Title: Foundations of Quantum Mechanics - Lecture 2

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Abstract:

perimeter scholars

international

Towards a purely operational formulation of quantum theory

Towards a purely operational formulation of quantum theory

“Orthodox” postulates of quantum theory

Representational completeness of ψ. The rays of Hilbert space correspond one-to-one with the physical states of the system.

Measurement. If the Hermitian operator A with spectral projectors $\{P_k\}$ is measured, the probability of outcome k is $\langle\psi|P_k|\psi\rangle$. These probabilities are objective -- indeterminism.

Evolution of isolated systems. It is unitary, $|\psi\rangle \rightarrow U|\psi\rangle = e^{-\frac{i}{\hbar}Ht}|\psi\rangle$ therefore deterministic and continuous.

Evolution of systems undergoing measurement. If Hermitian operator A with spectral projectors $\{P_k\}$ is measured and outcome k is obtained, the physical state of the system changes discontinuously,

$$|\psi\rangle \rightarrow |\psi_k\rangle = \frac{P_k|\psi\rangle}{\sqrt{\langle\psi|P_k|\psi\rangle}}$$

First problem: the term "measurement" is not defined in terms of the more primitive "physical states of systems". Isn't a measurement just another kind of physical interaction?

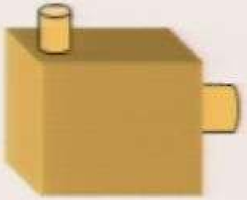
Two strategies:

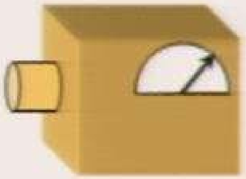
(1) Realist strategy: Eliminate measurement as a primitive concept and describe everything in terms of physical states

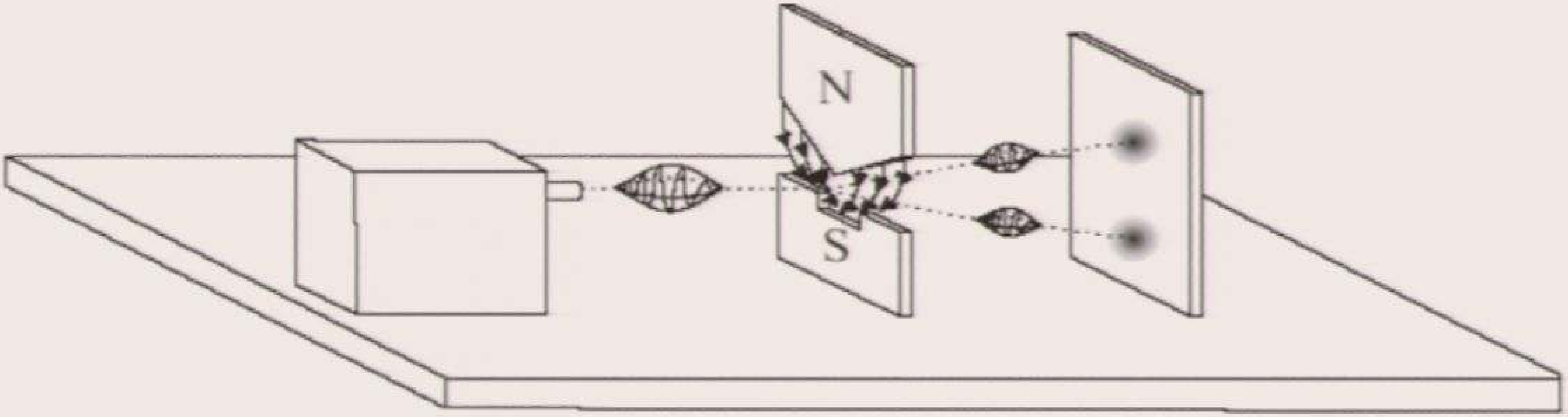
(2) Operational strategy: Eliminate "the physical state of a system" as a primitive concept and describe everything in terms of operational concepts

Elements of the formalism correspond to list of instructions of what to do in the laboratory

The operational strategy



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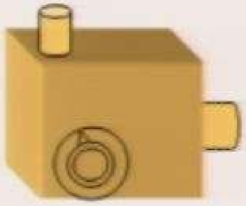
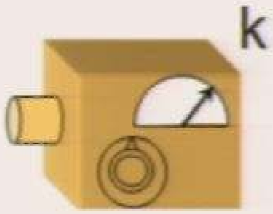
Operational Quantum Mechanics



