Title: A quantization quandary on the canonical road to quantum gravity

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Abstract: Karim Thebault
Canonical quantization techniques are generally considered to provide one of the most rigorous methodologies for passing from a classical to a quantum description of reality. For classical Hamiltonian systems with constraints a number of such techniques are available (i.e. gauge fixing, Dirac constraint quantization, BRST quantization and geometric quantization) but all are arguably equivalent to the quantization of an underlying reduced phase space that parameterizes the "true degrees of freedom" and displays a symplectic geometric structure. The philosophical coherence of making any ontological investment in such a space for the case of canonical general relativity will be questioned here. Further to this, the particular example of Dirac quantization will be critically examined. Under the Dirac scheme the classical constraint functions are interpreted as quantum constraint operators restricting the allowed state vectors. For canonical general relativity this leads to the Wheeler-de Witt equation and the infamous problem of time but, prima facie, seems to rely on our interpretation of the classical Poisson bracket algebra of constraints as the phase space realization of the theory's local symmetries (i.e. the group of space-time diffeomorphisms). As with the construction of an interpretively viable symplectic reduced phase space, this straight forward connection between constraints and local symmetry will be questioned for the case of GR. These issues cast doubt on the basis behind the derivation of the so-called wave function of the universe and give us some grounds for re-examining the entire canonical quantum gravity program as currently constituted.

A Quantisation Quandary on the Canonical Road to Quantum Gravity

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Overview

- In constrained Hamiltonian theory local symmetry transformations are represented in terms of the action of (first class) constraint functions on a physical phase space
- One approach to quantising constrained systems is the geometric quantisation procedure whereby we first reduce out the action of the constraints and then quantise the space which results
- Another is the Dirac procedure which involves first quantising an extended phase space and then imposing the constraints at the quantum level

Overview

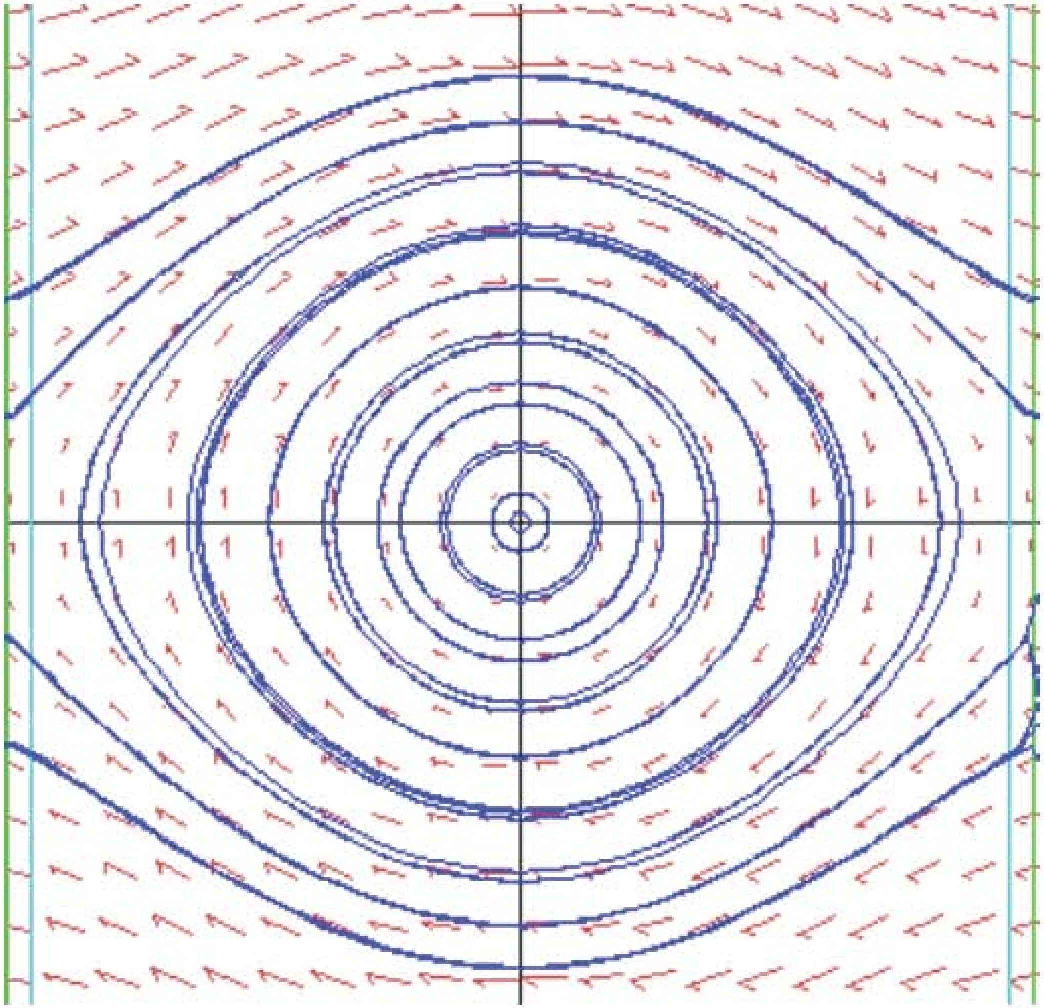
- When equivalence holds between these two procedures ('quantisation commutes with reduction') the desired quantisation of the 'true' classical degrees of freedom seems assured
- In canonical general relativity where the Hamiltonian function is itself a constraint, commutation between reduction and quantisation (if it can be shown) might actually be reason for doubting the conceptual foundations of Dirac quantised GR
- But if the imposition of a quantum Hamiltonian constraint (i.e. the WDE) isn't equivalent to a trivialising reduction procedure what does it mean in terms of the classical symmetries and degrees of freedom?

