Title: Counterterms in N=8 Supergravity

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Abstract: N=8 supergravity in 4 dimensions exhibits a surprisingly favorable UV behavior -- it is known from explicit computations that the 4-point amplitudes in N=8 supergravity are finite up to 4-loop order.

I explain how a "matrix-element approach" can be used to study candidate counterterms for UV divergences in this theory. This approach both demystifies the finiteness found in previous computations, and predicts finiteness of arbitrary n-point amplitudes in N=8 supergravity below the 7-loop order. It also points to the 7-loop 4-point amplitude as the first amplitude whose finiteness is not guaranteed by any known symmetry of the theory.

Counterterms in $\mathcal{N} = 8$ supergravity

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Perimeter Institute, Jan 25, 2011

based on work with

N. Beisert, H. Elvang, D.Z. Freedman, A. Morales, S. Stieberger

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Divergences of gravity theories in $D = 4$

power counting

- naive divergence gets worse for larger loop order
- higher-point amplitudes as bad as 4-point

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- pure gravity diverges at 2-loops (counterterm: R^3)
- R^3 not supersymmetrizable
  $\Rightarrow$ pure supergravity theories finite at 2-loops

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most symmetric 4D gravity theory: $\mathcal{N} = 8$ supergravity

- $\mathcal{N} = 8$ supersymmetry
- global $SU(8)_R$ symmetry
- in fact: global $E_{7(7)}$ symmetry

Can $\mathcal{N} = 8$ supergravity be perturbatively finite?

Explicit computations [Bern, Carrasco, Dixon, Johansson, Roiban] **show:**

- the 3-loop 4-point amplitude is finite
- the 4-loop 4-point amplitude is finite

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- desired pattern: external s^L in loop numerators $\Rightarrow D_c = 4 + 6/L$

Natural questions:

- are n-point amplitudes also finite at 3- and 4-loops?
- what can we expect at 5-loops and beyond?

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Methods used to study these questions:

- Berkovits' pure-spinor formalism [Björnsson, Green, Vanhove]
- (Harmonic) superspace [Bossard, Drummond, Howe, Stelle]
- chiral & real light-cone superspace [Kallosh, Ramond, Rube]
- String theory and automorphism analysis [Green, Miller, Russo, Vanhove]

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We take a more pedestrian approach, and see how far we get!

Main idea

Shift attention from operators to their matrix elements!

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- satisfy the $SU(8)$ R-symmetry Ward-Takahashi Identities
- are compatible with $E_{7(7)}$ - symmetry, in particular:

$$\lim_{p_1 \to 0} \langle \phi \ldots \rangle = 0 .$$

(Single-soft limit (SSL) of any scalar external line ϕ must vanish)

Counterterms: Overview of possible $D^{2k}R^n$ counterterms

	4-point	5	6	7	8	9
3-loop	R^4					
4	D^2R^4	R^5				
5	D^4R^4	D^2R^5	R^6			
6	D^6R^4	D^4R^5	D^2R^6	R^7		
7	D^8R^4	D^6R^5	D^4R^6	D^2R^7	R^8	
8	$D^{10}R^4$	D^8R^5	D^6R^6	D^4R^7	D^2R^8	R^9

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1. SUSY and R-symmetry constraints
2. $E_{7\,7}$ constraints at 3, 5 and 6 loops
3. SUSY & $E_{7\,7}$ constraints at 7 loops and beyond

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Constraints on n-point matrix elements of $D^{2k}R^n$

- must be local

$$\text{locality} \iff \text{polynomial in spinor brackets } \langle ij \rangle,\ [ij]$$

(using 4D spinor helicity formalism $p_i^{\alpha\dot{\alpha}} = |i\rangle[i|$)

- dimensional analysis: spinor bracket polynomial has degree $2n + 2k$

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$$|i\rangle \to t|i\rangle, \quad |i] \to t^{-1}|i] \quad \Rightarrow \quad \langle \ldots \rangle_{D^{2k}R^n} \to t^{-2h_i}\langle \ldots \rangle_{D^{2k}R^n}$$

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Example: R^4

Unique $--++$ matrix element: $\langle --++\rangle_{R^4} = \langle 12\rangle^4[34]^4$

MHV $\Rightarrow$ All others matrix elements fixed by SUSY

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MHV Counterterms [Elvang, Freedman, MK]

The 5-point superamplitude of R^5

- Pure-graviton matrix element $--+++$ unique:

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$$|1\rangle \to |1\rangle + z[23]|X\rangle \text{ and cyclic in } 1,2,3 .$$

Under this shift $+++--$ has a pole $\Rightarrow$ indeed non-local

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There is no $\mathcal{N}=8$ supersymmetrization of R^5!

$$s + t + u = 0$$

$$s^2 + t^2 + u^2 \neq 0$$

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MHV Counterterms [Elvang, Freedman, MK]

Generalization

- Recall R^5:

$$\langle --+++\rangle_{R^5} = \frac{\langle 45\rangle^8}{\langle 12\rangle^4} \times [34]^2[45]^2[53]^2 .$$

- Generalization to all $n \geq 5$ straight-forward:

$$\langle --+\cdots+\rangle_{R^n} \sim \frac{\langle n-1, n\rangle^8}{\langle 12\rangle^4} \times [..]^{2(n-2)} .$$

Three-line shift argument works!

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  $\Rightarrow$ Need at least $D^8 \sim \langle ..\rangle^4[..]^4$ to soak up non-locality

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No MHV R^n, D^2R^n, D^4R^n, D^6R^n operator for $n \geq 5$!

