

Title: Foundations of Quantum Mechanics - Lecture 11

Date: Jan 17, 2011 11:30 AM

URL: <http://pirsa.org/11010052>

Abstract:

# The generalized notion of noncontextuality

## Problems with the traditional definition of noncontextuality:

- applies only to projective measurements
- applies only to deterministic hidden variable models
- applies only to models of quantum theory

Problems with the traditional definition of noncontextuality:

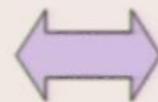
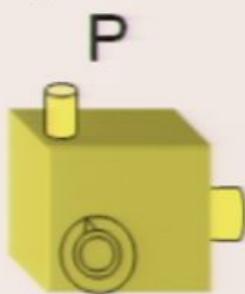
- applies only to projective measurements
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A better notion of noncontextuality would determine

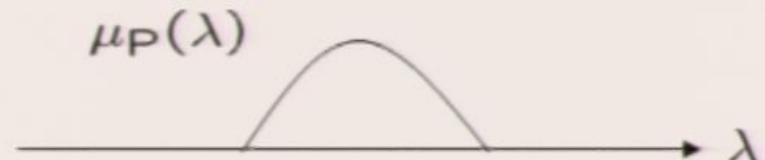
- whether any given theory admits a noncontextual model
- whether any given experimental data can be explained by a noncontextual model

## A realist model of an operational theory

Preparation

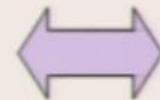
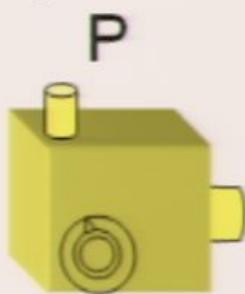


$$\int \mu_P(\lambda) d\lambda = 1$$

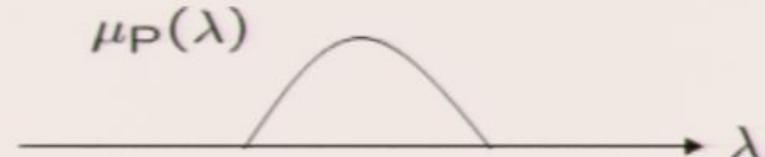


## A realist model of an operational theory

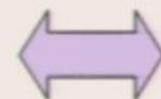
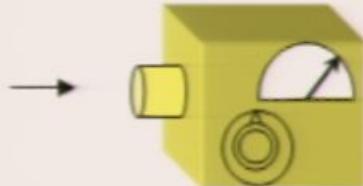
Preparation



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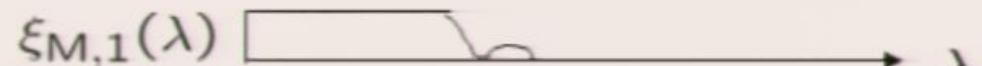


Measurement  
M



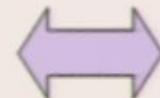
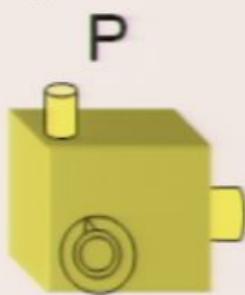
$$0 \leq \xi_{M,k} \leq 1$$

$$\sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda$$

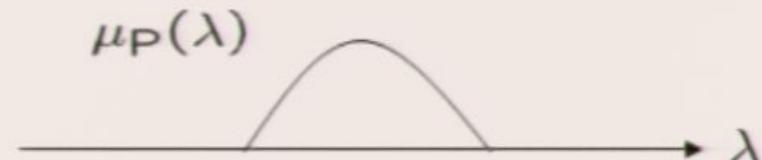


## A realist model of an operational theory

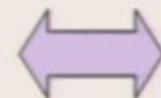
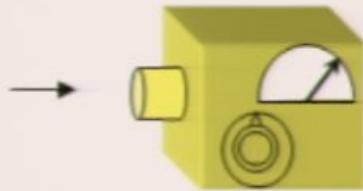
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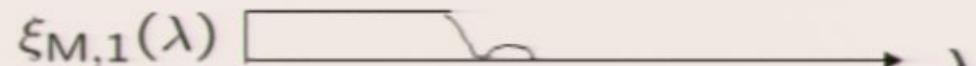


Measurement  
M



$$0 \leq \xi_{M,k} \leq 1$$

$$\sum_k \xi_{M,k}(\lambda) = 1 \text{ for all } \lambda$$



$$p(k|P, M) = \int d\lambda \xi_{M,k}(\lambda) \mu_P(\lambda)$$

## Generalized definition of noncontextuality:

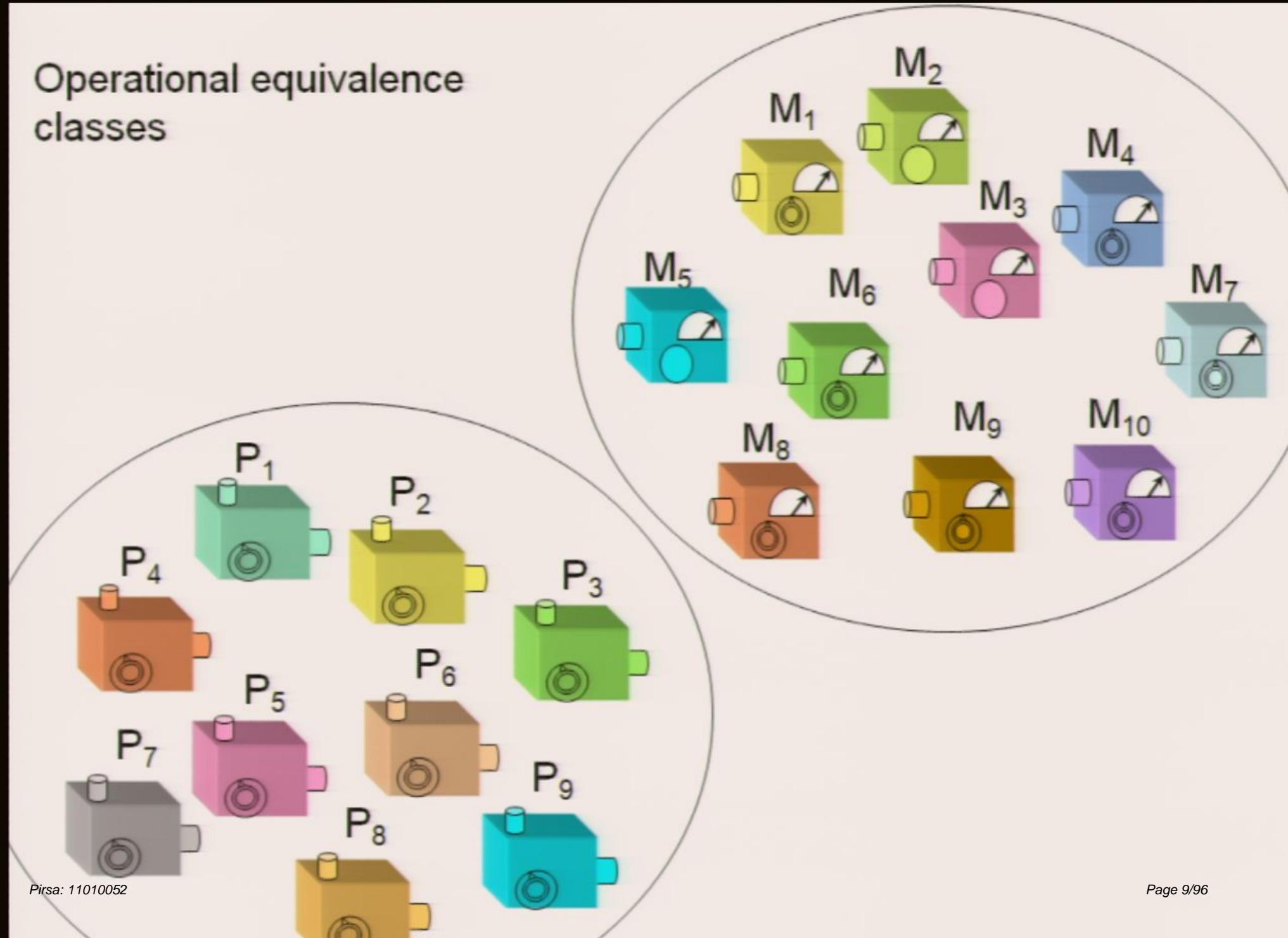
A realist model of an operational theory is **noncontextual** if

Operational equivalence  
of two experimental  
procedures

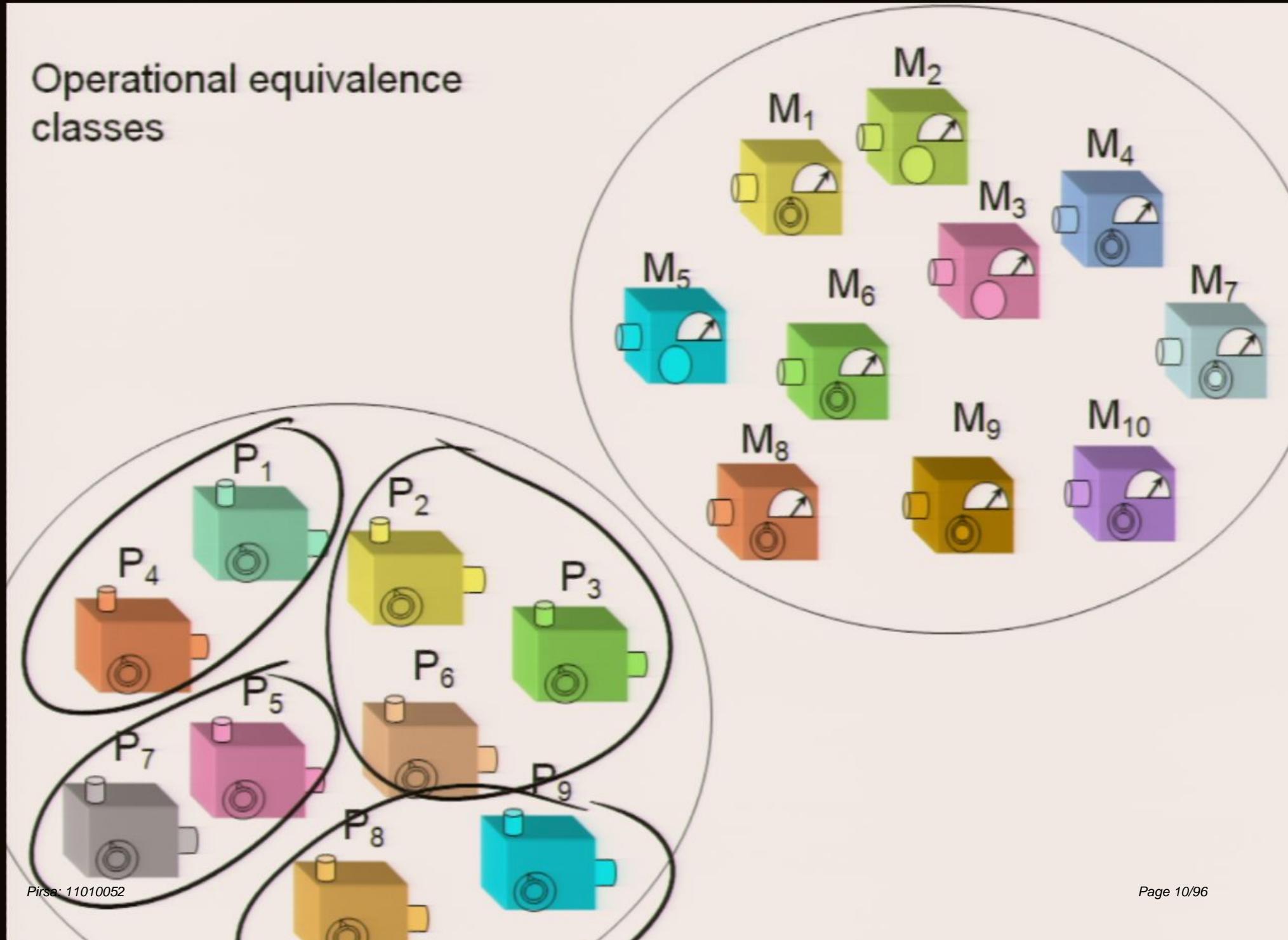


Equivalent  
representations  
in the realist model

## Operational equivalence classes

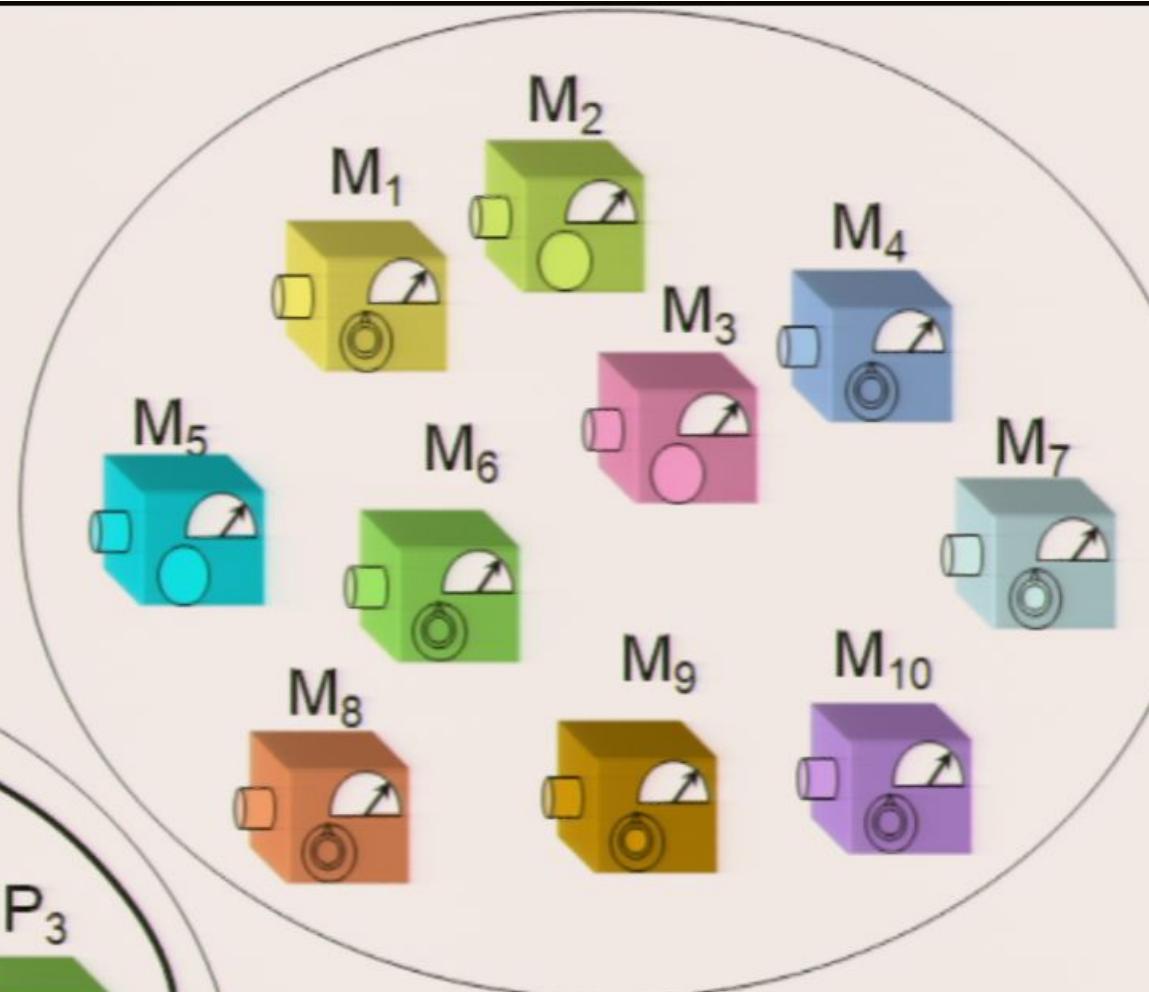
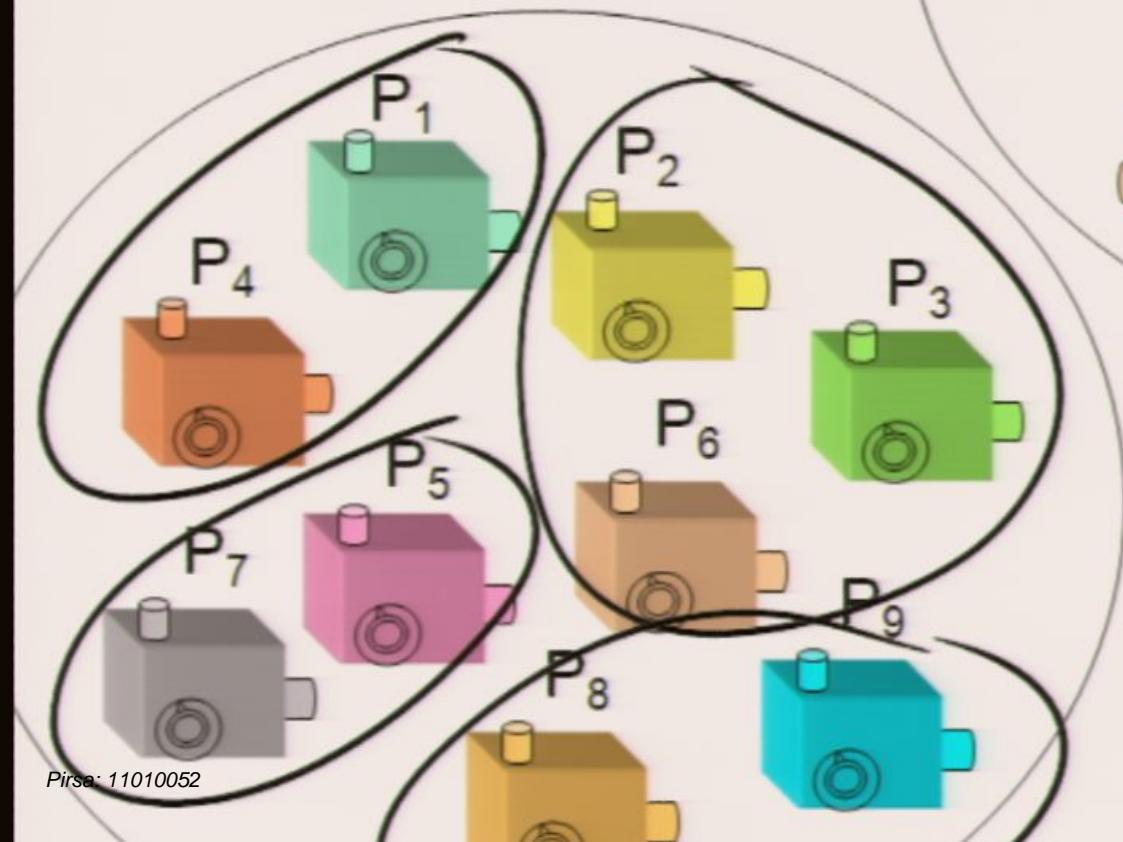


## Operational equivalence classes

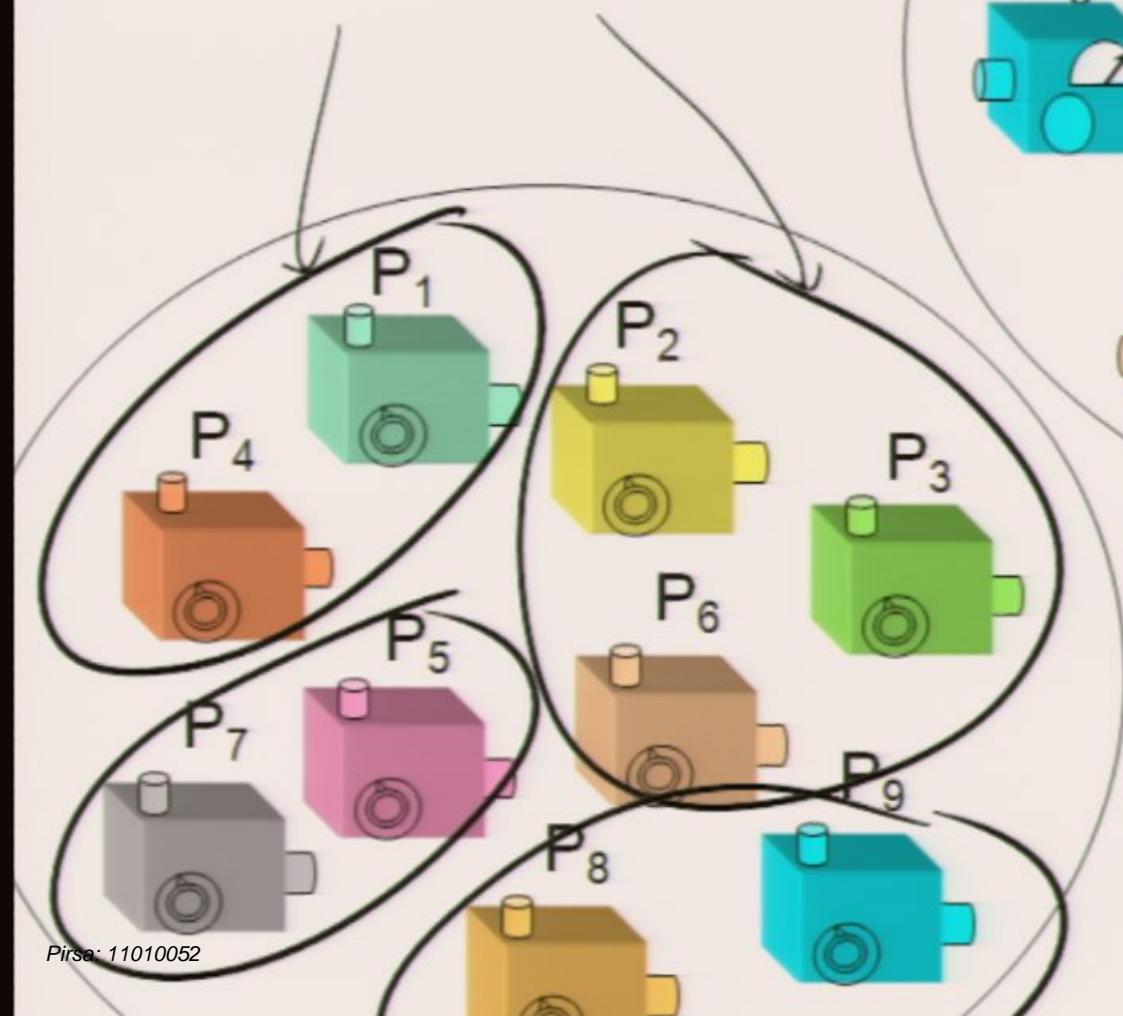


## Operational equivalence classes

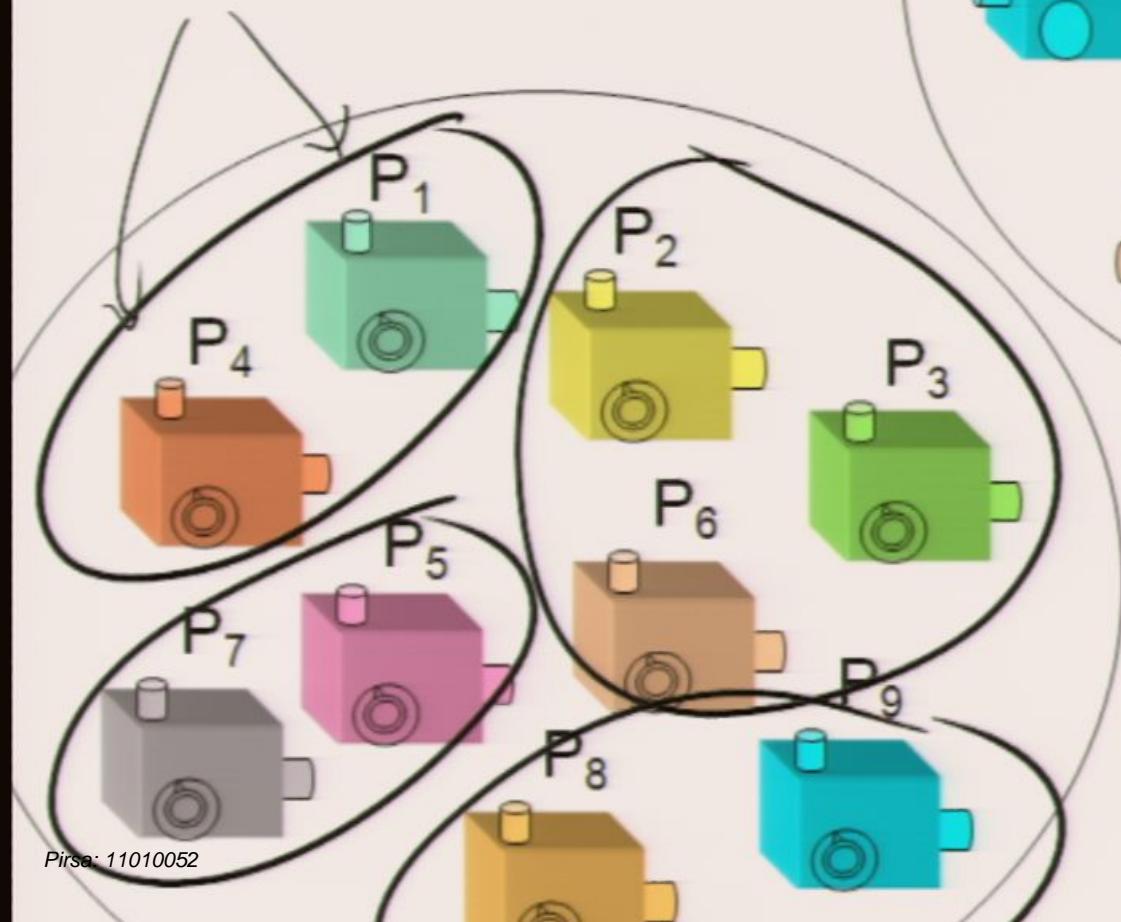
$P$  is equivalent to  $P'$  if  
 $\forall M \forall k : p(k|P, M) = p(k|P', M)$



