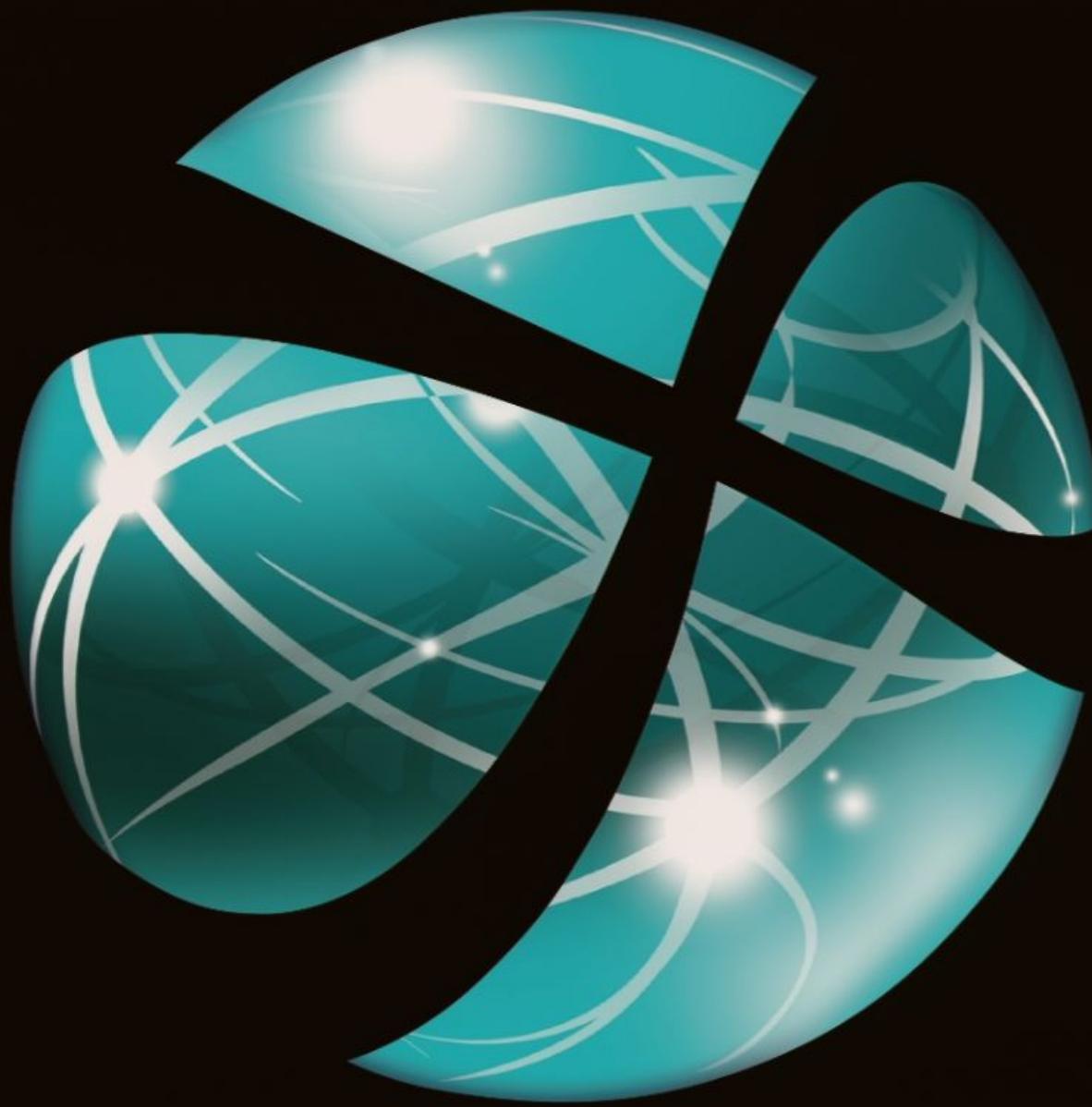


Title: Foundations of Quantum Mechanics - Lecture 2

Date: Jan 04, 2011 11:30 AM

URL: <http://pirsa.org/11010039>

Abstract:



perimeter scholars
international

Towards a purely operational formulation of quantum theory

Towards a purely operational formulation of quantum theory

“Orthodox” postulates of quantum theory

Representational completeness of ψ . The rays of Hilbert space correspond one-to-one with the **physical states** of the system.

Measurement. If the Hermitian operator A with spectral projectors $\{P_k\}$ is measured, the probability of outcome k is $\langle \psi | P_k | \psi \rangle$. These **probabilities are objective -- indeterminism**.

Evolution of isolated systems. It is unitary, $|\psi\rangle \rightarrow U|\psi\rangle = e^{-\frac{i}{\hbar}Ht}|\psi\rangle$ therefore **deterministic and continuous**.

Evolution of systems undergoing measurement. If Hermitian operator A with spectral projectors $\{P_k\}$ is measured and outcome k is obtained, the physical state of the system **changes discontinuously**,

