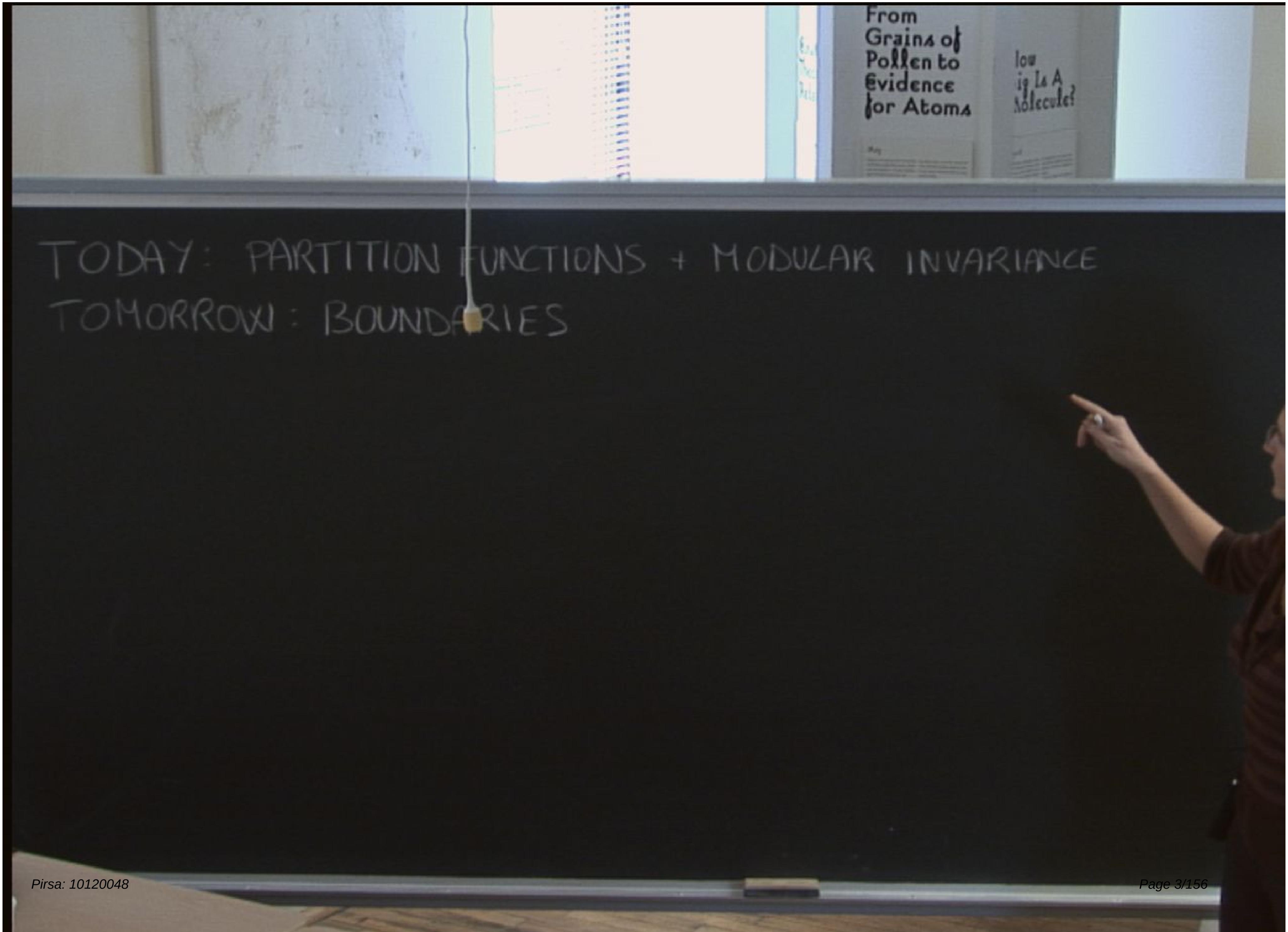
Title: Conformal Field Theory - Additional Lecture 1

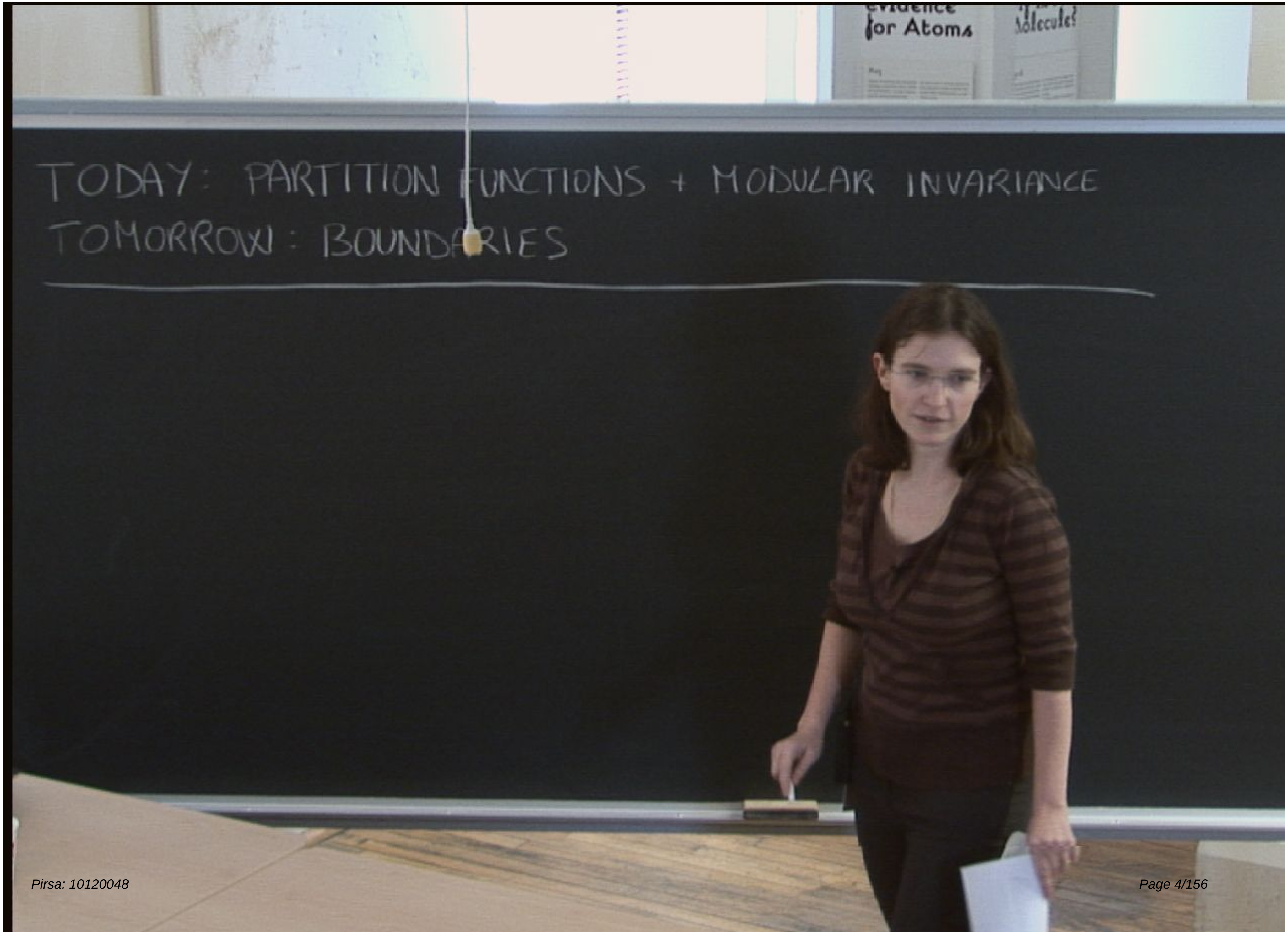
Date: Dec 13, 2010 10:00 AM

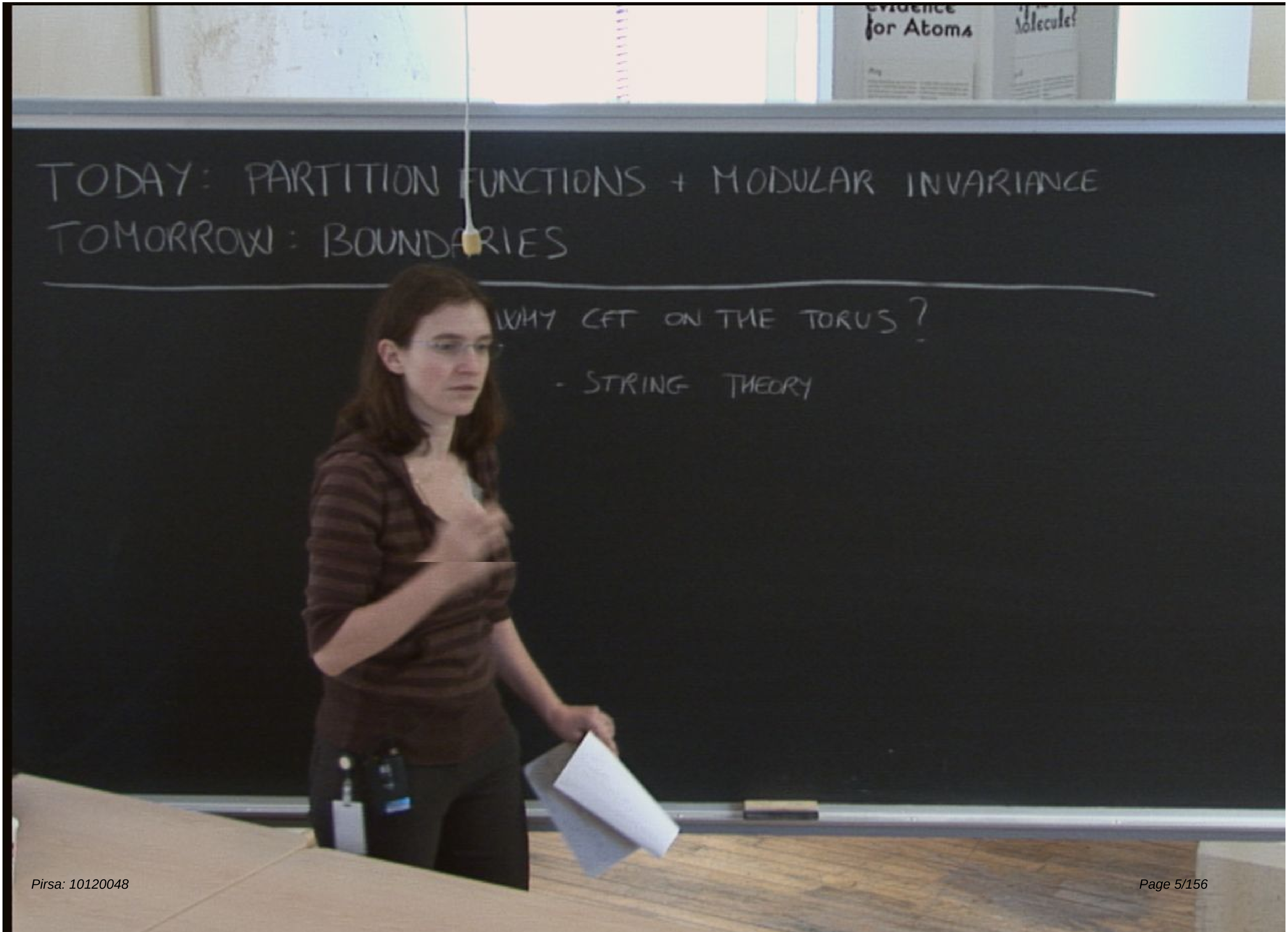
URL: http://pirsa.org/10120048

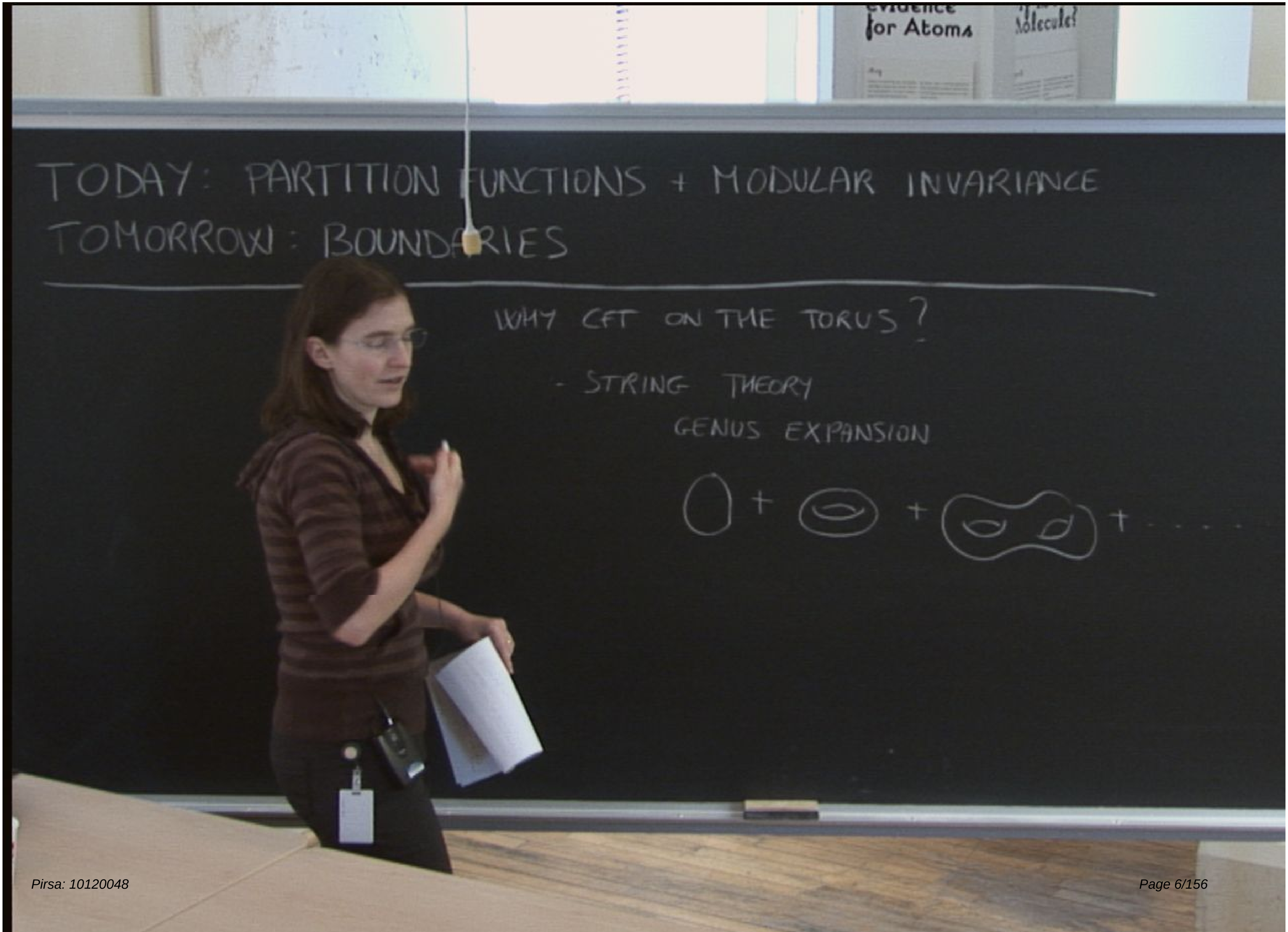
Abstract:

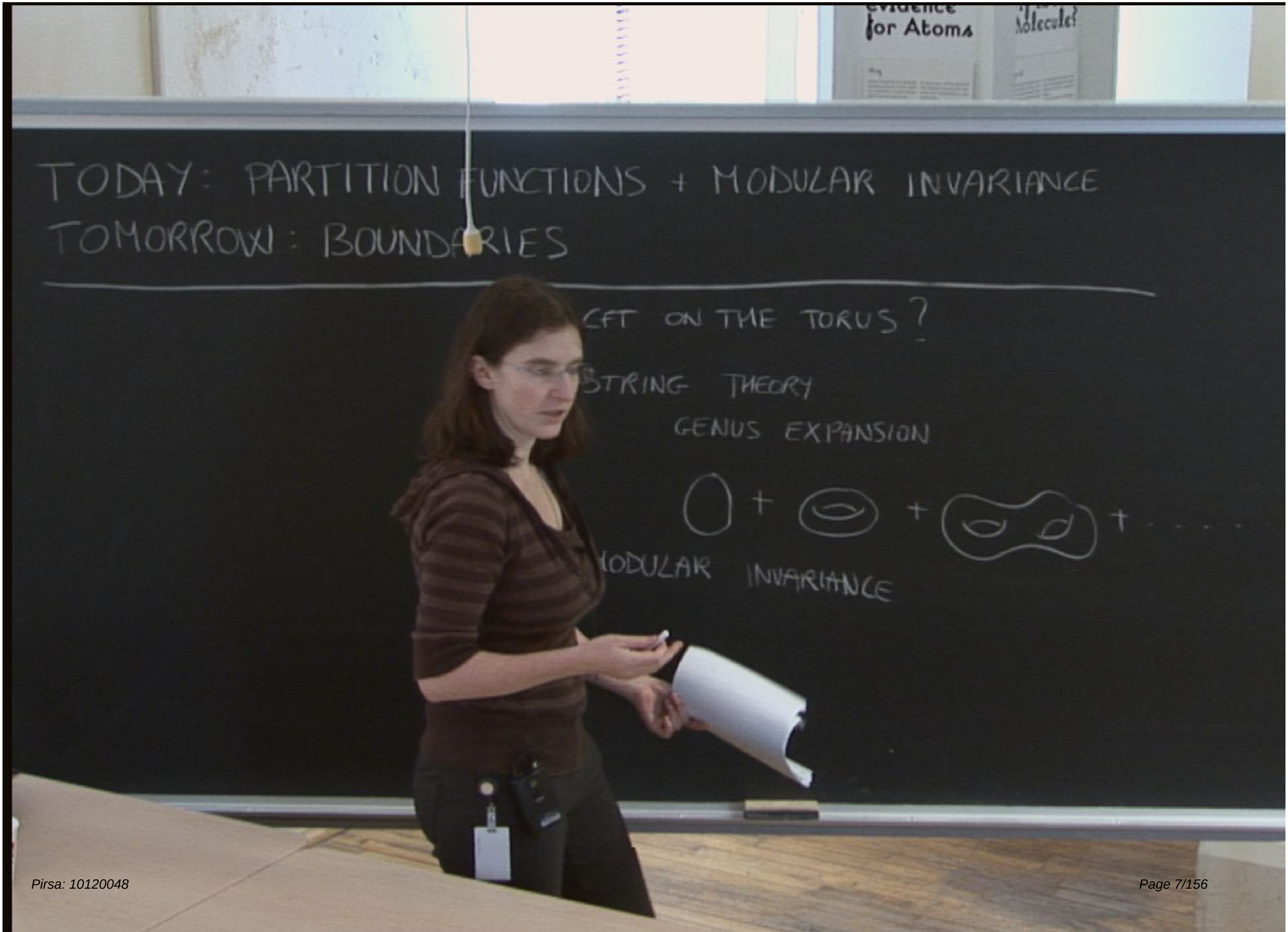
perimeter scholars
international


From Grains of Pollen to Evidence for Atoms
TODAY: PARTITION FUNCTIONS + MODULAR INVARIANCE
TOMORROW: BOUNDARIES


for Atoms
TODAY: PARTITION FUNCTIONS + MODULAR INVARIANCE
TOMORROW: BOUNDARIES


for Atoms
TODAY: PARTITION FUNCTIONS + MODULAR INVARIANCE
TOMORROW: BOUNDARIES
WHY CFT ON THE TORUS?
- STRING THEORY


TODAY: PARTITION FUNCTIONS + MODULAR INVARIANCE
TOMORROW: BOUNDARIES
WHY CFT ON THE TORUS?
- STRING THEORY
GENUS EXPANSION
for Atoms
Molecule!


for Atoms
TODAY: PARTITION FUNCTIONS + MODULAR INVARIANCE
TOMORROW: BOUNDARIES
CFT ON THE TORUS?
STRING THEORY
GENUS EXPANSION
+ + + ...
ODULAR INVARIANCE

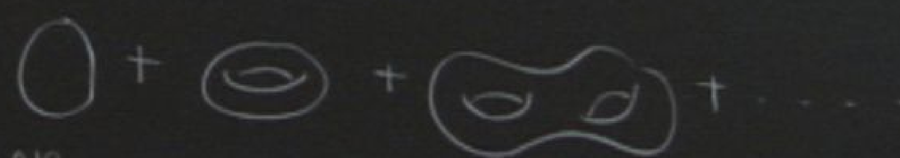
TODAY: PARTITION FUNCTIONS + MODULAR INVARIANCE

TOMORROW: BOUNDARIES

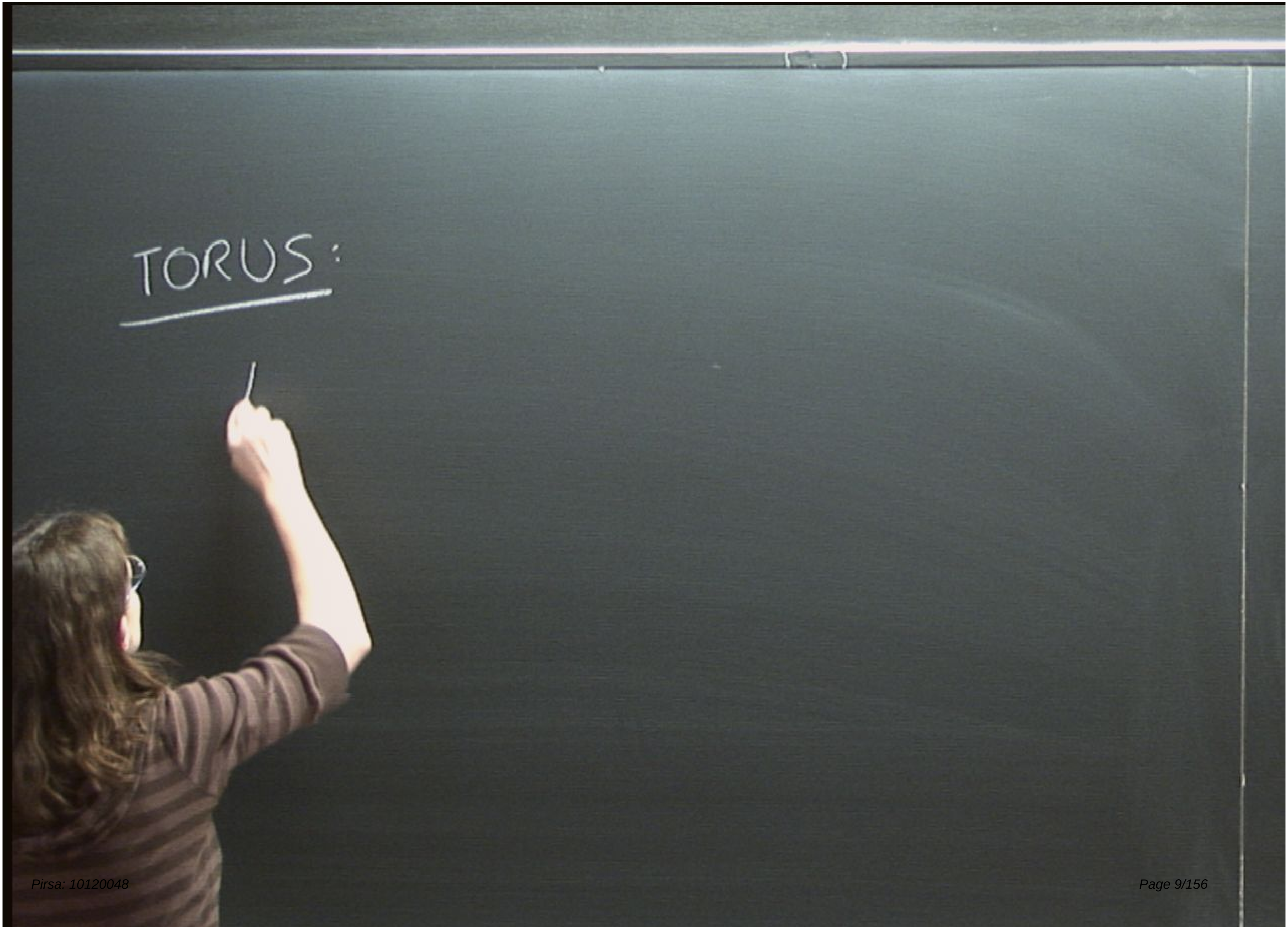
WHY CFT ON THE TORUS?

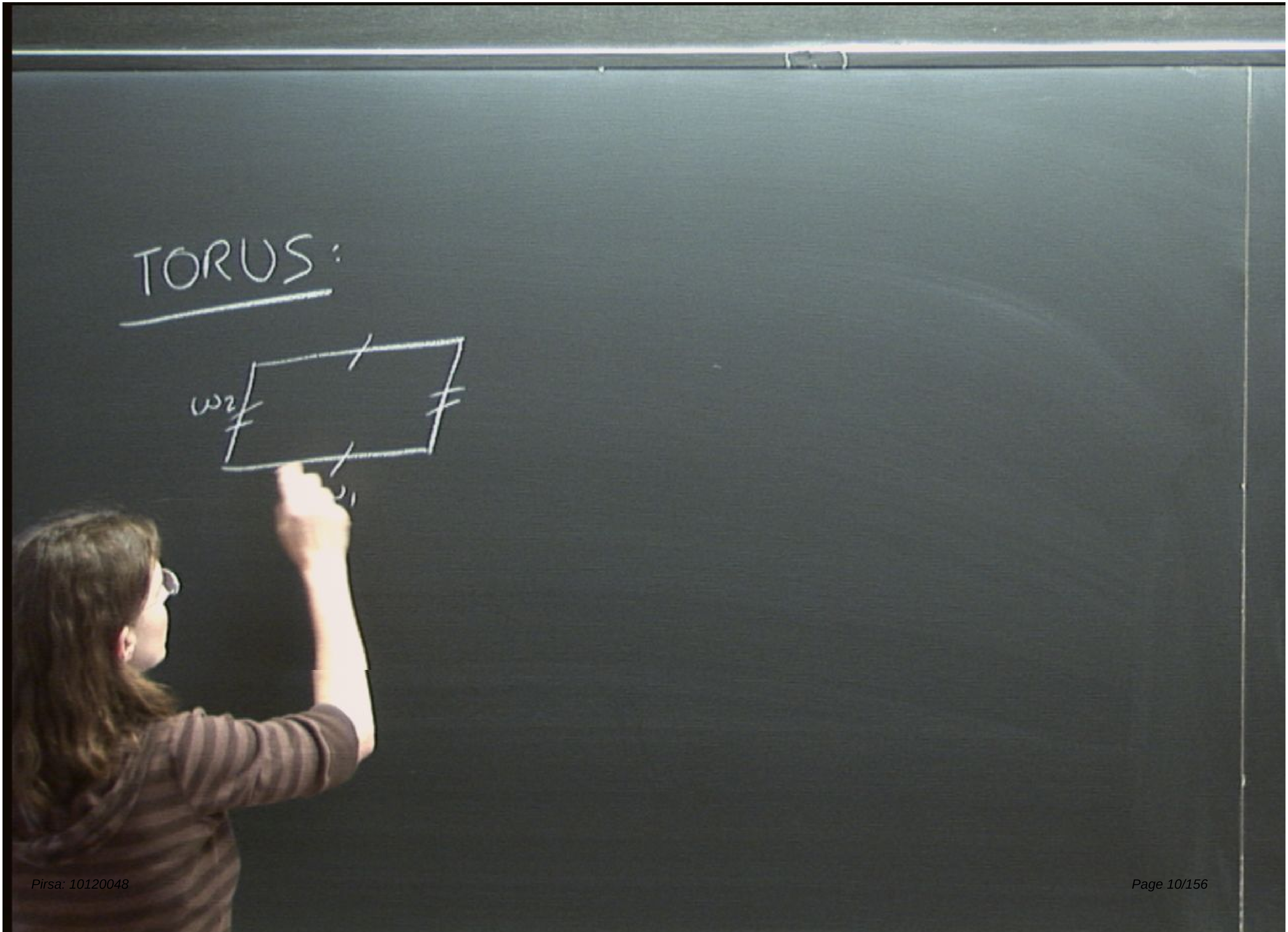
- STRING THEORY

GENUS EXPANSION

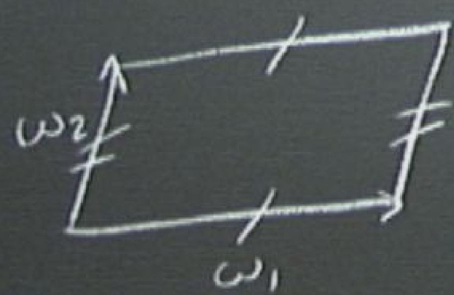


- MODULAR INVARIANCE CONSTRAINS SPECTRUM

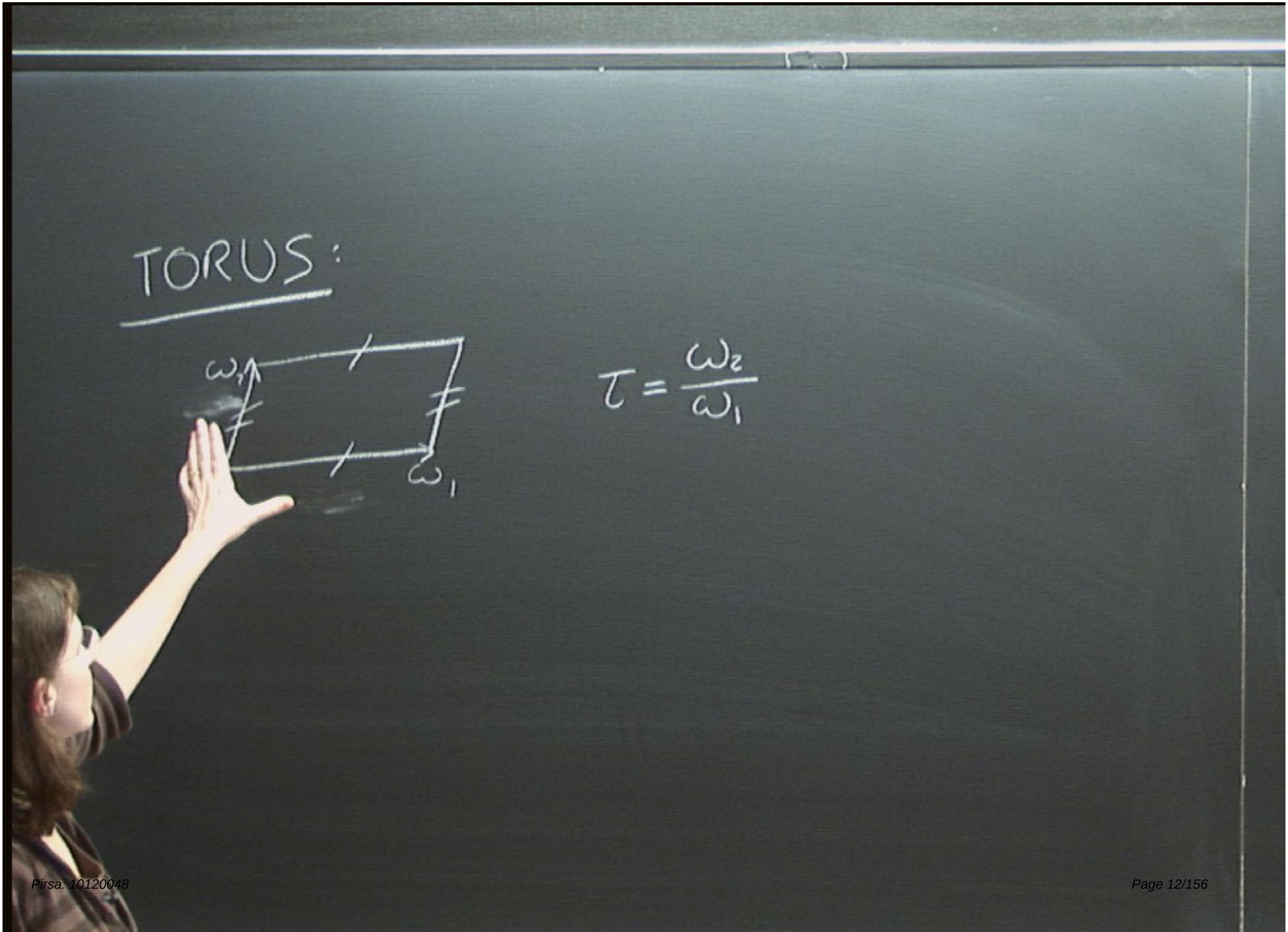

TORUS:

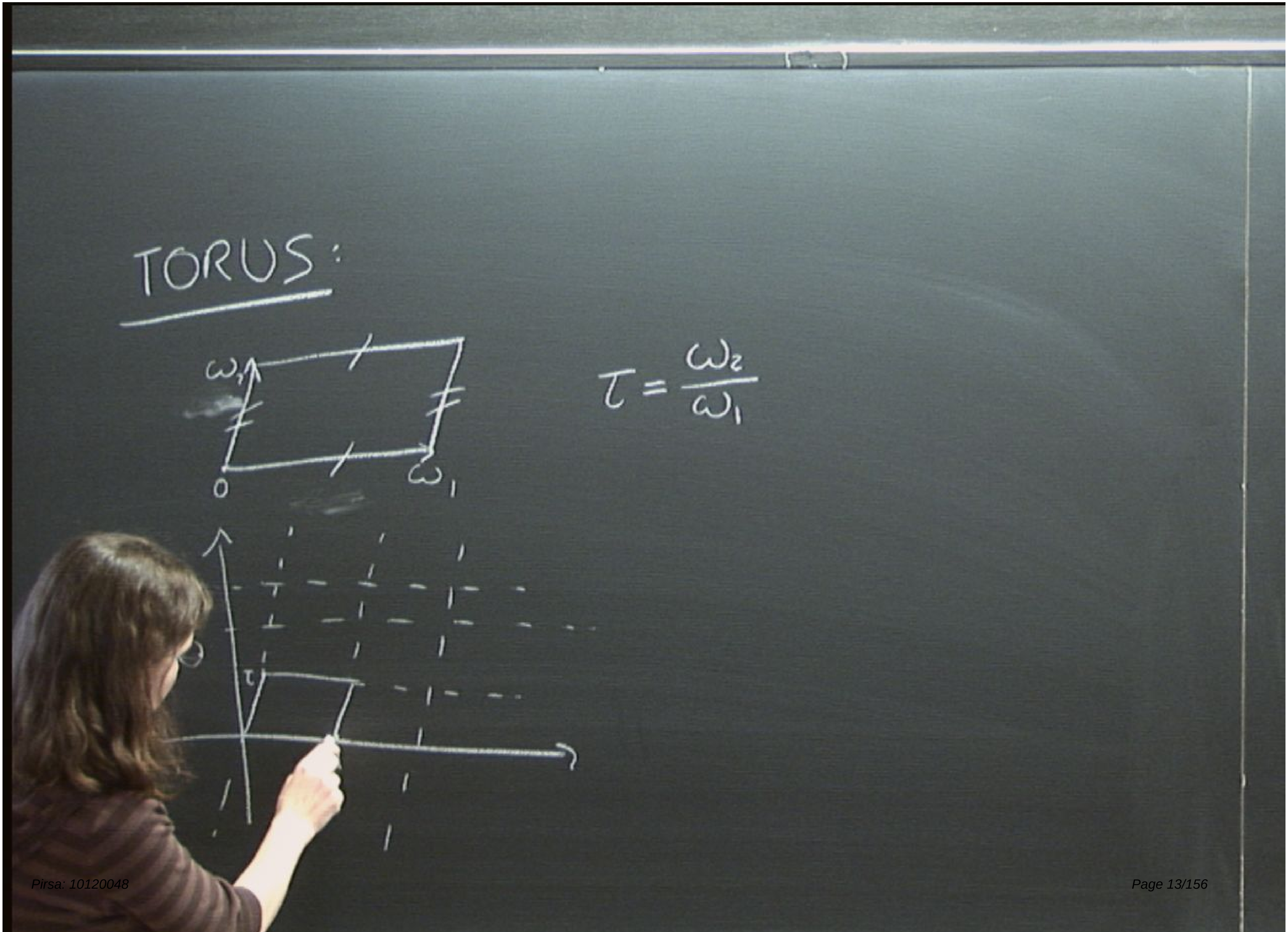

TORUS:
ω2

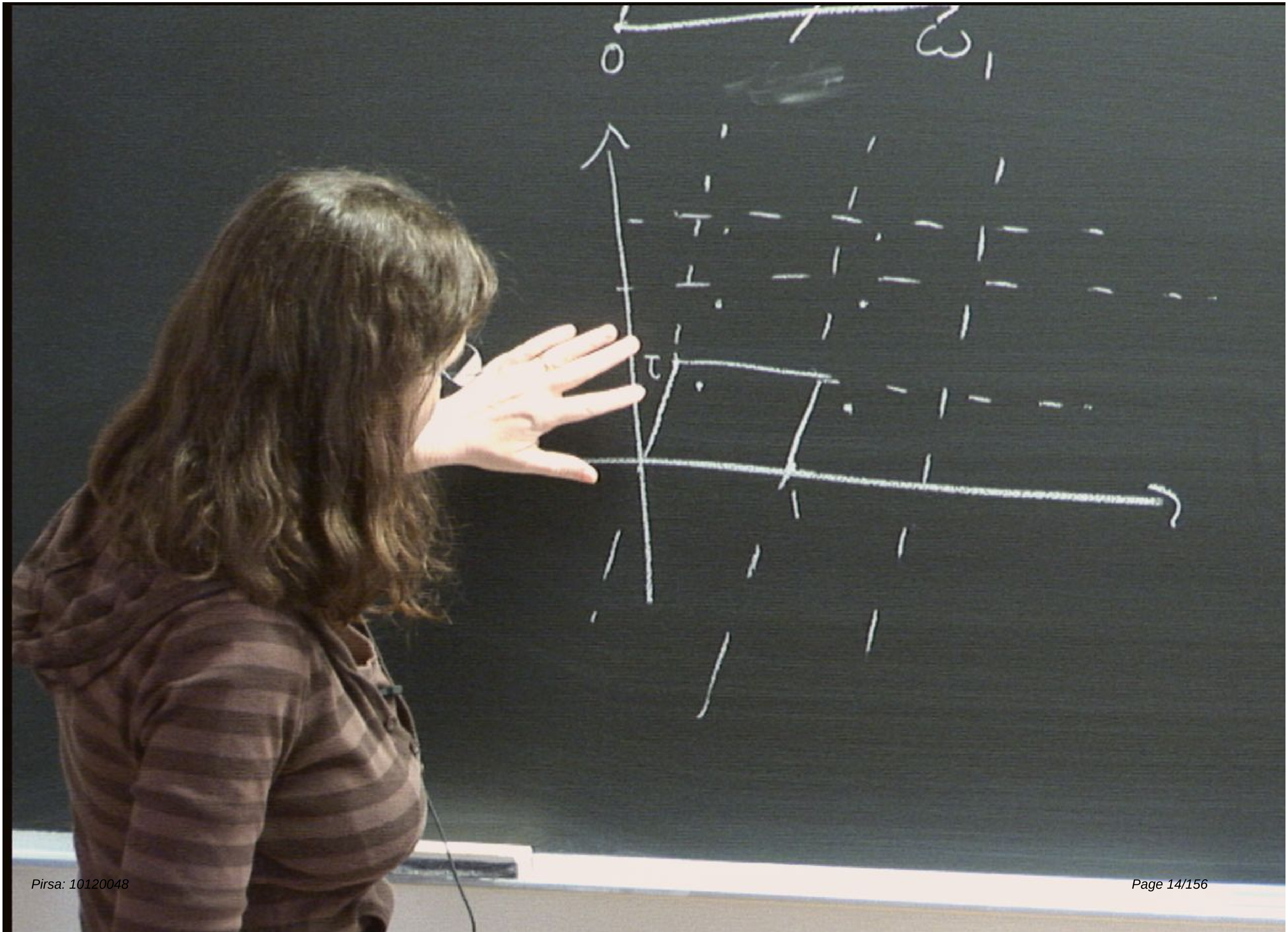
TORUS:

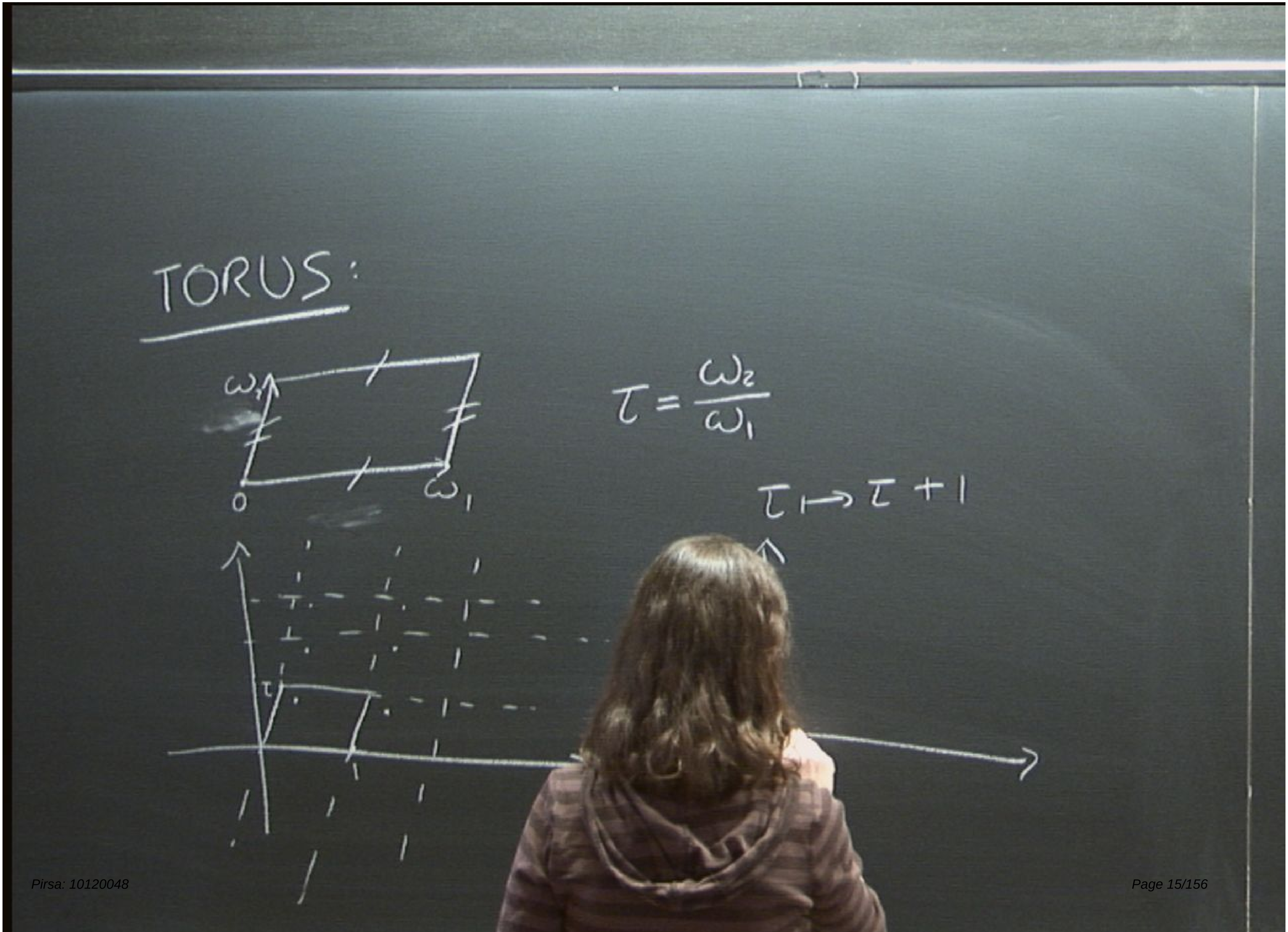


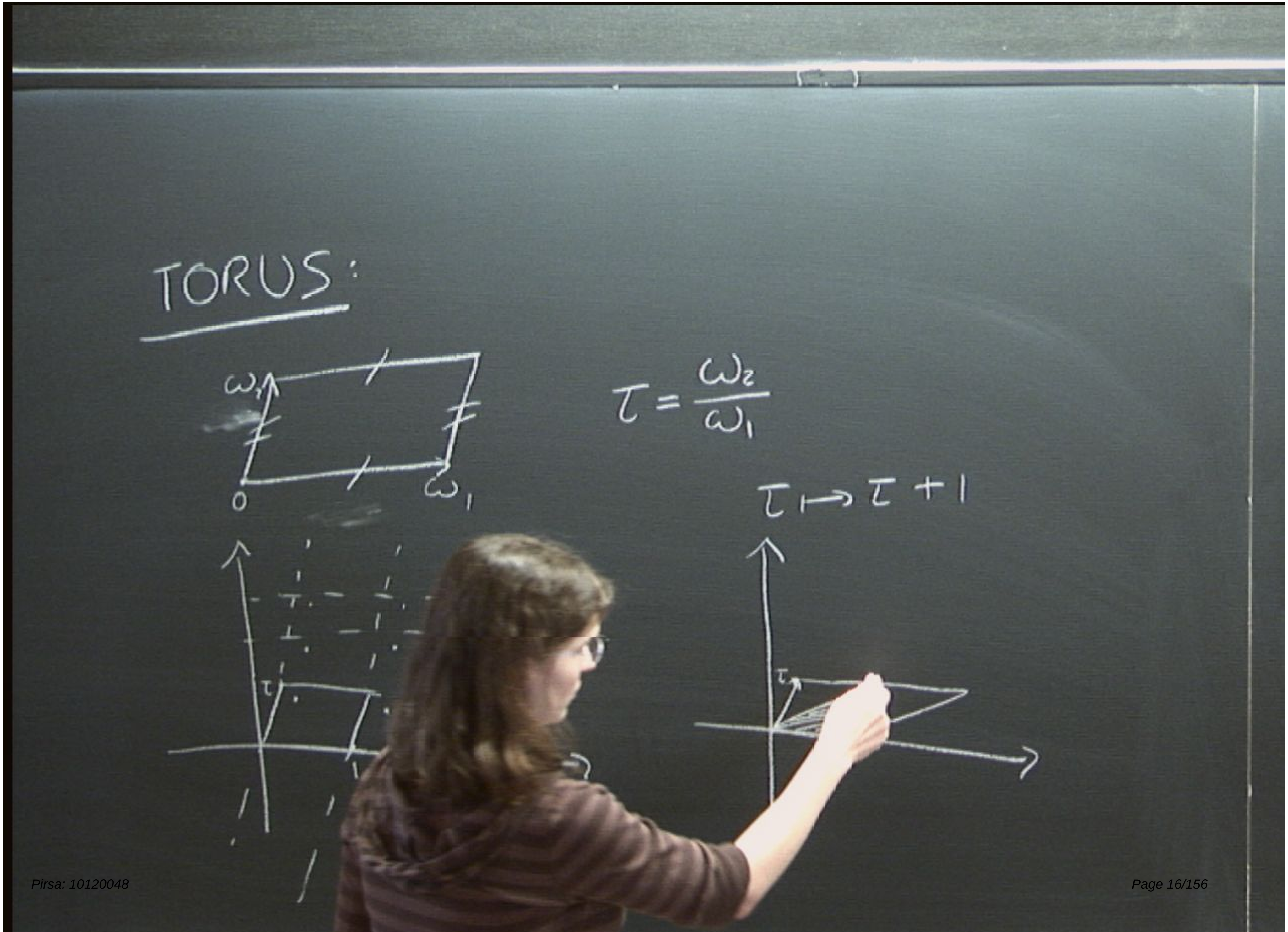
$$\tau = \frac{\omega_2}{\omega_1}$$


TORUS:
ω_2
ω_1
$\tau = \frac{\omega_2}{\omega_1}$


TORUS:
ω2
0
ω1
τ = ω2/ω1
τ




TORUS:
ω2
0
ω1
τ = ω2/ω1
τ ↦ τ + 1
τ


TORUS:
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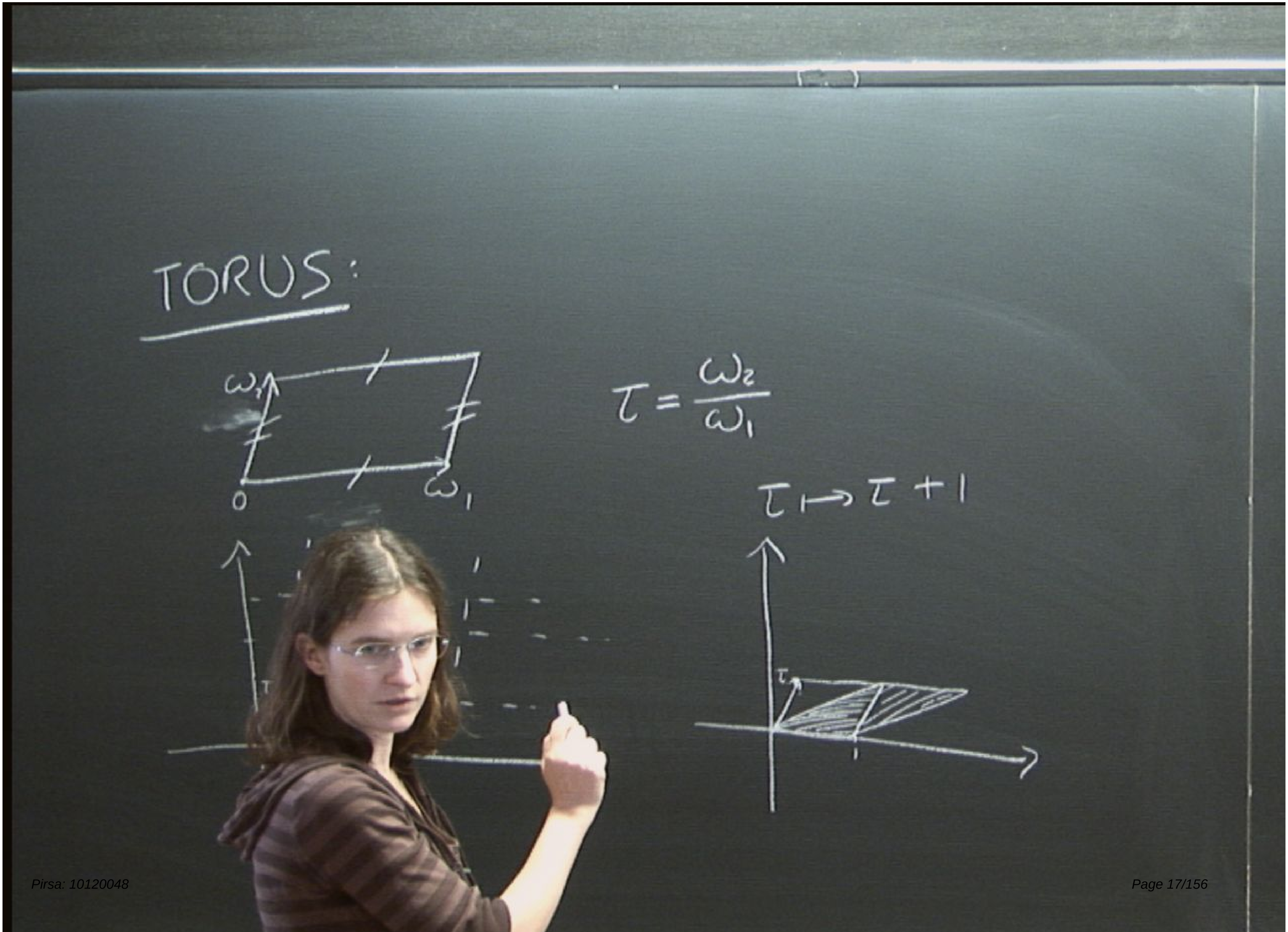
TORUS:
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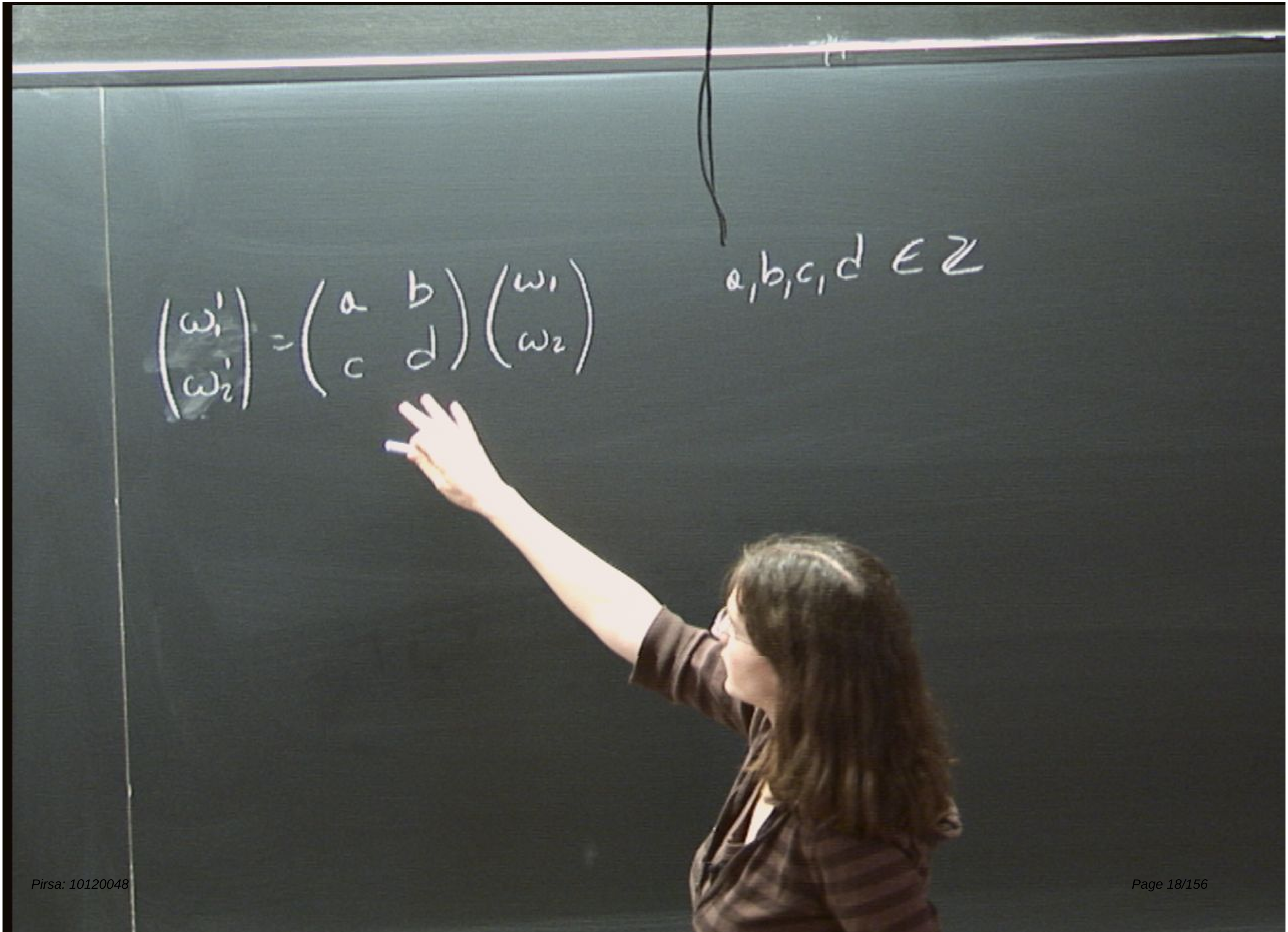
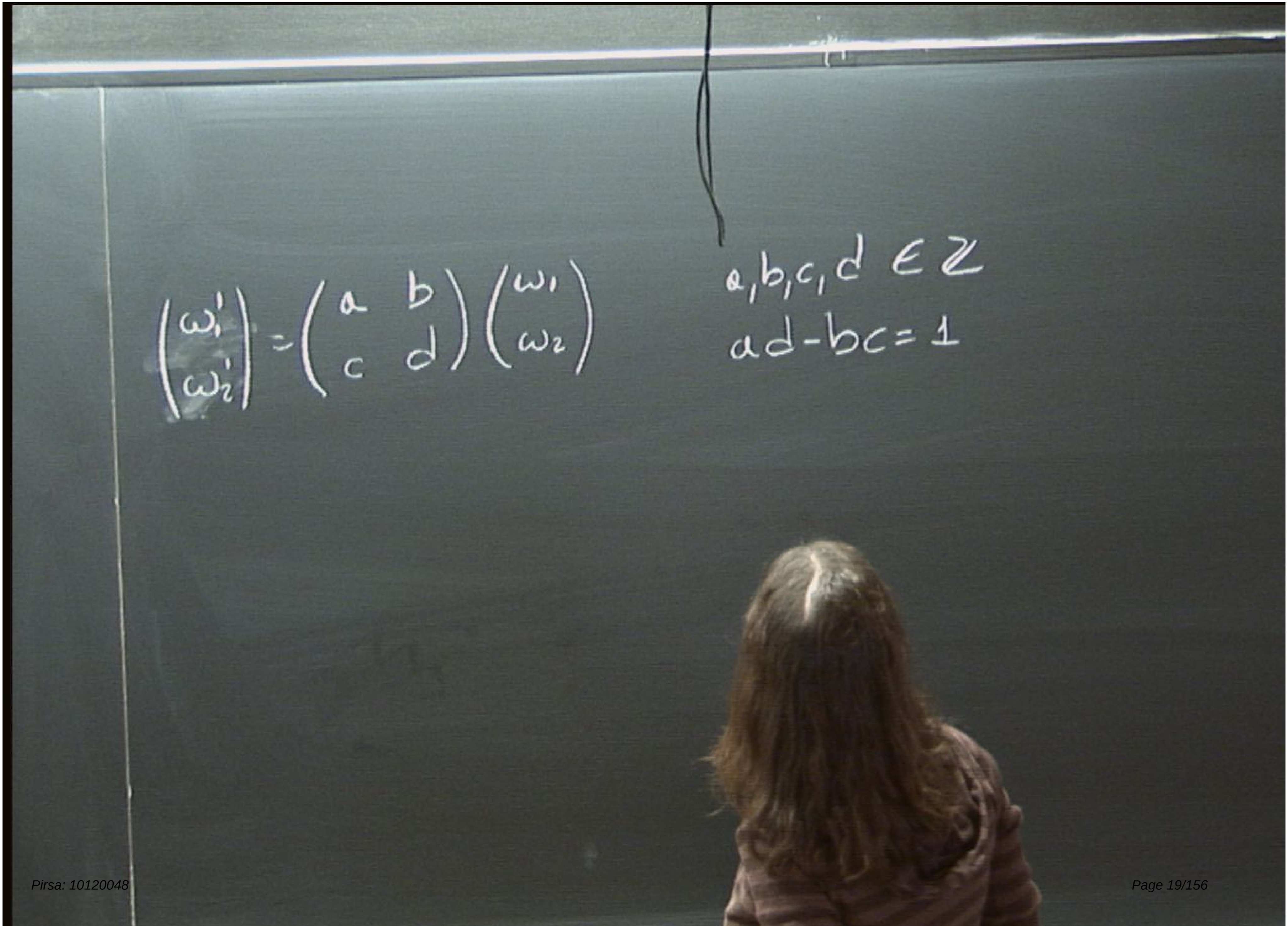
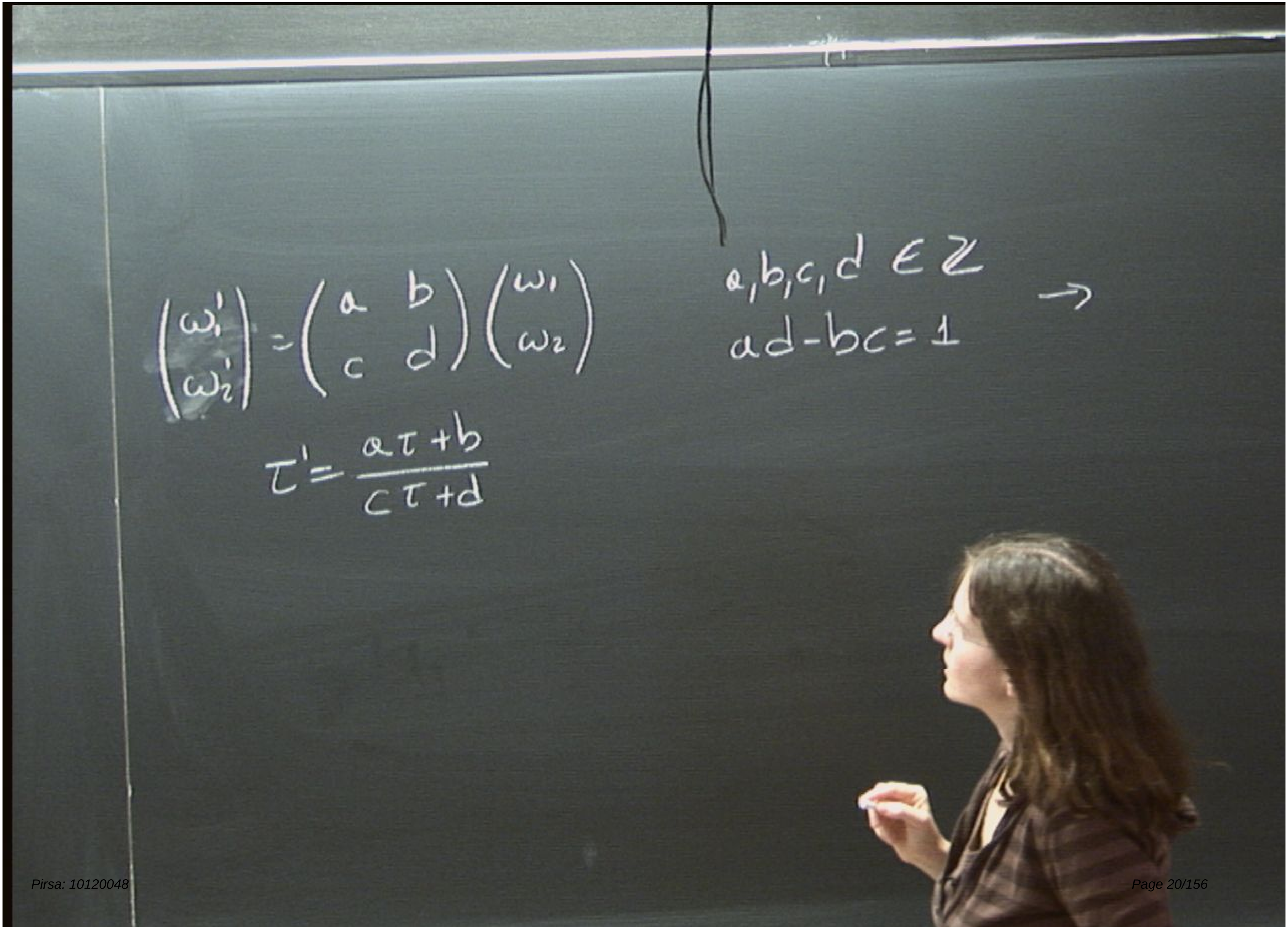