## Difference of Equivalence class

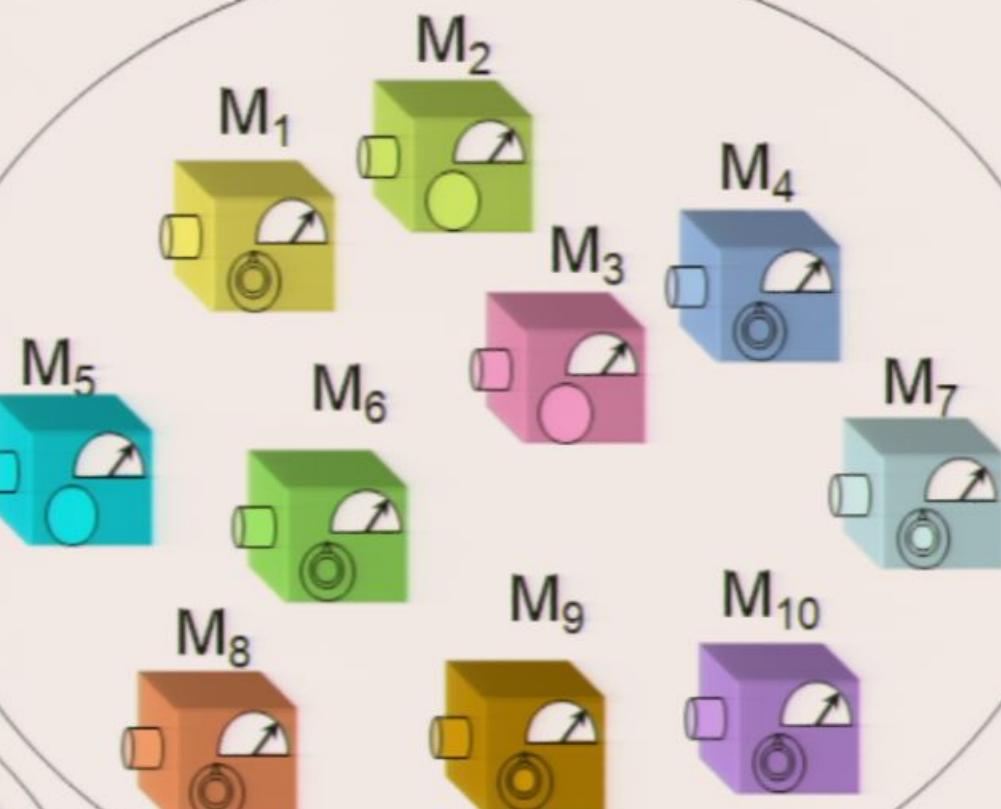
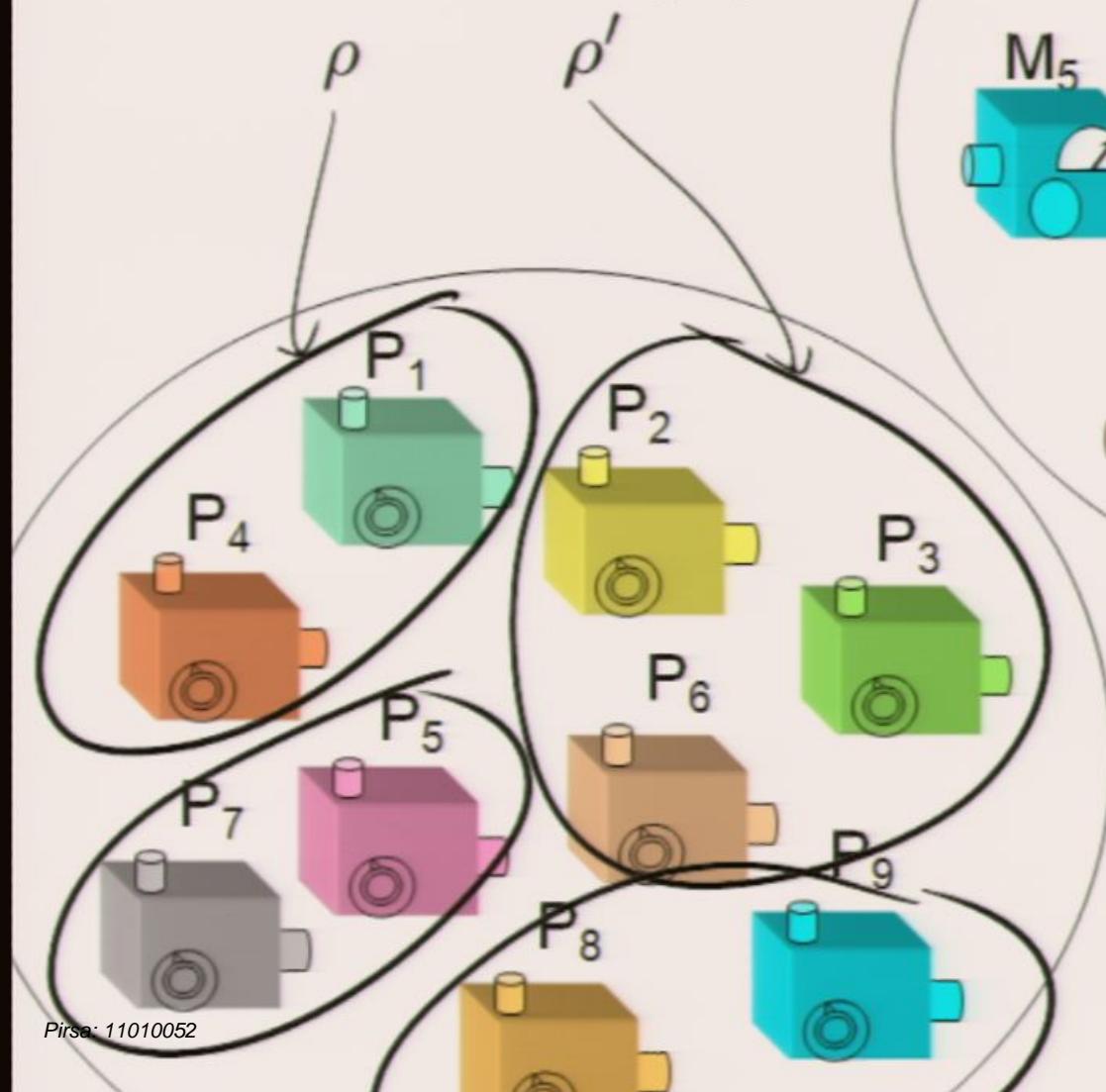


## Difference of context



## Example from quantum theory

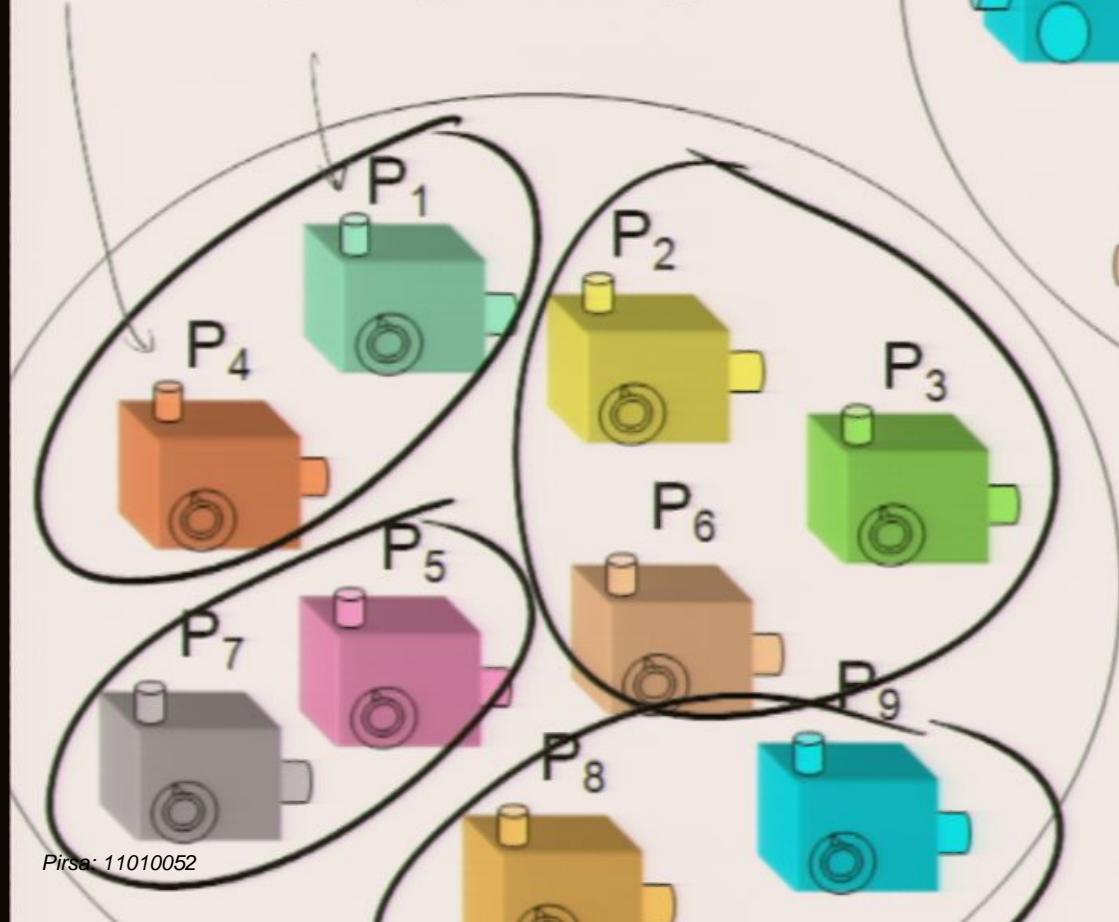
Different density op's



## Example from quantum theory

$$I = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

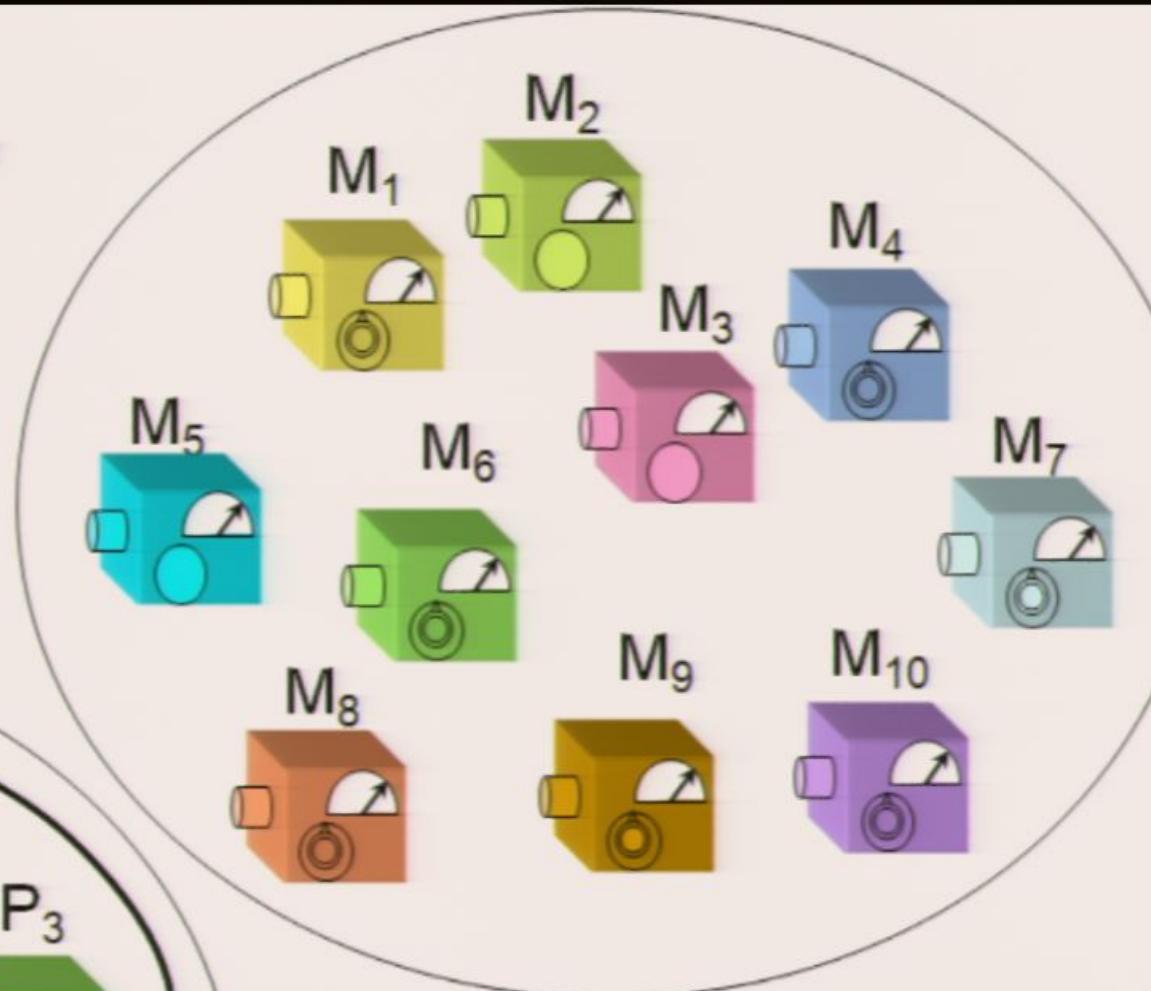
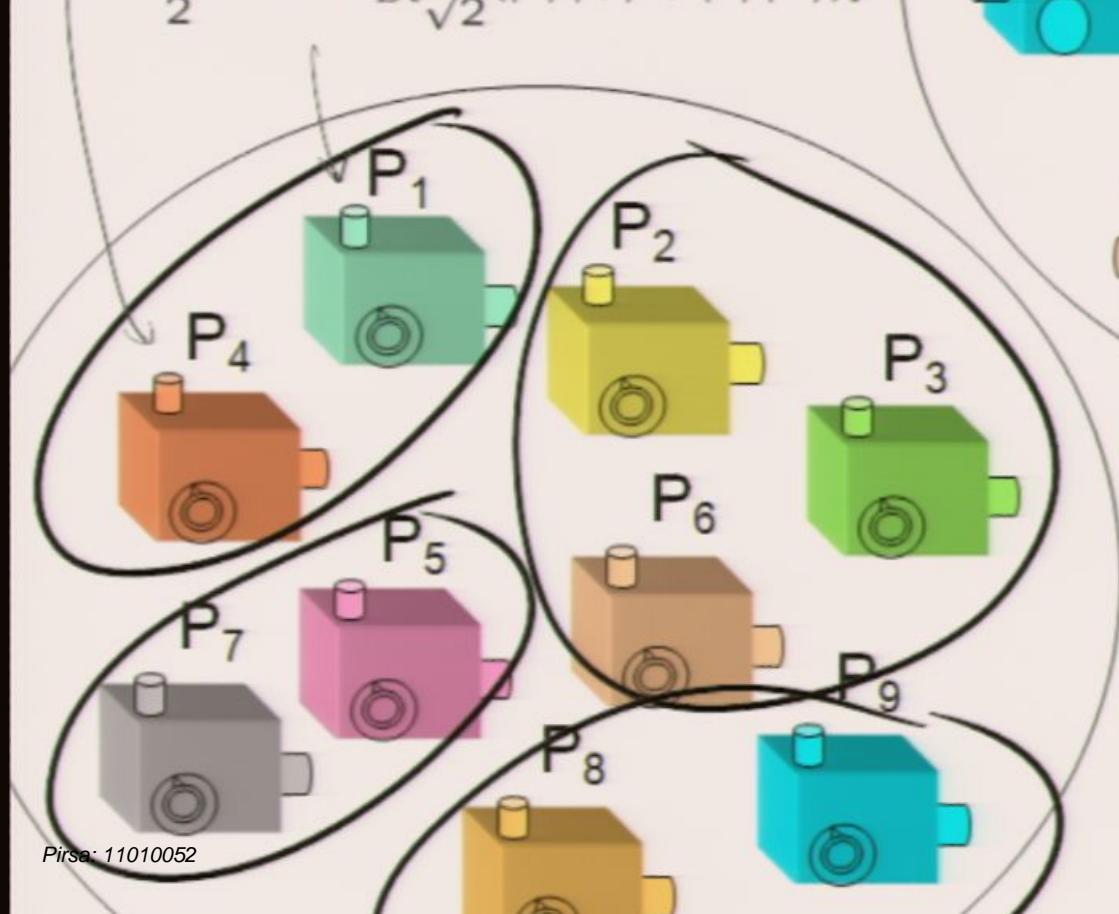
$$\frac{1}{2}I = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$



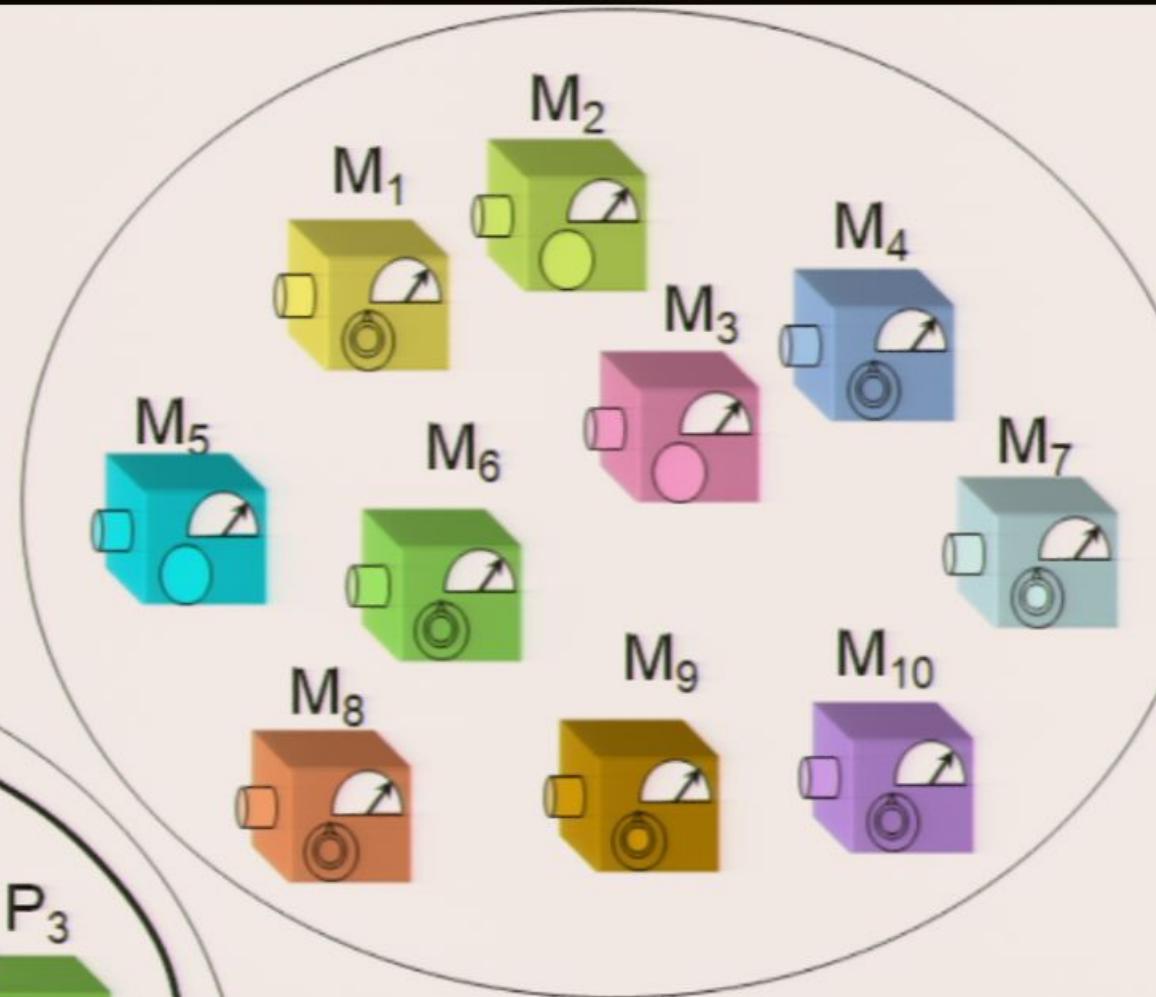
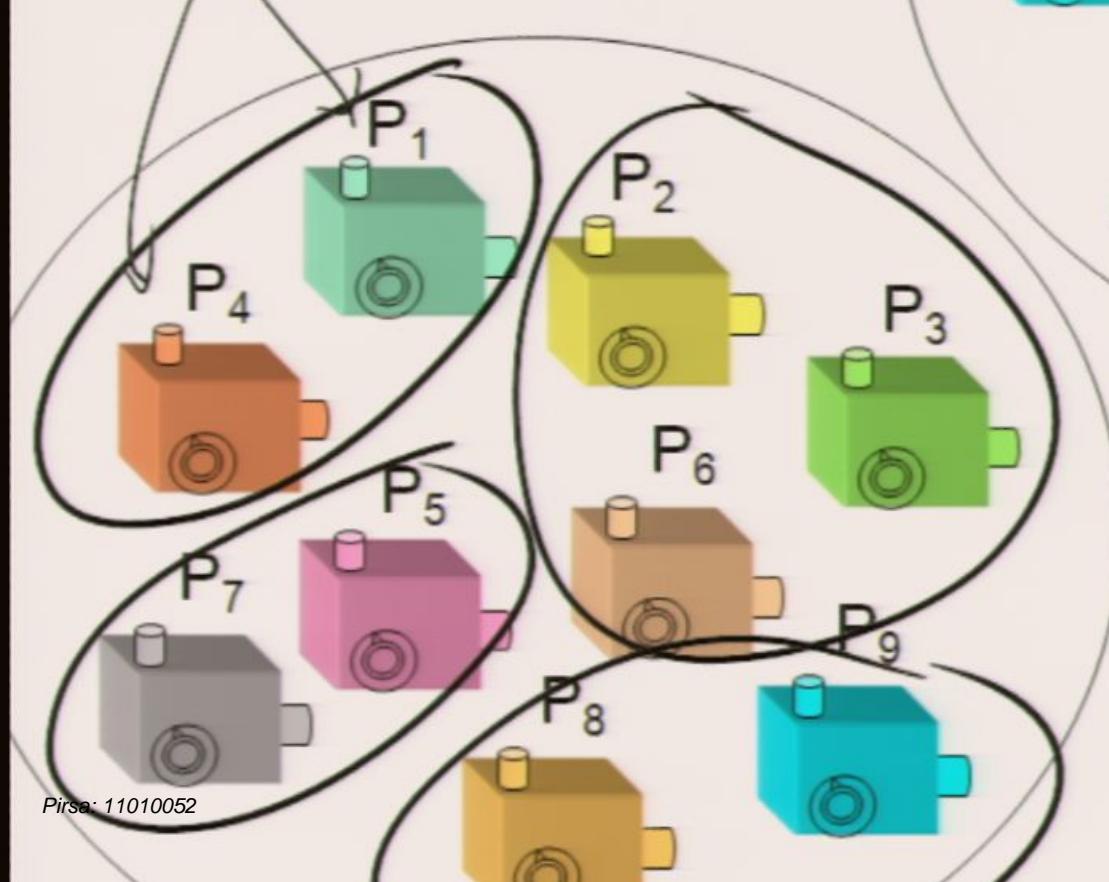
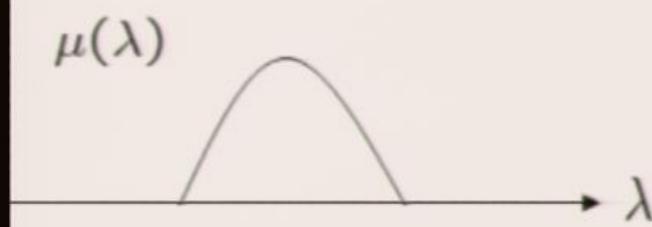
## Example from quantum theory

$$I = \text{Tr}_B\left[\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)\right]$$

$$\frac{1}{2}I = \text{Tr}_B\left[\frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle)\right]$$



