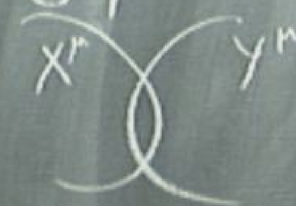
Title: Gravitational Physics Review - Lecture 2

Date: Jan 25, 2011 10:15 AM

URL: http://pirsa.org/11010066

Abstract:

$$T = T^{\mu}\frac{\partial}{\partial x^{\mu}} = T^{\alpha}\frac{\partial}{\partial y^{\alpha}}$$



$$T = T^{\mu} \frac{\partial}{\partial x^{\mu}} = T^{\alpha} \frac{\partial}{\partial y^{\alpha}}$$

x-coord basis

y-coord basis

x^{μ} y^{μ}

$$T^{\alpha'} = \frac{\partial y^{\alpha'}}{\partial x^{\alpha}} T^{\alpha} \leftarrow$$ contravariant vector transformation rules

$$\omega_{\alpha'} = \frac{\partial X^{\beta}}{\partial y^{\alpha'}}\, \omega_{\beta} \quad \text{- covariant rules}$$

y^{μ}

d basis

ravariant

tor

sformation

es

$$\omega_{\alpha'} = \frac{\partial X^{\beta}}{\partial Y^{\alpha'}} \omega_{\beta}$$ – covariant rules

