

Title: Foundations of Quantum Mechanics - Lecture 4

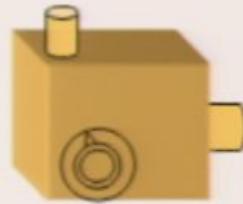
Date: Jan 06, 2011 11:30 AM

URL: <http://pirsa.org/11010041>

Abstract:

The most general types of transformations

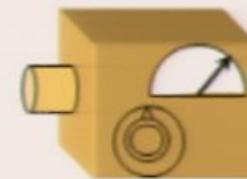
Operational Quantum Mechanics



Preparation
P



Transformation
T



Measurement
M

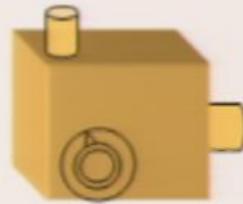
Vector
 $|\psi\rangle$

Unitary map
 U

Projection valued
measure (PVM)
 $\{\Pi_k\}$

$$Pr(k|P, T, M) = \langle \psi | U^\dagger \Pi_k U | \psi \rangle$$

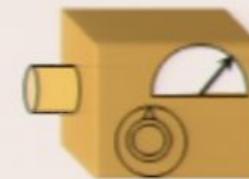
Operational Quantum Mechanics



Preparation
P



Transformation
T



Measurement
M

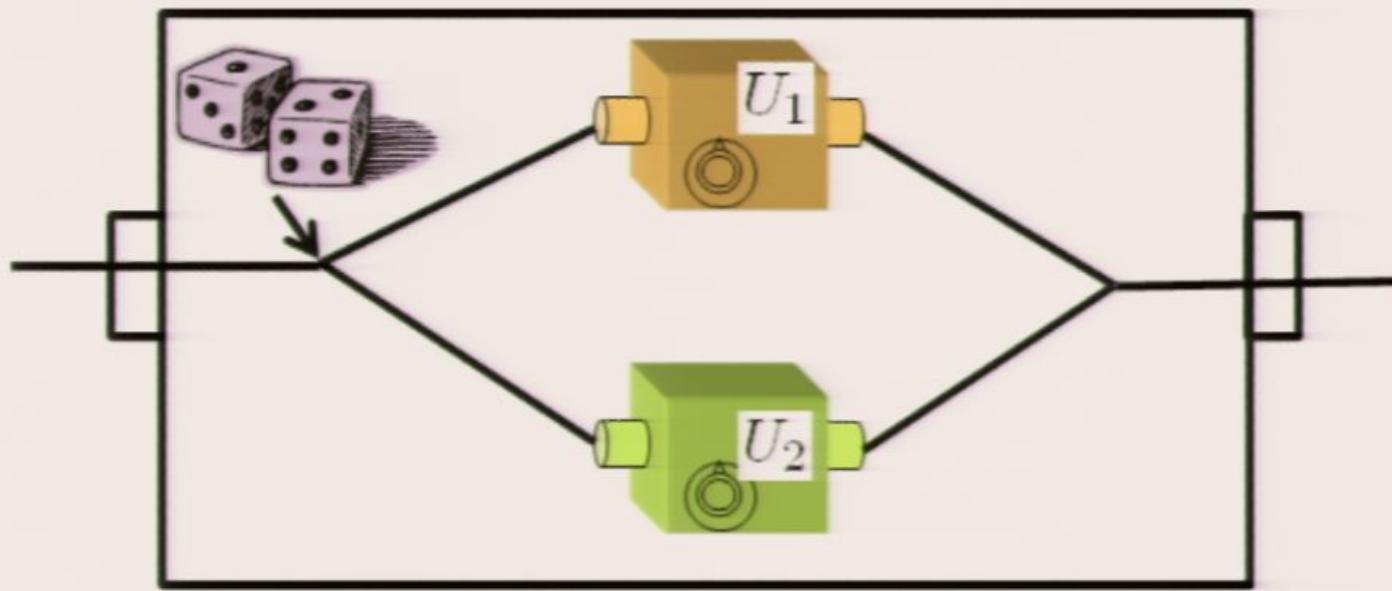
Density operator
 ρ

Unitary map
 U

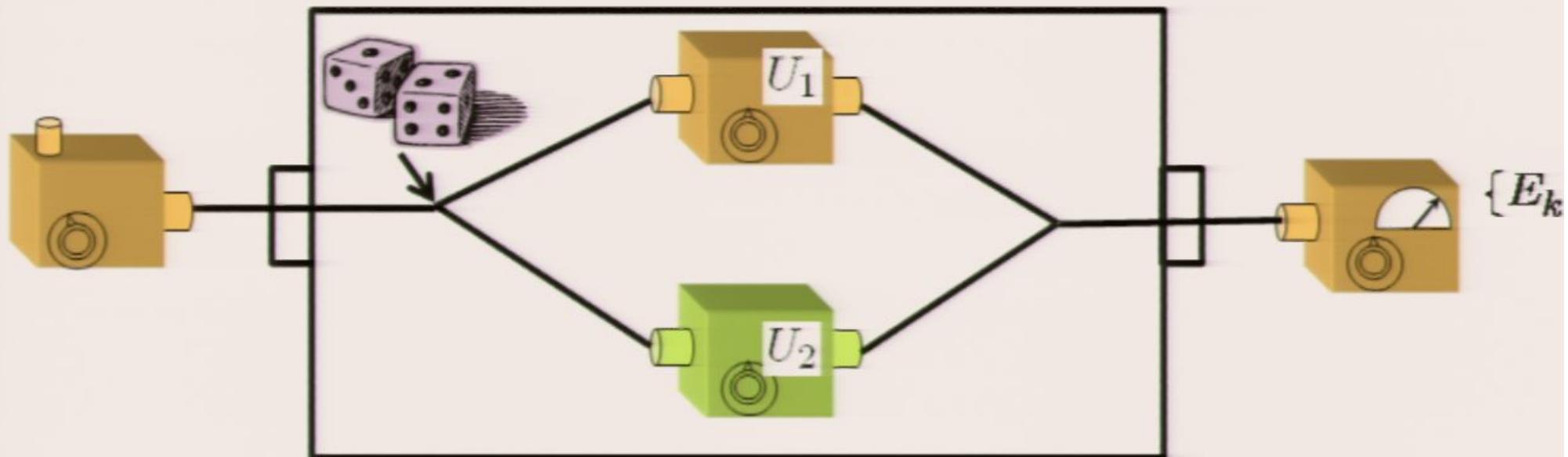
Position operator valued
measure (POVM)
 $\{E_k\}$

$$Pr(k|P, M) = \text{Tr}(U\rho U^\dagger E_k)$$

Mixture of unitaries

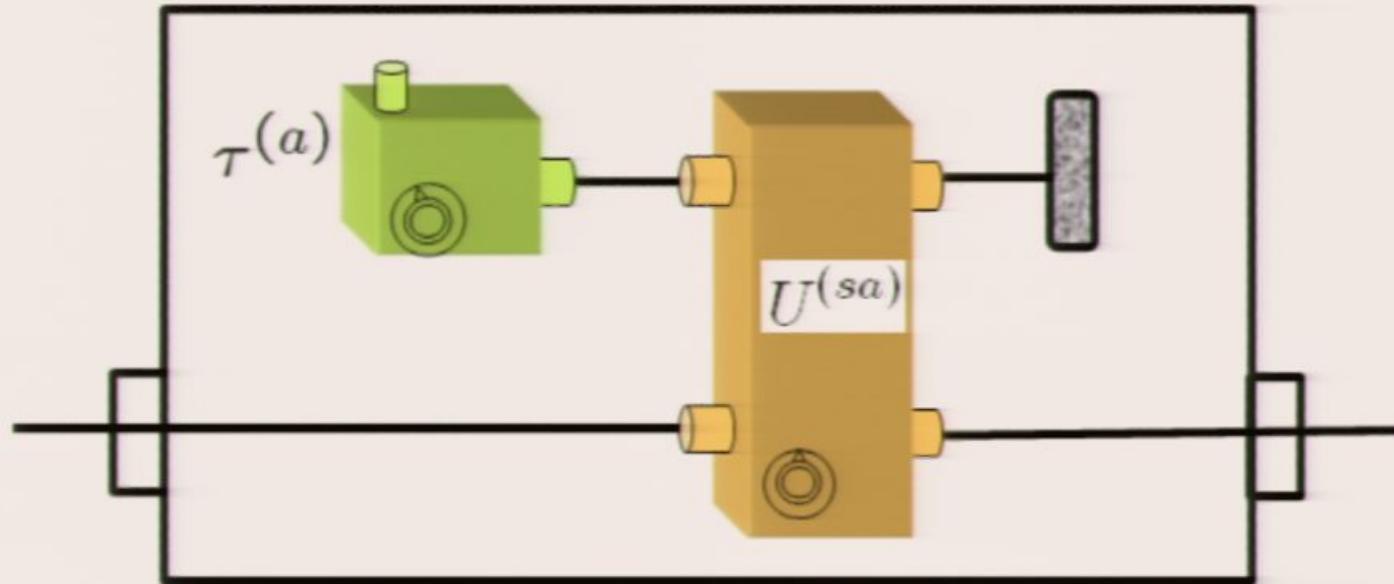


Mixture of unitaries

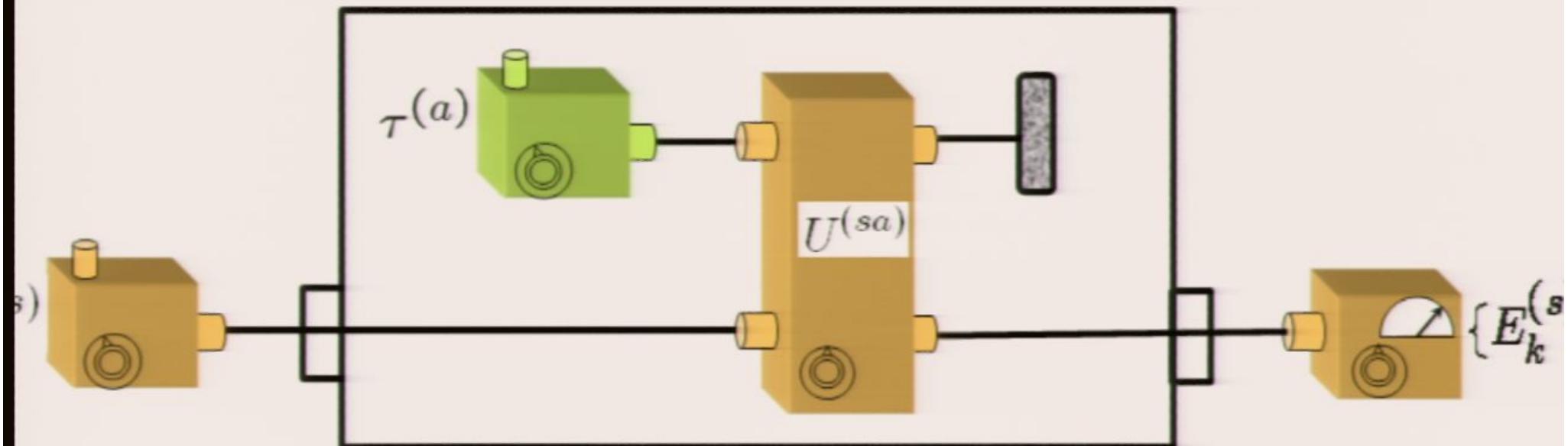


$$\begin{aligned} p(k) &= \sum_i p(k|i)p(i) \\ &= \sum_i \text{Tr}(U_i \rho U_i^\dagger E_k) p_i \\ &= \text{Tr}(\sum_i p_i U_i \rho U_i^\dagger E_k) \end{aligned}$$