$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$a, b, c, d \in \mathbb{Z}$


$\begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$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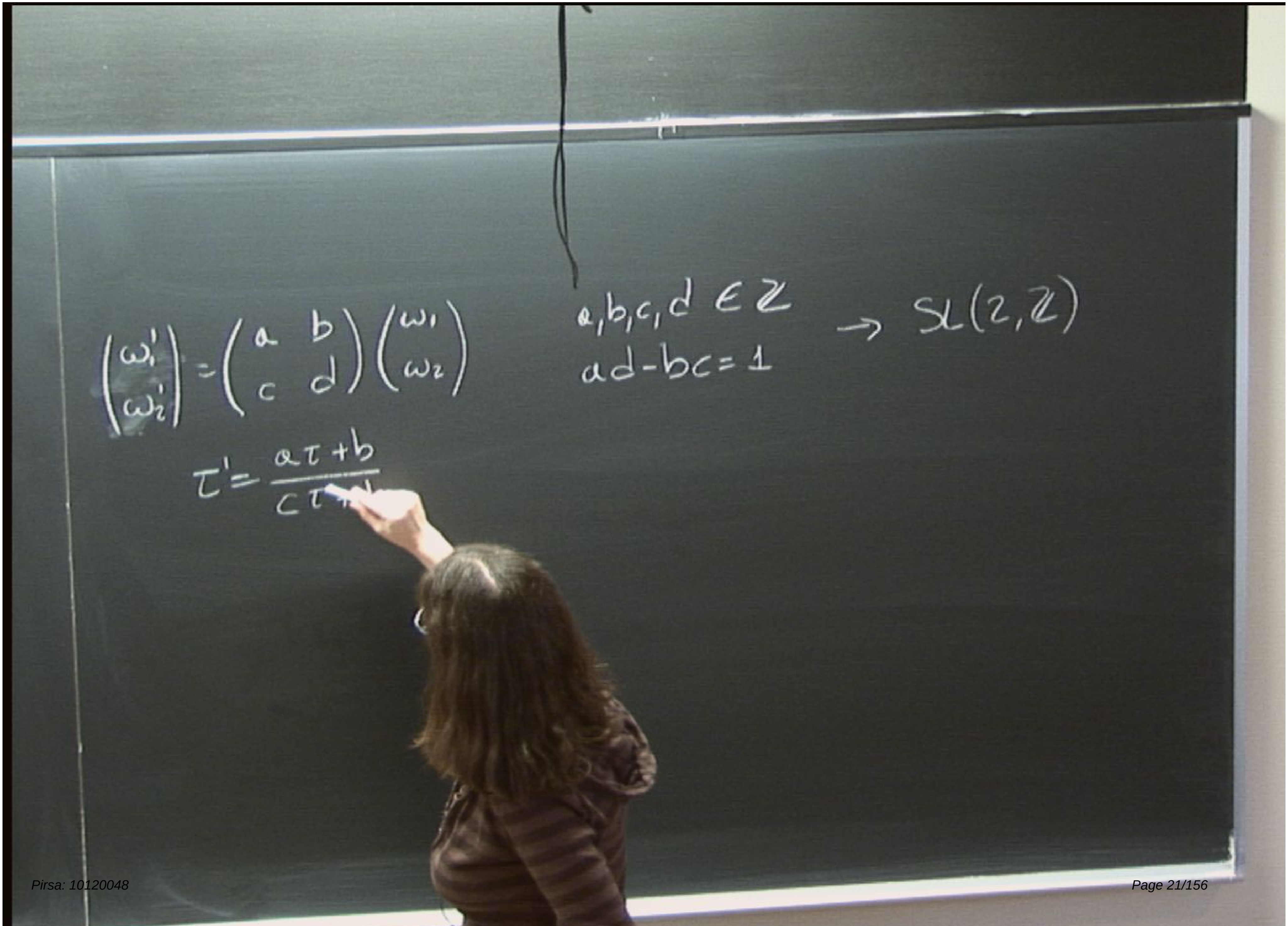
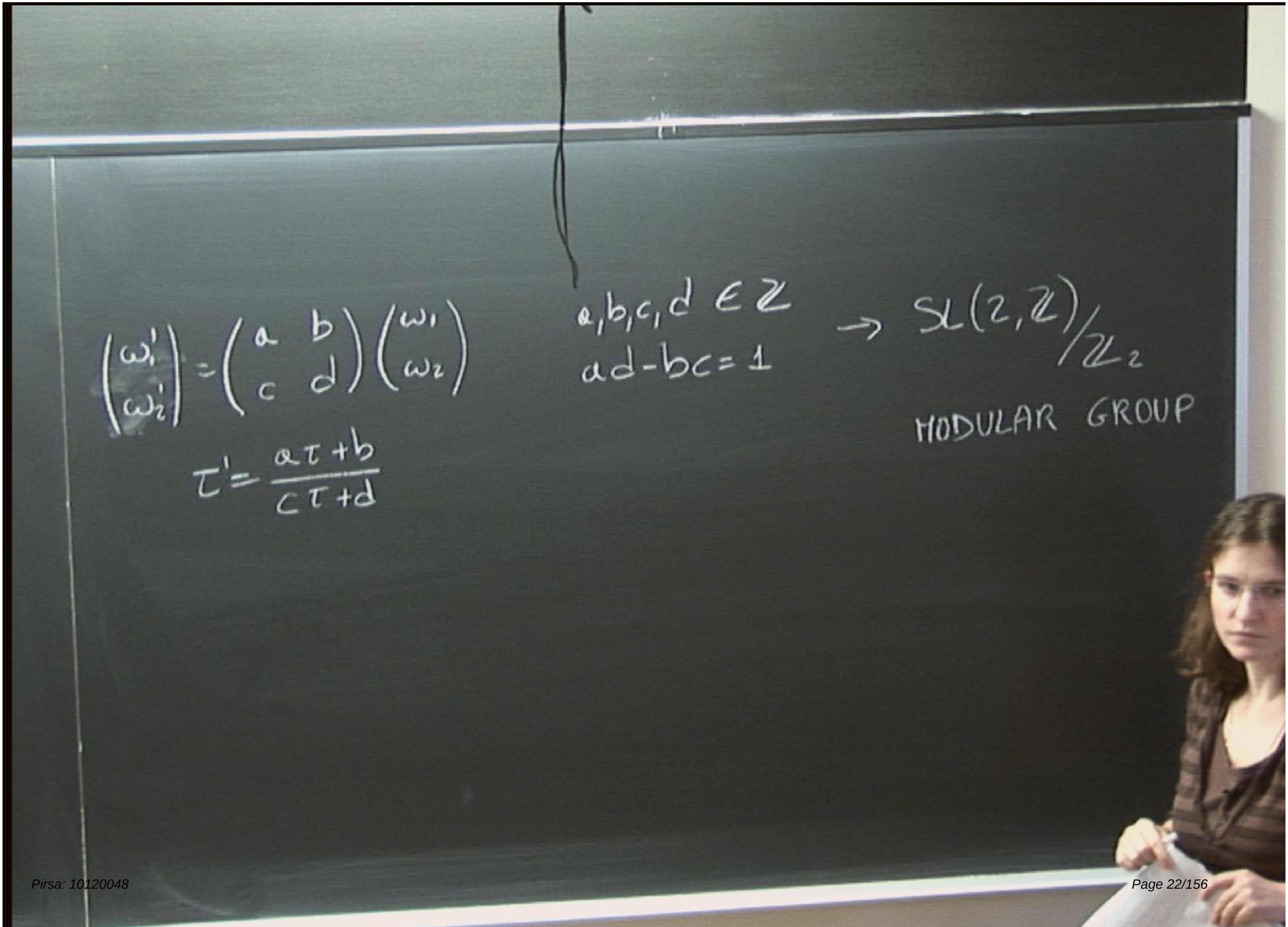
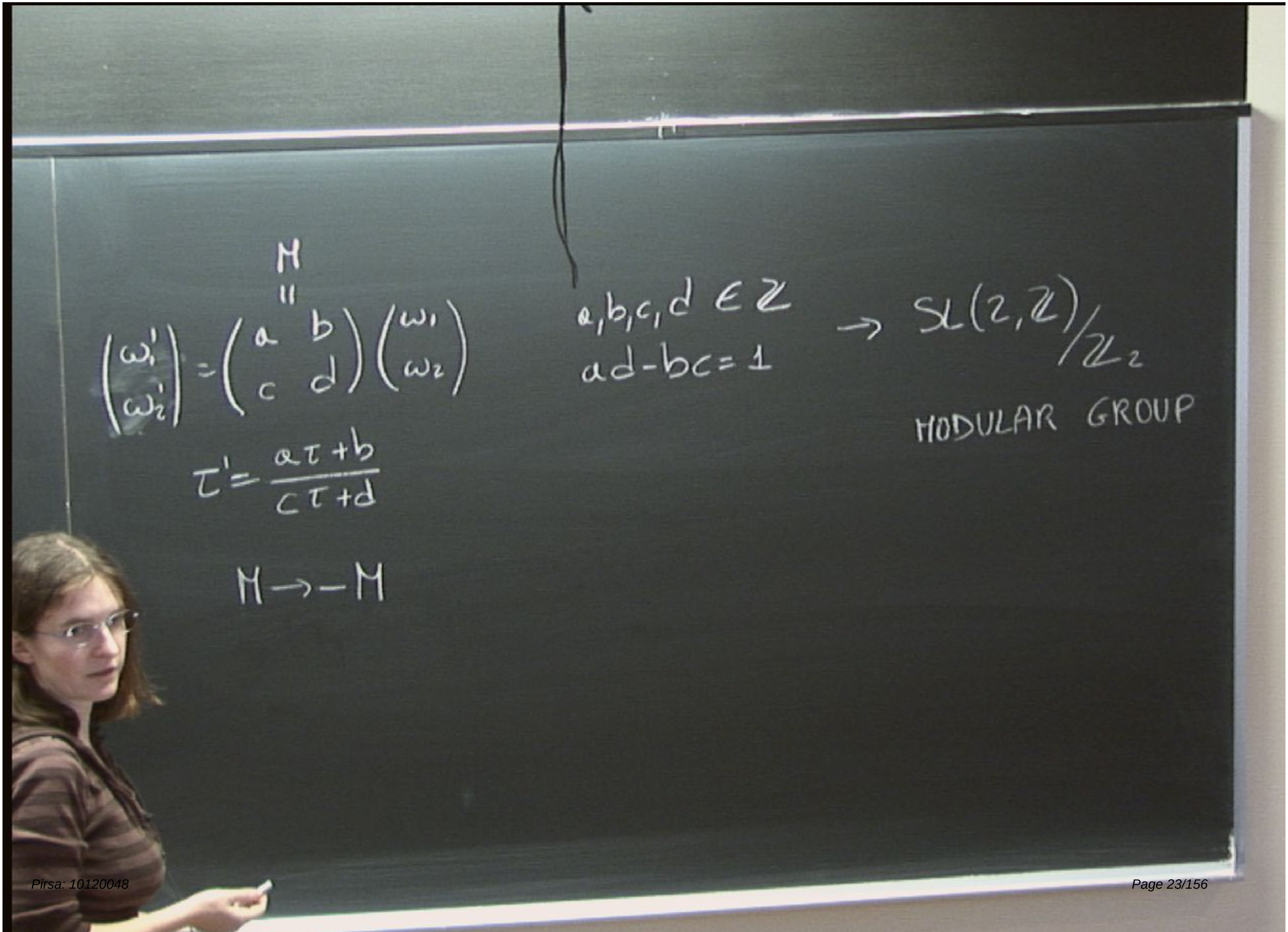
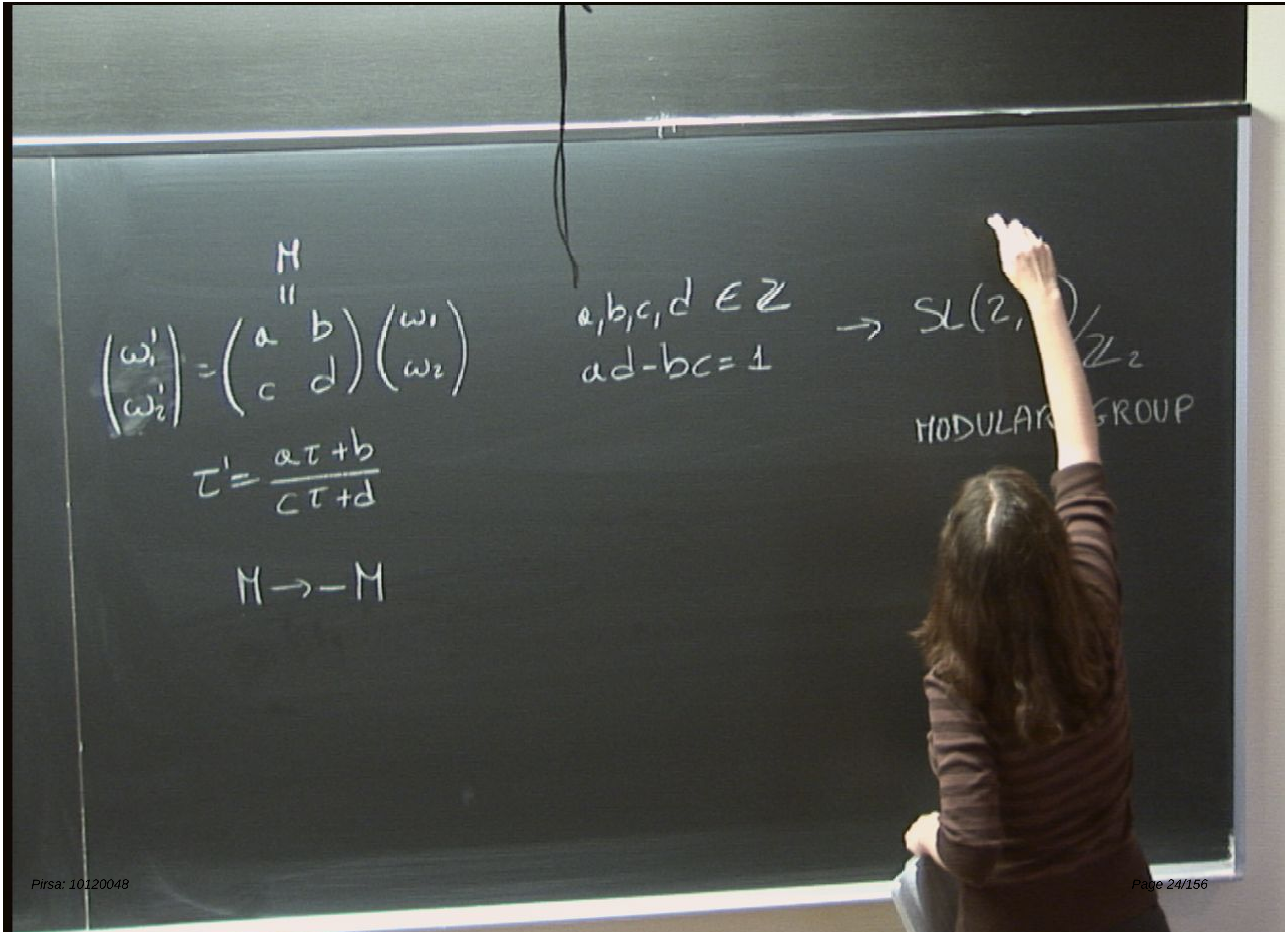
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\rightarrow$
$\tau' = \frac{a\tau + b}{c\tau + d}$

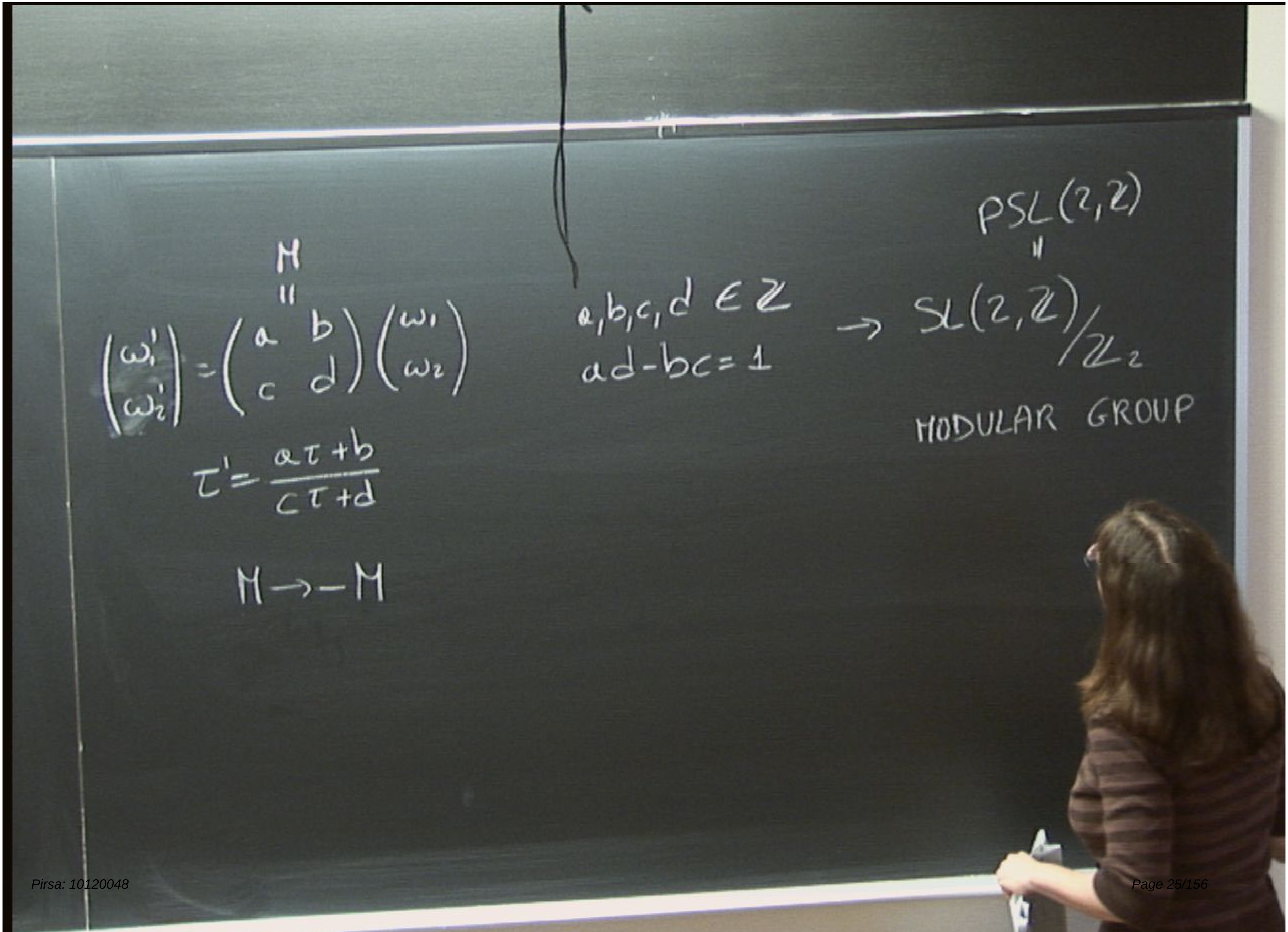

$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$a, b, c, d \in \mathbb{Z}$ $\rightarrow$ $SL(2,\mathbb{Z})$
$ad - bc = 1$
$\tau' = \frac{a\tau + b}{c\tau}$

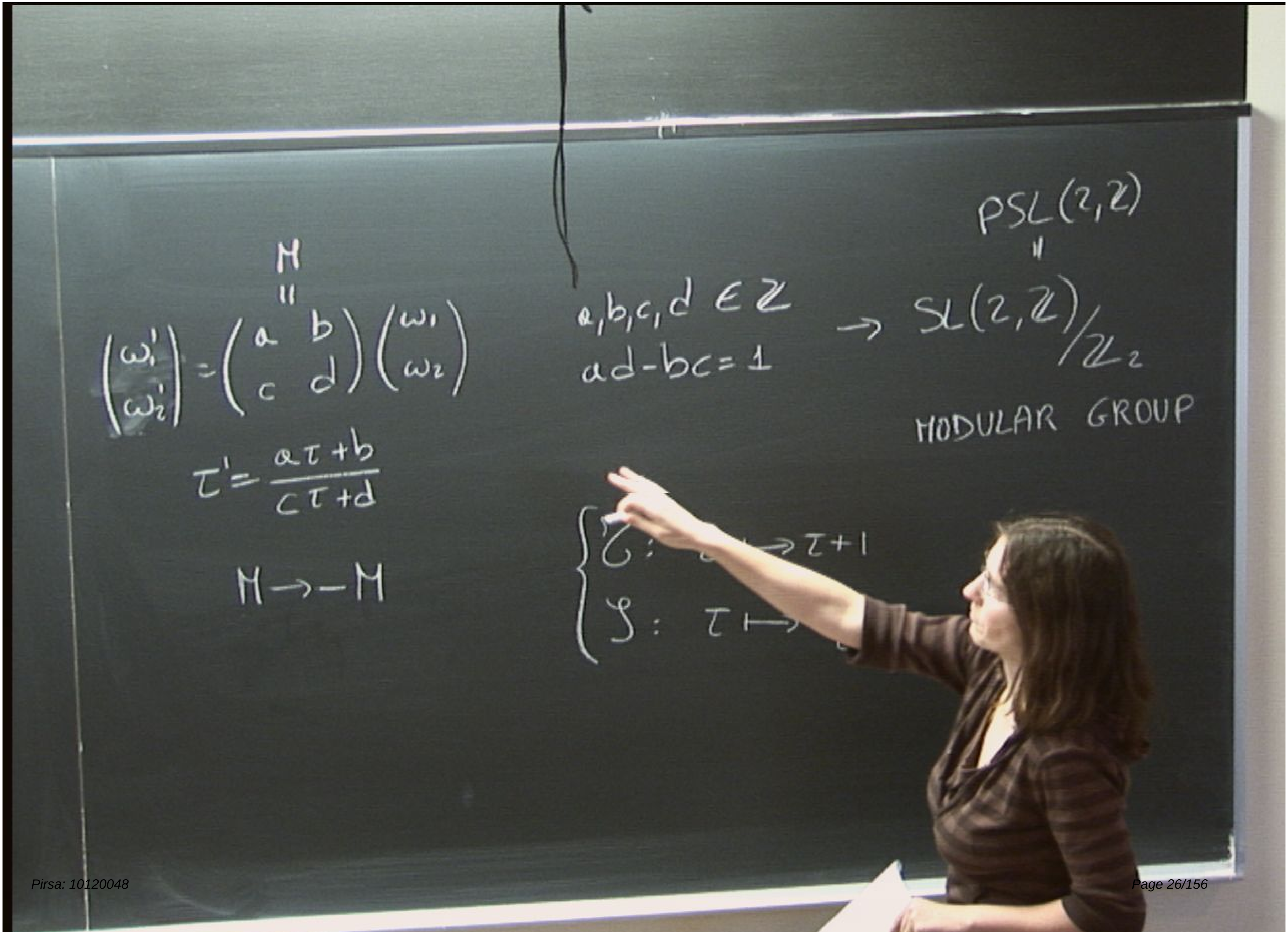

$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$\tau' = \frac{a\tau + b}{c\tau + d}$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\rightarrow SL(2,\mathbb{Z})/\mathbb{Z}_2$
MODULAR GROUP

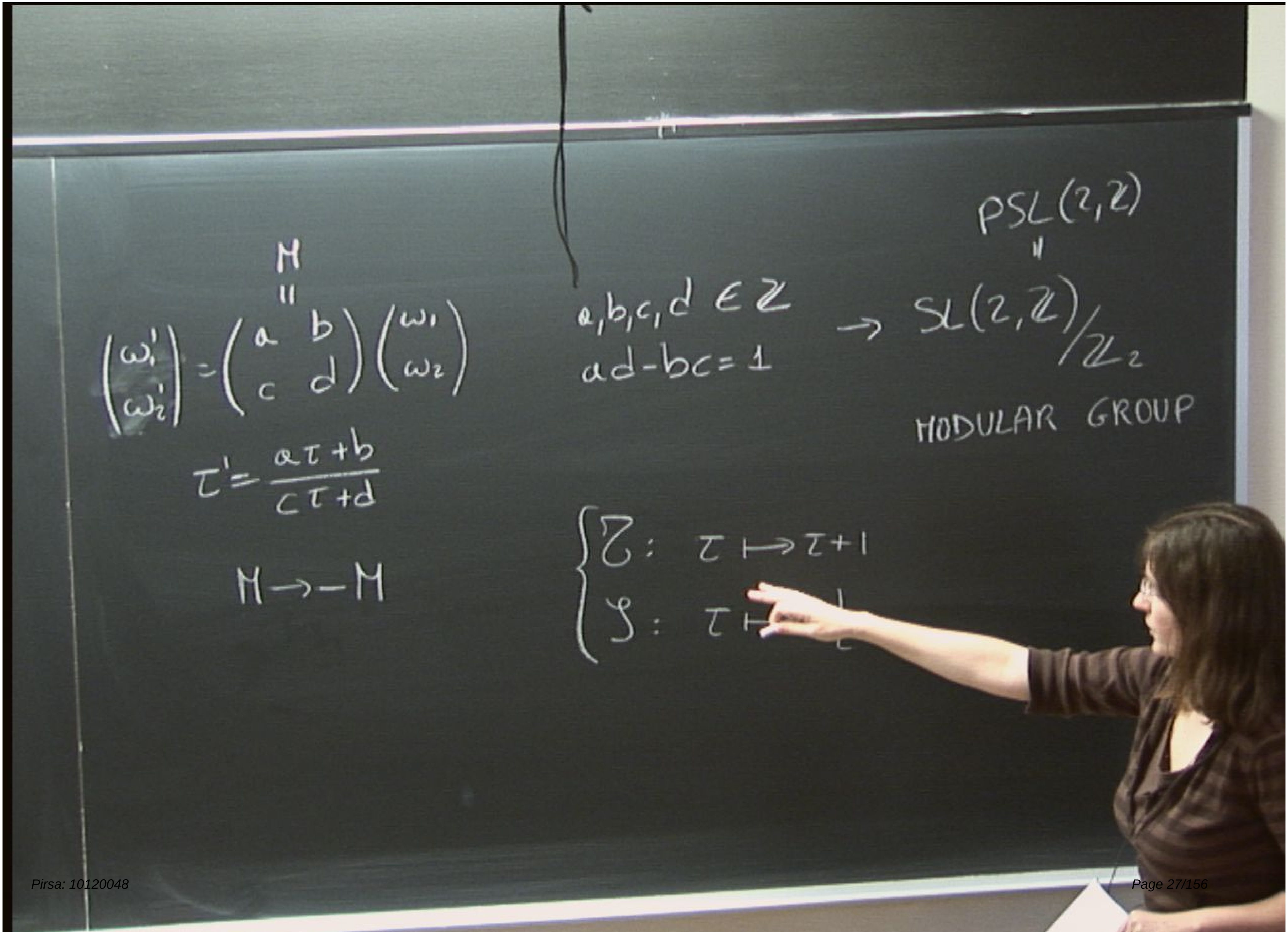

M
$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$\tau' = \frac{a\tau + b}{c\tau + d}$
$M \to -M$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\to SL(2,\mathbb{Z}) / \mathbb{Z}_2$
MODULAR GROUP