Counterterms: MHV exclusions

	4-point	5	6	7	8	9
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NMHV n-point matrix elements of $D^{2k}R^n$

- have more $\langle .. \rangle$ in numerator at NMHV than MHV level
- Use general solution to SUSY Ward identities [Elvang, Freedman, MK]
  - assume local basis amplitudes
  - then try to use shift show non-locality of non-basis amplitude

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No NMHV R^n and D^2R^n operators

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No MHV

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Counterterms: $L < 7$ exclusions

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No potential $D^{2k}R^n$ counterterms with $n>4$ below seven-loop order

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Exclusion bounds saturated!

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What about non-gravitational operators?

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	ϕR^4	$\phi^2 R^4$	$\phi^3 R^4$	$\phi^4 R^4$	$\phi^5 R^4$	$\phi^6 R^4$	$\phi^7 R^4$
4	~~$D^2 R^4$~~	~~R^5~~	ϕR^5	$\phi^2 R^5$	$\phi^3 R^5$	$\phi^4 R^5$	$\phi^5 R^5$	$\phi^6 R^5$
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	ϕR^6	$\phi^2 R^6$	$\phi^3 R^6$	$\phi^4 R^6$	$\phi^5 R^6$
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	ϕR^7	$\phi^2 R^7$	$\phi^3 R^7$	$\phi^4 R^7$
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	ϕR^8	$\phi^2 R^8$	$\phi^3 R^8$
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9	$\phi^2 R^9$

No MHV

No NMHV

non-gravitational counterterms: NMHV exclusions

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	$\phi^4 R^4$	$\phi^5 R^4$	$\phi^6 R^4$	$\phi^7 R^4$
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	$\phi^3 R^5$	$\phi^4 R^5$	$\phi^5 R^5$	$\phi^6 R^5$
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	$\phi^2 R^6$	$\phi^3 R^6$	$\phi^4 R^6$	$\phi^5 R^6$
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	ϕR^7	$\phi^2 R^7$	$\phi^3 R^7$	$\phi^4 R^7$
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	ϕR^8	$\phi^2 R^8$	$\phi^3 R^8$
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9	$\phi^2 R^9$

No MHV

No NMHV

Counterterms: N^2MHV exclusions

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	$\phi^6 R^4$	$\phi^7 R^4$
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	$\phi^5 R^5$	$\phi^6 R^5$
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	$\phi^4 R^6$	$\phi^5 R^6$
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	$\phi^3 R^7$	$\phi^4 R^7$
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	$\phi^3 R^8$
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

No MHV

No NMHV

No N^2MHV

The pattern continues!

Conj.: [Elvang, Freedman, MK]

Proof: [Drummond, Heslop, Howe]

Counterterms: beyond N^2MHV exclusions

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

No MHV

No NMHV

No N^2MHV

The pattern continues!

Conj.: [Elvang, Freedman, MK]

Proof: [Drummond, Heslop, Howe]

Counterterms: below 7-loop level

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

No potential counterterms with $n > 4$ **below 7 loops**!

Counterterms: below 7-loop level

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

1. SUSY and R-symmetry constraints

2. $E_{7\,7}$ constraints at 3, 5 and 6 loops

3. SUSY & $E_{7\,7}$ constraints at 7 loops and beyond

Counterterms: below 7-loop level

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

Non-linear supersymmetrizations of R^4, D^4R^4 and D^6R^4 are UNIQUE!

$E_{7(7)}$ symmetry and the operators R^4, D^4R^4, D^6R^4

$E_{7(7)}$ symmetry in $\mathcal{N}=8$ supergravity

- In $\mathcal{N}=8$ sugra, $E_{7(7)}$ is spontaneously-broken to $SU(8)_R$
- The 70 scalars of $\mathcal{N}=8$ are the Goldstone bosons
  $\Longrightarrow$ amplitudes must vanish in soft-scalar limit
- Continuous $E_{7(7)}$ in $\mathcal{N}=8$ is preserved at loop level!
  [Bossard, Hillmann, Nicolai]

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- Continuous $E_{7(7)}$ in $\mathcal{N}=8$ is preserved at loop level! [Bossard, Hillmann, Nicolai]

4-point matrix elements

$$\begin{aligned}
\langle ....\rangle_{R^4} &= stu \times M_4^{\text{sugra}}(--++) \\
\langle ....\rangle_{D^4R^4} &= (s^2+t^2+u^2)\, stu \times M_4^{\text{sugra}}(--++) \\
\langle ....\rangle_{D^6R^4} &= (s^3+t^3+u^3)\, stu \times M_4^{\text{sugra}}(--++)
\end{aligned}$$

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\end{aligned}$$

SSL of 4-point matrix elements vanish $\Rightarrow$ go to 6-point NMHV

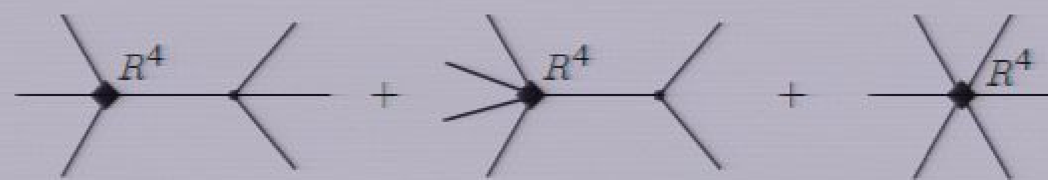
Counterterms: R^4

	4-point	5	6	7	8	9	10	11
3-loop	R^4 ✓	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