Preparation

P



Measurement

M

Vector

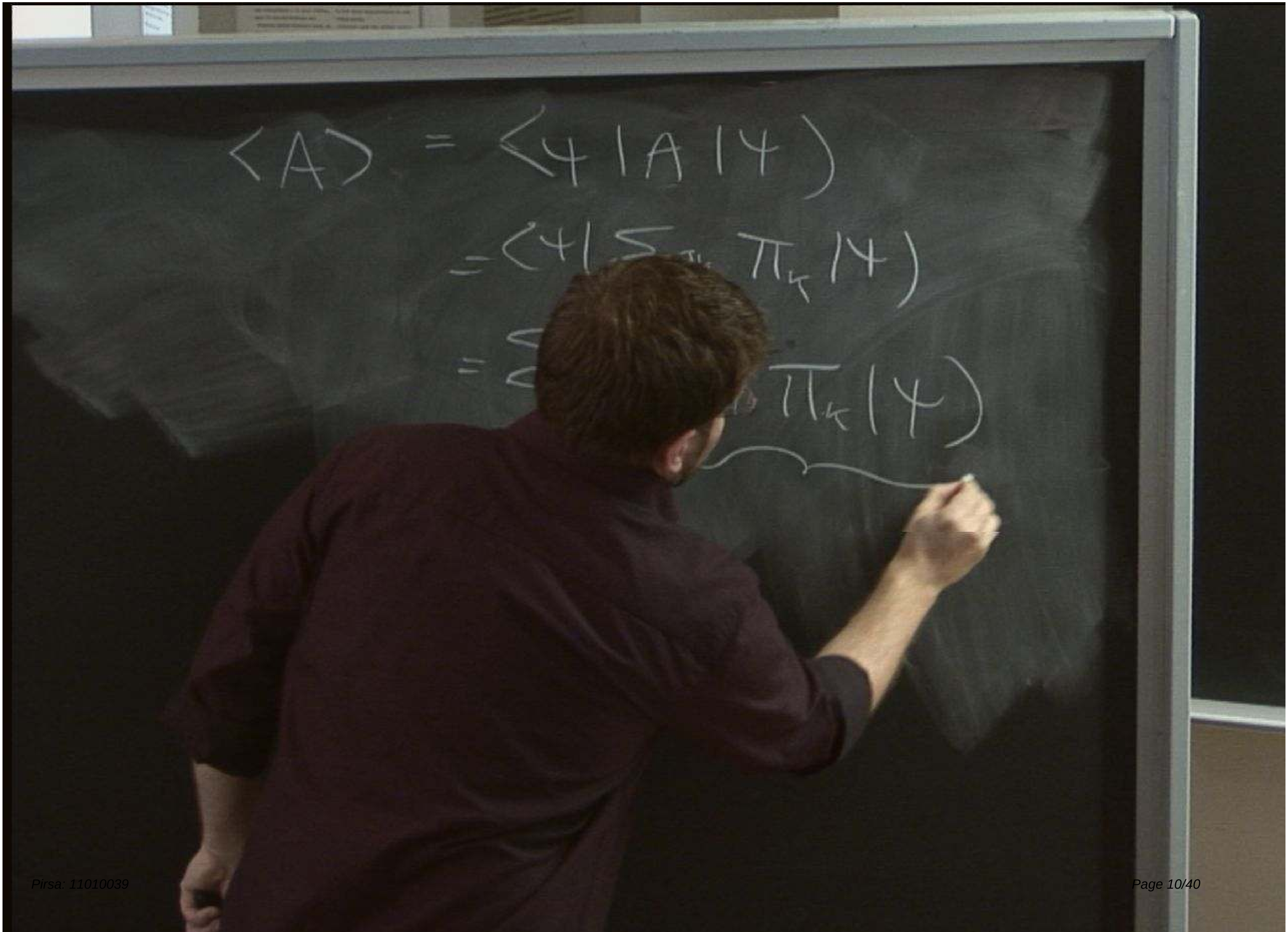
$|\psi\rangle$

Hermitian operator

A

$A = \sum_k a_k \Pi_k$

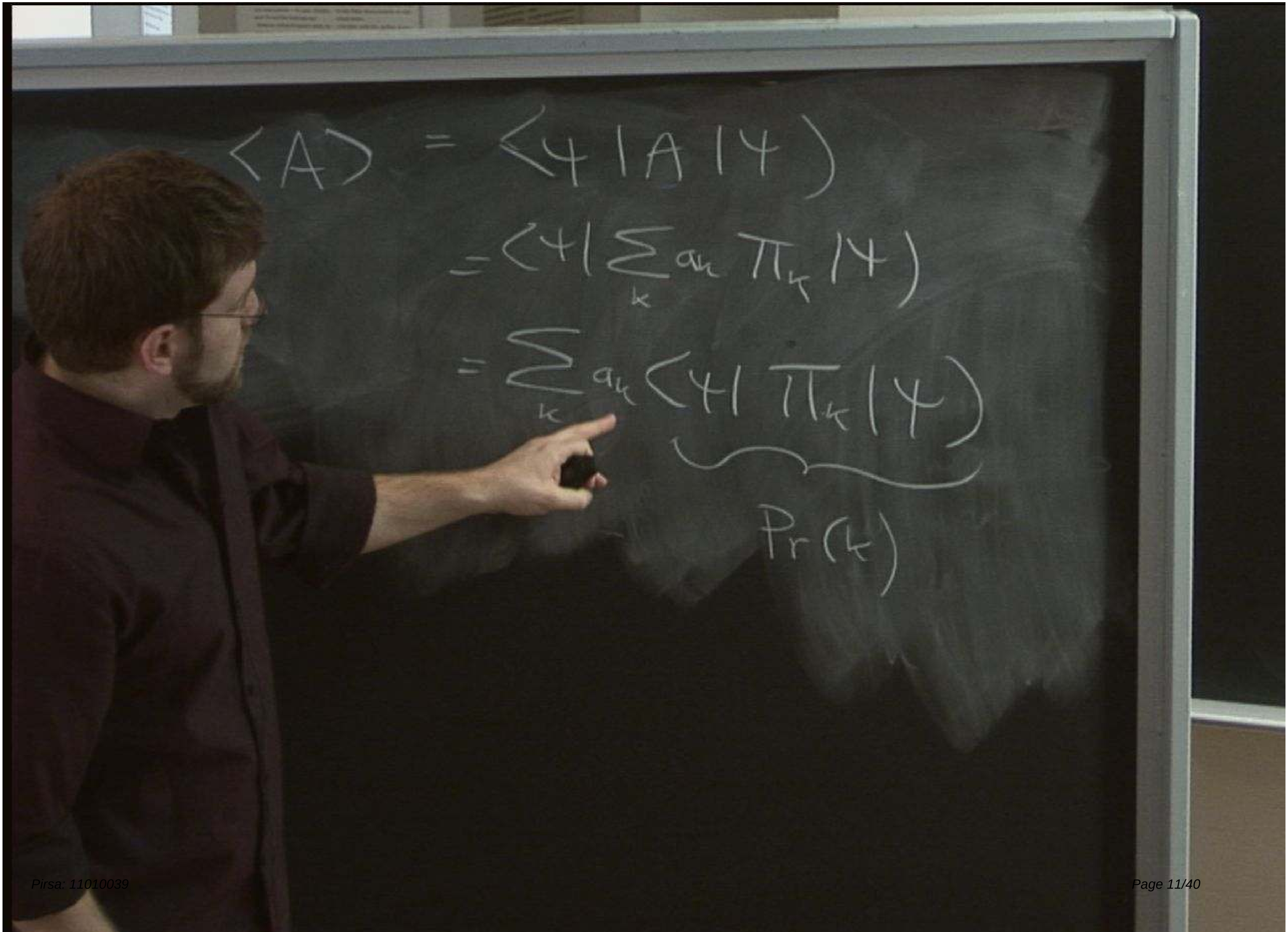
$$Pr(k|\mathsf{P},\mathsf{M}) = \langle\psi|\Pi_k|\psi\rangle$$