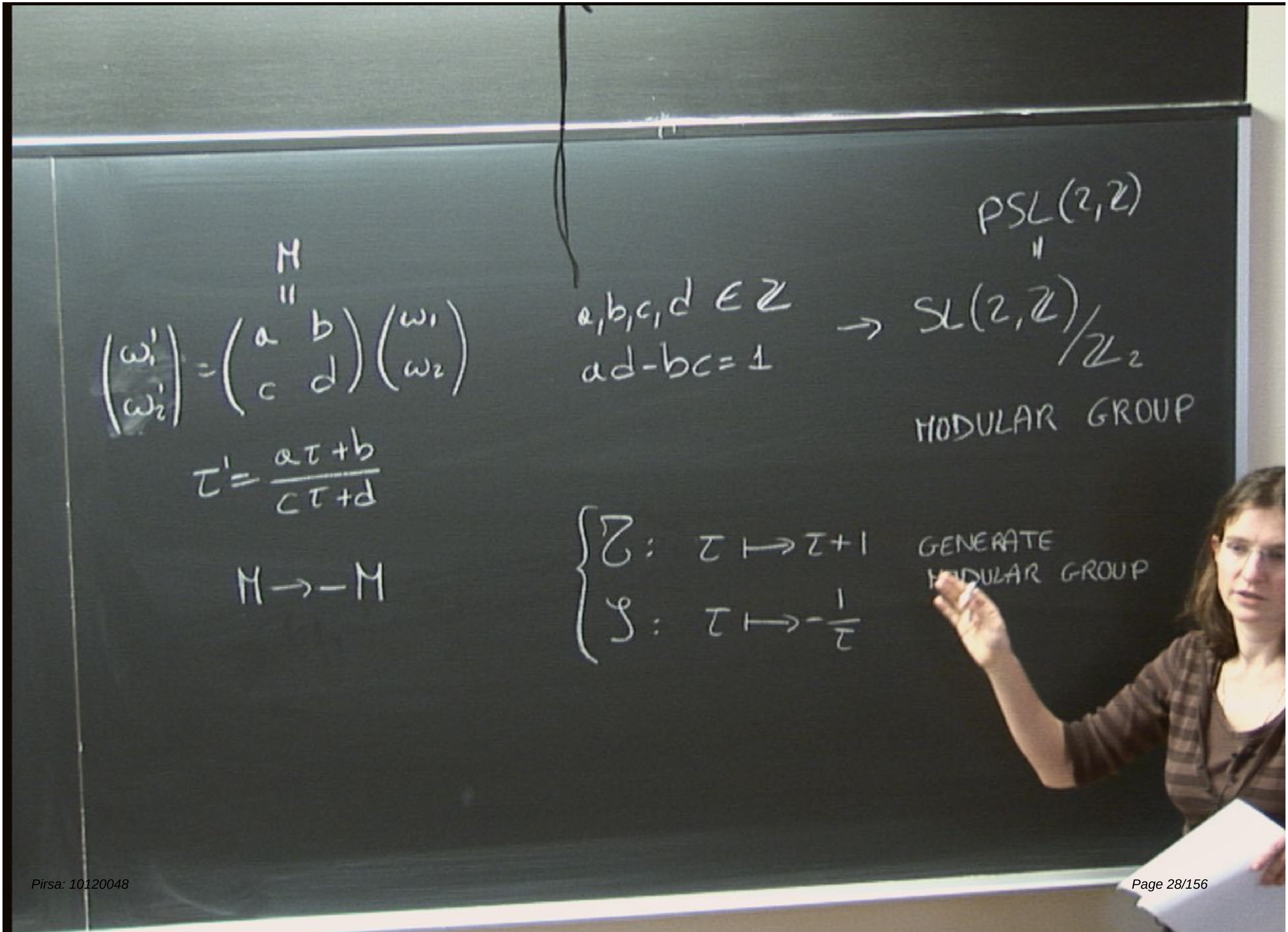


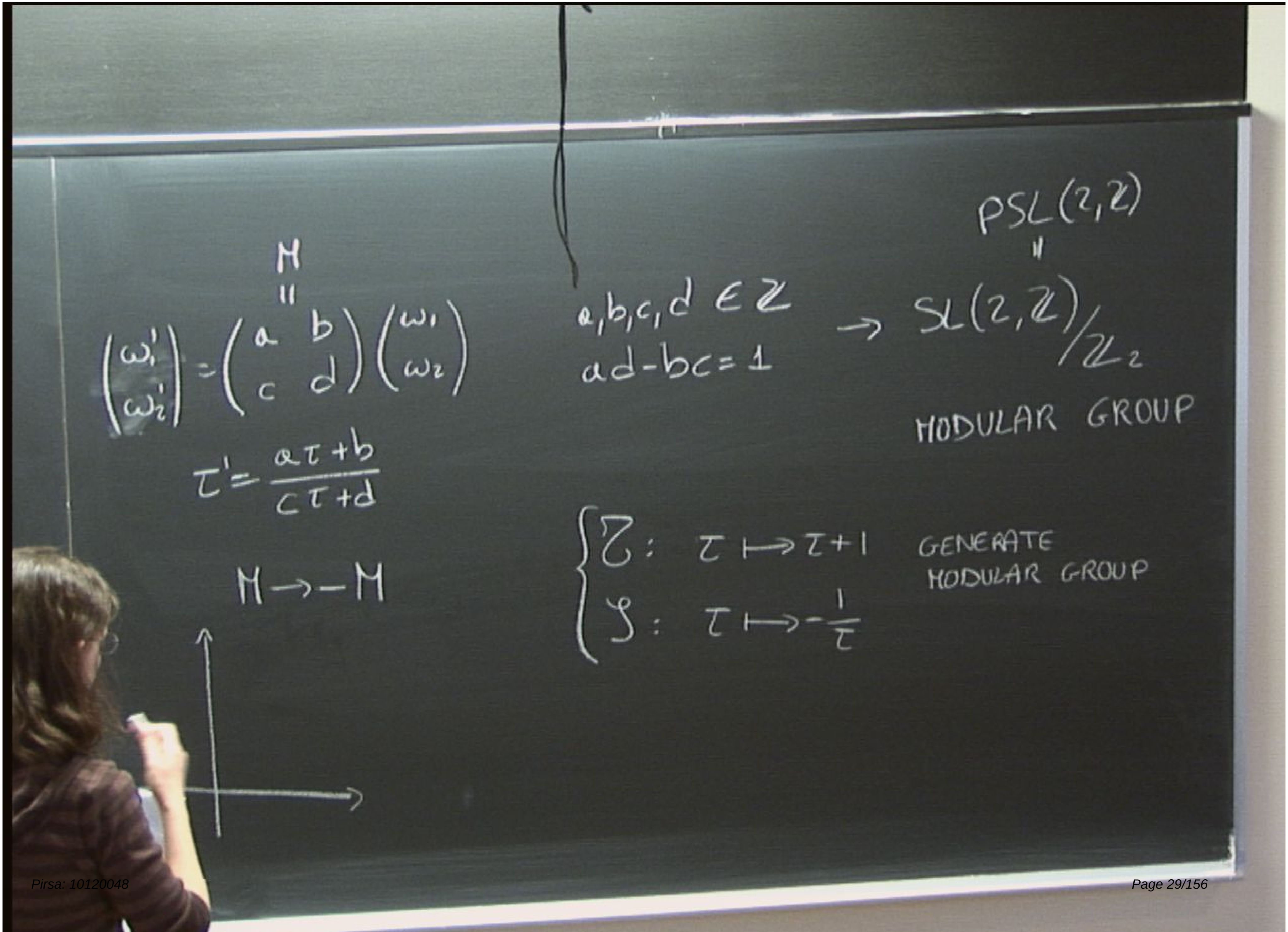
M
$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$\tau' = \frac{a\tau + b}{c\tau + d}$
$M \to -M$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\to SL(2, \quad) / \mathbb{Z}_2$
MODULAR GROUP

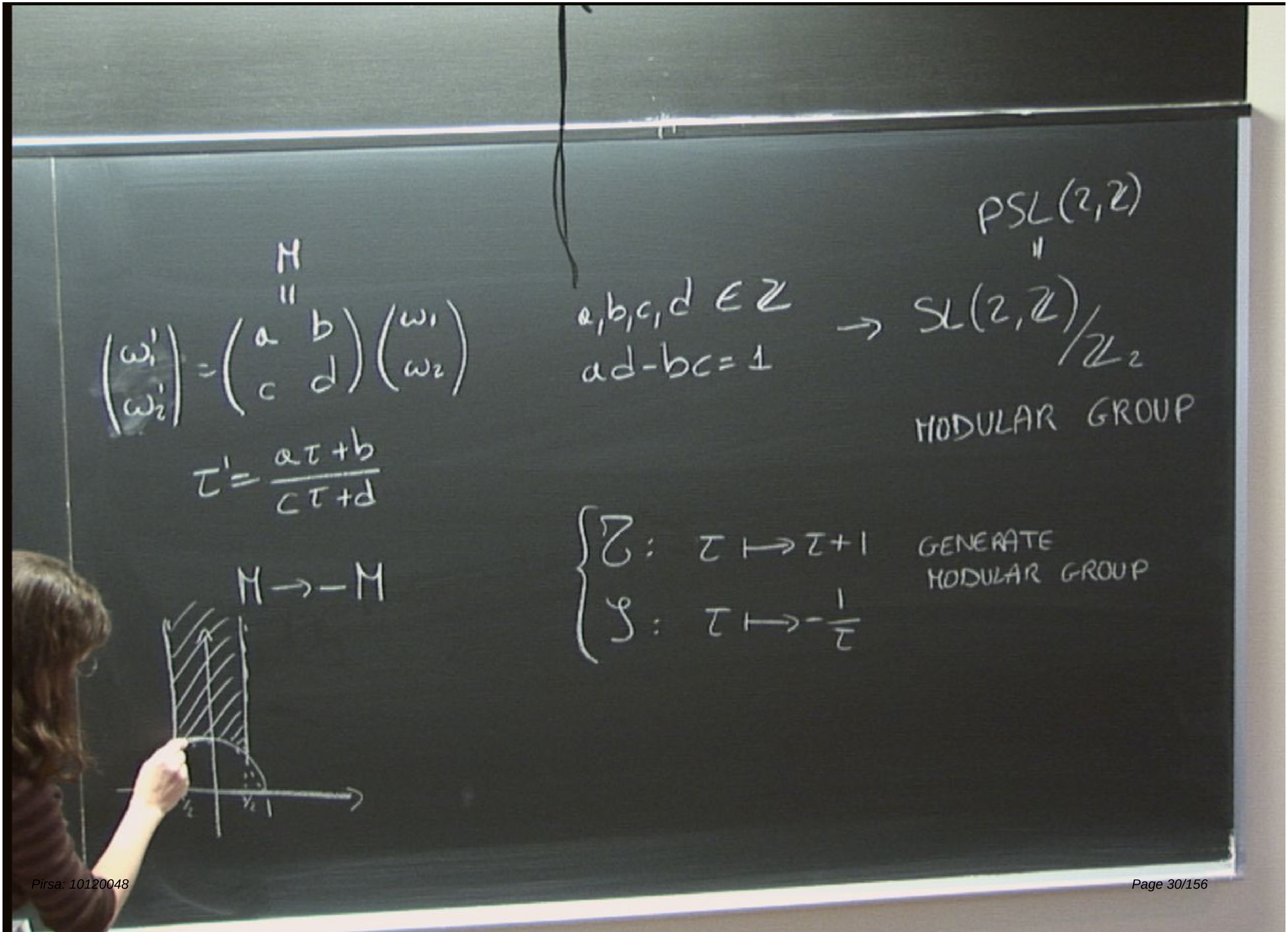

$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$
M
$\tau' = \frac{a\tau + b}{c\tau + d}$
$M \to -M$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\to SL(2,\mathbb{Z})/\mathbb{Z}_2$
$PSL(2,\mathbb{Z})$
MODULAR GROUP


M
$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$\tau' = \frac{a\tau + b}{c\tau + d}$
$M \to -M$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\to SL(2,\mathbb{Z})/\mathbb{Z}_2$
$PSL(2,\mathbb{Z})$
MODULAR GROUP
$\tau + 1$
$S : \tau \mapsto$


M
$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$\tau' = \frac{a\tau + b}{c\tau + d}$
$M \to -M$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\to SL(2,\mathbb{Z})/\mathbb{Z}_2$
$PSL(2,\mathbb{Z})$
MODULAR GROUP
$\mathcal{T}: \tau \mapsto \tau + 1$


M
$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
$\tau' = \frac{a\tau + b}{c\tau + d}$
$M \to -M$
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\to SL(2,\mathbb{Z})/\mathbb{Z}_2$
$PSL(2,\mathbb{Z})$
MODULAR GROUP
$\mathcal{T}: \tau \mapsto \tau + 1$
$\mathcal{S}: \tau \mapsto -\frac{1}{\tau}$
GENERATE
MODULAR GROUP


$PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$
$\begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}$, $M = \begin{pmatrix}a & b \\ c & d\end{pmatrix}$
$a,b,c,d \in \mathbb{Z}$
$ad-bc=1$
MODULAR GROUP
$\tau' = \frac{a\tau+b}{c\tau+d}$
$M \to -M$
$\mathcal{T}: \tau \mapsto \tau+1$
$\mathcal{S}: \tau \mapsto -\frac{1}{\tau}$
GENERATE MODULAR GROUP

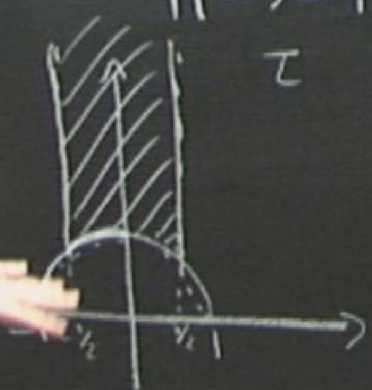

M
$\begin{pmatrix}\omega_1' \\ \omega_2'\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}$
$\tau' = \frac{a\tau + b}{c\tau + d}$
$M \to -M$
$\frac{1}{2}$ $\frac{1}{2}$ 1
$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$
$\to SL(2,\mathbb{Z})/\mathbb{Z}_2 = PSL(2,\mathbb{Z})$
MODULAR GROUP
$\mathcal{T}: \tau \mapsto \tau + 1$
$\mathcal{S}: \tau \mapsto -\frac{1}{\tau}$
GENERATE MODULAR GROUP

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

$$M \to -M$$



$a, b, c, d \in \mathbb{Z}$

$ad - bc = 1$ $\rightarrow$ $SL(2,\mathbb{Z})/\mathbb{Z}_2 = PSL(2,\mathbb{Z})$

MODULAR GROUP

$$\begin{cases} \mathcal{T}: & \tau \mapsto \tau + 1 \\ \mathcal{S}: & \tau \mapsto -\frac{1}{\tau} \end{cases}$$

GENERATE MODULAR GROUP

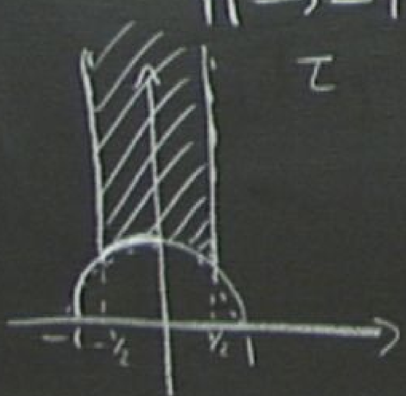
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

$$M \to -M$$

τ



$a, b, c, d \in \mathbb{Z}$

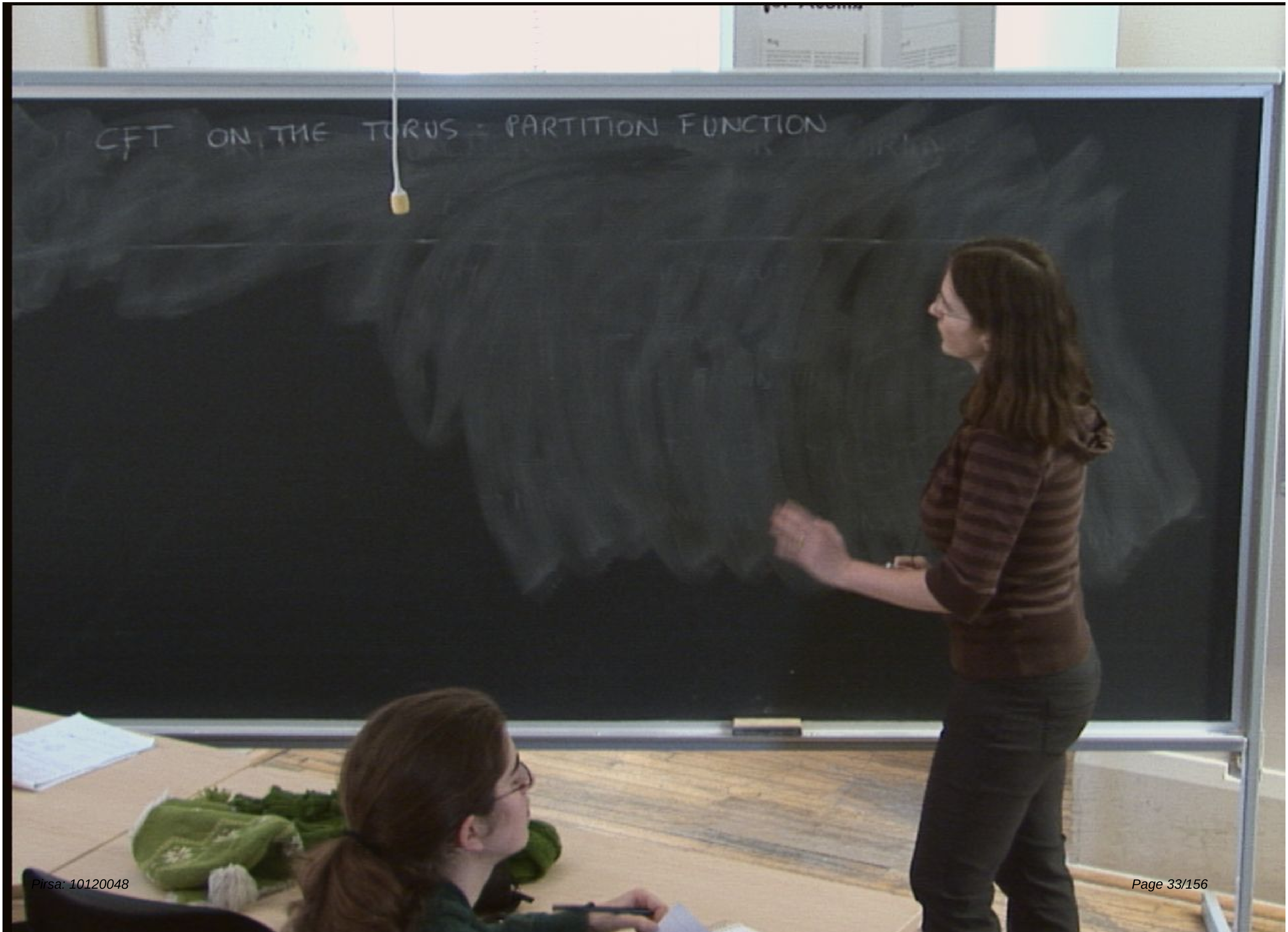
$ad - bc = 1$

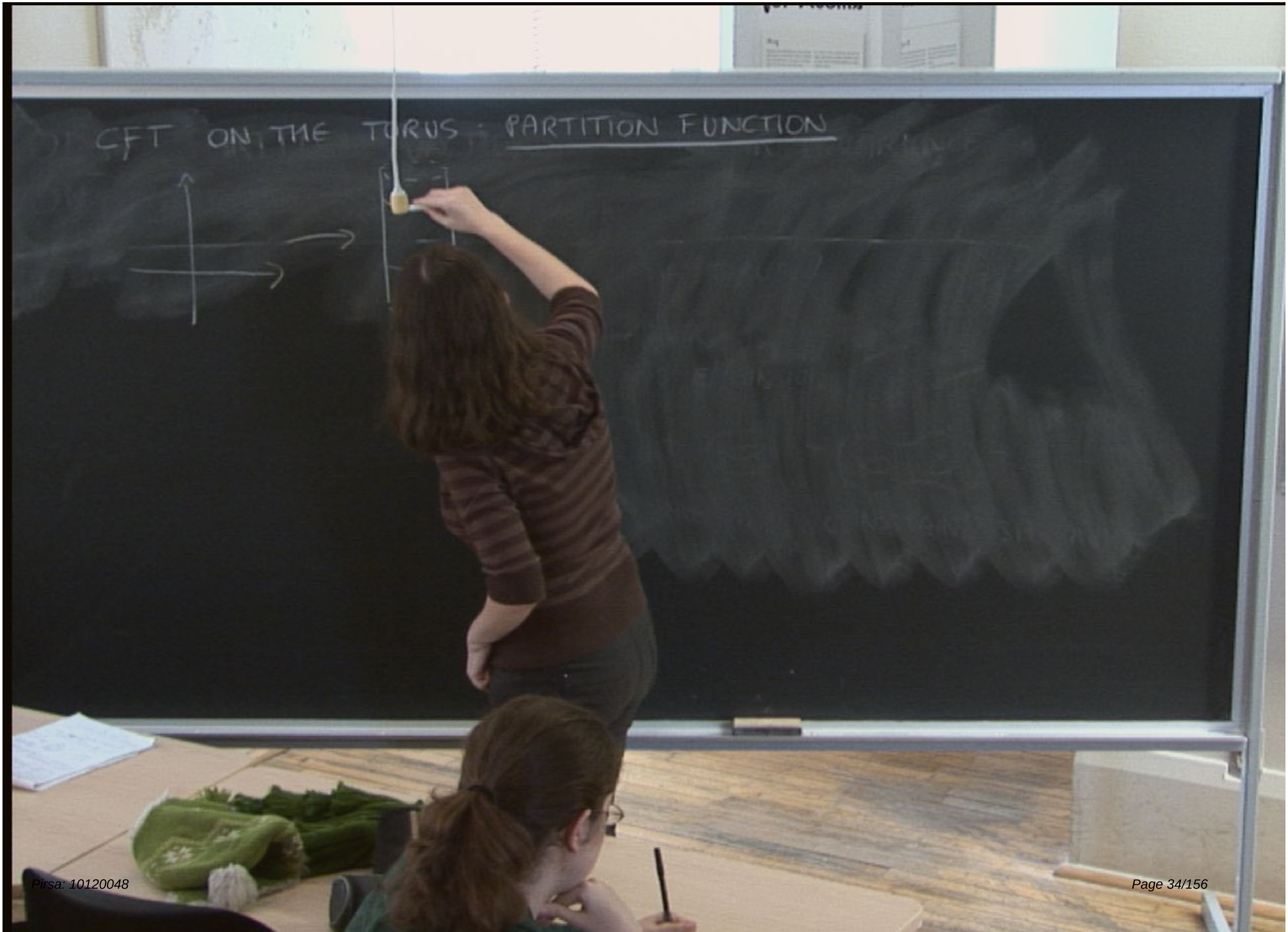
$$\to \quad SL(2,\mathbb{Z}) / \mathbb{Z}_2 = PSL(2,\mathbb{Z})$$

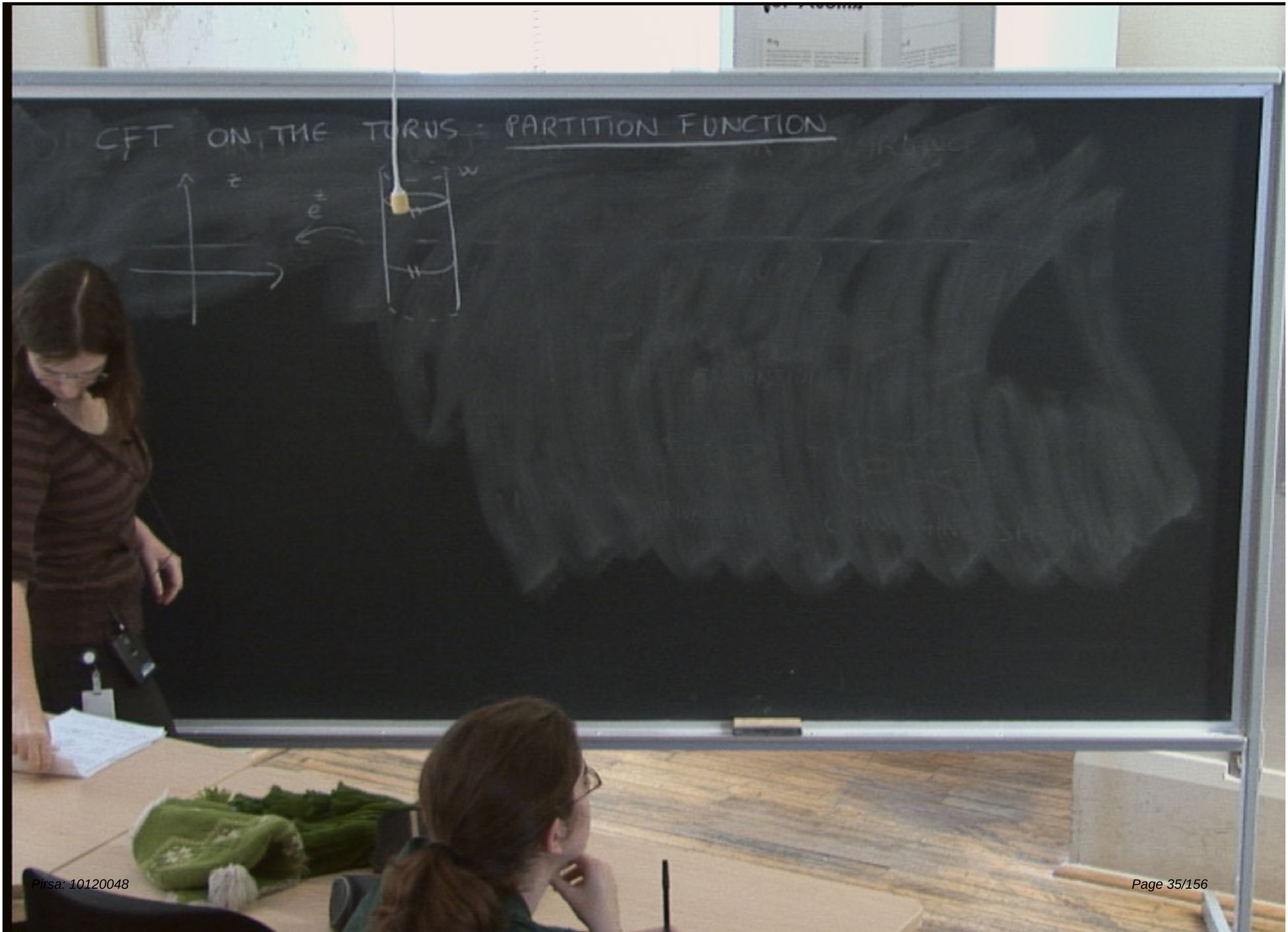
MODULAR GROUP

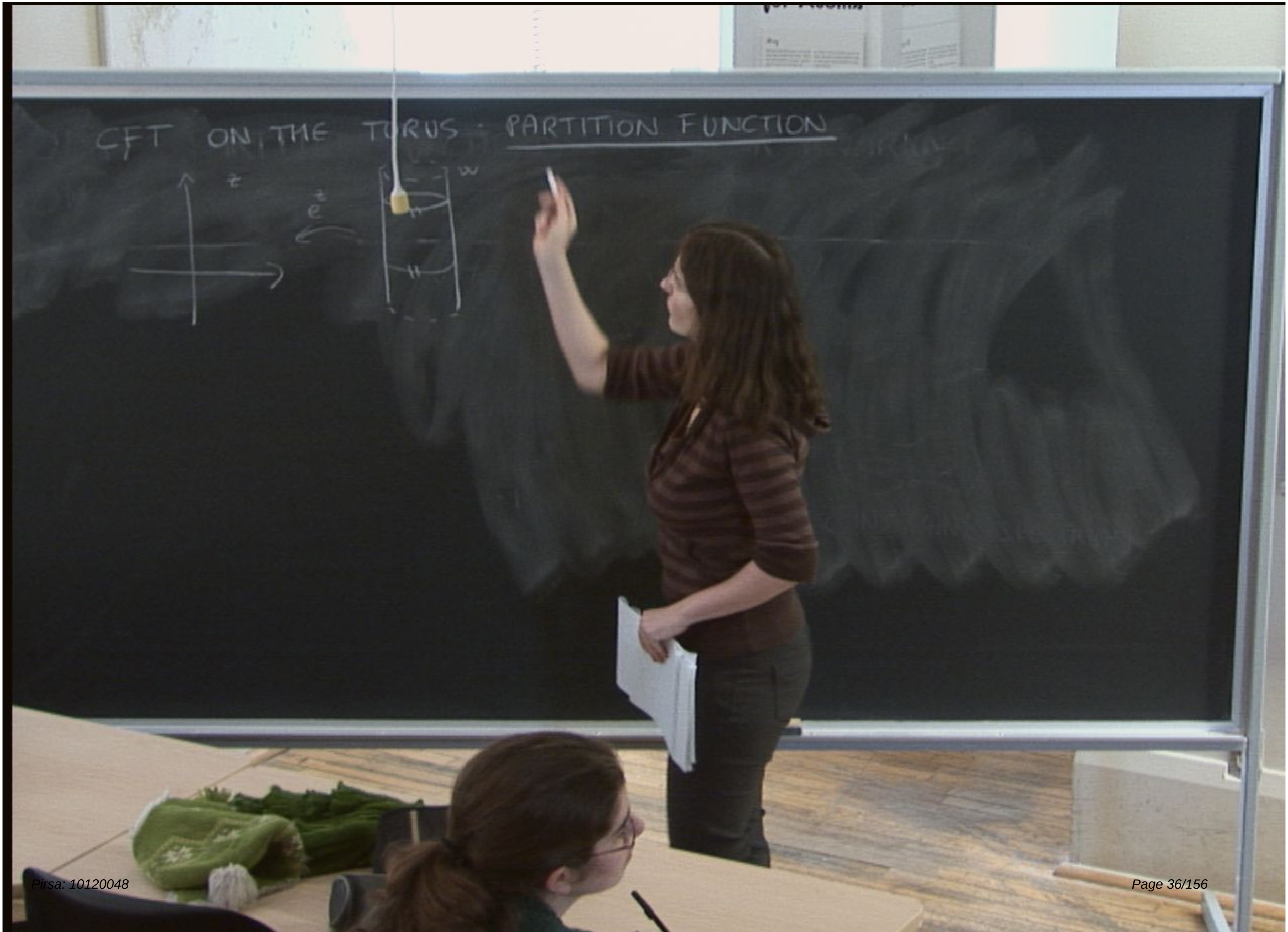
$$\begin{cases} \mathcal{T}: & \tau \mapsto \tau + 1 \\ \mathcal{S}: & \tau \mapsto -\frac{1}{\tau} \end{cases}$$