Symplectic Mechanics

First of all lets consider the physical phase space of a non-gauge theory:

- Phase space generically has a symplectic geometry (Γ, Ω)
- Dynamics is characterised by the geometric form of Hamilton's equations $\Omega(X_H, \cdot) = dH$ where H is the Hamiltonian and X_H is a vector field the integral curves of which are dynamical solutions in phase space γ
- There is a one-to-one representational correspondence between both points and solutions in the formalism and the instantaneous states and histories of the physical system being represented

Symplectic Mechanics



Presymplectic Mechanics

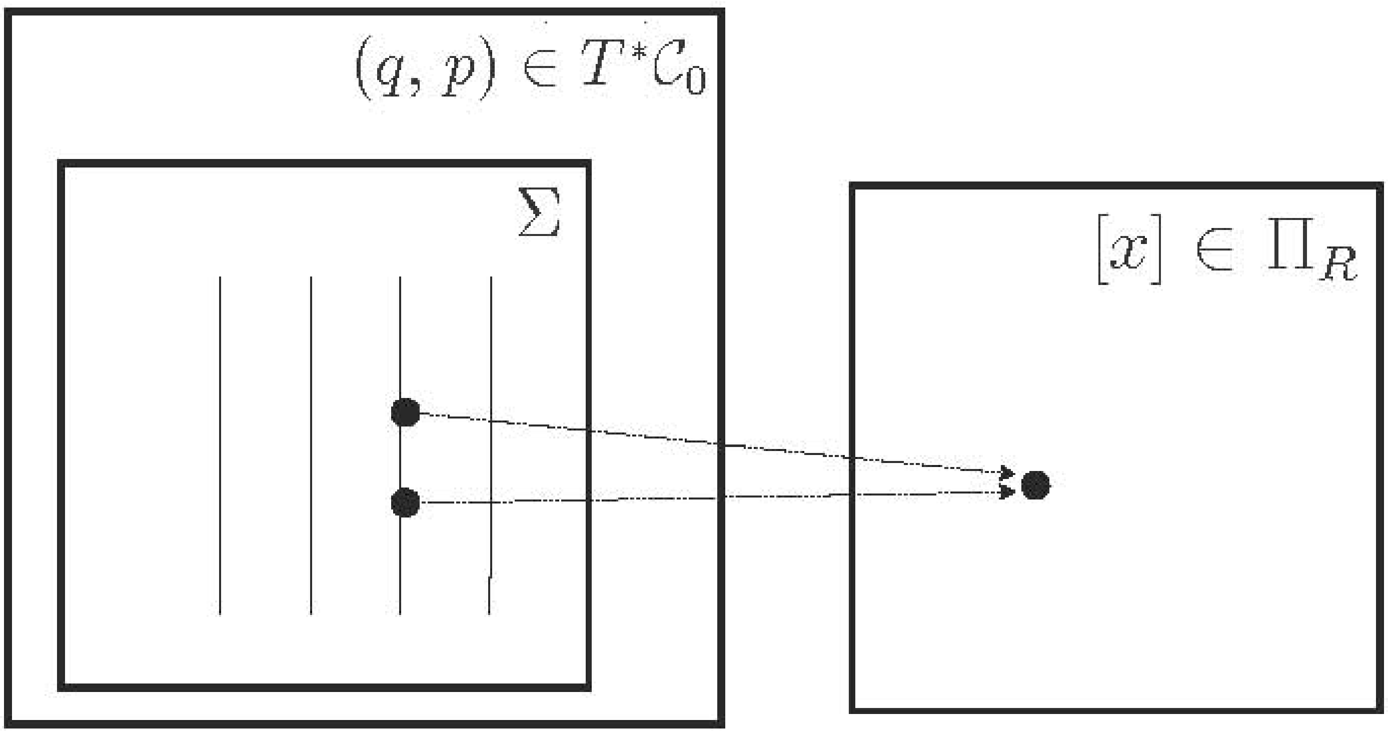
The physical phase space of a gauge theory on the other hand does not provide such a straight forward representation of mechanics