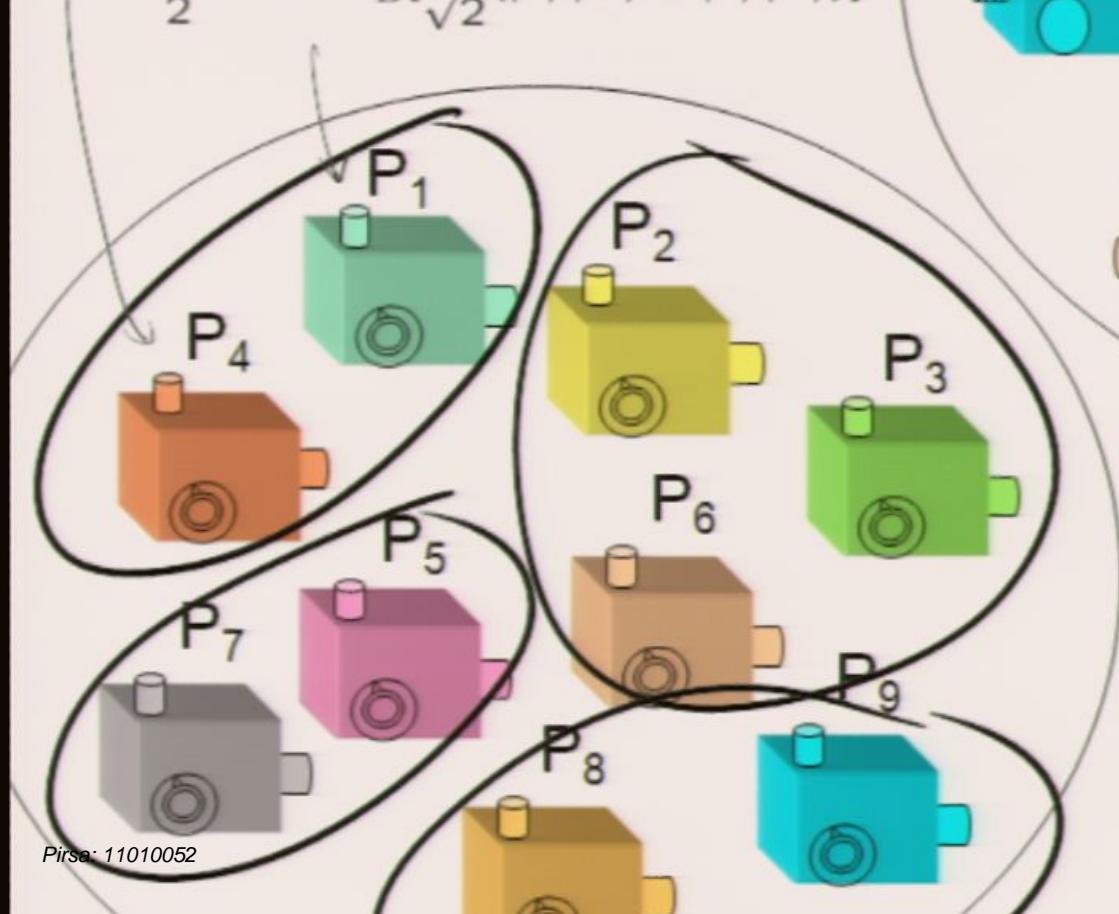
## Preparation noncontextual model



## Example from quantum theory

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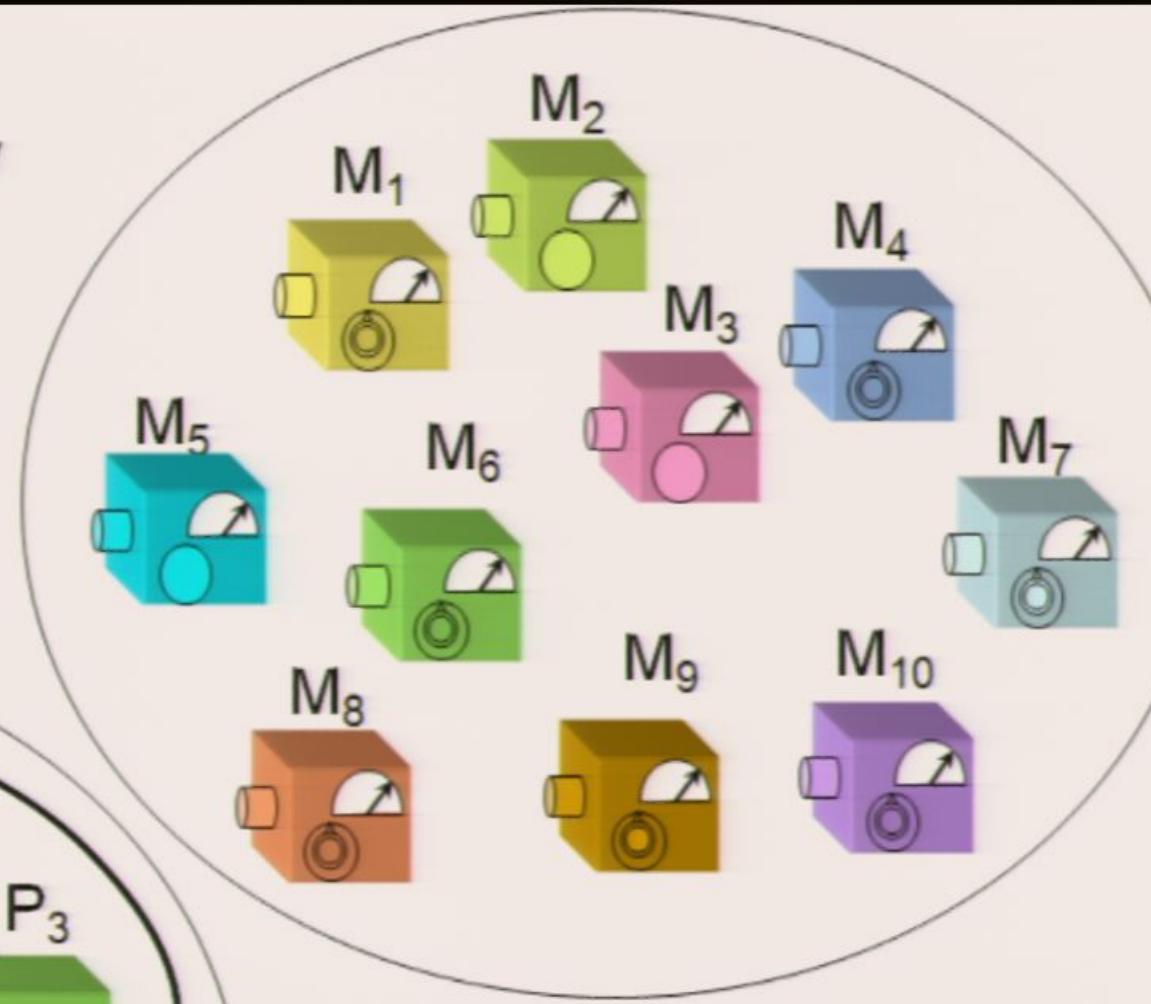
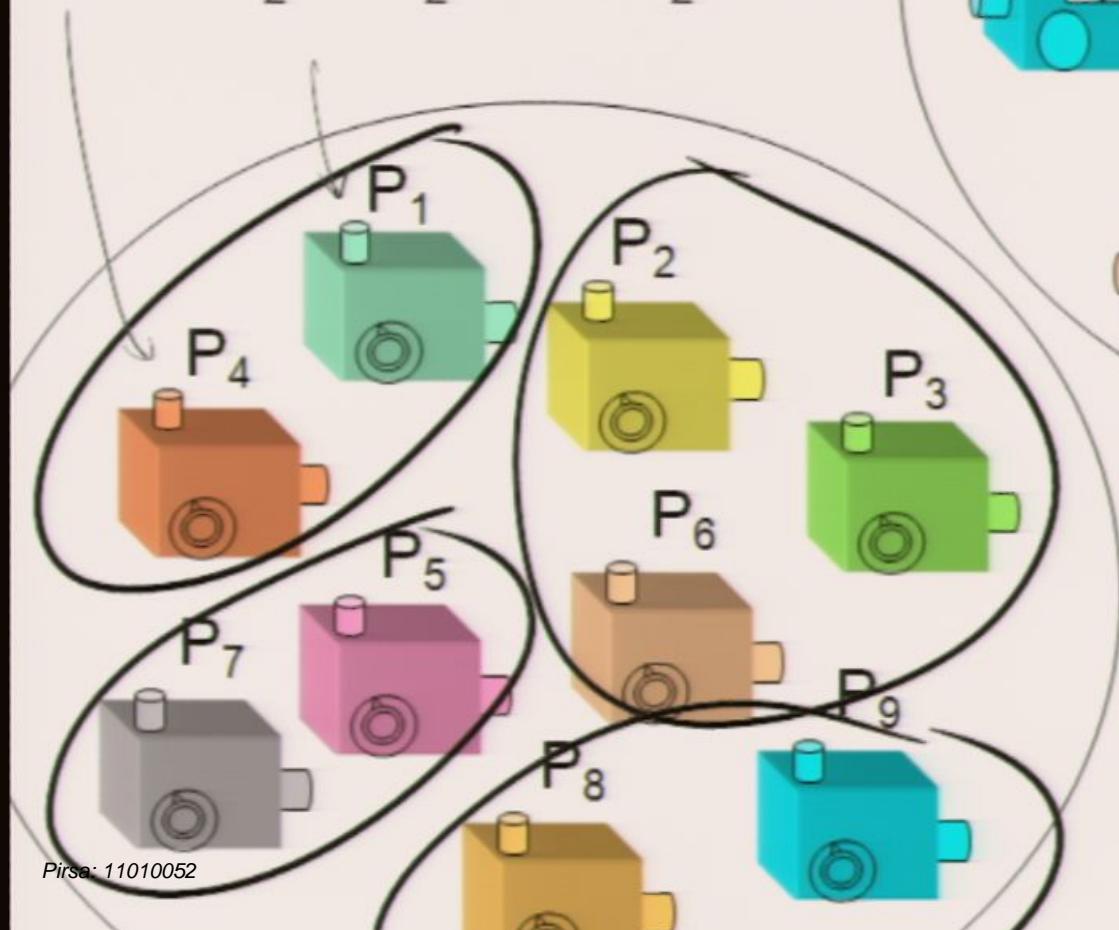
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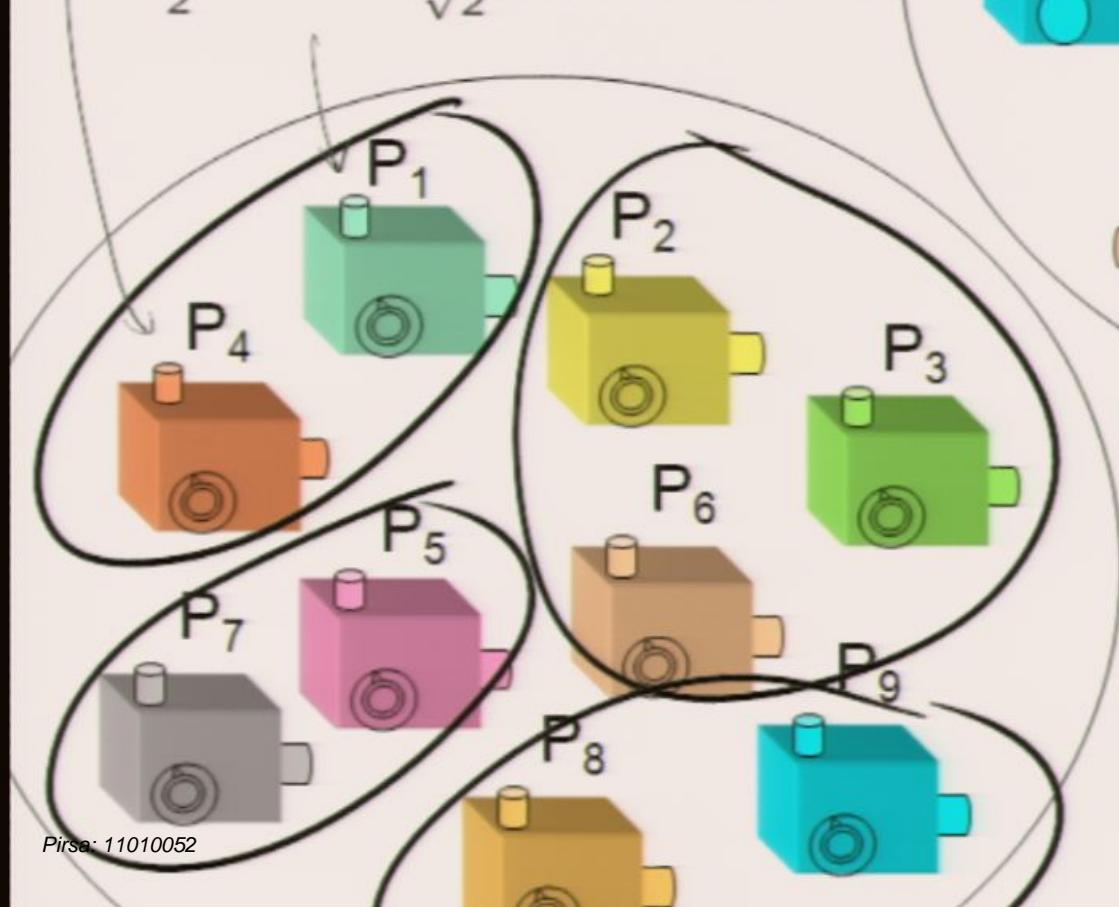
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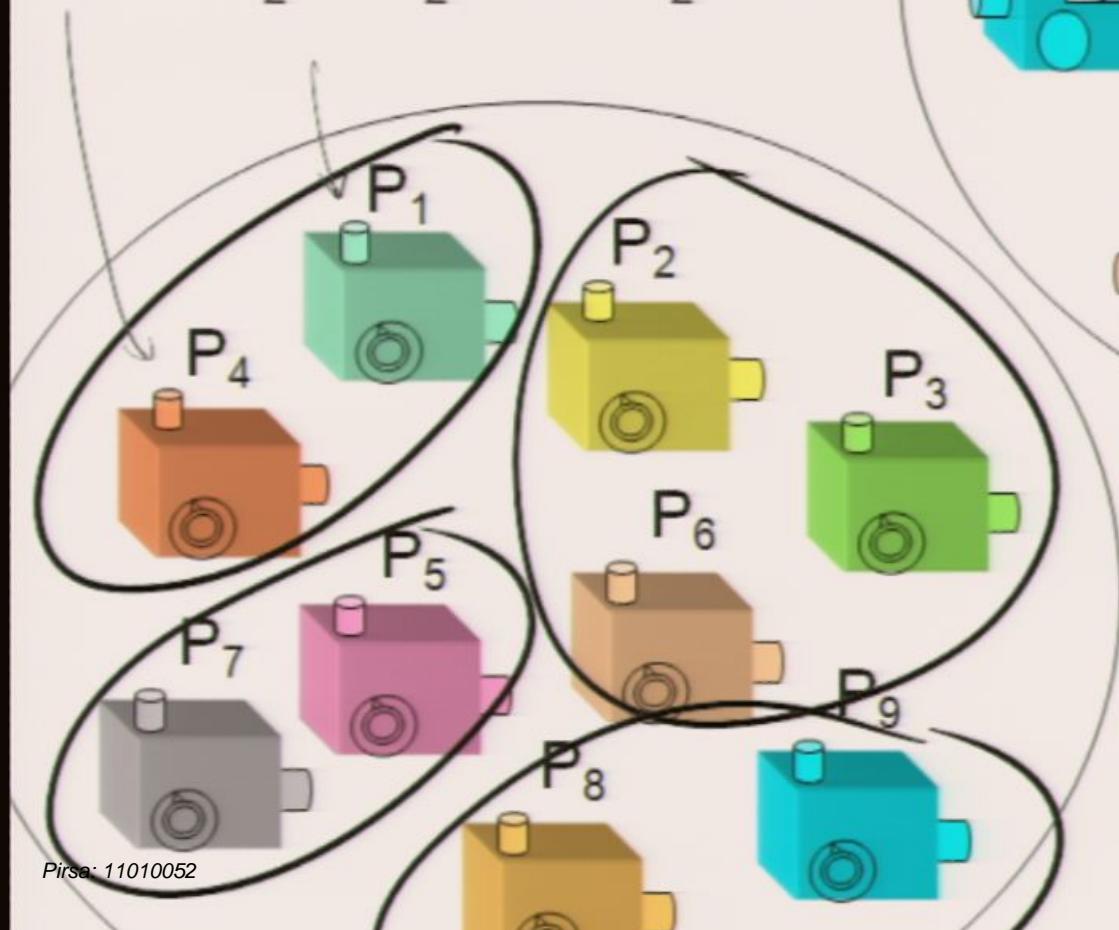
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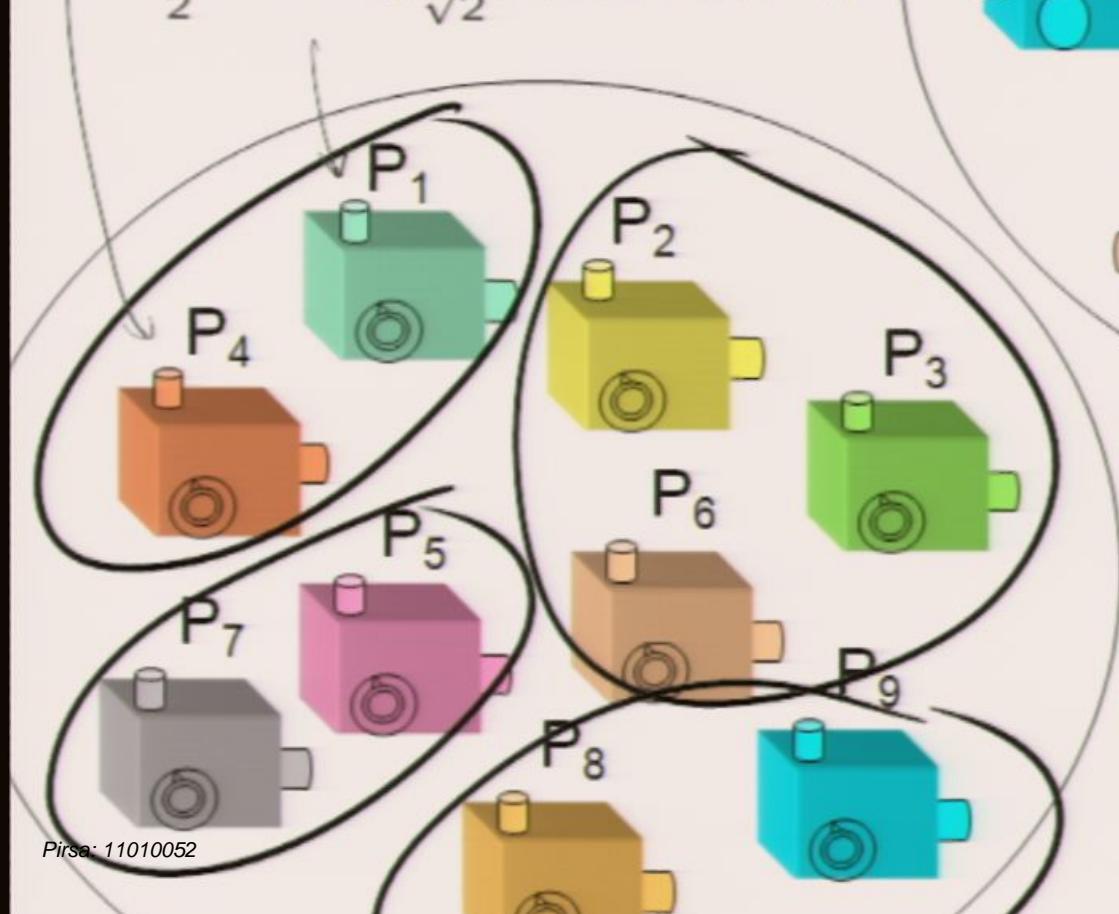
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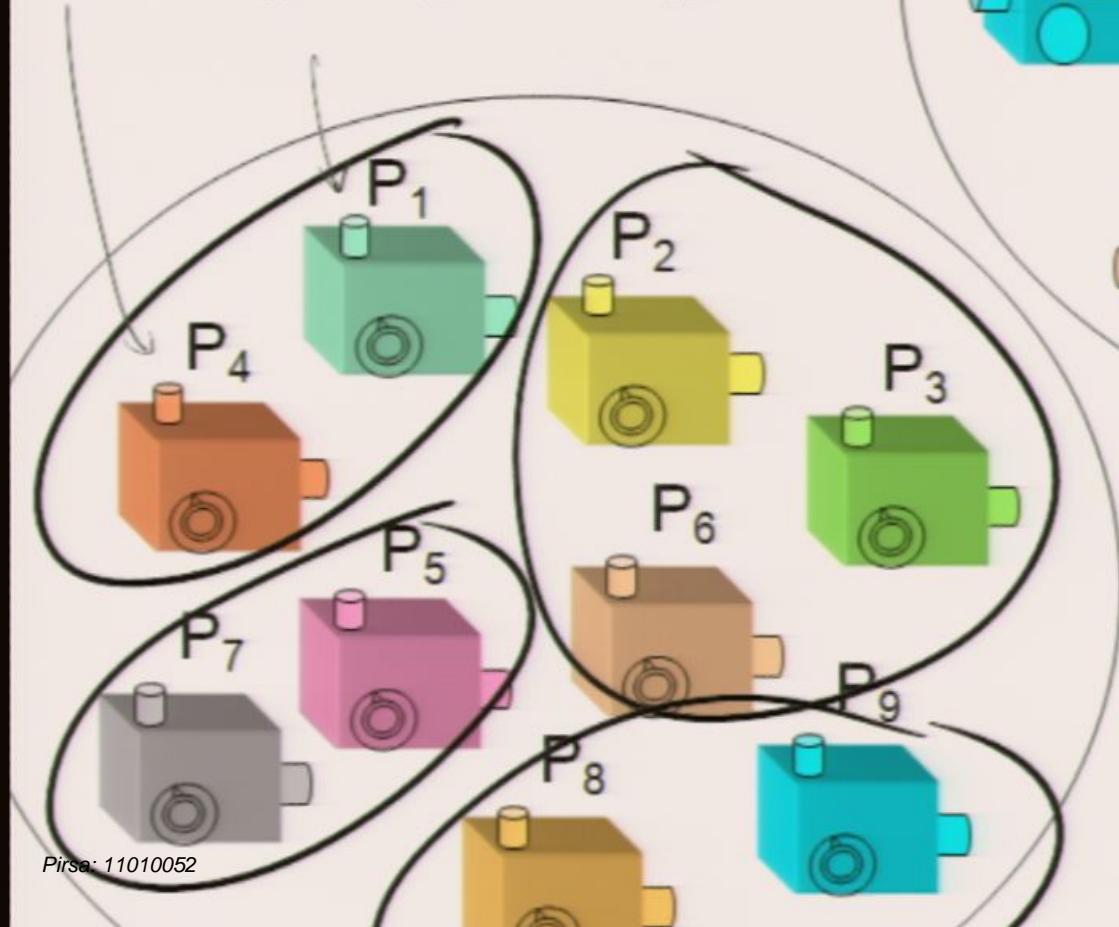
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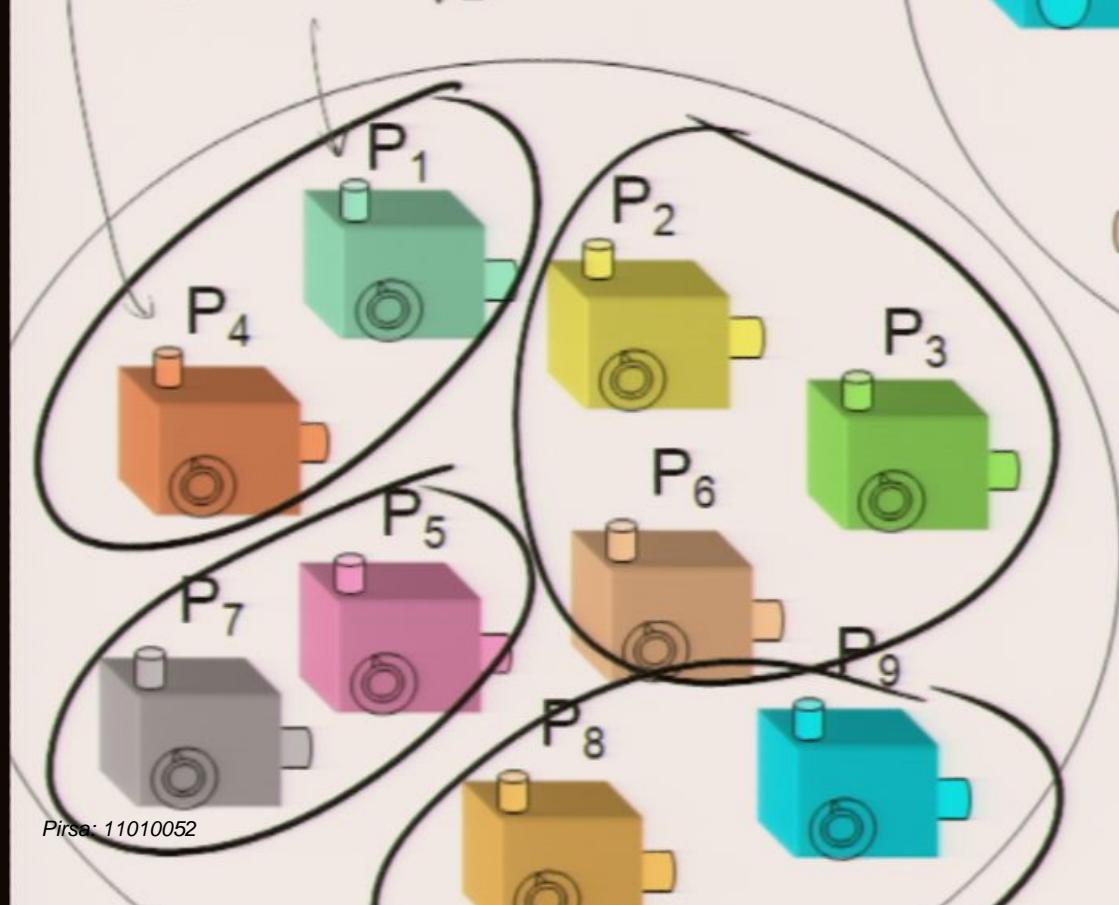
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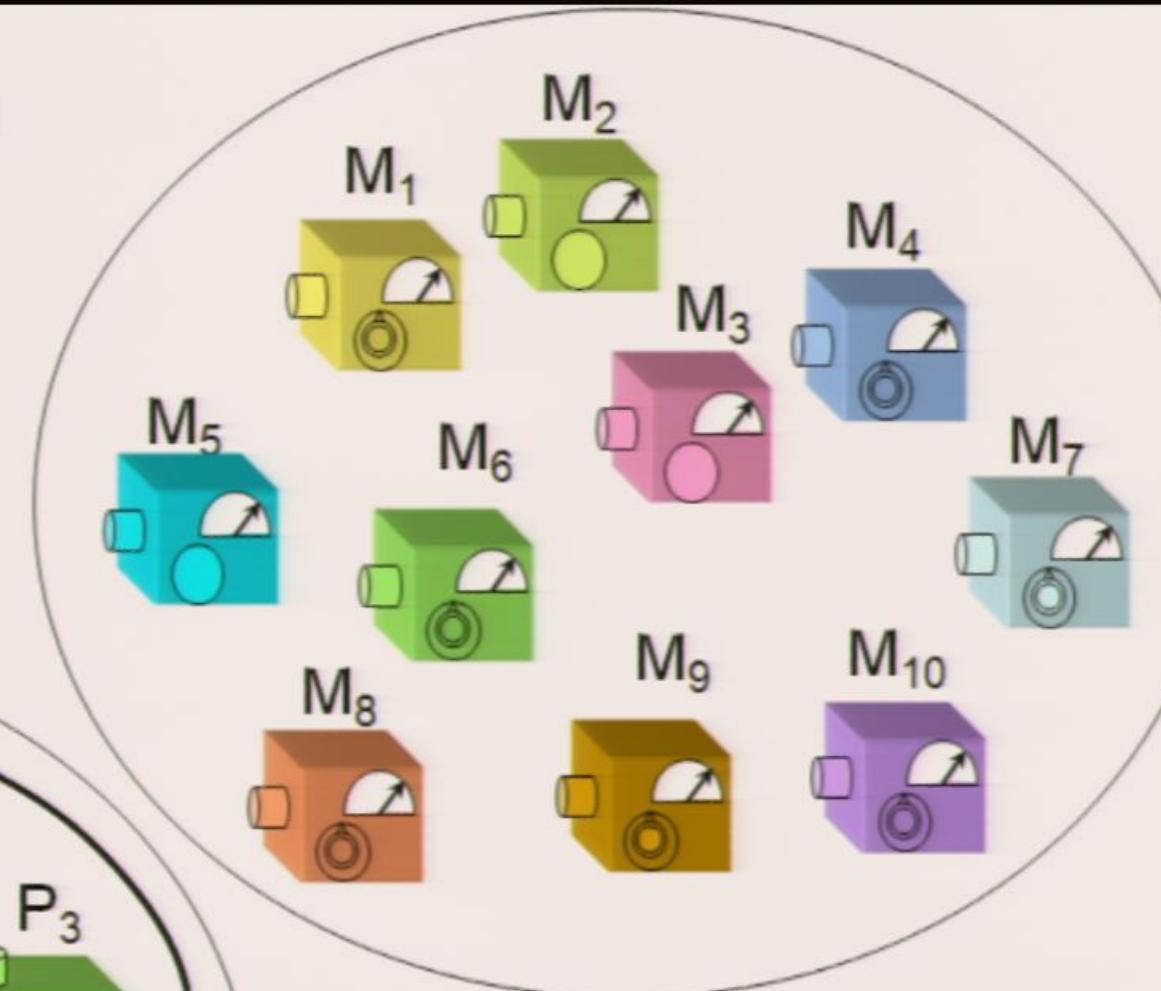
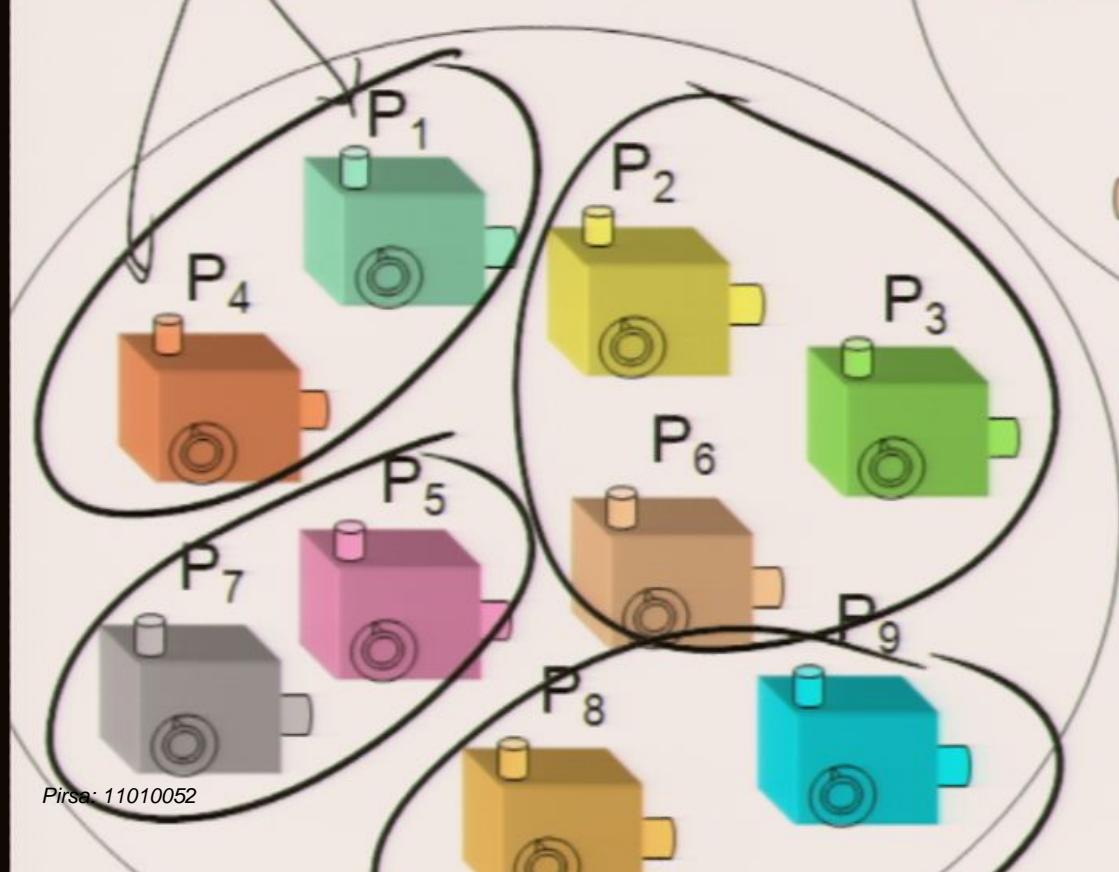
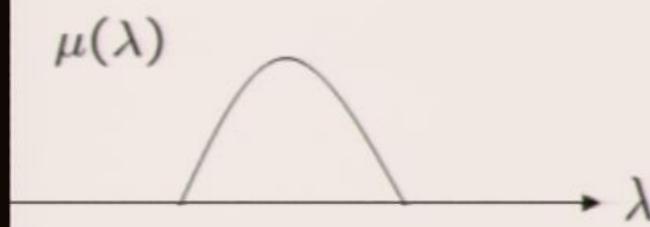
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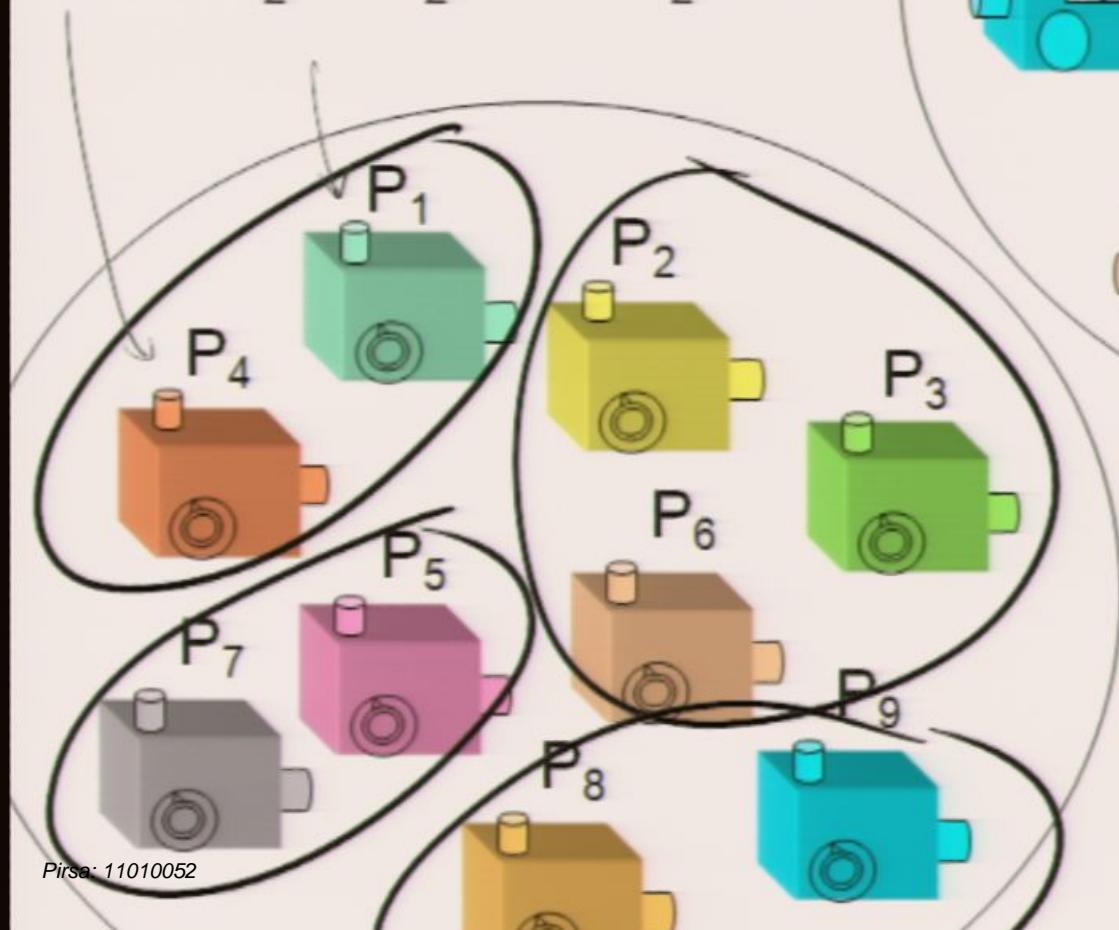
## Preparation noncontextual model