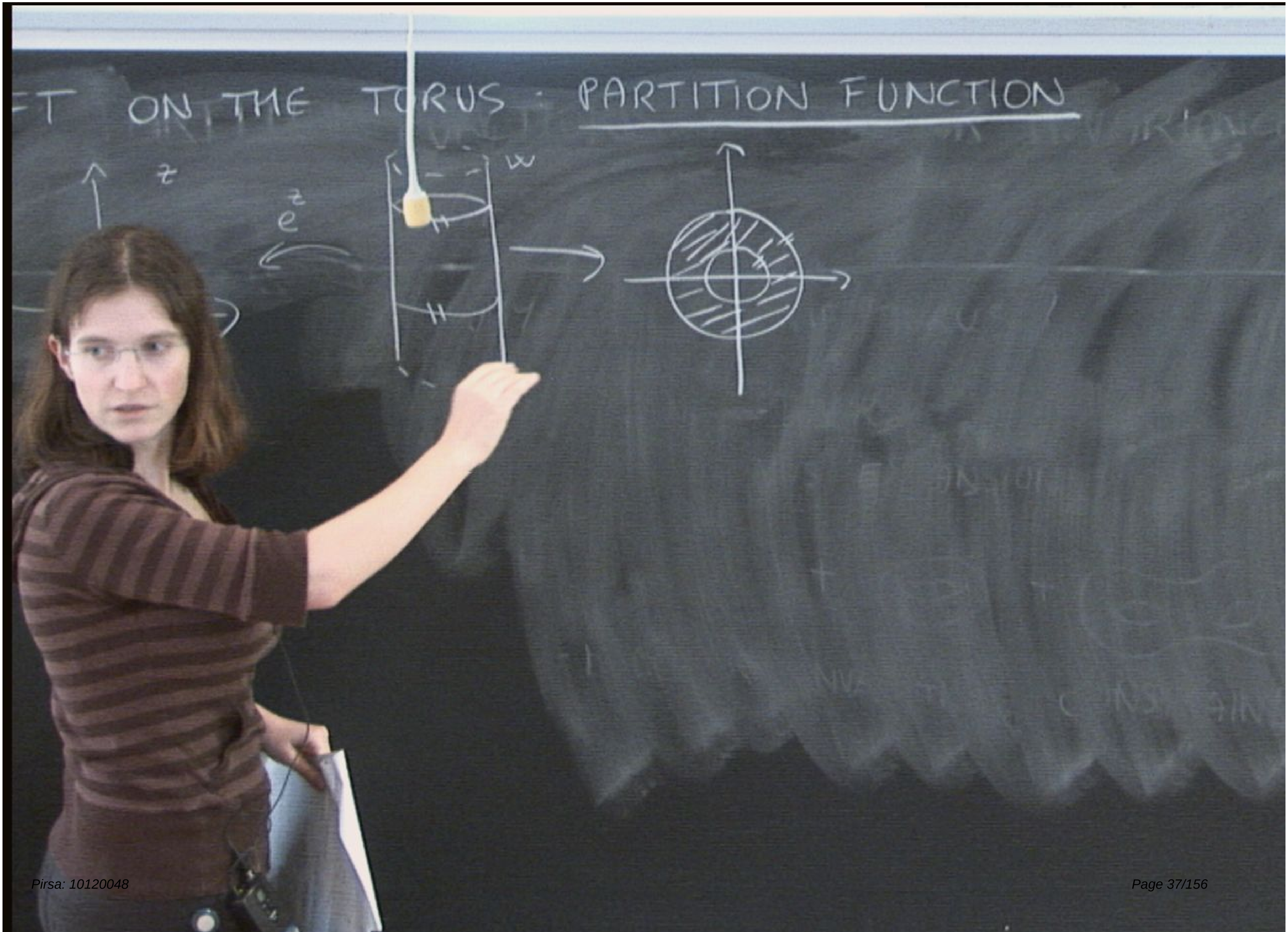
GENERATE MODULAR GROUP

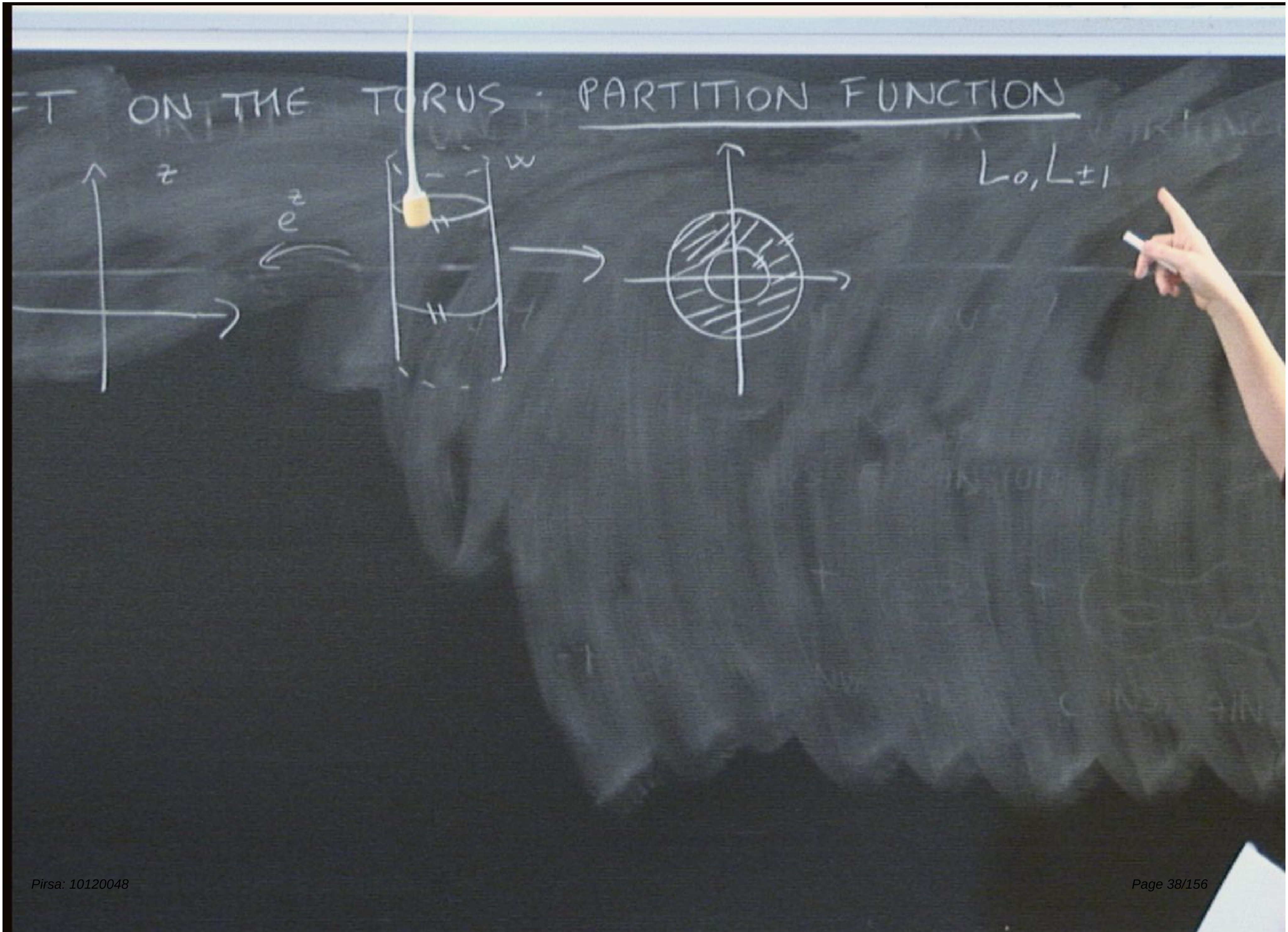

CFT ON THE TORUS : PARTITION FUNCTION

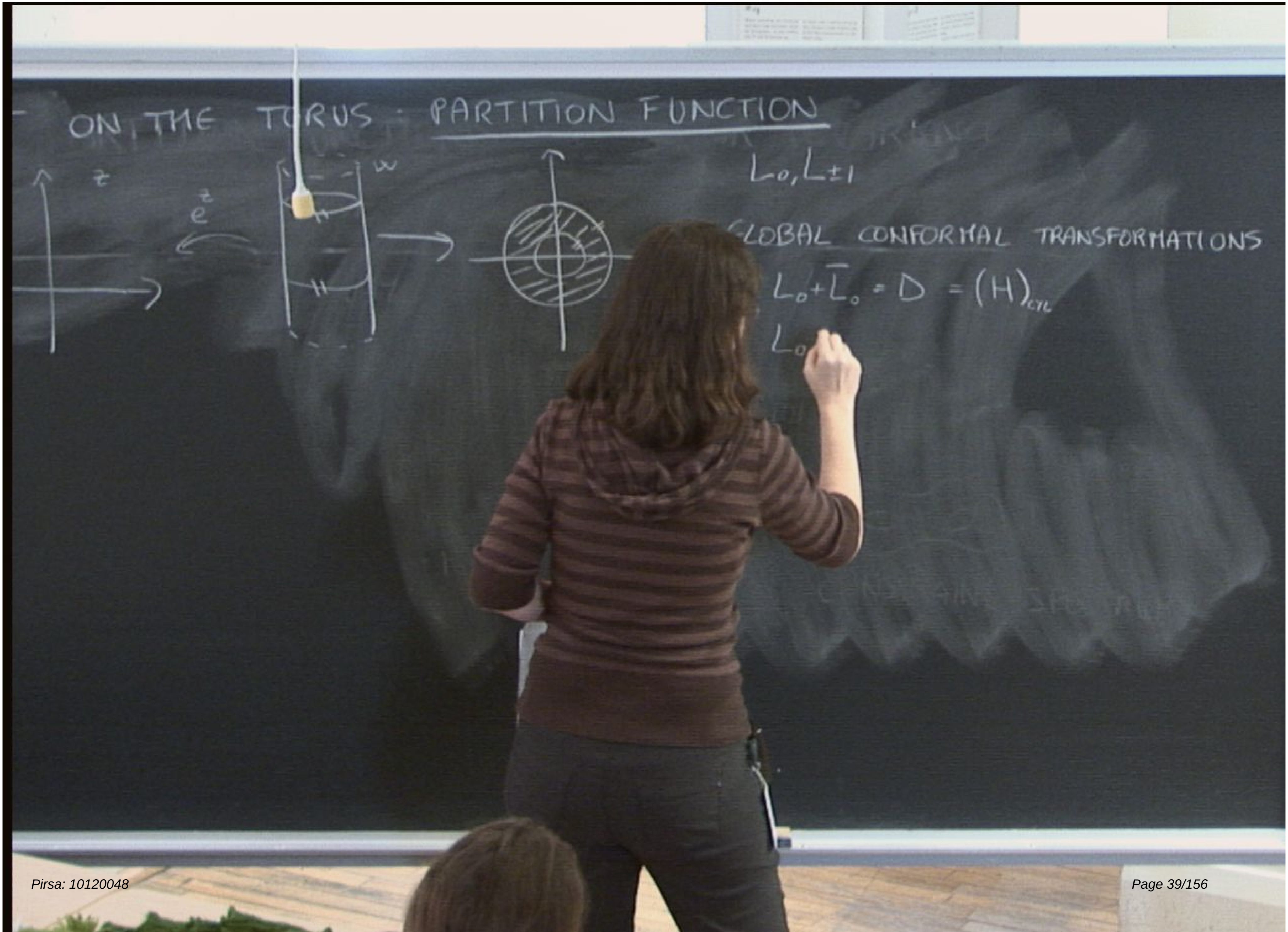

CFT ON THE TORUS : PARTITION FUNCTION

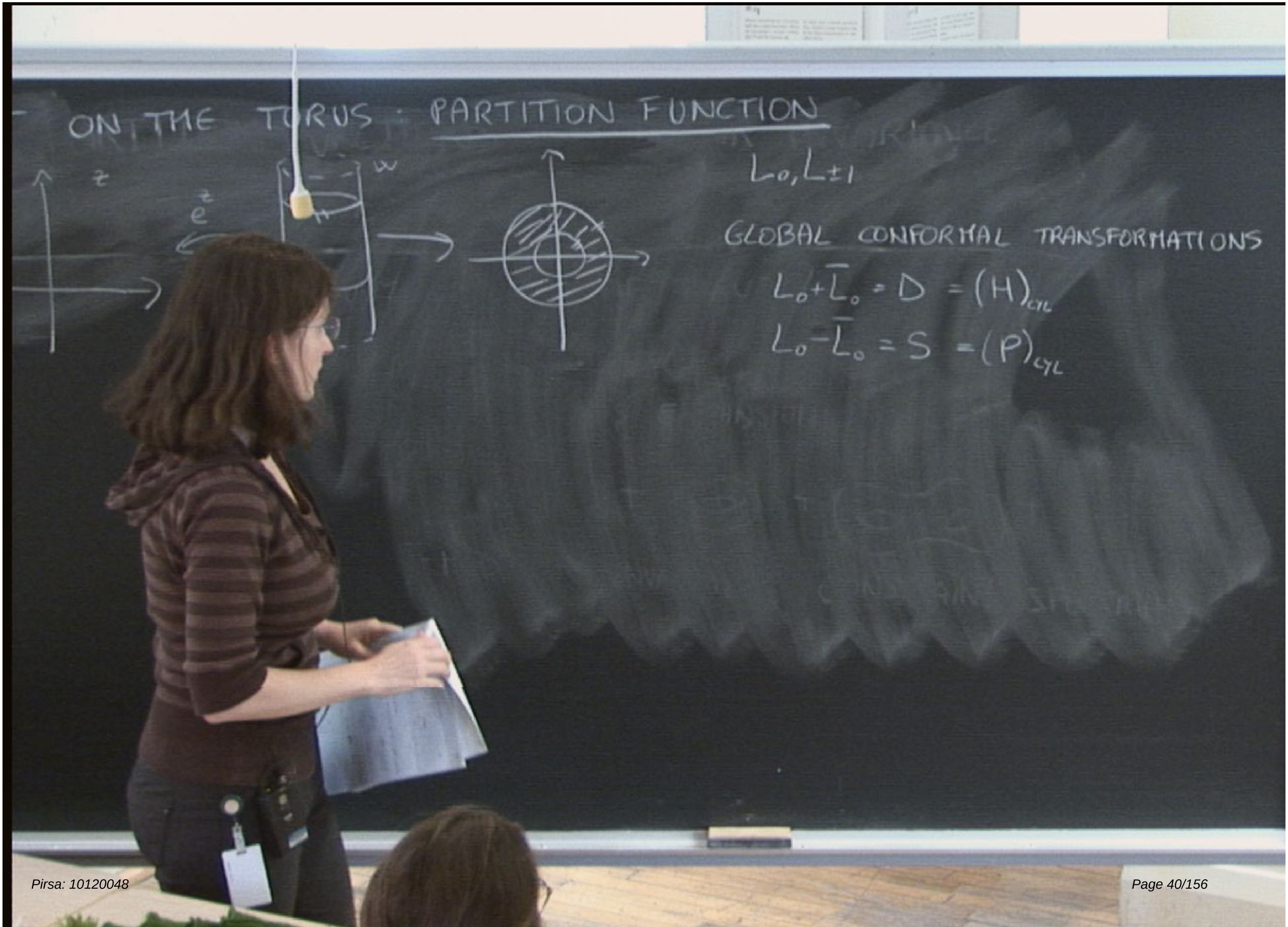

CFT ON THE TORUS : PARTITION FUNCTION
z
w

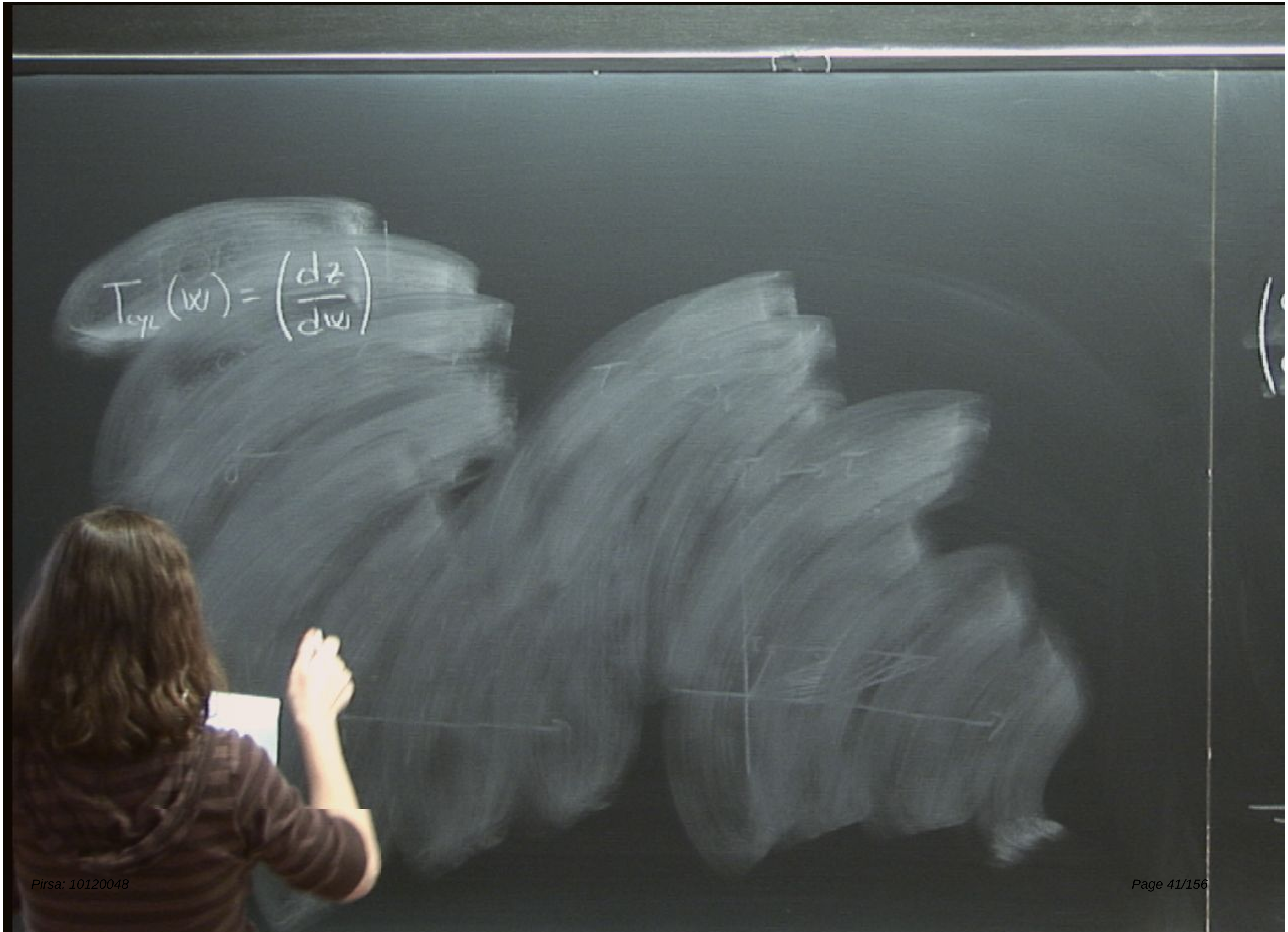

CFT ON THE TORUS : PARTITION FUNCTION
z
w


FT ON THE TORUS : PARTITION FUNCTION
z
e^z
w

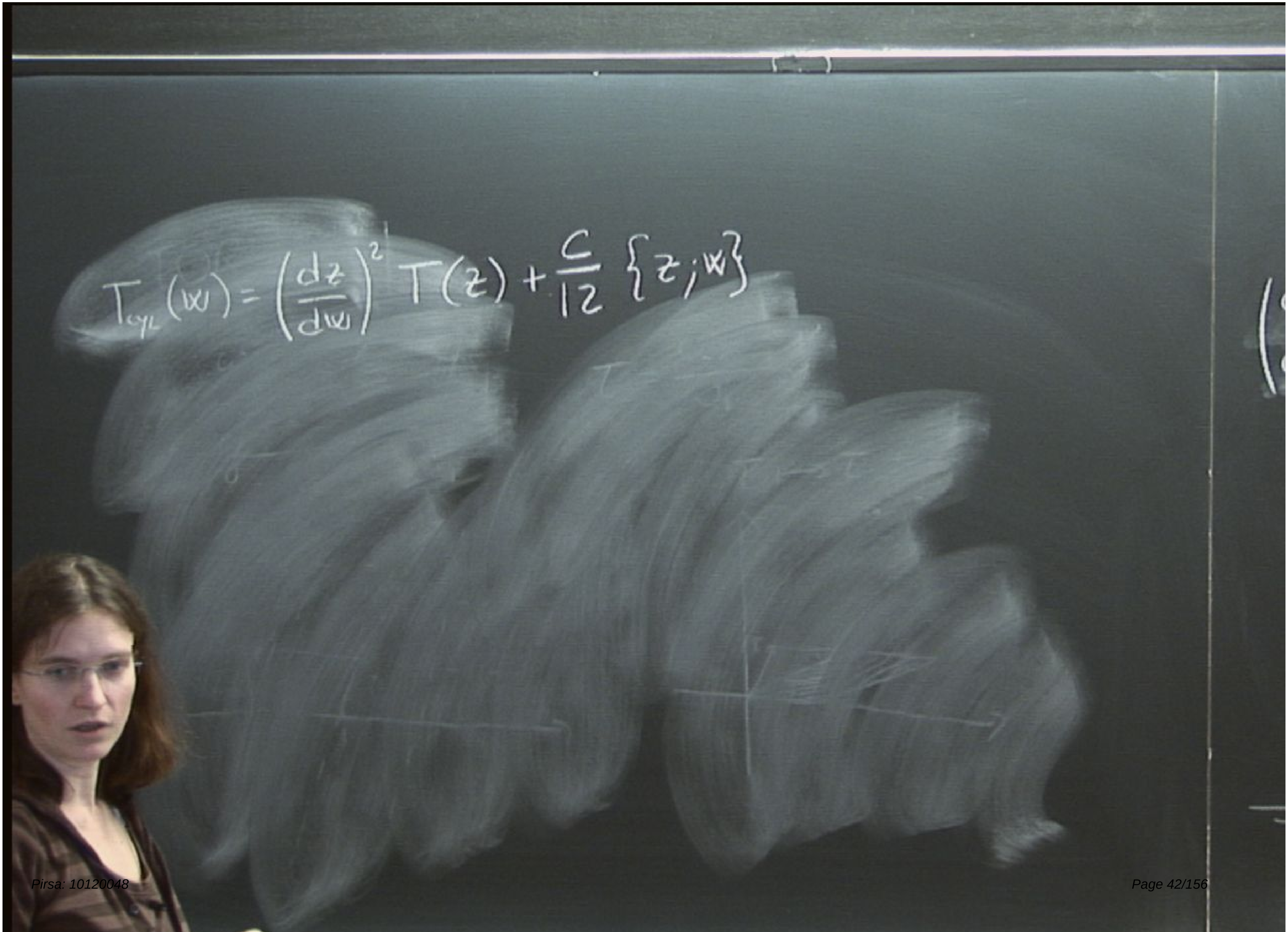

FT ON THE TORUS : PARTITION FUNCTION
z
e^z
w
L0, L±1

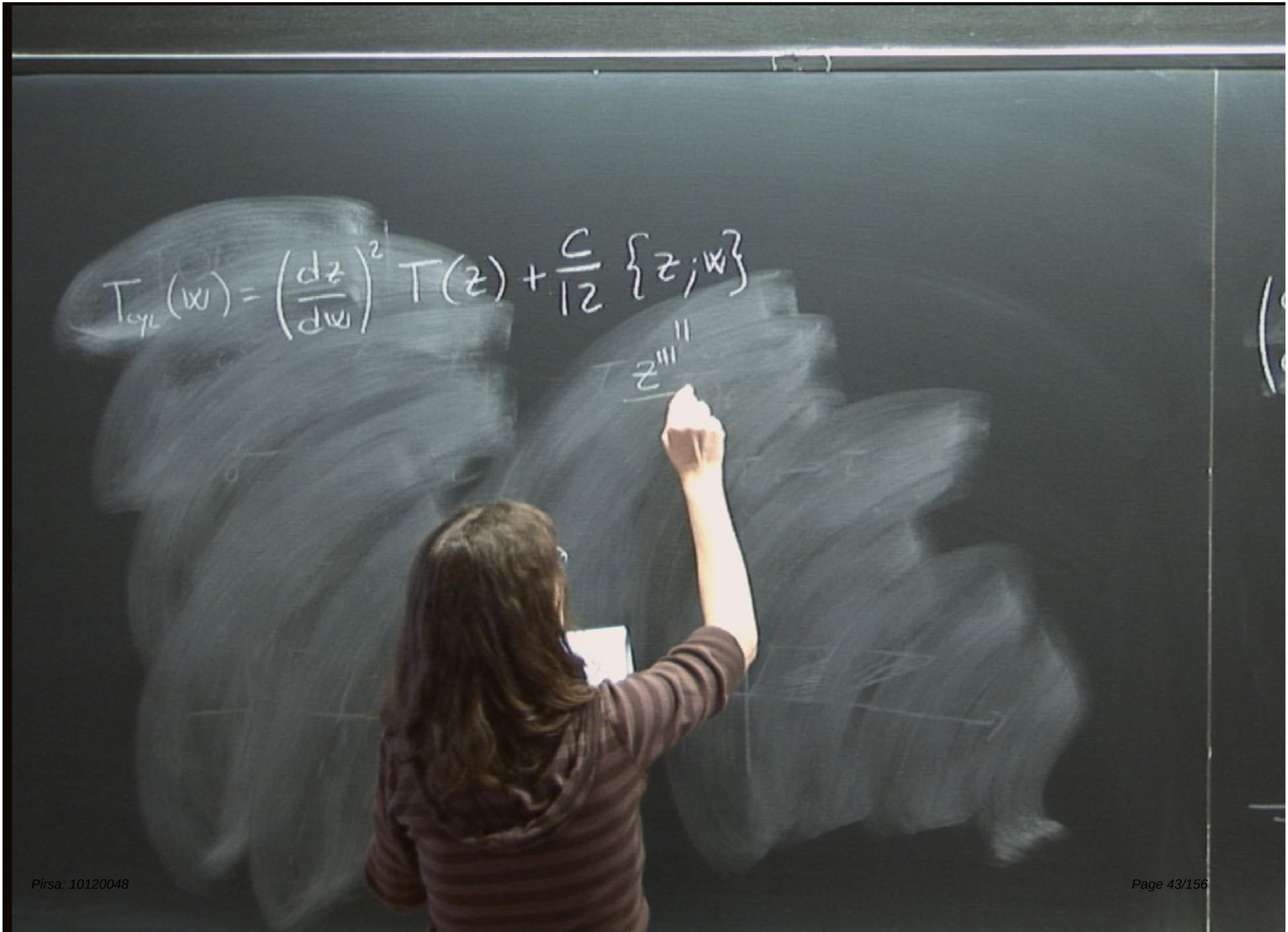

ON THE TORUS : PARTITION FUNCTION
z
w
e^z
$L_0, L_{\pm 1}$
GLOBAL CONFORMAL TRANSFORMATIONS
$L_0 + \bar{L}_0 = D = (H)_{cyl}$
L_0

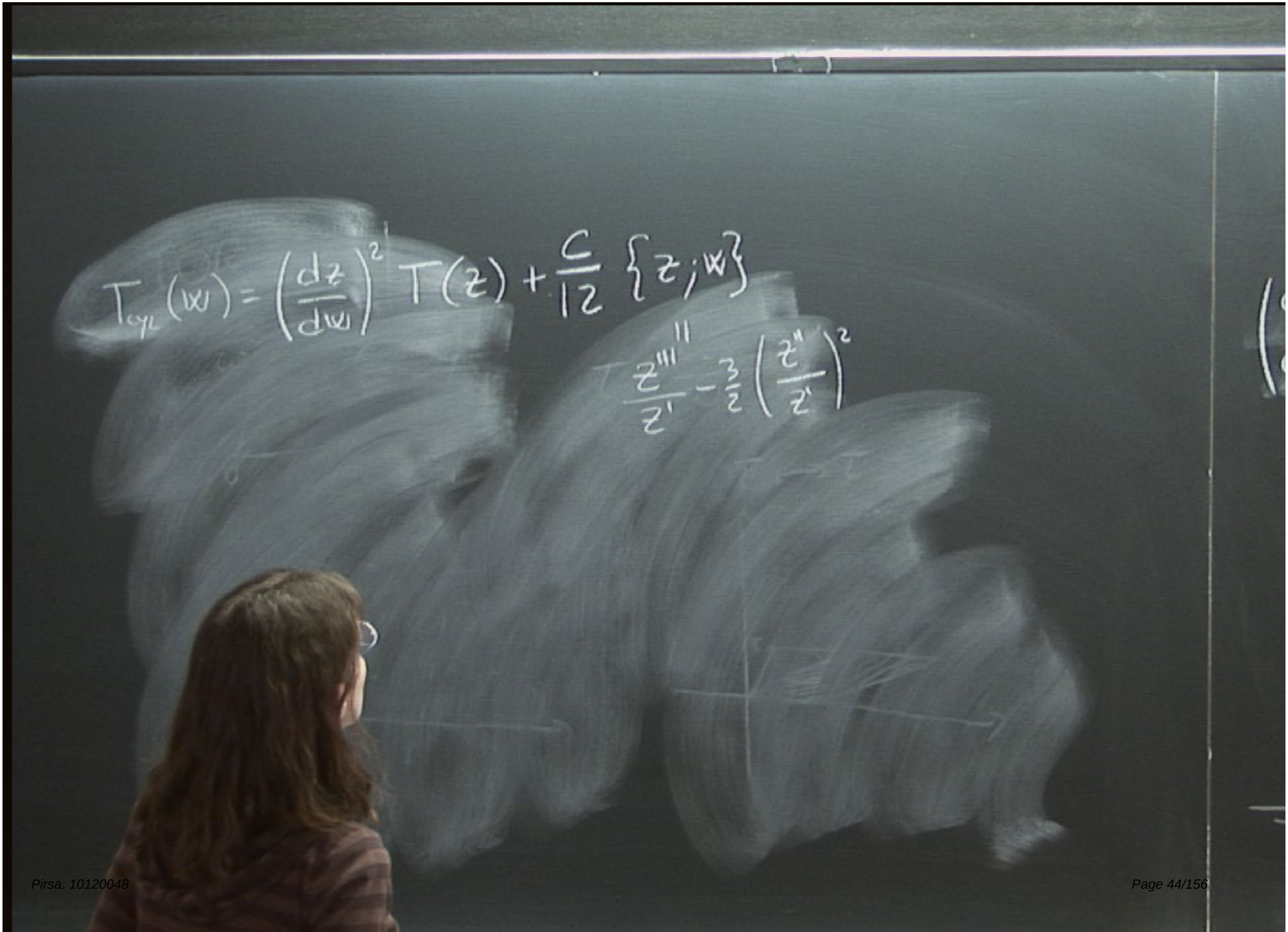

ON THE TORUS : PARTITION FUNCTION
z
w
$L_0, L_{\pm 1}$
GLOBAL CONFORMAL TRANSFORMATIONS
$L_0 + \bar{L}_0 = D = (H)_{CYL}$
$L_0 - \bar{L}_0 = S = (P)_{CYL}$

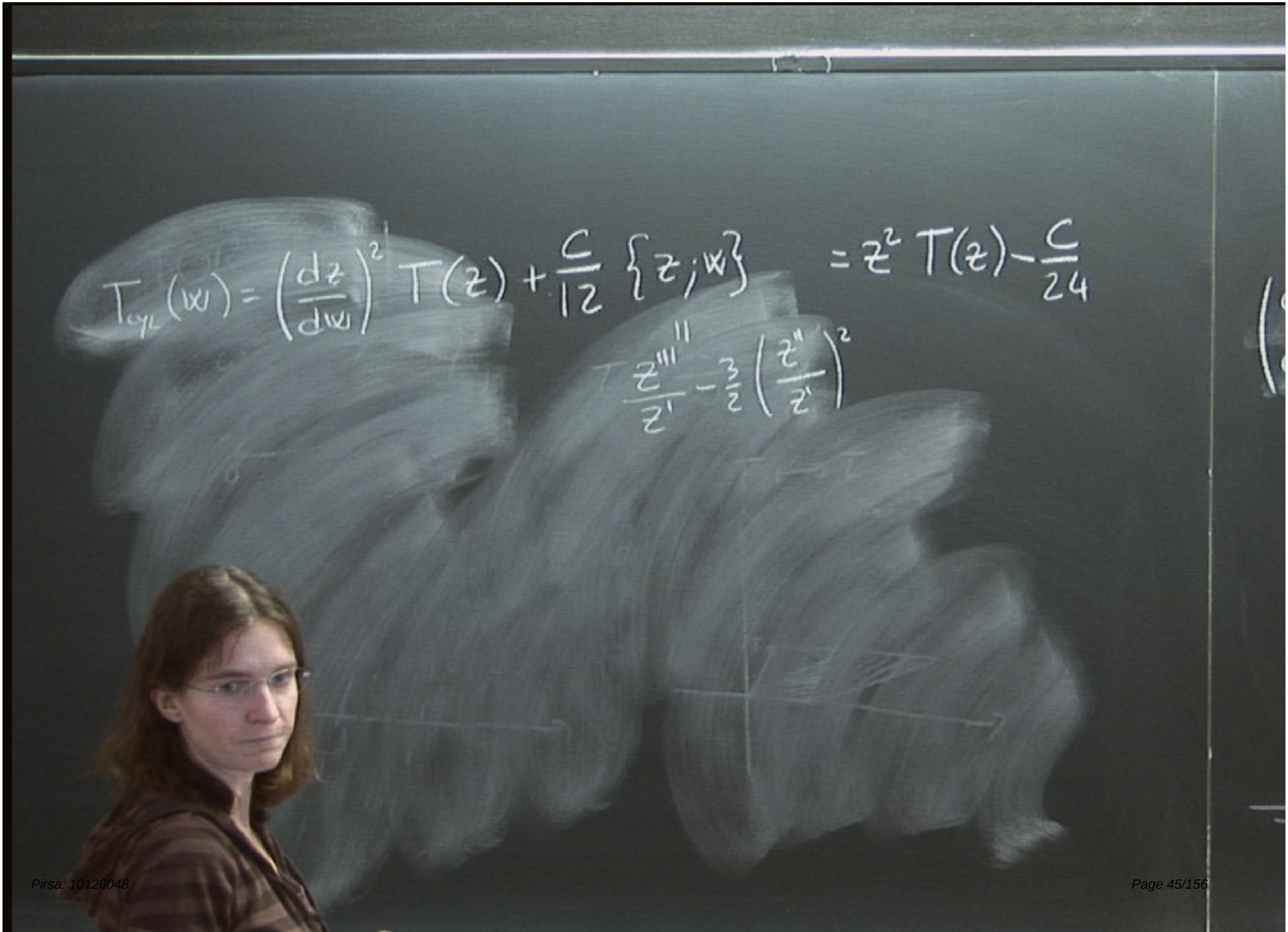


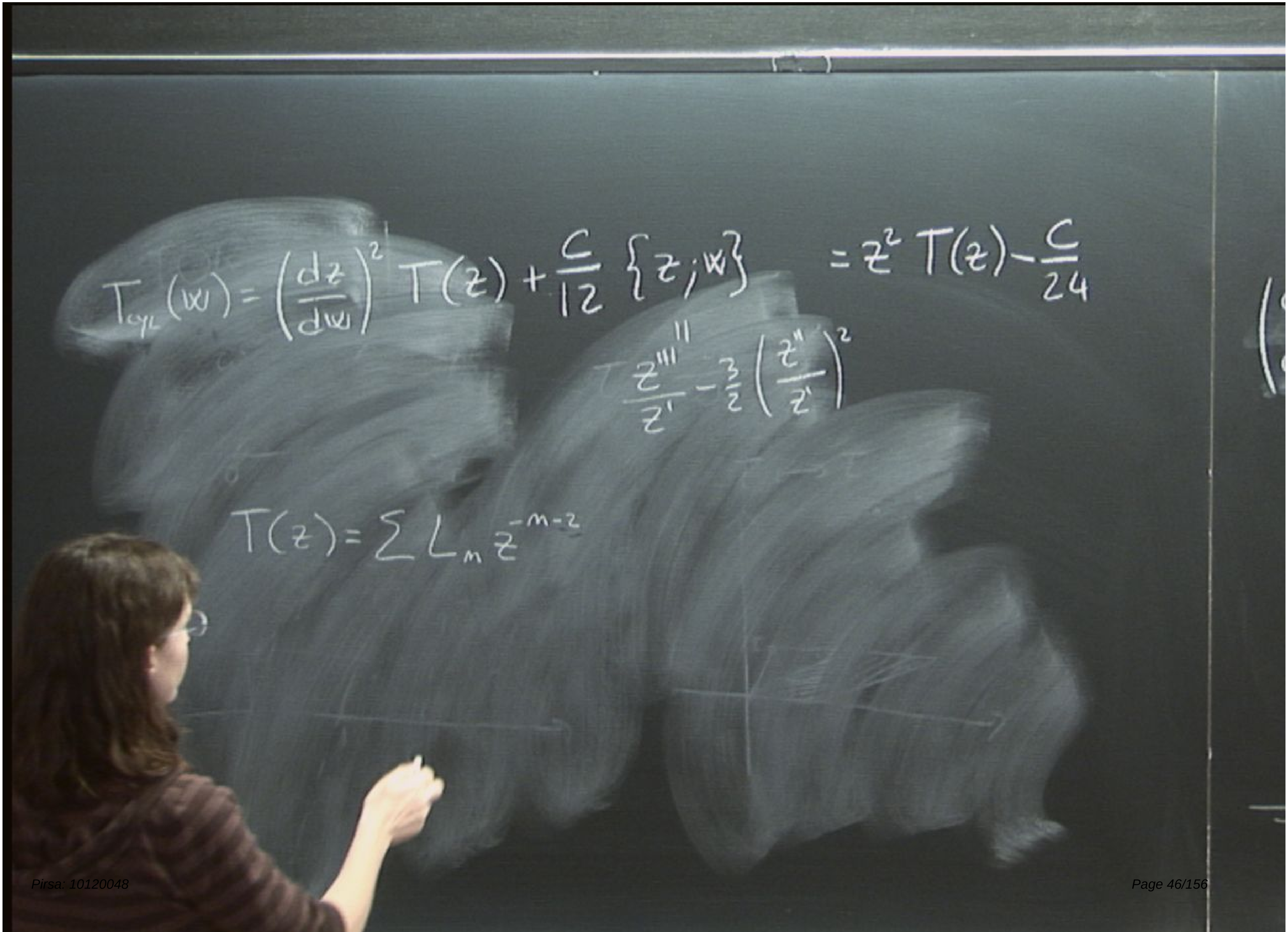
$T_{cyl}(w) = \left(\frac{dz}{dw}\right)$