– For differentiation, to see the issues involved consider

$$\frac{\partial V^{\mu}}{\partial X^{\nu}}$$

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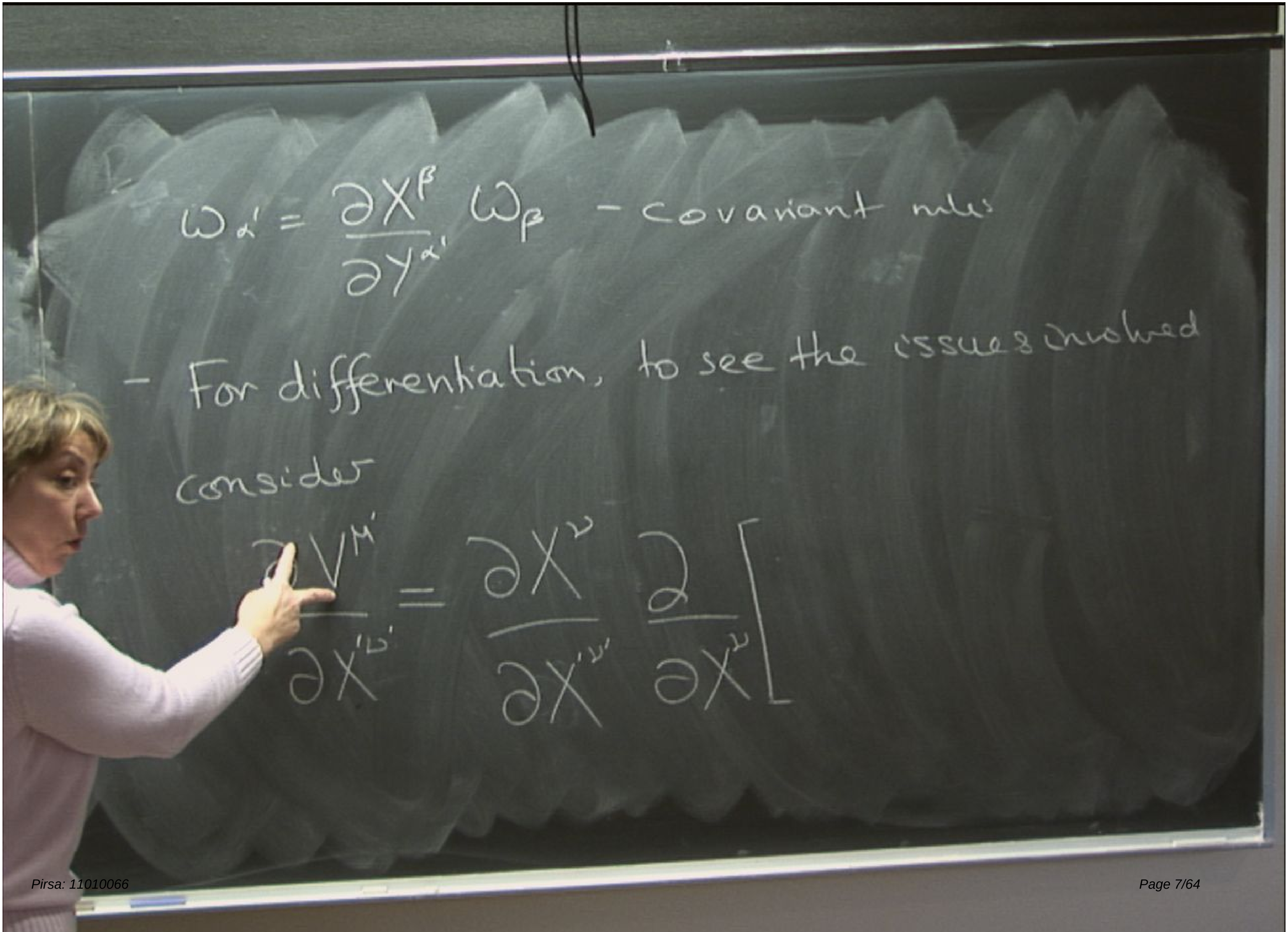
$$\frac{\partial V^{\mu'}}{\partial X'^{\nu'}} = \frac{\partial X^{\nu}}{\partial X'^{\nu'}} \frac{\partial}{\partial X^{\nu}}$$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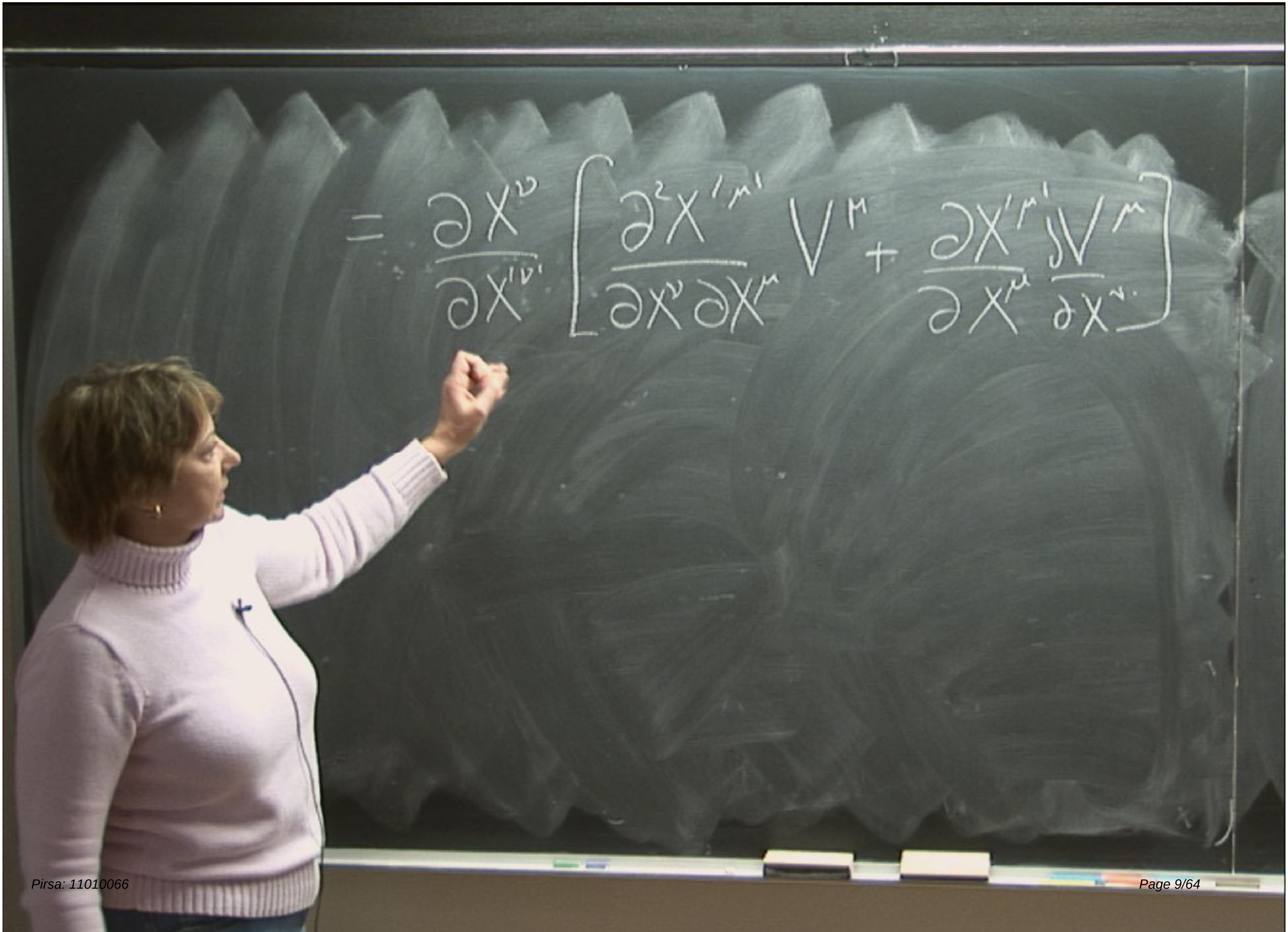
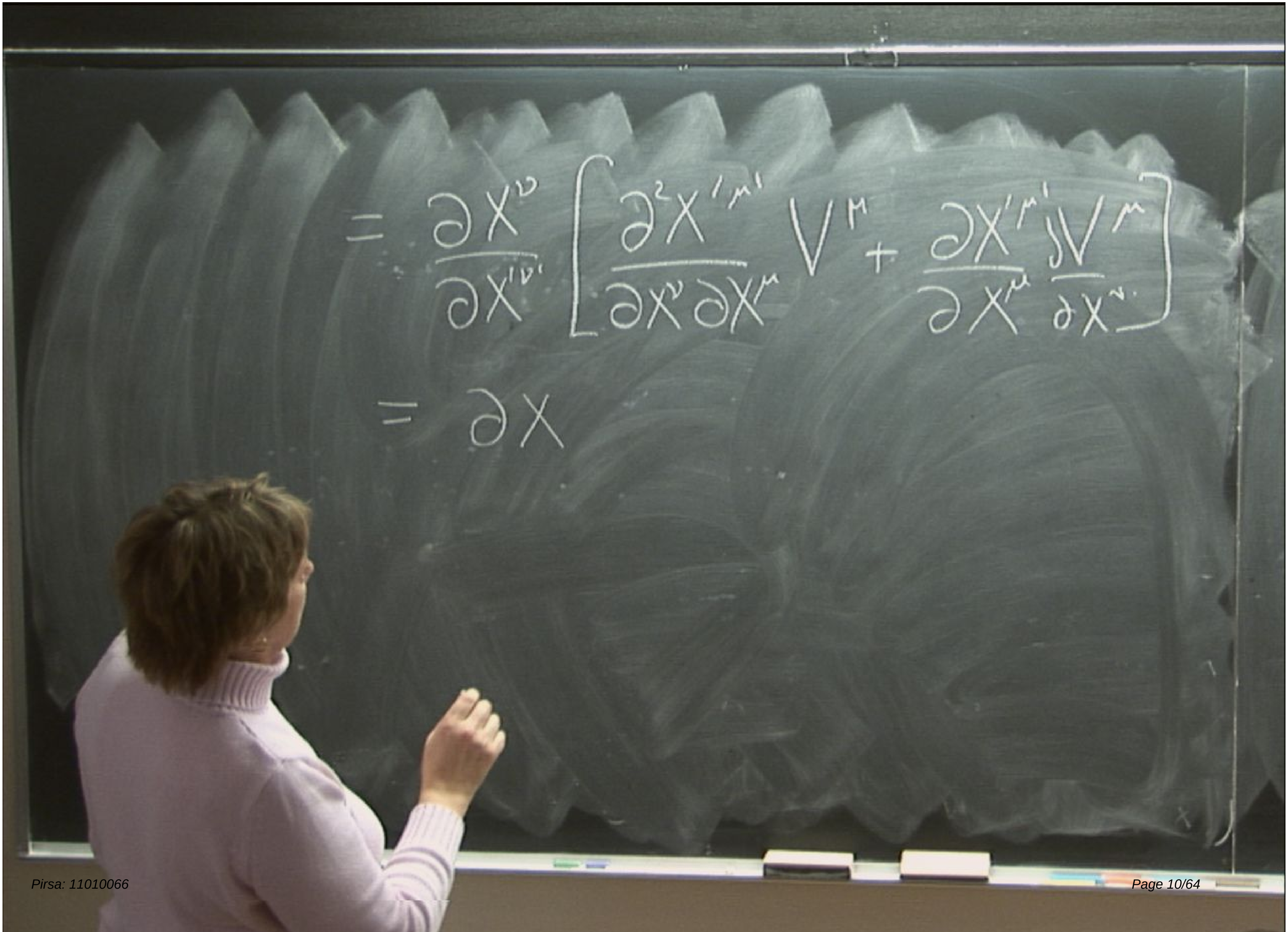
consider

$$\frac{\partial V^{\mu'}}{\partial X'^{\nu'}} = \frac{\partial X^{\nu}}{\partial X'^{\nu'}}\,\frac{\partial}{\partial X^{\nu}}\left[\frac{\partial X'^{\mu'}}{\partial X^{\mu}}\,V^{\mu}\right]$$