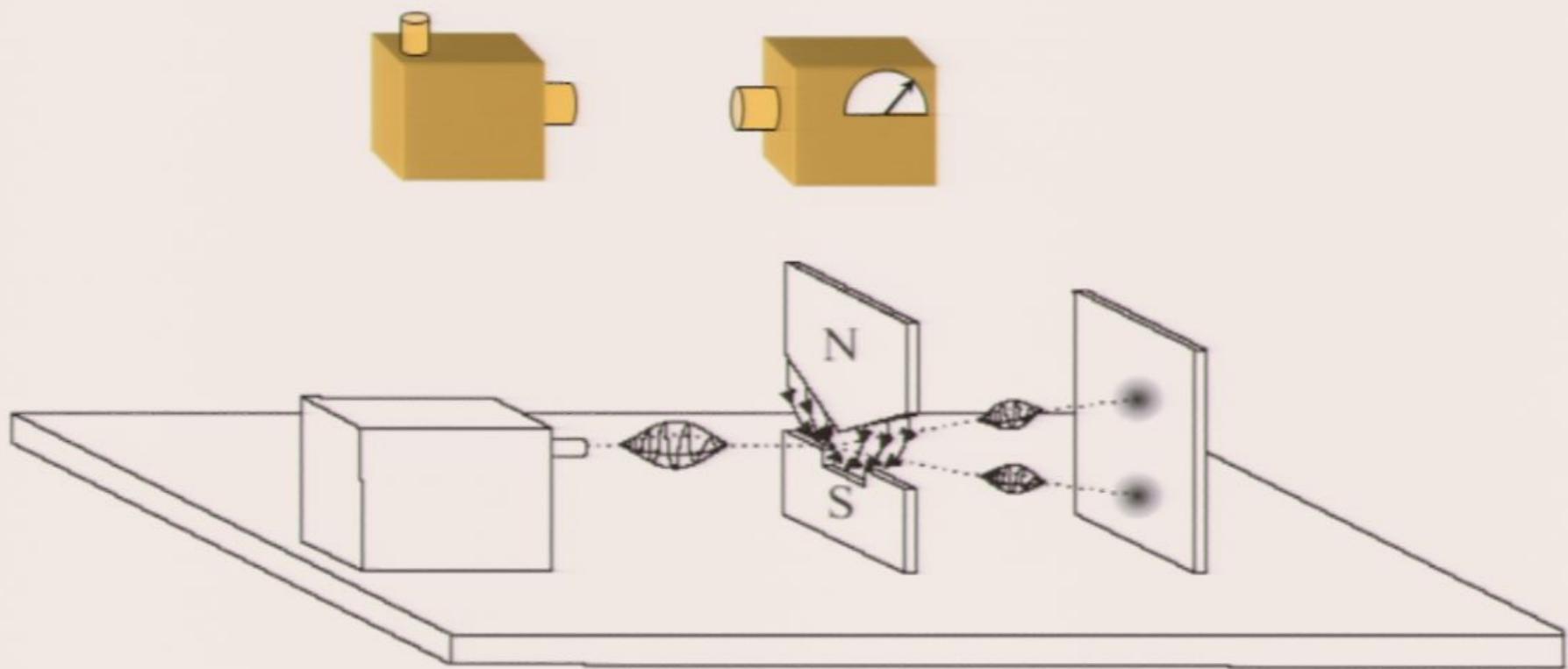
$$|\psi\rangle \rightarrow |\psi_k\rangle = \frac{P_k|\psi\rangle}{\sqrt{\langle\psi|P_k|\psi\rangle}}$$

First problem: the term “measurement” is not defined in terms of the more primitive “physical states of systems”. Isn’t a measurement just another kind of physical interaction?

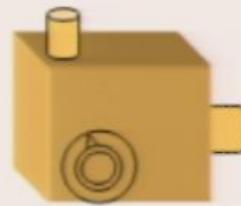
Two strategies:

- (1) **Realist strategy:** Eliminate measurement as a primitive concept and describe everything in terms of physical states
- (2) **Operational strategy:** Eliminate “the physical state of a system” as a primitive concept and describe everything in terms of operational concepts
Elements of the formalism correspond to list of instructions of what to do in the laboratory

The operational strategy



Operational Quantum Mechanics



Preparation

P



Measurement

M

Vector

$|\psi\rangle$

Hermitian operator

A

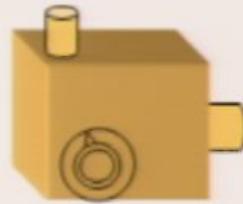
$$A = \sum_k a_k \Pi_k$$

$$Pr(k|P, M) = \langle \psi | \Pi_k | \psi \rangle$$

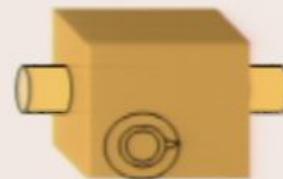
$$\begin{aligned}\langle A \rangle &= \langle + | A | + \rangle \\&= \langle + | \sum_{\pi_k} \pi_k | + \rangle \\&= \langle + | \underbrace{\pi_k}_{\text{operator}} | + \rangle\end{aligned}$$

$$\begin{aligned}\langle A \rangle &= \langle + | A | + \rangle \\&= \langle + | \sum_k a_k \pi_k | + \rangle \\&= \sum_k a_k \underbrace{\langle + | \pi_k | + \rangle}_{\Pr(k)}\end{aligned}$$

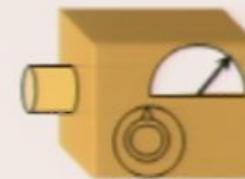
Operational Quantum Mechanics



Preparation
P



Transformation
T



Measurement
M

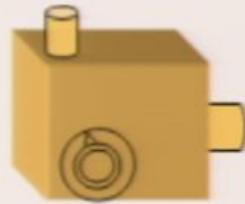
Vector
 $|\psi\rangle$

Unitary map
 U

Hermitian operator
 A
 $A = \sum_k a_k \Pi_k$

$$Pr(k|P, T, M) = \langle \psi | U^\dagger \Pi_k U | \psi \rangle$$

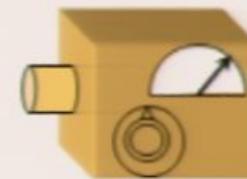
Operational Quantum Mechanics



Preparation
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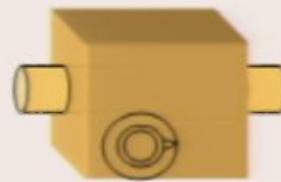
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Operational Quantum Mechanics



