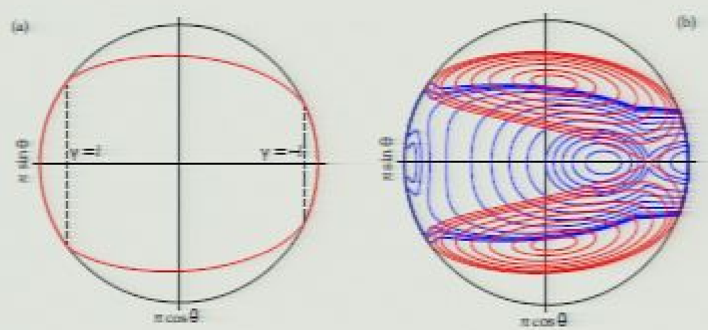
arXiv:0805.1538v2 [quant-ph] 17 Sep 2008

Built in the standard formalism of quantum mechanics, there are mutually exclusive but equally real aspects of quantum systems, as summarized
ciple of Bohr [1]. Mutually
hibited via noncommuting ob
plementarity is quantitatively
uncertainty relationships, nam
relationships (PURs) and the
tionships (MURs).

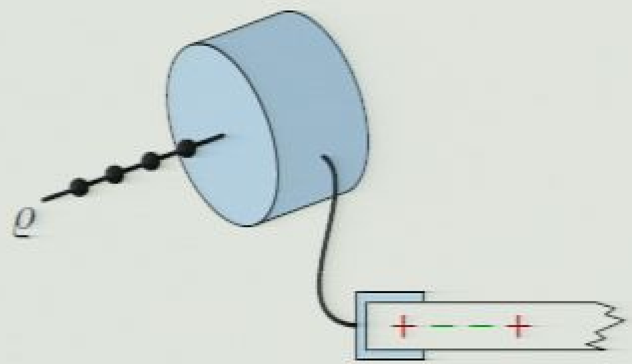
The PURs stem from the
density matrix describing the
the predictability of two nonco
quantum state. To test PURs tw
ments will be performed on t
bles of the quantum system a
be performed within one expe
semble.

On the other hand MURs ch
the precisions of unsharp mea
ing observables in a single ex
effort of Heisenberg [2] in de
ships was based on a simulta
sition and momentum, with t
tablished recently by Werner
wave-particle duality between

and $\sum_k O_k = I$) with K being the number of outcomes. By definition, a *joint measurement* of two observables $\{O_k\}$ and

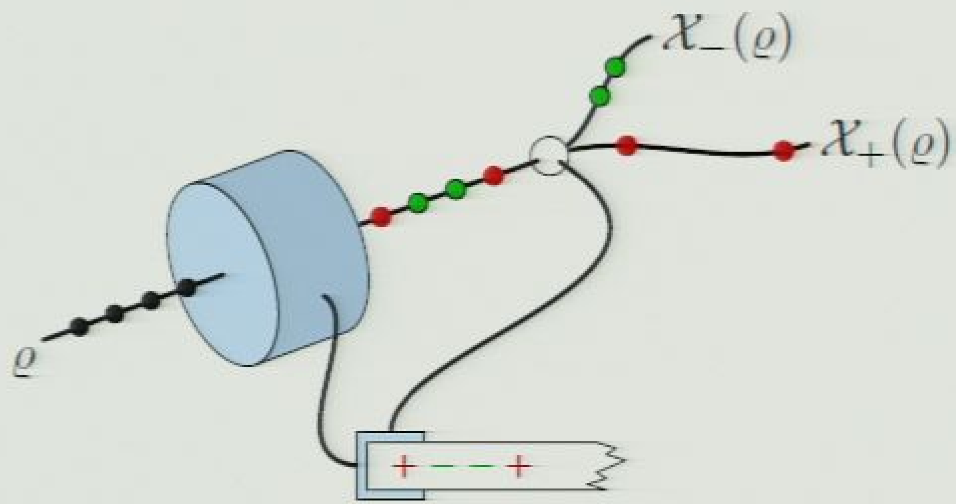


Instruments



- observables have quantum state on input and classical output
- instruments are measurement devices that have quantum state on input and in addition to classical output they have also a quantum output — state of the system changed by measurement

Instruments



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Instruments

- *operations* are completely positive trace non-increasing maps — probabilistic state change
- *channels* are completely positive trace preserving maps — state changes
- instrument $\mathcal{I}$ is a mapping from 2^{Ω} to the set of operations with specific properties:
  - $\mathcal{I}_j$ is an operation for all $j \in \Omega$
  - state ρ, after getting outcome j is obtained with probability $p_j = \mathrm{Tr}[\mathcal{I}_j(\rho)]$ and is changed to state $\mathcal{I}_j(\rho)/p_j$
  - $\sum_{j\in\Omega} \mathcal{I}_j \equiv \mathcal{I}_\Omega$ is a channel, as $\mathrm{Tr}[\mathcal{I}_\Omega(\rho)] = 1$
  - every $\mathcal{I}$ uniquely determines an observable A through $p_j = \mathrm{Tr}[\mathcal{I}_j(\rho)] = \mathrm{Tr}[\hat{A}_j \rho]$ — this will be depicted by superscipt $\mathcal{I}^{\mathsf{A}}$ — but for every A there are many corresponding instruments

Examples

- *trivial instrument:* $\mathcal{I}_j^{\mathsf{A}}(\rho) = \mathrm{Tr}[\hat{A}_j\rho]\xi$
- *conditional state preparation:* $\mathcal{I}_j^{\mathsf{A}}(\rho) = \mathrm{Tr}[\hat{A}_j\rho]\xi_j$
- *von Neumann instruments* are specific conditional state preparators where $\hat{A}_j = \xi_j \equiv \hat{P}_j$ and

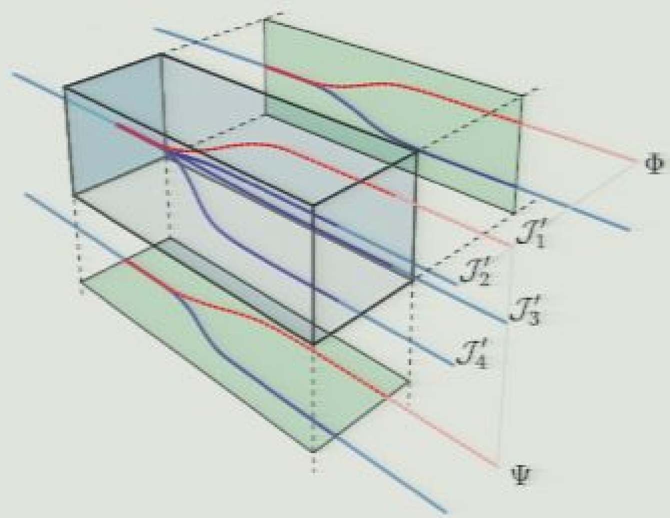
$$\mathcal{I}_j^{\mathsf{A}}(\rho) = \mathrm{Tr}[\hat{P}_j\rho]\hat{P}_j \equiv \hat{P}_j\rho\hat{P}_j$$

- von Neumann instruments are also a special case of so called *Lüders instruments*

$$\mathcal{I}_j^{\mathsf{A}}(\rho) = A_j^{1/2}\rho A_j^{1/2}$$

since $\hat{P}_j = \hat{P}_j^{1/2}$

Coexistence of Instruments



Definition

Operations Φ and Ψ are coexistent if they are in the range of one instrument $\mathcal{I}$.