$$\langle A\rangle = \langle\psi|A|\psi\rangle$$

$$= \langle\psi|\sum_{\pi_k} \pi_k|\psi\rangle$$

$$= \sum \quad \pi_k|\psi\rangle$$

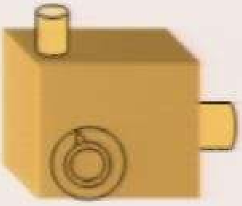


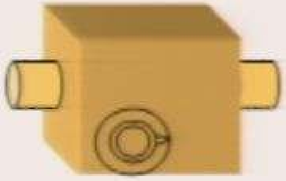
$$\langle A \rangle = \langle \psi | A | \psi \rangle$$

$$= \langle \psi | \sum_k a_k \Pi_k | \psi \rangle$$

$$= \sum_k a_k \underbrace{\langle \psi | \Pi_k | \psi \rangle}_{\Pr(k)}$$

Operational Quantum Mechanics

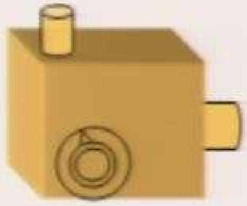


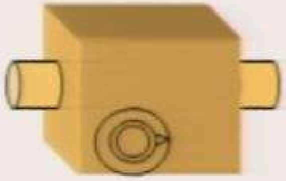


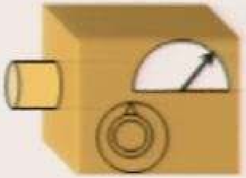
Preparation	Transformation	Measurement
P	T	M
Vector	Unitary map	Hermitian operator
$\|\psi\rangle$	U	A
		$A = \sum_k a_k \Pi_k$

$$\boxed{Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \langle\psi|U^\dagger \Pi_k U|\psi\rangle}$$

Operational Quantum Mechanics







Preparation

P

Transformation

T

Measurement

M

Vector

$|\psi\rangle$

Unitary map

U

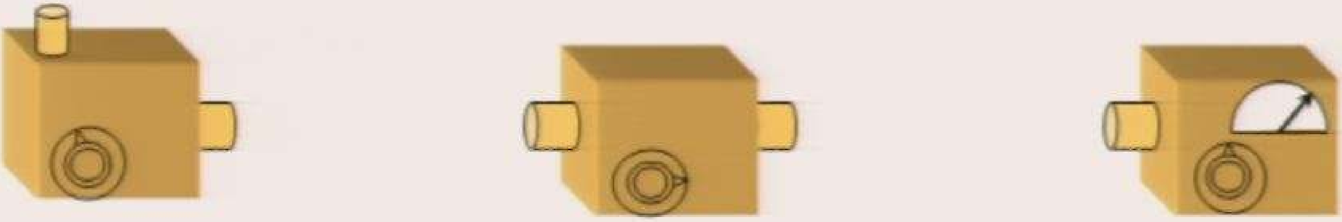
Hermitian operator

A

$A = \sum_k a_k \Pi_k$

$$Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \langle\psi|U^\dagger \Pi_k U|\psi\rangle$$

Operational Quantum Mechanics



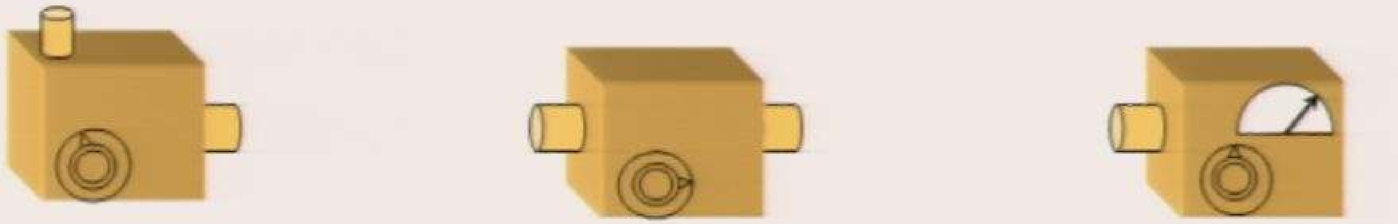
Preparation Transformation Measurement

$$\underbrace{\text{P} \qquad\qquad \text{T}}_{\substack{\text{Effective preparation} \\ \text{P}'}} \qquad\qquad \text{M}$$

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$$

$$\boxed{Pr(k|\text{P}',\text{M}) = \langle\psi'|\Pi_k|\psi'\rangle = \langle\psi|U^\dagger\Pi_k U|\psi\rangle}$$

Operational Quantum Mechanics



Preparation Transformation Measurement

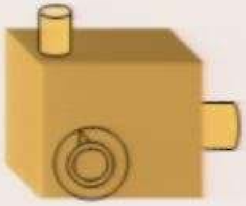
$$\underbrace{\mathsf{P} \qquad \mathsf{T}}_{\text{Effective preparation } \mathsf{P}'} \qquad \mathsf{M}$$

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$$

$$\boxed{Pr(k|\mathsf{P}',\mathsf{M}) = \langle\psi'|\Pi_k|\psi'\rangle = \langle\psi|U^\dagger\Pi_k U|\psi\rangle}$$

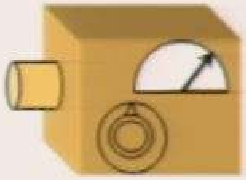
Is the operational interpretation satisfactory?