Interaction with another system

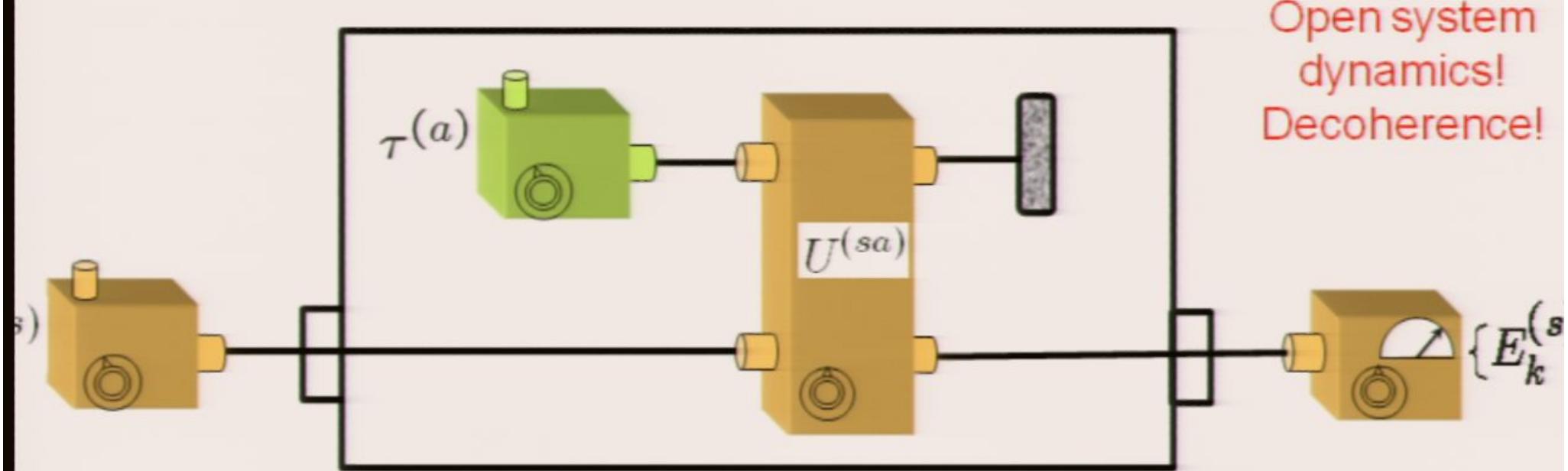


Interaction with another system



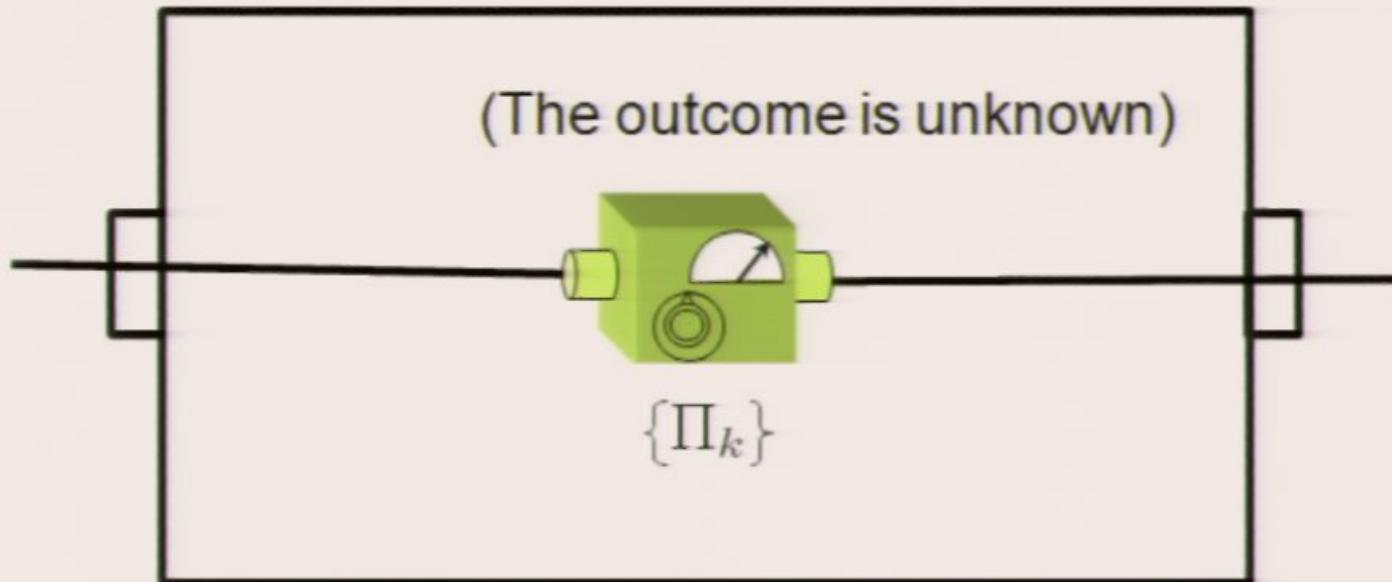
$$\begin{aligned} p(k) &= \text{Tr}_{sa} \left[(E_k^{(s)} \otimes I^{(a)}) U^{(sa)} (\rho^{(s)} \otimes \tau^{(a)}) U^{(sa)\dagger} \right] \\ &= \text{Tr}_s \left[E_k^{(s)} \text{Tr}_a \left(U^{(sa)} (\rho^{(s)} \otimes \tau^{(a)}) U^{(sa)\dagger} \right) \right] \end{aligned}$$

Interaction with another system



$$\begin{aligned} p(k) &= \text{Tr}_{sa} \left[(E_k^{(s)} \otimes I^{(a)}) U^{(sa)} (\rho^{(s)} \otimes \tau^{(a)}) U^{(sa)\dagger} \right] \\ &= \text{Tr}_s \left[E_k^{(s)} \text{Tr}_a \left(U^{(sa)} (\rho^{(s)} \otimes \tau^{(a)}) U^{(sa)\dagger} \right) \right] \\ &= \text{Tr}_s \left[E_k^{(s)} \mathcal{T}(\rho^{(s)}) \right] \end{aligned}$$

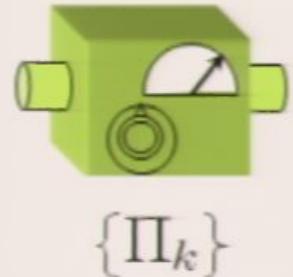
A nonselective measurement





$\{\Pi_k\}$

$$|\psi\rangle \rightarrow \frac{\Pi_k |\psi\rangle}{\sqrt{\langle\psi|\Pi_k|\psi\rangle}} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$



$$|\psi\rangle \rightarrow \frac{\Pi_k |\psi\rangle}{\sqrt{\langle\psi|\Pi_k|\psi\rangle}} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$

$$|\psi\rangle\langle\psi| \rightarrow \frac{\Pi_k |\psi\rangle\langle\psi|\Pi_k}{\langle\psi|\Pi_k|\psi\rangle} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$



$\{\Pi_k\}$

$$|\psi\rangle \rightarrow \frac{\Pi_k |\psi\rangle}{\sqrt{\langle\psi|\Pi_k|\psi\rangle}} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$

$$|\psi\rangle\langle\psi| \rightarrow \frac{\Pi_k |\psi\rangle\langle\psi|\Pi_k}{\langle\psi|\Pi_k|\psi\rangle} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$

$$\rho = \sum_i p(i) |\psi_i\rangle\langle\psi_i|$$

$$\rho \rightarrow \sum_i p(i|k) \frac{\Pi_k |\psi_i\rangle\langle\psi_i|\Pi_k}{\langle\psi_i|\Pi_k|\psi_i\rangle} \text{ with probability } p(k) = \text{Tr}(\rho\Pi_k)$$

but $p(i|k) = \frac{p(k|i)p(i)}{p(k)}$

 $\{\Pi_k\}$

$$|\psi\rangle \rightarrow \frac{\Pi_k |\psi\rangle}{\sqrt{\langle\psi|\Pi_k|\psi\rangle}} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$

$$|\psi\rangle\langle\psi| \rightarrow \frac{\Pi_k |\psi\rangle\langle\psi|\Pi_k}{\langle\psi|\Pi_k|\psi\rangle} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$

$$\rho = \sum_i p(i) |\psi_i\rangle\langle\psi_i|$$

$$\rho \rightarrow \sum_i p(i|k) \frac{\Pi_k |\psi_i\rangle\langle\psi_i|\Pi_k}{\langle\psi_i|\Pi_k|\psi_i\rangle} \text{ with probability } p(k) = \text{Tr}(\rho\Pi_k)$$

$$\begin{aligned} \text{but } p(i|k) &= \frac{p(k|i)p(i)}{p(k)} \\ &= \frac{\langle\psi_i|\Pi_k|\psi_i\rangle p(i)}{p(k)} \end{aligned}$$

 $\{\Pi_k\}$

$$|\psi\rangle \rightarrow \frac{\Pi_k |\psi\rangle}{\sqrt{\langle\psi|\Pi_k|\psi\rangle}} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$

$$|\psi\rangle\langle\psi| \rightarrow \frac{\Pi_k |\psi\rangle\langle\psi|\Pi_k}{\langle\psi|\Pi_k|\psi\rangle} \text{ with probability } p(k) = \langle\psi|\Pi_k|\psi\rangle$$

$$\rho = \sum_i p(i) |\psi_i\rangle\langle\psi_i|$$

$$\rho \rightarrow \sum_i p(i|k) \frac{\Pi_k |\psi_i\rangle\langle\psi_i|\Pi_k}{\langle\psi_i|\Pi_k|\psi_i\rangle} \text{ with probability } p(k) = \text{Tr}(\rho\Pi_k)$$

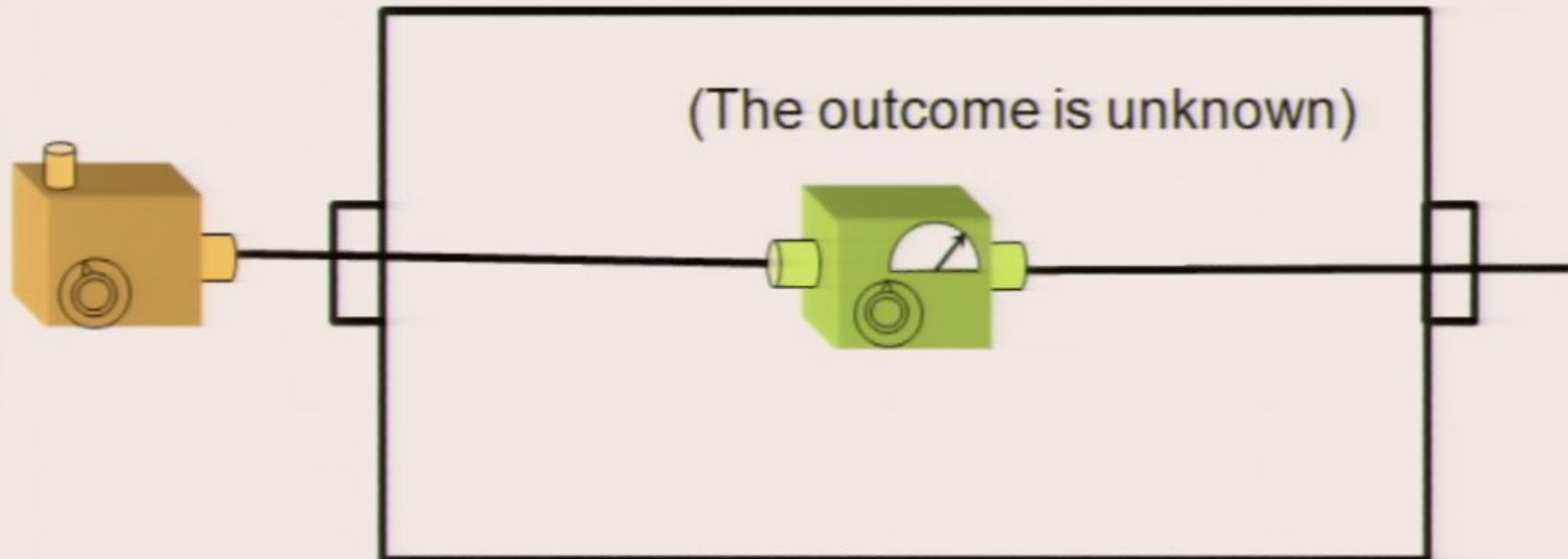
$$\begin{aligned} \text{but } p(i|k) &= \frac{p(k|i)p(i)}{p(k)} \\ &= \frac{\langle\psi_i|\Pi_k|\psi_i\rangle p(i)}{p(k)} \end{aligned}$$

$$\rho \rightarrow \sum_i \frac{\langle\psi_i|\Pi_k|\psi_i\rangle p(i)}{p(k)} \frac{\Pi_k |\psi_i\rangle\langle\psi_i|\Pi_k}{\langle\psi_i|\Pi_k|\psi_i\rangle} \text{ with probability } p(k) = \text{Tr}(\rho\Pi_k)$$

$$= \sum_i p(i) \frac{\Pi_k |\psi_i\rangle\langle\psi_i|\Pi_k}{\text{Tr}(\Pi_k|\rho)}$$