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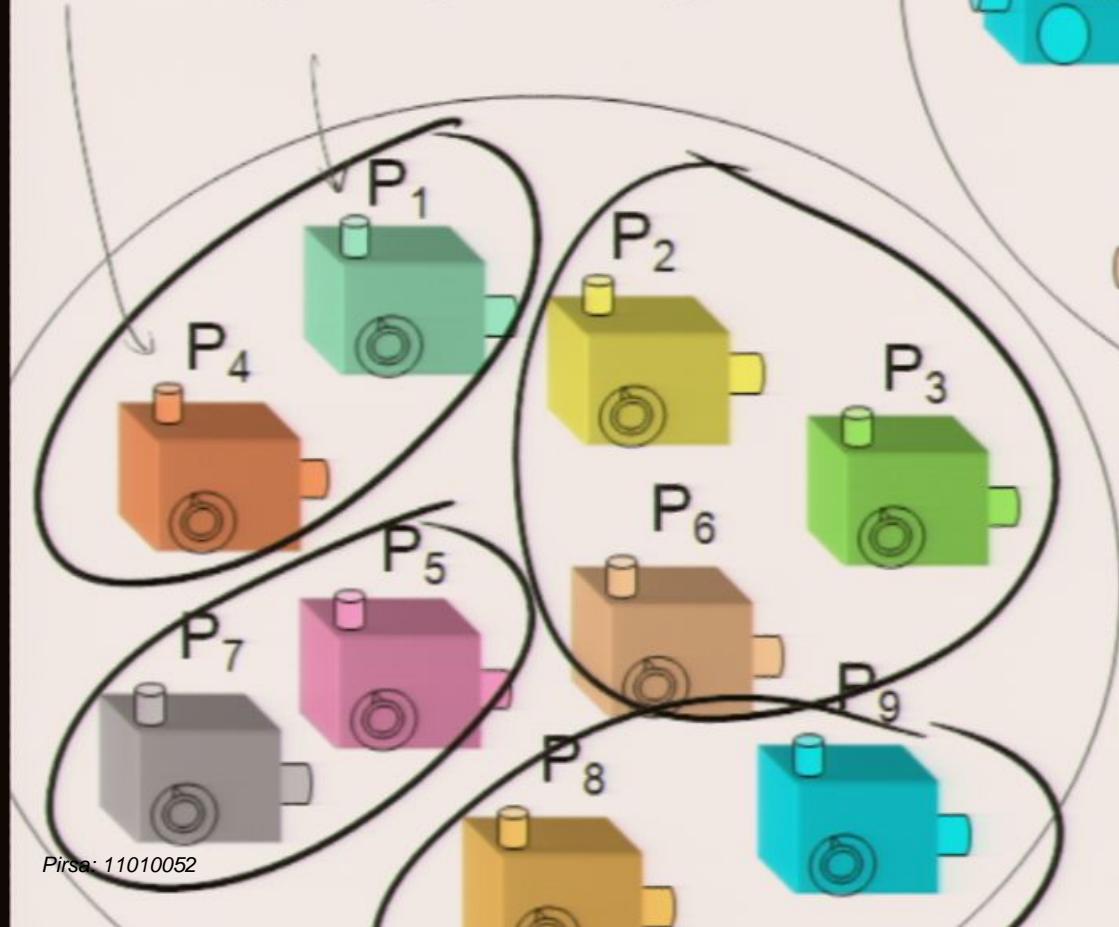
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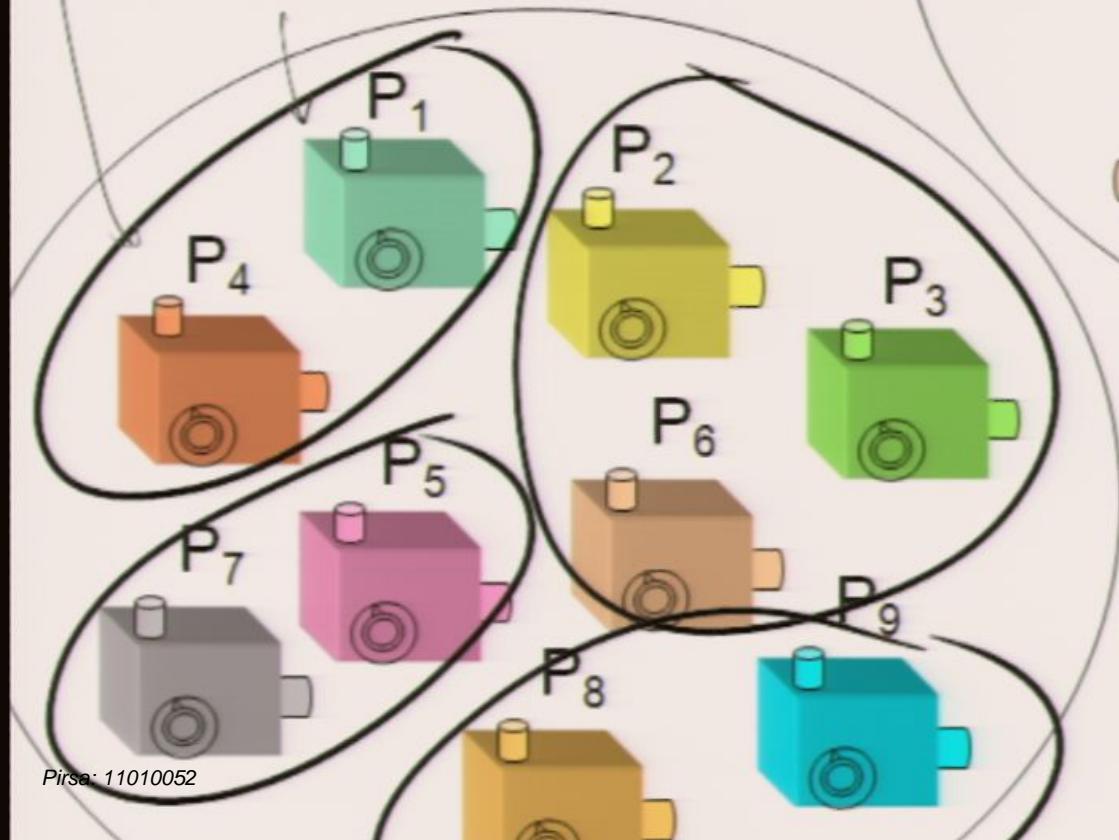
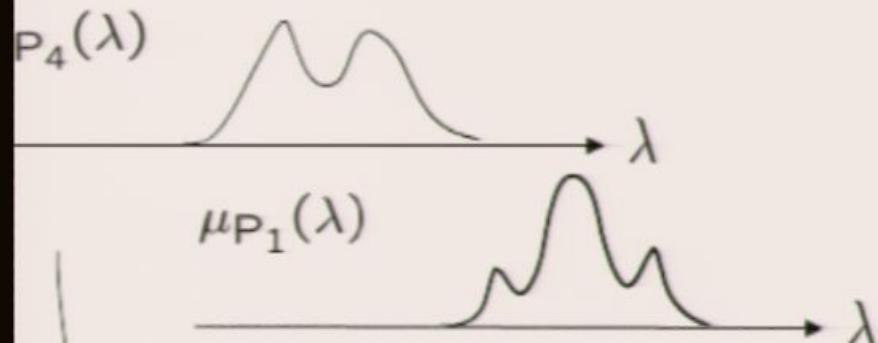
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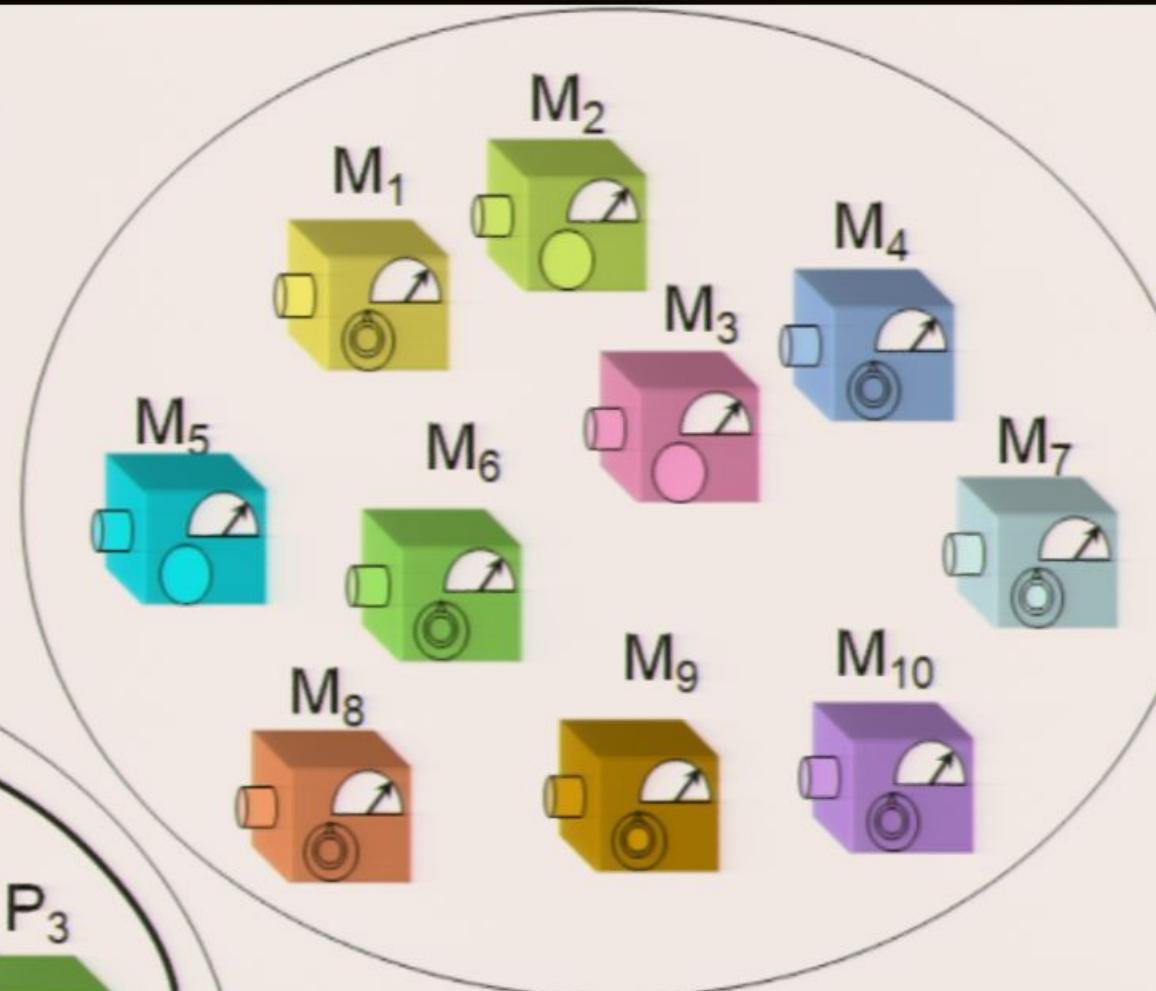
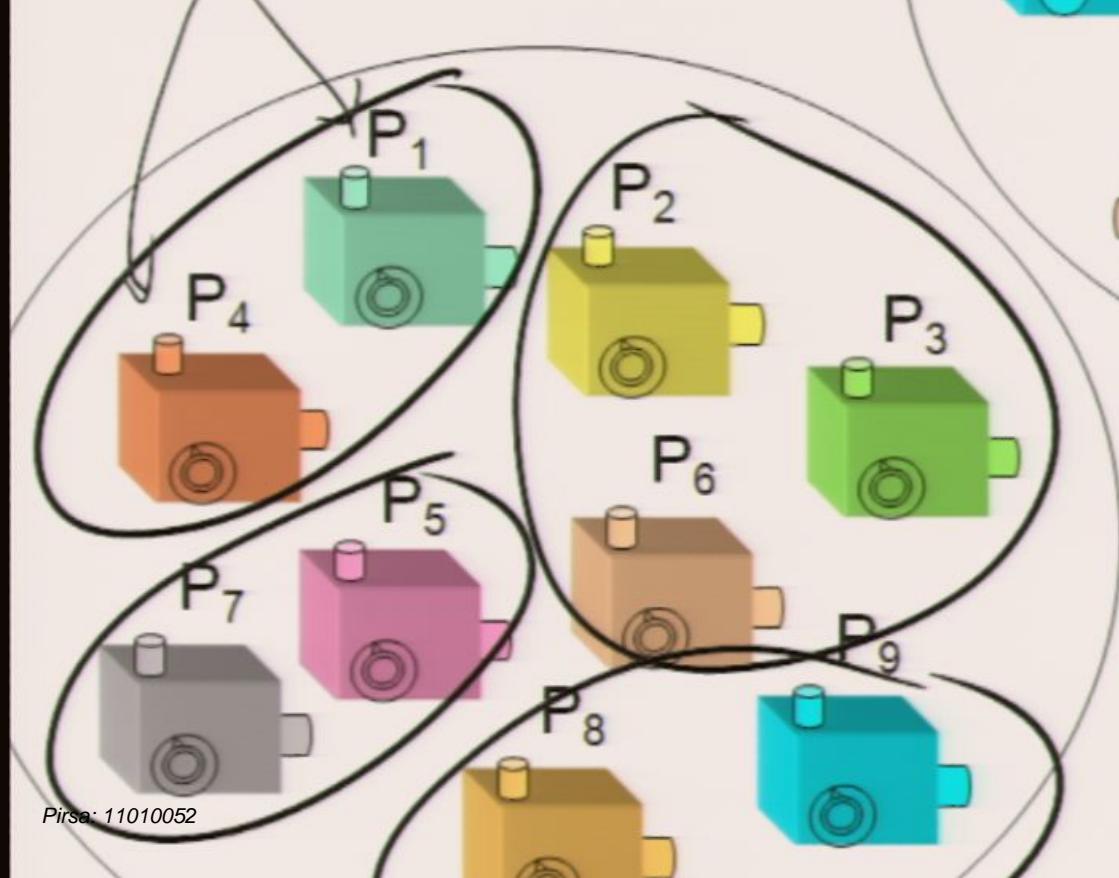
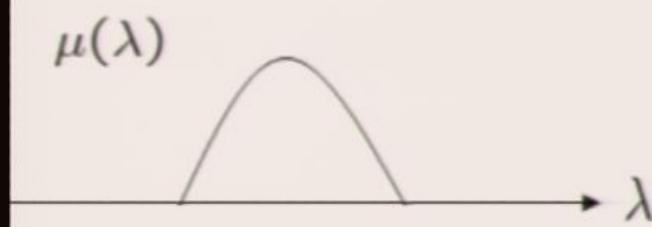
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## Preparation contextual model



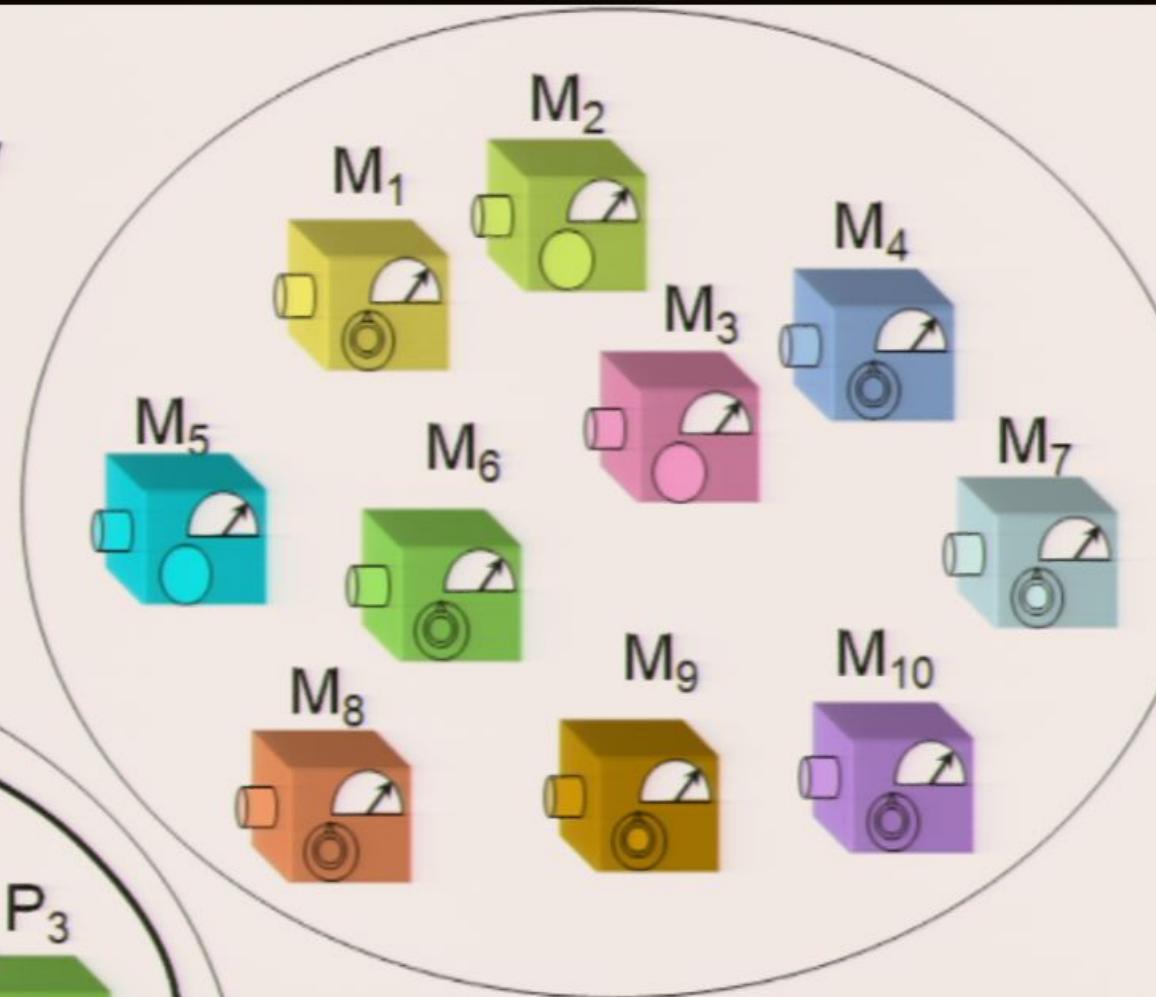
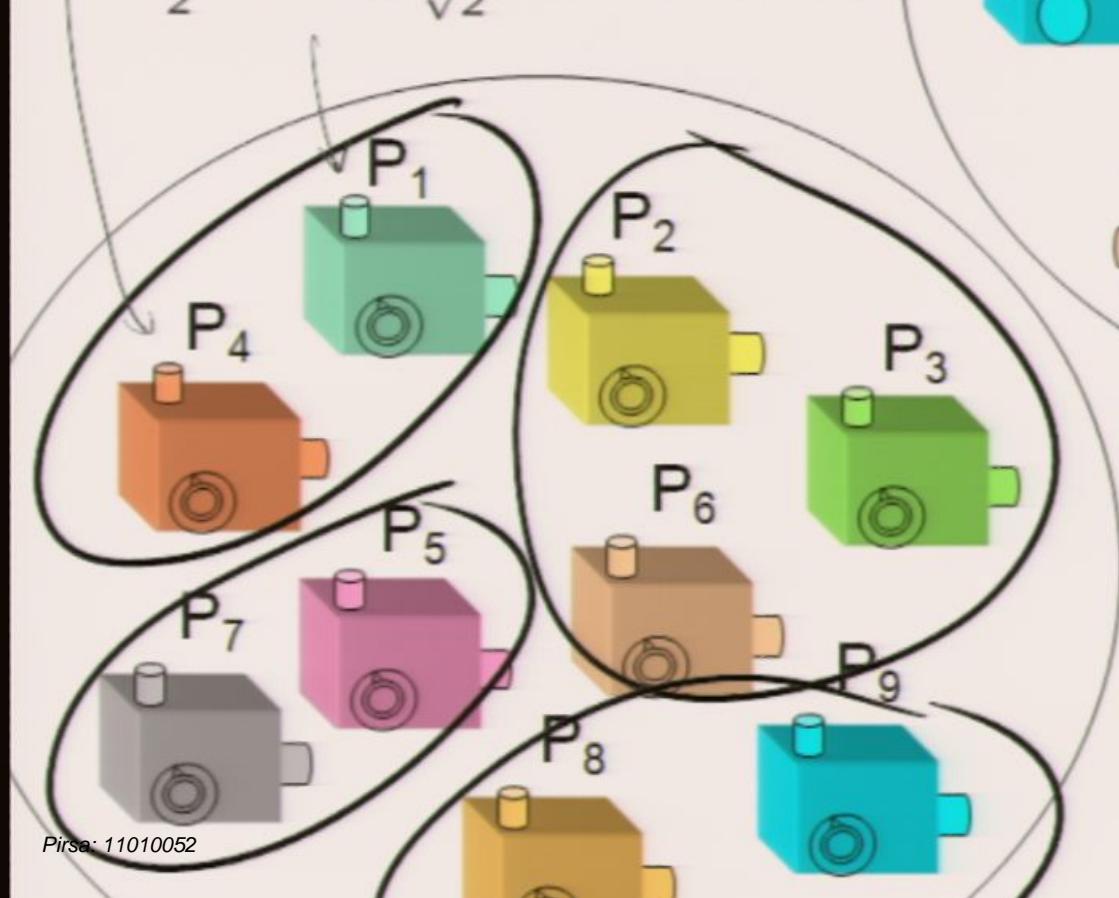
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## Example from quantum theory