Operational Quantum Mechanics



Preparation

P



Measurement

M

Vector

$|\psi\rangle$

Hermitian operator

A

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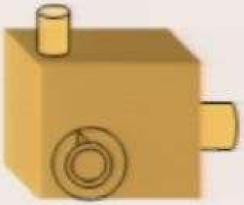
$$\boxed{Pr(k|\mathsf{P},\mathsf{M}) = \langle\psi|\Pi_k|\psi\rangle}$$

Operational Quantum Mechanics

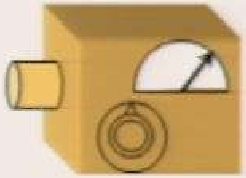


Preparation

P

Vector

$|\psi\rangle$



Measurement

M

Hermitian operator

A

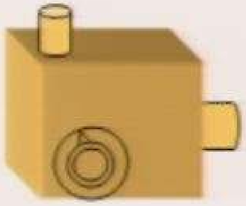
$A = \sum_k a_k \Pi_k$

$$\boxed{Pr(k|\mathsf{P},\mathsf{M}) = \langle\psi|\Pi_k|\psi\rangle}$$

The trace operation

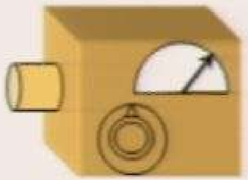
$$\mathrm{Tr}(A) \equiv \sum_k \langle k|A|k\rangle$$

Operational Quantum Mechanics



Preparation

P



Measurement

M

Density operator

ρ

Hermitian operator

A

$A = \sum_k a_k \Pi_k$

$$\boxed{Pr(k|\mathsf{P},\mathsf{M}) = \mathrm{Tr}(\rho\Pi_k)}$$

The trace operation

$$\mathrm{Tr}(A) \equiv \sum_k \langle k|A|k\rangle$$

The trace operation

$$\mathrm{Tr}(A) \equiv \sum_k \langle k|A|k\rangle$$

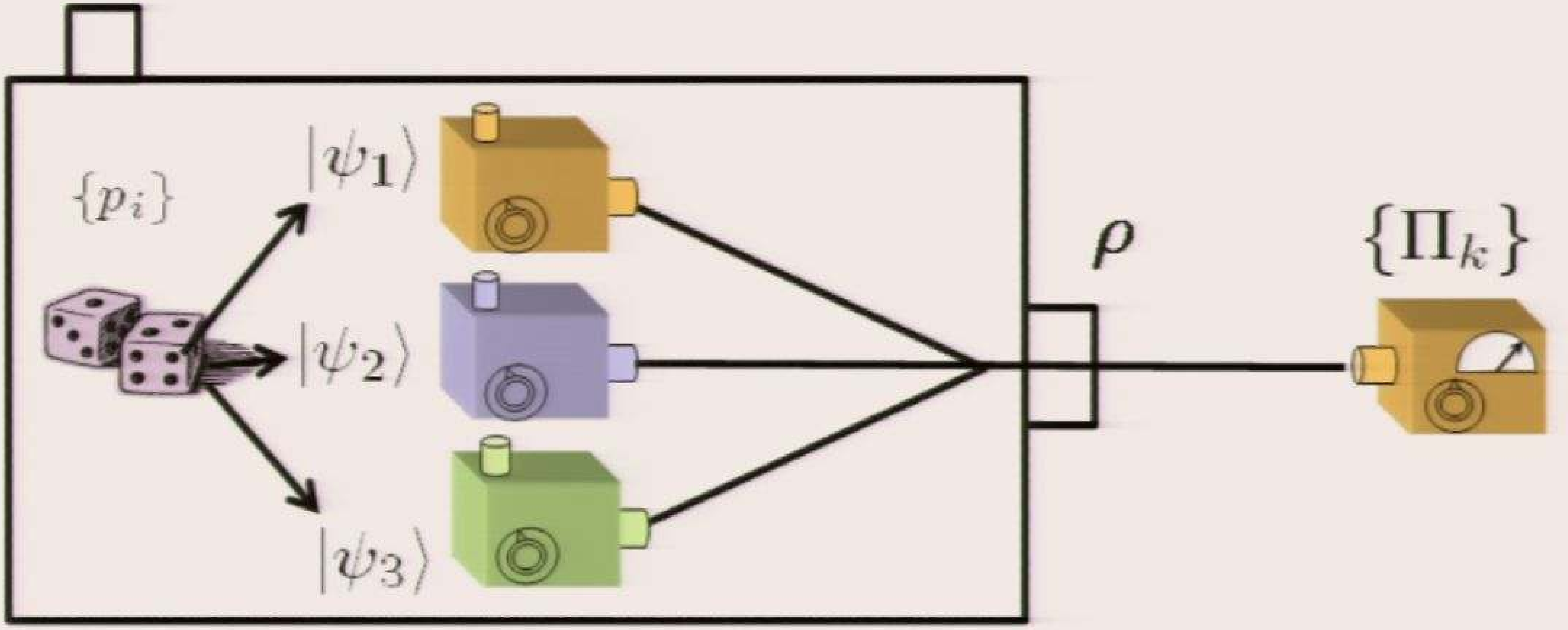
A useful identity $\mathrm{Tr}(A|\psi\rangle\langle\psi'|) = \langle\psi'|A|\psi\rangle$

Proof: $$\mathrm{Tr}(A|\psi\rangle\langle\psi'|) = \sum_k \langle k|A|\psi\rangle\langle\psi'|k\rangle$$
$$= \sum_k \langle\psi'|k\rangle\langle k|A|\psi\rangle$$
$$= \langle\psi'|\left(\sum_k |k\rangle\langle k|\right)A|\psi\rangle$$
$$= \langle\psi'|A|\psi\rangle$$