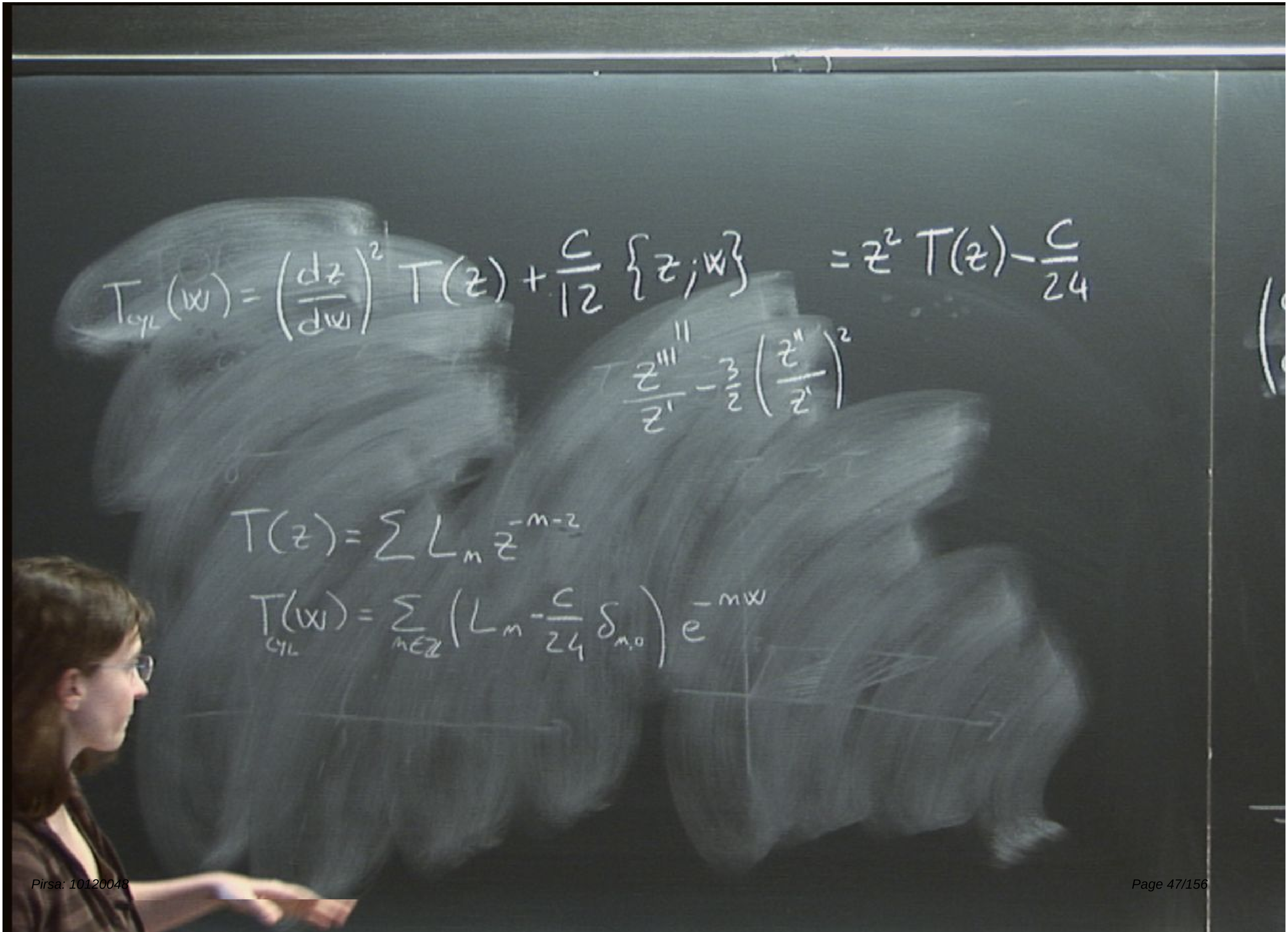

$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\}$


$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\}$
z'''


$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\}$
$\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$


$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\} = z^2 T(z) - \frac{c}{24}$
$= \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$


$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\} = z^2 T(z) - \frac{c}{24}$
$\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$
$T(z) = \sum L_n z^{-n-2}$

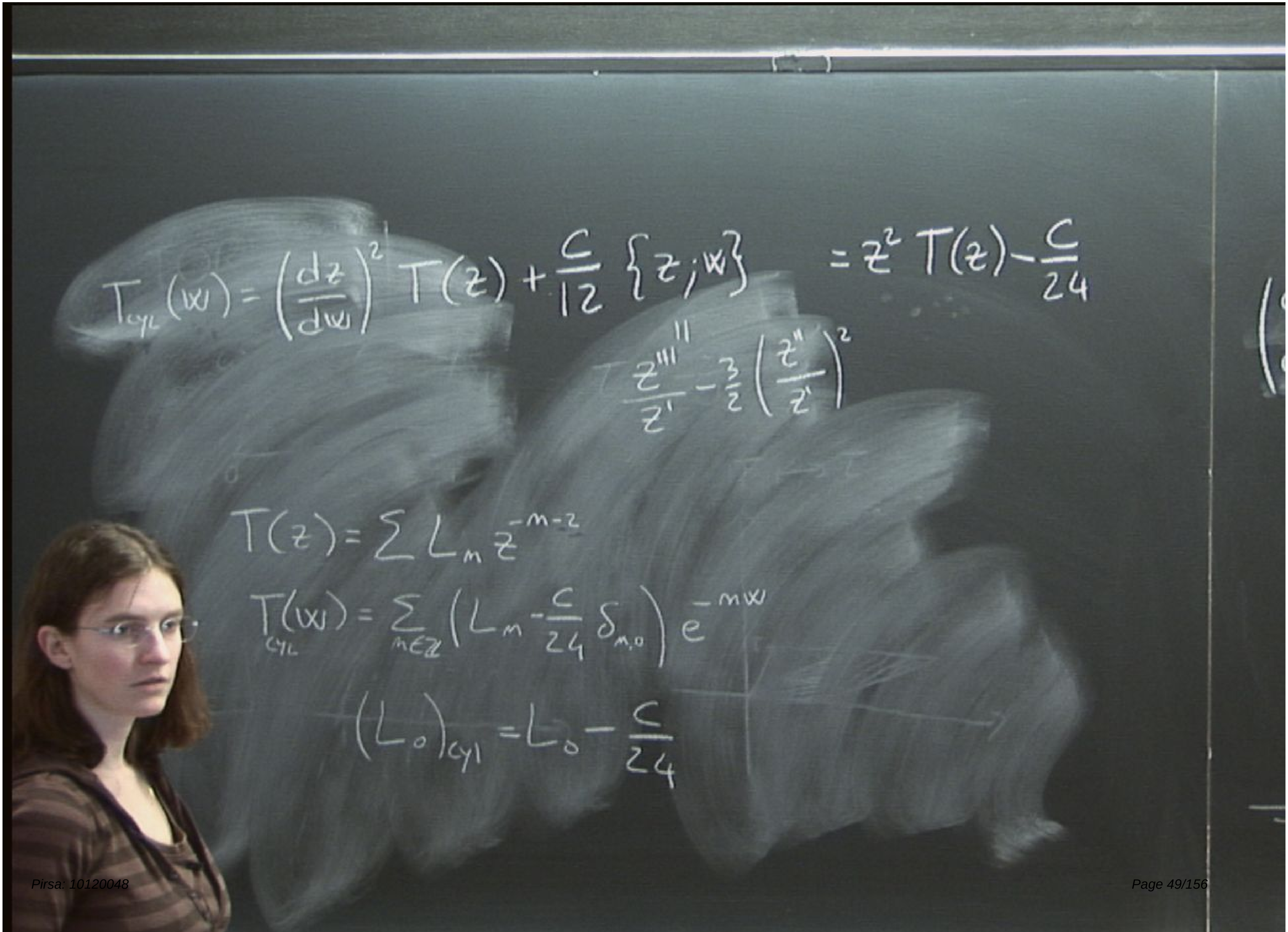

$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\} = z^2 T(z) - \frac{c}{24}$
$\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$
$T(z) = \sum L_m z^{-m-2}$
$T_{cyl}(w) = \sum_{m\in\mathbb{Z}} \left(L_m - \frac{c}{24}\delta_{m,0}\right) e^{-mw}$

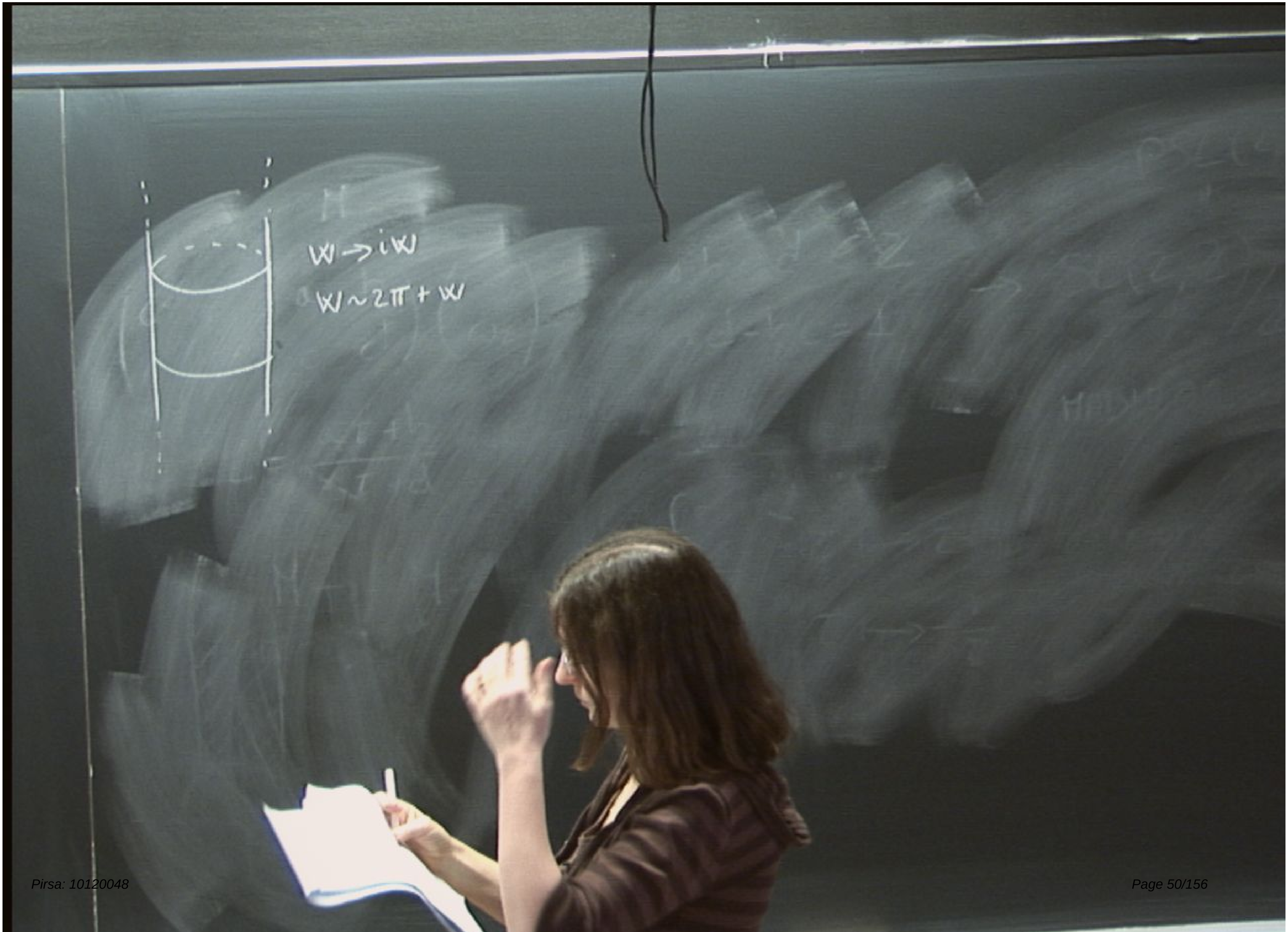
$$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\} \quad = z^2\, T(z) - \frac{c}{24}$$

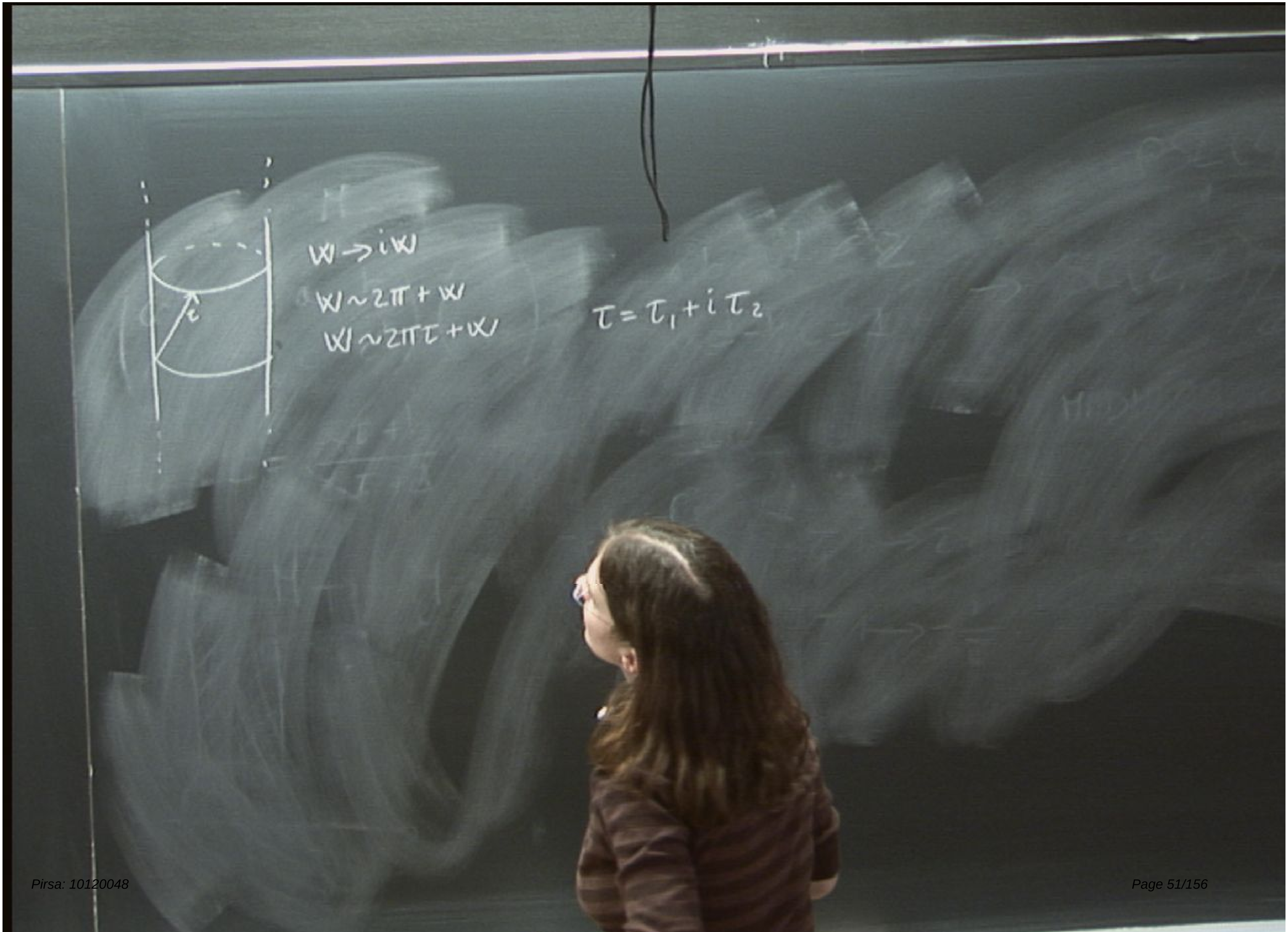
$$\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

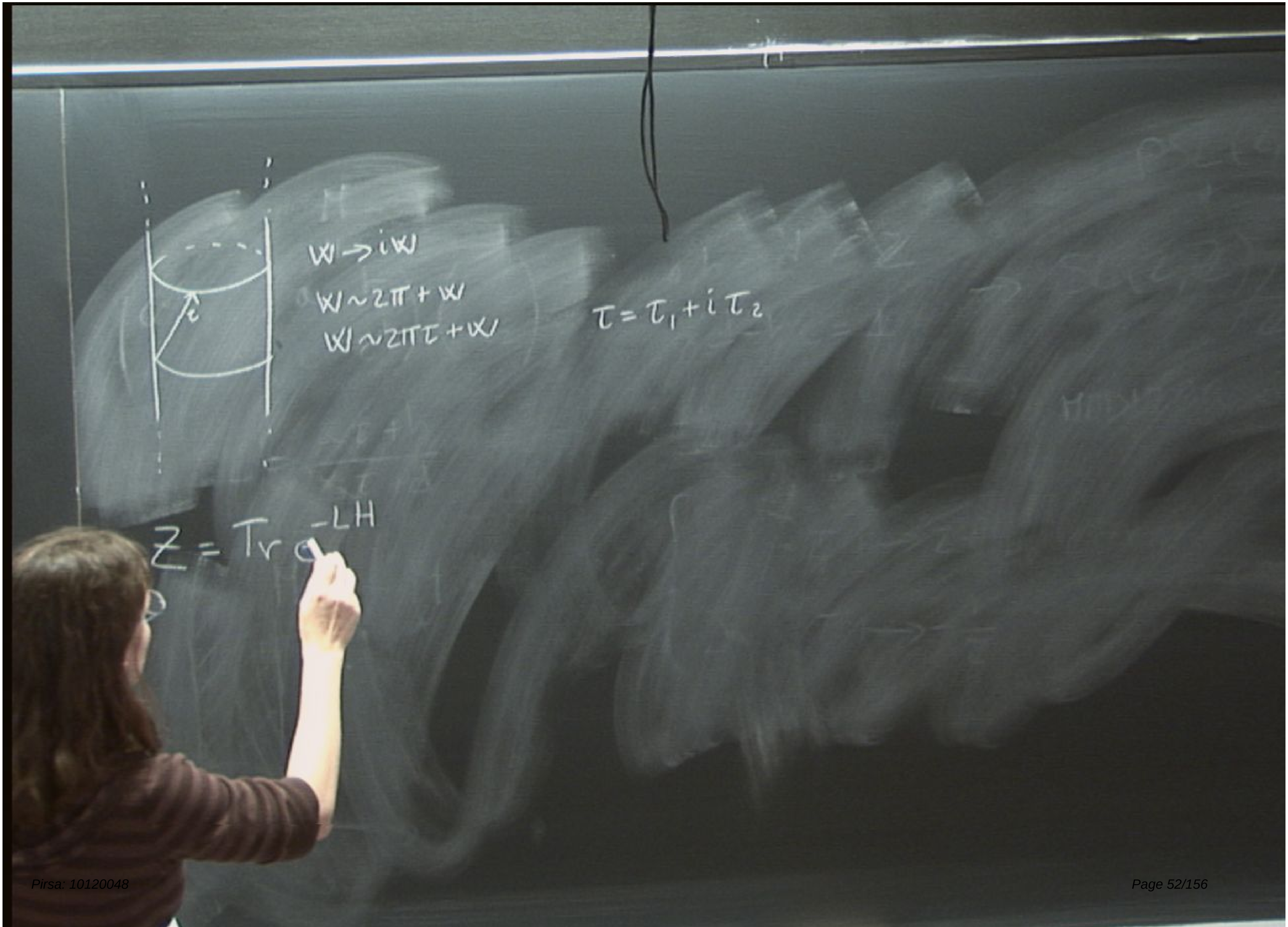
$$T(z) = \sum L_m z^{-m-2}$$

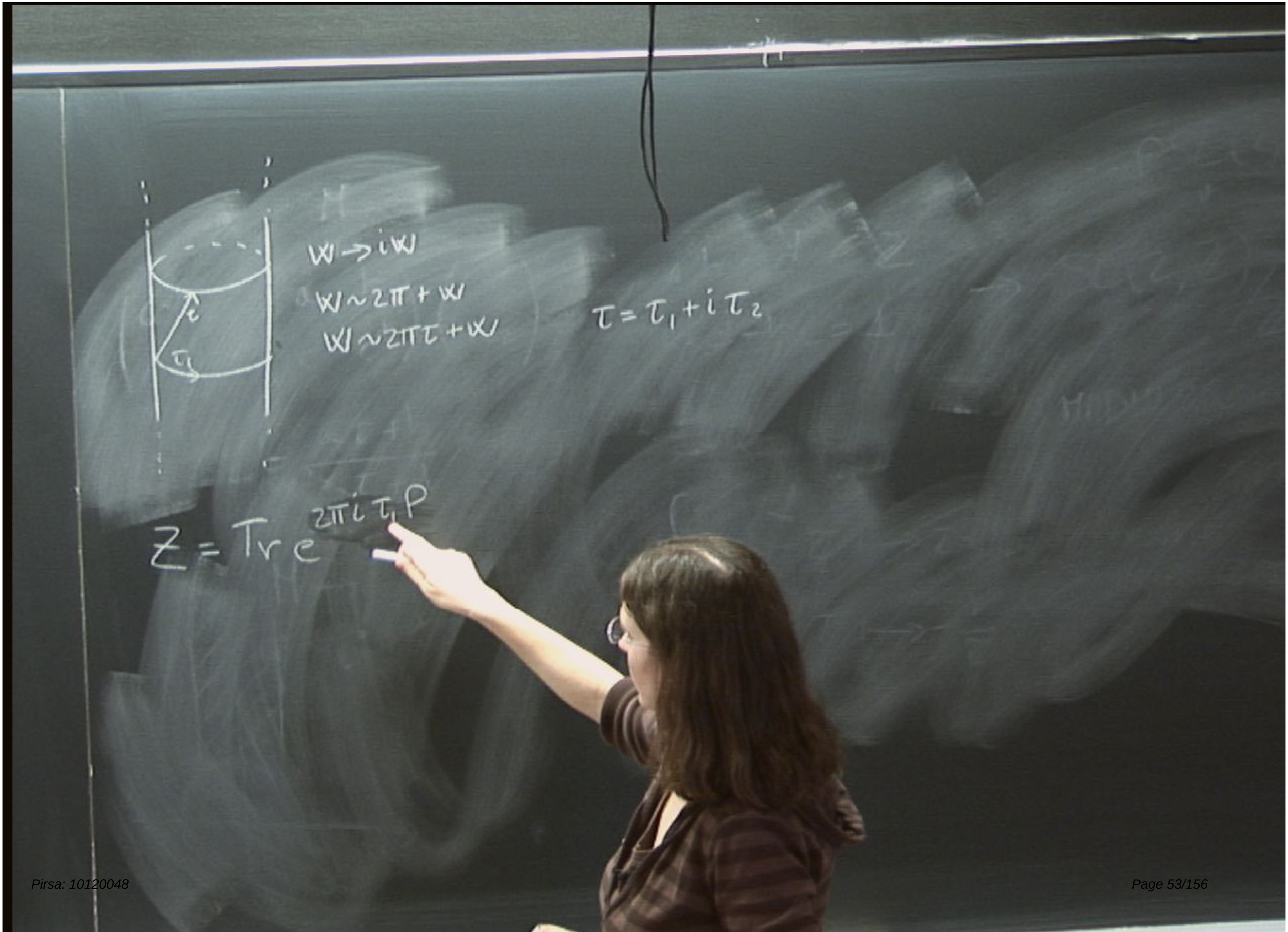
$$T_{cyl}(w) = \sum_{m\in\mathbb{Z}} \left(L_m - \frac{c}{24}\delta_{m,0}\right) e^{-mw}$$


$T_{cyl}(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\} = z^2 T(z) - \frac{c}{24}$
$\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$
$T(z) = \sum L_m z^{-m-2}$
$T_{cyl}(w) = \sum_{m\in\mathbb{Z}} \left(L_m - \frac{c}{24}\delta_{m,0}\right) e^{-mw}$
$(L_0)_{cyl} = L_0 - \frac{c}{24}$