Again it is sufficient to consider $\Omega = \{1, 2, 3, 4\}$ with conditions

$$\Phi = \mathcal{I}_1 + \mathcal{I}_2 \qquad \text{and} \qquad \Psi = \mathcal{I}_1 + \mathcal{I}_3$$

Observations

Suppose we are dealing with simple instruments having $\Omega = \{-1, +1\}$

Theorem

If observables A *and* B *are jointly measurable, then there exist coexistent instruments* $\mathcal{I}^{\mathsf{A}}$ *and* $\mathcal{I}^{\mathsf{B}}$.

Trivial instruments always exist for jointly measurable observables.

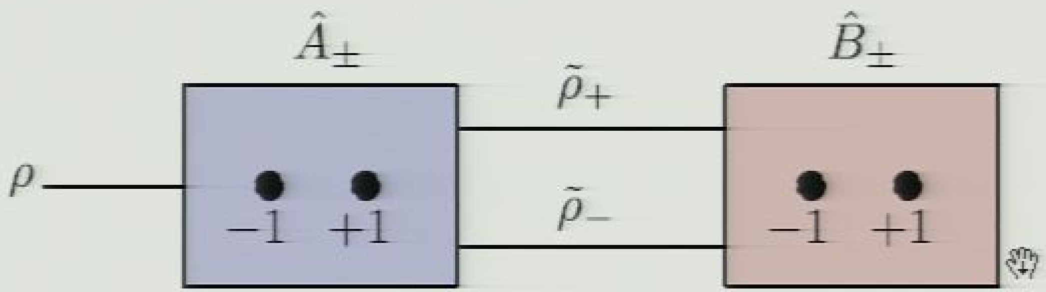
Theorem

If instruments $\mathcal{I}^{\mathsf{A}}$ *and* $\mathcal{I}^{\mathsf{B}}$ *are coexistent, then also their corresponding observables* A *and* B *are coexistent.*

T. Heinosaari, D. Reitzner, P. Stano, M. Ziman, J. Phys. A **42**, *365302 (2009),*
T. Heinosaari, M.A. Jivulescu, D. Reitzner, M. Ziman, Phys. Rev. A **82**, *032328 (2010).*

Example

Consecutive Measurements I



Suppose we first perform measurement A by the means of Lüders instrument $\tilde{\rho}_\pm \equiv \mathcal{I}^{\mathsf{A}}_\pm(\rho) = \hat{A}_\pm^{1/2}\rho A_\pm^{1/2}$, where

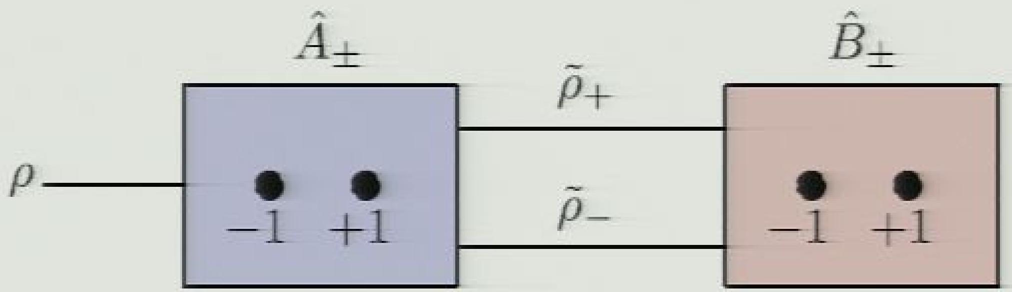
$$\hat{A}_\pm = \frac{1}{2}(\hat{I} \pm \mathbf{a}\cdot\boldsymbol{\sigma})$$

and subsequently we perform projective measurement B,

$$\hat{B}_\pm = \frac{1}{2}(\hat{I} \pm \hat{\mathbf{b}}\cdot\boldsymbol{\sigma}),$$

Example

Consecutive Measurements II



As $\hat{A}$ and $\hat{B}$ are not coexistent, so are not $\mathcal{I}^{\mathsf{A}}$ and $\mathcal{I}^{\mathsf{B}}$. However they perfom measurement G:

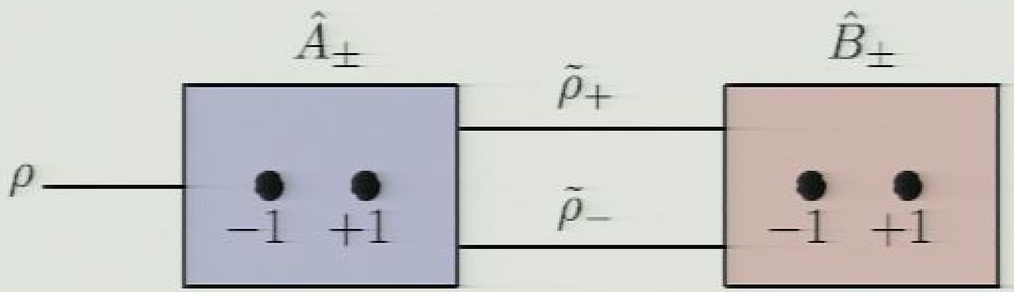
$$\hat{G}(\pm 1, \pm 1) = \frac{1}{4}\left[\hat{I} \pm \mathbf{a}\cdot\boldsymbol{\sigma} \pm \sqrt{1-a^2}\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}\right]$$

where

$$\begin{aligned}\hat{G}(+1,+1)+\hat{G}(+1,-1) &= \hat{A}_+ \text{ but} \\ \hat{G}(+1,+1)+\hat{G}(-1,+1) &= \frac{1}{2}\left[\hat{I}+\sqrt{1-a^2}\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}\right] \neq \hat{B}_+\end{aligned}$$

Example

Consecutive Measurements I



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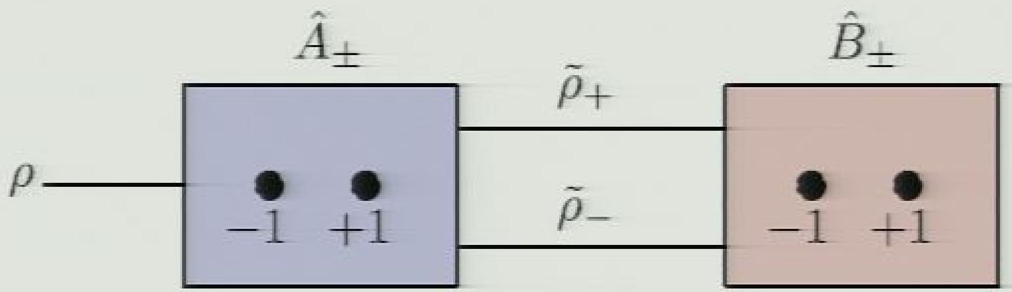