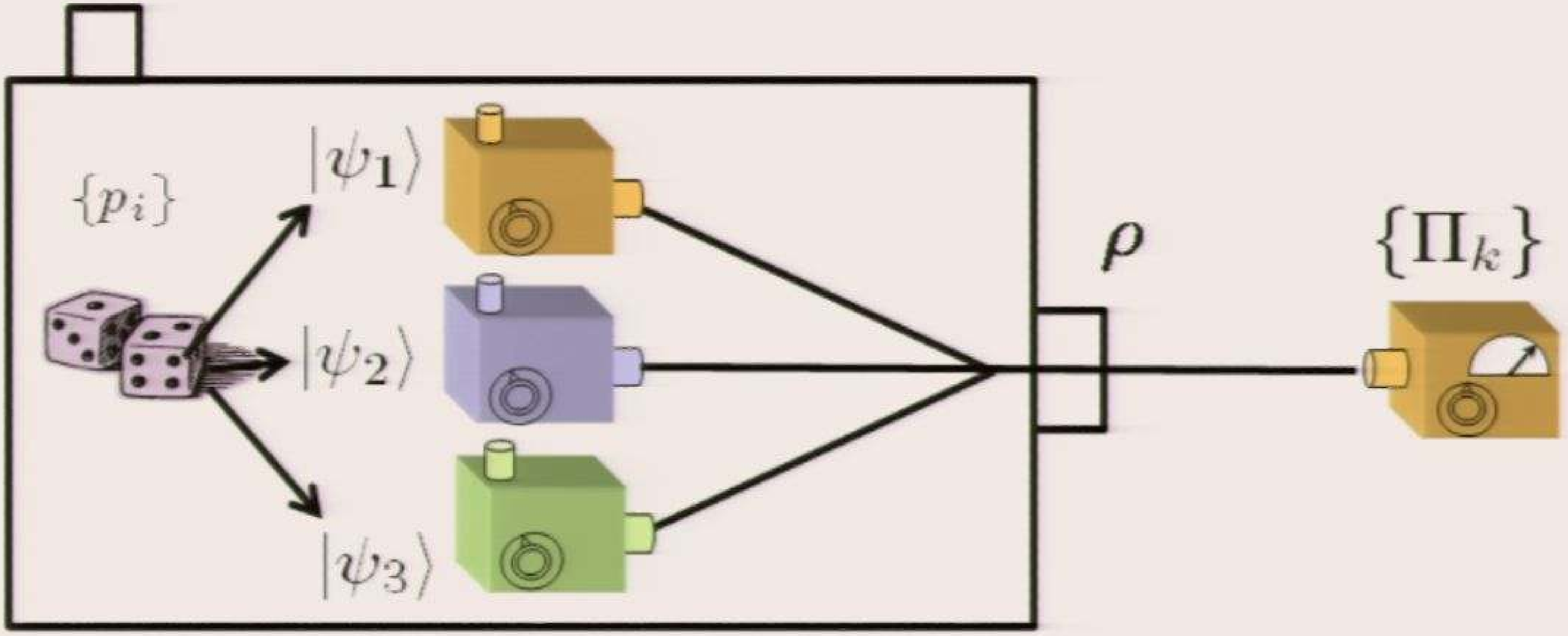
In particular $\mathrm{Tr}(|\psi\rangle\langle\psi'|) = \langle\psi'|\psi\rangle$

Proof: $\mathrm{Tr}(|\psi\rangle\langle\psi'|) = \mathrm{Tr}(I|\psi'\rangle\langle\psi|)$

Ensembles of quantum states



Ensembles of quantum states



$$p(k) = \textstyle\sum_i p(k|i)p(i)$$
$$= \textstyle\sum_i \langle\psi_i|\Pi_k|\psi_i\rangle p_i$$
$$= \textstyle\sum_i \mathrm{Tr}(\Pi_k|\psi_i\rangle\langle\psi_i|)p_i$$
$$= \mathrm{Tr}\left[\Pi_k \left(\textstyle\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)\right]$$

An ensemble of states $\{(p_i, |\psi_i\rangle)\}$

is a set of normalized vectors $\langle\psi_i|\psi_i\rangle = 1$ (not necessarily orthogona

and probabilistic weights $p_i \geq 0$ and $\sum_i p_i = 1$

Every ensemble defines a density operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

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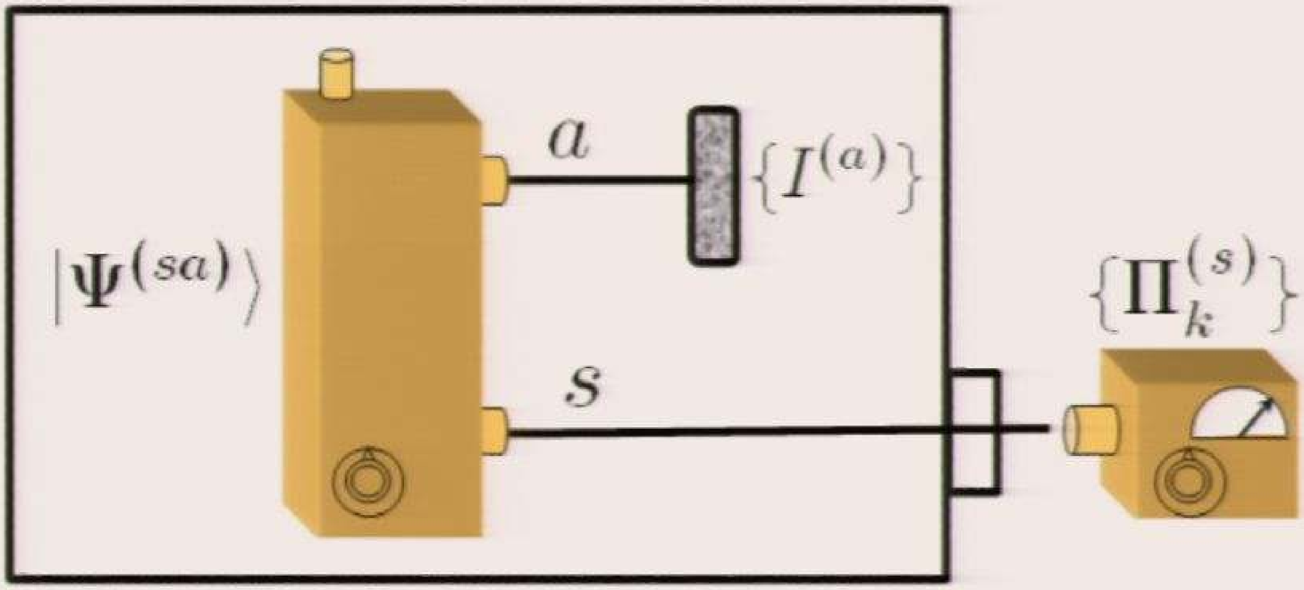
For arbitrary $|\psi\rangle$

$$\begin{aligned}\langle\psi|\rho|\psi\rangle &= \sum_i p_i \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle \\ &= \sum_i p_i |\langle\psi|\psi_i\rangle|^2 \\ &\geq 0\end{aligned}$$

Therefore $\langle\psi|\rho|\psi\rangle \geq \mathbf{0} \ \forall |\psi\rangle \in \mathcal{H}$ Positivity