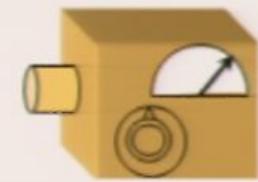
Preparation

P



Transformation

T



Measurement

M

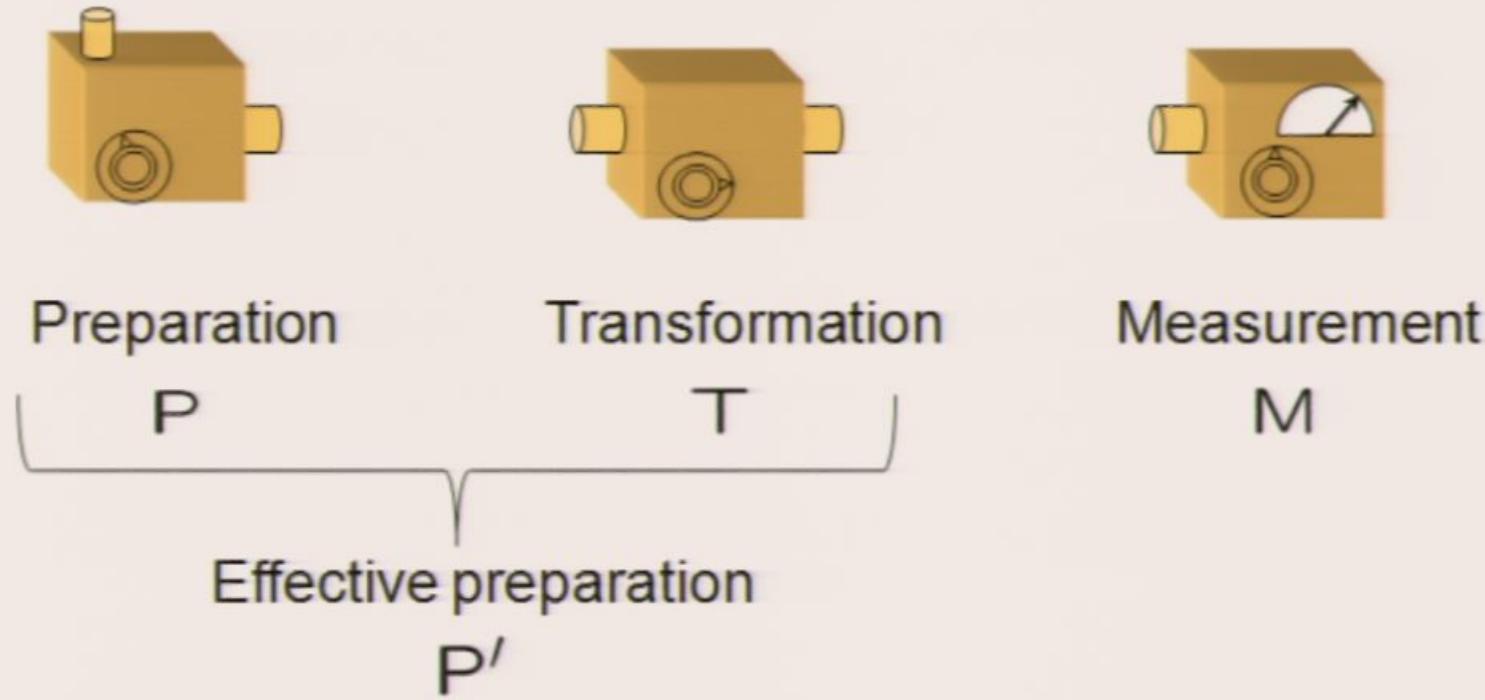
Effective preparation

P'

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$$

$$Pr(k|P' M) = \langle \psi' | \Pi_k | \psi' \rangle = \langle \psi | U^\dagger \Pi_k U | \psi \rangle$$

Operational Quantum Mechanics



$$Pr(k|P' M) = \langle \psi' | \Pi_k | \psi' \rangle = \langle \psi | U^\dagger \Pi_k U | \psi \rangle$$

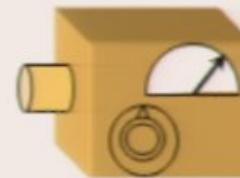
Is the operational interpretation satisfactory?

Operational Quantum Mechanics



Preparation

P



Measurement

M

Vector

$|\psi\rangle$

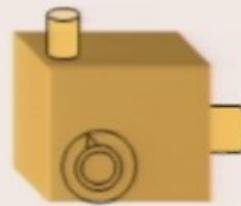
Hermitian operator

A

$$A = \sum_k a_k \Pi_k$$

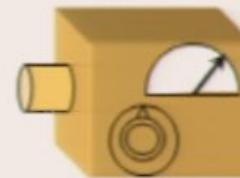
$$Pr(k|P, M) = \langle \psi | \Pi_k | \psi \rangle$$

Operational Quantum Mechanics



Preparation

P



Measurement

M

Vector

$|\psi\rangle$

Hermitian operator

A

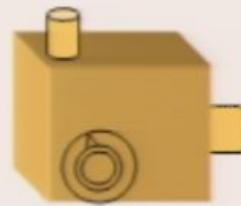
$$A = \sum_k a_k \Pi_k$$

$$Pr(k|P, M) = \langle \psi | \Pi_k | \psi \rangle$$

The trace operation

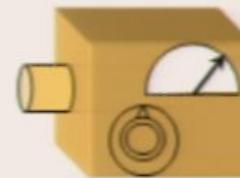
$$\text{Tr}(A) \equiv \sum_k \langle k | A | k \rangle$$