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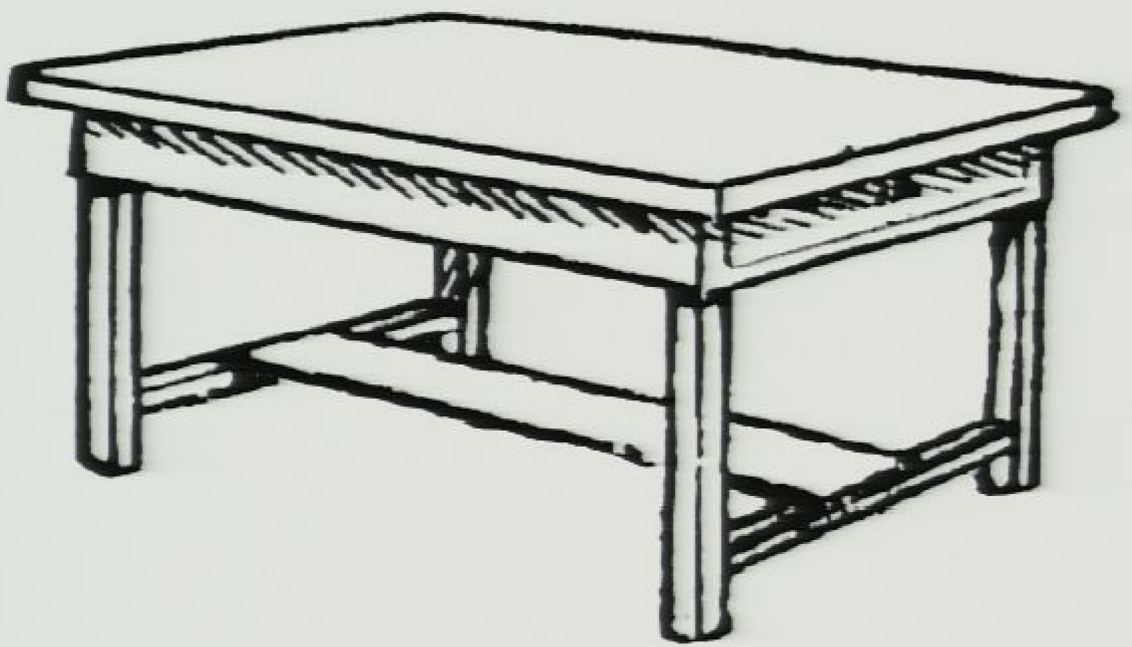
Conclusion

- one of the consequences of QM is, that not every pair of observables is jointly measurable — we can measure the height and width of a table (classically), but we cannot measure two spin components of one system precisely

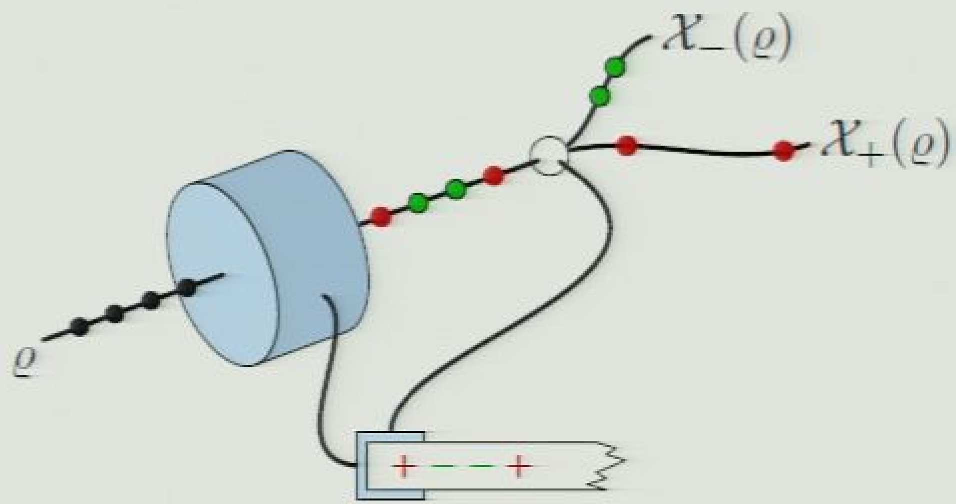


Conclusion

- one of the consequences of QM is, that not every pair of observables is jointly measurable — we can measure the height and width of a table (classically), but we cannot measure two spin components of one system precisely
- on a simple example of simple qubit observables we see:
  - unbiased observables have to have some unsharpness (imprecision) to be jointly measurable
  - biased observables show also "asymmetry" in the condition of joint measurability — the bias can allow us extract more information, in which case the other observable has to be more restricted
- as in the case of observables, coexistence can be studied on more involved measurement models — *instruments:*
  - if two instruments are coexistent, then also their corresponding observables are
  - if two observables are coexistent, then there exist corresponding instruments that also are coexistent

Thank You for attention!

Instruments



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Sufficient condition

Definition

Function $\mathfrak{S}$ from $\mathcal{E}(\mathcal{H}_2)$ to $[0;1]$,

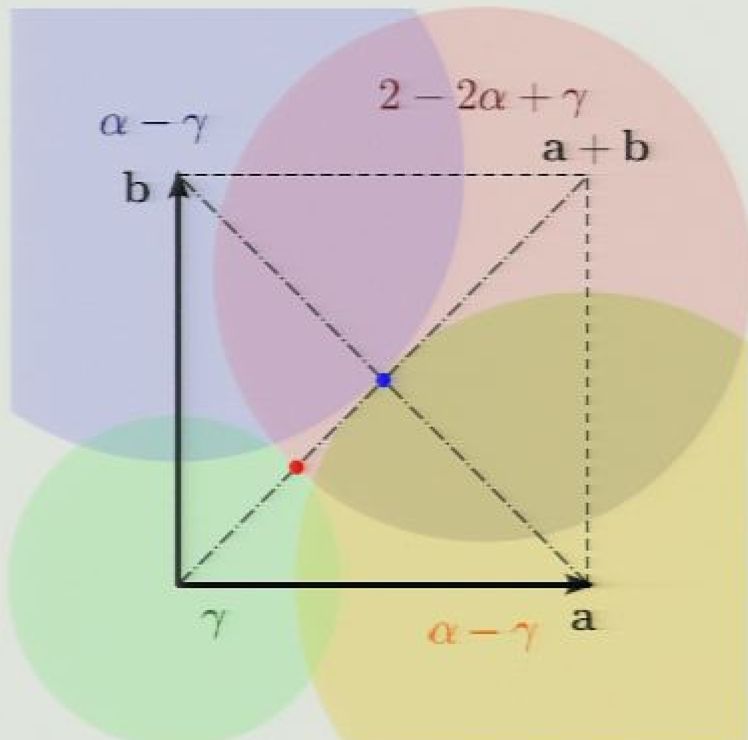
$$\mathfrak{S}(\hat{A}) = \frac{1}{2}\left\{a^2 + \alpha(2-\alpha) - \sqrt{(\alpha^2 - a^2)\left[(2-\alpha)^2 - a^2\right]}\right\}$$

will be called sharpness.