$= \frac{\partial x^{\nu}}{\partial x'^{\nu'}} \left[\frac{\partial^2 x'^{\mu'}}{\partial x^{\nu} \partial x^{\mu}} V^{\mu} + \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial V^{\mu}}{\partial x^{\nu}} \right]$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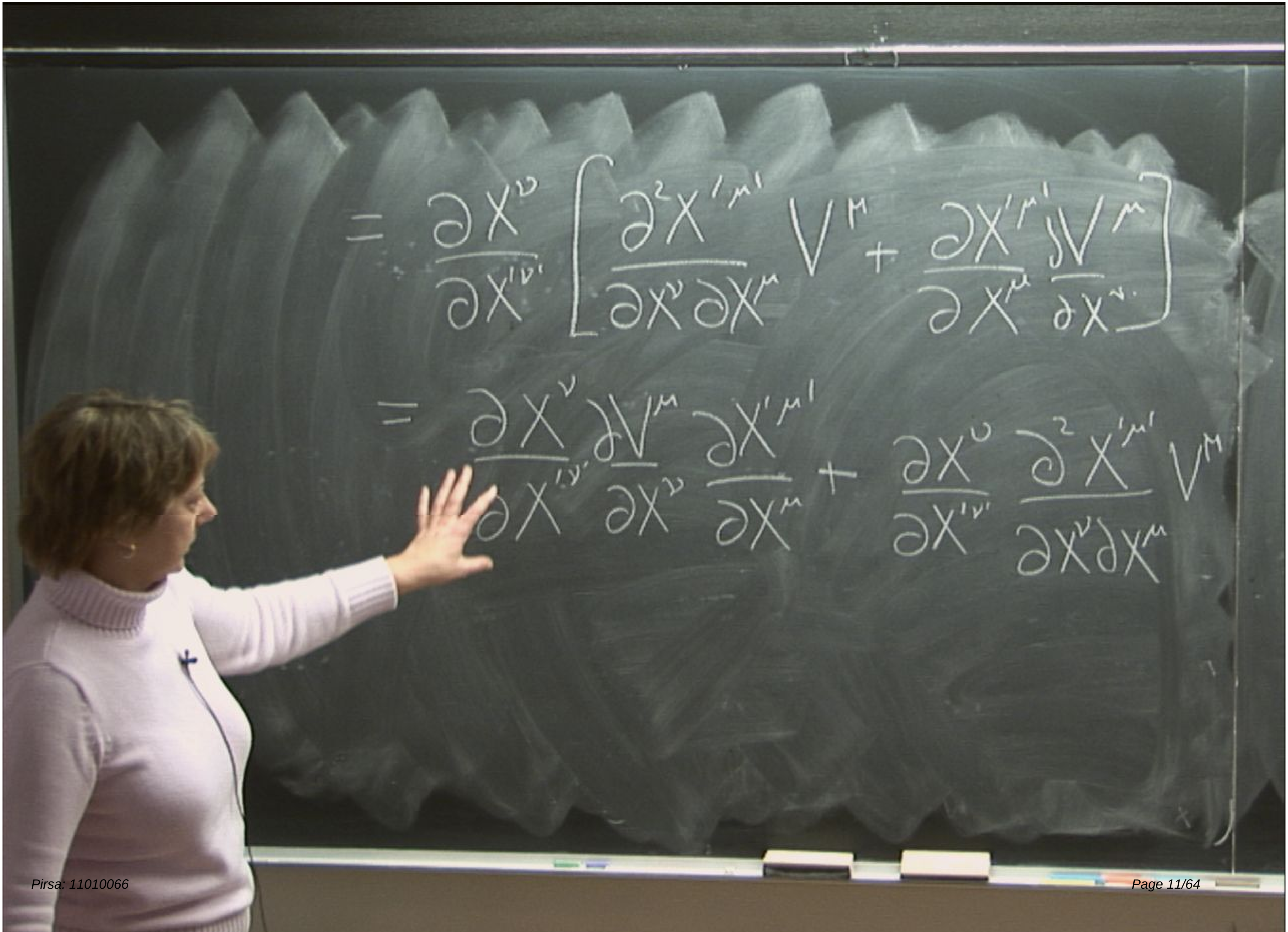
$= \partial x$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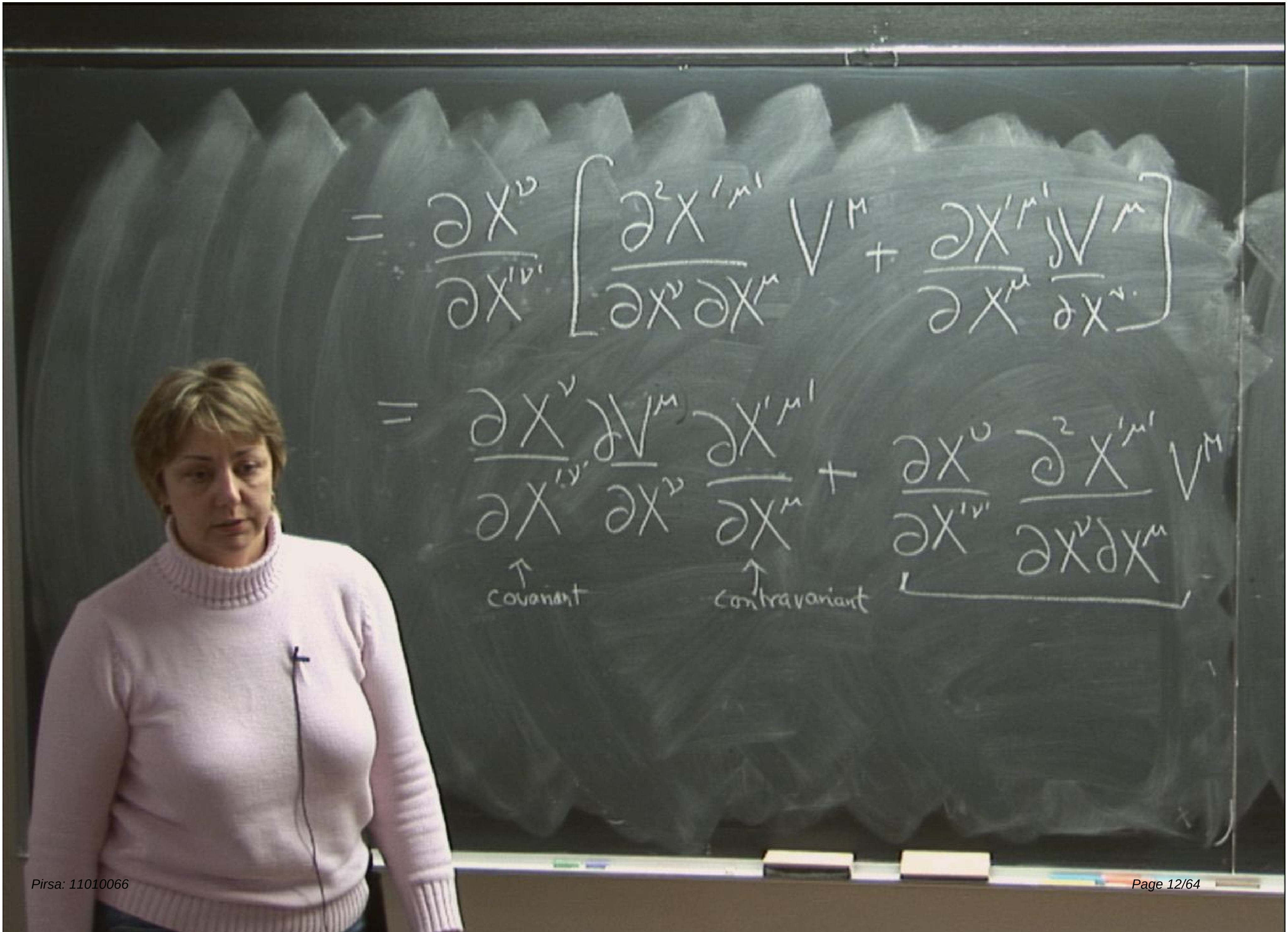