- It is a sub-manifold within the extended phase space $T^{\star}\mathscr{C}$ defined by satisfaction of the constraint functions $\Sigma = \{x \in \Gamma | \forall_i : \phi_i(x) = 0\}$
- The presymplectic geometry (Σ, ω) provides us with a degenerate dynamical structure since the null directions (defined by the constraints) mean that the Hamiltonian function no longer defines a unique dynamical vector filed
- There is now a many-to-one representative relationship between points/solutions and states/histories

Reduced Mechanics

- Points connected by the integral curves of the null vector fields form equivalence classes called gauge orbits $[x]$ - each point on a gauge orbit is understood as representing the same physical state
- We can construct a quotient space $[x] \in \Pi_R$ with each point corresponding to a gauge orbit
- If all goes well this space will be a symplectic manifold with a unique representation of physical states and dynamics

Reduced Mechanics



Gauge Symmetry and Constraints

- Dirac presumes that the first class constraints (those that commute with all the others) generate 'infinitesimal transformations that do not change the physical state'
- If this presumption is accepted then the reduced phase space (where the action of the constraints has been removed) should faithfully parameterise the 'true degrees of freedom' in our theory and we can interpret it as the true physical phase space
- However, it is the details of each theory on its own terms that dictate whether the Dirac interpretation of constraints is correct - if it is not then any reduced space will be problematic

Geometric Quantisation

- The objective of the geometric quantisation programme is to find a correspondence between the set of pairs: Symplectic manifolds $(\mathscr{M}, \Omega)$, smooth real functions $C^{\infty}(\mathscr{M})$ on the one hand; and complex Hilbert spaces $\mathscr{H}$, self-adjoint operators $\mathscr{O}(\mathscr{H})$ on the other
- We define the full quantisation of a classical system $(\mathscr{M}, \Omega)$ as a pairing of a Hilbert space, $\mathscr{H}_Q$ and a one to one map, O, which takes the classical obervables $f \in \Omega^0(\mathscr{M})$ to the self adjoint operators O_f on $\mathscr{H}_Q$

Geometric Quantisation

Explicitly we require that [Echeverría-Enríquez 1999]:

1. $\mathscr{H}_Q$ is a separable complex Hilbert space. The elements $|\psi\rangle \in \mathscr{H}_Q$ are the quantum wavefunctions and the elements $|\psi\rangle_{\mathbb{C}} \in \mathbf{P}\mathscr{H}_Q$ are the quatum states where $\mathbf{P}\mathscr{H}_Q$ is the projective Hilbert space
2. O is such that: i) $O_{f+g} = O_f + O_g$ ii) $O_{\lambda f} = \lambda O_f \ \forall \lambda \in \mathbb{C}$ iii) $O_1 = Id_{\mathscr{H}_Q}$
3. $[O_f, O_g] = i\hbar O_{\{f,g\}}$ (i.e. O is a Lie algebra morphism up to a factor)
4. For a complete set of classical observables $\{f_j\}$, $\mathscr{H}_Q$ is irreducible under the action of the set $\{O_{f_j}\}$

Geometric Quantisation

- To geometrically quantise a canonical gauge theory we first reduce the presymplectic physical phase space to construct a symplectic reduced phase space. We can then use ideas above to quantise this space and get a Hilbert space $\bar{\mathscr{H}}$
- Both the constraints and symmetries are incorporated and divided out at the classical level so we don't have to represent them at the quantum level at all. Furthermore a well defined Hilbert space structure with an inner product is guaranteed

Dirac quantisation

The first stage in the Dirac approach to quantisation is to quantise the extended phase space. We can formalise this step in abstract algebraic terms [Thiemann 2007]:

1. Define a classical Poisson *-subalgebra $\mathscr{B}$ in terms of (sub-set) of functions on phase space with the symplectic structure of phase space providing the Poisson bracket
2. Then define (based on $\mathscr{B}$) a quantum *-algebra $\mathscr{A}$ which implements $i\hbar$ times the Poisson bracket of $\mathscr{B}$ as commutation relations
3. Next, find a representation of $\mathscr{A}$ in terms of a subalgebra of linear operators on a Hilbert space $\mathscr{L}(\mathscr{H}_{aux})$ such that the constraints are supported as operators on $\mathscr{H}_{aux}$

Dirac quantisation

In the second stage of Dirac quantisation we seek to construct a *physical* Hilbert space by imposing the quantum constraints. Informally, this amounts to:

- Treating the (first class) constraint functions operators restricting the physical state vectors $O(\phi_i) \mid \psi\rangle_{phys} = 0$
- The Hilbert space that is constructed by taking the physical states is the physical Hilbert space $\mathscr{H}_{phys}$ of the quantum theory
- The quantum observables are then taken to be self-adjoint operators which commute with the constraints

Dirac quantisation

- Formally, however this second stage in the Dirac procedure suffers from a number of problems: i) ambiguity in the operator ordering; ii) lack of a rigours procedure for defining an inner product structure on the space of physical states; and iii) non-triviality of solving the constraints at a quantum level
- The extent that these difficulties can be overcome depends on the theory involved and in particular the structure of the constraint algebra (when it fails to be a Lie algebra the situation becomes far more complex)

Dirac quantisation

- But, for the purposes of this talk let us make the (highly nontrivial) assumption that these problems have been solved - our preoccupation will be with the conceptual rather than formal foundations of Dirac quantisation

Does quantisation commute with reduction?

- We have considered two methods of quantisation for constrained systems - an immediate question is whether the they are equivalent - i.e. does quantisation commutes with reduction?
- Strictly this question is entwined with the formalisation issue of the last slide - in particular the precise structure of the physical Hilbert space, $\mathscr{H}_{phys}$, and whether it is suitably isomorphic to that reached by geometric methods, $\bar{\mathscr{H}}$

Does quantisation commute with reduction?

- However, what we are interested in is the extent to which the two procedures are *representatively equivalent* in terms of the structure they use to describe quantum systems displaying gauge symmetry
- More precisely, should we think of the classical projection from physical to reduced phase space as equivalent to the quantum projection from the auxiliary to physical Hilbert space in terms of the elimination of the same otiose representational structure and the isolation of the same 'true' degrees of freedom

Does quantisation commute with reduction?

- On one level there is a clear inequivalence; whereas classically the symplectic reduction procedure takes us from a representation of physical states in terms of an equivalence class to one which is unique, the Dirac procedure takes us from an unphysical and otiose representation to one which is both unique and physical
- Thus it is important that we should not think think of the action of the constraints on the auxiliary Hilbert space as generating gauge orbits which are equivalencies classes in the classical sense

Does quantisation commute with reduction?