- $\mathfrak{S}(\hat{I} - \hat{A}) = \mathfrak{S}(\hat{A})$ and it does not depend on the choice of the basis
- $\mathfrak{S}(\hat{A}) = 1$ if and only if $\hat{A}$ is a non-trivial projection
- $\mathfrak{S}(\hat{A}) = 0$ if and only if $\hat{A} = \lambda I$ for some $\lambda \in [0;1]$
- $0 \leq \mathfrak{S}(\hat{A}) \leq \alpha$ and if $\alpha = 1$, then $\mathfrak{S}(\hat{A}) = a$

Not a sufficient condition!

Suppose $\alpha = \beta < 1$, $a = b = 1/\sqrt{2}$ and $\mathbf{a} \perp \mathbf{b}$



If we choose γ such that $2\alpha - 2\gamma = 1$, one common point of opposite circles lies in the "centre", while at the same time the other circles touch, but they do not have the same radiuses, so

Necessary Condition

Theorem (P. Busch, Phys. Rev. D 33, 2253 (1986))

If $\hat{A}$ and $\hat{B}$ are jointly measurable, then

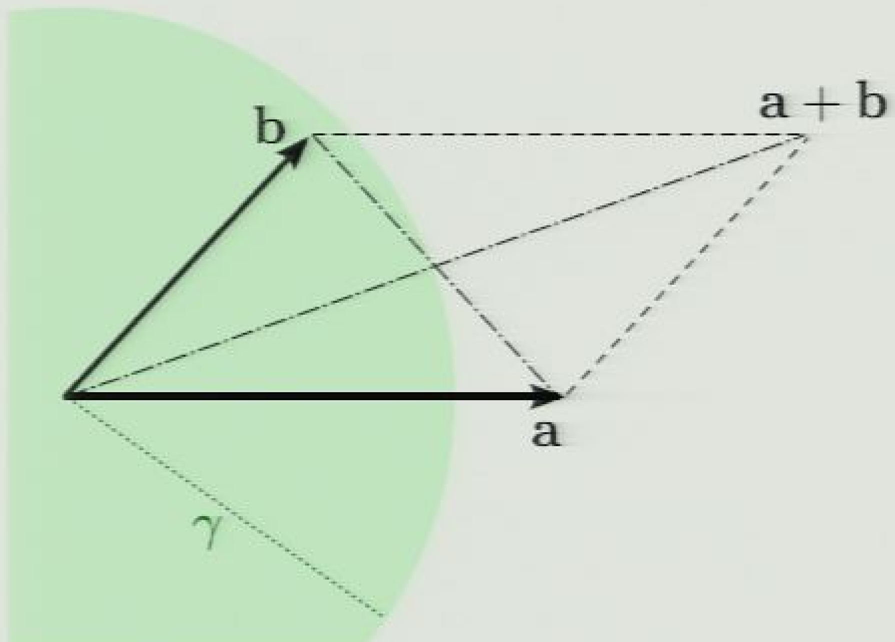
$$\|\mathbf{a}+\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\| \leq 2$$

- if $\alpha = \beta = 1$, then this is also sufficient condition ($\gamma = 1/2$) and $\mathbf{g} = (\mathbf{a}+\mathbf{b})/2$
- moreover if $\mathbf{a} \perp \mathbf{b}$, $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\| \leq 1$
- and moreover if $a = b$, $\|\mathbf{a}\| \leq 1/\sqrt{2}$ — previous example
- it is not a sufficient condition in general!

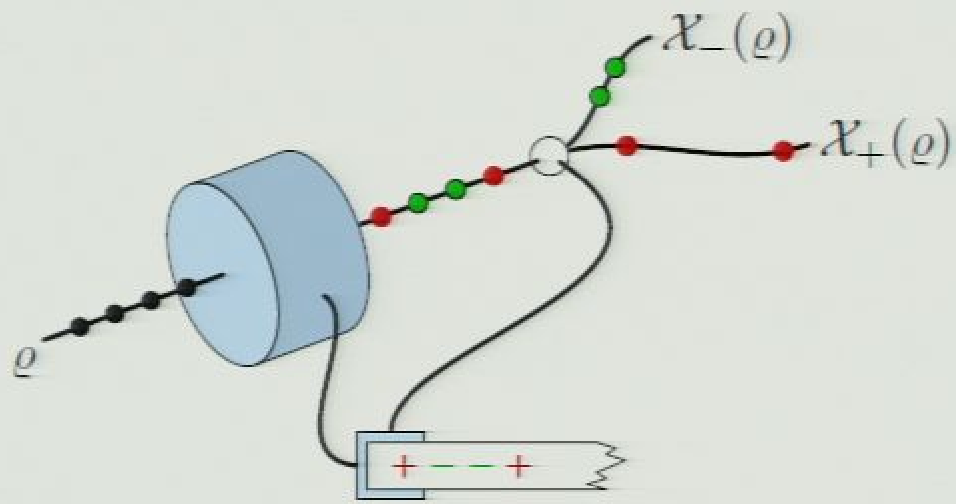
Geometrical Conditions

Taking $\hat{G} = \frac{1}{2}(\gamma\hat{I} + \mathbf{g}\cdot\boldsymbol{\sigma})$ we have following conditions:

$$\|\mathbf{g}\| \leq \gamma \qquad \hat{G} \leq \hat{A}$$
$$\hat{A} + \hat{B} \leq \hat{I} + \hat{G} \qquad \hat{G} \leq \hat{B}$$



Instruments



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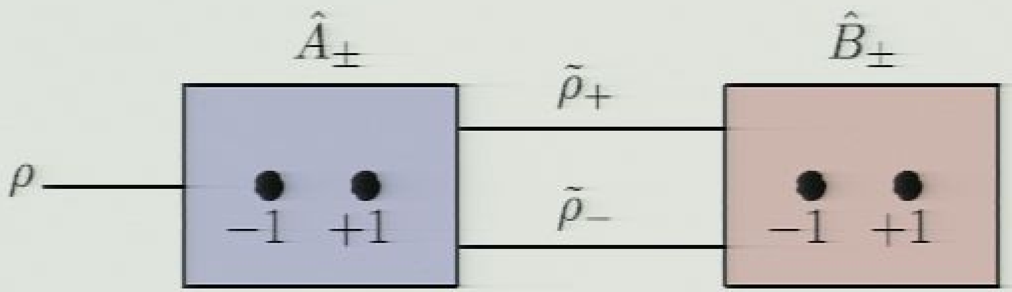
Example