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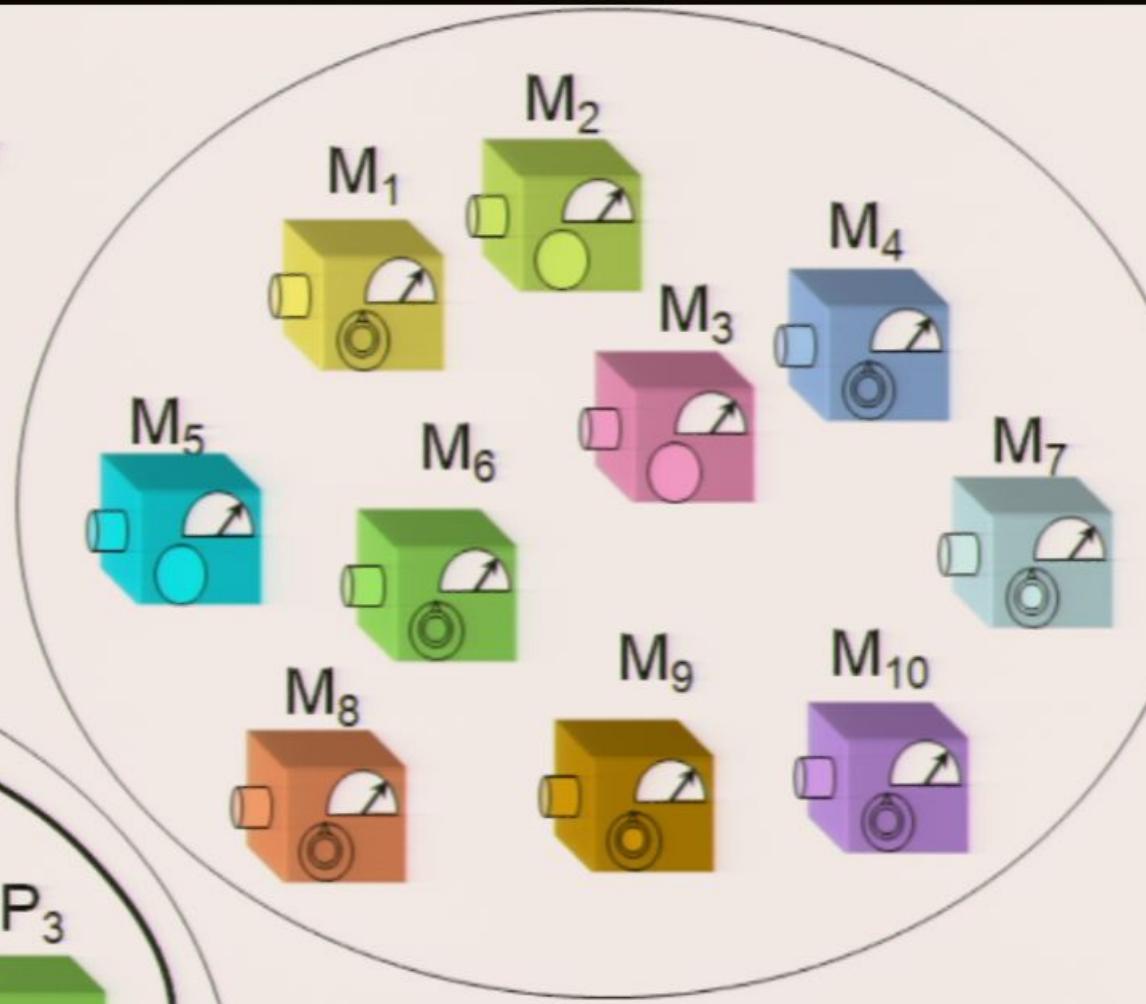
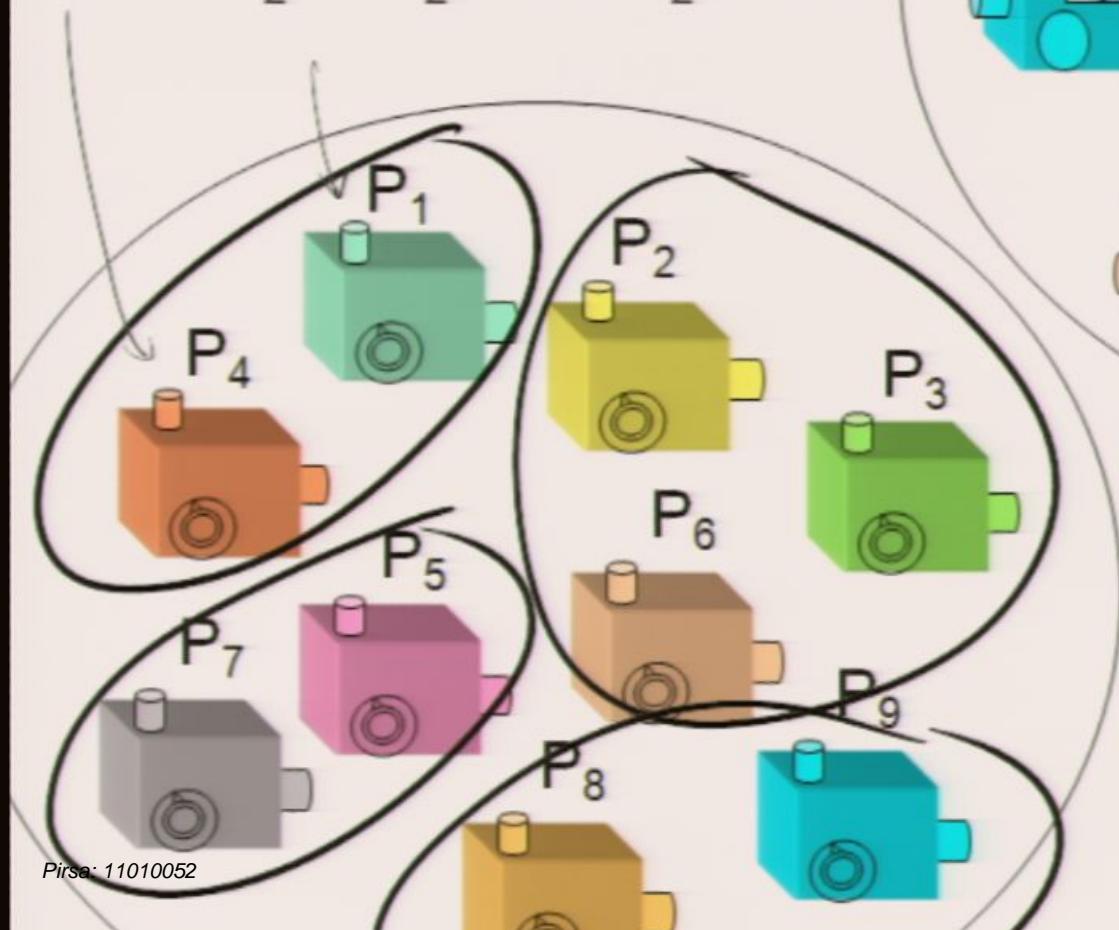
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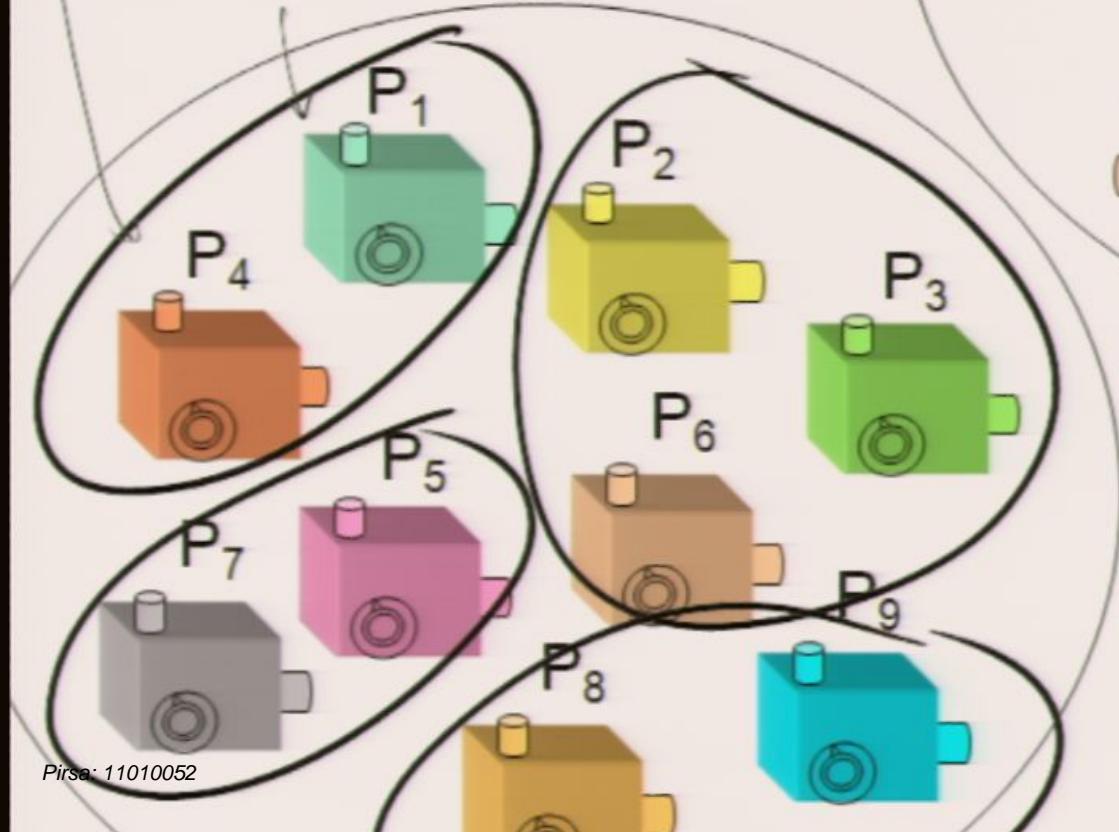
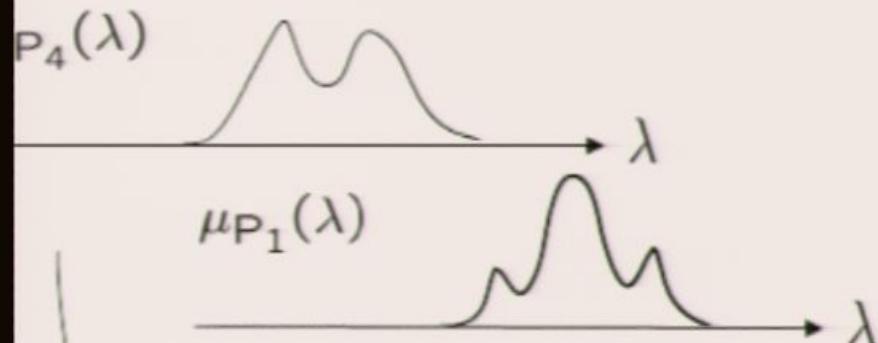
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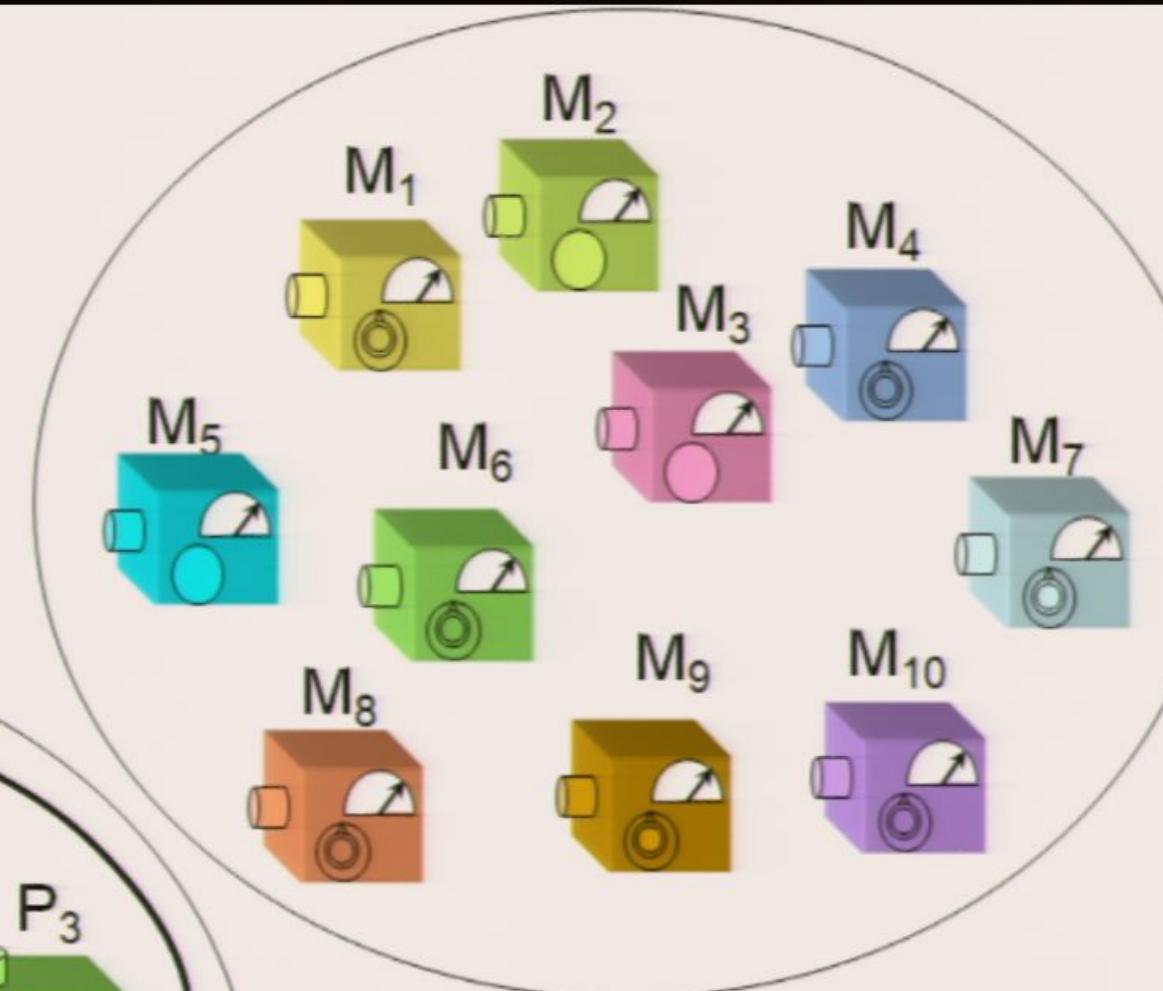
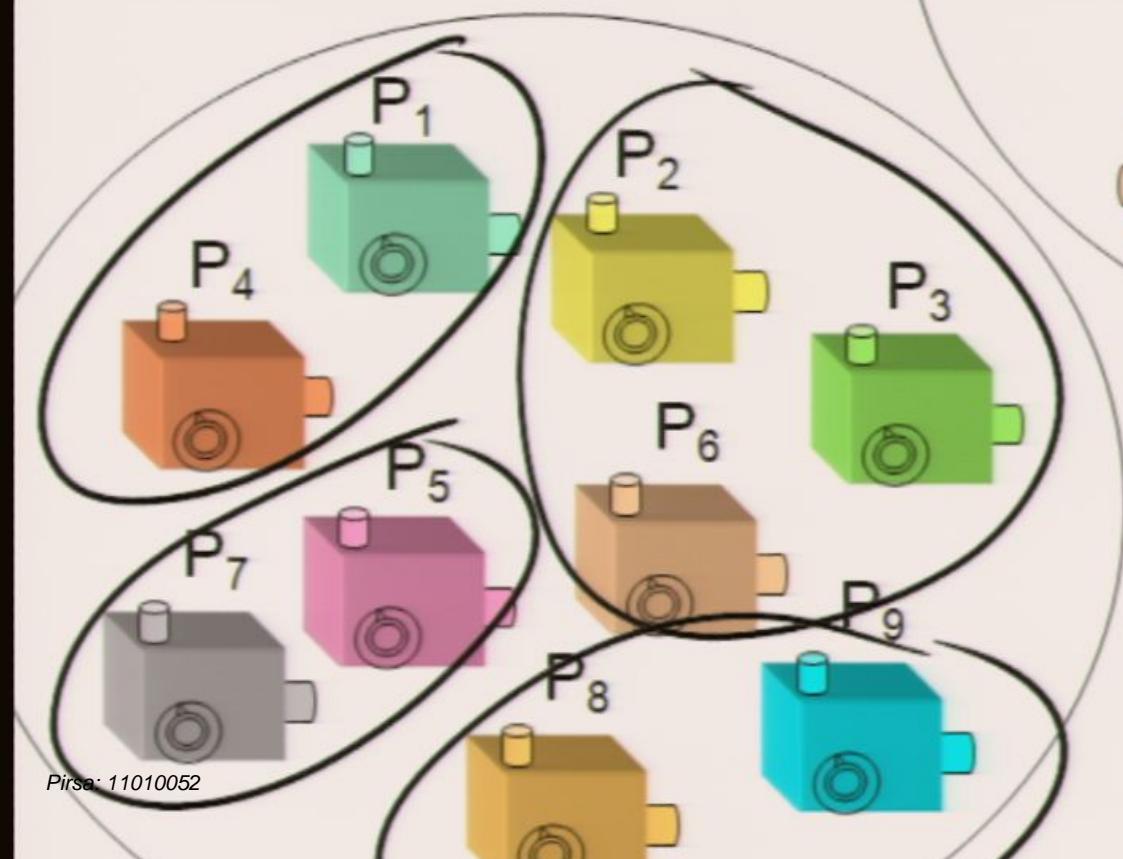
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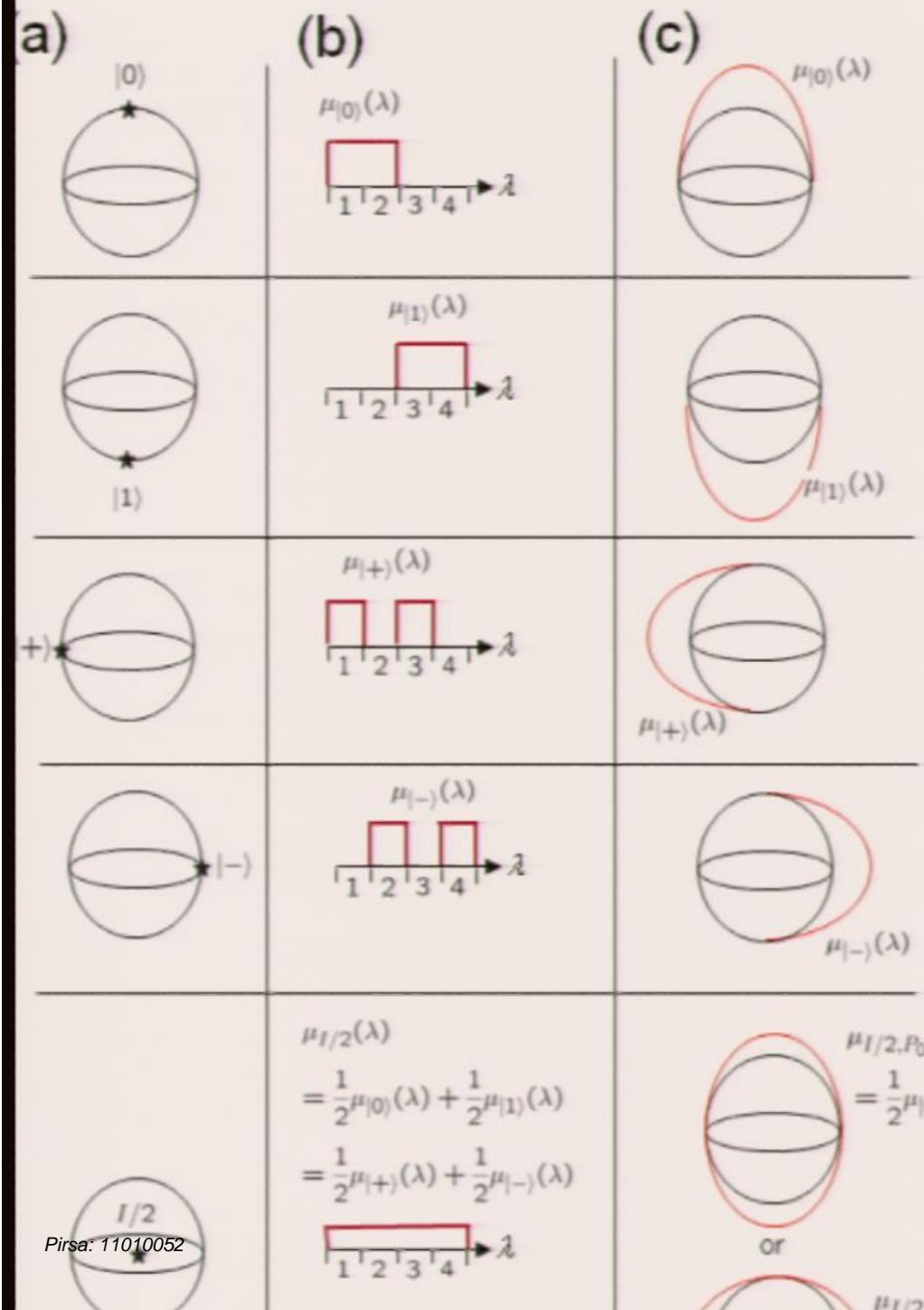


## Definition of preparation noncontextual model:

$$\forall M : p(k|P, M) = p(k|P', M)$$

→  $p(\lambda|P) = p(\lambda|P')$

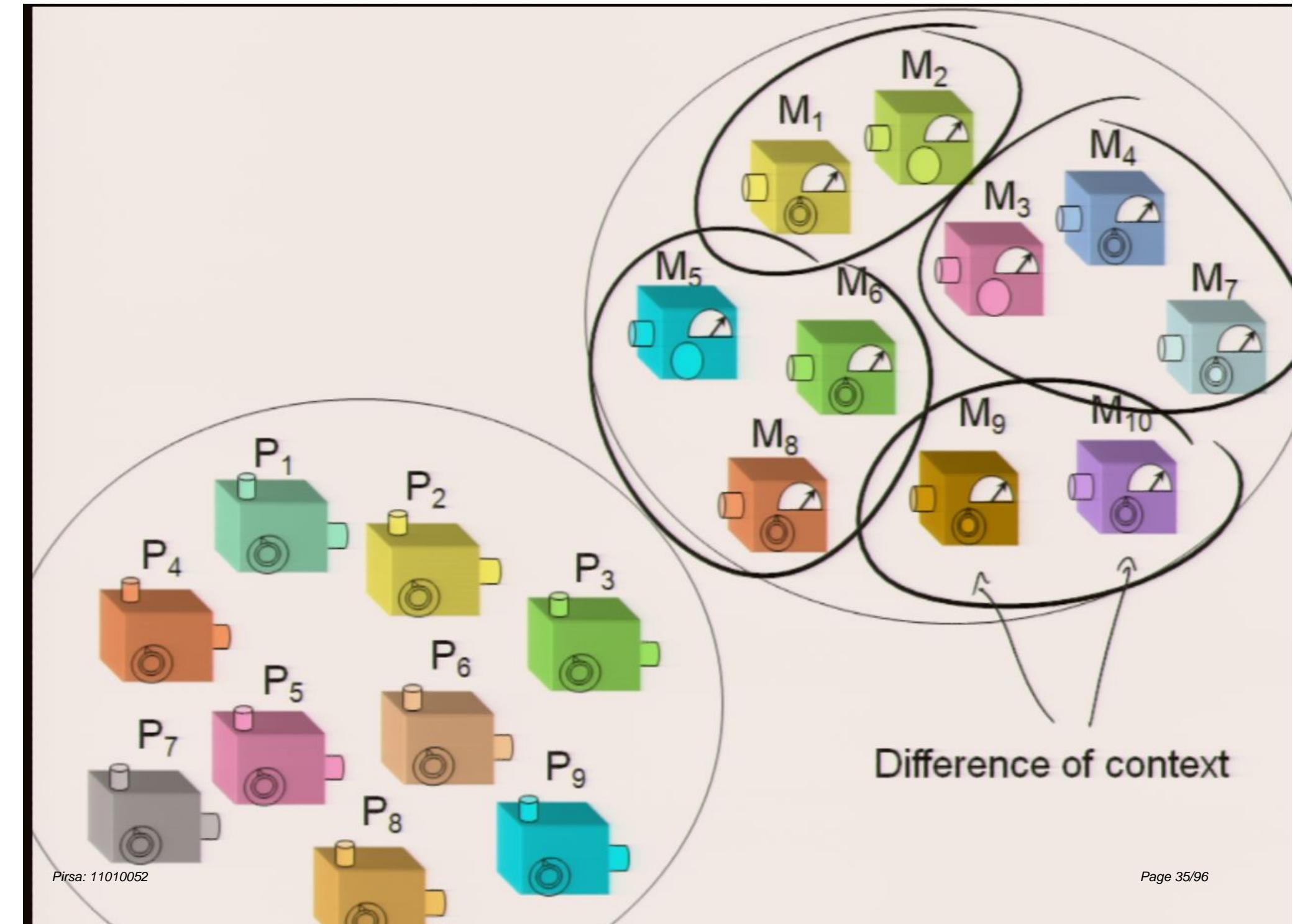


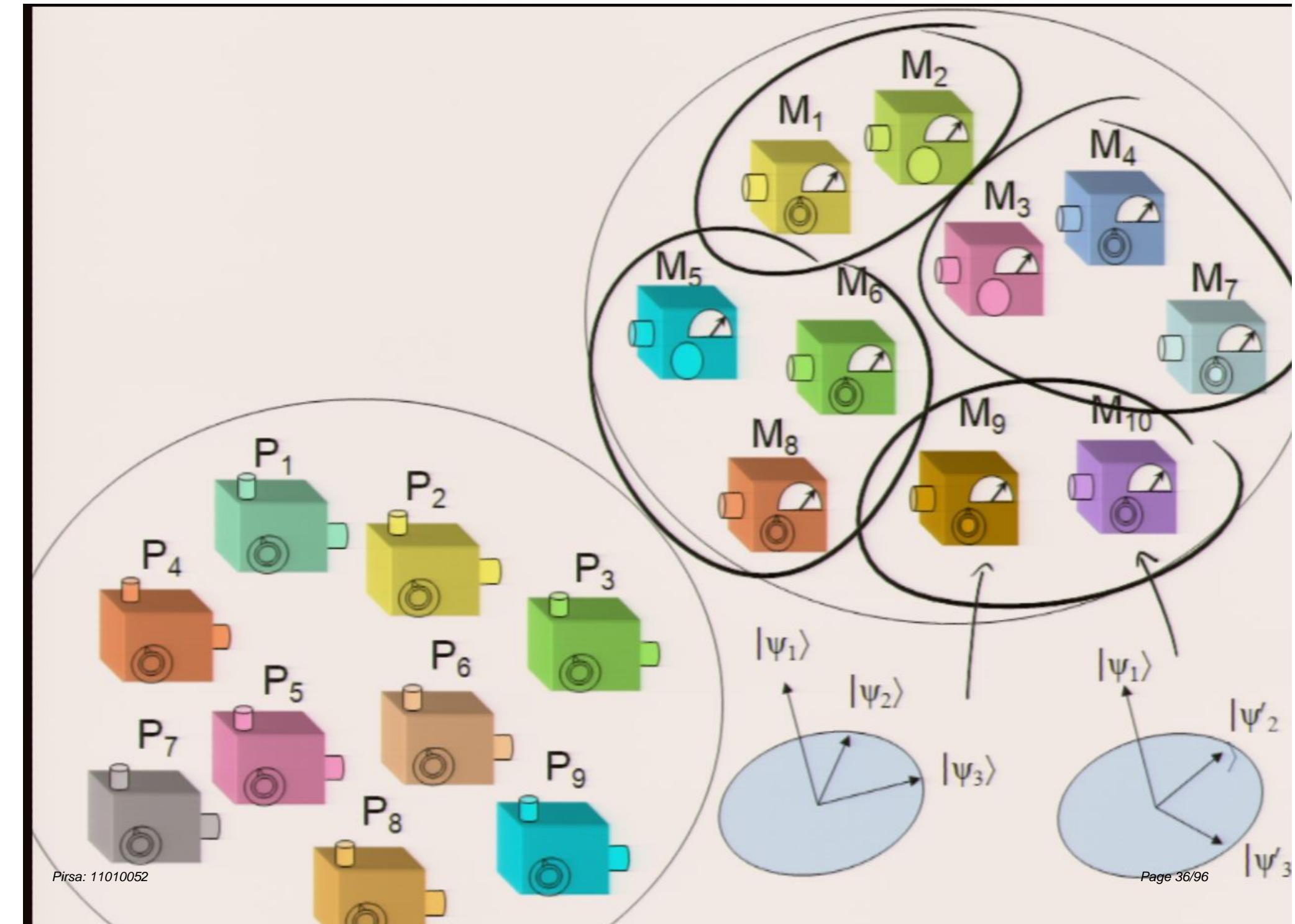


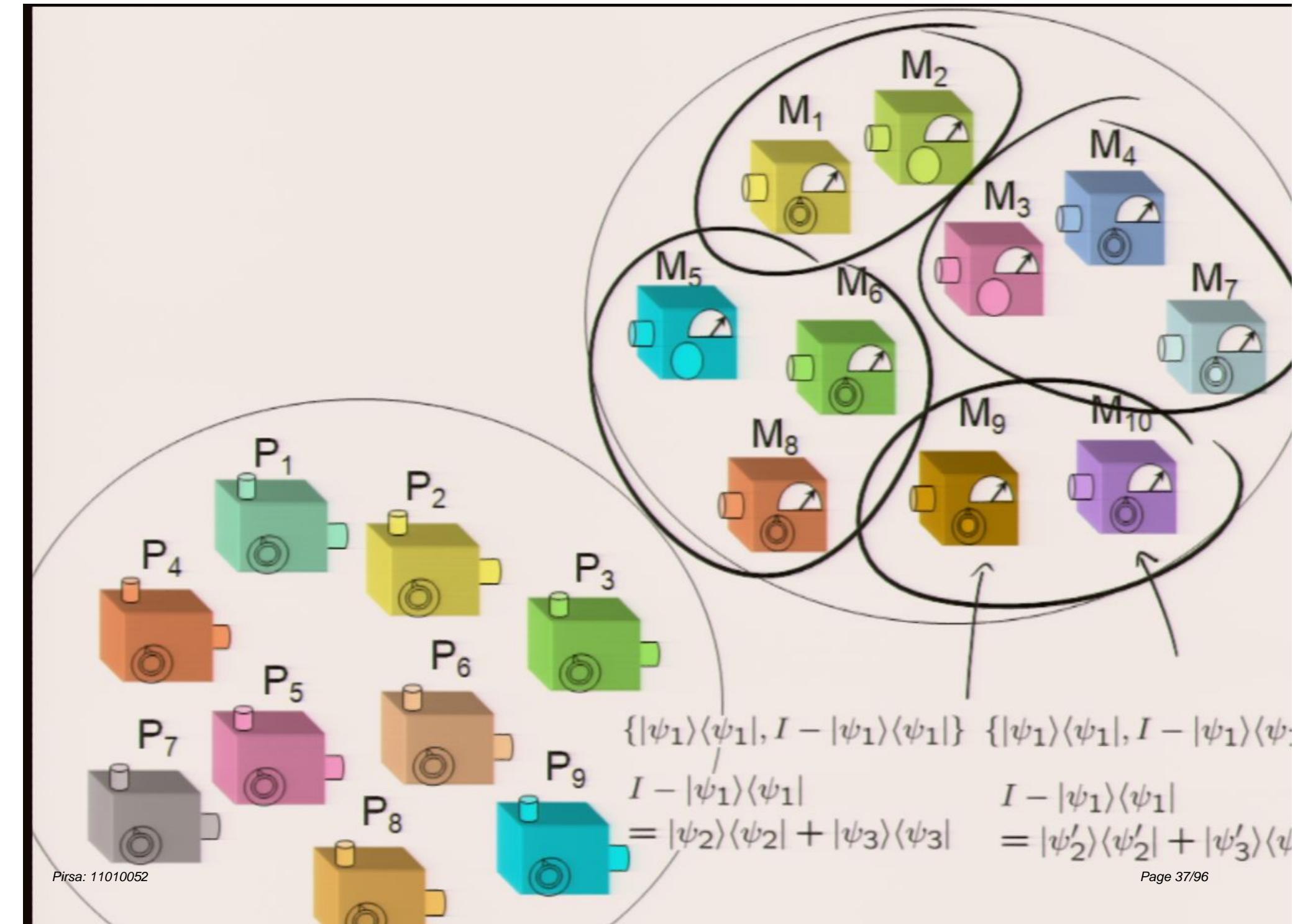
(a) Some states of a qubit

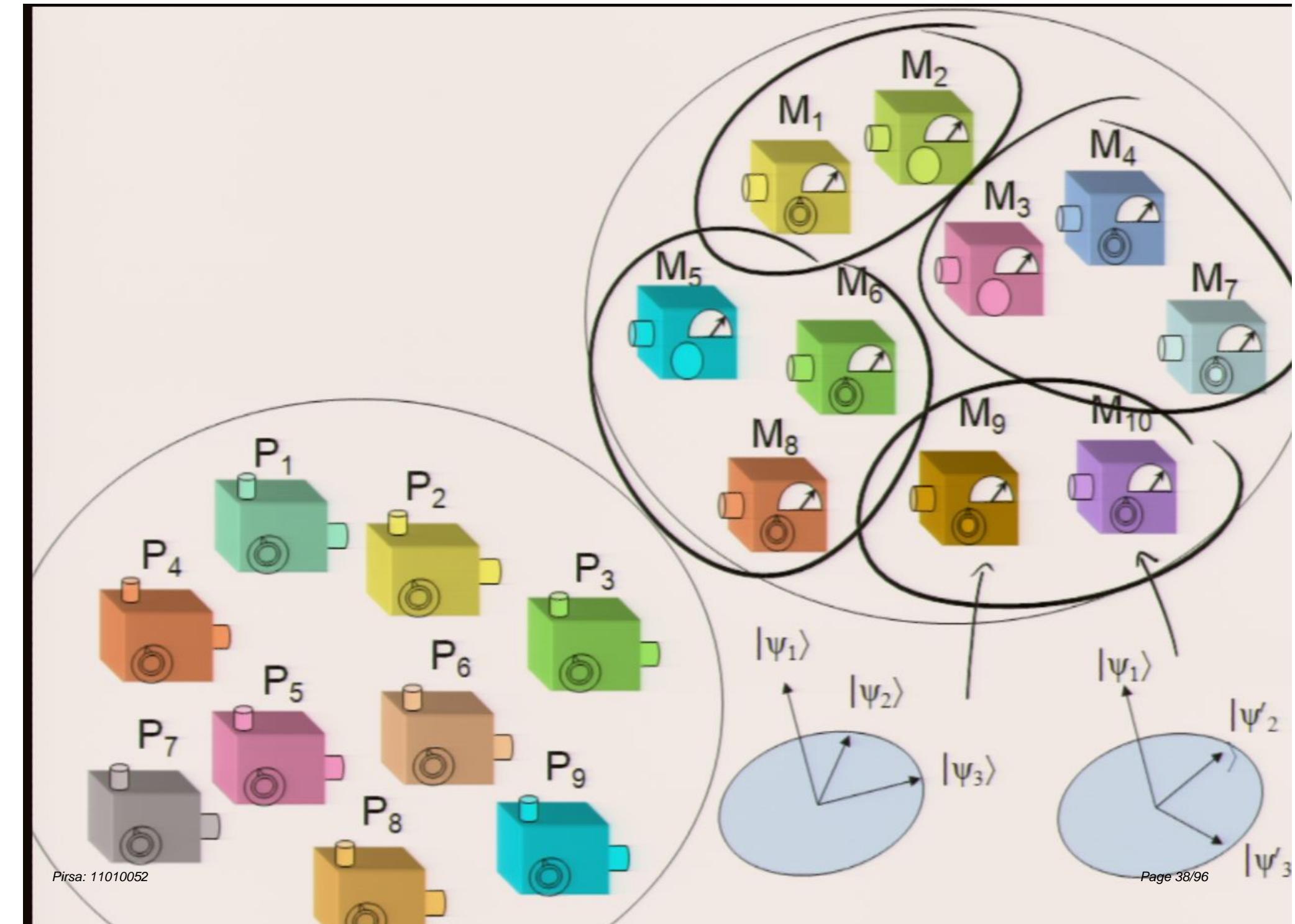
(b) A preparation **noncontextual** model of these