R^4 at 6-point NMHV level [Elvang, MK]

Desired matrix element

$$\langle \varphi \bar{\varphi} + + - - \rangle_{R^4} =$$



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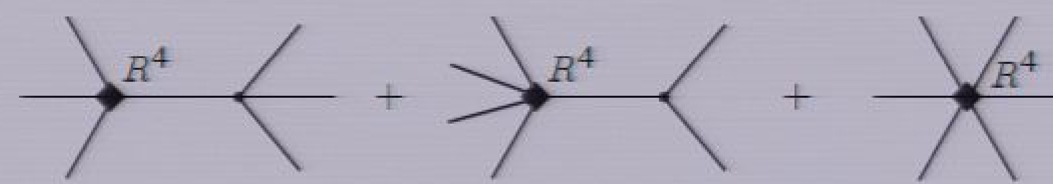

Direct computation

need non-linear $\mathcal{N} = 8$ SUSY completion of $R^4 \Rightarrow$ very hard problem!

R^4 at 6-point NMHV level [Elvang, MK]

Trick: Use closed string tree amplitude [see also: Dixon, Brödel]

- Recall effective action

$$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3\zeta(3)e^{-6\phi}R^4 - \zeta(5)\alpha'^5 e^{-10\phi}D^4R^4 + \frac{2}{3}\alpha'^6\zeta(3)^2 e^{-12\phi}D^6R^4 + \dots$$

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- BUT: global symmetries of $\mathcal{N}=8$ supergravity broken for $\alpha' \neq 0$

$$\begin{array}{ccc} E_{7(7)}\ \text{(133 generators)} & \xrightarrow{\text{explicit}} & O(6,6)\quad \text{(66 generators)} \\ \text{spontaneous} \downarrow\ \text{70 GSB} & & \text{36 GSB} \downarrow \text{spontaneous} \\ SU(8) & \xrightarrow{\text{explicit}} & SU(4)\times SU(4) \end{array}$$

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- $e^{-6\phi}R^4$ coincides with R^4 at 4-point, but not for $n > 4$

R^4 at 6-point NMHV level [Elvang, MK]

$SU(8)$ broken $\Rightarrow$ Three distinct soft scalar limits of $e^{-6\phi}R^4$

$$\lim_{p_1\to 0}\left\langle \varphi_g\bar{\varphi}_g ++--\right\rangle_{e^{-6\phi}R^4} = 6\times[34]^4\langle 56\rangle^4\,,\qquad \lim_{p_1\to 0}\left\langle \varphi_s\bar{\varphi}_s ++--\right\rangle_{e^{-6\phi}R^4} = 0\,,$$

$$\lim_{p_1\to 0}\left\langle \varphi_f\bar{\varphi}_f ++--\right\rangle_{e^{-6\phi}R^4} = 3\times[34]^4\langle 56\rangle^4\,.$$

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Essential ingredients [see also: Dixon, Brödel]

- 6-point open-string tree amplitudes [Stieberger]
- KLT to obtain closed-string tree ampltiudes

R^4 at 6-point NMHV level [Elvang, MK]

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$$\lim_{p_1\to 0} \left\langle \varphi_g \bar{\varphi}_g + + - - \right\rangle_{e^{-6\phi}R^4} = 6 \times [34]^4 \langle 56 \rangle^4 , \quad \lim_{p_1\to 0} \left\langle \varphi_s \bar{\varphi}_s + + - - \right\rangle_{e^{-6\phi}R^4} = 0 ,$$

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Puzzle

tree-level $SU(8)$-violating operator vs. 3-loop $SU(8)$-invariant operator

How can this possibly help to extract soft-limits of R^4 ?

R^4 at 6-point NMHV level [Elvang, MK]

Solution

- decompose matrix elements by irreps of $SU(8)$
- take singlet $\Rightarrow$ matrix elements of SUSY and $SU(8)$-invariant op.
- R^4 is the unique such operator of right dimension

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Single-scalar soft limit of R^4

singlet contribution from $SU(8)$-averaging:

$$\langle \varphi \overline{\varphi} {+}{+}{-}{-} \rangle_{R^4} = \frac{1}{35} \langle \varphi_g \bar{\varphi}_g {+}{+}{-}{-} \rangle_{e^{-6\phi} R^4} - \frac{16}{35} \langle \varphi_f \bar{\varphi}_f {+}{+}{-}{-} \rangle_{e^{-6\phi} R^4} + \frac{18}{35} \langle \varphi_s \bar{\varphi}_s {+}{+}{-}{-} \rangle_{e^{-6\phi} R^4}$$

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singlet contribution from $SU(8)$-averaging:

$$\langle \varphi \overline{\varphi}{+}{+}{-}{-} \rangle_{R^4} = \frac{1}{35} \langle \varphi_g \bar{\varphi}_g {+}{+}{-}{-} \rangle_{e^{-6\phi} R^4} - \frac{16}{35} \langle \varphi_f \bar{\varphi}_f {+}{+}{-}{-} \rangle_{e^{-6\phi} R^4} + \frac{18}{35} \langle \varphi_s \bar{\varphi}_s {+}{+}{-}{-} \rangle_{e^{-6\phi} R^4}$$

$$\Rightarrow \quad \lim_{p_1 \to 0} \langle \varphi \overline{\varphi} {+}{+}{-}{-} \rangle_{R^4} = -\frac{6}{5} [34]^4 \langle 56 \rangle^4 \quad .$$

R^4 at 6-point NMHV level [Elvang, MK]