Consecutive Measurements I



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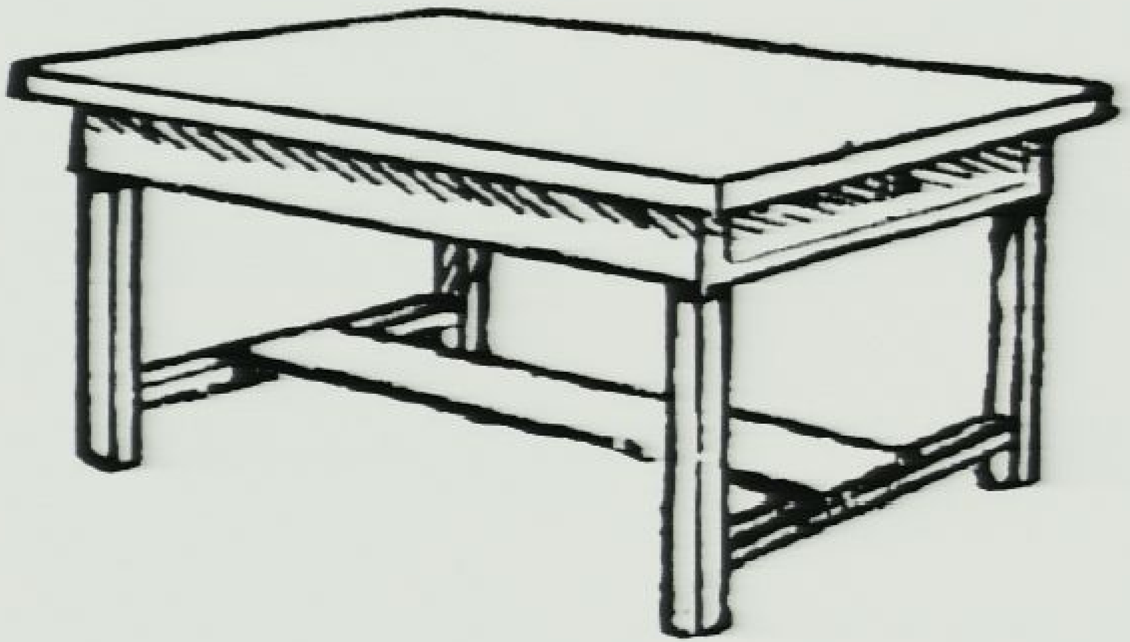

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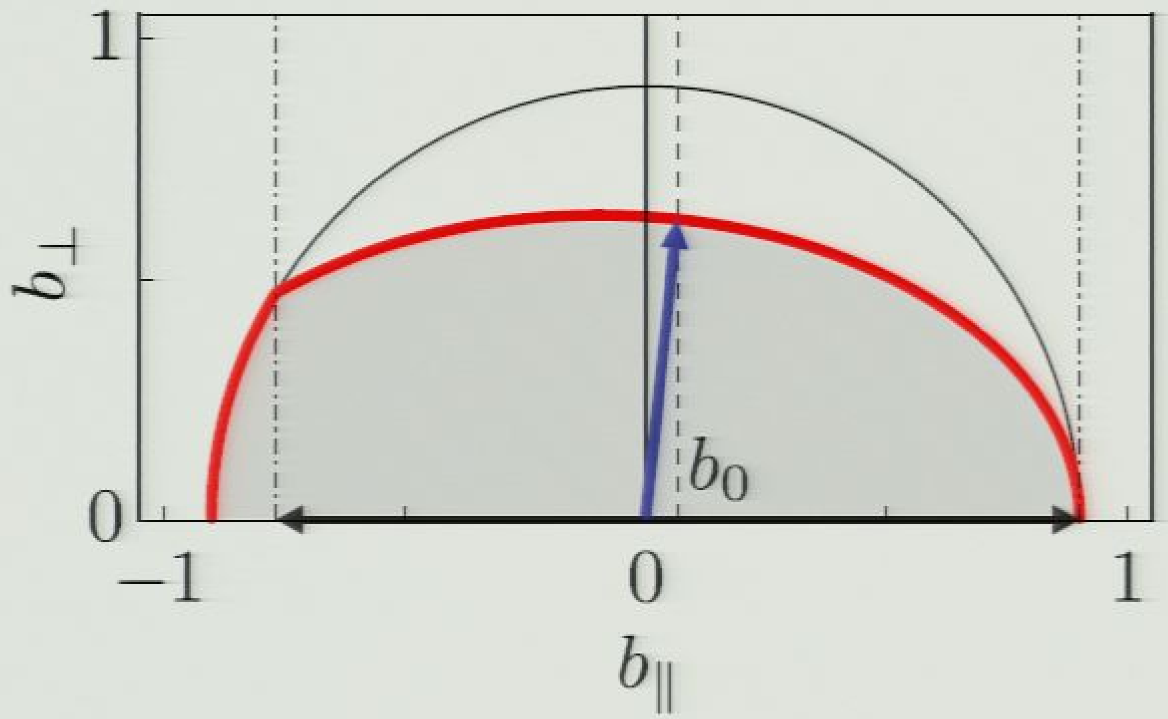

$\alpha = 0.6 = a$ and $\beta = 0.9$, when $b_0 + w = \beta$

Theorem (P. Stano, D. Reitzner and T. Heinosaari, PRA 78, 012315 (2008))

An effect $\hat{B}$ is coexistent with $\hat{A}$ if and only if it falls into one of the following three disjoint cases:

1. *if $\beta \leq 1 - \mathfrak{S}(\hat{A})$, irrespectively of $\mathbf{b}$,*
2. *if $\beta > 1 - \mathfrak{S}(\hat{A})$ and $|b_{\|} - b_0| \geq w$,*
3. *if $\beta > 1 - \mathfrak{S}(\hat{A})$, $|b_{\|} - b_0| < w$ and if and only if $b_{\perp} \leq b_{\perp}^{\max}$,*