Operational Quantum Mechanics



Preparation

P



Measurement

M

Density operator

ρ

Hermitian operator

A

$$A = \sum_k a_k \Pi_k$$

$$Pr(k|P, M) = \text{Tr}(\rho \Pi_k)$$

The trace operation

$$\text{Tr}(A) \equiv \sum_k \langle k | A | k \rangle$$

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$$\text{Tr}(A) \equiv \sum_k \langle k | A | k \rangle$$

A useful identity $\text{Tr}(A|\psi\rangle\langle\psi'|) = \langle\psi'|A|\psi\rangle$

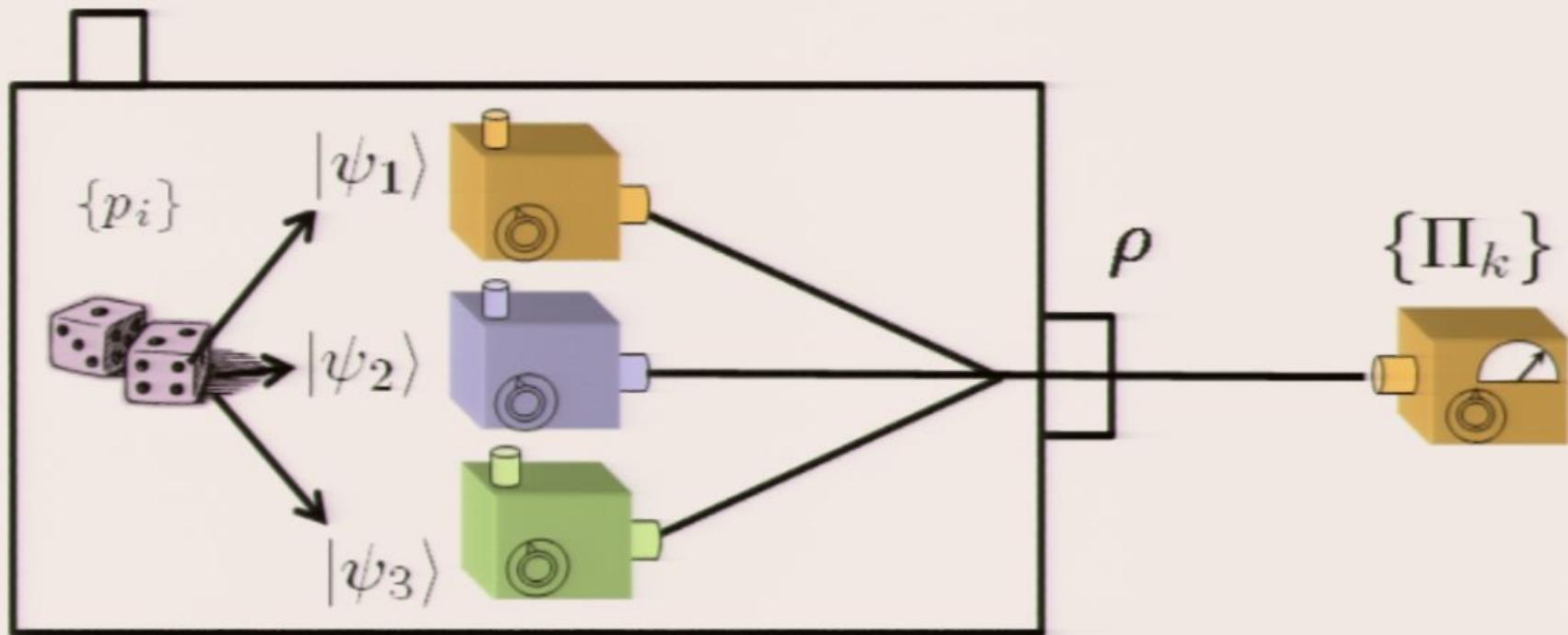
Proof:

$$\begin{aligned}\text{Tr}(A|\psi\rangle\langle\psi'|) &= \sum_k \langle k | A | \psi \rangle \langle \psi' | k \rangle \\&= \sum_k \langle \psi' | k \rangle \langle k | A | \psi \rangle \\&= \langle \psi' | \left(\sum_k |k\rangle\langle k| \right) A | \psi \rangle \\&= \langle \psi' | A | \psi \rangle\end{aligned}$$