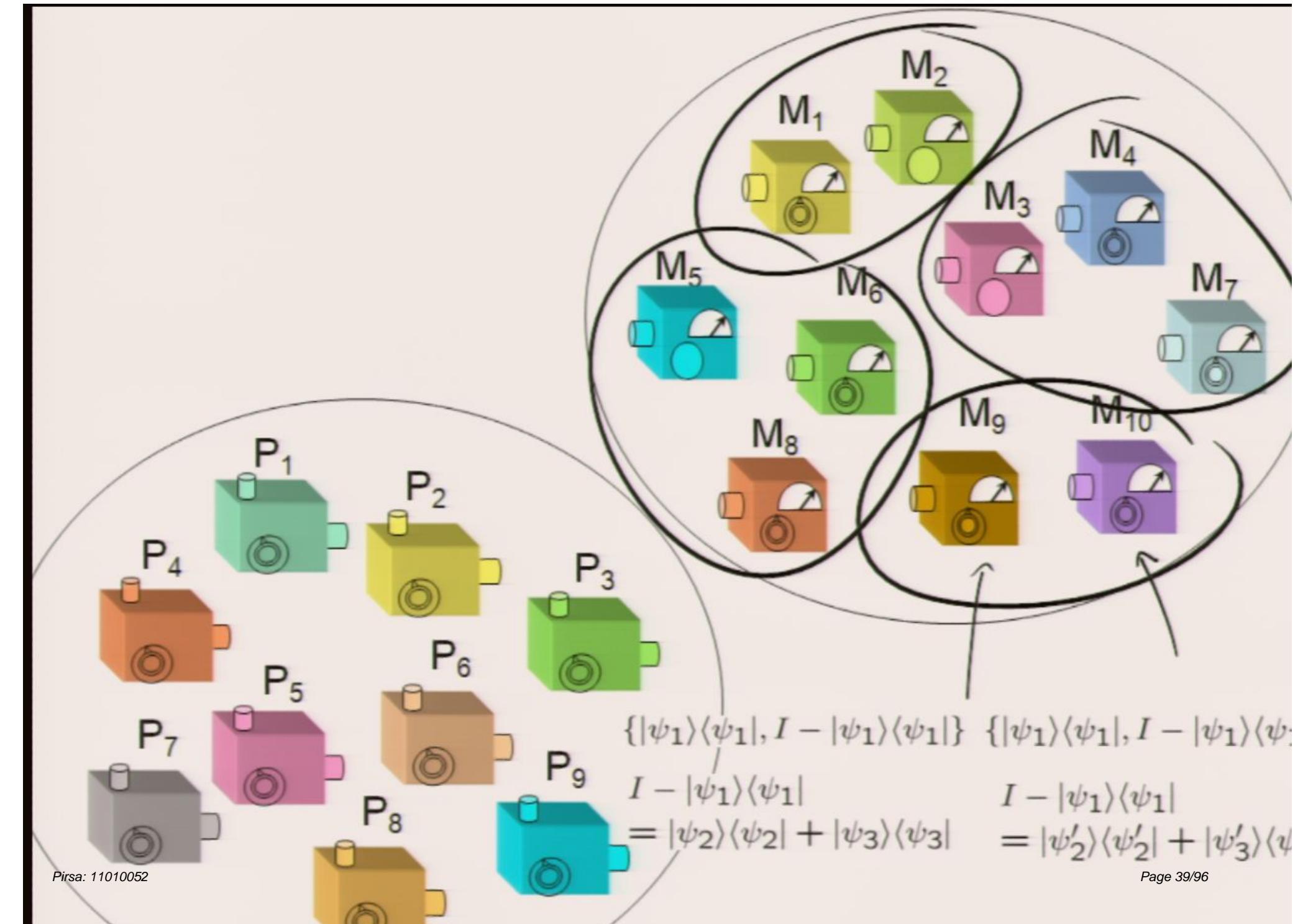
(c) A preparation **contextual** model of these  
(Kochen-Specker, 1967)