where $b_0 = \frac{1}{a}(1-\alpha)(1-\beta)$,

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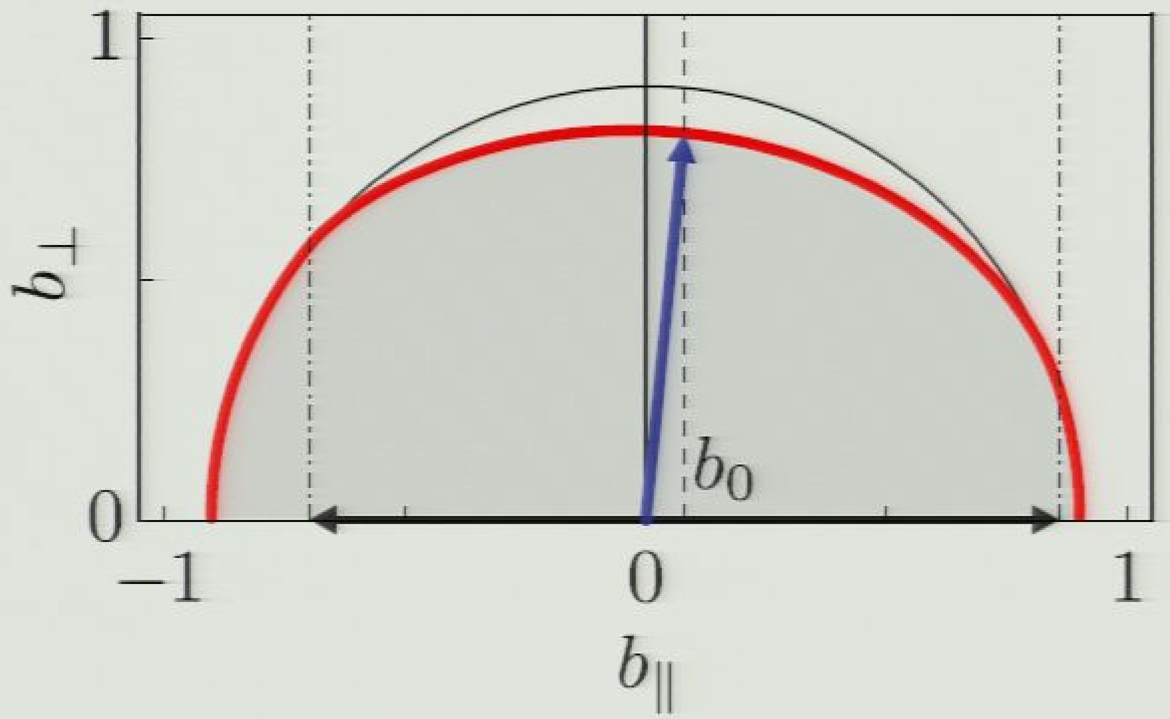

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Theorem (P. Stano, D. Reitzner and T. Heinosaari, PRA **78**, 012315 (2008))

An effect $\hat{B}$ *is coexistent with* $\hat{A}$ *if and only if it falls into one of the following three disjoint cases:*

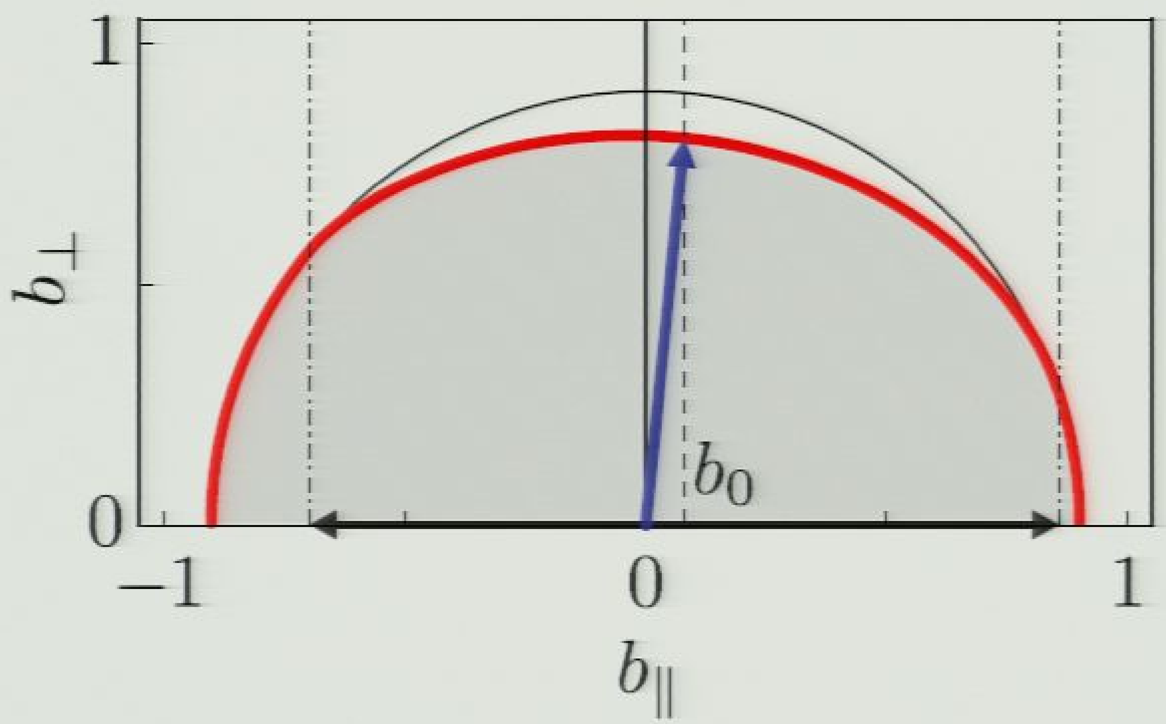
1. *if* $\beta \leq 1 - \mathfrak{S}(\hat{A})$*, irrespectively of* $\mathbf{b}$*,*
2. *if* $\beta > 1 - \mathfrak{S}(\hat{A})$ *and* $|b_{\|} - b_0| \geq w$*,*
3. *if* $\beta > 1 - \mathfrak{S}(\hat{A})$*,* $|b_{\|} - b_0| < w$ *and if and only if* $b_{\perp} \leq b_{\perp}^{\max}$*,*

where $b_0 = \frac{1}{a}(1-\alpha)(1-\beta)$*,*

$$w = \frac{1}{a}\sqrt{(1-\alpha)^2 - \beta[(1-\alpha)^2 + 1 - a^2] + \beta^2}$$

$$b_{\perp}^{\max} = \frac{1}{2a}\sqrt{[(2-\alpha)^2 - a^2]\{a^2 - [a(b_{\|} - b_0) + (1-\beta)^2\}} + \frac{1}{2a}\sqrt{[\alpha^2 - a^2]\{a^2 - [a(b_{\|} - b_0) - (1-\beta)^2\}}$$

Examples



$\alpha = 0.6$, $a = 0.5$ and $\beta = 0.9$; note: $b_0 + w \leq \beta$ (if applicable)

Other results

Joint measurement of two unsharp observables of a qubit

Sixia Yu,[1,2] Nai-le Liu,[1] Li Li,[1] and C. H. Oh[2]

[1]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China*
[2]*Centre for Quantum Technologies and Physics Department, National University of Singapore, 2 Science Drive 3, Singapore 117542*

We present a single inequality as the necessary and sufficient condition for two unsharp observables of a two-level system to be jointly measurable in a single apparatus and construct explicitly the joint observables. A complementarity inequality arising from the condition of joint measurement, which generalizes Englert's duality inequality, is derived as the trade-off between the unsharpnesses of two jointly measurable observables.

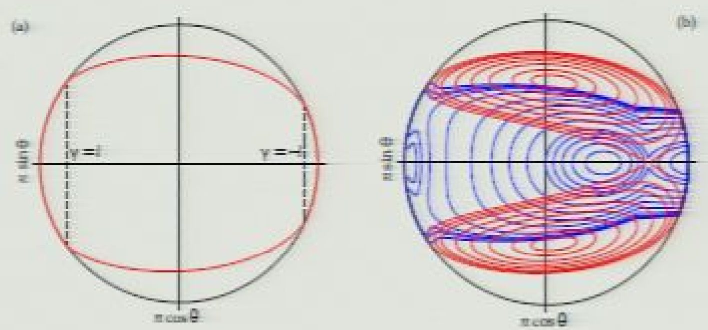
PACS numbers: 03.65.Ta, 03.67.-a

Built in the standard formalism of quantum mechanics, there are mutually exclusive but equally real aspects of quan-
tum systems, as summarized
ciple of Bohr [1]. Mutually
hibited via noncommuting ob
plementarity is quantitatively
uncertainty relationships, nam
relationships (PURs) and the
tionships (MURs).