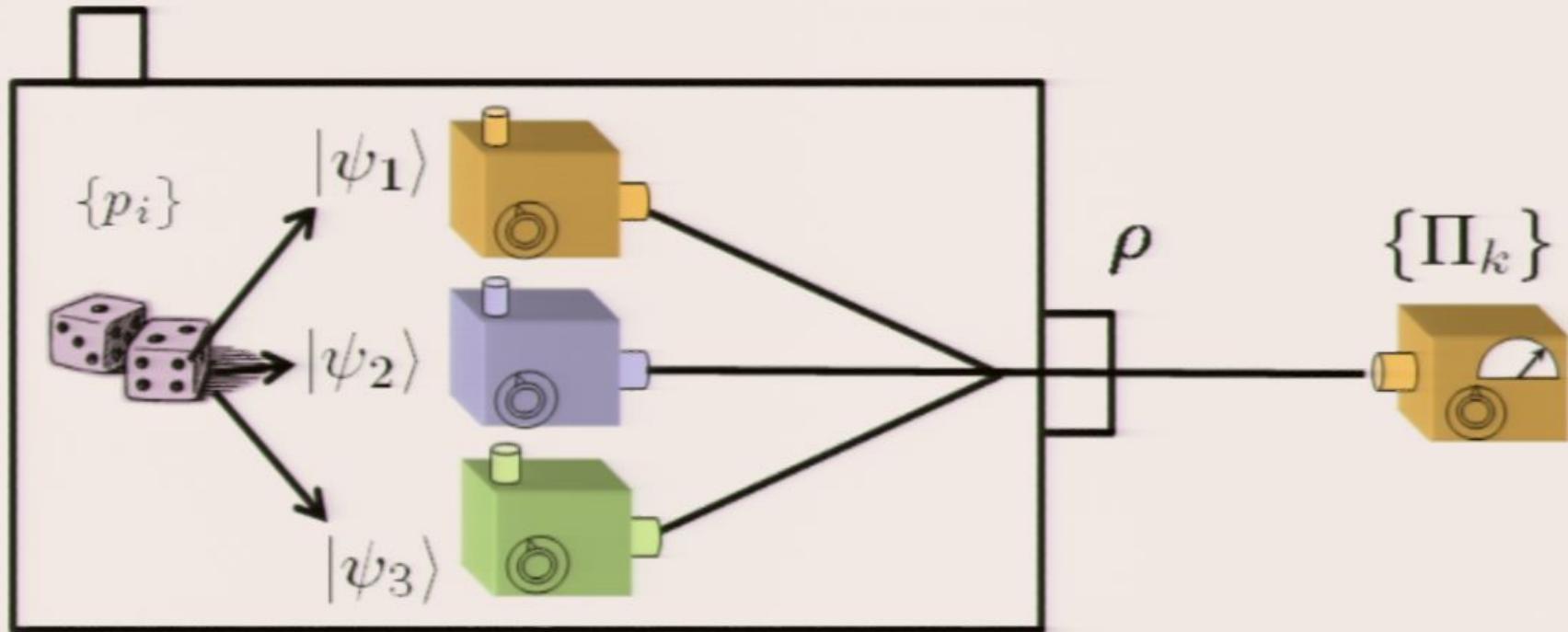
In particular $\text{Tr}(|\psi\rangle\langle\psi'|) = \langle\psi'|\psi\rangle$

Proof: $\text{Tr}(|\psi\rangle\langle\psi'|) = \text{Tr}(I|\psi'\rangle\langle\psi|)$

Ensembles of quantum states



Ensembles of quantum states



$$\begin{aligned} p(k) &= \sum_i p(k|i)p(i) \\ &= \sum_i \langle \psi_i | \Pi_k | \psi_i \rangle p_i \\ &= \sum_i \text{Tr}(\Pi_k | \psi_i \rangle \langle \psi_i |) p_i \\ &= \text{Tr} [\Pi_k (\sum_i p_i | \psi_i \rangle \langle \psi_i |)] \end{aligned}$$

An **ensemble** of states $\{(p_i, |\psi_i\rangle)\}$

is a set of normalized vectors $\langle\psi_i|\psi_i\rangle = 1$ (not necessarily orthogonal)

and probabilistic weights $p_i \geq 0$ and $\sum_i p_i = 1$

Every ensemble defines a **density operator**

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

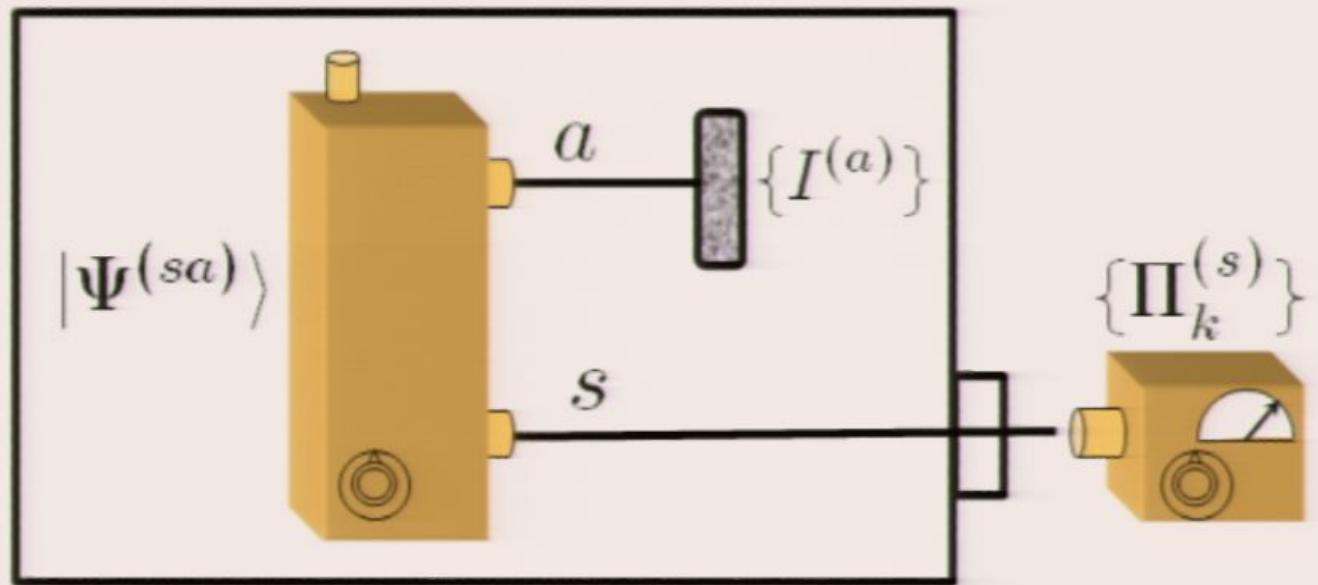
$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