Reduced density operators



The partial trace operation

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First recall the partial inner product

$$|\psi^{(s)}\rangle \in \mathcal{H}_s$$

The partial trace operation

First recall the partial inner product

$$|\psi^{(s)}\rangle \in \mathcal{H}_s$$

$$|\mathbf{\Psi}^{(sa)}\rangle \in \mathcal{H}_s \otimes \mathcal{H}_a$$

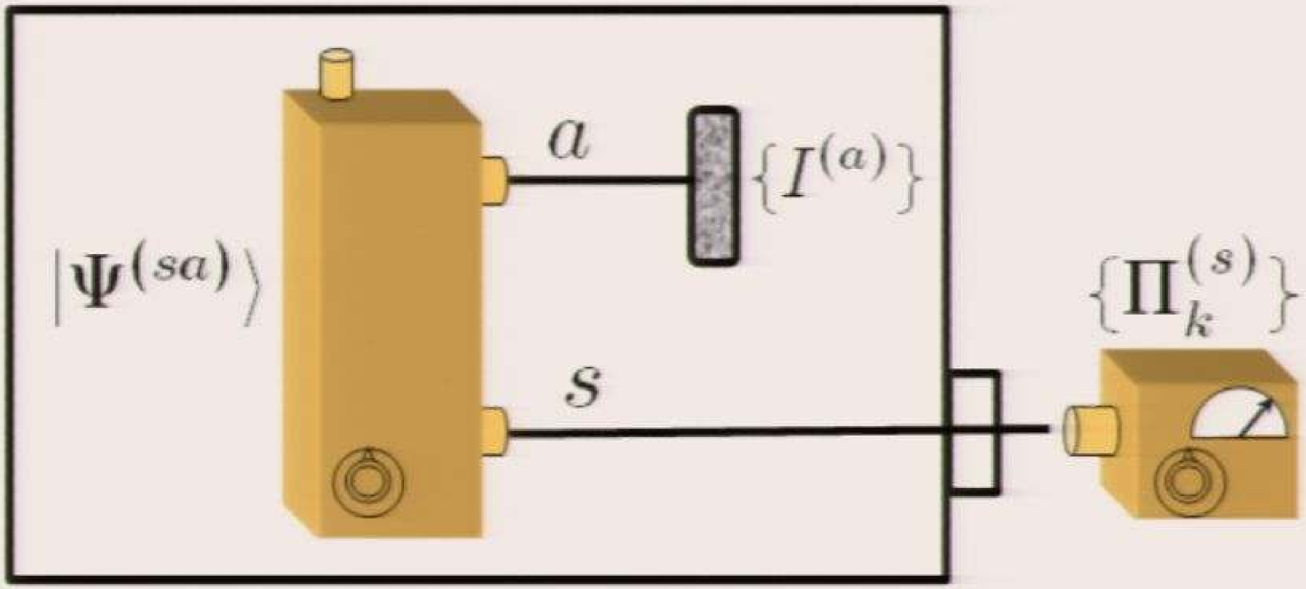
$$\langle\psi^{(s)}|\Psi^{(sa)}\rangle \equiv |\chi^{(a)}\rangle \in \mathcal{H}_a$$

For instance, if $|\Psi^{(sa)}\rangle = \sum_k \sqrt{\lambda_k}\, |\mu_k^{(s)}\rangle \otimes |\nu_k^{(a)}\rangle$

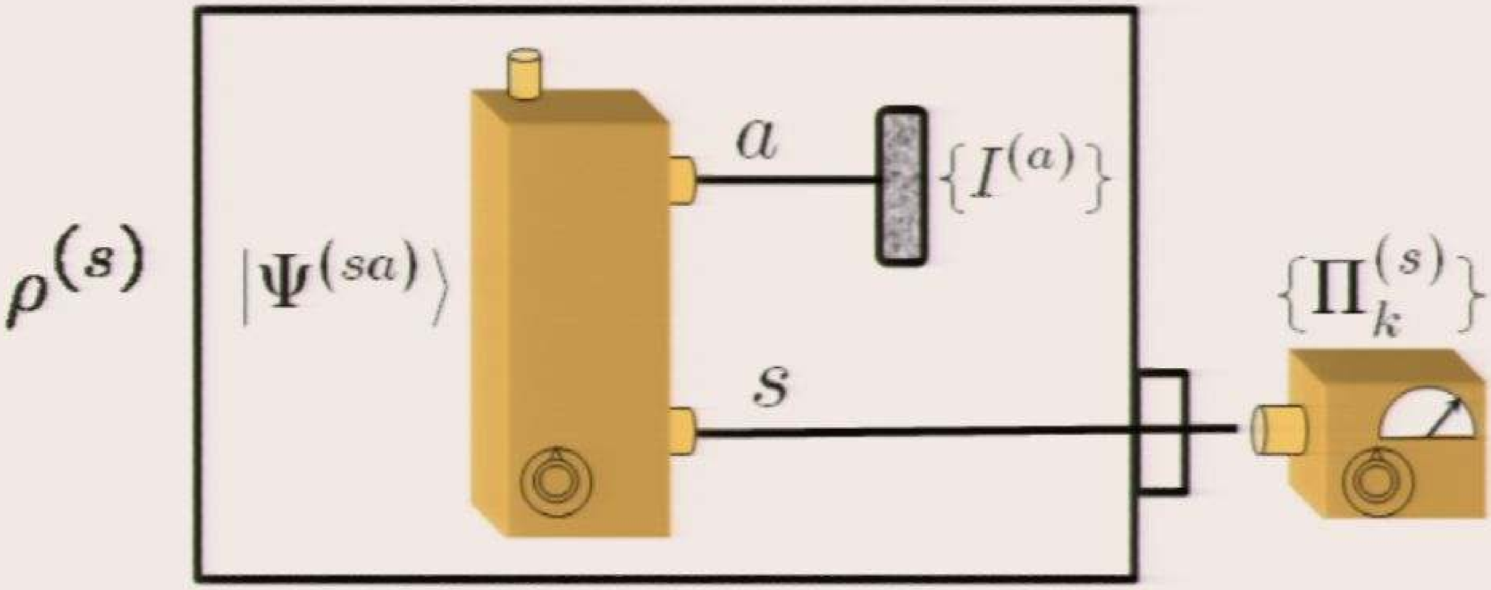
Then $\langle\psi^{(s)}|\Psi^{(sa)}\rangle = \sum_k \sqrt{\lambda_k}\, \left(\langle\psi^{(s)}|\mu_k^{(s)}\rangle\right) |\nu_k^{(a)}\rangle$