- For the case that the classical constraints form a Lie algebra we can identify the quantum constraints with self adjoint operators on $\mathscr{H}_{aux}$. These operators can then be understood as defining a unitary representation of the relevant canonical gauge group
- This means that we can define a precise connection between the classical and quantum quotienting procedures in group theoretic terms and thus confer representational equivalence on the physical Hilbert space and the quantised, reduced phase space - they both represent the same reduced set of degrees of freedom

Does quantisation commute with reduction?

- However, when the classical constraint algebra fails to be a Lie algebra this group theoretic basis for interpreting the quantum constraints is no longer available to us
- We must still surely hope that $\mathscr{H}_{phys}$ and $\bar{\mathscr{H}}$ are representationally equivalent not least because representational equivalence to reduced space quantisation would seem the best methodology for solidifying the conceptual basis of the Dirac methodology as correctly isolating the 'true' degrees of freedom of the system
- But we have no hard and fast arguments, let alone proofs of this equivalence...

The ADM formalism

We can re-cast the original Lagrangian formulation of general relativity due to Einstein into a constrained Hamiltonian formalism (Arnowitt et al. [1962]):

$$S = \frac{1}{\kappa}\int_{\mathbb{R}} dt \int_{\sigma} d^3 \{\dot{q}_{ab}P^{ab} - [N^a H_a + |N| H]\}$$

where σ is a three dimensional manifold of arbitrary topology, q_{ab} and P^{ab} are tensor fields defined on σ and N and N^a are arbitrary multipliers called the lapse and shift

The ADM formalism

H_a and H are constraint functions of the form:

$$\begin{aligned} H_a &:= -2q_{ac}D_bP^{bc} \\ H &:= -\frac{s\kappa}{\sqrt{det(q)}}[q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd}] \end{aligned}$$

These are called the momentum and Hamiltonian constraints respectively.

The ADM formalism

The physical phase space Σ is a sub-manifold within the extended phase space Γ defined by the constraints:

$$\Sigma = \{(q_{ab}, P^{ab}) = x \in \Gamma | H_a(x) = 0; H(x) = 0\}$$

Like in a typical gauge theory it has a presymplectic geometry (Σ, ω)

Degrees of freedom and the gravitational field

- The need for the imposition of these constraints is easily understood on an intuitive basis since we know that the physical modes of the classical gravitational field should correspond to a canonical representation with $4 \times \infty^3$ degrees of freedom and these constraints serve to cut the $12 \times \infty^3$ variables of the extended phase space down to $8 \times \infty^3$
- But this still leaves another $4 \times \infty^3$ unphysical degrees of freedom on the physical phase space - how exactly we can eliminate this remaining degeneracy without simultaneously interfering with the dynamical degrees is the essential problem of canonical gravity
- Naively, we would simply press ahead and try and reduced them out using the symplectic reduction procedure...

Reduction

- If we define the orbits of ω to be four dimensional surfaces $\bar{\gamma}$ in Σ such that the quadritangent to the orbit X is in the kernel of ω (i.e. $\omega(X) = 0$) then we can identify the $\bar{\gamma}$ with the set of (globally hyperbolic) solutions of the Einstein field equations
- This poses an immediate problem since these orbits are precisely those which we would normally classify as gauge equivalence classes - thus a symplectic reduction procedure would (in principle) lead to a reduced phase space within which, prima facie, dynamics has been gauged out

A space of histories?

- In light of this one might then attempt to re-interpret the reduced space as a space of histories with each point taken to represent a solution invariant under the class of four-dimensional diffeomorphisms
- This could be justified on the basis that there exists a single canonical isomorphism from our reduced phase space points to the space of gauge invariant solutions in a Lagrangian formalism
- However, the existence of an isomorphism does not automatically confer representational equivalence and if we read the reduced space in such a manner then it has problematic consequences for how we view the unreduced phase space