For arbitrary $|\psi\rangle$

$$\begin{aligned}\langle\psi|\rho|\psi\rangle &= \sum_i p_i \langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle \\ &= \sum_i p_i |\langle\psi|\psi_i\rangle|^2 \\ &\geq 0\end{aligned}$$

Therefore $\langle\psi|\rho|\psi\rangle \geq 0 \quad \forall |\psi\rangle \in \mathcal{H}$ Positivity

Reduced density operators



The partial trace operation

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First recall the **partial inner product**

$$|\psi^{(s)}\rangle \in \mathcal{H}_s$$

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$$|\psi^{(s)}\rangle \in \mathcal{H}_s$$

$$|\Psi^{(sa)}\rangle \in \mathcal{H}_s \otimes \mathcal{H}_a$$

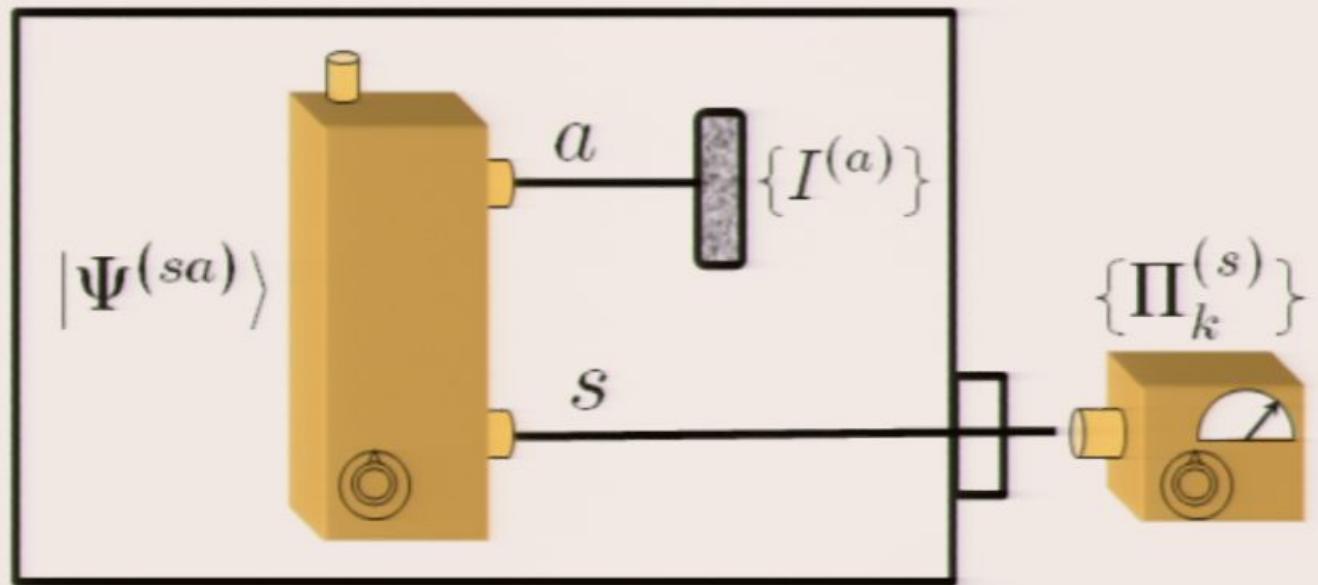
$$\langle \psi^{(s)} | \Psi^{(sa)} \rangle \equiv |\chi^{(a)}\rangle \in \mathcal{H}_a$$

For instance, if $|\Psi^{(sa)}\rangle = \sum_k \sqrt{\lambda_k} |\mu_k^{(s)}\rangle \otimes |\nu_k^{(a)}\rangle$

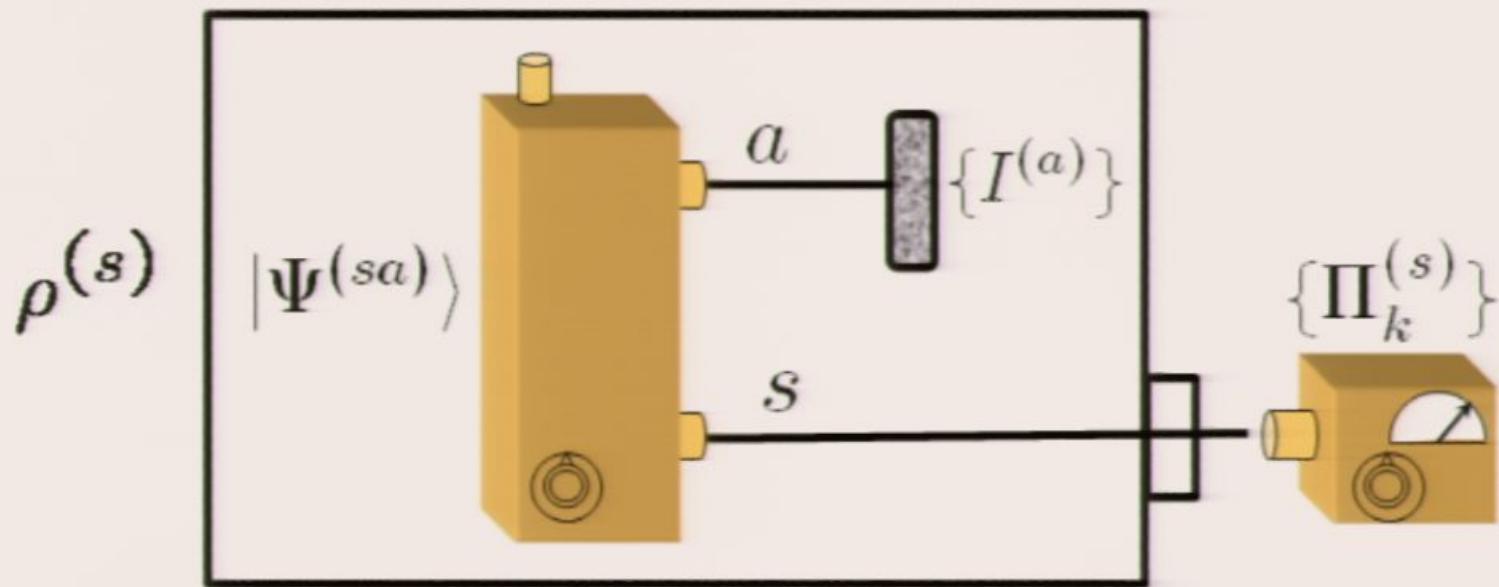
$$\text{Then } \langle \psi^{(s)} | \Psi^{(sa)} \rangle = \sum_k \sqrt{\lambda_k} \left(\langle \psi^{(s)} | \mu_k^{(s)} \rangle \right) |\nu_k^{(a)}\rangle$$