Solution

- decompose matrix elements by irreps of $SU(8)$
- take singlet $\Rightarrow$ matrix elements of SUSY and $SU(8)$-invariant op.
- R^4 is the unique such operator of right dimension

Single-scalar soft limit of R^4

singlet contribution from $SU(8)$-averaging:

$$\langle \varphi\overline{\varphi}++--\rangle_{R^4} = \frac{1}{35}\langle \varphi_g\bar{\varphi}_g++--\rangle_{e^{-6\phi}R^4} - \frac{16}{35}\langle \varphi_f\bar{\varphi}_f++--\rangle_{e^{-6\phi}R^4} + \frac{18}{35}\langle \varphi_s\bar{\varphi}_s++--\rangle_{e^{-6\phi}R^4}$$

$$\Rightarrow \quad \lim_{p_1\to 0}\langle \varphi\overline{\varphi}++--\rangle_{R^4} = -\frac{6}{5}[34]^4\langle 56\rangle^4 \neq 0\,.$$

R^4 violates continuous $E_{7(7)}$

$\Rightarrow$ R^4 ruled out as counterterm

$\Rightarrow$ explains 3-loop finiteness of $\mathcal{N}=8$ supergravity!

R^4 at 6-point NMHV level [Elvang, MK]

Automorphism analysis of [Green, Miller, Russo, Vanhove]

- SUSY constraints on moduli dependence of $f_{R^4}(\varphi)R^4$ in string theory
- simplest in $D = 10$, infer lower D through consistency conditions
- leads to Laplace equation for $f_{R^4}(\varphi)$ on $E_{7(7)}/SU(8)$, in $D = 4$:

$$(\Delta + 42) f_{R^4}(\varphi) = 0\,, \quad \Delta = \Big[\frac{\partial}{\partial\varphi^{1234}}\frac{\partial}{\partial\varphi^{5678}} + \text{ 34 inequiv. perms}\Big] + \dots\ .$$

R^4 at 6-point NMHV level [Elvang, MK]

Automorphism analysis of [Green, Miller, Russo, Vanhove]

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Matching soft-limits

use soft-limit result $\quad \langle\varphi\,\overline{\varphi}++--\rangle_{R^4}\;=\;-\frac{6}{5}[34]^4\langle 56\rangle^4$

to reconstruct $SU(8)$-invariant f_{R^4}:

$$f_{R^4}(\varphi)=1-\frac{6}{5}\big[\varphi^{1234}\varphi^{5678}+34\text{ inequiv. perms}\big]+\dots\ .$$

R^4 at 6-point NMHV level [Elvang, MK]

Automorphism analysis of [Green, Miller, Russo, Vanhove]

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Matching soft-limits

use soft-limit result $\langle \varphi\overline{\varphi} + + - - \rangle_{R^4} = -\frac{6}{5}[34]^4\langle 56\rangle^4$

to reconstruct $SU(8)$-invariant f_{R^4}:

$$f_{R^4}(\varphi) = 1 - \frac{6}{5}\left[\varphi^{1234}\varphi^{5678} + 34 \text{ inequiv. perms}\right] + \dots\ .$$

$f_{R^4}(\varphi)$ precisely satisfies Laplace equation $\Rightarrow$ valuable consistency check!

$E_{7(7)}$ excludes 3-loop counterterm R^4

	4-point	5	6	7	8	9	10	11
3-loop	~~R^4~~ E_{77}	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

Counterterms: $D^4 R^4$

	4-point	5	6	7	8	9	10	11
3-loop	~~R^4~~ E_{77}	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	$D^4 R^4$ ✓	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

$D^4 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

$D^4 R^4$: same trick works again!

Recall:

$$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3 \zeta(3) e^{-6\phi} R^4 - \zeta(5)\alpha'^5 e^{-10\phi} D^4 R^4 + \tfrac{2}{3}\alpha'^6 \zeta(3)^2 e^{-12\phi} D^6 R^4 + \ldots$$

D^4R^4 at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

D^4R^4: same trick works again!

Recall:

$$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3\zeta(3)e^{-6\phi}R^4 - \zeta(5)\alpha'^5 e^{-10\phi}D^4R^4 + \frac{2}{3}\alpha'^6\zeta(3)^2 e^{-12\phi}D^6R^4 + \dots$$

Using

- Stieberger's open string 6-point amplitudes
- KLT
- SUSY Ward identities
- $SU(8)$-averaging
- uniqueness of D^4R^4

gives:

$$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{D^4R^4} = -\frac{6}{7}[12]^4\langle 34\rangle^4 \sum_{i<j} s_{ij}^2$$

D^4R^4 at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

D^4R^4 violates continuous $E_{7(7)}$

$\Rightarrow$ D^4R^4 ruled out as counterterm

$\Rightarrow$ predicts 5-loop finiteness of $\mathcal{N}=8$ supergravity!

soft-scalar limit consistent with automorphism analysis

$$f_{D^4R^4}(\varphi) = 1 - \frac{12}{7}\varphi^{1234}\varphi^{5678} + \ldots \qquad \text{satisfies} \qquad (\Delta + 60) f_{D^4R^4}(\varphi) = 0\,.$$

Indication of 5-loop finiteness from different considerations

- explicit superspace construction of $E_{7(7)}$-invariant ops [Howe, Lindstrom; Kallosh; Bossard, Howe, Stelle]
- pure-spinor formalism [Green, Russo, Vanhove]
- light-cone superspace [Kallosh, Ramond, Rube]

$E_{7(7)}$ excludes 5-loop counterterm D^4R^4

	4-point	5	6	7	8	9	10	11
3-loop	~~R^4~~ E_{77}	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~D^2R^4~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	~~D^4R^4~~ E_{77}	~~D^2R^5~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	D^6R^4 ✓	~~D^4R^5~~	~~D^2R^6~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	D^8R^4 ✓	~~D^6R^5~~	D^4R^6 ✓	~~D^2R^7~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10}R^4$ ✓	D^8R^5 ✓	D^6R^6 ✓	D^4R^7 ✓	D^2R^8 ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

Counterterms: $D^6 R^4$

	4-point	5	6	7	8	9	10	11
3-loop	~~R^4~~ E_{77}	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	~~$D^4 R^4$~~ E_{77}	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	$D^6 R^4$ ✓	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

$D^6 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

$D^6 R^4$: cannot naively apply same approach!