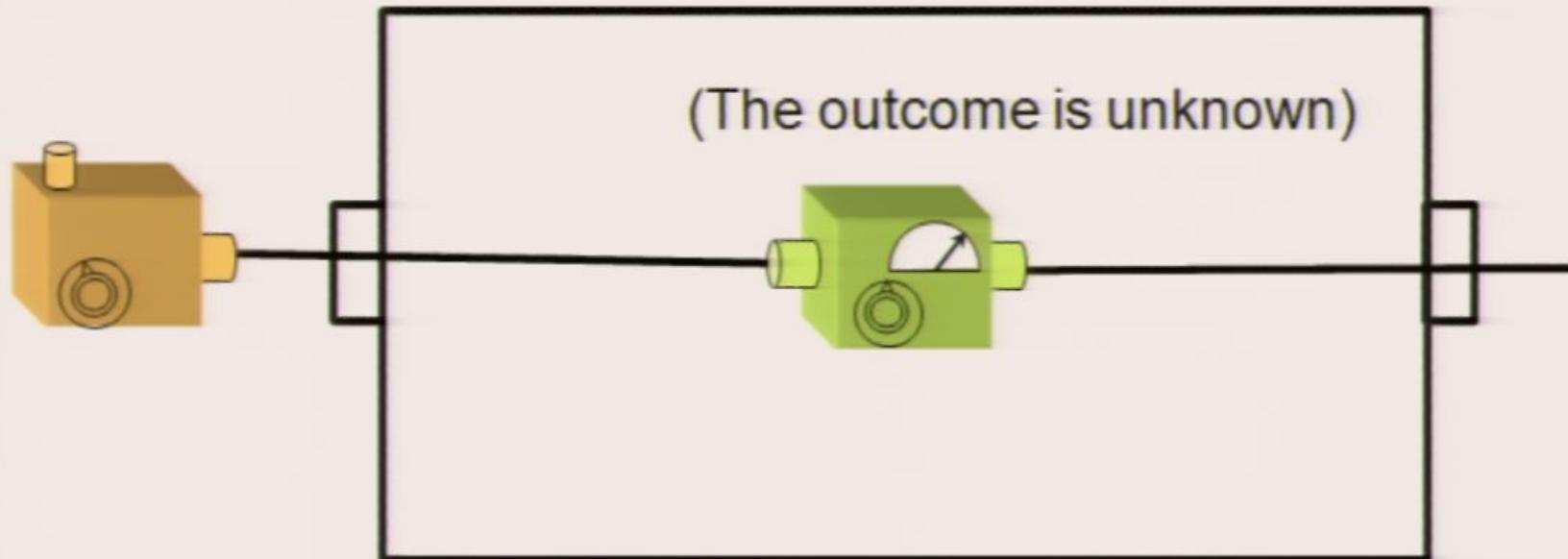
$$= \frac{\Pi_k \rho \Pi_k}{\text{Tr}(\rho\Pi_k)}$$

A nonselective measurement



$$\rho \rightarrow \frac{\Pi_k \rho \Pi_k}{\text{Tr}(\rho \Pi_k)} \quad \text{with probability} \quad p(k) = \text{Tr}(\rho \Pi_k)$$

A nonselective measurement



$$\rho \rightarrow \frac{\Pi_k \rho \Pi_k}{\text{Tr}(\rho \Pi_k)} \quad \text{with probability} \quad p(k) = \text{Tr}(\rho \Pi_k)$$

$$\rho \rightarrow \left\{ \left(p(k), \frac{\Pi_k \rho \Pi_k}{\text{Tr}(\rho \Pi_k)} \right) \right\}$$

Intrinsic characterization of transformation maps:

Intrinsic characterization of transformation maps:

1. $\mathcal{T}(\sum_i w_i \rho_i) = \sum_i w_i \mathcal{T}(\rho_i)$

\mathcal{T} is convex-linear

Intrinsic characterization of transformation maps:

1. $\mathcal{T}(\sum_i w_i \rho_i) = \sum_i w_i \mathcal{T}(\rho_i)$

\mathcal{T} is convex-linear

2. $\text{Tr}(\rho) = 1 \rightarrow \text{Tr}(\mathcal{T}(\rho)) = 1$

\mathcal{T} is trace-preserving

3. ρ is positive $\rightarrow \mathcal{T}(\rho)$ is positive

\mathcal{T} is positive (i.e. positivity-preserving)

Intrinsic characterization of transformation maps:

1. $\mathcal{T}(\sum_i w_i \rho_i) = \sum_i w_i \mathcal{T}(\rho_i)$

\mathcal{T} is convex-linear

2. $\text{Tr}(\rho) = 1 \rightarrow \text{Tr}(\mathcal{T}(\rho)) = 1$

\mathcal{T} is trace-preserving

3. ρ is positive $\rightarrow \mathcal{T}(\rho)$ is positive

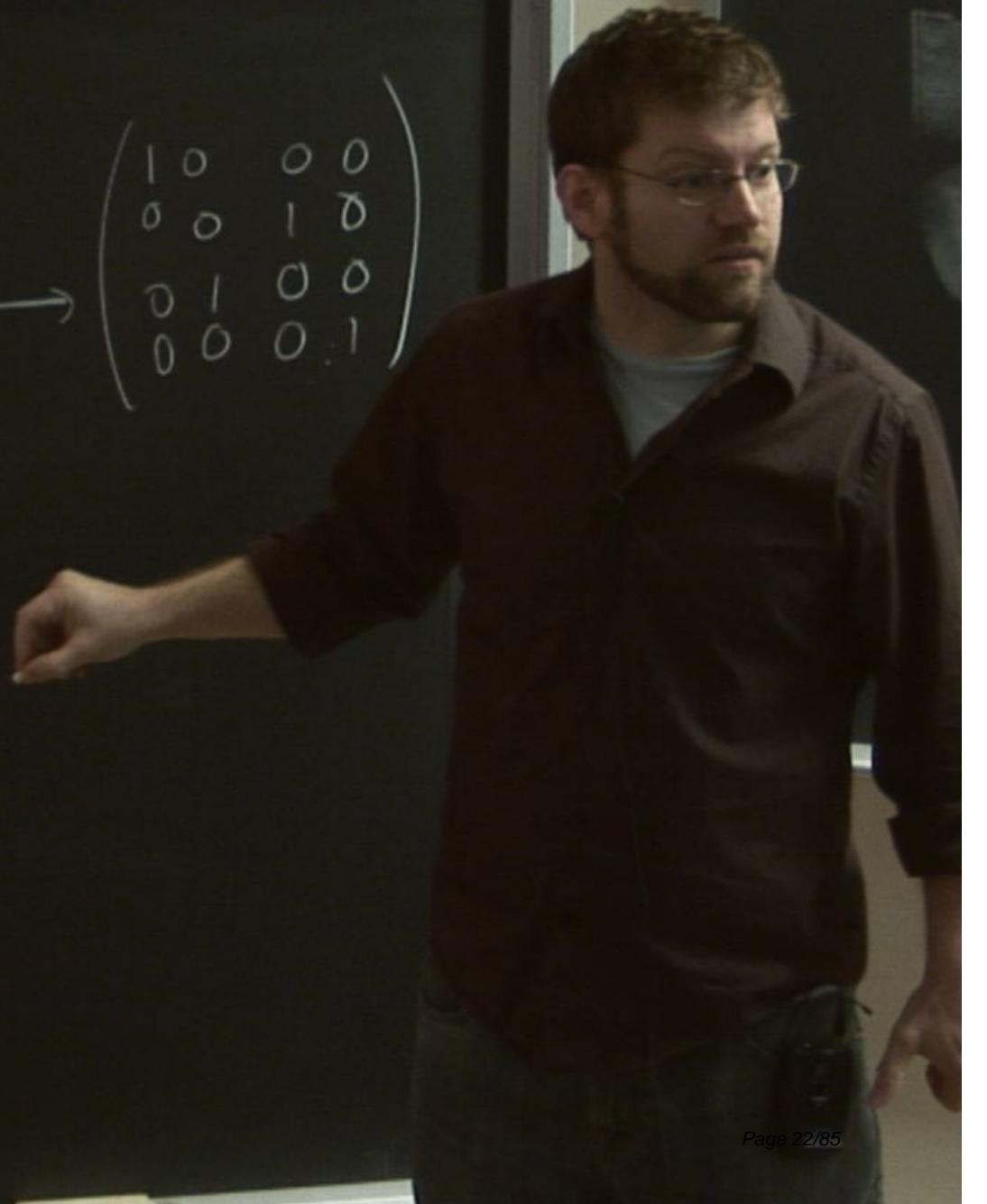
\mathcal{T} is positive (i.e. positivity-preserving)

$\rho^{(sa)}$ is positive $\rightarrow (\mathcal{T}^{(s)} \otimes \mathcal{I}^{(a)})(\rho^{(sa)})$ is positive

(even when $\rho^{(sa)}$ is entangled)

\mathcal{T} is completely positive

$$|0010\rangle + |1111\rangle$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$(|0\rangle|0\rangle + |1\rangle|1\rangle)$$
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$A_{H\nu} \rightarrow A_{\nu H}$

(10)10

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A_{H\nu} \rightarrow A_{\nu H} \quad (10)10$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Intrinsic characterization of transformation maps:

1. $\mathcal{T}(\sum_i w_i \rho_i) = \sum_i w_i \mathcal{T}(\rho_i)$

\mathcal{T} is convex-linear

2. $\text{Tr}(\rho) = 1 \rightarrow \text{Tr}(\mathcal{T}(\rho)) = 1$

\mathcal{T} is trace-preserving

3. ρ is positive $\rightarrow \mathcal{T}(\rho)$ is positive

\mathcal{T} is positive (i.e. positivity-preserving)

$\rho^{(sa)}$ is positive $\rightarrow (\mathcal{T}^{(s)} \otimes \mathcal{I}^{(a)})(\rho^{(sa)})$ is positive

(even when $\rho^{(sa)}$ is entangled)

\mathcal{T} is completely positive

All maps satisfying these assumptions can be written as:

$$T(\rho) = \sum_k A_k \rho A_k^\dagger$$

where the A_k are linear operators satisfying

$$\sum_k A_k^\dagger A_k = I$$

(operator-sum or 'Kraus' decomposition)

All maps satisfying these assumptions can be written as:

$$\mathcal{T}(\rho) = \sum_k A_k \rho A_k^\dagger$$

where the A_k are linear operators satisfying

$$\sum_k A_k^\dagger A_k = I$$

(operator-sum or 'Kraus' decomposition)

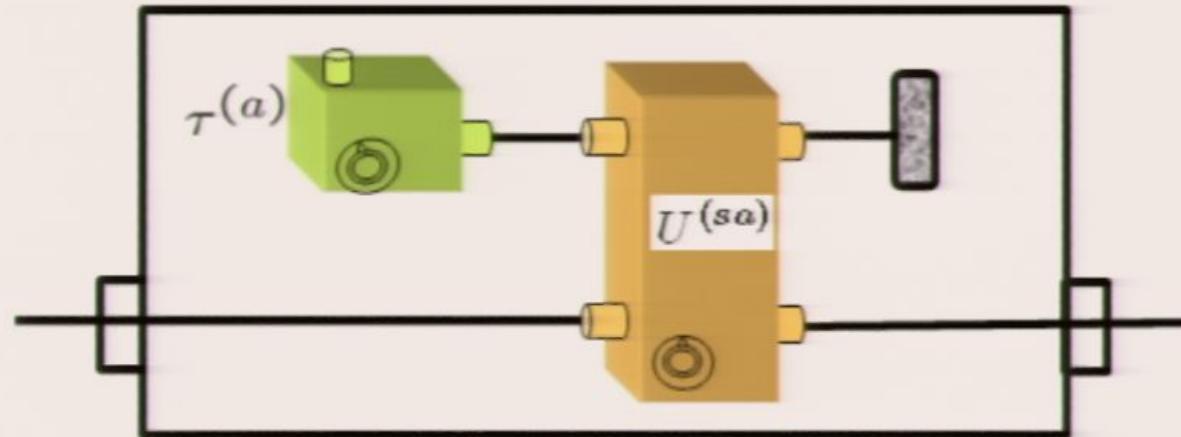
$$\mathcal{T}(\cdot) \equiv \sum_k \Pi_k(\cdot) \Pi_k$$

$$\begin{aligned}\mathcal{T}(\cdot) &= \sum_i p_i U_i(\cdot) U_i^\dagger \\ &= \sum_i \sqrt{p_i} U_i(\cdot) U_i^\dagger \sqrt{p_i}\end{aligned}$$

$$\mathcal{T}(\cdot) = \text{Tr}_a \left(U^{(sa)} (\cdot \otimes \tau^{(a)}) U^{(sa)\dagger} \right)$$