Note that $\mathrm{Tr}_{sa}(A^{(sa)}) = \mathrm{Tr}_s[\mathrm{Tr}_a(A^{(sa)})]$

Reduced density operators



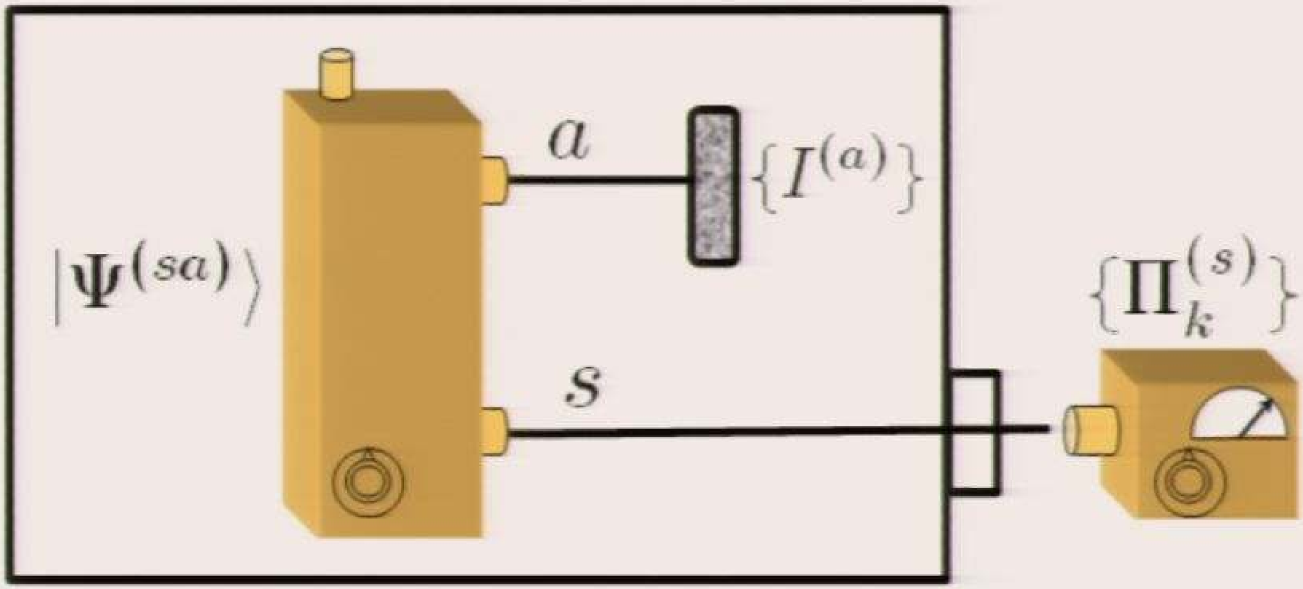
Reduced density operators



$$\begin{aligned}\mathrm{Pr}(k) &= \mathrm{Tr}_{sa}[(\Pi_k^{(s)} \otimes I^{(a)})|\Psi^{(sa)}\rangle\langle\Psi^{(sa)}|] \\ &= \mathrm{Tr}_s\left[\Pi_k^{(s)}\left[\mathrm{Tr}_a(|\Psi^{(sa)}\rangle\langle\Psi^{(sa)}|)\right]\right] \\ &= \mathrm{Tr}_s(\Pi_k^{(s)}\rho^{(s)})\end{aligned}$$

where $\rho^{(s)} \equiv \mathrm{Tr}_a(|\Psi^{(sa)}\rangle\langle\Psi^{(sa)}|)$

Reduced density operators



Note that $\mathrm{Tr}_{sa}(A^{(sa)}) = \mathrm{Tr}_s[\mathrm{Tr}_a(A^{(sa)})]$

Proof: $\mathrm{Tr}_{sa}(A^{(sa)}) = \sum_{j,k}(\langle j^{(s)}|\otimes\langle k^{(a)}|)A^{(sa)}(|j^{(s)}\rangle\otimes|k^{(a)}\rangle)$

The partial trace operation

First recall the partial inner product

$$|\psi^{(s)}\rangle \in \mathcal{H}_s$$

$$|\Psi^{(sa)}\rangle \in \mathcal{H}_s \otimes \mathcal{H}_a$$

$$\langle\psi^{(s)}|\Psi^{(sa)}\rangle \equiv |\chi^{(a)}\rangle \in \mathcal{H}_a$$

For instance, if $|\Psi^{(sa)}\rangle = \sum_k \sqrt{\lambda_k}\, |\mu_k^{(s)}\rangle \otimes |\nu_k^{(a)}\rangle$

Then $\langle\psi^{(s)}|\Psi^{(sa)}\rangle = \sum_k \sqrt{\lambda_k}\, \left(\langle\psi^{(s)}|\mu_k^{(s)}\rangle\right) |\nu_k^{(a)}\rangle$