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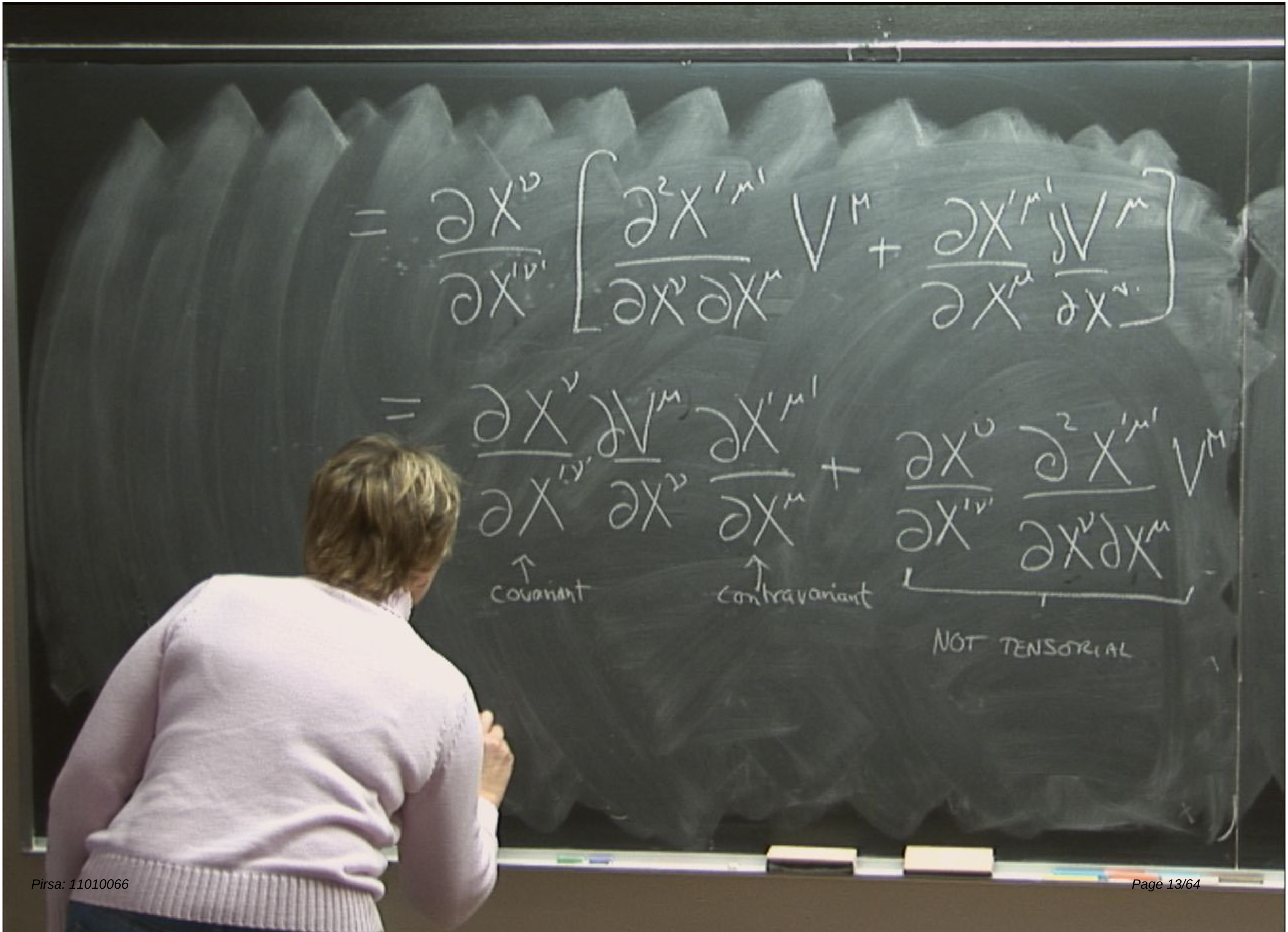
covariant
contravariant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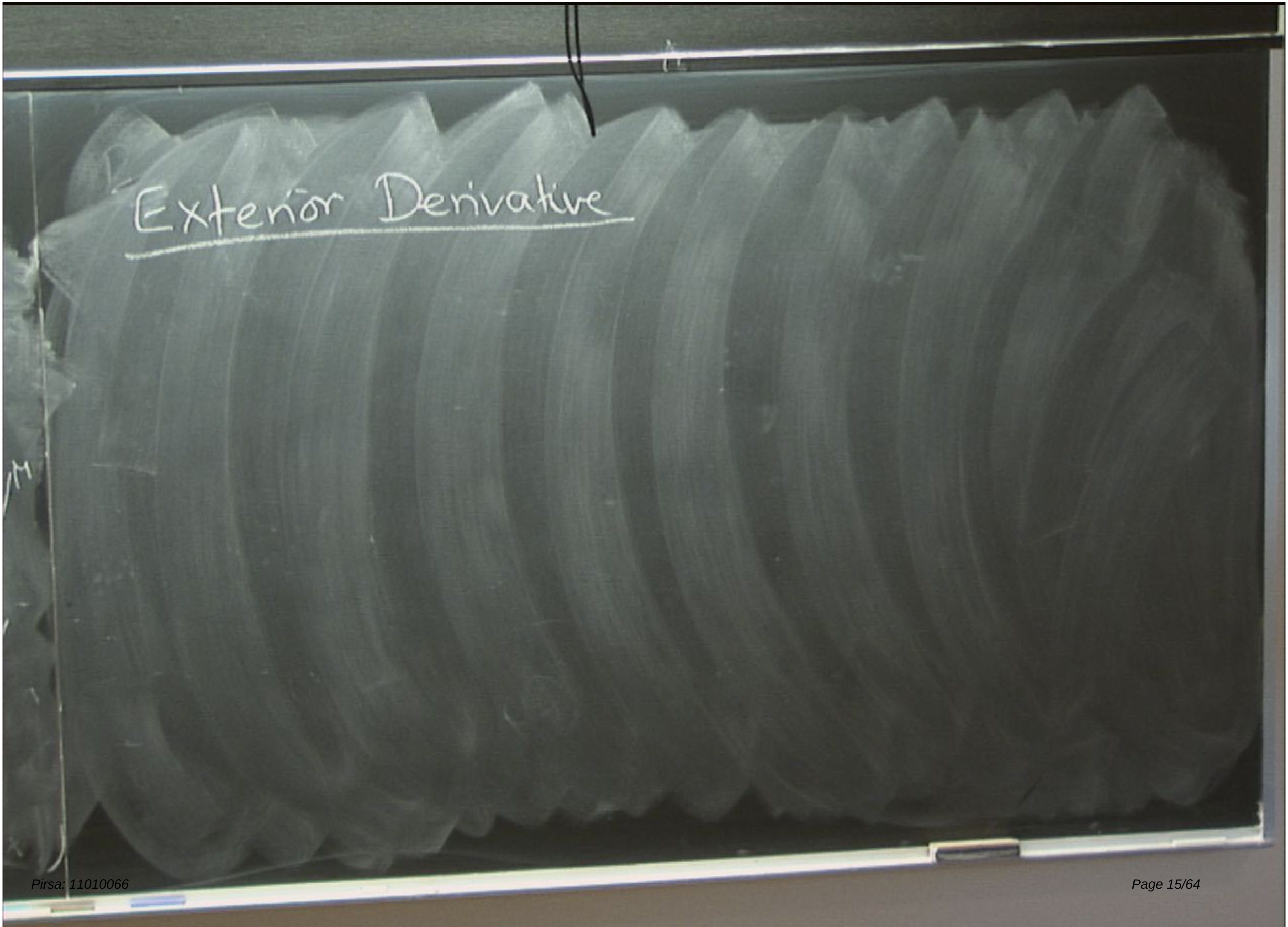
covariant
contravariant
NOT TENSORIAL