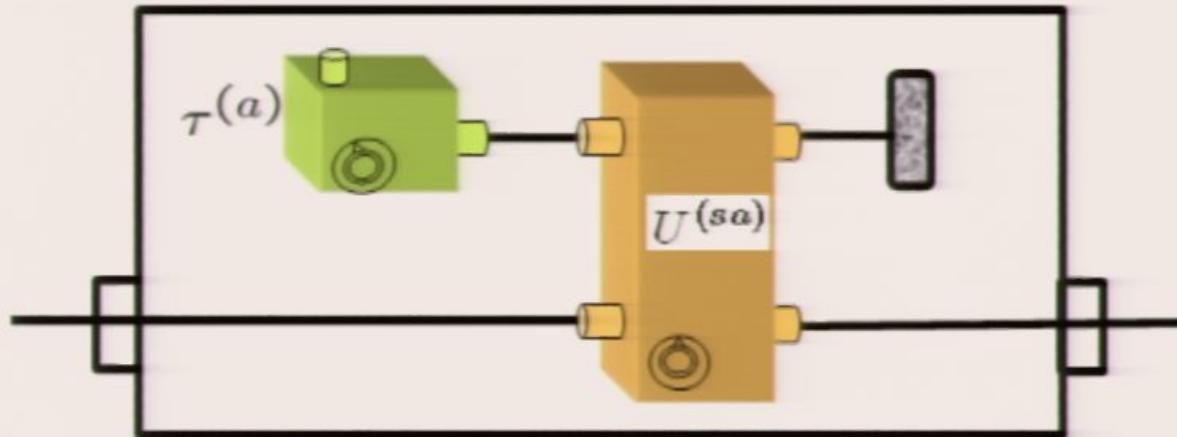
$$= \sum \langle j^{(a)} | U^{(sa)} | \phi^{(a)} \rangle (\cdot) \langle \phi^{(a)} | U^{(sa)\dagger} | j^{(a)} \rangle \quad \text{if } \tau^{(a)} = |\phi^{(a)}\rangle \langle \phi^{(a)}|$$

All maps satisfying these assumptions can be implemented by interaction with another system



(Stinespring dilation theorem)

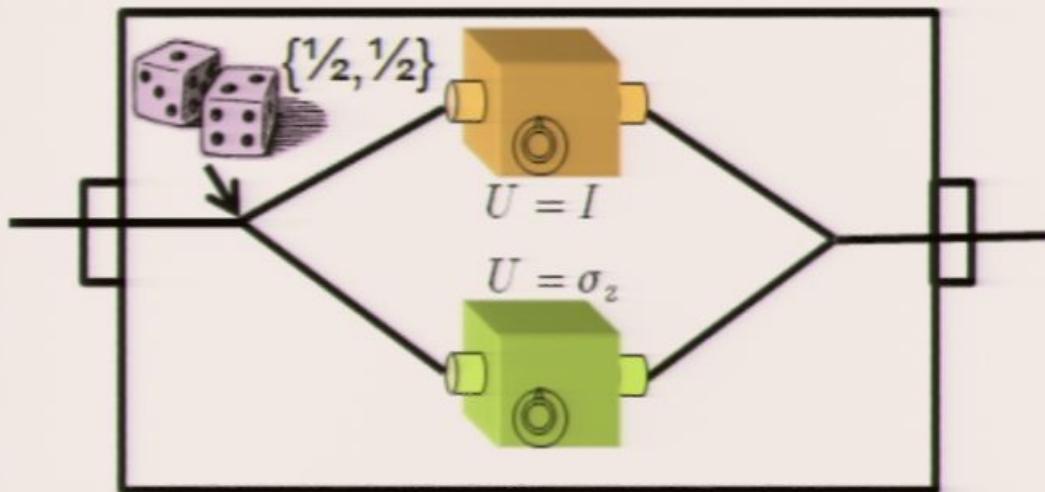
All maps satisfying these assumptions can be implemented by interaction with another system



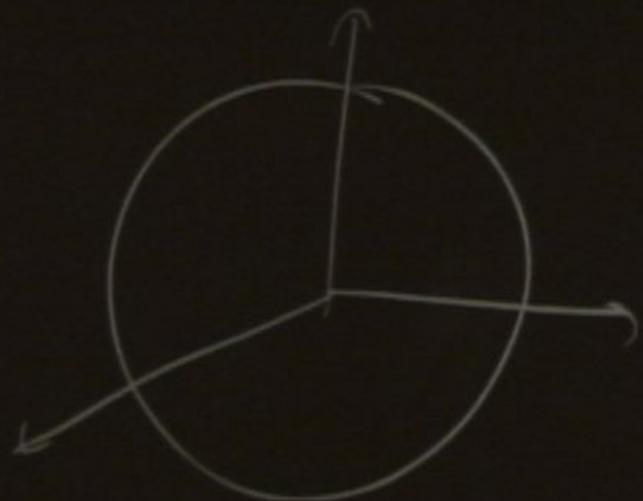
(Stinespring dilation theorem)

The Church of the larger Hilbert space

A given TPCP map can be implemented in many ways



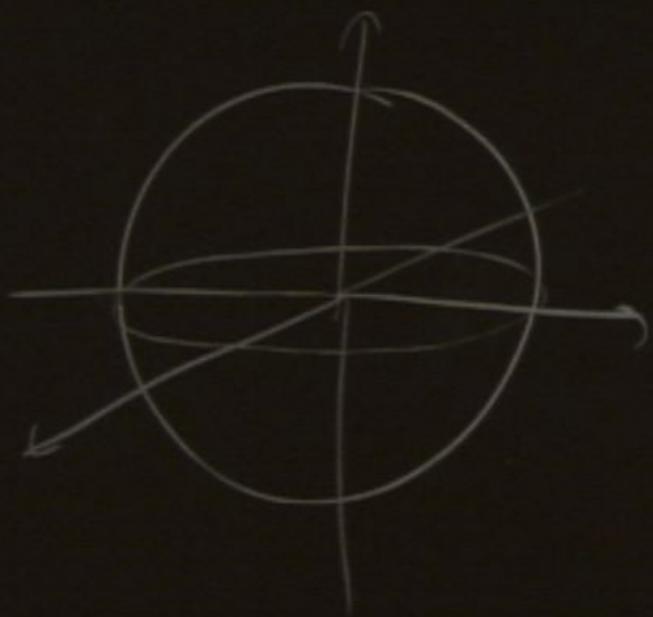
$$\mathcal{T}(\rho) = \frac{1}{2}\rho + \frac{1}{2}\sigma_z\rho\sigma_z$$



$A_{H\nu} \rightarrow A_{V\nu}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

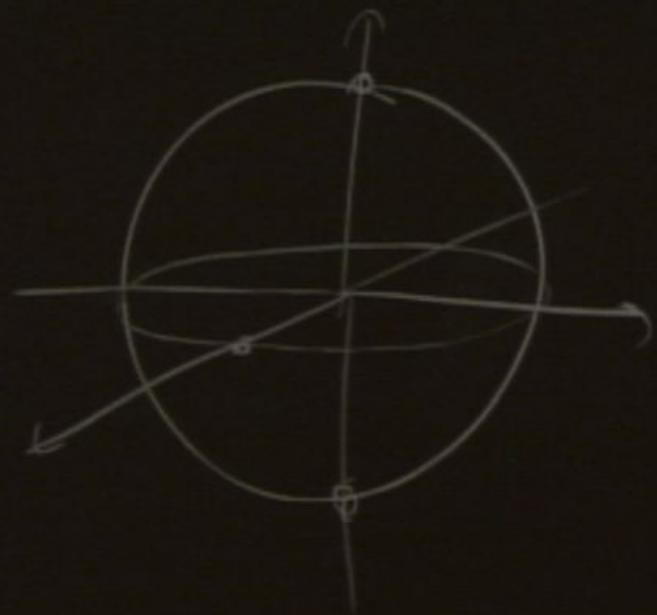
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



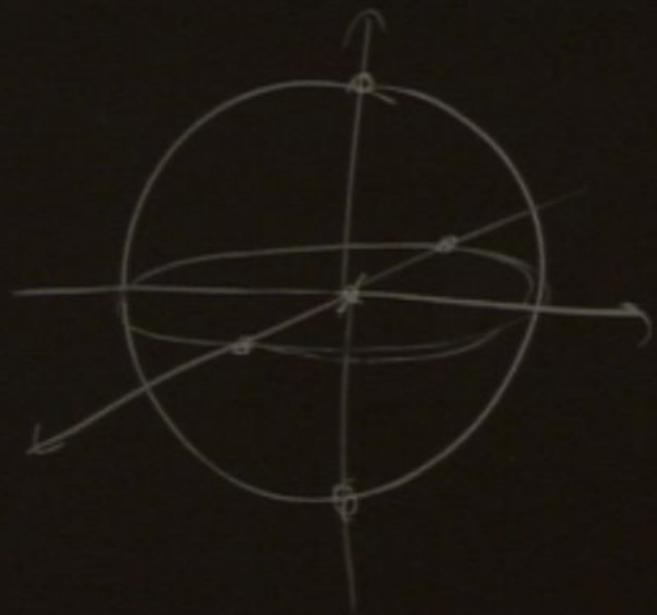
$$A_{H^V} \rightarrow A_{V^H}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

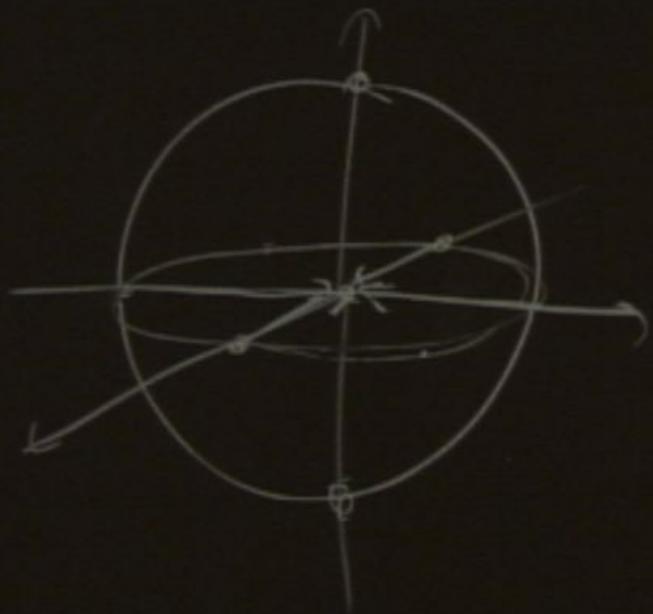
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

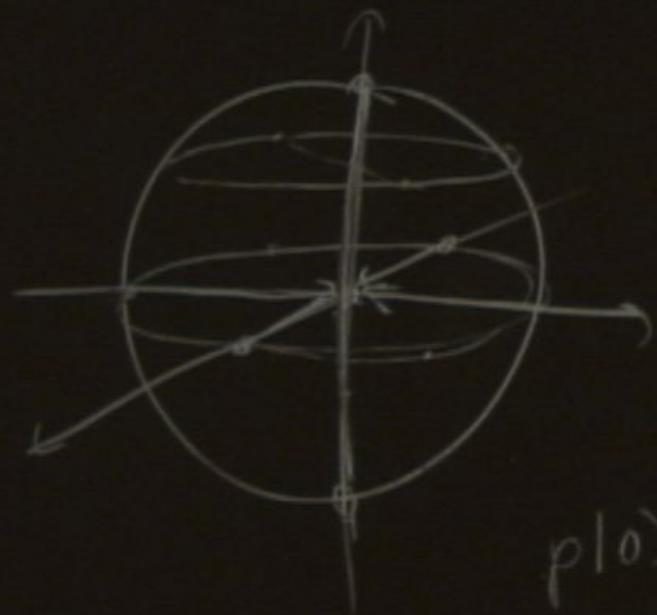


$$A_{\text{MH}} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} |0\rangle|0\rangle + |1\rangle|1\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$A_{H\nu} \rightarrow A_{\nu H}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