W → iW
W ~ 2π + W


w → iw
w ~ 2π + w
w ~ 2πτ + w
τ = τ₁ + i τ₂


w → iw
w ~ 2π + w
w ~ 2πτ + w
τ = τ₁ + i τ₂
Z = Tr e^{-LH}

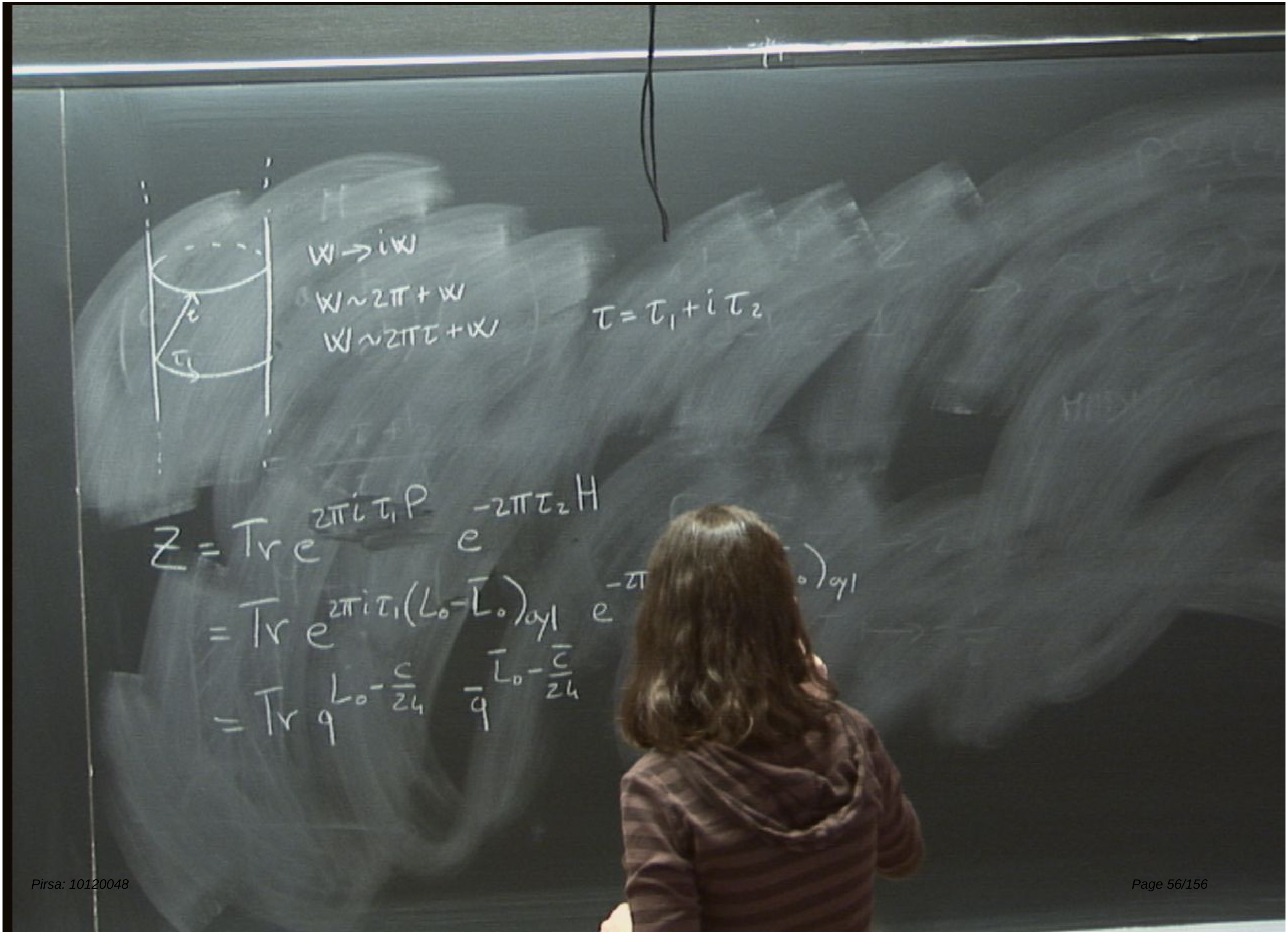

w → iw
w ~ 2π + w
w ~ 2πτ + w
τ = τ₁ + iτ₂
Z = Tr e^{2πiτ₁P}


w → iw
w ~ 2π + w
w ~ 2πτ + w
τ = τ1 + i τ2
Z = Tr e^{2πi τ1 P} e^{−2π τ2 H}

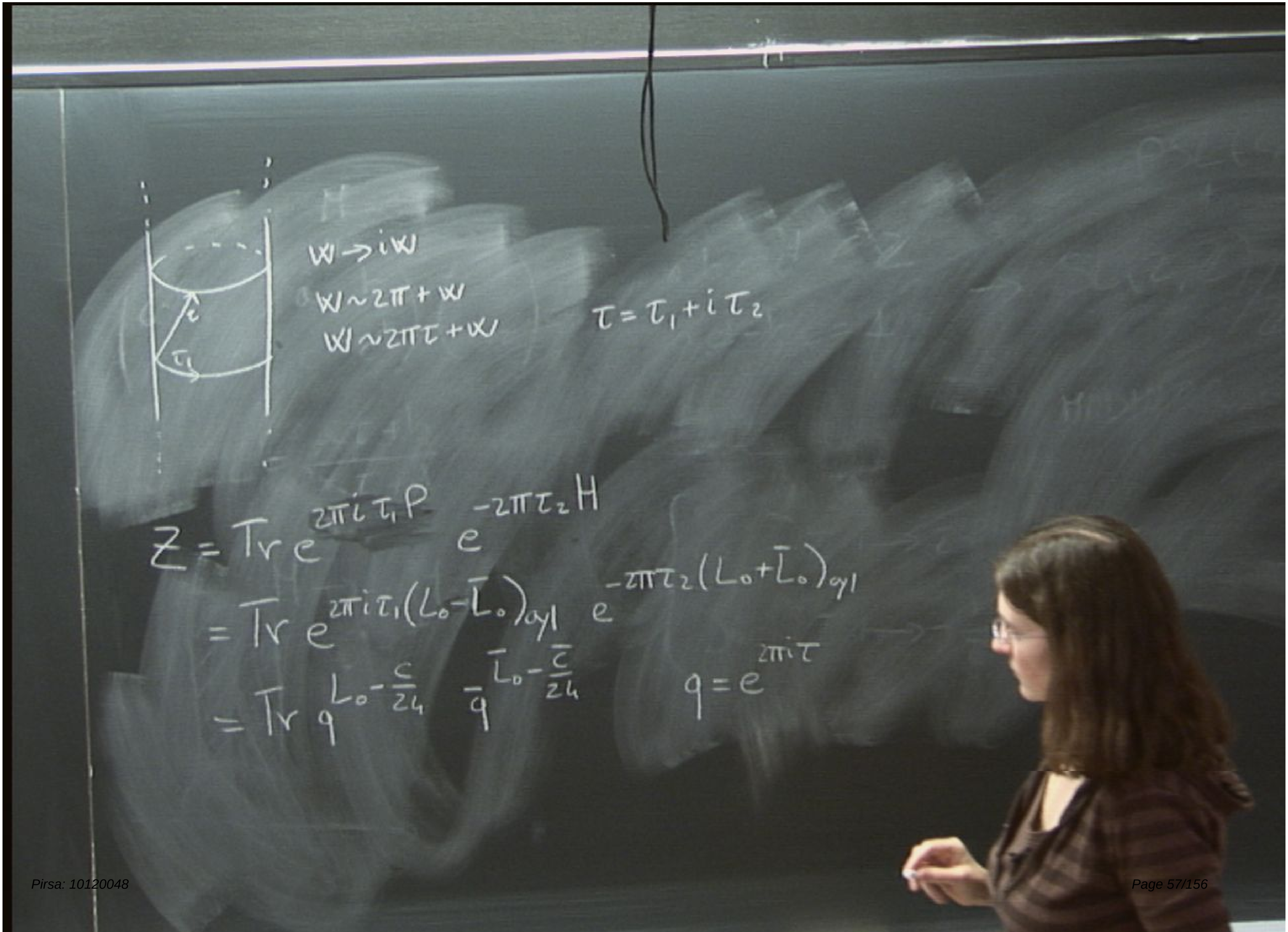
$$w \to i w$$
$$w \sim 2\pi + w$$
$$w \sim 2\pi\tau + w$$

$$\tau = \tau_1 + i\tau_2$$

$$Z = \mathrm{Tr}\, e^{2\pi i \tau_1 P}\, e^{-2\pi \tau_2 H}$$
$$= \mathrm{Tr}\, e^{2\pi i \tau_1 (L_0 - \bar{L}_0)_{cyl}}\, e^{-2\pi \tau_2 (L_0 + \bar{L}_0)_{cyl}}$$
$$=$$



τ
τ_1
$w \to iw$
$w \sim 2\pi + w$
$w \sim 2\pi\tau + w$
$\tau = \tau_1 + i\tau_2$
$Z = \mathrm{Tr}\, e^{2\pi i \tau_1 P} e^{-2\pi \tau_2 H}$
$= \mathrm{Tr}\, e^{2\pi i \tau_1 (L_0 - \bar{L}_0)_{cyl}} e^{-2\pi}$
$)_{cyl}$
$= \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$


w → iw
w ~ 2π + w
w ~ 2πτ + w
τ = τ1 + i τ2
Z = Tr e^{2πi τ1 P} e^{-2πτ2 H}
= Tr e^{2πiτ1(L0 - L̄0)cyl} e^{-2πτ2(L0 + L̄0)cyl}
= Tr q^{L0 - c/24} q̄^{L̄0 - c̄/24}
q = e^{2πiτ}

$w \to i w$

$w \sim 2\pi + w$

$w \sim 2\pi\tau + w$

$\tau = \tau_1 + i\tau_2$

$$Z = \mathrm{Tr}\, e^{2\pi i \tau_1 P}\, e^{-2\pi\tau_2 H}$$

$$= \mathrm{Tr}\, e^{2\pi i \tau_1 (L_0 - \bar{L}_0)_{cyl}}\, e^{-2\pi\tau_2 (L_0 + \bar{L}_0)_{cyl}}$$

$$= \mathrm{Tr}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \qquad q = e^{2\pi i \tau}$$

$$\boxed{Z = q^{-\frac{c}{24}}\, \bar{q}^{-c/24}\, \mathrm{Tr}\, q^{L_0}\, \bar{q}^{\bar{L}_0}}$$

$w \to i w$

$w \sim 2\pi + w$

$w \sim 2\pi\tau + w$

$\tau = \tau_1 + i\tau_2$

$$Z = \mathrm{Tr}\, e^{2\pi i \tau_1 P}\, e^{-2\pi \tau_2 H}$$

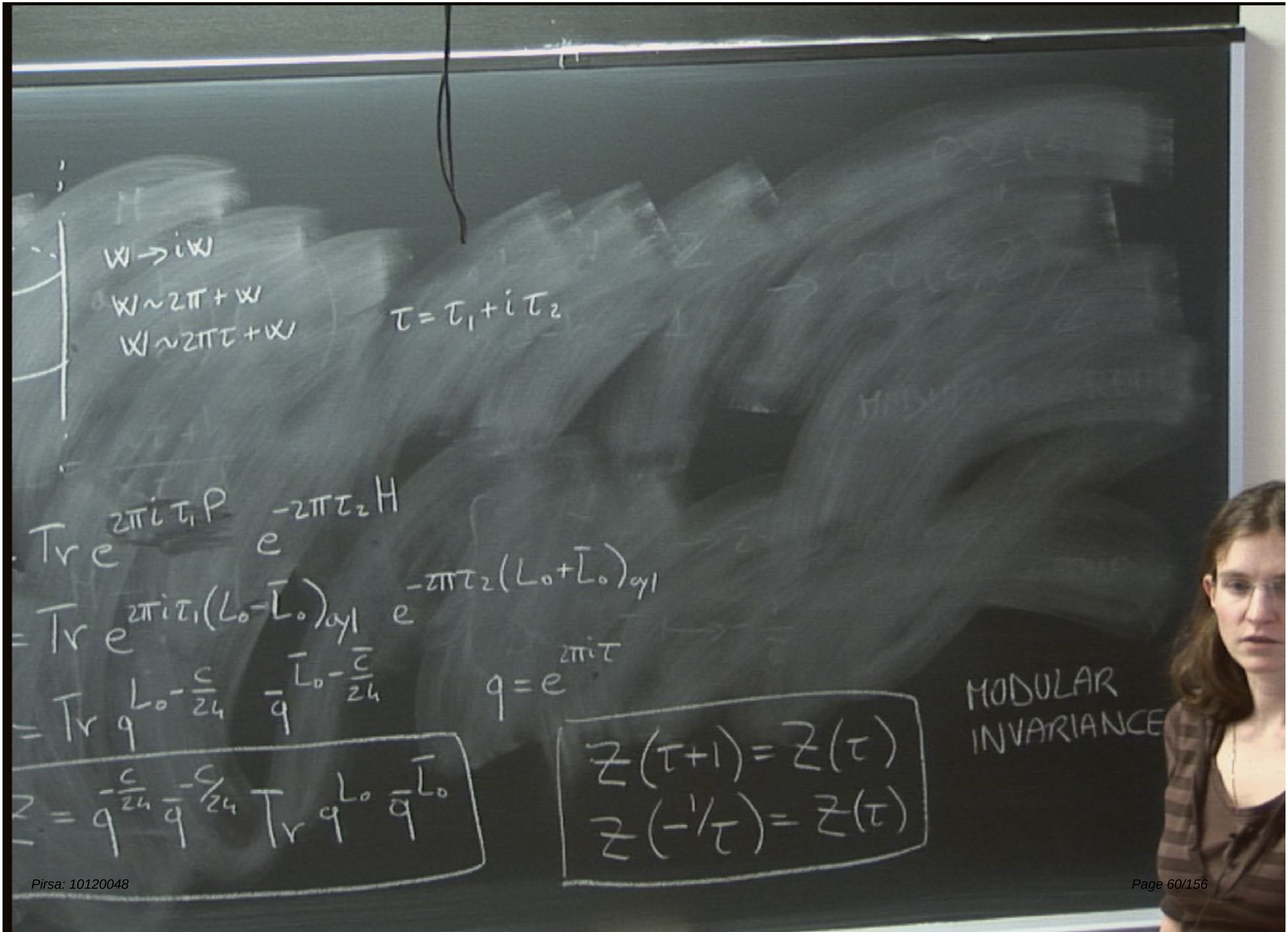
$$= \mathrm{Tr}\, e^{2\pi i \tau_1 (L_0 - \bar{L}_0)_{cyl}}\, e^{-2\pi \tau_2 (L_0 + \bar{L}_0)_{cyl}}$$

$$= \mathrm{Tr}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \qquad q = e^{2\pi i \tau}$$

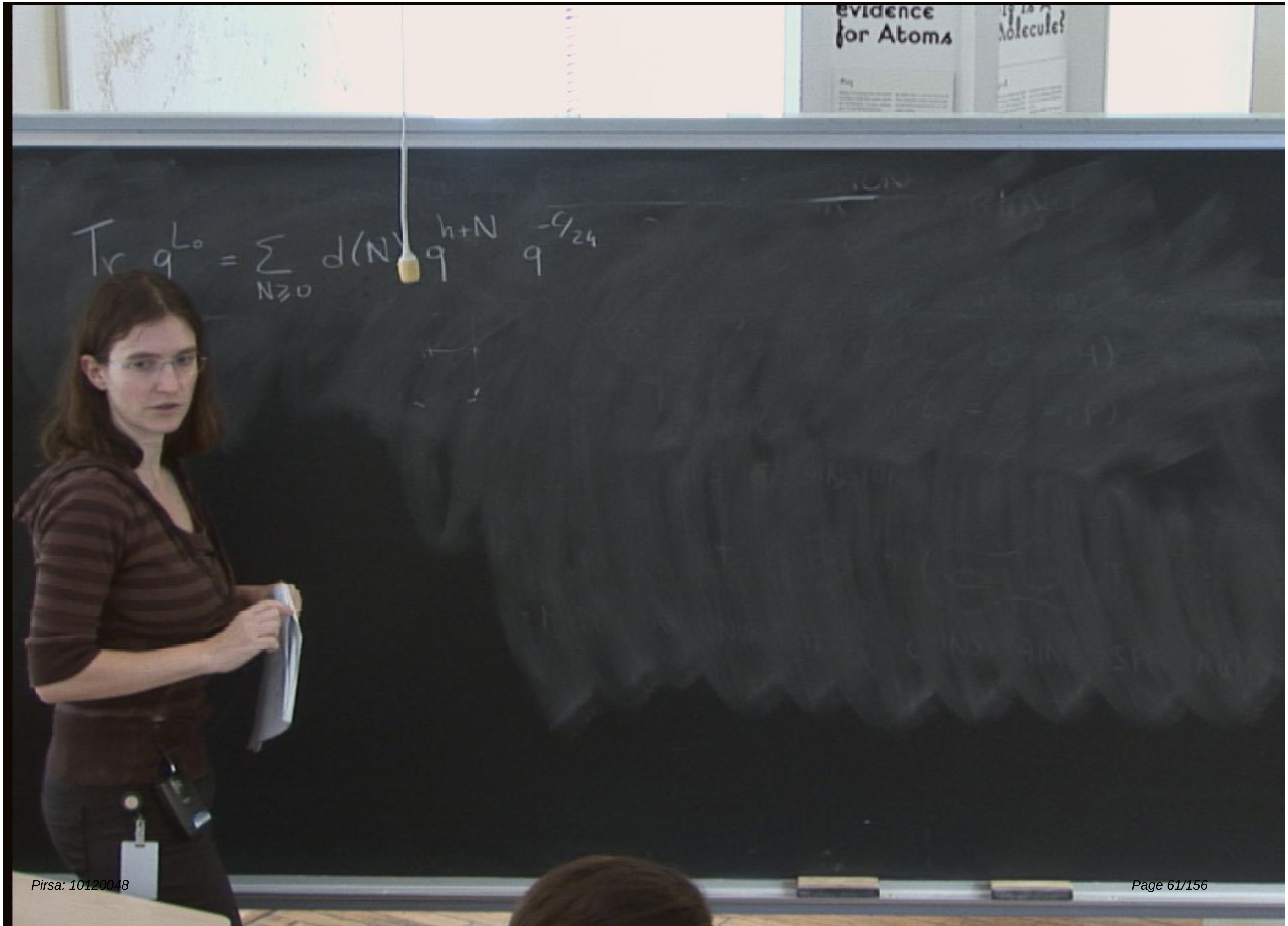
$$Z = q^{-\frac{c}{24}}\, \bar{q}^{-c/24}\, \mathrm{Tr}\, q^{L_0}\, \bar{q}^{\bar{L}_0}$$

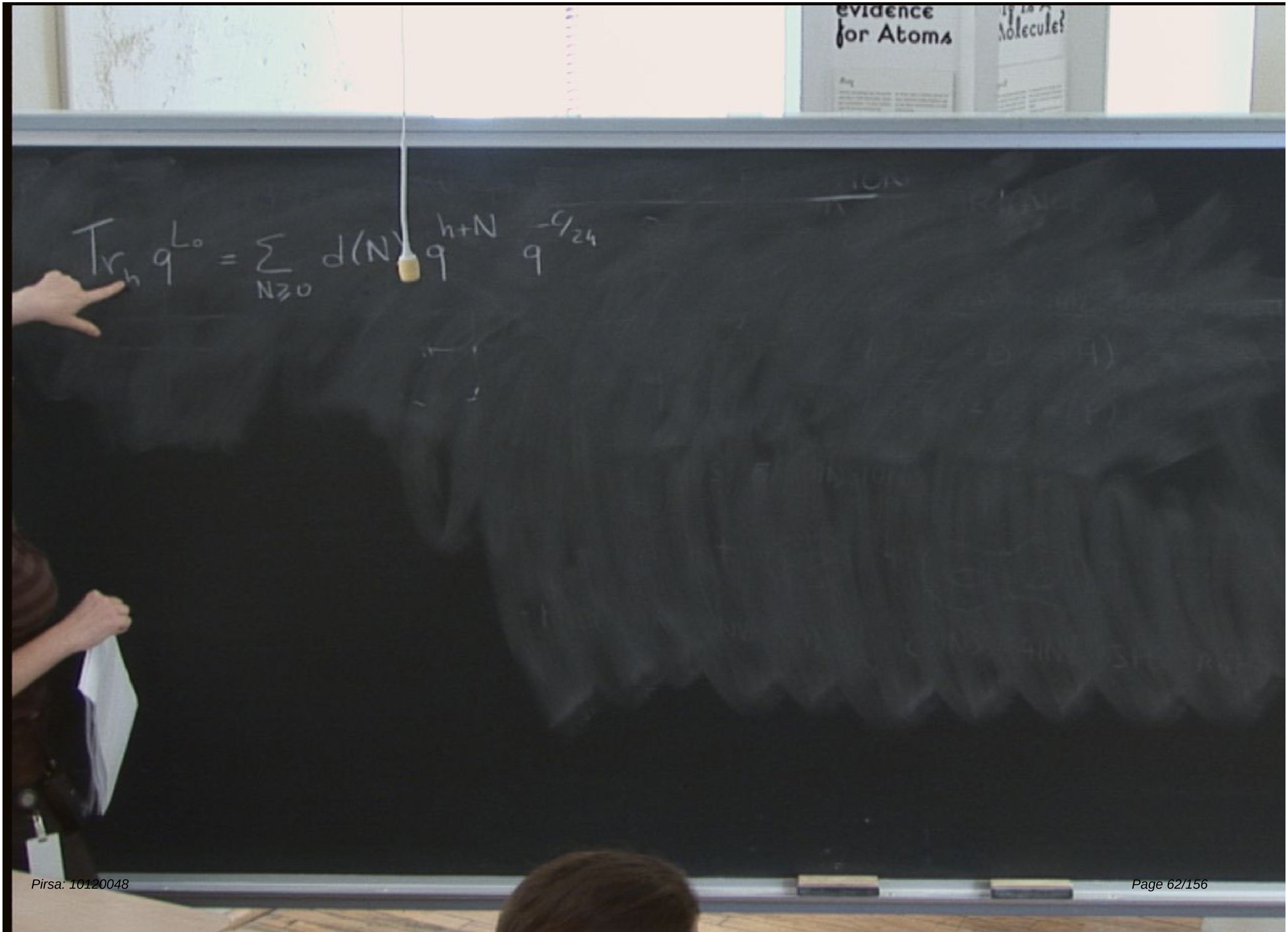
$$Z(\tau + 1) = Z(\tau)$$
$$Z(-1/\tau) = Z(\tau)$$

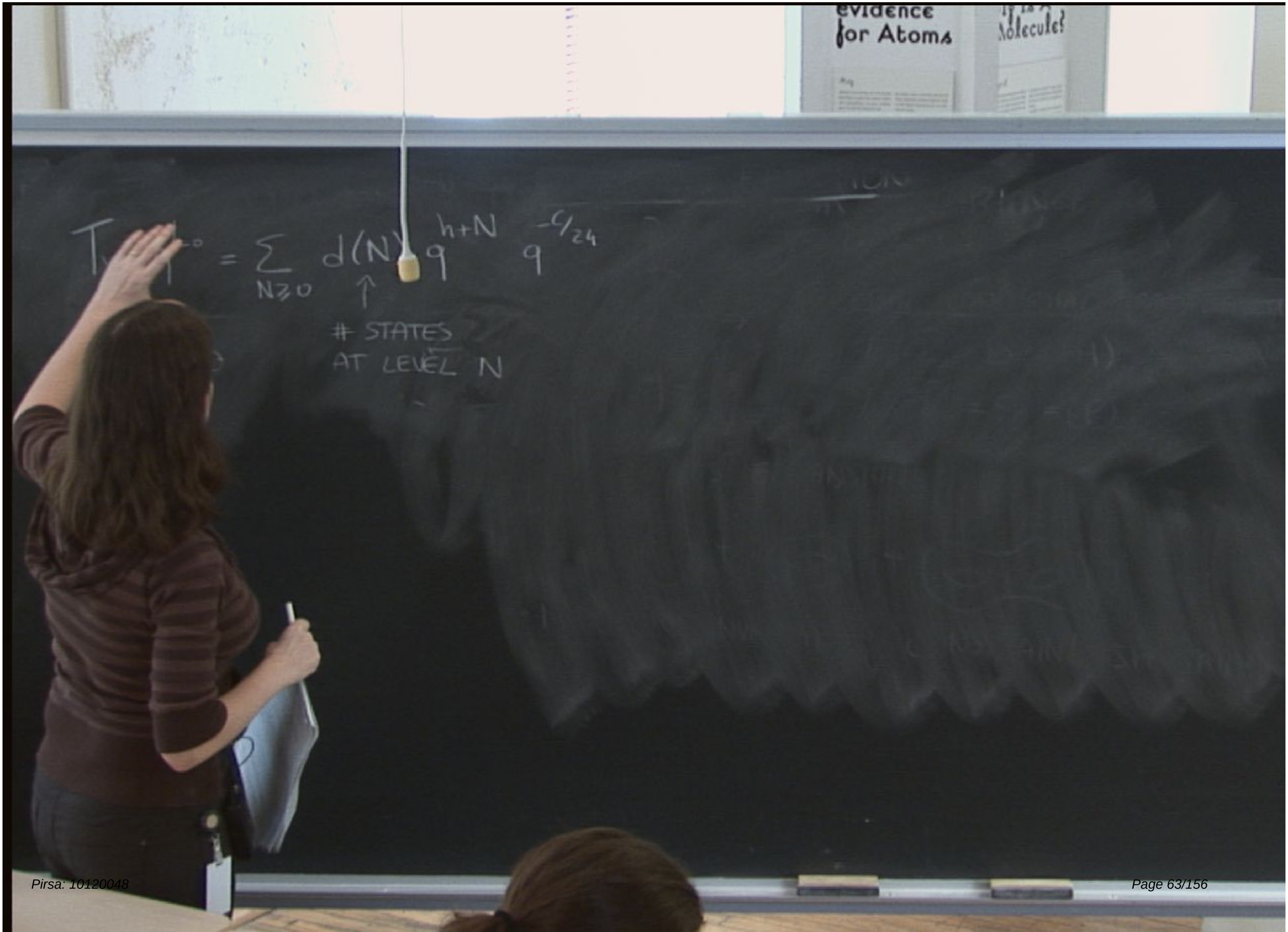
MODUL
INVAR

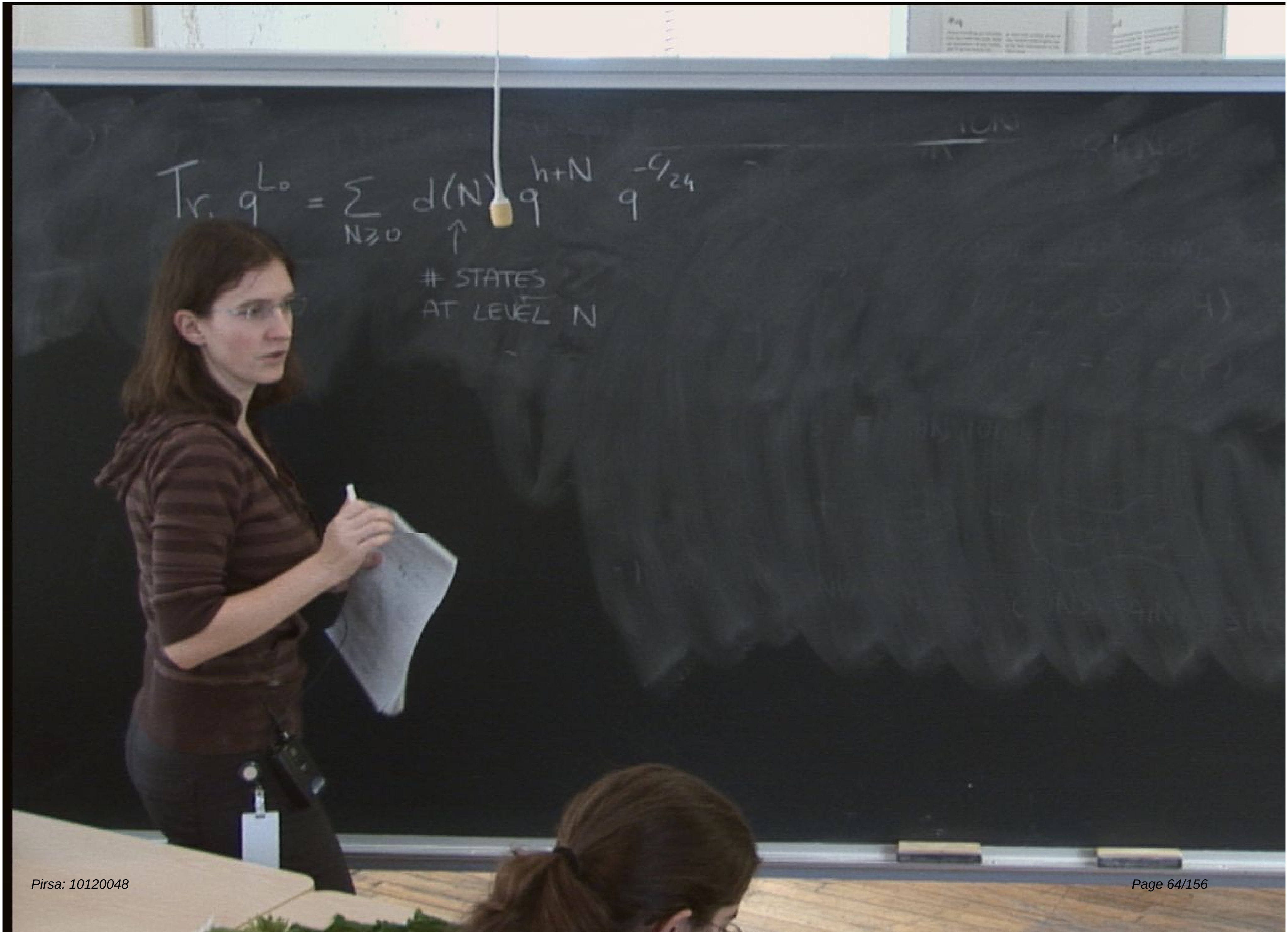


$w \to iw$
$w \sim 2\pi + w$
$w \sim 2\pi\tau + w$
$\tau = \tau_1 + i\tau_2$
$\mathrm{Tr}\, e^{2\pi i \tau_1 P} e^{-2\pi\tau_2 H}$
$= \mathrm{Tr}\, e^{2\pi i \tau_1 (L_0 - \bar{L}_0)_{cyl}} e^{-2\pi\tau_2 (L_0 + \bar{L}_0)_{cyl}}$
$= \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$
$q = e^{2\pi i \tau}$
$Z = q^{-\frac{c}{24}} \bar{q}^{-c/24} \mathrm{Tr}\, q^{L_0} \bar{q}^{\bar{L}_0}$
$Z(\tau+1) = Z(\tau)$
$Z(-1/\tau) = Z(\tau)$
MODULAR
INVARIANCE

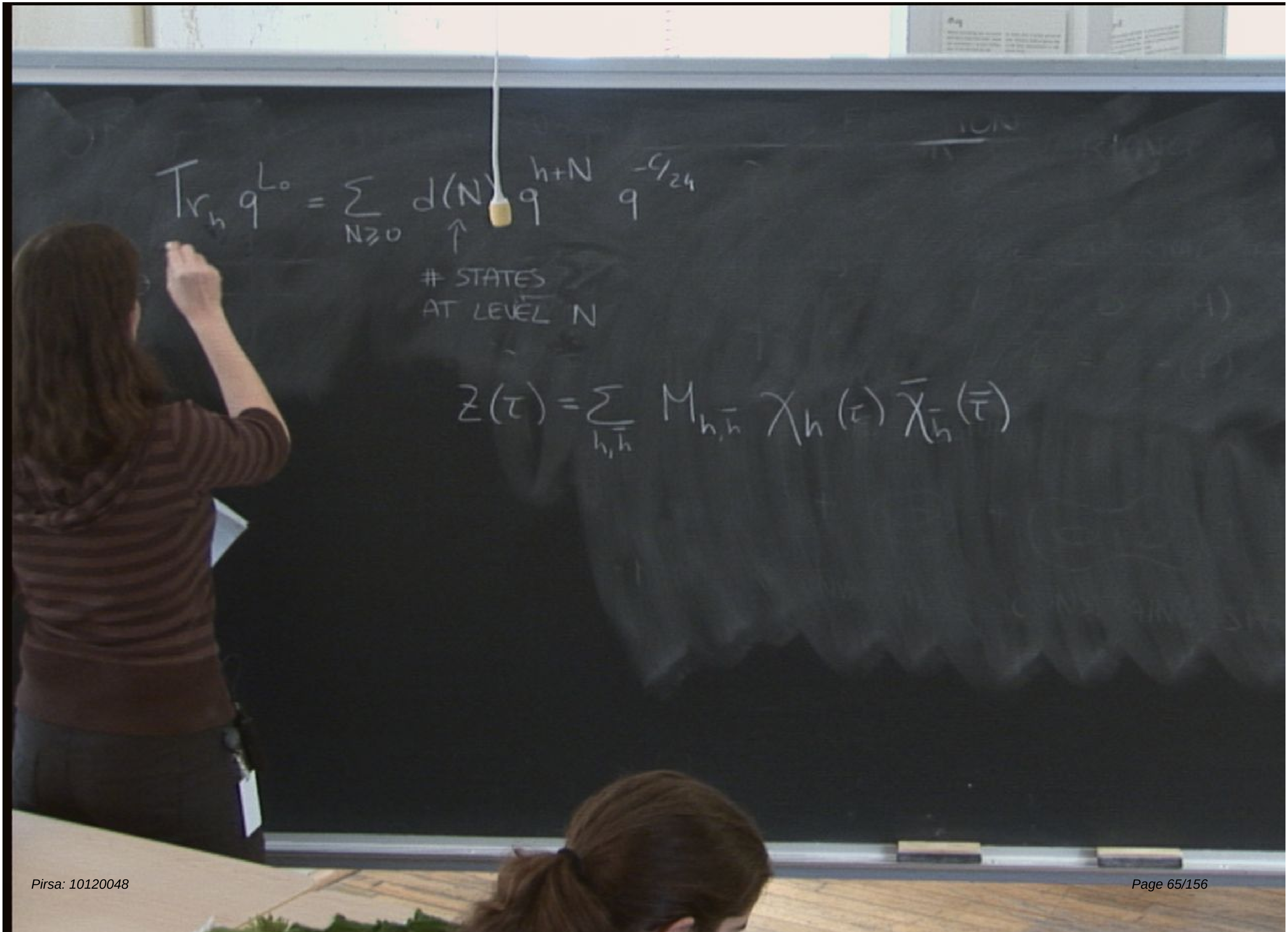

$\mathrm{Tr}\, q^{L_0} = \sum_{N \geq 0} d(N)\, q^{h+N}\, q^{-c/24}$
evidence
for Atoms