Note that $\text{Tr}_{sa}(A^{(sa)}) = \text{Tr}_s[\text{Tr}_a(A^{(sa)})]$

Reduced density operators

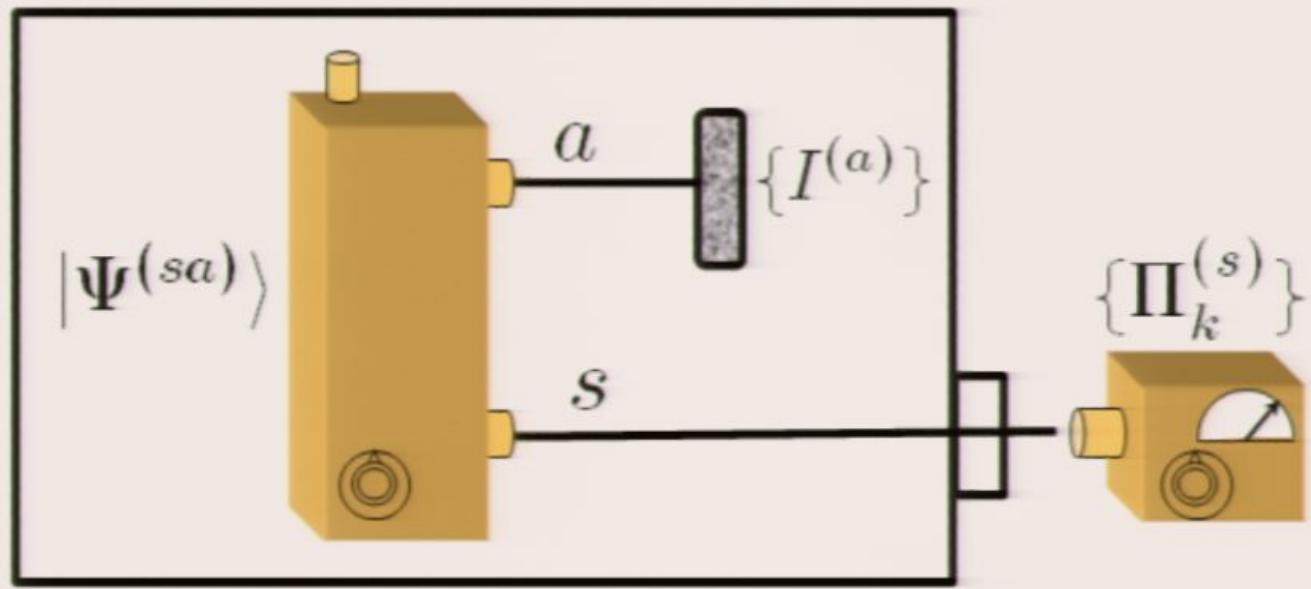


Reduced density operators



$$\begin{aligned}\Pr(k) &= \text{Tr}_{sa}[(\Pi_k^{(s)} \otimes I^{(a)}) |\Psi^{(sa)}\rangle \langle \Psi^{(sa)}|] \\ &= \text{Tr}_s \left[\Pi_k^{(s)} \left[\text{Tr}_a (|\Psi^{(sa)}\rangle \langle \Psi^{(sa)}|) \right] \right] \\ &= \text{Tr}_s (\Pi_k^{(s)} \rho^{(s)})\end{aligned}$$

Reduced density operators



Note that $\text{Tr}_{sa}(A^{(sa)}) = \text{Tr}_s[\text{Tr}_a(A^{(sa)})]$

Proof: $\text{Tr}_{sa}(A^{(sa)}) = \sum_{j,k} (\langle j^{(s)} | \otimes \langle k^{(a)} |) A^{(sa)} (| j^{(s)} \rangle \otimes | k^{(a)} \rangle)$

The partial trace operation

First recall the **partial inner product**

$$|\psi^{(s)}\rangle \in \mathcal{H}_s$$

$$|\Psi^{(sa)}\rangle \in \mathcal{H}_s \otimes \mathcal{H}_a$$

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The **partial trace** on \mathcal{H}_s