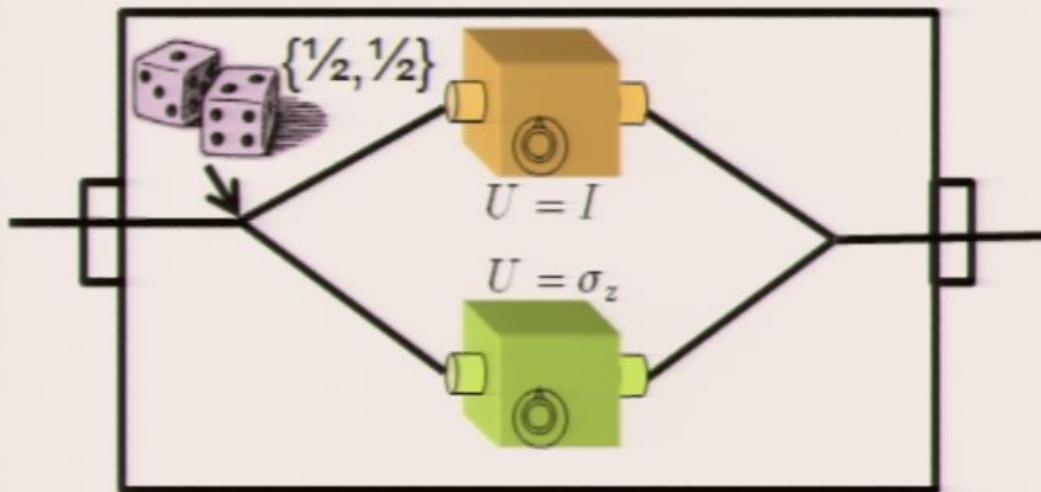


$A_{H\nu} \rightarrow A_{\nu H}$

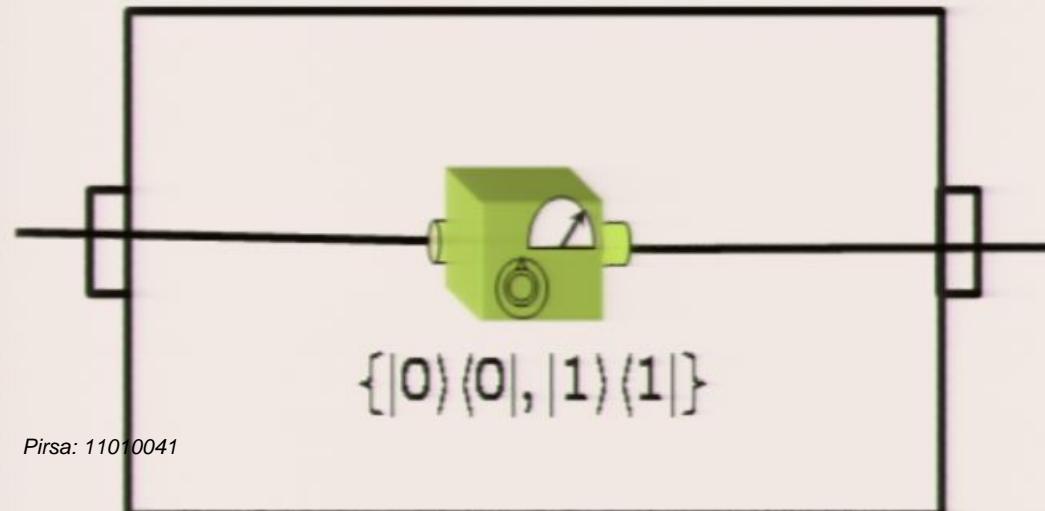
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$$

A given TPCP map can be implemented in many ways



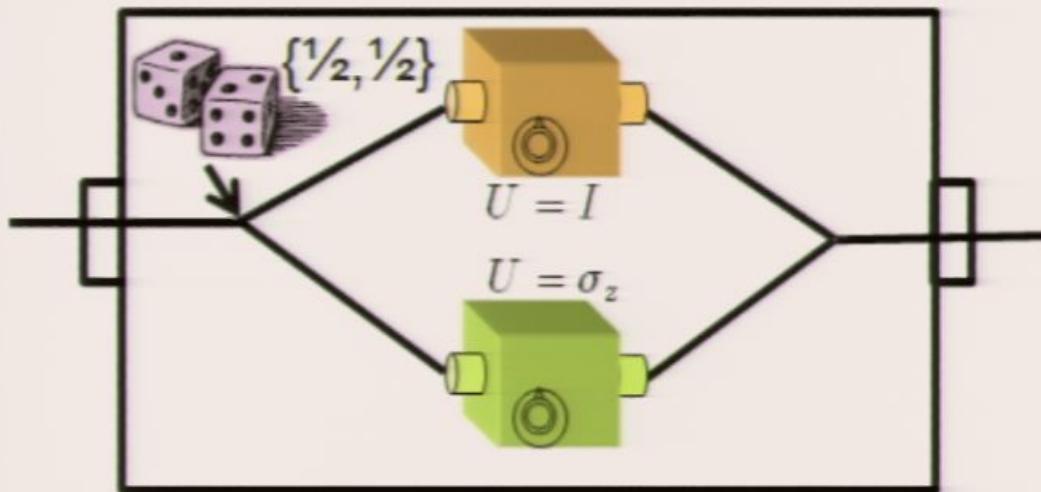
$$\mathcal{T}(\rho) = \frac{1}{2}\rho + \frac{1}{2}\sigma_z\rho\sigma_z$$



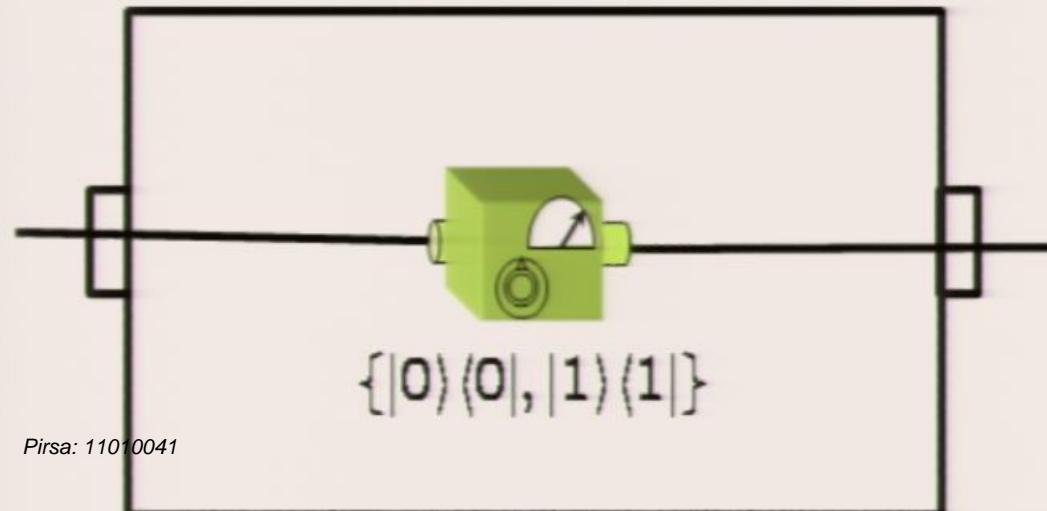
$$\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$$

$$\begin{aligned}\mathcal{T}(\rho) &= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| \\ &= \langle 0|\rho|0\rangle|0\rangle\langle 0| + \langle 1|\rho|1\rangle|1\rangle\langle 1|\end{aligned}$$

A given TPCP map can be implemented in many ways



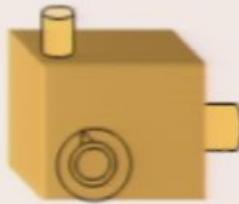
$$\mathcal{T}(\rho) = \frac{1}{2}\rho + \frac{1}{2}\sigma_z\rho\sigma_z$$



$$\begin{aligned}\mathcal{T}(\rho) &= |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1| \\ &= \langle 0|\rho|0\rangle|0\rangle\langle 0| + \langle 1|\rho|1\rangle|1\rangle\langle 1|\end{aligned}$$

The complete formalism of operational quantum theory

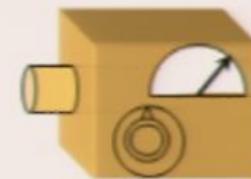
Operational Quantum Theory



Preparation
P



Transformation
T



Measurement
M

Density operator
 ρ

Trace-preserving
completely positive
linear map (CP map)

T

Positive operator-valued
measure (POVM)

$\{E_k\}$

$$Pr(k|P, T, M) = \text{Tr}[E_k T(\rho)]$$

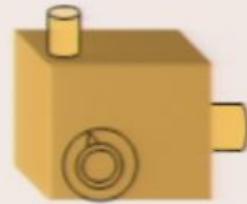
Operational Quantum Theory



$$\rho \rightarrow \rho' = \mathcal{T}(\rho)$$

$$Pr(k|P' M) = \text{Tr}(E_k \rho') = \text{Tr}(E_k \mathcal{T}(\rho))$$

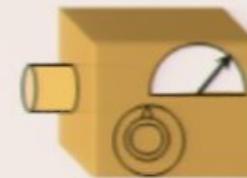
Operational Quantum Theory



Preparation
P



Transformation
T

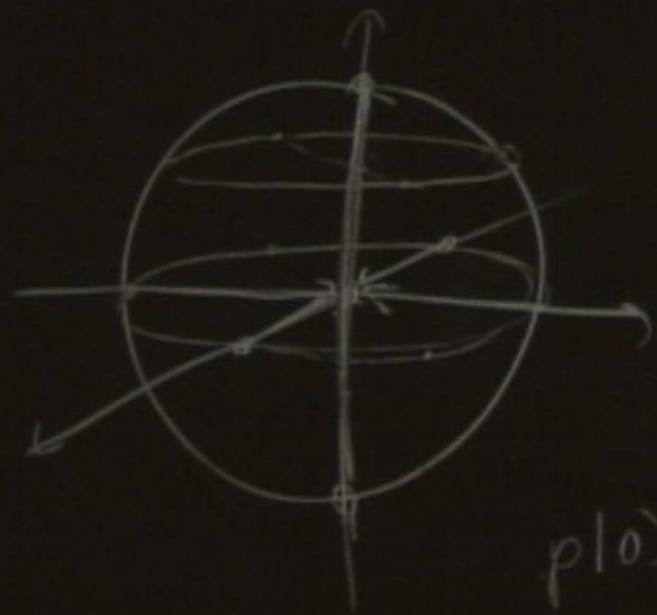


Measurement
M

Effective Measurement
 M'

$$E_k \rightarrow E'_k = T^\dagger(E_k)$$

$$Pr(k|P', M) = \text{Tr}(E'_k \rho) = \text{Tr}(T^\dagger(E_k) \rho)$$



$$A_{H\nu} \rightarrow A_{\nu H}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\rho |0\rangle\langle 0| + (1-\rho) |1\rangle\langle 1|$$

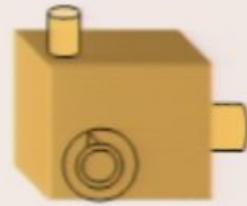
$$(A, B) = \text{Tr}(AB)$$

$$(A, \varepsilon B)$$

T

2)