The PURs stem from the s
density matrix describing the
the predictability of two nonco
quantum state. To test PURs tw
ments will be performed on t
bles of the quantum system a
be performed within one expe
semble.

On the other hand MURs ch
the precisions of unsharp mea
ing observables in a single ex
effort of Heisenberg [2] in de
ships was based on a simulta
sition and momentum, with t
tablished recently by Werner
wave-particle duality betwee

and $\sum_k O_k = I$) with K being the number of outcomes. By definition, a *joint measurement* of two observables $\{O_k\}$ and



Necessary Condition

Theorem (P. Busch, Phys. Rev. D 33, 2253 (1986))

If $\hat{A}$ and $\hat{B}$ are jointly measurable, then

$$\|\mathbf{a}+\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\| \leq 2$$

- if $\alpha = \beta = 1$, then this is also sufficient condition ($\gamma = 1/2$) and $\mathbf{g} = (\mathbf{a}+\mathbf{b})/2$
- moreover if $\mathbf{a} \perp \mathbf{b}$, $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}-\mathbf{b}\| \leq 1$
- and moreover if $a = b$, $\|\mathbf{a}\| \leq 1/\sqrt{2}$ — previous example
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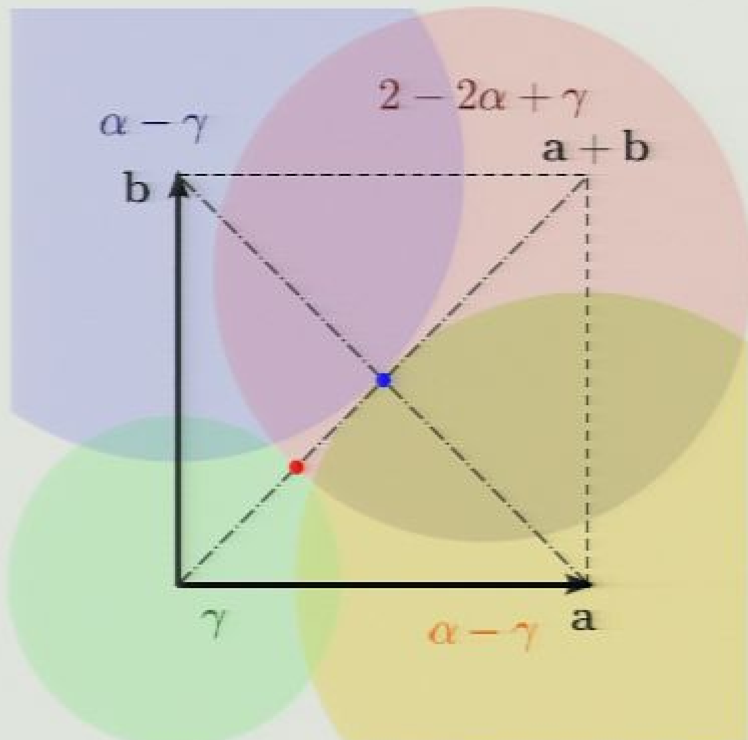
Not a sufficient condition!

Suppose $\alpha = \beta < 1$, $a = b = 1/\sqrt{2}$ and $\mathbf{a} \perp \mathbf{b}$



If we choose γ such that $2\alpha - 2\gamma = 1$, one common point of opposite circles lies in the "centre", while at the same time the other circles touch, but they do not have the same radiuses, so

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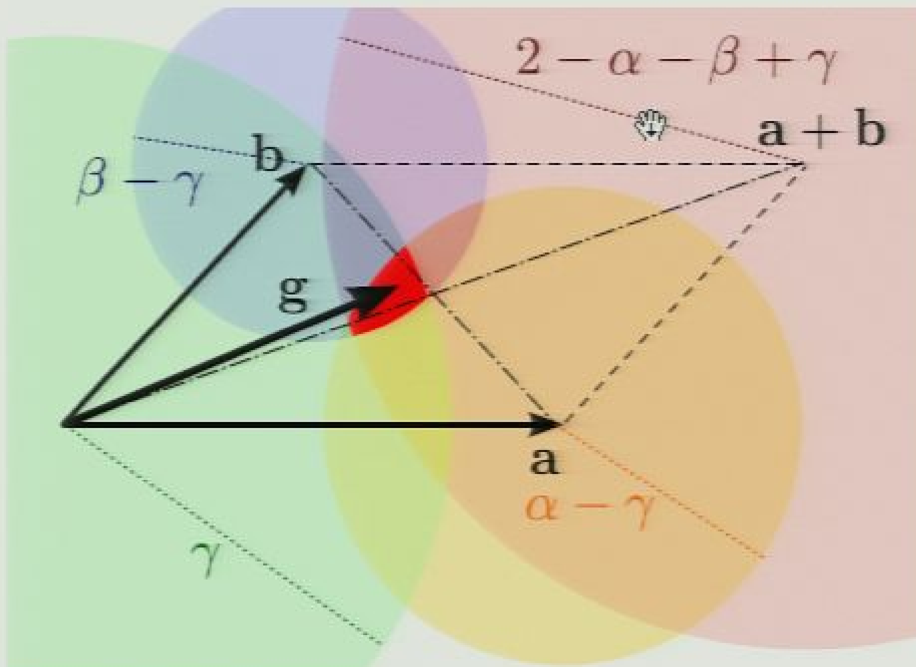
Necessary Condition



we see that

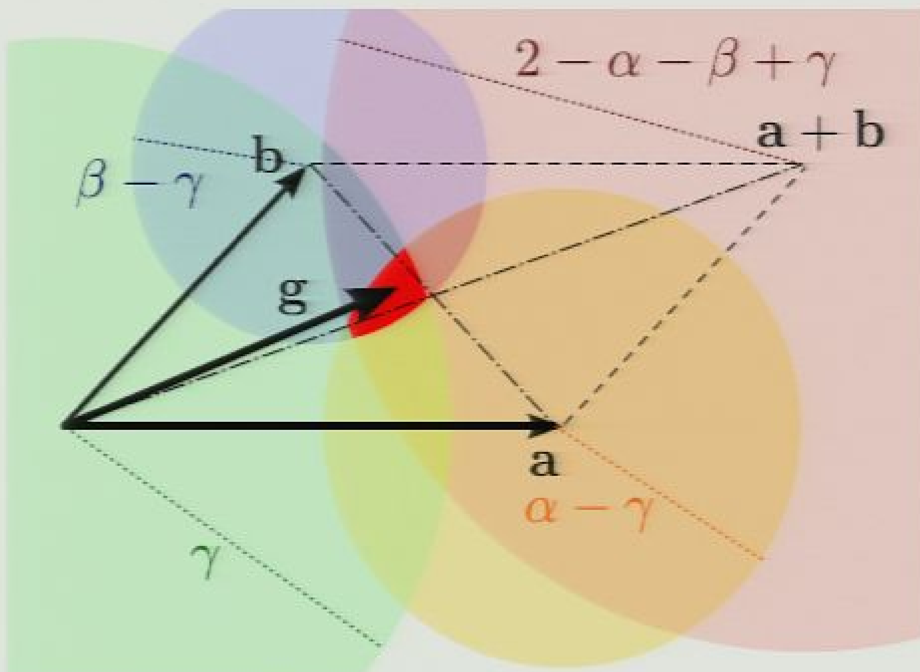
$$\|\mathbf{a}+\mathbf{b}\| + \|\mathbf{a}-\mathbf{b}\| \leq 2$$