$$= \frac{\partial x^{\nu}}{\partial x'^{\nu'}}\left[\frac{\partial^2 x'^{\mu'}}{\partial x^{\nu}\partial x^{\mu}} V^{\mu} + \frac{\partial x'^{\mu'}}{\partial x^{\mu}}\frac{\partial V^{\mu}}{\partial x^{\nu}}\right]$$

$$= \underset{\substack{\uparrow \\ \text{covariant}}}{\frac{\partial x^{\nu}}{\partial x'^{\nu'}}} \frac{\partial V^{\mu}}{\partial x^{\nu}} \underset{\substack{\uparrow \\ \text{contravariant}}}{\frac{\partial x'^{\mu'}}{\partial x^{\mu}}} + \underbrace{\frac{\partial x^{\nu}}{\partial x'^{\nu'}} \frac{\partial^2 x'^{\mu'}}{\partial x^{\nu}\partial x^{\mu}} V^{\mu}}_{\text{NOT TENSORIAL}}$$

$\rightarrow$ 2^{nd} piece symmetric under $\mu \leftrightarrow \nu$.


Exterior Derivative

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$$2\omega_{[\mu,\nu]} = \omega_{\mu,\nu} - \omega_{\nu,\mu}$$

Under a co-ord transfm, this is a tensor

$$2\omega_{[\mu',\nu']} = 2\omega_{[\mu,\nu]} \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}}$$

For a function, define

$$\underset{\sim}{d} f = \frac{\partial f}{\partial x^\mu} dx^\mu$$

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Can generalize to any anti-symmetric tensor. Completely antisymmetric (covariant) tensors are forms.

A form of rank p is an antisymmetric tensor of rank p (# indices) & $p \leq n$

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in general

$$(A_{(p)} \wedge B_{(q)})_{a_1 \dots a_p b_1 \dots b_q} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \dots a_p} B_{b_1 \dots b_q]}$$

$\wedge$ not commutative: $A_{(p)} \wedge B_{(q)} = (-)^{pq} B_{(q)} \wedge A_{(p)}$

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in cpts.