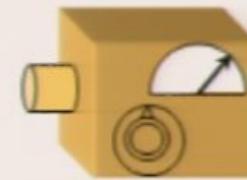
Operational Quantum Theory



Preparation
P



Transformation
T



Measurement
M

Effective Measurement
 M'

$$E_k \rightarrow E'_k = T^\dagger(E_k)$$

$$P_{\pi}(k|P', M) = \text{Tr}(E'_k \rho) = \text{Tr}(T^\dagger(E_k) \rho)$$

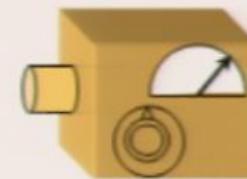
Operational Quantum Theory



Preparation
P



Transformation
T



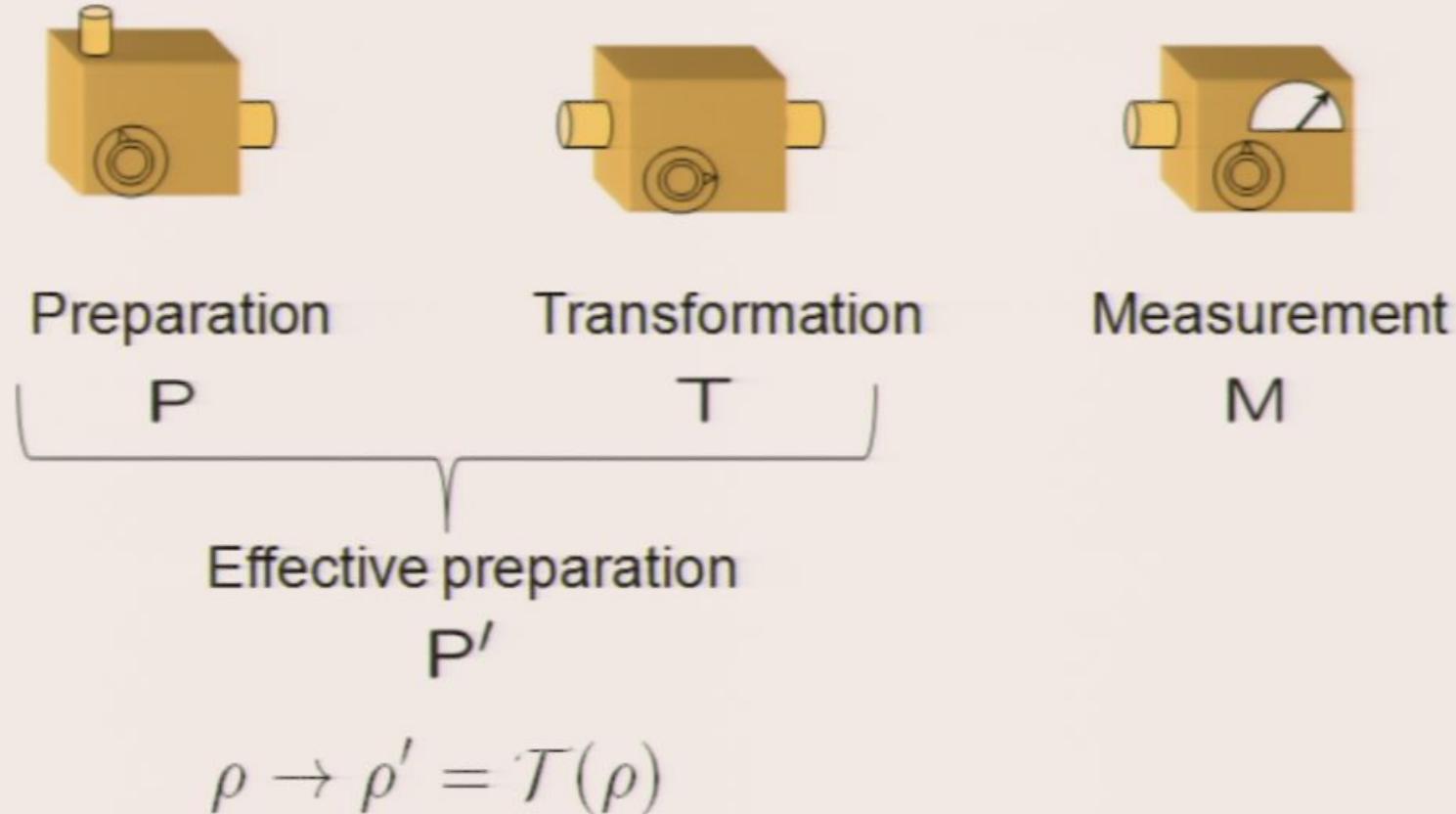
Measurement
M

Effective Measurement
 M'

$$E_k \rightarrow E'_k = T^\dagger(E_k)$$

$$Pr(k|P', M) = \text{Tr}(E'_k \rho) = \text{Tr}(T^\dagger(E_k) \rho)$$

Operational Quantum Theory



$$Pr(k|P' M) = \text{Tr}(E_k \rho') = \text{Tr}(E_k \mathcal{T}(\rho))$$

Operational Quantum Theory



Preparation

P



Measurement

M

Effective preparation

P_k

Update map

$$\rho \rightarrow \rho_k = \frac{\tau_k(\rho)}{\text{Tr}[\tau_k(\rho)]}$$

Trace-nonincreasing
completely positive
linear map

Operational formulation of quantum theory

Every preparation P is associated with a density operator ρ

Every measurement M is associated with a positive operator-valued measure $\{E_k\}$. The probability of M yielding outcome k given a preparation P is $Pr(k|P, M) = \text{Tr}(\rho E_k)$

Every transformation is associated with a trace-preserving completely-positive linear map $\rho \rightarrow \rho' = \mathcal{T}(\rho)$

Every measurement outcome k is associated with a trace-nonincreasing completely-positive linear map T_k such that

$$\rho \rightarrow \rho_k = \frac{\mathcal{T}_k(\rho)}{\text{Tr}[\mathcal{T}_k(\rho)]} \quad \text{where} \quad \mathcal{T}_k^\dagger(I) = E_k$$

Towards an operational axiomatization of quantum theory

Operational formulation of quantum theory

Every preparation P is associated with a density operator ρ

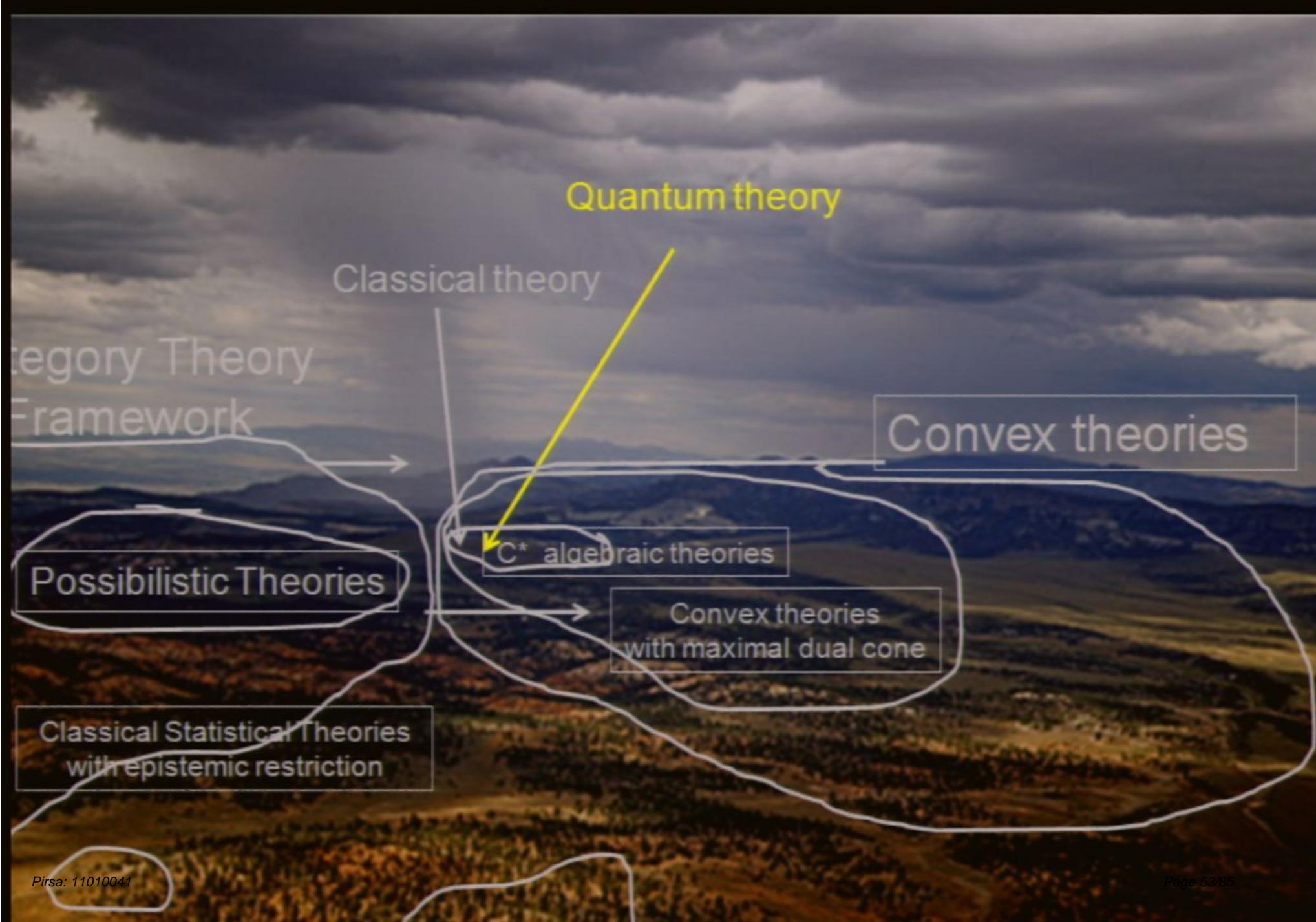
Every measurement M is associated with a positive operator-valued measure $\{E_k\}$. The probability of M yielding outcome k given a preparation P is $Pr(k|P, M) = \text{Tr}(\rho E_k)$

Every transformation is associated with a trace-preserving completely-positive linear map $\rho \rightarrow \rho' = \mathcal{T}(\rho)$

Every measurement outcome k is associated with a trace-nonincreasing completely-positive linear map T_k such that

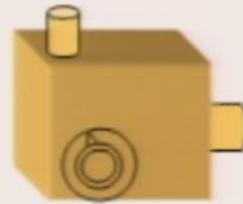
$$\rho \rightarrow \rho_k = \frac{\mathcal{T}_k(\rho)}{\text{Tr}[\mathcal{T}_k(\rho)]} \quad \text{where} \quad \mathcal{T}_k^\dagger(I) = E_k$$

Towards an operational axiomatization of quantum theory



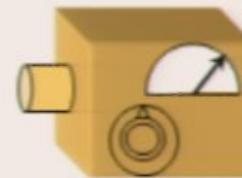
A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



Preparation

P

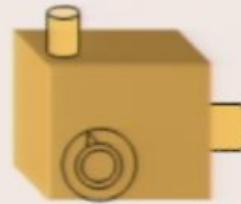


Measurement

M

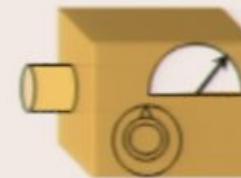
A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



Preparation

P



Measurement

M

$$\mathbf{s}_P = \begin{pmatrix} \Pr(1|M_1, P) \\ \Pr(2|M_1, P) \\ \Pr(1|M_2, P) \\ \Pr(2|M_2, P) \\ \Pr(3|M_2, P) \\ \vdots \end{pmatrix}$$

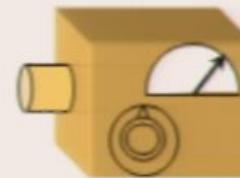
A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



Preparation

P



Measurement

M

$$\mathbf{s}_P = \begin{pmatrix} \Pr(1|M_1, P) \\ \Pr(2|M_1, P) \\ \Pr(1|M_2, P) \\ \Pr(2|M_2, P) \\ \Pr(3|M_2, P) \\ \vdots \end{pmatrix}$$

$$\mathbf{r}_{M,k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

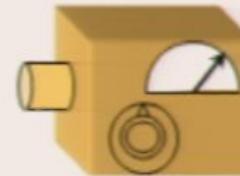
A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



Preparation

P



Measurement

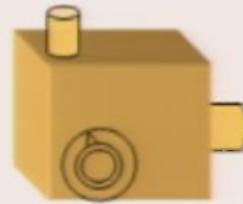
M

$$\mathbf{s}_P = \begin{pmatrix} \Pr(1|M_1, P) \\ \Pr(2|M_1, P) \\ \Pr(1|M_2, P) \\ \Pr(2|M_2, P) \\ \Pr(3|M_2, P) \\ \vdots \end{pmatrix}$$

$$\mathbf{r}_{M,k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$$

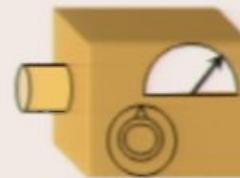
A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