The partial trace on $\mathcal{H}_s$

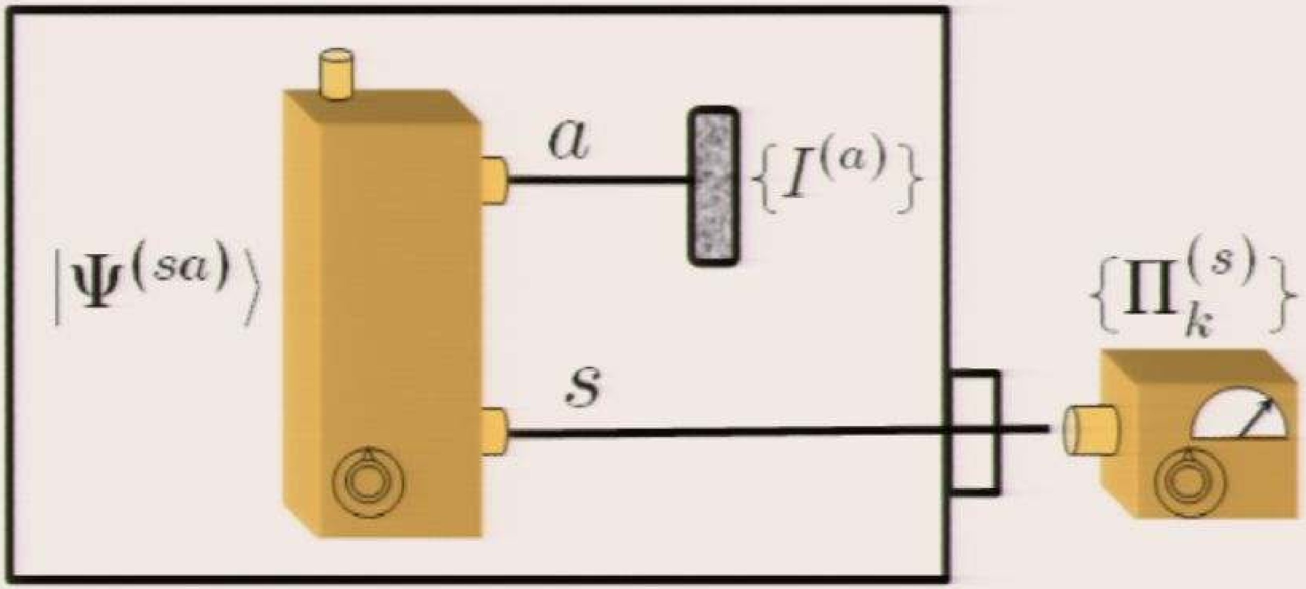
$$\mathrm{Tr}_s(A^{(sa)}) \equiv \sum_k \langle k^{(s)}|A^{(sa)}|k^{(s)}\rangle$$

The partial trace operation

First recall the partial inner product

$$|\psi^{(s)}\rangle \in \mathcal{H}_s$$

Reduced density operators



$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

For arbitrary $|\psi\rangle$

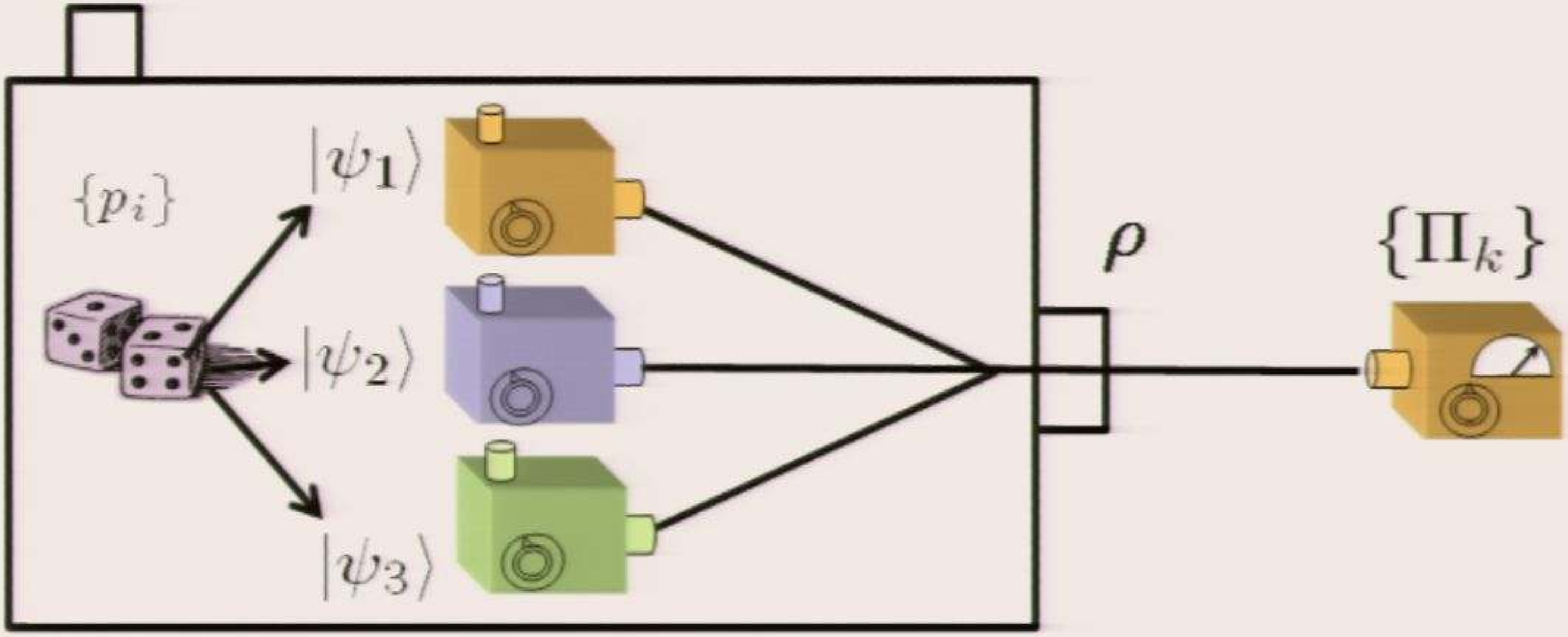
$$\langle\psi|\rho|\psi\rangle = \sum_i p_i \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle$$

$$= \sum_i p_i |\langle\psi|\psi_i\rangle|^2$$

$$\geq 0$$

Therefore $\langle\psi|\rho|\psi\rangle \geq \mathbf{0} \ \forall|\psi\rangle \in \mathcal{H}$ Positivity

Ensembles of quantum states



$$
\begin{aligned}
p(k) &= \textstyle\sum_i p(k|i)p(i) \\
&= \textstyle\sum_i \langle\psi_i|\Pi_k|\psi_i\rangle p_i \\
&= \textstyle\sum_i \mathrm{Tr}(\Pi_k|\psi_i\rangle\langle\psi_i|)p_i \\
&= \mathrm{Tr}\left[\Pi_k\left(\textstyle\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)\right]
\end{aligned}
$$