Using

- Stieberger's open string 6-point amplitudes
- KLT
- SUSY Ward identities
- $SU(8)$-averaging

gives:

$$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{(e^{-12\phi} D^6 R^4)_{\mathrm{avg}}} = -\frac{33}{35}[12]^4 \langle 34\rangle^4 \sum_{i<j} s_{ij}^3 \,.$$

BUT: $$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{(e^{-12\phi} D^6 R^4)_{\mathrm{avg}}} \neq \lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{D^6 R^4} \,.$$

$D^6 R^4$ at 6-point NMHV level

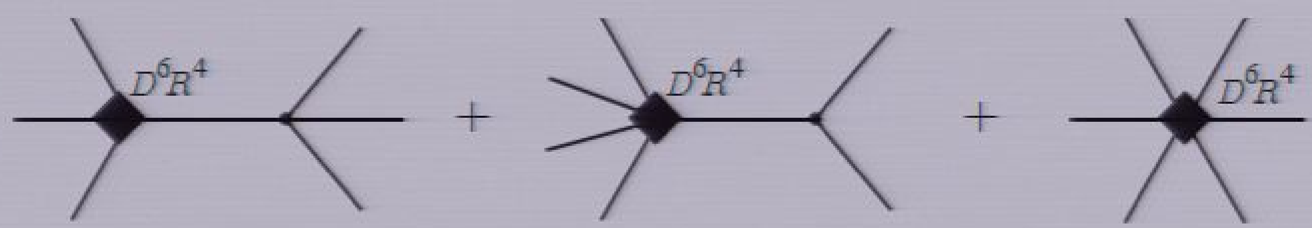
[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The problem

$$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3 \zeta(3) e^{-6\phi} R^4 - \zeta(5)\alpha'^5 e^{-10\phi} D^4 R^4 + \frac{2}{3}\alpha'^6 \zeta(3)^2 e^{-12\phi} D^6 R^4 + \dots$$

Various contributions to 6-point closed string amplitudes at $O(\alpha'^6)$:

$$\langle + + - - \varphi\bar{\varphi}\rangle\Big|_{\alpha'^6} =$$



D^6R^4 at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The problem

$$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3\zeta(3)e^{-6\phi}R^4 - \zeta(5)\alpha'^5 e^{-10\phi}D^4R^4 + \tfrac{2}{3}\alpha'^6\zeta(3)^2 e^{-12\phi}D^6R^4 + \dots$$

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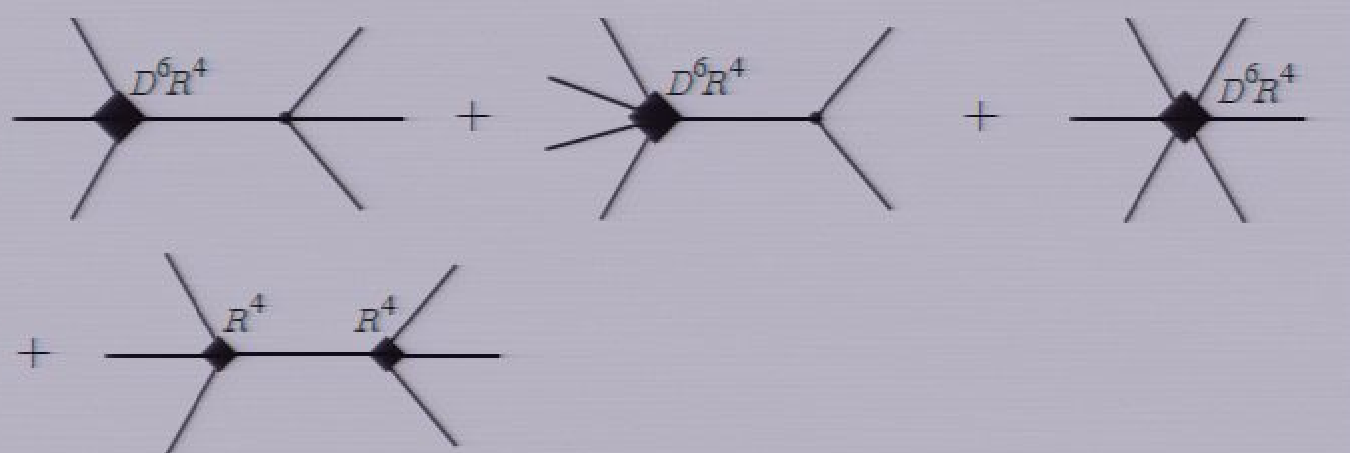

D^6R^4 at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The problem

$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3\zeta(3)e^{-6\phi}R^4 - \zeta(5)\alpha'^5 e^{-10\phi}D^4R^4 + \frac{2}{3}\alpha'^6\zeta(3)^2 e^{-12\phi}D^6R^4 + \ldots$