Preparation

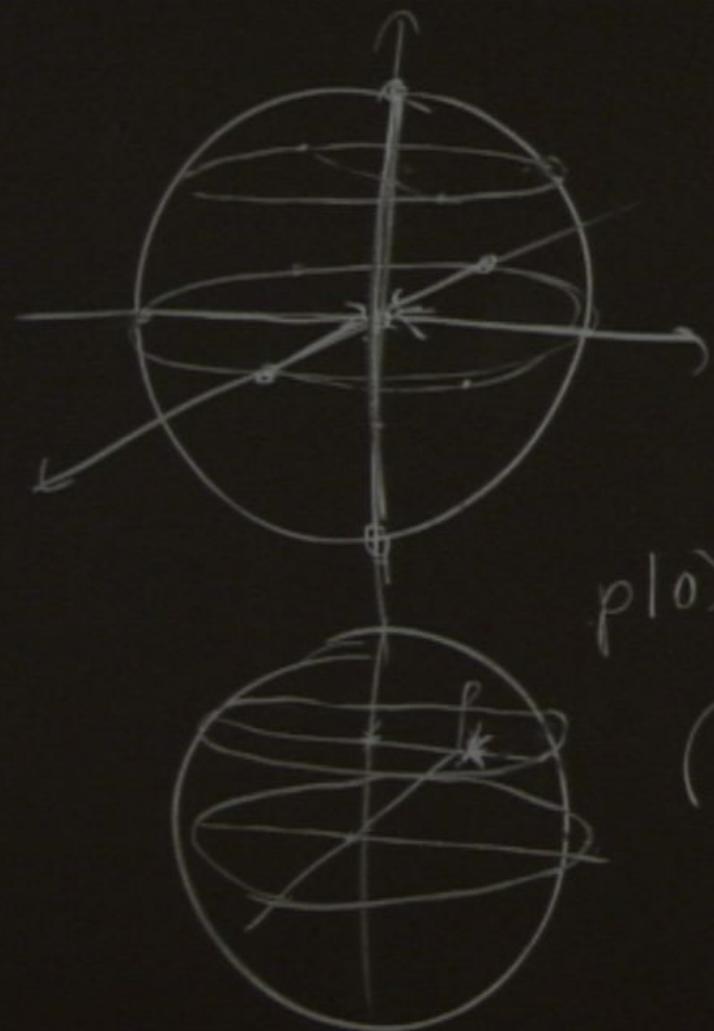
P



Measurement

M

Suppose there are K fiducial measurements (pass-fail mmts from which one can infer the statistics for all mmts)



$$A_{\mu\nu} \rightarrow A_{\nu\mu}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$(10)(10) + 1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$$

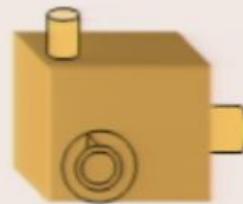
$$(A, B) = \text{Tr}(AB)$$

$$(A, \varepsilon B) = (\varepsilon^+ A | \varepsilon^- B)$$

$$\text{Tr}(A\varepsilon(B)) = F(\varepsilon^+(A))$$

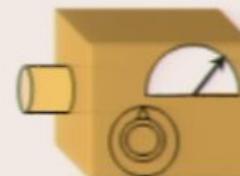
A framework for convex operational theories

See: L. Hardy, quant-ph/0101012



Preparation

P



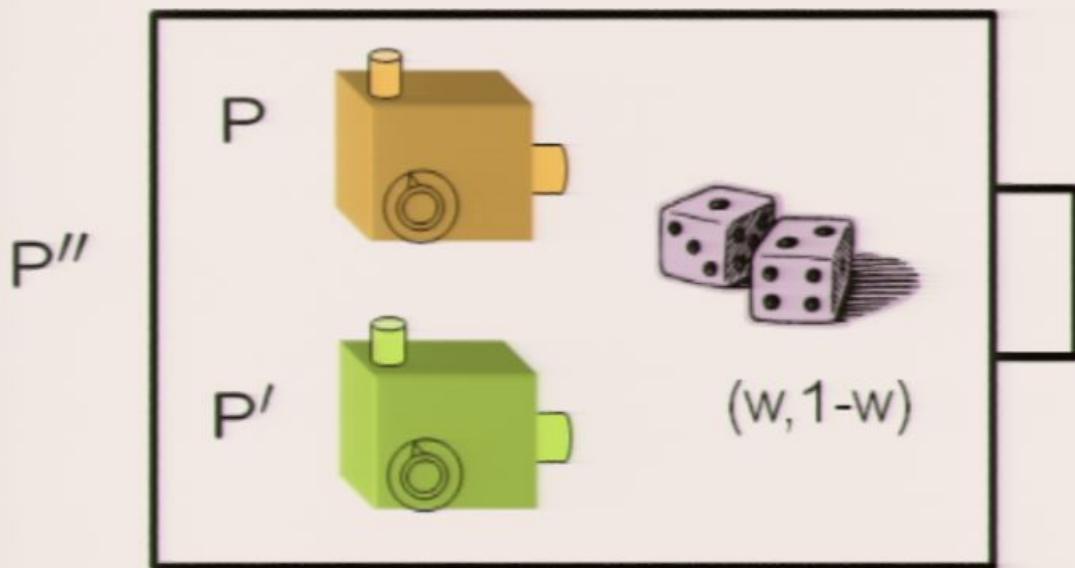
Measurement

M

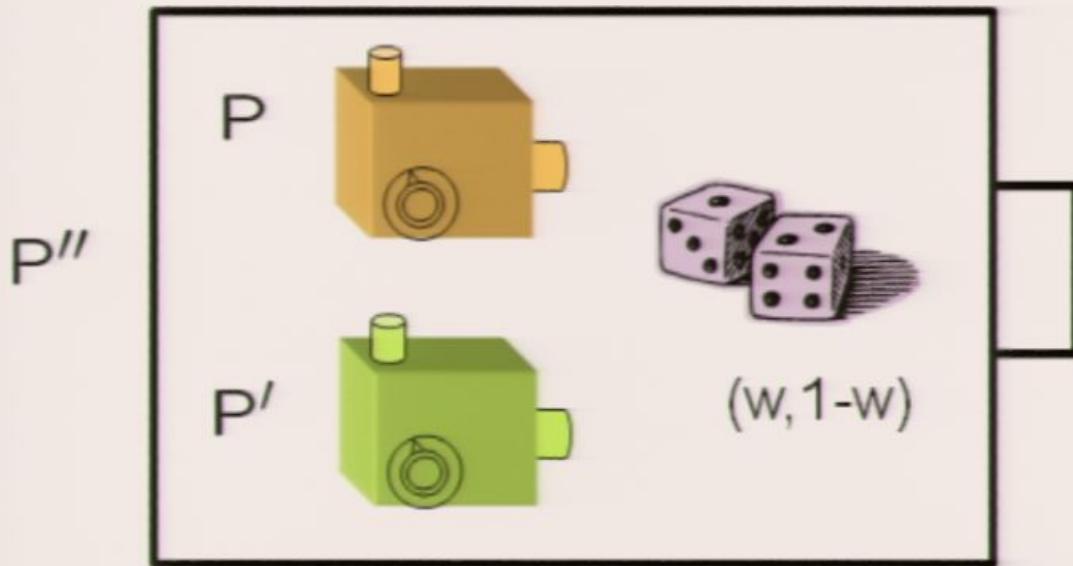
Suppose there are K fiducial measurements (pass-fail mmts from which one can infer the statistics for all mmts)

$$\mathbf{s}_P = \begin{pmatrix} \Pr(\text{pass} | M_1, P) \\ \Pr(\text{pass} | M_2, P) \\ \vdots \\ \Pr(\text{pass} | M_K, P) \end{pmatrix}$$