The dynamical role of the Hamiltonian constraints

The Hamiltonian constraints of canonical gravity are of an unusual *dynamical* type such that the transformations they effect on the physical phase space cannot be understood purely as gauge:

$$\{H(N), P^{\mu\nu}\} = \frac{q^{\mu\nu}NH}{2} - N\sqrt{\|q\|}[q^{\mu\rho}q^{\nu\sigma} - q^{\mu\nu}q^{\rho\sigma}]R^{D+1}_{\rho\sigma} + \mathfrak{L}_{Nn}p^{\mu\nu}$$

where $H(N) := \int_\sigma d^3xNH$ and $X^\mu = (t, x^a)$

Since the second term on the right is non-zero on the constraint surface - contra Dirac's presumption - we cannot view the role of the Hamiltonian constraint as purely producing infinitesimal diffeomorphisms

The dynamical role of the Hamiltonian constraints

- In fact, we can only view the constraints of canonical GR as collectively producing four dimensional diffeomorphisms once a dynamical solution (generated by the Hamiltonian constraint) has already be defined
- Thus we can assert that passing to the reduced phase space of canonical gravity (where the action of the Hamiltonian constraint is treated as pure gauge) will involving throwing the dynamical baby out with the gauge symmetry bathwater

Dirac quantisation of canonical gravity

- The principle reason general relativity was cast into canonical form in the first place was because it was thought that the application of canonical quantisation techniques would then provide a natural path towards a theory of quantum gravity
- Such a procedure (modulo numerous technical issues) leads to an auxiliary Hilbert space with the appropriate commutation relations between the self-adjoint operators $\hat{q}_{ab}$ and $\hat{P}^{ab}$

Dirac quantisation of canonical gravity

As we have seen the standard procedure would then involve imposing both Hamiltonian and momentum constraints as restraints on the allowed physical states

$$\hat{H}(N) \mid \psi\rangle_{phys} = 0$$
$$\hat{H}_a(N_a) \mid \psi\rangle_{phys} = 0$$

Quantum degrees of freedom counting

- The imposition of these quantum constraints seems well motivated since it corresponds to exactly the reduction to the required two degrees of freedom that we were interested in classically.
- However, we need to be sure that we have isolated the correct, dynamical degrees - and we can only be sure of this by making an identification between these quantum constraints and the appropriate classical symmetries

The quantum momentum constraints

- With regard to the momentum constraints this is can be done since the quantum momentum operator algebra can be understood as a natural extension of constrains role in the classical theory as generators of Poisson bracket algebra homomorphic to the Lie algebra of infinitesimal diffeomorphism of σ
- This close connection between quantum and classical algebraic/group theoretic structures motivates us to posit that the imposition of the momentum constraints will lead to a representative structure equivalent to that reached by first reducing out their action at a classical level and then quantising

The Wheeler-de Witt equation

- The imposition of the quantum Hamiltonian constraints on the other hand has no simple group theoretic interpretation since the classical counterparts close only with structure functions
- Thus, - assuming $\hat{H}_a(N_a) \mid \psi\rangle_{phys} = 0$ does at least allow us to construct a viable physical Hilbert space - we have no straight forward basis to argue that this Hilbert space will be representationally equivalent to that which would result from reducing and then quantising

Does reduction commute with quantisation for canonical quantum gravity?

So let us not try and make a strong argument either way - rather just explore the two possible answers to the question - should we think of the Dirac quantisation of canonical gravity as representationally equivalent to a geometric quantisation of the reduced phase space of the classical theory?

Does reduction commute with quantisation for canonical quantum gravity?

- Assuming that we should then it appears that the only viable interpretation of the Wheeler-de Witt equation is as imposing at a quantum level the dynamically trivialising reduction that we argued against for the case of the classical theory
- Assuming that we we shouldn't, we then must then try and understand exactly which degrees of freedom are we quantising? How can we be sure they are the correct ones if the conceptual foundations of the Dirac approach cannot be anchored in its geometric reduction correlate?

More questions than answers...

- If we are to provide the Wheeler-de Witt equation with a solid conceptual basis it is essential to understand exactly which classical symmetries its imposition at a quantum level is connected to
- Should we think of it as representationally equivalent to the application of full symplectic reduction? If so it seems to lead us into difficult territory
- If not, then it is important to ensure that its imposition can be understood in terms of the isolation of the true dynamical degrees of freedom of classical canonical gravity - and we currently lack a basis for doing this

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