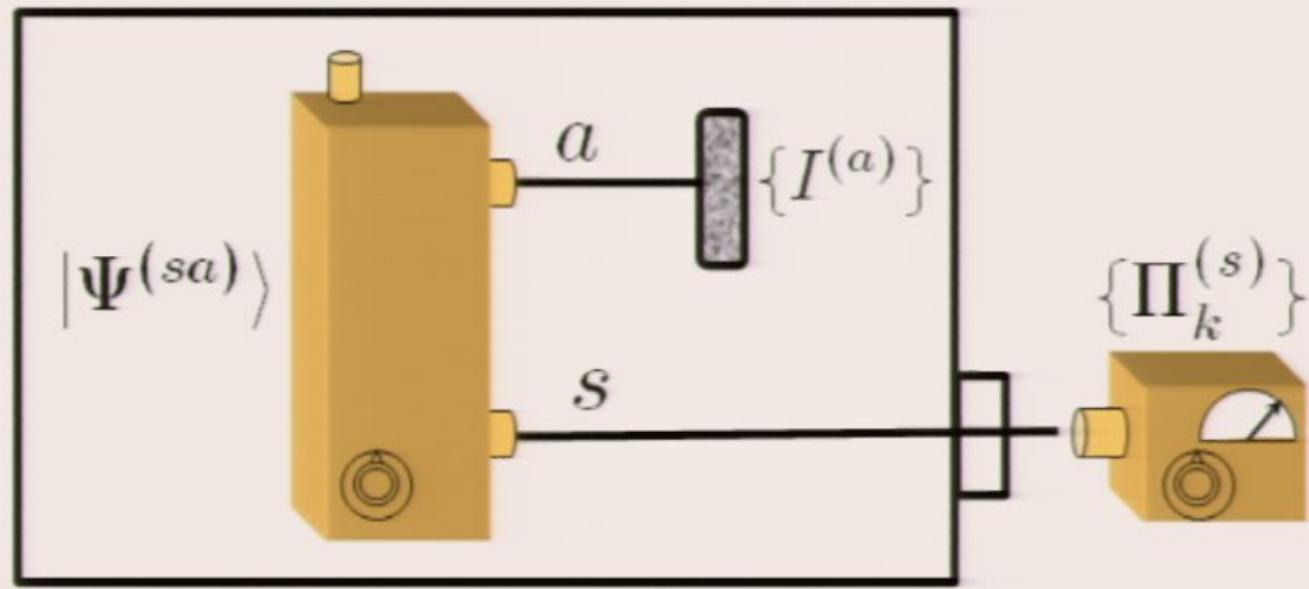
$$\text{Tr}_s(A^{(sa)}) \equiv \sum_k \langle k^{(s)} | A^{(sa)} | k^{(s)} \rangle$$

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Reduced density operators



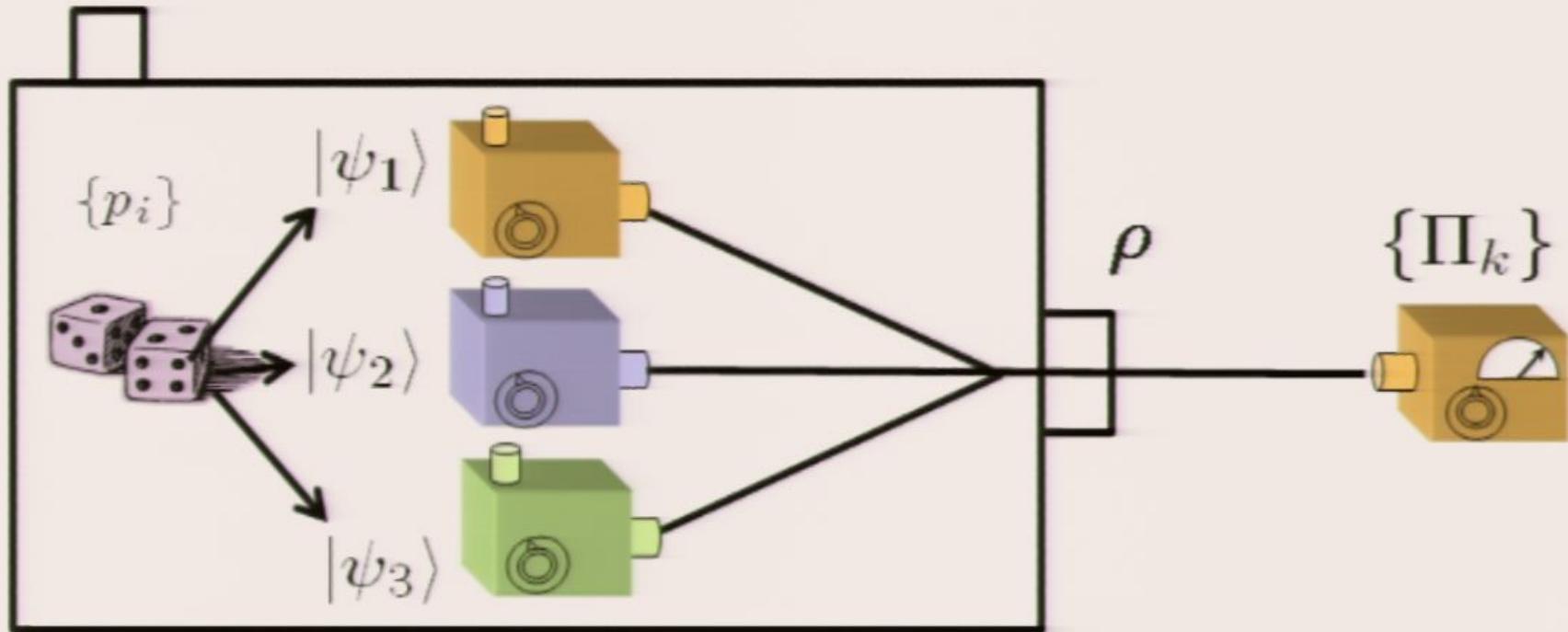
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