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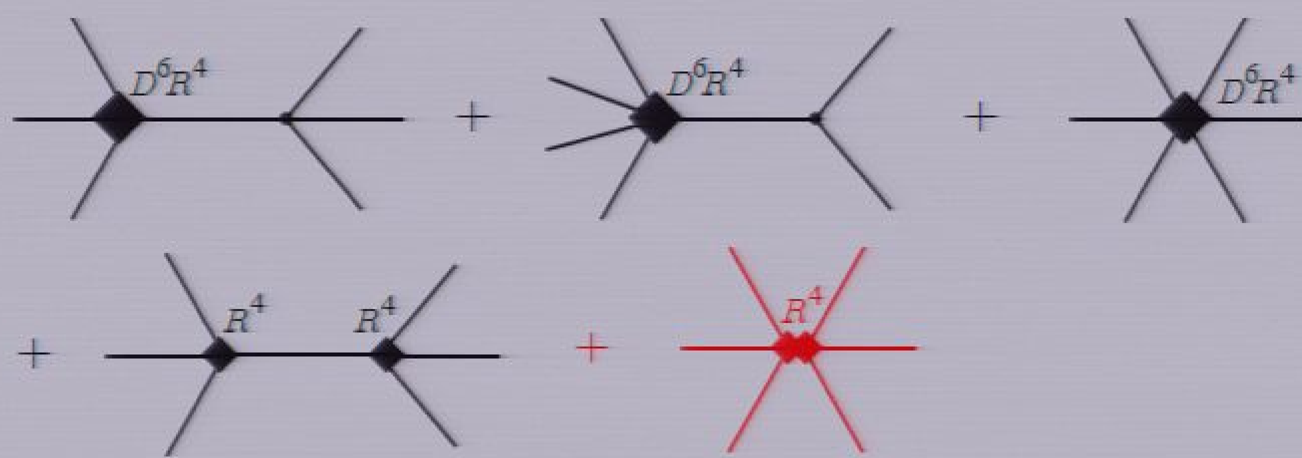
$\langle ++--\varphi\bar{\varphi}\rangle\Big|_{\alpha'^6} =$



local 6-point $O(\alpha'^6)$ terms in the SUSY completion of R^4 are problematic!

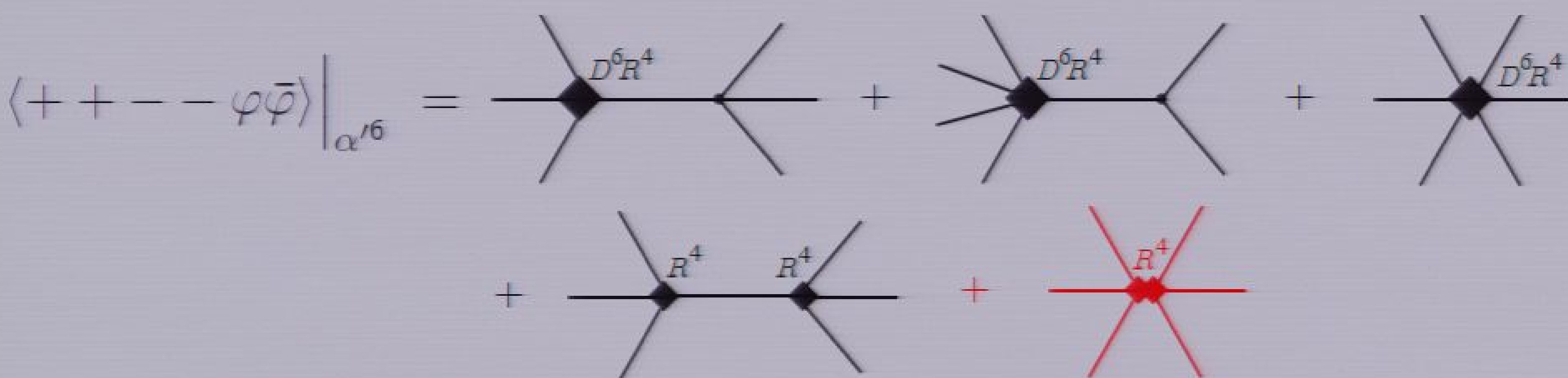
$D^6 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The problem

$$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3\zeta(3)e^{-6\phi}R^4 - \zeta(5)\alpha'^5 e^{-10\phi}D^4R^4 + \frac{2}{3}\alpha'^6\zeta(3)^2 e^{-12\phi}D^6R^4 + \ldots$$

Various contributions to 6-point closed string amplitudes at $O(\alpha'^6)$:



local 6-point $O(\alpha'^6)$ terms in the SUSY completion of R^4 are problematic!

The goal

Need to extract soft-limit of independent D^6R^4 operator only!