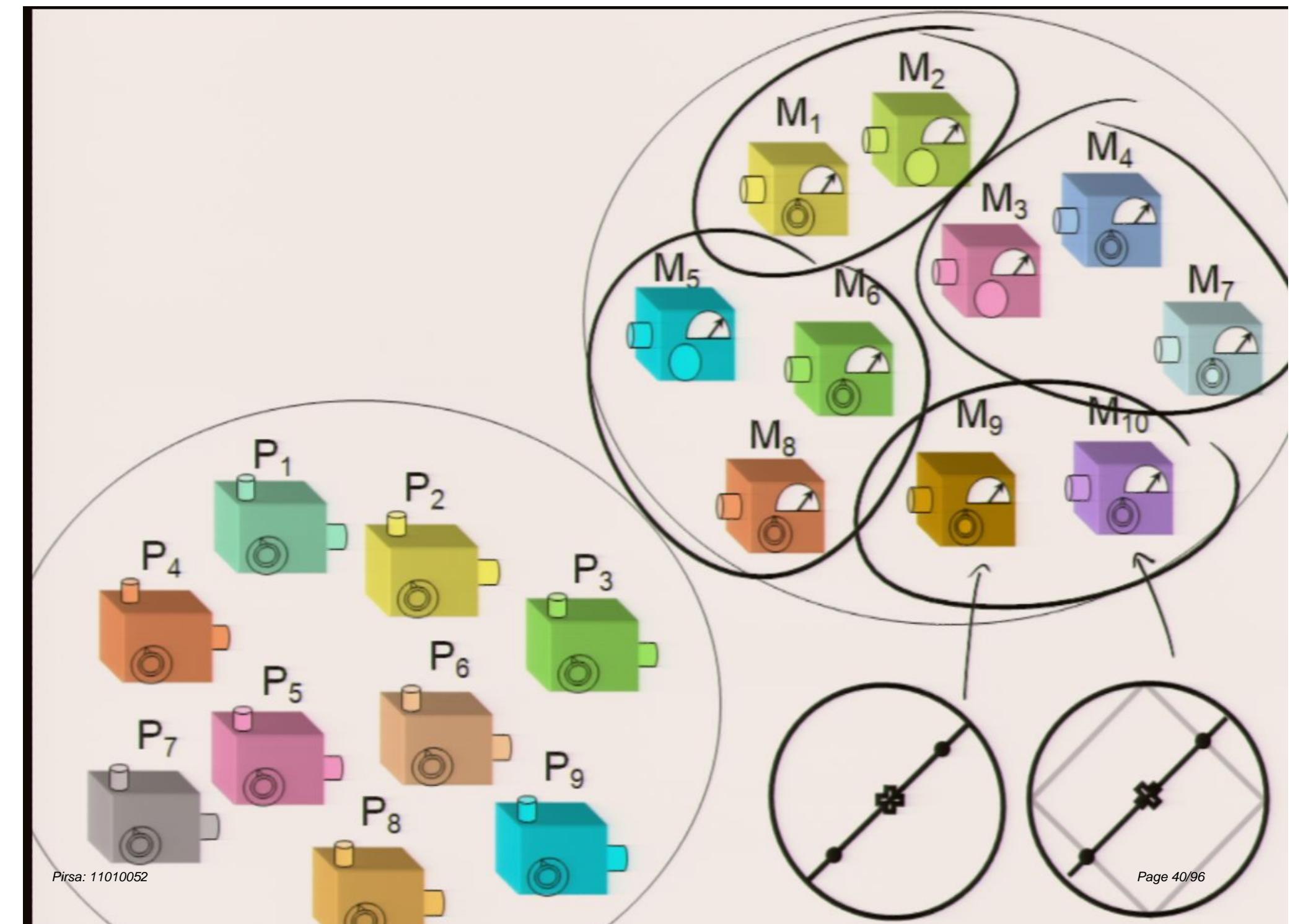


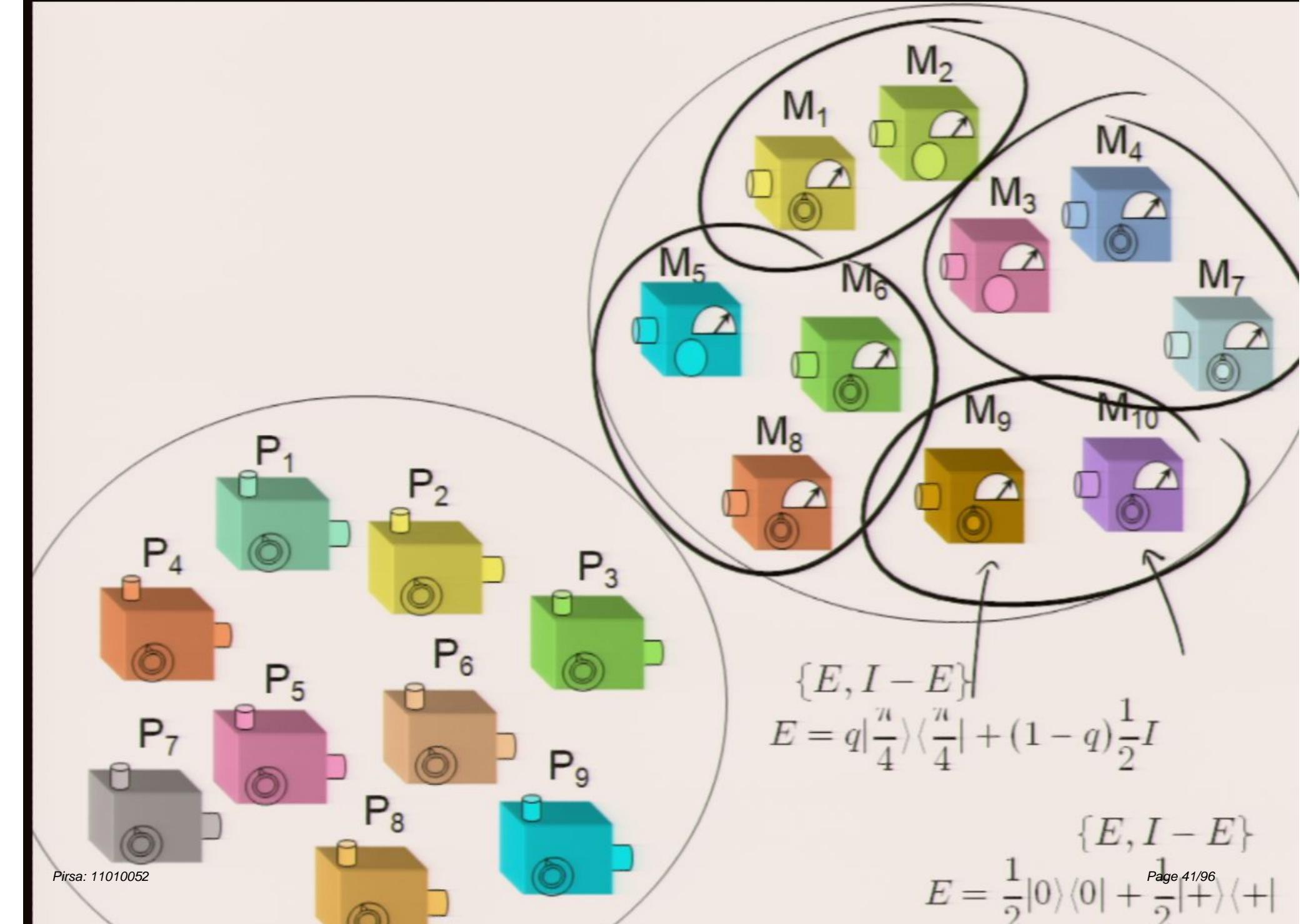


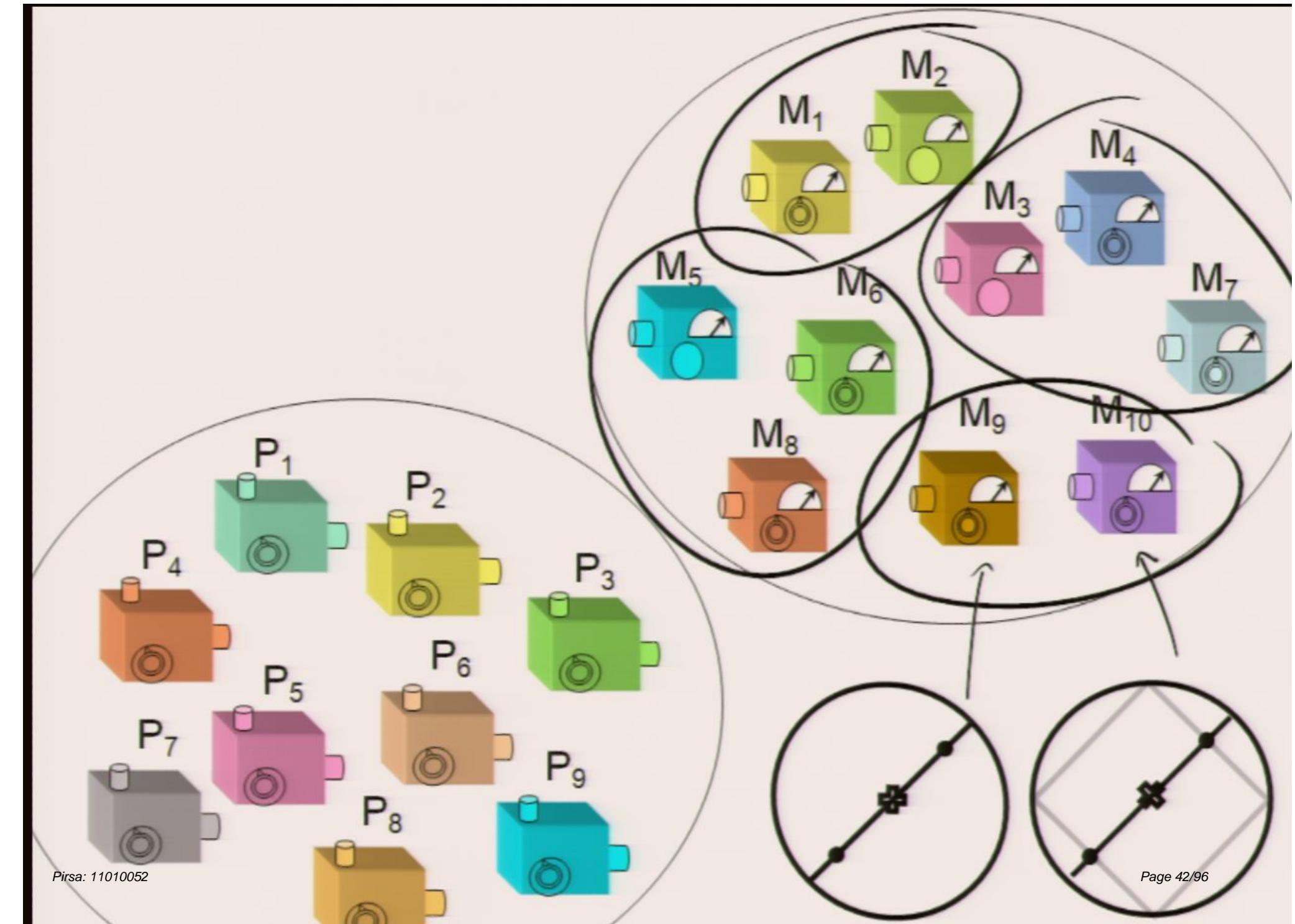


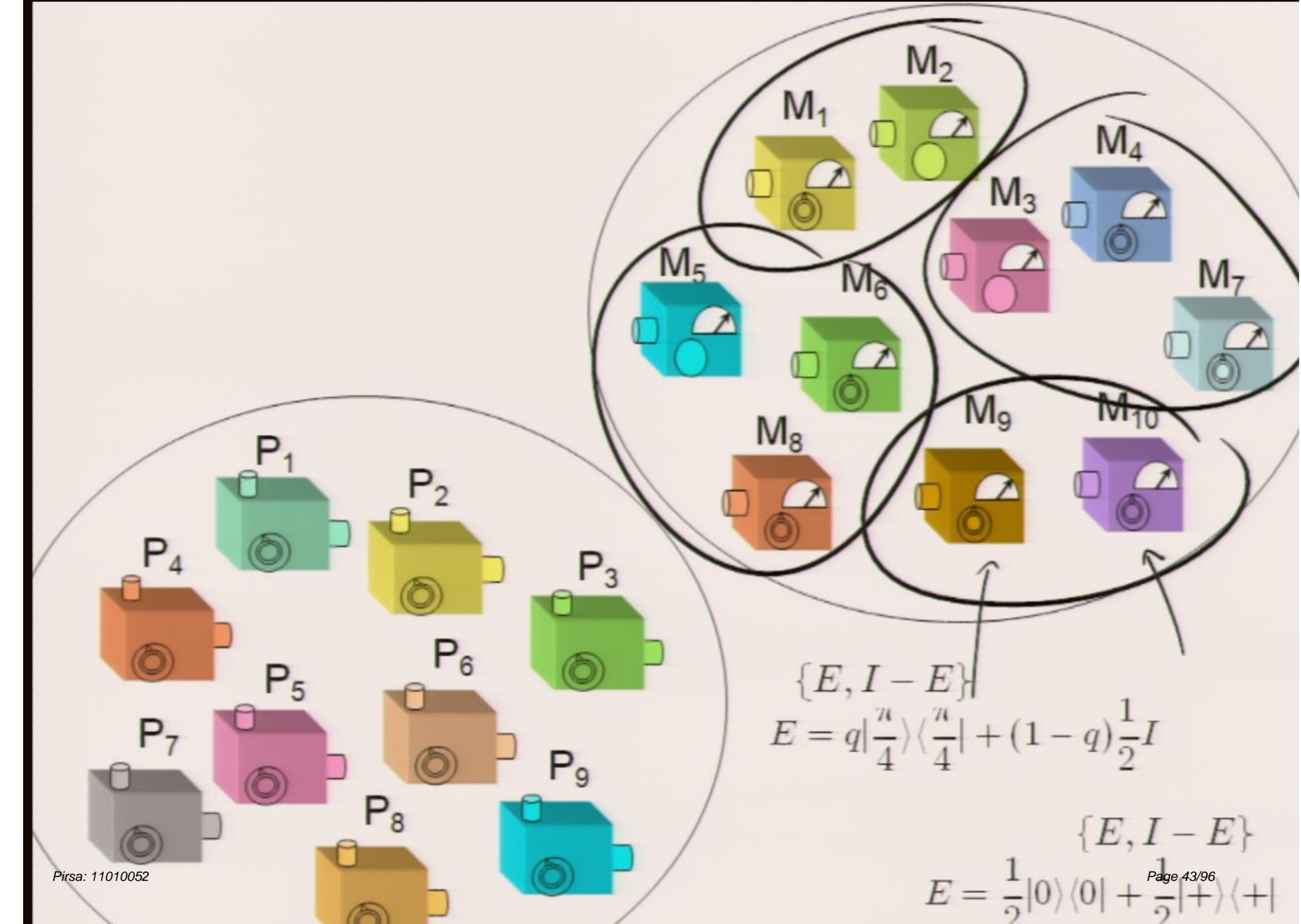


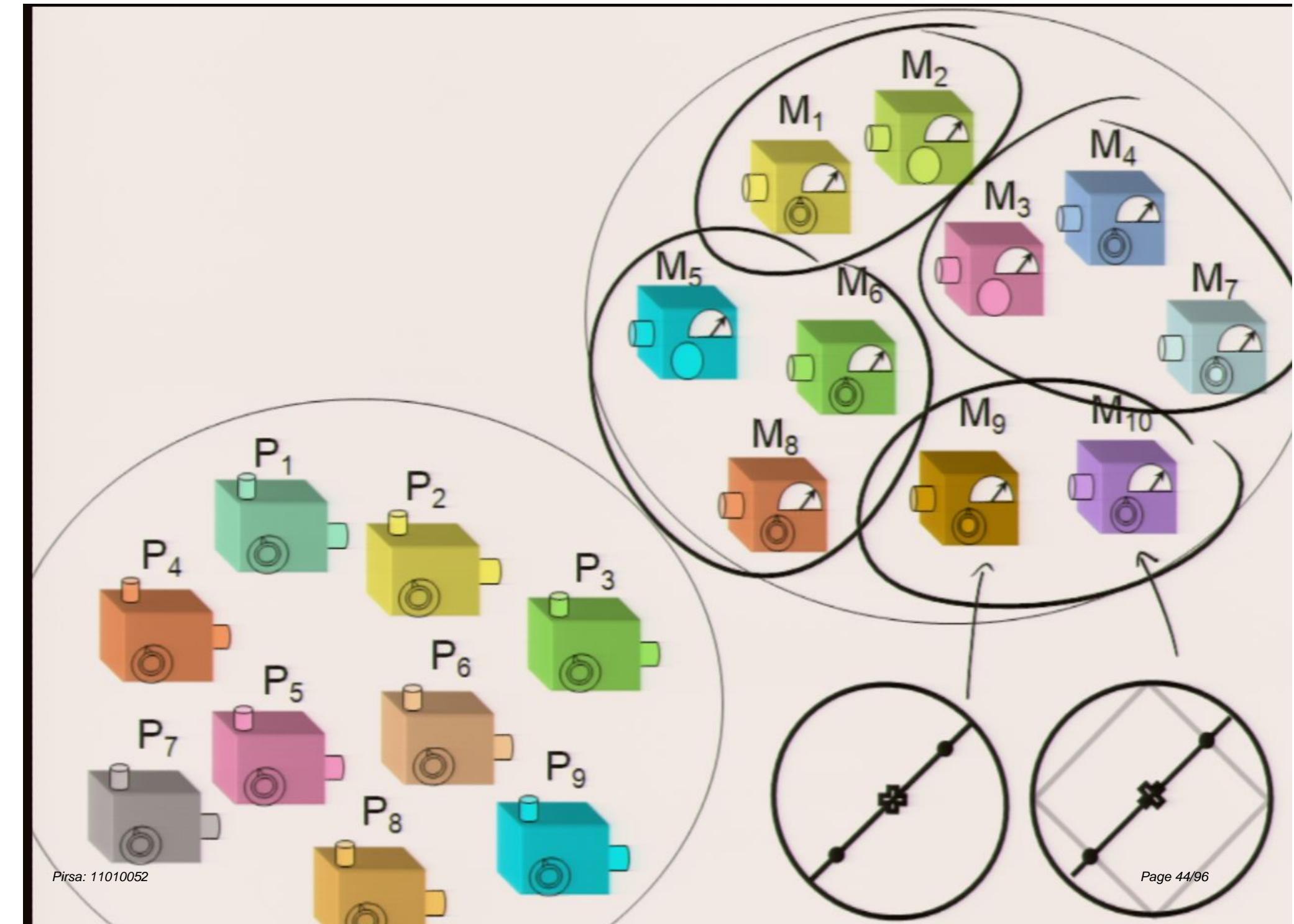


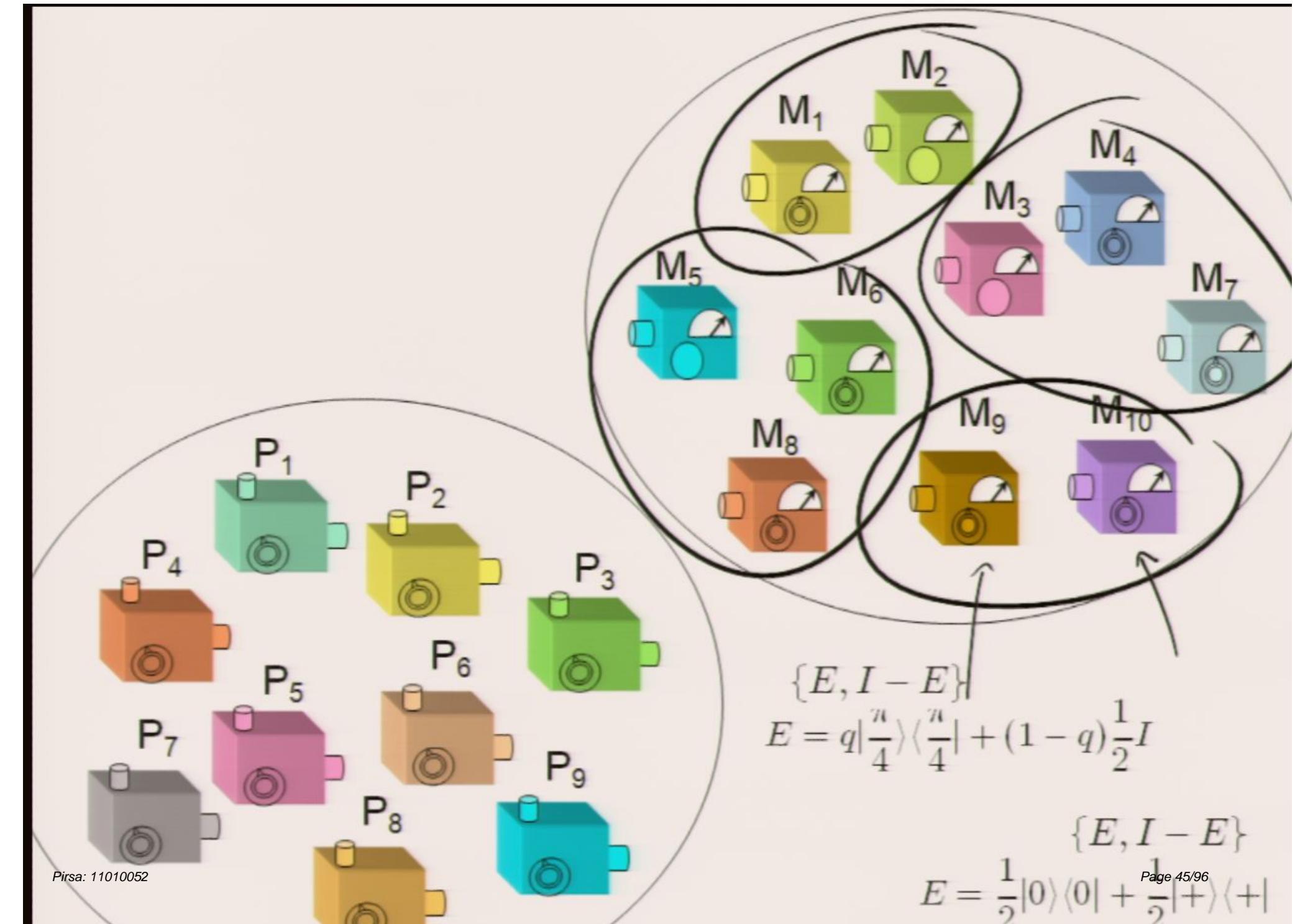


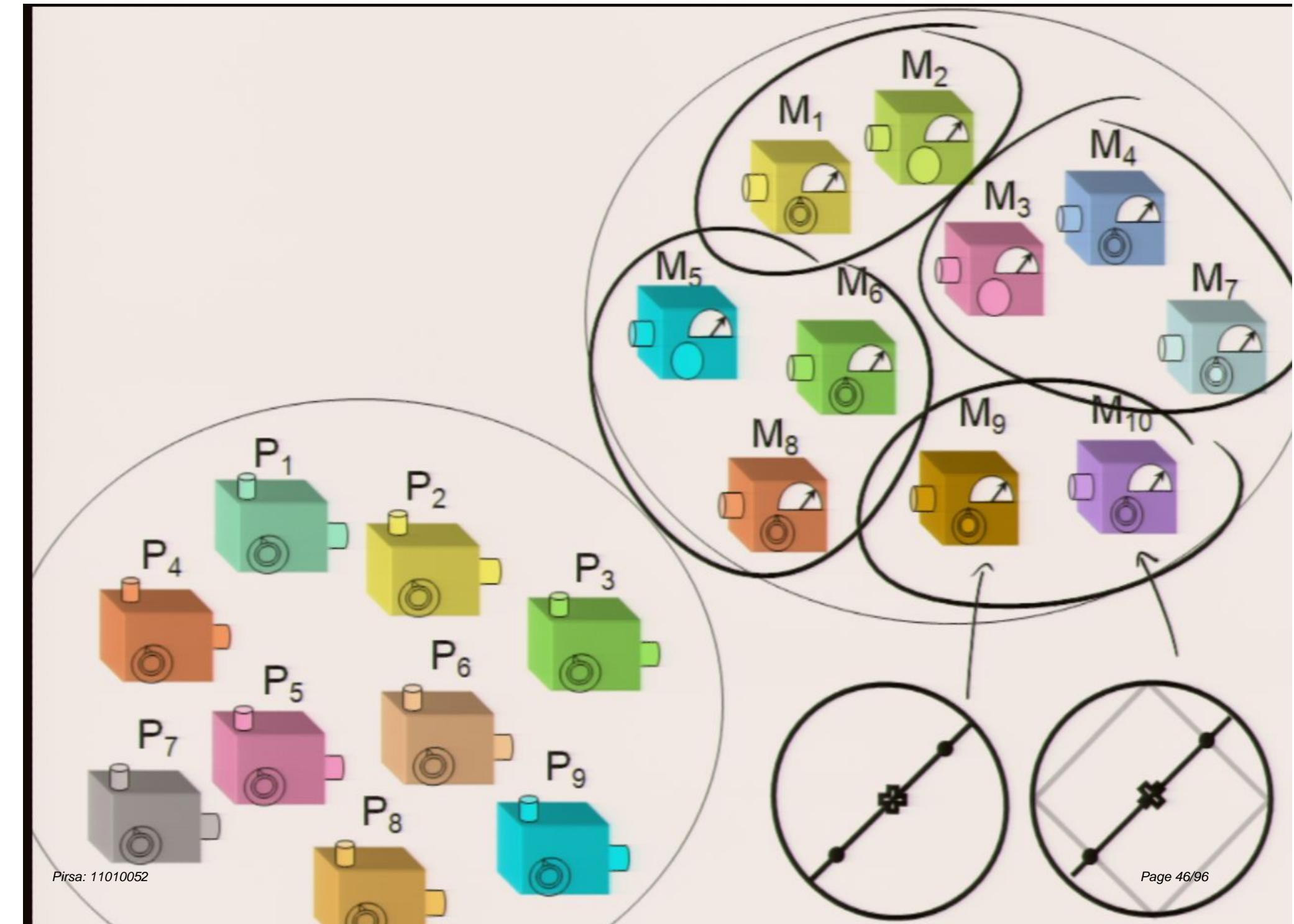


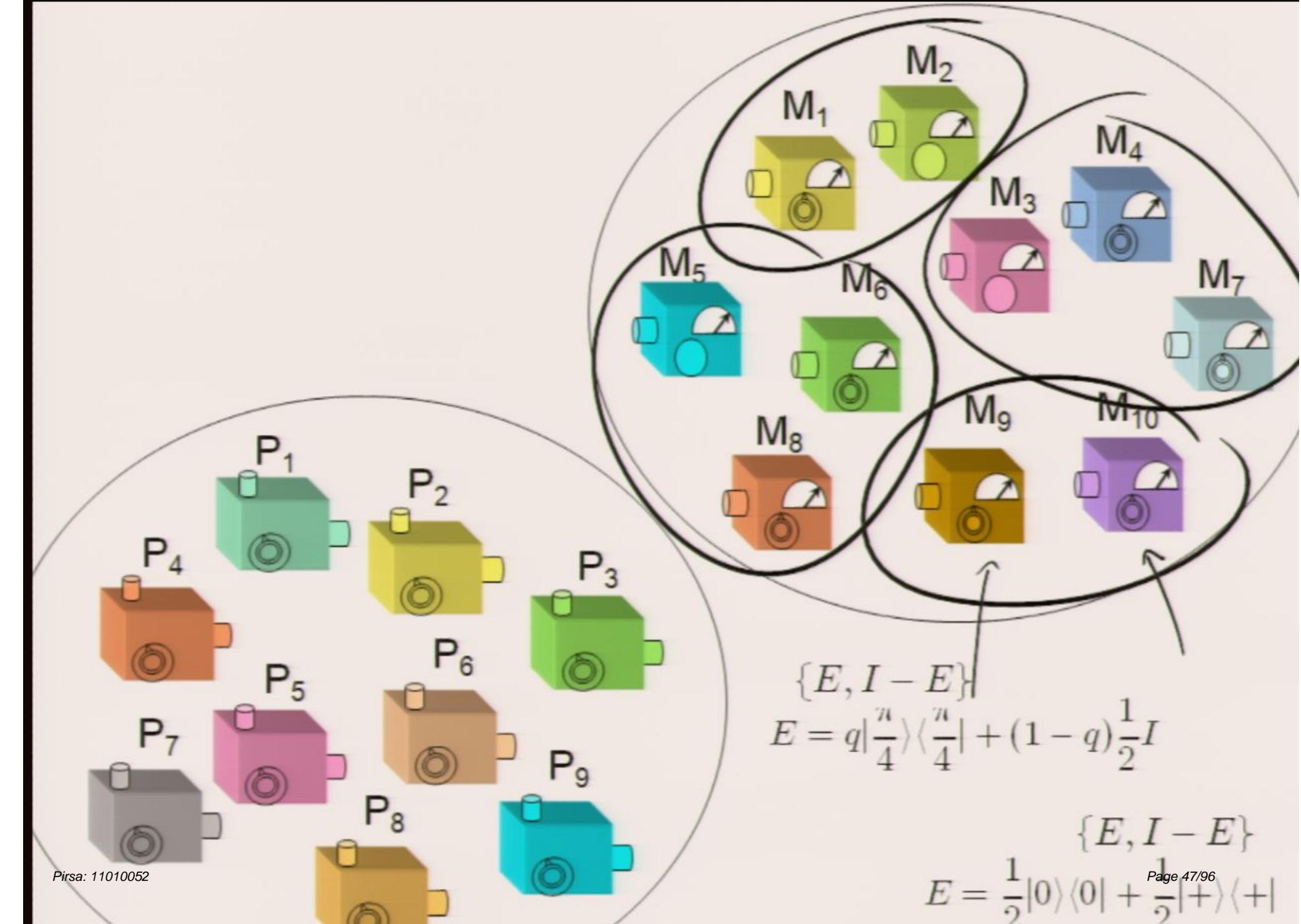


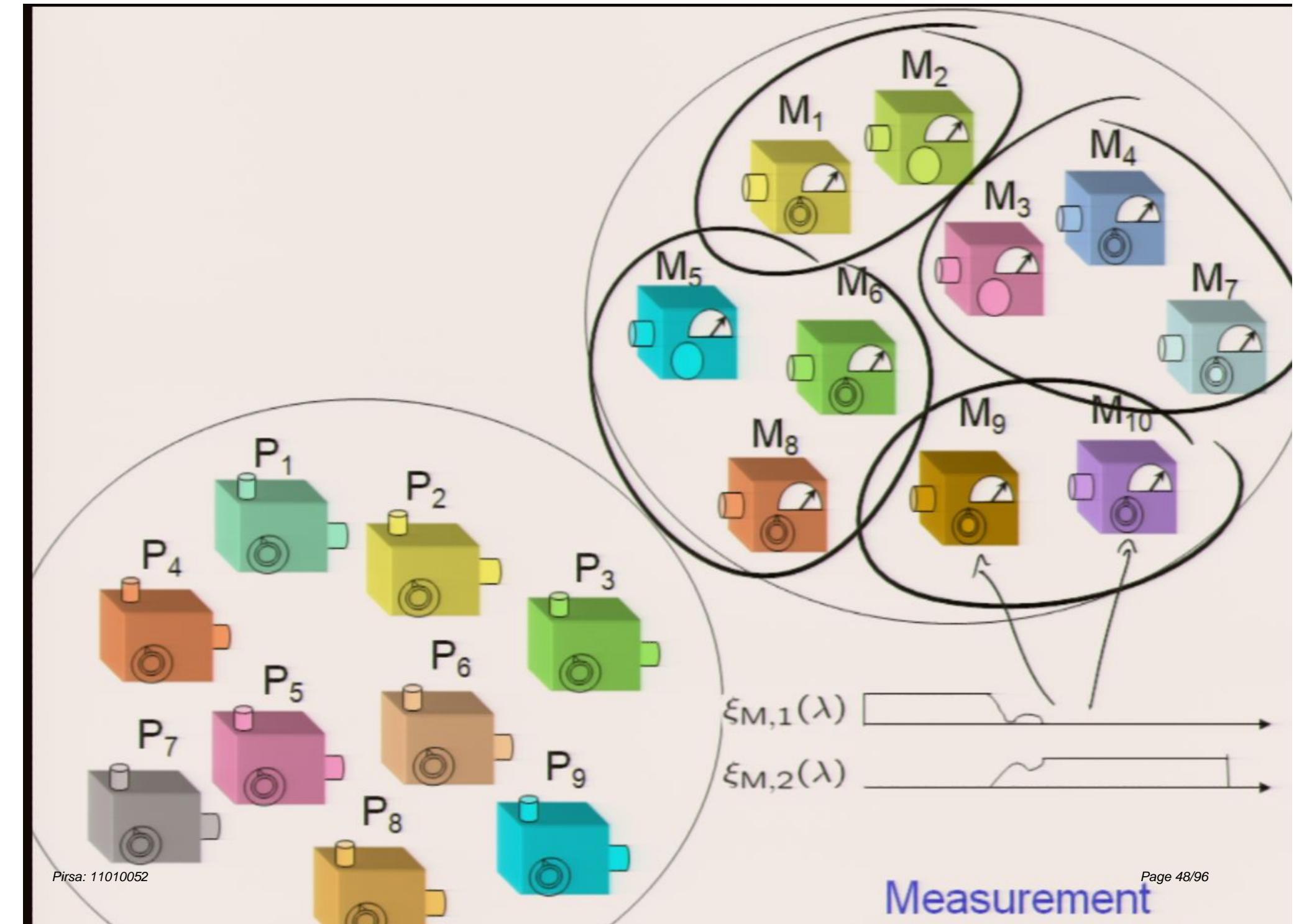


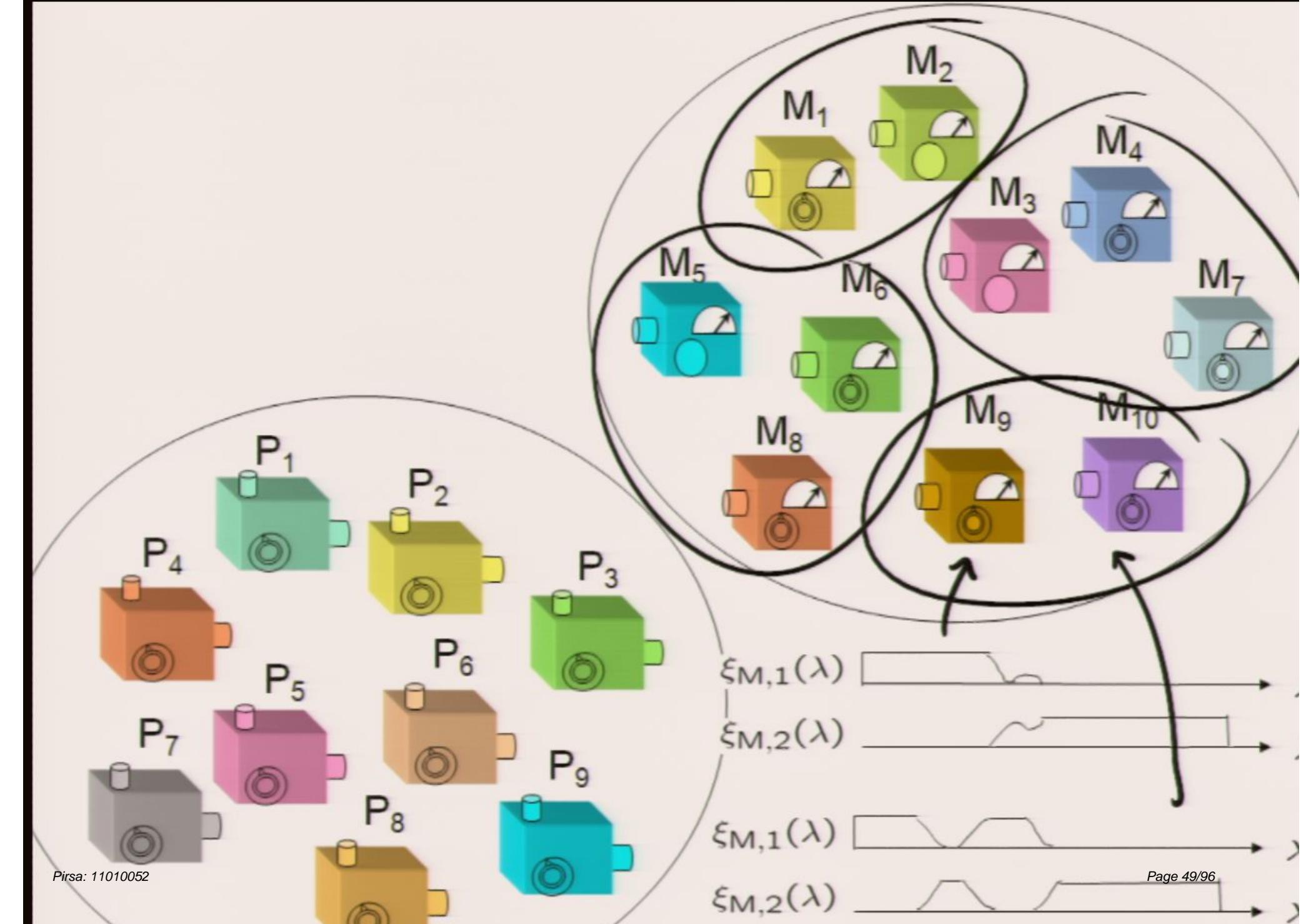












**universal noncontextuality**

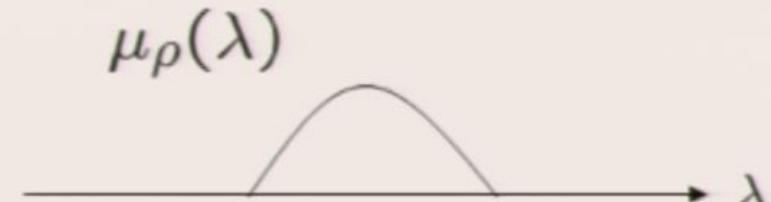
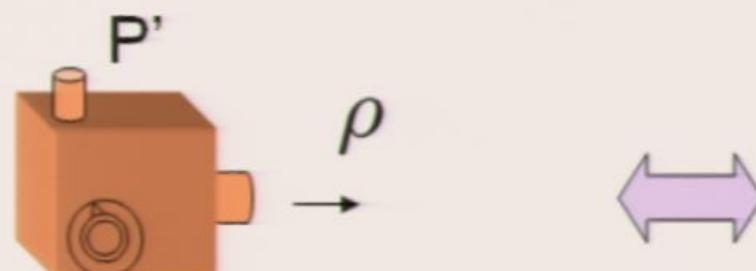
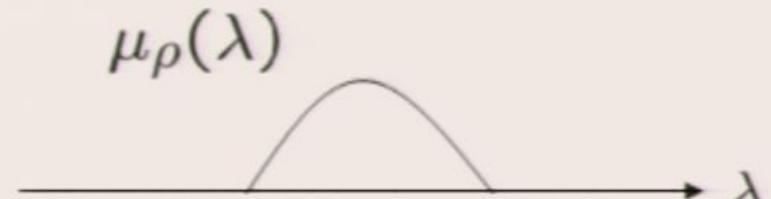
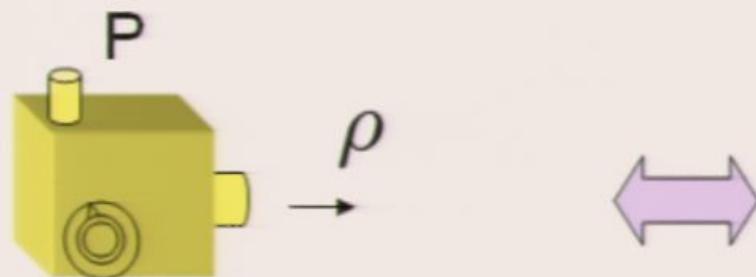
= noncontextuality for preparations *and* measurements

# Generalized noncontextuality in quantum theory

# Defining noncontextuality in quantum theory

## Preparation Noncontextuality in QT

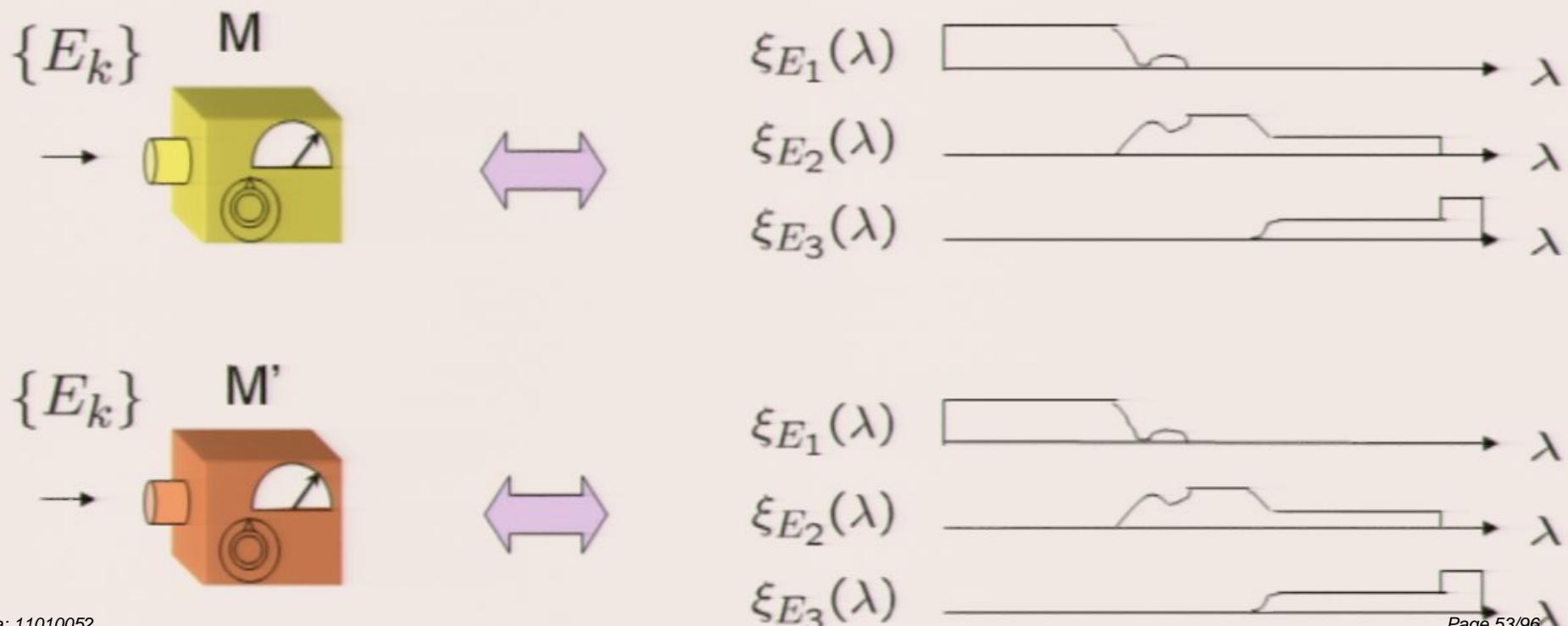
if  $P, P' \rightarrow \rho$  then  $\mu_P(\lambda) = \mu_{P'}(\lambda) = \mu_\rho(\lambda)$



# Defining noncontextuality in quantum theory

## Measurement Noncontextuality in QT

if  $M, M' \rightarrow \{E_k\}$  then  $\xi_{M,k}(\lambda) = \xi_{M',k}(\lambda) = \xi_{E_k}(\lambda)$



# Preparation-based proof of contextuality

(i.e. of the impossibility of a noncontextual  
realist model of quantum theory)

# Important features of realist models

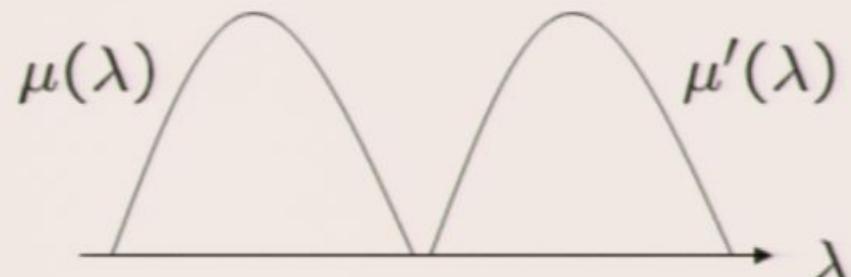
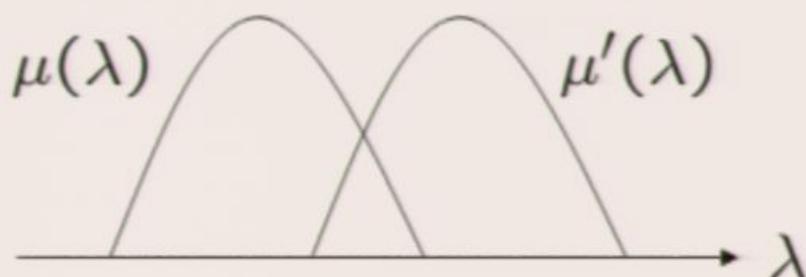
Let  $P \leftrightarrow \mu(\lambda)$

$P' \leftrightarrow \mu'(\lambda)$

Representing one-shot distinguishability:

If  $P$  and  $P'$  are distinguishable with certainty

then  $\mu(\lambda) \mu'(\lambda) = 0$



## Important features of realist models

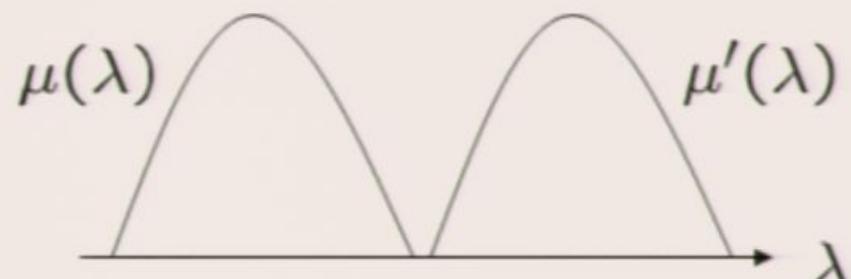
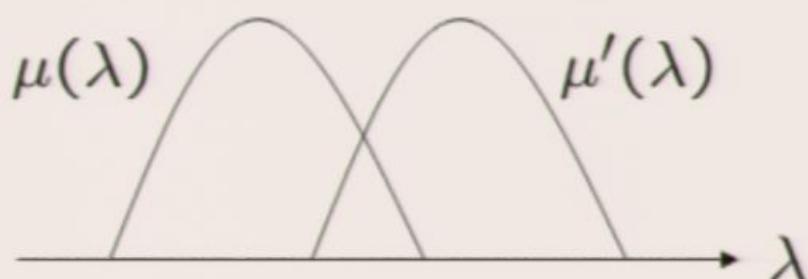
Let  $P \leftrightarrow \mu(\lambda)$

$P' \leftrightarrow \mu'(\lambda)$

Representing one-shot distinguishability:

If  $P$  and  $P'$  are distinguishable with certainty

then  $\mu(\lambda) \mu'(\lambda) = 0$



Representing convex combination:

If  $P'' = P$  with prob.  $p$  and  $P'$  with prob.  $1 - p$

Then  $\mu''(\lambda) = p \mu(\lambda) + (1 - p) \mu'(\lambda)$

# Proof based on finite construction in 2d

$$P_a \leftrightarrow \psi_a = (1, 0)$$

$$P_A \leftrightarrow \psi_A = (0, 1)$$

$$P_b \leftrightarrow \psi_b = (1/2, \sqrt{3}/2)$$

$$P_B \leftrightarrow \psi_B = (\sqrt{3}/2, -1/2)$$

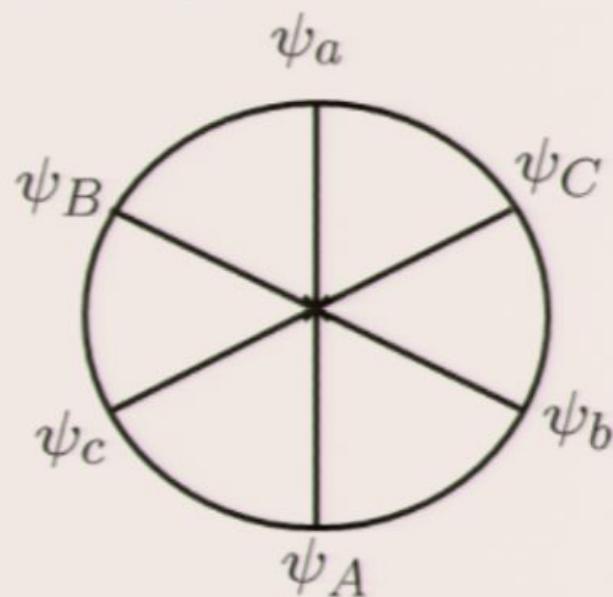
$$P_c \leftrightarrow \psi_c = (1/2, -\sqrt{3}/2)$$

$$P_C \leftrightarrow \psi_C = (\sqrt{3}/2, 1/2)$$

$$\psi_a \perp \psi_A$$

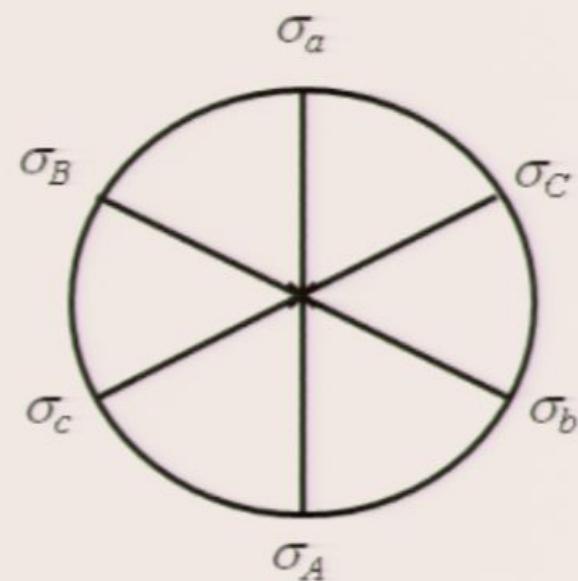
$$\psi_b \perp \psi_B$$

$$\psi_c \perp \psi_C$$



# Proof based on finite construction in 2d

$$\begin{array}{ll}
 P_a \leftrightarrow \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \sigma_a \sigma_A = 0 \\
 P_A \leftrightarrow \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \sigma_b \sigma_B = 0 \\
 P_b \leftrightarrow \sigma_b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} & \sigma_c \sigma_C = 0 \\
 P_B \leftrightarrow \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} & \\
 P_c \leftrightarrow \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} & \\
 P_C \leftrightarrow \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} &
 \end{array}$$



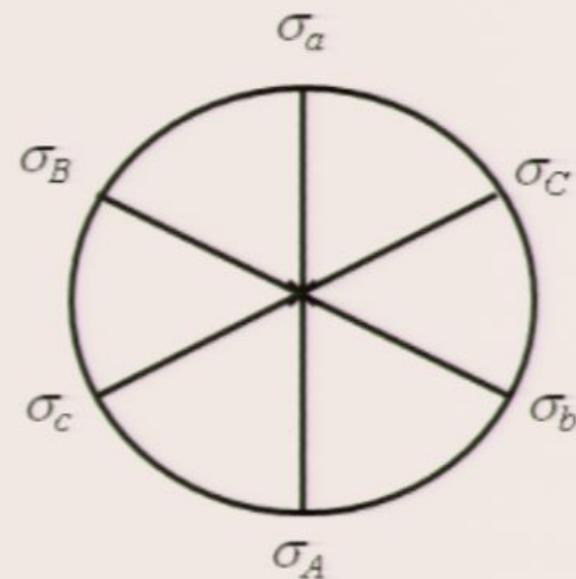
# Proof based on finite construction in 2d

$$\begin{array}{ll}
 P_a \leftrightarrow \sigma_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \sigma_a \sigma_A = 0 \\
 P_A \leftrightarrow \sigma_A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \sigma_b \sigma_B = 0 \\
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 P_B \leftrightarrow \sigma_B = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} & \\
 P_c \leftrightarrow \sigma_c = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4}\sqrt{3} \\ -\frac{1}{4}\sqrt{3} & \frac{3}{4} \end{pmatrix} & \\
 P_C \leftrightarrow \sigma_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}\sqrt{3} \\ \frac{1}{4}\sqrt{3} & \frac{1}{4} \end{pmatrix} &
 \end{array}$$

$P_a$  and  $P_A$  are distinguishable with certainty

$P_b$  and  $P_B$  are distinguishable with certainty

$P_c$  and  $P_C$  are distinguishable with certainty



$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\rightarrow \mu_b(\lambda) \mu_B(\lambda) = 0$$