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$$d(A \wedge B) = dA \wedge B - A \wedge dB$$

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$$d : \Lambda^{(p)} \to \Lambda^{(p+1)}$$

$$(\partial_c A_a) B_b + A_a (\partial_c B_b)$$

$$\uparrow$$

in cpts

$$\partial_c (A_a B_b - A_b B_a)$$

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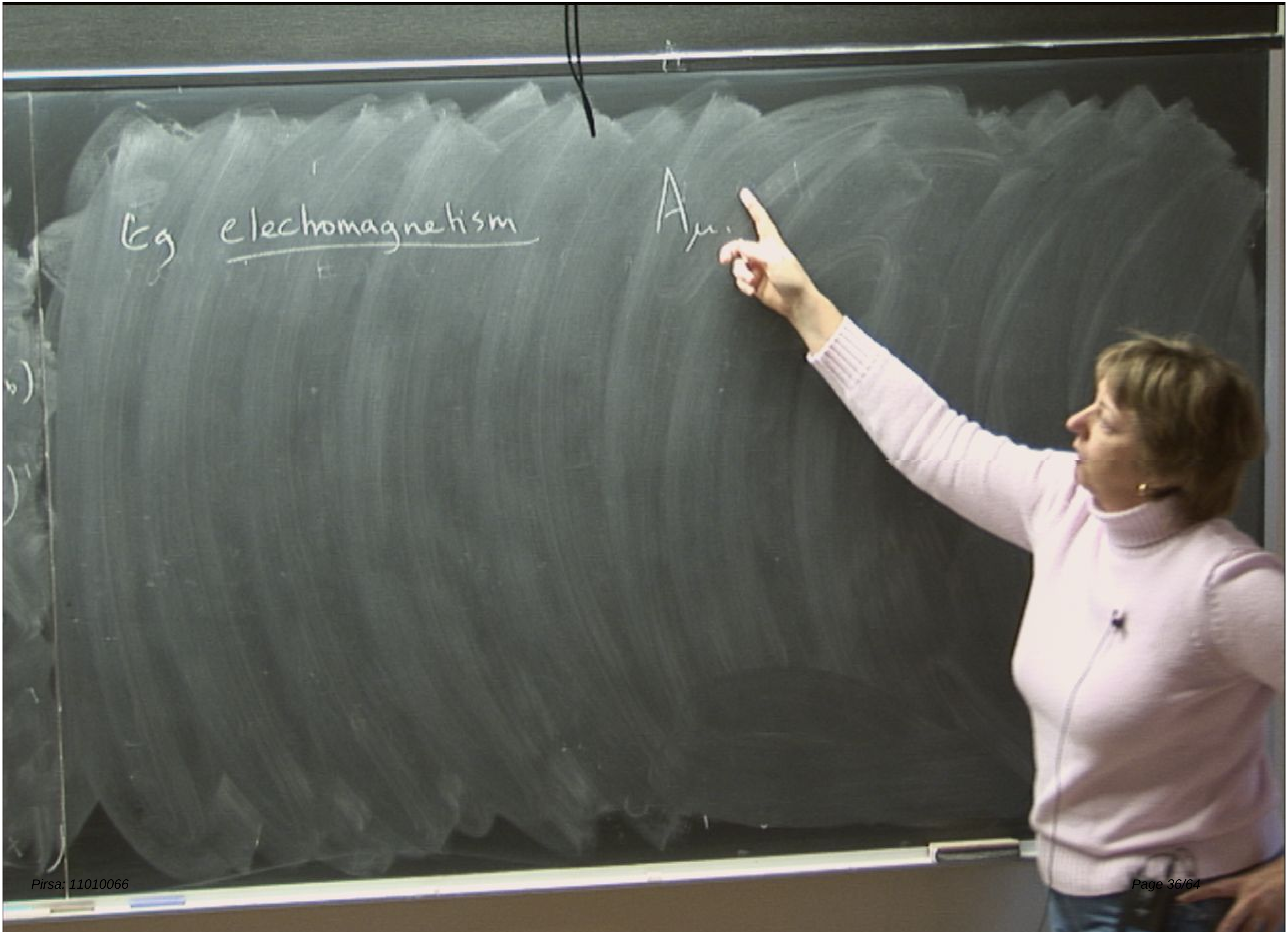
Eg electromagnetism
A_μ.

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Same reason gives gauge inv.

$$A \to A + df \qquad dA \to dA$$

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("nice" space)

Same reason gives gauge inv.

$$A \to A + df \qquad dA \to dA.$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

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F_{0i} electric field.

$\underline{B} = \nabla \times \underline{A}$

With metric can relate $\Lambda^{(p)}$ to $\Lambda^{(n-p)}$

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[With metric can relate $\Lambda^{(p)}$ to $\Lambda^{(n-p)}$]

e-m $U(1)$ gauge theory, because can couple to ($\mathbb{C}$) scalar fields. $\Phi \to e^{if}\Phi$

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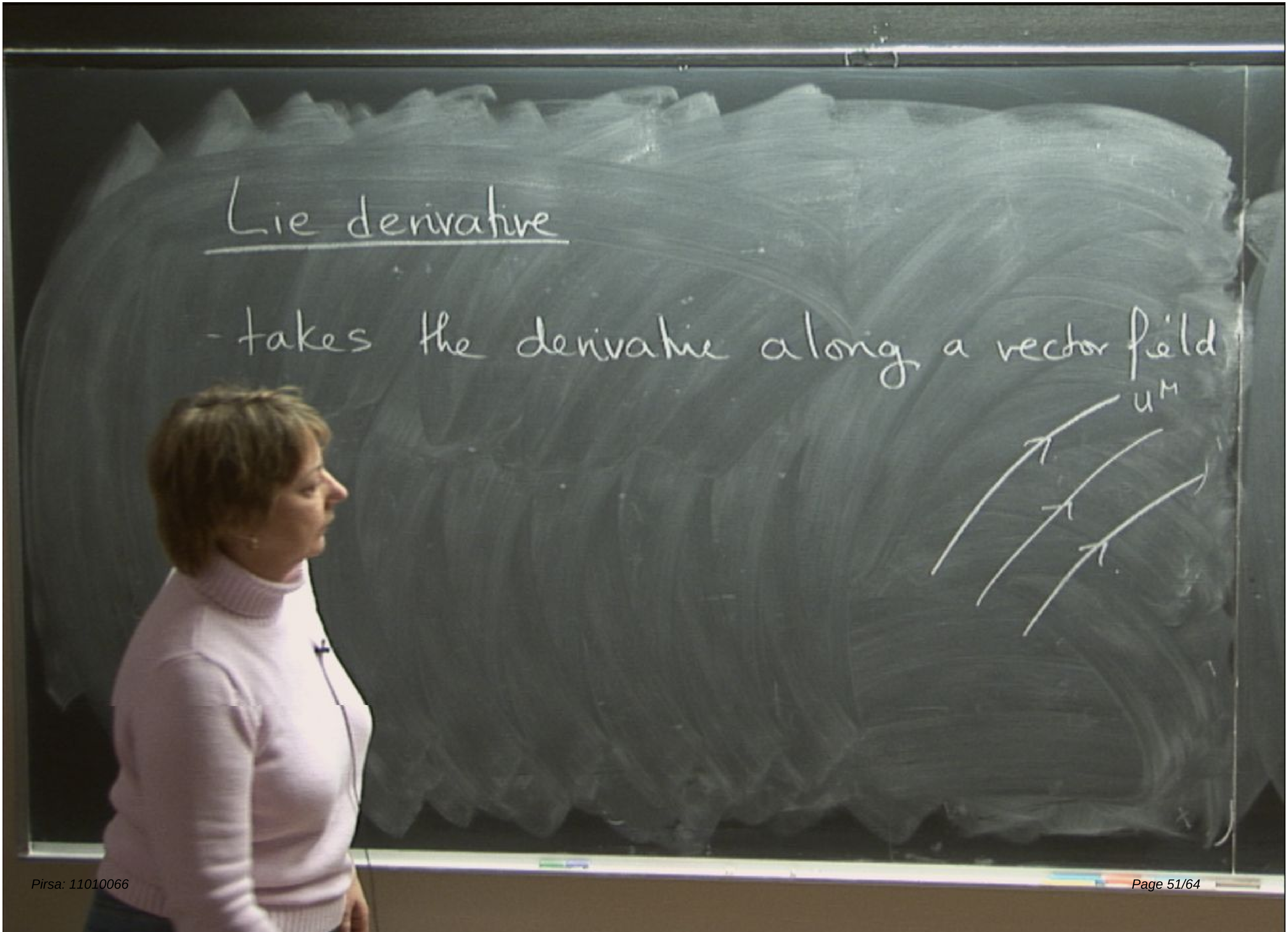
$$\underline{H} = d\underline{B} \qquad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}$$