$D^6 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The solution

Compute dependent $(R^4)^2$ contribution at $O(\alpha'^6)$ first:

$$\langle \cdots\cdots \rangle_{(R^4)^2} = \text{pole terms} + \text{local}\,.$$

D^6R^4 at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The solution

Compute dependent $(R^4)^2$ contribution at $O(\alpha'^6)$ first:

$$\langle \ldots\ldots \rangle_{(R^4)^2} \;=\; \text{pole terms} + \text{local}\,.$$

- use Feynman diagrams to compute pole terms
- fix local terms by imposing SUSY WI's and Bose symmetry
  $\Rightarrow$ uniquely determines all 6-point matrix elements
- take single-soft scalar limit of $\langle ++--\varphi\bar{\varphi}\rangle_{(R^4)^2}$

$D^6 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The result

The dependent $(R^4)^2$ contribution to the soft limit:

$$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{(R^4)^2} = -\tfrac{1}{70}[12]^4\langle 34\rangle^4 \sum_{i<j} s_{ij}^3 .$$

Recall:

$$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{(e^{-12\phi} D^6 R^4)_{\rm avg}} = -\tfrac{33}{35}[12]^4\langle 34\rangle^4 \sum_{i<j} s_{ij}^3 .$$

Subtracting the dependent contribution, we obtain:

$$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{D^6 R^4} = -\tfrac{30}{35}[12]^4\langle 34\rangle^4 \sum_{i<j} s_{ij}^3 \neq 0 .$$

$D^6 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

The result

The dependent $(R^4)^2$ contribution to the soft limit:

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Subtracting the dependent contribution, we obtain:

$$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{D^6R^4} = -\tfrac{30}{35}[12]^4\langle 34\rangle^4 \sum_{i<j} s_{ij}^3 \neq 0 .$$

- $D^6 R^4$ violates continuous $E_{7(7)} \Rightarrow$ ruled out as counterterm
- predicts 6-loop finiteness of $\mathcal{N} = 8$ supergravity!
- consistent with automorphism analysis [Green, Miller, Russo, Vanhove]

$E_{7(7)}$ excludes 6-loop counterterm D^6R^4

	4-point	5	6	7	8	9	10	11
3-loop	~~R^4~~ E_{77}	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~D^2R^4~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	~~D^4R^4~~ E_{77}	~~D^2R^5~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	~~D^6R^4~~ E_{77}	~~D^4R^5~~	~~D^2R^6~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	D^8R^4 ✓	~~D^6R^5~~	D^4R^6 ✓	~~D^2R^7~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10}R^4$ ✓	D^8R^5 ✓	D^6R^6 ✓	D^4R^7 ✓	D^2R^8 ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

Counterterms: $E_{7(7)}$ exclusions for $L < 7$

	4-point	5	6	7	8	9	10	11
3-loop	~~R^4~~ E_{77}	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	~~$D^4 R^4$~~ E_{77}	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	~~$D^6 R^4$~~ E_{77}	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	$D^4 R^6$ ✓	~~$D^2 R^7$~~	R^8 ✓	~~ϕR^8~~	$\phi^2 R^8$ ✓	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

No potential counterterms below 7 loops!

1. SUSY and R-symmetry constraints

2. $E_{7(7)}$ constraints at 3, 5 and 6 loops

3. SUSY & $E_{7(7)}$ constraints at 7 loops and beyond

Counterterms: $L \geq 7$, multiplicities

	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt
7-loop	D^8R^4 $1\times$MHV	~~D^6R^5~~	D^4R^6 $2\times$NMHV	~~D^2R^7~~	R^8 $3\times$N^{2}MHV	~~ϕR^8~~	$\phi^2 R^8$ $4\times$N^{3}MHV
8-loop	$D^{10}R^4$ $1\times$MHV	D^8R^5 $1\times$MHV	D^6R^6 $3\times$NMHV	D^4R^7 $3\times$NMHV	D^2R^8 $8\times$N^{2}MHV	R^9 $8\times$N^{2}MHV	$\phi^2 D^2R^8$ $25\times$N^{3}MHV
9-loop	$D^{12}R^4$ $2\times$MHV	$D^{10}R^5$ $1\times$MHV	D^8R^6 $12\times$NMHV $2\times$MHV	D^6R^7 $14\times$NMHV	D^4R^8 $117\times$N^{2}MHV $7\times$NMHV	D^2R^9 $123\times$N^{2}MHV	R^{10} $780\times$N^{3}MHV $36\times$N^{2}MHV

Two methods to determine multiplicities

- explicit construction of general superamplitudes
  - gives all matrix elements explicitly
  - BUT: inefficient, runs out of steam for high dim or high n

Counterterms: $L \geq 7$, multiplicities

	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt
7-loop	D^8R^4 1×MHV	~~D^6R^5~~	D^4R^6 2×NMHV	~~D^2R^7~~	R^8 3×N^2MHV	~~ϕR^8~~	$\phi^2 R^8$ 4×N^3MHV
8-loop	$D^{10}R^4$ 1×MHV	D^8R^5 1×MHV	D^6R^6 3×NMHV	D^4R^7 3×NMHV	D^2R^8 8×N^2MHV	R^9 8×N^2MHV	$\phi^2 D^2R^8$ 25×N^3MHV
9-loop	$D^{12}R^4$ 2×MHV	$D^{10}R^5$ 1×MHV	D^8R^6 12×NMHV 2×MHV	D^6R^7 14×NMHV	D^4R^8 117×N^2MHV 7×NMHV	D^2R^9 123×N^2MHV	R^{10} 780×N^3MHV 36×N^2MHV

Two methods to determine multiplicities

- explicit construction of general superamplitudes
  - gives all matrix elements explicitly
  - BUT: inefficient, runs out of steam for high dim or high n
- BETTER: adapt methods of `[Beisert, Bianchi et al.]` to $\mathcal{N} = 8$
  - enumerate irreps in symmetric tensor products of field multiplet
  - gives multiplicities of local operators, but not their detailed form
  - also works for ops in the **70** of $SU(8)$ $\Rightarrow$ candidate soft limits!