$P_{aA} \equiv P_a$  and  $P_A$  with prob. 1/2 each

$P_{bB} \equiv P_b$  and  $P_B$  with prob. 1/2 each

$P_{cC} \equiv P_c$  and  $P_C$  with prob. 1/2 each

$P_{abc} \equiv P_a, P_b$  and  $P_c$  with prob. 1/3 each

$P_{ABC} \equiv P_A, P_B$  and  $P_C$  with prob. 1/3 each

$P_{aA} \equiv P_a$  and  $P_A$  with prob. 1/2 each

$P_{bB} \equiv P_b$  and  $P_B$  with prob. 1/2 each

$P_{cC} \equiv P_c$  and  $P_C$  with prob. 1/2 each

$P_{abc} \equiv P_a, P_b$  and  $P_c$  with prob. 1/3 each

$P_{ABC} \equiv P_A, P_B$  and  $P_C$  with prob. 1/3 each



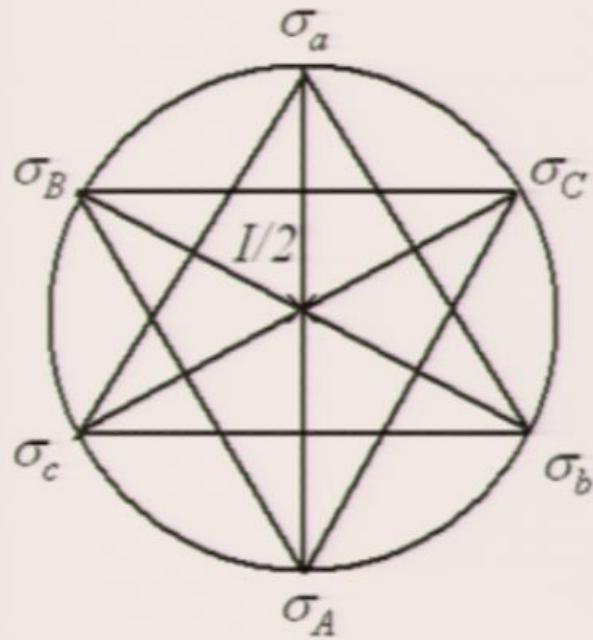
$$\mu_{aA}(\lambda) = \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)$$

$$\mu_{bB}(\lambda) = \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)$$

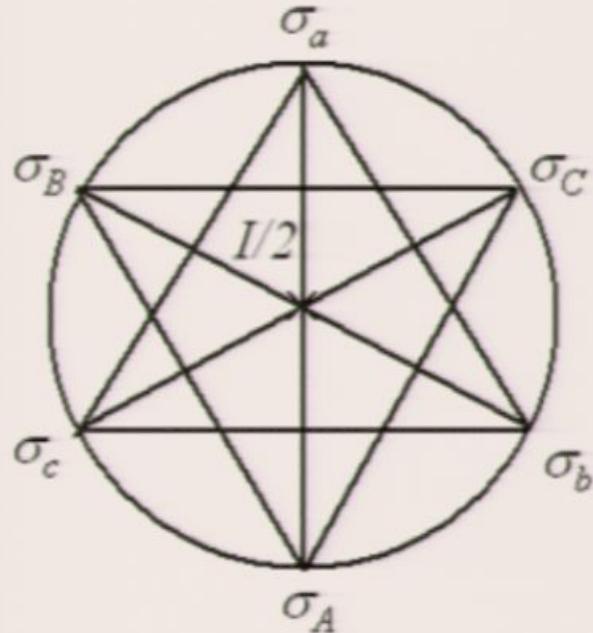
$$\mu_{cC}(\lambda) = \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)$$

$$\mu_{abc}(\lambda) = \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)$$

$$\mu_{ABC}(\lambda) = \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)$$



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

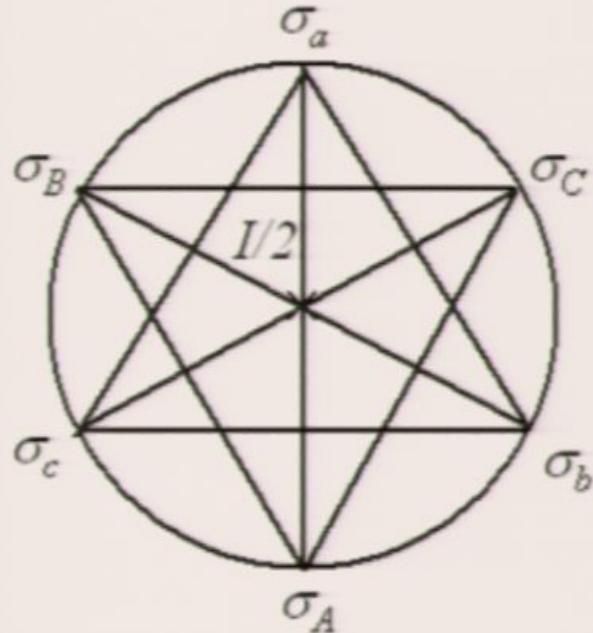


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 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

$$\begin{aligned}
 P_{aA} &\simeq P_{bB} \simeq P_{cC} \\
 &\simeq P_{abc} \simeq P_{ABC}
 \end{aligned}$$

By preparation noncontextuality

$$\begin{aligned}
 \mu_{aA}(\lambda) &= \mu_{bB}(\lambda) = \mu_{cC}(\lambda) \\
 &= \mu_{abc}(\lambda) = \mu_{ABC}(\lambda) \\
 &\equiv \nu(\lambda)
 \end{aligned}$$



$$\begin{aligned}
 I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\
 &= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\
 &= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\
 &= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\
 &= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.
 \end{aligned}$$

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$$\begin{aligned}
 \nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\
 &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\
 &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\
 &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\
 &= \frac{1}{3}\nu(\lambda) + \frac{1}{3}\nu(\lambda) + \frac{1}{3}\nu(\lambda)
 \end{aligned}$$

Our task is to find

$$\mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda),$$

$$\mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda),$$

and  $\nu(\lambda)$  such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).\end{aligned}$$

Our task is to find

$\mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda),$   
 $\mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda),$   
 and  $\nu(\lambda)$  such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\nu(\lambda) = \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda)$$

$$= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda)$$

$$= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)$$

$$= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)$$

$$= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda).$$

i.e., paralleling the quantum structure:

$$\sigma_a \sigma_A = 0$$

$$\sigma_b \sigma_B = 0$$

$$\sigma_c \sigma_C = 0$$

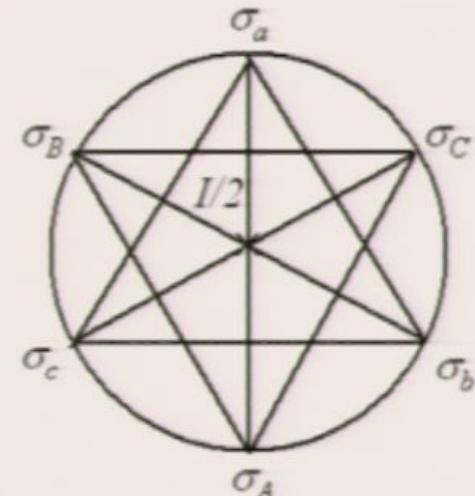
$$I/2 = \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A$$

$$= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B$$

$$= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C$$

$$= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c$$

$$= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.$$



Our task is to find

$$\mu_a(\lambda), \mu_A(\lambda), \mu_b(\lambda),$$

$$\mu_B(\lambda), \mu_c(\lambda), \mu_C(\lambda),$$

and  $\nu(\lambda)$  such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

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$$= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda)$$

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$$= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)$$

Our task is to find  
 $\mu_a(\lambda)$ ,  $\mu_A(\lambda)$ ,  $\mu_b(\lambda)$ ,  
 $\mu_B(\lambda)$ ,  $\mu_c(\lambda)$ ,  $\mu_C(\lambda)$ ,  
and  $\nu(\lambda)$  such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

Consider  $\lambda'$  such that  $\nu(\lambda') \neq 0$

From decompositions (1)-(3)

$$\mu_a(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\mu_b(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\mu_c(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\ &= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\ &= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\ &= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\ &= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

Our task is to find  
 $\mu_a(\lambda)$ ,  $\mu_A(\lambda)$ ,  $\mu_b(\lambda)$ ,  
 $\mu_B(\lambda)$ ,  $\mu_c(\lambda)$ ,  $\mu_C(\lambda)$ ,  
and  $\nu(\lambda)$  such that

$$\mu_a(\lambda) \mu_A(\lambda) = 0$$

$$\mu_b(\lambda) \mu_B(\lambda) = 0$$

$$\mu_c(\lambda) \mu_C(\lambda) = 0$$

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda) \\&= \frac{1}{3}\mu_A(\lambda) + \frac{1}{3}\mu_B(\lambda) + \frac{1}{3}\mu_C(\lambda)\end{aligned}$$

Consider  $\lambda'$  such that  $\nu(\lambda') \neq 0$

From decompositions (1)-(3)

$$\mu_a(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

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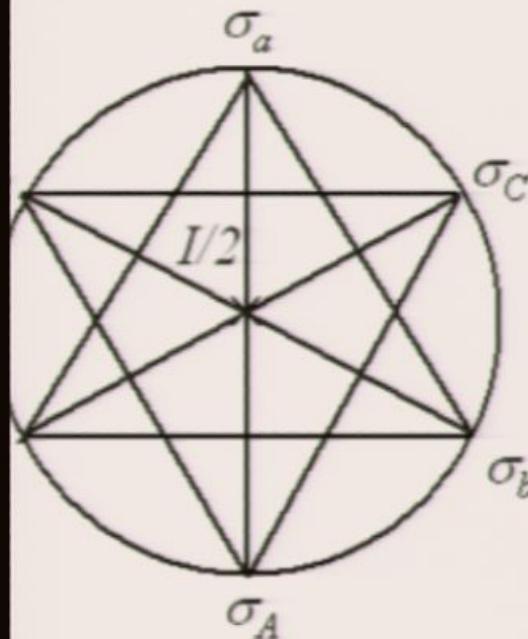
$$\mu_c(\lambda') = 0 \text{ or } 2\nu(\lambda')$$

But then the RHS of decomposition (4) is

$$0, \frac{2}{3}\nu(\lambda'), \frac{4}{3}\nu(\lambda'), 2\nu(\lambda') \neq \nu(\lambda')$$

**CONTRADICTION**

## Aside: justifying preparation noncontextuality by local causality



$$\begin{aligned}I/2 &= \frac{1}{2}\sigma_a + \frac{1}{2}\sigma_A \\&= \frac{1}{2}\sigma_b + \frac{1}{2}\sigma_B \\&= \frac{1}{2}\sigma_c + \frac{1}{2}\sigma_C \\&= \frac{1}{3}\sigma_a + \frac{1}{3}\sigma_b + \frac{1}{3}\sigma_c \\&= \frac{1}{3}\sigma_A + \frac{1}{3}\sigma_B + \frac{1}{3}\sigma_C.\end{aligned}$$

By preparation noncontextuality

$$\begin{aligned}\nu(\lambda) &= \frac{1}{2}\mu_a(\lambda) + \frac{1}{2}\mu_A(\lambda) \\&= \frac{1}{2}\mu_b(\lambda) + \frac{1}{2}\mu_B(\lambda) \\&= \frac{1}{2}\mu_c(\lambda) + \frac{1}{2}\mu_C(\lambda) \\&= \frac{1}{3}\mu_a(\lambda) + \frac{1}{3}\mu_b(\lambda) + \frac{1}{3}\mu_c(\lambda)\end{aligned}$$

PNC for  $I/2$  can be justified by local causality

But PNC for  $\sigma_x$  cannot be justified by local causality

Also,

Any bipartite Bell-type  
proof of nonlocality



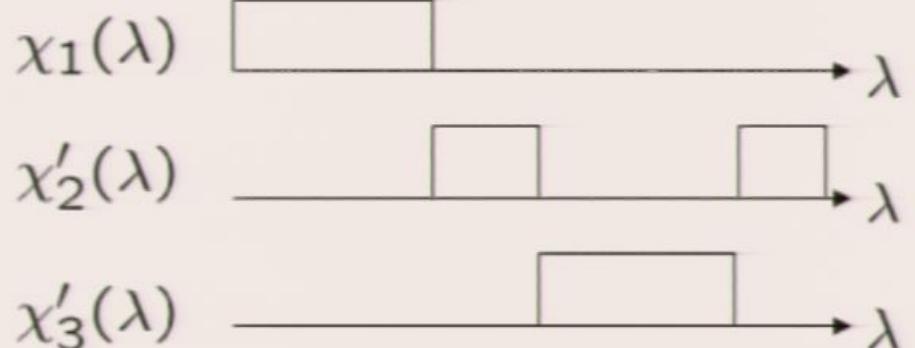
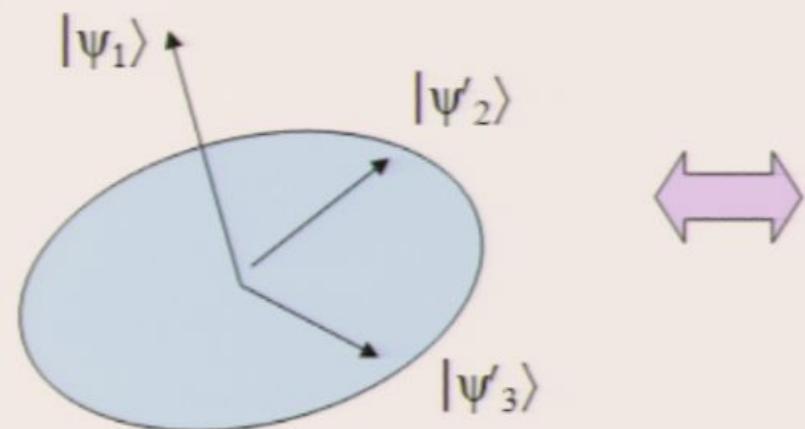
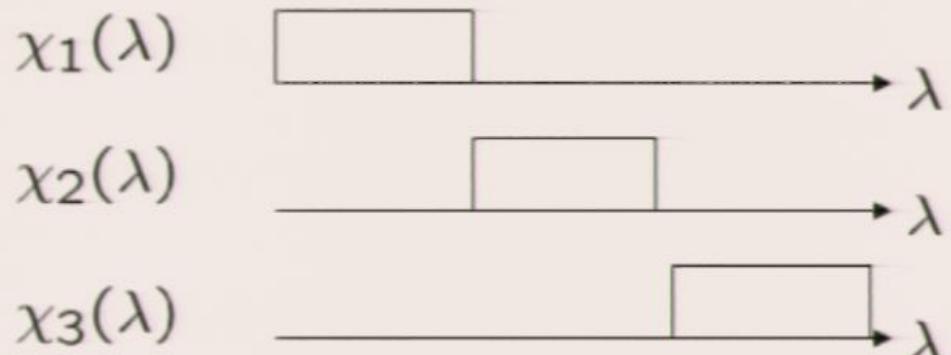
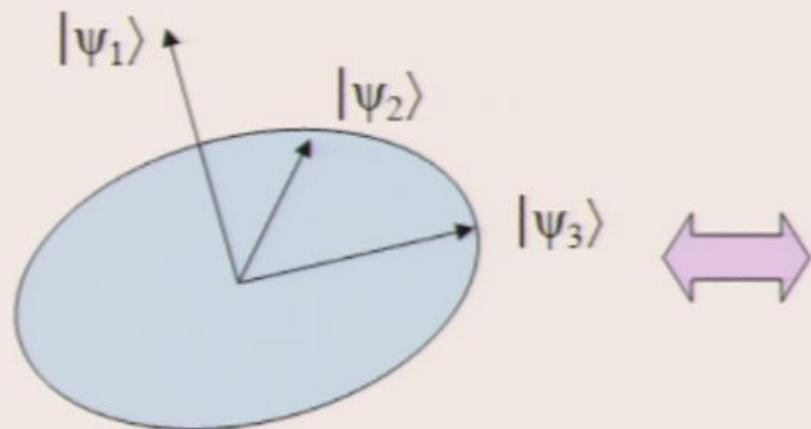
proof of preparation  
contextuality

(proof due to Jon Barrett)

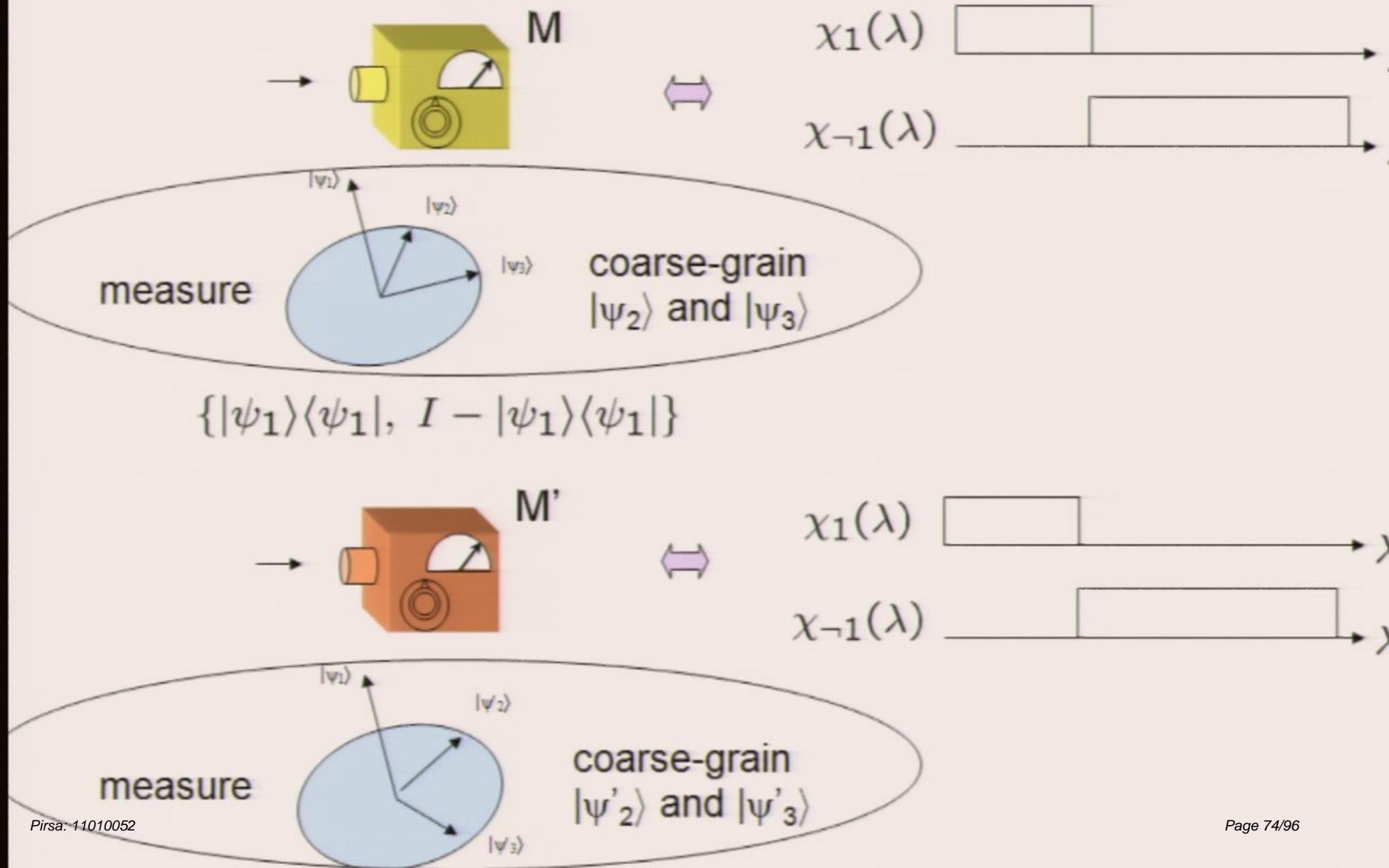
# Measurement contextuality

New definition versus traditional definition

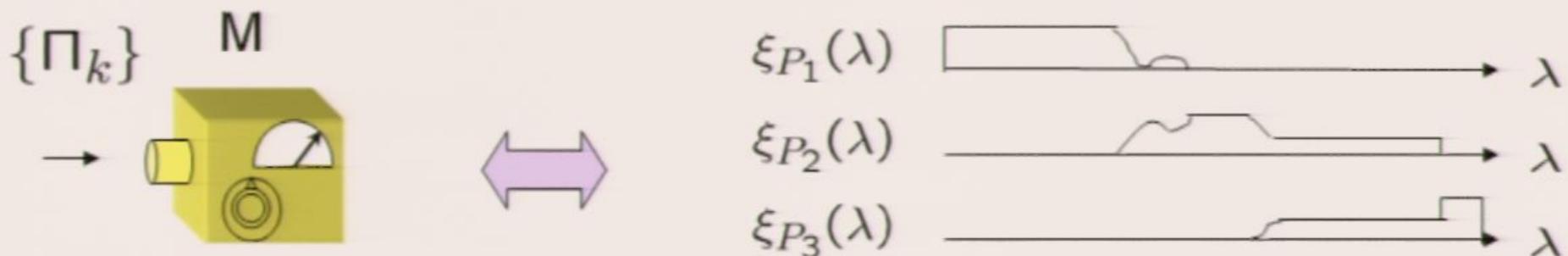
Recall: the traditional notion of noncontextuality:



This is equivalent to assuming:



But recall that the most general representation was



Therefore:

traditional notion of  
noncontextuality      =      revised notion of  
noncontextuality for projective  
measurements  
  
and  
outcome determinism for  
projective measurements

So, the new definition of noncontextuality is not simply a generalization of the traditional notion

For sharp measurements, it is a revision of the traditional notion

### Local determinism:

We ask: Does **the outcome** depend on space-like separated events  
(in addition to local settings and  $\lambda$ )?

### Local causality:

We ask: Does **the probability of the outcome** depend on space-like separated events (in addition to local settings and  $\lambda$ )?

---

### Local determinism:

We ask: Does **the outcome** depend on space-like separated events  
(in addition to local settings and  $\lambda$ )?

### Local causality:

We ask: Does **the probability of the outcome** depend on space-like  
separated events (in addition to local settings and  $\lambda$ )?

---

### Traditional notion of measurement noncontextuality:

We ask: Does **the outcome** depend on the measurement context  
(in addition to the observable and  $\lambda$ )?

### The revised notion of measurement noncontextuality:

We ask: Does **the probability of the outcome** depend on the  
measurement context (in addition to the observable and  $\lambda$ )?

### Local determinism:

We ask: Does **the outcome** depend on space-like separated events  
(in addition to local settings and  $\lambda$ )?

### Local causality:

We ask: Does **the probability of the outcome** depend on space-like separated events (in addition to local settings and  $\lambda$ )?

---

### Traditional notion of measurement noncontextuality:

We ask: Does **the outcome** depend on the measurement context  
(in addition to the observable and  $\lambda$ )?

### The revised notion of measurement noncontextuality:

We ask: Does **the probability of the outcome** depend on the measurement context (in addition to the observable and  $\lambda$ )?

traditional notion of noncontextuality = revised notion of noncontextuality for projective measurements  
and  
outcome determinism for projective measurements

No-go theorems for previous notion are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up outcome determinism

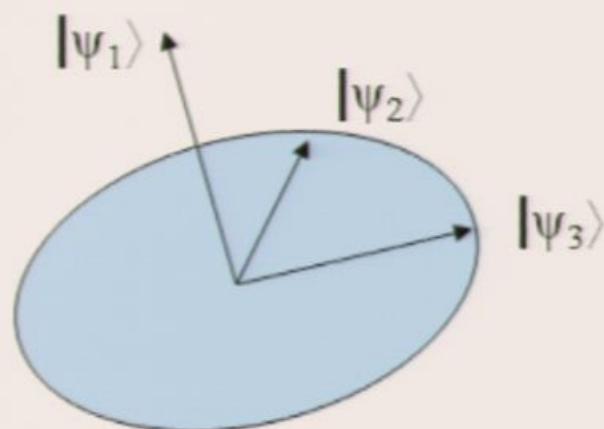
However, one can prove that

preparation  
noncontextuality  outcome determinism for  
projective measurements

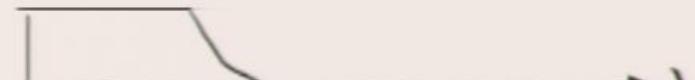
However, one can prove that

preparation  
noncontextuality  $\longrightarrow$  outcome determinism for  
projective measurements

Proof



$$\chi_{\psi_1}(\lambda)$$



$$\chi_{\psi_2}(\lambda)$$



$$\chi_{\psi_3}(\lambda)$$



We've established that

preparation  
noncontextuality → outcome determinism for  
sharp measurements

Therefore:

measurement  
noncontextuality → measurement  
noncontextuality  
and  
preparation  
noncontextuality → and  
outcome determinism for  
sharp measurements

We've established that

preparation  
noncontextuality → outcome determinism for  
sharp measurements

Therefore:

measurement  
noncontextuality → Traditional notion of  
noncontextuality  
and  
preparation  
noncontextuality

We've established that

preparation  
noncontextuality → outcome determinism for  
sharp measurements

Therefore:

measurement  
noncontextuality → Traditional notion of  
noncontextuality  
and  
preparation  
noncontextuality

no-go theorems for the traditional notion of noncontextuality can  
be salvaged as no-go theorems for the generalized notion

# Measurement-based proof of contextuality

(i.e. of the impossibility of a noncontextual  
realist model of quantum theory)

# Proof of contextuality for unsharp measurements in 2d

$$M_a \leftrightarrow \{\Pi_a, \Pi_A\}$$

$$M_b \leftrightarrow \{\Pi_b, \Pi_B\}$$

$$M_c \leftrightarrow \{\Pi_c, \Pi_C\}$$

$\Pi_x$  projects onto  $\psi_x$

$$\Pi_a + \Pi_A = I$$

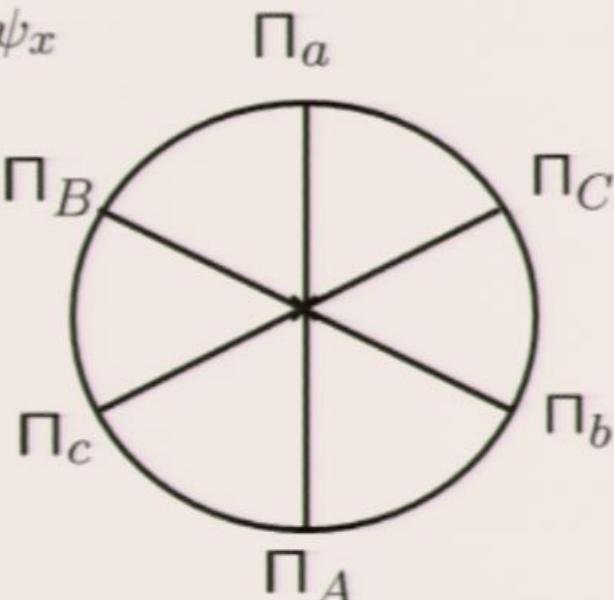
$$\Pi_b + \Pi_B = I$$

$$\Pi_c + \Pi_C = I$$

$$\Pi_a \Pi_A = 0$$

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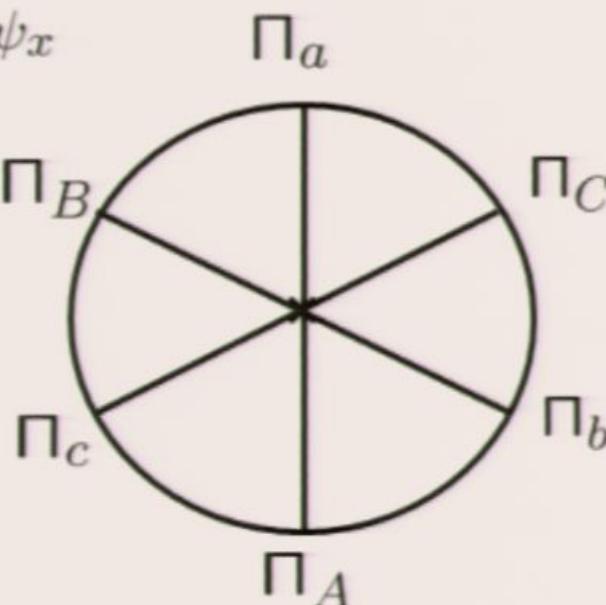
$$M_c \leftrightarrow \{\chi_c(\lambda), \chi_C(\lambda)\}$$

By definition

$$\chi_a(\lambda) + \chi_A(\lambda) = 1$$

$$\chi_b(\lambda) + \chi_B(\lambda) = 1$$

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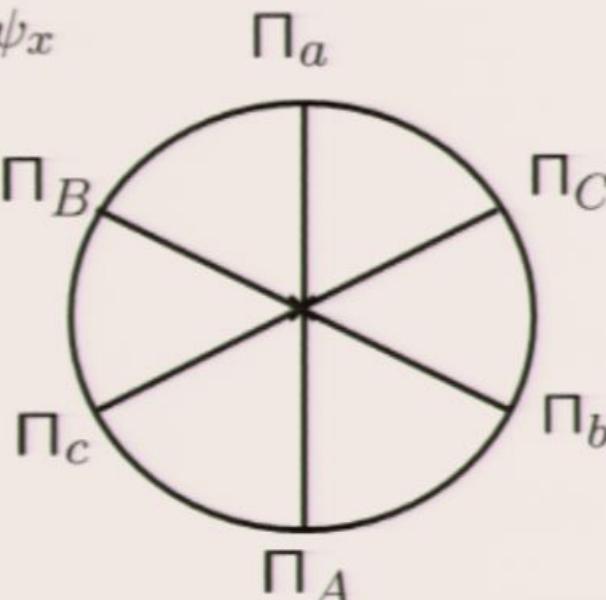
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By definition

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By outcome determinism for sharp measurements

$$\chi_a(\lambda)\chi_A(\lambda) = 0$$

$$\chi_b(\lambda)\chi_B(\lambda) = 0$$

$$\chi_c(\lambda)\chi_C(\lambda) = 0$$

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By the assumption of **measurement noncontextuality**

$$M \simeq \tilde{M} \longrightarrow \left\{ \frac{1}{3}\chi_a + \frac{1}{3}\chi_b + \frac{1}{3}\chi_c, \frac{1}{3}\chi_A + \frac{1}{3}\chi_B + \frac{1}{3}\chi_C \right\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\}$$

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$$\text{But } \{0, 1\}, \{\frac{1}{3}, \frac{2}{3}\}, \{1, 0\}, \{\frac{2}{3}, \frac{1}{3}\} \neq \left\{ \frac{1}{2}, \frac{1}{2} \right\}$$

# The mystery of contextuality

There is a tension between

- 1) the dependence of representation on certain details of the experimental procedure
- and
- 2) the independence of outcome statistics on those details of the experimental procedure