Operational states form a convex set

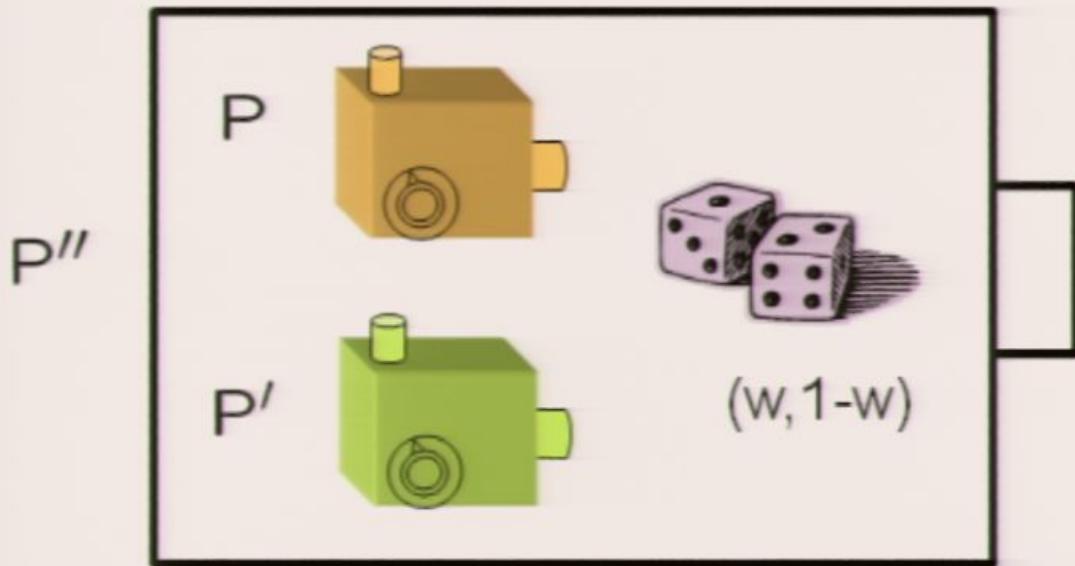


Operational states form a convex set



$$\forall M, k : p(k|M, P'') = w p(k|M, P) + (1-w) p(k|M, P')$$

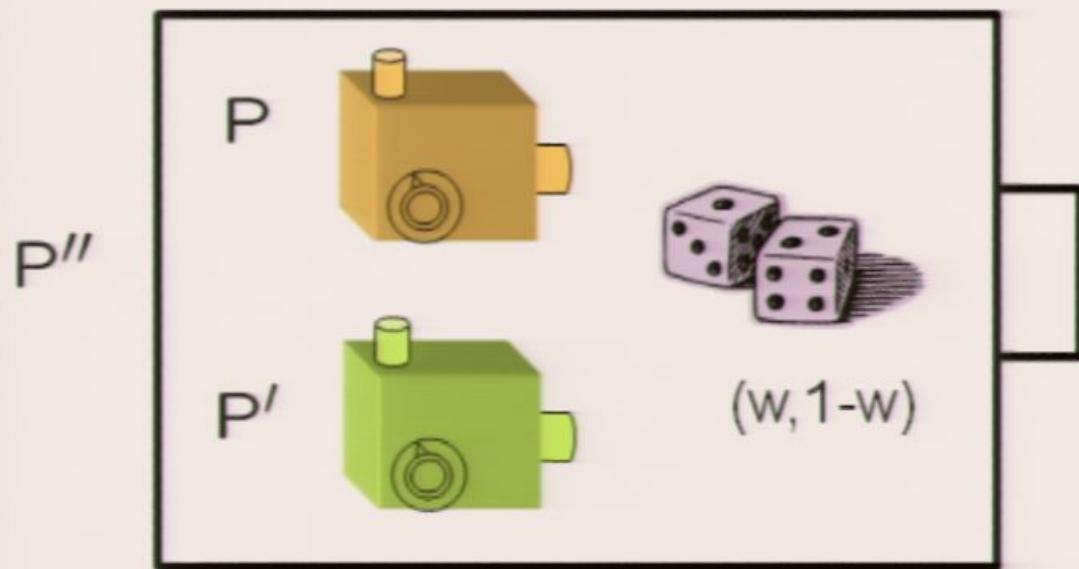
Operational states form a convex set



$$\forall M, k : p(k|M, P'') = w p(k|M, P) + (1-w) p(k|M, P')$$

$$f(\mathbf{s}_{P''}) = w f(\mathbf{s}_P) + (1 - w) f(\mathbf{s}_{P'})$$

Operational states form a convex set

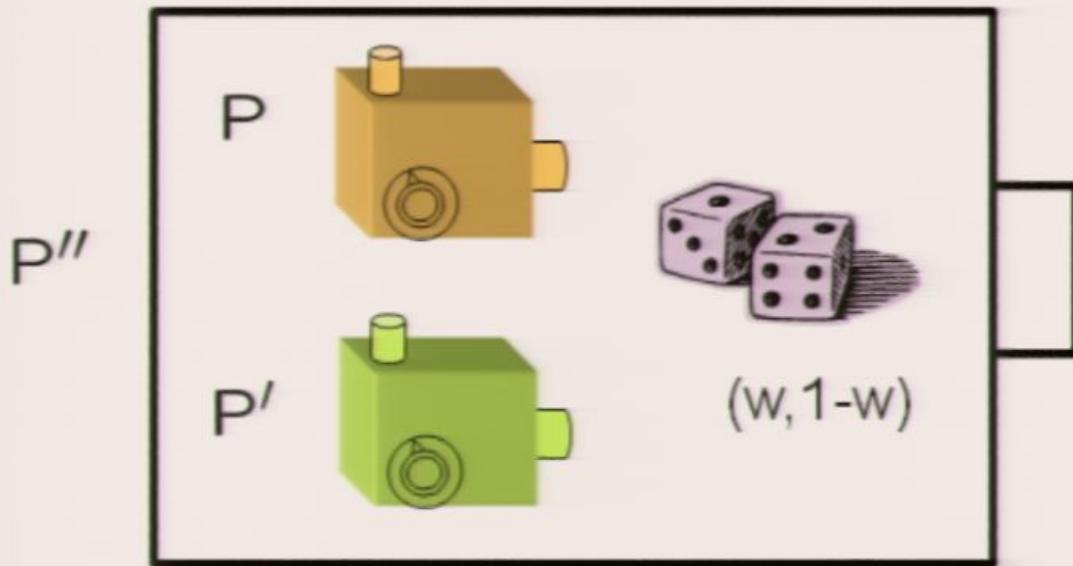


$$\forall M, k : p(k|M, P'') = w p(k|M, P) + (1-w) p(k|M, P')$$

$$f(\mathbf{s}_{P''}) = w f(\mathbf{s}_P) + (1 - w) f(\mathbf{s}_{P'})$$

Also true for fiducial mmts, so $\mathbf{s}_{P''} = w \mathbf{s}_P + (1 - w) \mathbf{s}_{P'}$

Operational states form a convex set



$$\forall M, k : p(k|M, P'') = w p(k|M, P) + (1-w) p(k|M, P')$$

$$f(\mathbf{s}_{P''}) = w f(\mathbf{s}_P) + (1 - w) f(\mathbf{s}_{P'})$$

Also true for fiducial mmts, so $\mathbf{s}_{P''} = w \mathbf{s}_P + (1 - w) \mathbf{s}_{P'}$

Closed under convex combination \rightarrow a convex set

Convex linearity implies linearity

If f is convex linear on opt'l states

$$s = \sum_i w_i s_i \Rightarrow f(s) = \sum_i w_i f(s_i) \quad 0 \leq w_i \leq 1 \text{ and } \sum_i w_i = 1$$

Then f is linear on opt'l states

$$s = \sum_i \alpha_i s_i \Rightarrow f(s) = \sum_i \alpha_i f(s_i) \quad \alpha_i \in \mathbb{R}$$

Convex linearity implies linearity

If f is convex linear on opt'l states

$$s = \sum_i w_i s_i \Rightarrow f(s) = \sum_i w_i f(s_i) \quad 0 \leq w_i \leq 1 \text{ and } \sum_i w_i = 1$$

Then f is linear on opt'l states

$$s = \sum_i \alpha_i s_i \Rightarrow f(s) = \sum_i \alpha_i f(s_i) \quad \alpha_i \in \mathbb{R}$$

Convex linearity implies linearity

If f is convex linear on opt'l states

$$\mathbf{s} = \sum_i w_i \mathbf{s}_i \Rightarrow f(\mathbf{s}) = \sum_i w_i f(\mathbf{s}_i) \quad 0 \leq w_i \leq 1 \text{ and } \sum_i w_i = 1$$

Then f is linear on opt'l states

$$\mathbf{s} = \sum_i \alpha_i \mathbf{s}_i \Rightarrow f(\mathbf{s}) = \sum_i \alpha_i f(\mathbf{s}_i) \quad \alpha_i \in \mathbb{R}$$

Proof: $\mathbf{s} = \sum_i \alpha_i \mathbf{s}_i$

$$\mathbf{s} + \sum_{j \in I_-} |\alpha_j| \mathbf{s}_j = \sum_{i \in I_+} |\alpha_i| \mathbf{s}_i$$

If 1st component is the $1 = \sum_i \alpha_i$

trivial fiducial mmt, then: $1 + \sum_{j \in I_-} |\alpha_j| = \sum_{i \in I_+} |\alpha_i| \equiv N$

Thus: $\frac{1}{N} \mathbf{s} + \sum_{j \in I_-} \frac{|\alpha_j|}{N} \mathbf{s}_j = \sum_{i \in I_+} \frac{|\alpha_i|}{N} \mathbf{s}_i$

$$\frac{1}{N} f(\mathbf{s}) + \sum_{j \in I_-} \frac{|\alpha_j|}{N} f(\mathbf{s}_j) = \sum_{i \in I_+} \frac{|\alpha_i|}{N} f(\mathbf{s}_i)$$

$$f(\mathbf{s}) = \sum_i \alpha_i f(\mathbf{s}_i)$$

Convex linearity implies linearity

If f is convex linear on opt'l states

$$s = \sum_i w_i s_i \Rightarrow f(s) = \sum_i w_i f(s_i) \quad 0 \leq w_i \leq 1 \text{ and } \sum_i w_i = 1$$

Then f is linear on opt'l states

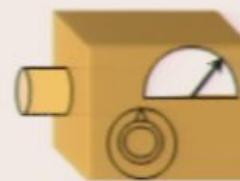
$$s = \sum_i \alpha_i s_i \Rightarrow f(s) = \sum_i \alpha_i f(s_i) \quad \alpha_i \in \mathbb{R}$$

A convex operational theory



Preparation

P



Measurement

M

$$\mathbf{s}_P \in S$$

“operational states”

S = Convex set

$$\mathbf{r}_{M,k} \in R$$

“operational effects”

R = Interval of
positive cone

$$Pr(k|P, M) = \mathbf{r}_{M,k} \cdot \mathbf{s}_P$$

Operational classical theory

s can be any probability distribution

$S = \text{a simplex}$

Operational classical theory

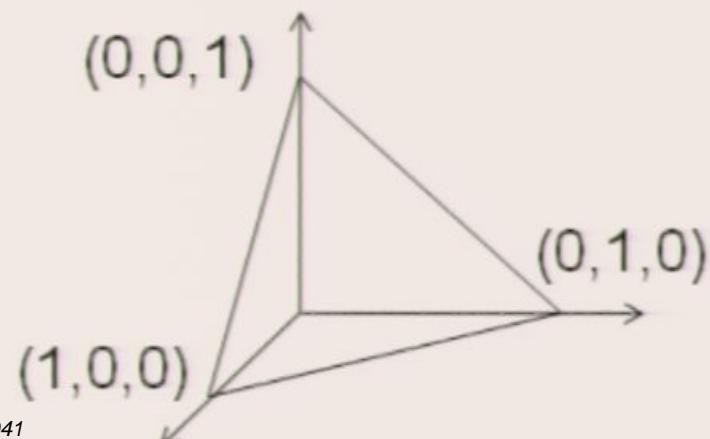
s can be any probability distribution

$S = \text{a simplex}$

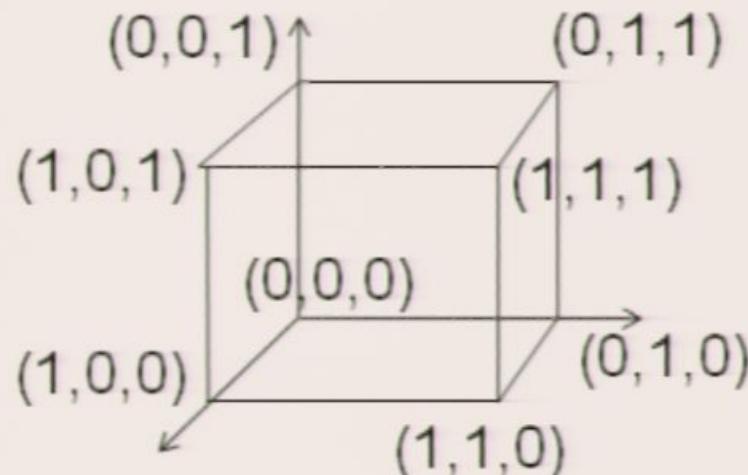
r can be any vector of conditional probabilities

$R = \text{the unit hypercube}$

$$(p(1), p(2), p(3))$$



$$(p(\text{pass}|1), p(\text{pass}|2), p(\text{pass}|3))$$



Operational quantum theory

Recall: The Hermitian operators on a Hilbert space of dimension d form a real Euclidean vector space of dimension d^2

- s** can be any trace one positive operator
- S** = the convex set of such operators

Operational quantum theory

Recall: The Hermitian operators on a Hilbert space of dimension d form a real Euclidean vector space of dimension d^2

- s** can be any trace one positive operator
- S** = the convex set of such operators

A little bit of axiomatics

Suppose one takes as given that

$S = \text{the convex set of positive trace-one operators}$

A little bit of axiomatics

Suppose one takes as given that

$S = \text{the convex set of positive trace-one operators}$

Suppose one assumes that **every logically possible measurement is physically possible**

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$

A little bit of axiomatics

Suppose one takes as given that

$S = \text{the convex set of positive trace-one operators}$

Suppose one assumes that **every logically possible measurement is physically possible**

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

The real vector space is the space of Hermitian operators

The inner product is $(A, B) = \text{Tr}(AB)$

Each \mathbf{s} is a density operator ρ

A little bit of axiomatics

Suppose one takes as given that

$S = \text{the convex set of positive trace-one operators}$

Suppose one assumes that **every logically possible measurement is physically possible**

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

The real vector space is the space of Hermitian operators

The inner product is $(A, B) = \text{Tr}(AB)$

Each \mathbf{s} is a density operator ρ

Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$

$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \text{Tr}(E_k \rho)$ \leftarrow the form of the Born rule

A little bit of axiomatics

Suppose one takes as given that

$S = \text{the convex set of positive trace-one operators}$

Suppose one assumes that **every logically possible measurement is physically possible**

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

The real vector space is the space of Hermitian operators

The inner product is $(A, B) = \text{Tr}(AB)$

Each \mathbf{s} is a density operator ρ

Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$

$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \text{Tr}(E_k \rho) \quad \leftarrow \text{the form of the Born rule}$

Allow all $\{E_k\}$ such that $\text{Tr}(E_k \rho) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$

$$\sum_k \text{Tr}(E_k \rho) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

A little bit of axiomatics

Suppose one takes as given that

$S = \text{the convex set of positive trace-one operators}$

Suppose one assumes that **every logically possible measurement is physically possible**

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

The real vector space is the space of Hermitian operators

The inner product is $(A, B) = \text{Tr}(AB)$

Each \mathbf{s} is a density operator ρ

Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$

$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \text{Tr}(E_k \rho) \quad \leftarrow \text{the form of the Born rule}$

Allow all $\{E_k\}$ such that $\text{Tr}(E_k \rho) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$

$$\sum_k \text{Tr}(E_k \rho) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\text{Tr}(\rho E_k) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\sum_k \text{Tr}(\rho E_k) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\text{Tr}(\rho E_k) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\sum_k \text{Tr}(\rho E_k) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\text{Tr}(\rho E_k) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\rightarrow \langle \psi | E_k | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$$

E_k is a positive operator

$$\sum_k \text{Tr}(\rho E_k) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\rightarrow \langle \psi | (\sum_k E_k) | \psi \rangle = 1 \quad \forall |\psi\rangle \in \mathcal{H}$$

$$\rightarrow \sum_k E_k = I$$

$$\text{Tr}(\rho E_k) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\rightarrow \langle \psi | E_k | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$$

E_k is a positive operator

$$\sum_k \text{Tr}(\rho E_k) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$

$$\rightarrow \langle \psi | (\sum_k E_k) | \psi \rangle = 1 \quad \forall |\psi\rangle \in \mathcal{H}$$

$$\rightarrow \sum_k E_k = I$$

The logically possible measurements correspond to the POVMs!

$$A_{\mu\nu} \rightarrow A_{\nu\mu}$$

$$(0)(0) + (1)(1)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$$

$$(A, B) = \text{Tr}(AB)$$

$$(A, \mathcal{E}B) = (\mathcal{E}^T A | B)$$

$$\text{Tr}(A \mathcal{E}(B)) = \mathcal{E}(\mathcal{E}^T A | B)$$