Lie derivative


Lie derivative
- takes the derivative along a vector field
u^μ

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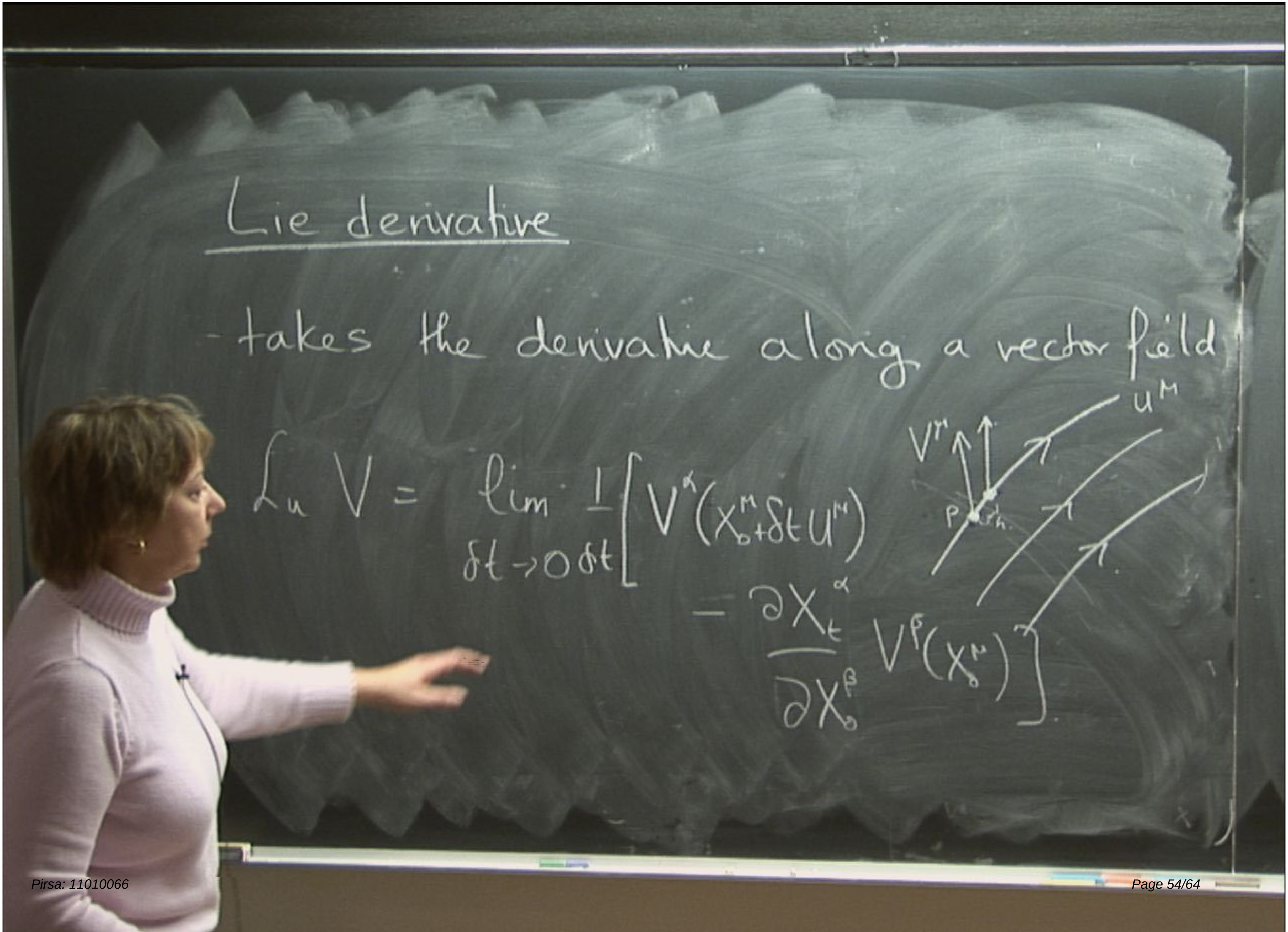
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u^μ
V^μ
P