Geometrical Conditions

Taking $\hat{G} = \frac{1}{2}(\gamma \hat{I} + \mathbf{g} \cdot \boldsymbol{\sigma})$ we have following conditions:

$$\|\mathbf{g}\| \leq \gamma \qquad \|\mathbf{g} - \mathbf{a}\| \leq \alpha - \gamma$$

$$\|\mathbf{g} - (\mathbf{a} + \mathbf{b})\| \leq 2 - \alpha - \beta + \gamma \qquad \|\mathbf{g} - \mathbf{b}\| \leq \beta - \gamma$$



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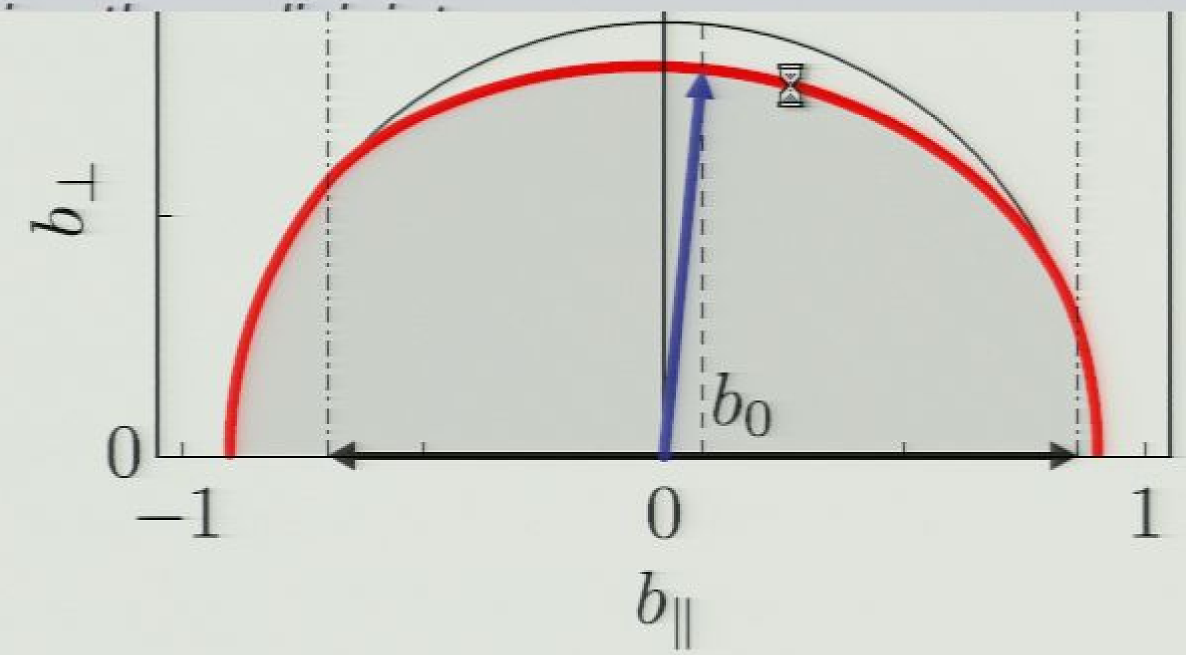
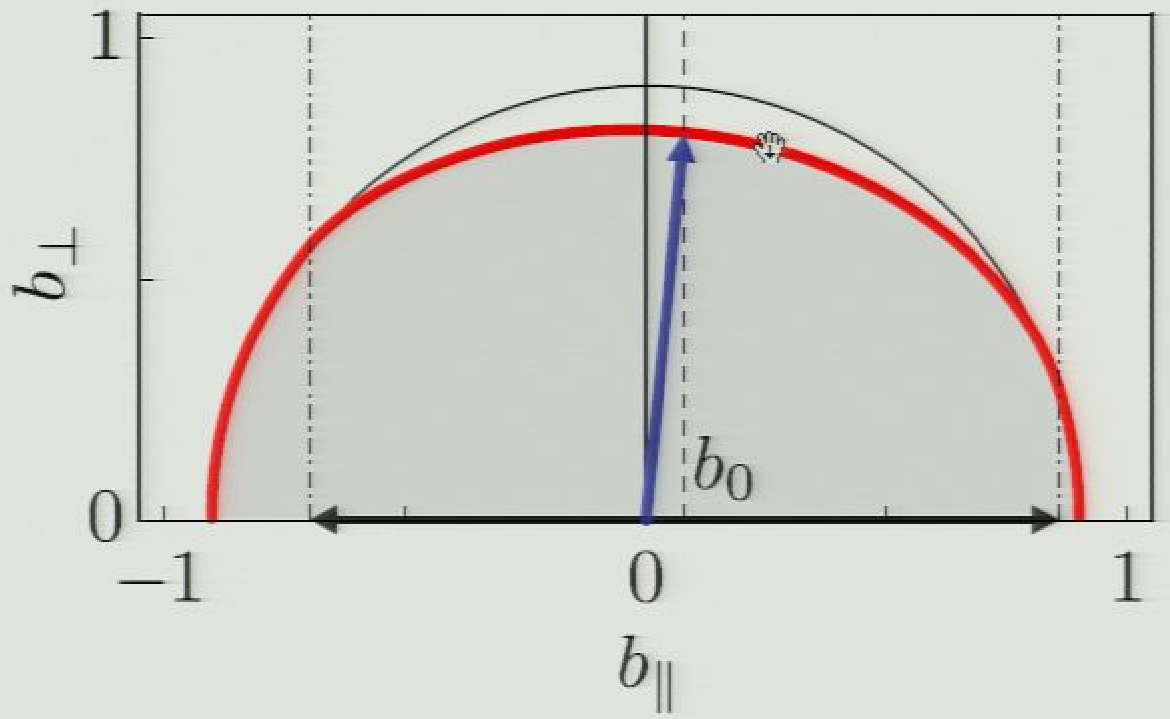
An effect $\hat{B}$ is coexistent with $\hat{A}$ if and only if it falls into one of



$\alpha = 0.6$, $a = 0.5$ and $\beta = 0.9$; note: $b_0 + w \leq \beta$ (if applicable)

Examples



$\alpha = 0.6$, $a = 0.5$ and $\beta = 0.9$; note: $b_0 + w \leq \beta$ (if applicable)