Counterterms: $L \geq 7$, multiplicities

	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt
7-loop	D^8R^4 1×MHV	~~D^6R^5~~	D^4R^6 2×NMHV	~~D^2R^7~~	R^8 3×N^{2}MHV	~~ϕR^8~~	$\phi^2 R^8$ 4×N^{3}MHV
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- there are infinitely many local supersymmetric operators for $L = 7$
- one operator for each singlet in the symmetric product of n **70**'s

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$\Rightarrow$ D^8R^4 is the only potential 7-loop counterterm

Counterterms: $E_{7(7)}$ exclusions for $L \leq 7$

	4-point	5	6	7	8	9	10	11
3-loop	~~R^4~~ E_{77}	~~ϕR^4~~	~~$\phi^2 R^4$~~	~~$\phi^3 R^4$~~	~~$\phi^4 R^4$~~	~~$\phi^5 R^4$~~	~~$\phi^6 R^4$~~	~~$\phi^7 R^4$~~
4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	~~$D^4 R^4$~~ E_{77}	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
6	~~$D^6 R^4$~~ E_{77}	~~$D^4 R^5$~~	~~$D^2 R^6$~~	~~R^7~~	~~ϕR^7~~	~~$\phi^2 R^7$~~	~~$\phi^3 R^7$~~	~~$\phi^4 R^7$~~
7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	~~$D^4 R^6$~~ E_{77}	~~$D^2 R^7$~~	~~R^8~~ E_{77}	~~ϕR^8~~	~~$\phi^2 R^8$~~ E_{77}	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

$D^8 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

$D^8 R^4$ from closed-string tree-amplitude at $O(\alpha'^7)$?

Recall:

$$S_{\text{eff}} = S_{\text{SG}} - 2\alpha'^3 \zeta(3) e^{-6\phi} R^4 - \zeta(5)\alpha'^5 e^{-10\phi} D^4 R^4 + \tfrac{2}{3}\alpha'^6 \zeta(3)^2 e^{-12\phi} D^6 R^4$$
$$- \tfrac{1}{2}\alpha'^7 \zeta(7) e^{-14\phi} D^8 R^4 + \dots .$$

$SU(8)$-averaging the closed-string amplitude gives:

$$\lim_{p_6 \to 0} \langle ++--\varphi\bar{\varphi}\rangle_{(e^{-14\phi} D^8 R^4)_{\text{avg}}} = -2[12]^4 \langle 34\rangle^4 \Big[\tfrac{3}{4} \sum_{i<j} s_{ij}^4 + \tfrac{1}{16} \Big(\sum_{i<j} s_{ij}^2 \Big)^2 \Big] .$$

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4	~~$D^2 R^4$~~	~~R^5~~	~~ϕR^5~~	~~$\phi^2 R^5$~~	~~$\phi^3 R^5$~~	~~$\phi^4 R^5$~~	~~$\phi^5 R^5$~~	~~$\phi^6 R^5$~~
5	~~$D^4 R^4$~~ E_{77}	~~$D^2 R^5$~~	~~R^6~~	~~ϕR^6~~	~~$\phi^2 R^6$~~	~~$\phi^3 R^6$~~	~~$\phi^4 R^6$~~	~~$\phi^5 R^6$~~
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7	$D^8 R^4$ ✓	~~$D^6 R^5$~~	~~$D^4 R^6$~~ E_{77}	~~$D^2 R^7$~~	~~R^8~~ E_{77}	~~ϕR^8~~	~~$\phi^2 R^8$~~ E_{77}	~~$\phi^3 R^8$~~
8	$D^{10} R^4$ ✓	$D^8 R^5$ ✓	$D^6 R^6$ ✓	$D^4 R^7$ ✓	$D^2 R^8$ ✓	R^9 ✓	ϕR^9 ✓	$\phi^2 R^9$ ✓

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- Unlike D^6R^4: no contamination from lower orders in α'
- BUT: D^8R^4 is not unique at 6-point! $\exists$ 2 independent D^4R^6 ops!

$$\lim_{p_6\to 0}\langle ++--\varphi\bar{\varphi}\rangle_{D^4R^6} = [12]^4\langle 34\rangle^4\Big[c_1\sum_{i<j} s_{ij}^4 + c_2\Big(\sum_{i<j} s_{ij}^2\Big)^2\Big].$$

$D^8 R^4$ at 6-point NMHV level

[Beisert, Elvang, Freedman, MK, Morales, Stieberger]

$D^8 R^4$ and $E_{7(7)}$

- adding $D^4 R^6$ appropriately gives vanishing 6-point soft-scalar limits
- consistent with continuous $E_{7(7)}$ up to 6-points

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- consistent with continuous $E_{7(7)}$ up to 6-points
- next test: 8-point

7-loop	4-pt	5-pt	6-pt	7-pt	8-pt	9-pt	10-pt
singlet	D^8R^4 1×MHV	~~D^6R^5~~	D^4R^6 2×NMHV	~~D^2R^7~~	R^8 3×N^2MHV	~~ϕR^8~~	$\phi^2 R^8$ 4×N^3MHV
			↙ soft		↙ soft		↙ soft
70		ϕD^8R^4 2×		ϕD^4R^6 4×		ϕR^8 6×	

Can one add R^8 to make 8-point soft limits vanish?

- non-linear 7-loop $E_{7(7)}$-invariant may or may not exist

[Bossard, Howe, Stelle; Howe, Lindstrom; Kallosh]

Summary and Outlook

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  BUT they violate $E_{7(7)}$ at leading order, except for $D^8 R^4$
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Open problems

- Is $D^8 R^4$ consistent with $E_{7(7)}$ up to all orders?
- for $L \geq 7$: new mechanism needed for finiteness