$P \leftrightarrow X_0^\mu$

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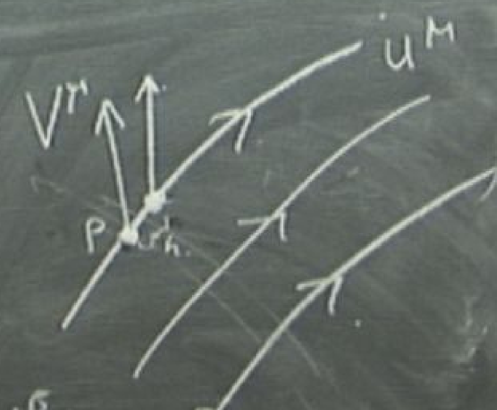
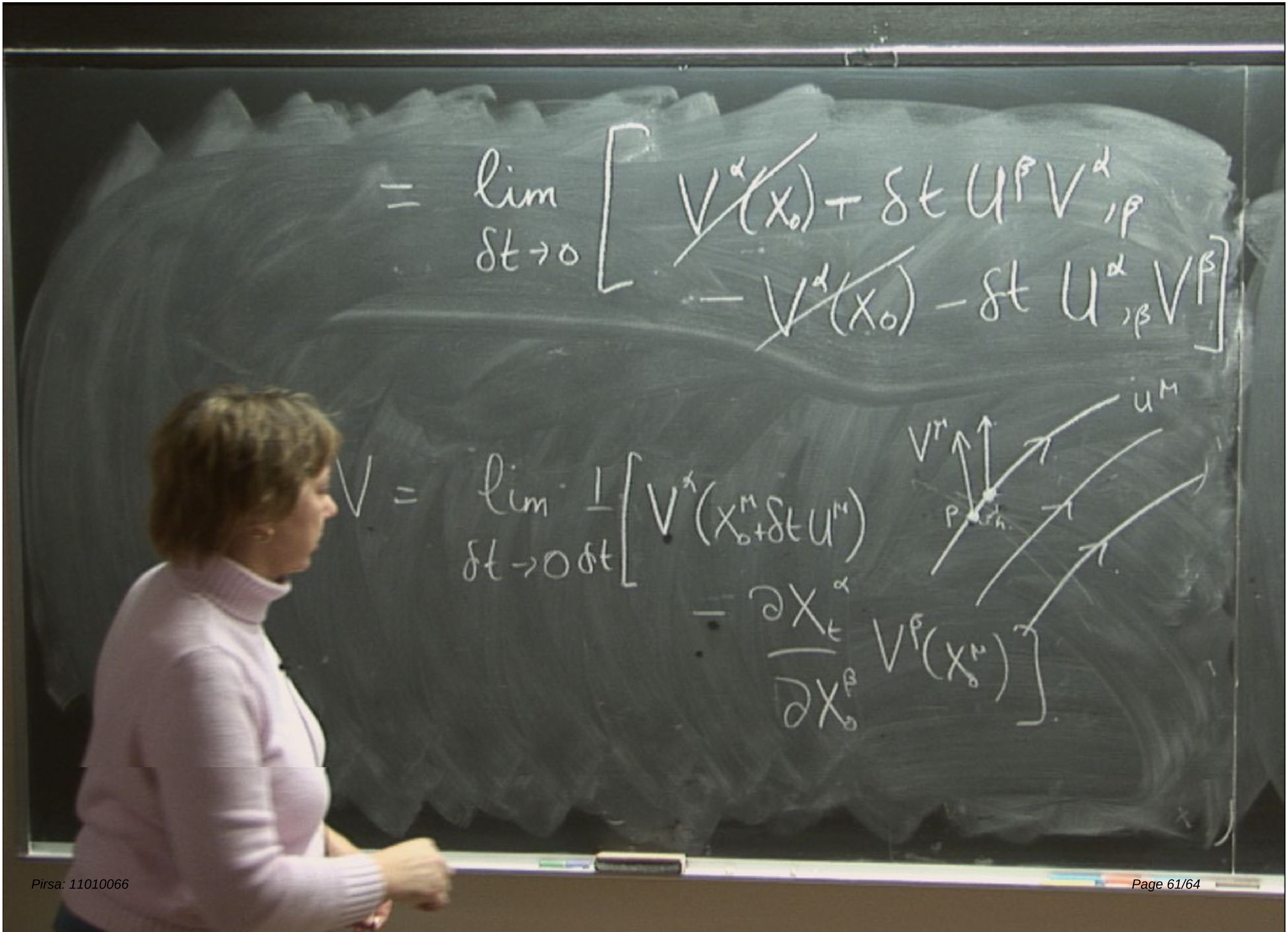
$$V^\alpha(X_t^\mu) = V^\alpha(X_0^\mu) + \delta t\, U^\beta\, V^\alpha{}_{,\beta}$$

$$= \lim_{\delta t \to 0} \Big[V^\alpha(x_0) + \delta t\, U^\beta V^\alpha{}_{,\beta} - V^\alpha(x_0) - \delta t\, U^\alpha{}_{,\beta} V^\beta \Big]$$

$$\mathcal{L}_u V = \lim_{\delta t \to 0} \frac{1}{\delta t} \Big[V^\alpha(x_0^\mu + \delta t\, u^\mu) - \frac{\partial X_t^\alpha}{\partial X_0^\beta} V^\beta(x_0^\mu) \Big]$$




$= \lim_{\delta t \to 0} \left[\cancel{V^\alpha(x_0)} + \delta t\, U^\beta V^\alpha{}_{,\beta} - \cancel{V^\alpha(x_0)} - \delta t\, U^\alpha{}_{,\beta} V^\beta \right]$
$V = \lim_{\delta t \to 0} \frac{1}{\delta t} \left[V^\alpha(x_0^\mu + \delta t\, U^\mu) - \frac{\partial X_t^\alpha}{\partial X_0^\beta} V^\beta(x_0^\mu) \right]$
V^μ
u^μ
p

$$= \lim_{\delta t \to 0} \Big[\cancel{V^\alpha(x_0)} + \delta t\, U^\beta V^\alpha{}_{,\beta} - \cancel{V^\alpha(x_0)} - \delta t\, U^\alpha{}_{,\beta} V^\beta \Big]$$

$$(\mathcal{L}_u V)^\alpha = U^\beta V^\alpha{}_{,\beta} - U^\alpha{}_{,\beta} V^\beta$$

$$= \lim_{\delta t \to 0} \left[\cancel{V^\alpha(x_0)} + \delta t\, U^\beta V^\alpha{}_{,\beta} - \cancel{V^\alpha(x_0)} - \delta t\, U^\alpha{}_{,\beta} V^\beta \right]$$

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$$(\mathcal{L}_V \omega)_\alpha = \omega_{\alpha,\beta} U^\beta + U^\beta{}_{,\alpha} \omega_\beta$$

$$= \lim_{\delta t \to 0} \Big[\cancel{V^\alpha(x_0)} + \delta t\, U^\beta V^\alpha{}_{,\beta} - \cancel{V^\alpha(x_0)} - \delta t\, U^\alpha{}_{,\beta} V^\beta \Big]$$

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$$(\mathcal{L}_v \omega)_\alpha = \omega_{\alpha,\beta} U^\beta + U^\beta{}_{,\alpha} \omega_\beta$$