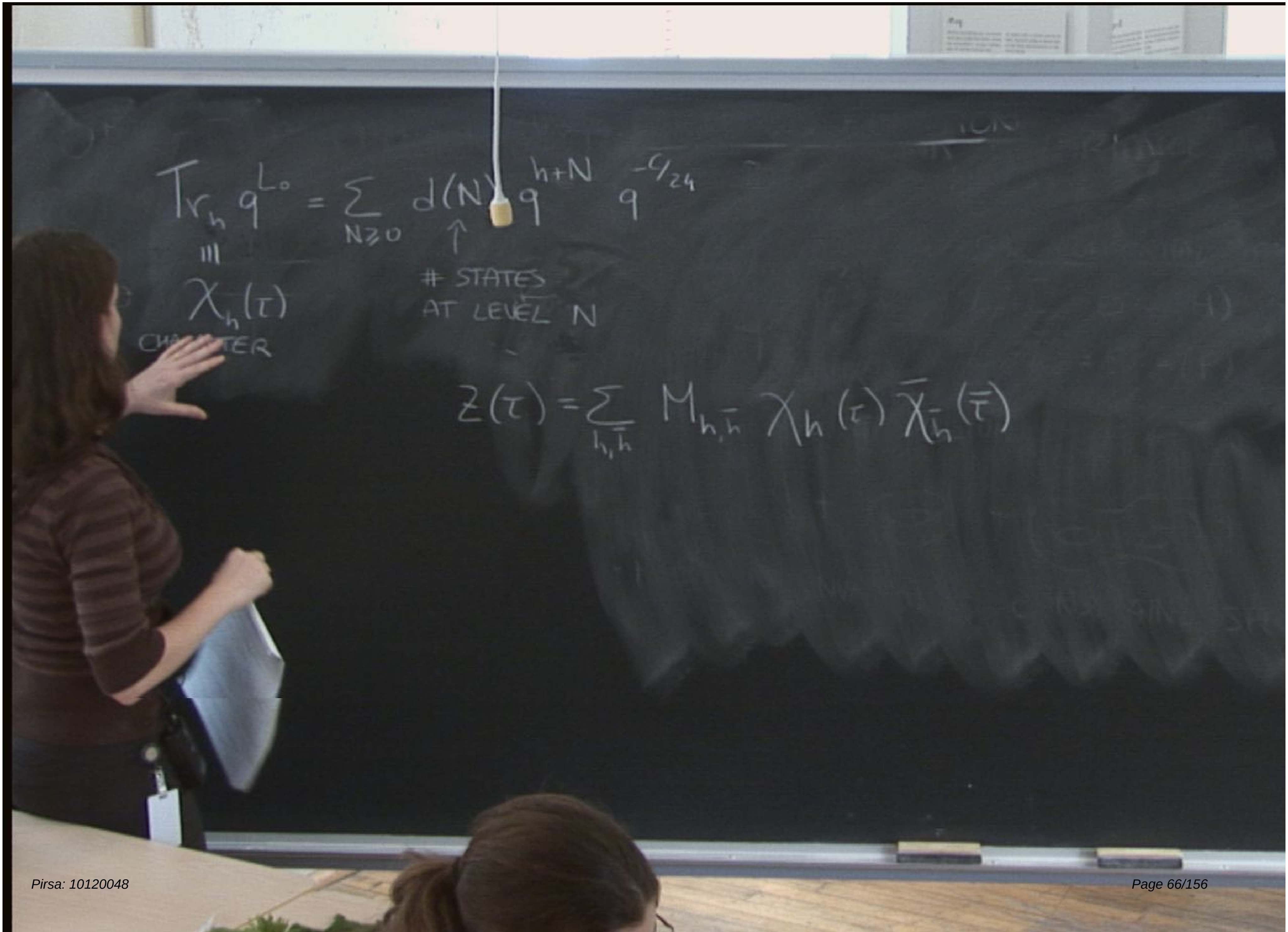

evidence
for Atoms
$Tr_h \, q^{L_0} = \sum_{N \geq 0} d(N) \, q^{h+N} \, q^{-c/24}$

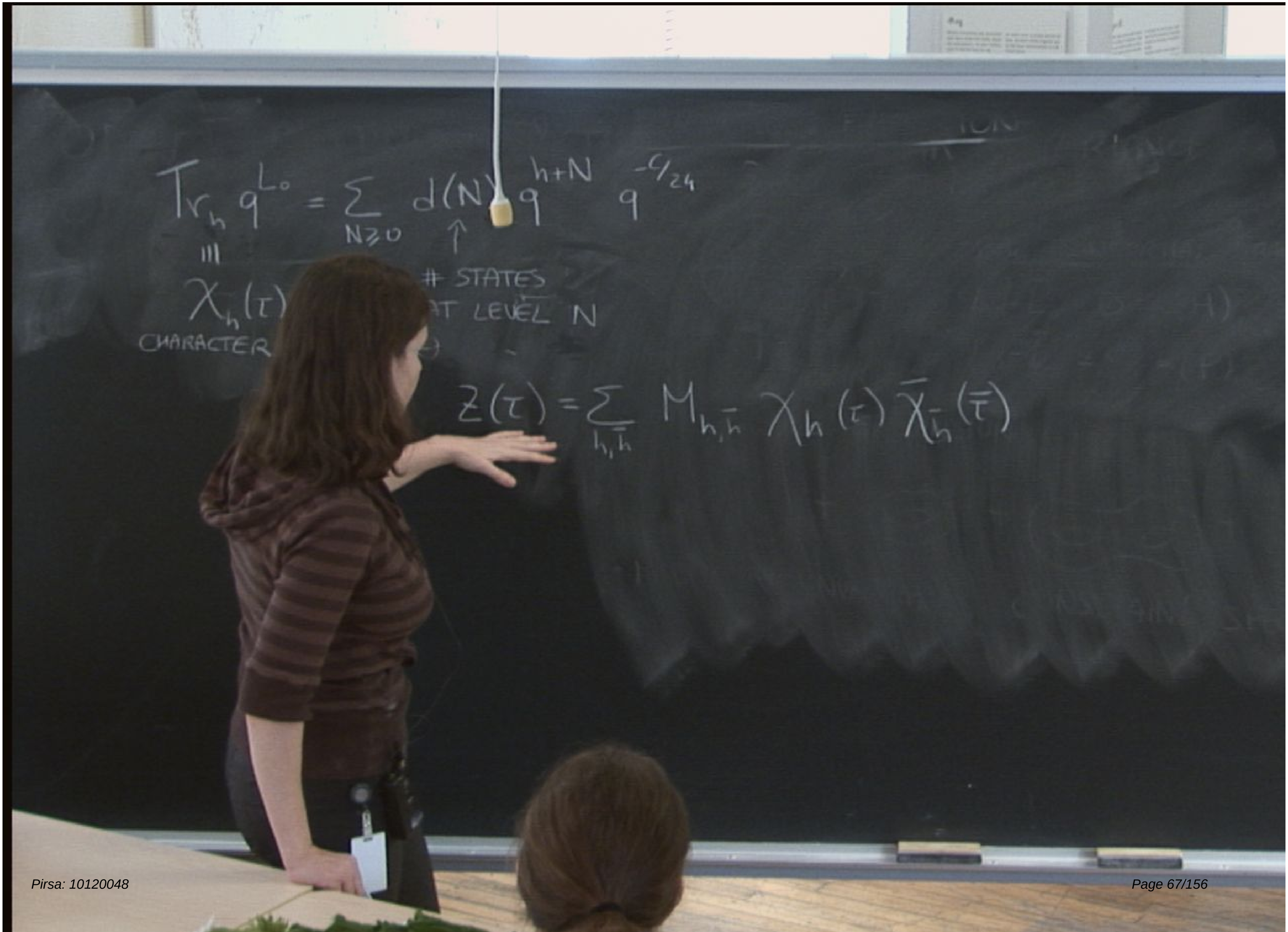

evidence
for Atoms
Molecule?
= Σ d(N) q h+N q -c/24
N≥0
STATES
AT LEVEL N

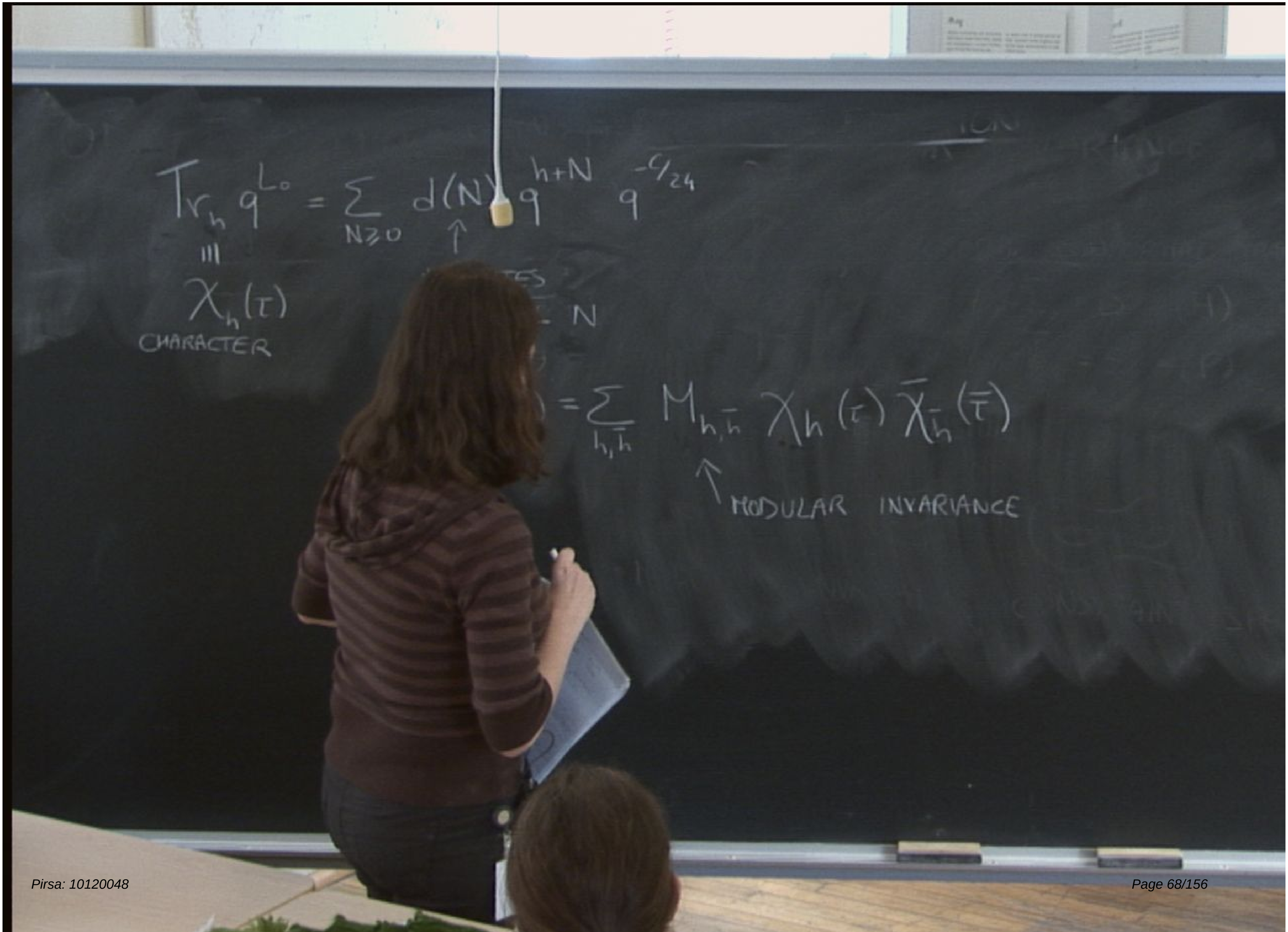


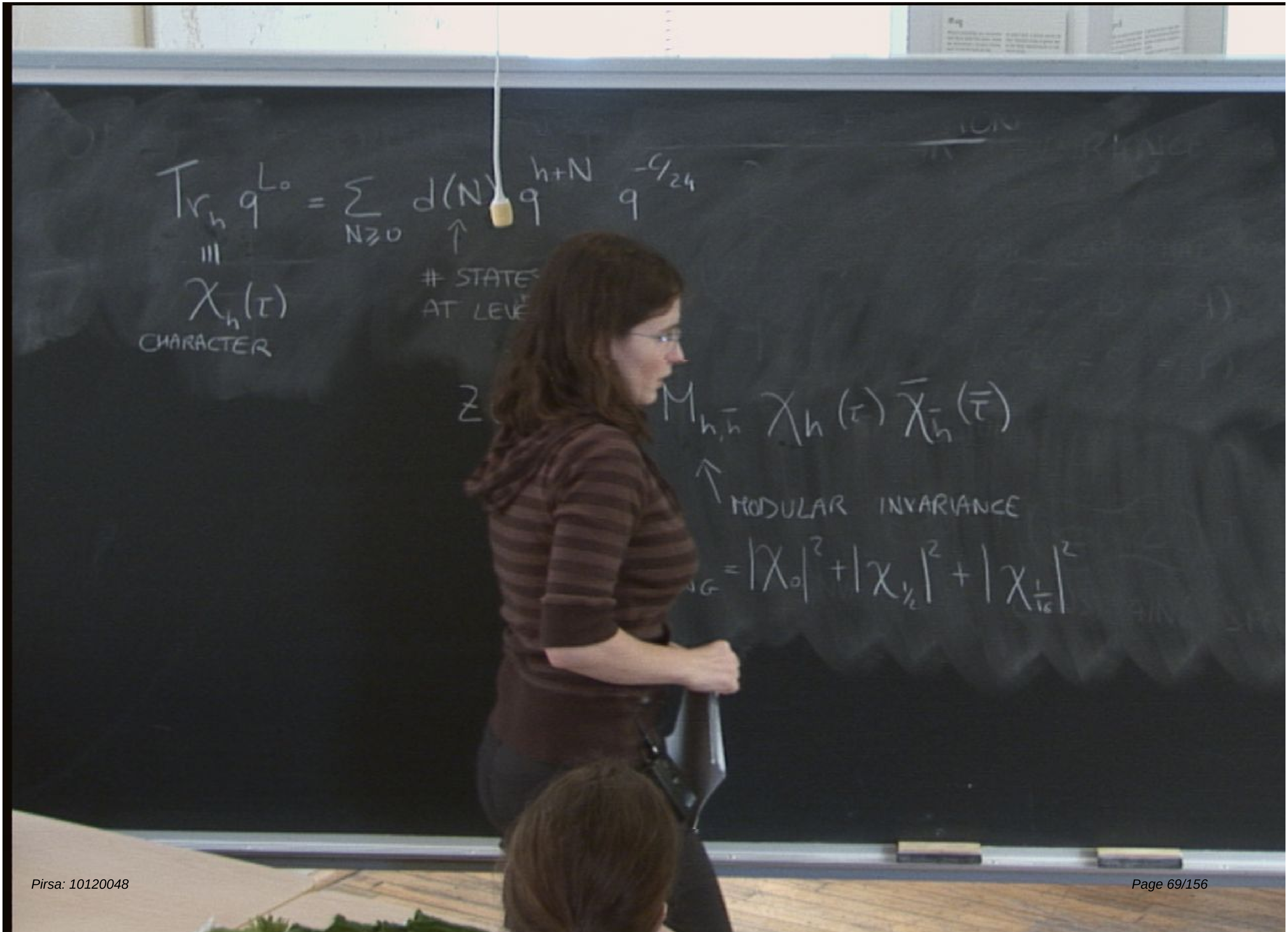
$Tr.\ q^{L_0} = \sum_{N \geq 0} d(N)\, q^{h+N}\, q^{-c/24}$
STATES
AT LEVEL N

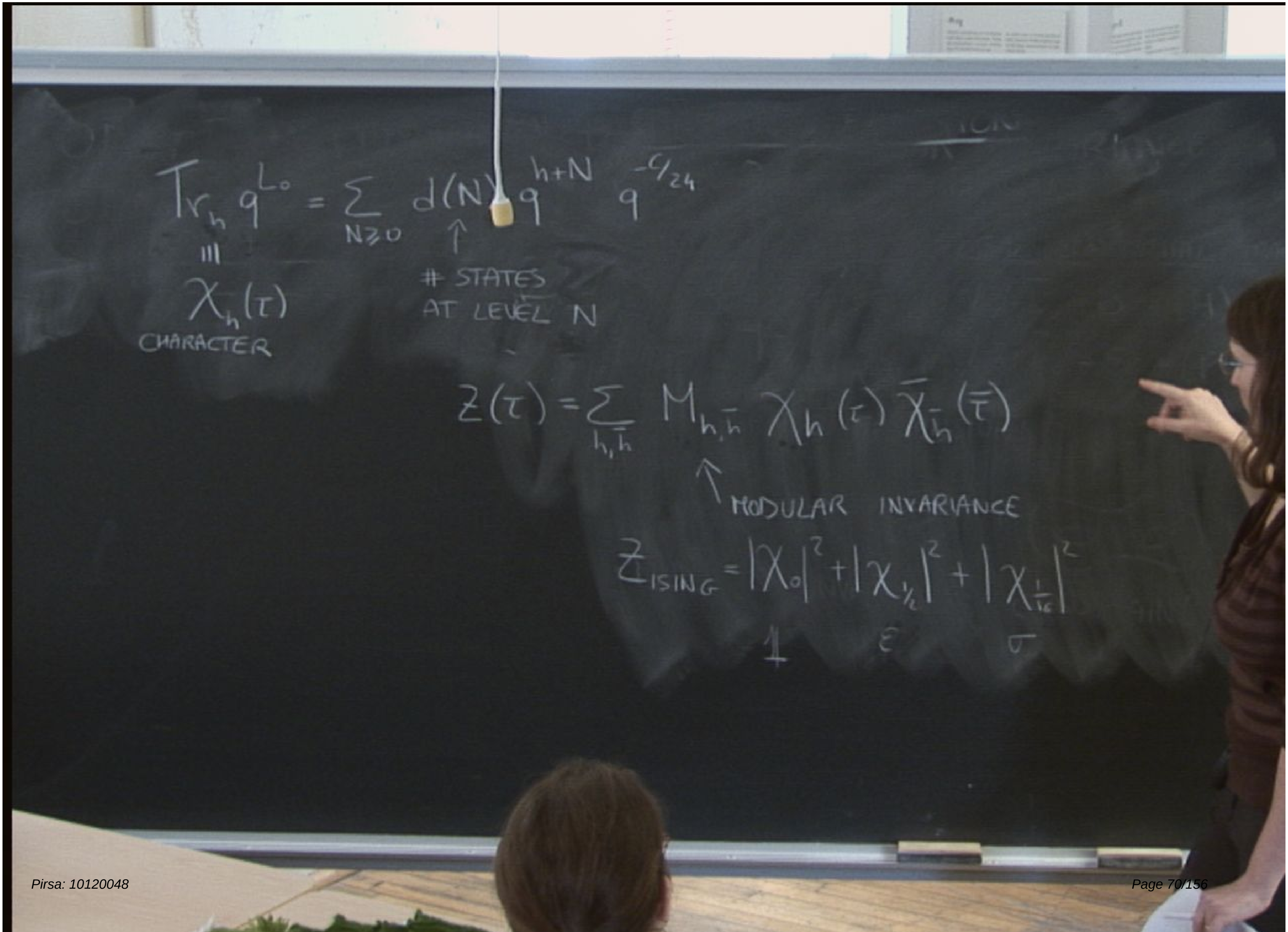

$\mathrm{Tr}_h\, q^{L_0} = \sum_{N \geq 0} d(N)\, q^{h+N}\, q^{-c/24}$
STATES AT LEVEL N
$Z(\tau) = \sum_{h,\bar{h}} M_{h,\bar{h}}\, \chi_h(\tau)\, \bar{\chi}_{\bar{h}}(\bar{\tau})$

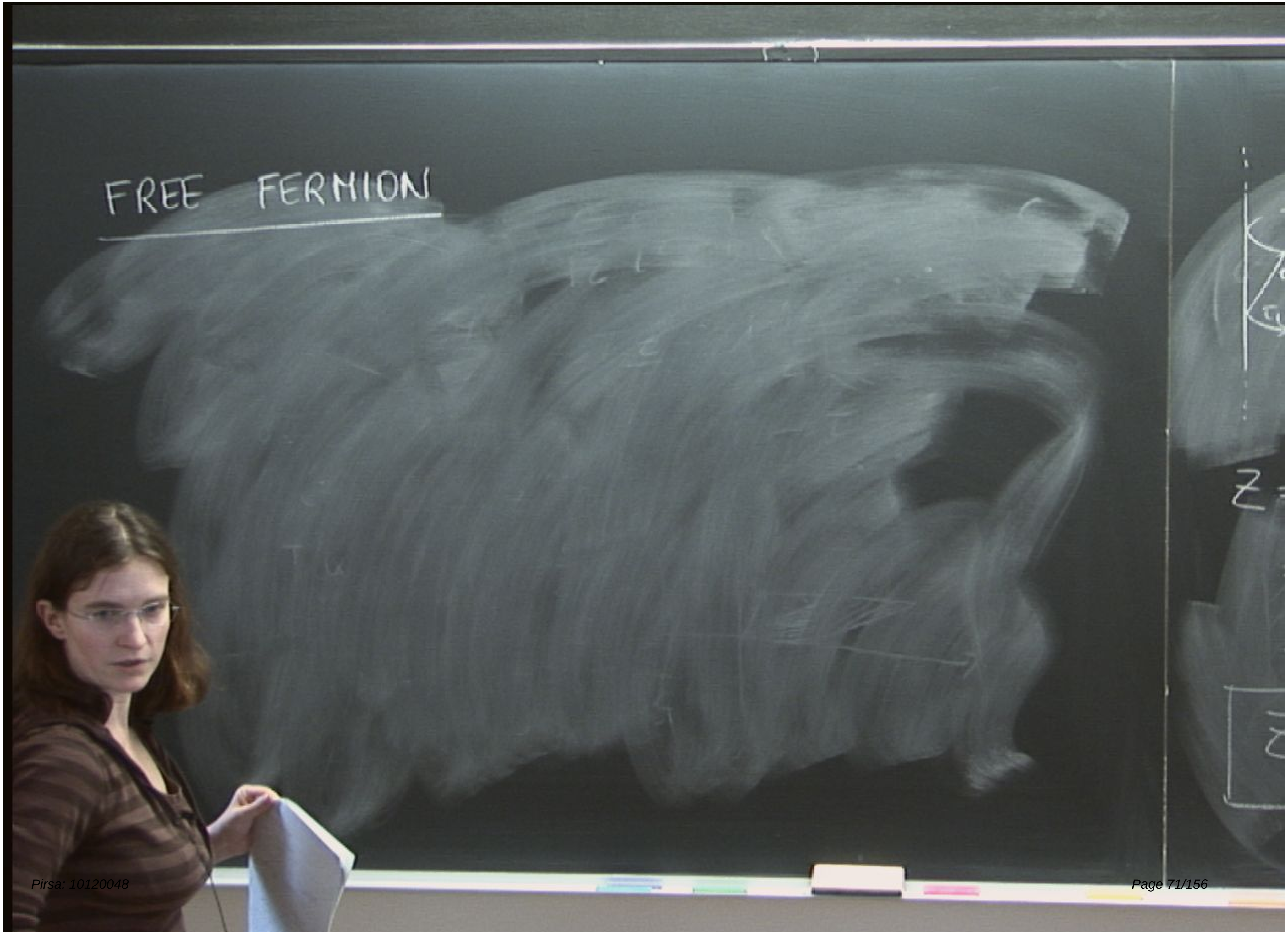

$\mathrm{Tr}_h\, q^{L_0} = \sum_{N \geq 0} d(N)\, q^{h+N}\, q^{-c/24}$
$\chi_h(\tau)$
CHARACTER
STATES
AT LEVEL N
$Z(\tau) = \sum_{h,\bar{h}} M_{h,\bar{h}}\, \chi_h(\tau)\, \bar{\chi}_{\bar{h}}(\bar{\tau})$

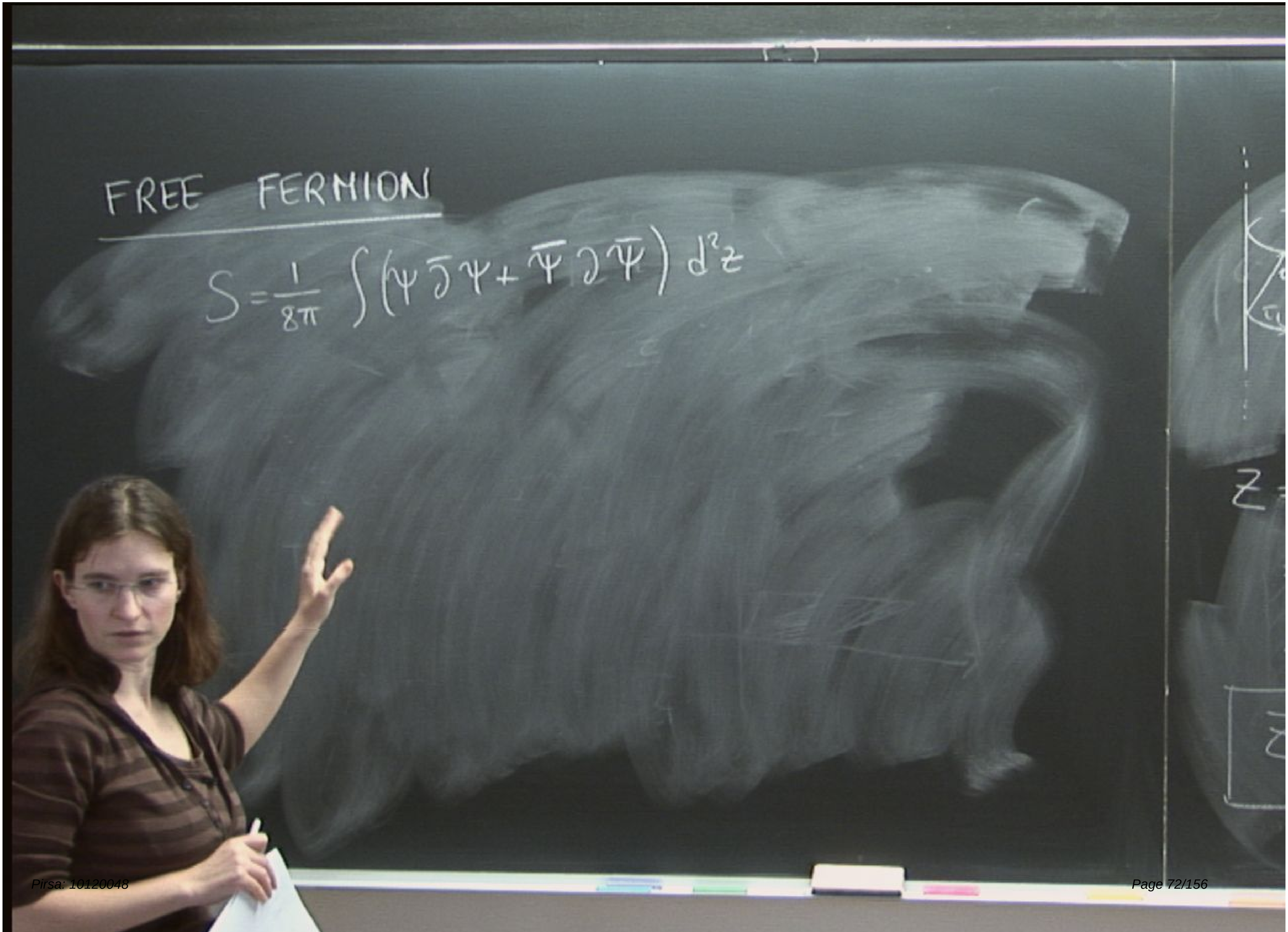

$\mathrm{Tr}_h\, q^{L_0} = \sum_{N \geq 0} d(N)\, q^{h+N}\, q^{-c/24}$
STATES
T LEVEL N
$\chi_h(\tau)$
CHARACTER
$Z(\tau) = \sum_{h,\bar{h}} M_{h,\bar{h}}\, \chi_h(\tau)\, \bar{\chi}_{\bar{h}}(\bar{\tau})$

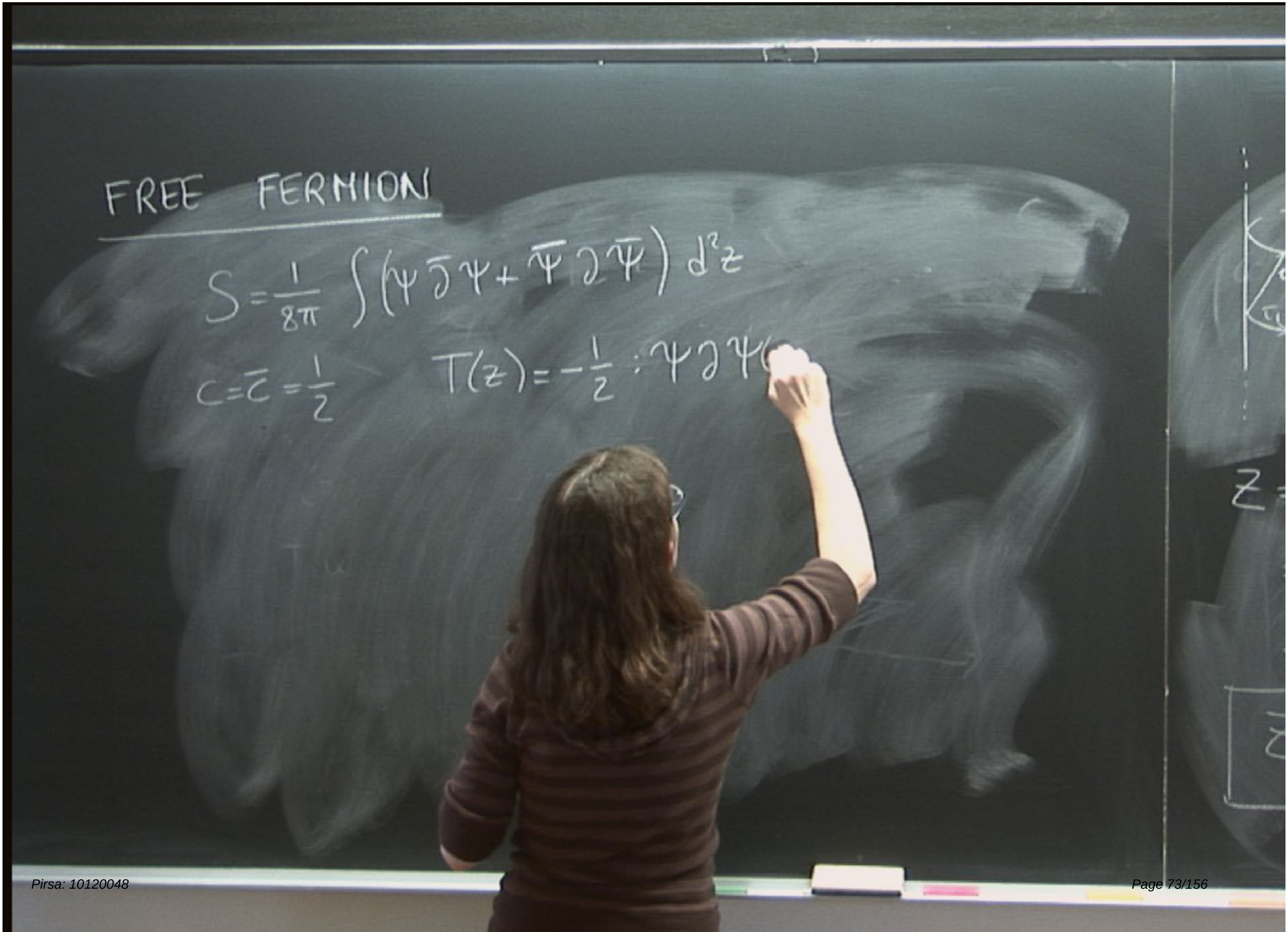

Tr_h q^{L_0} = Σ_{N≥0} d(N) q^{h+N} q^{-c/24}
χ_h(τ)
CHARACTER
= Σ_{h,h̄} M_{h,h̄} χ_h(τ) χ̄_h̄(τ̄)
MODULAR INVARIANCE

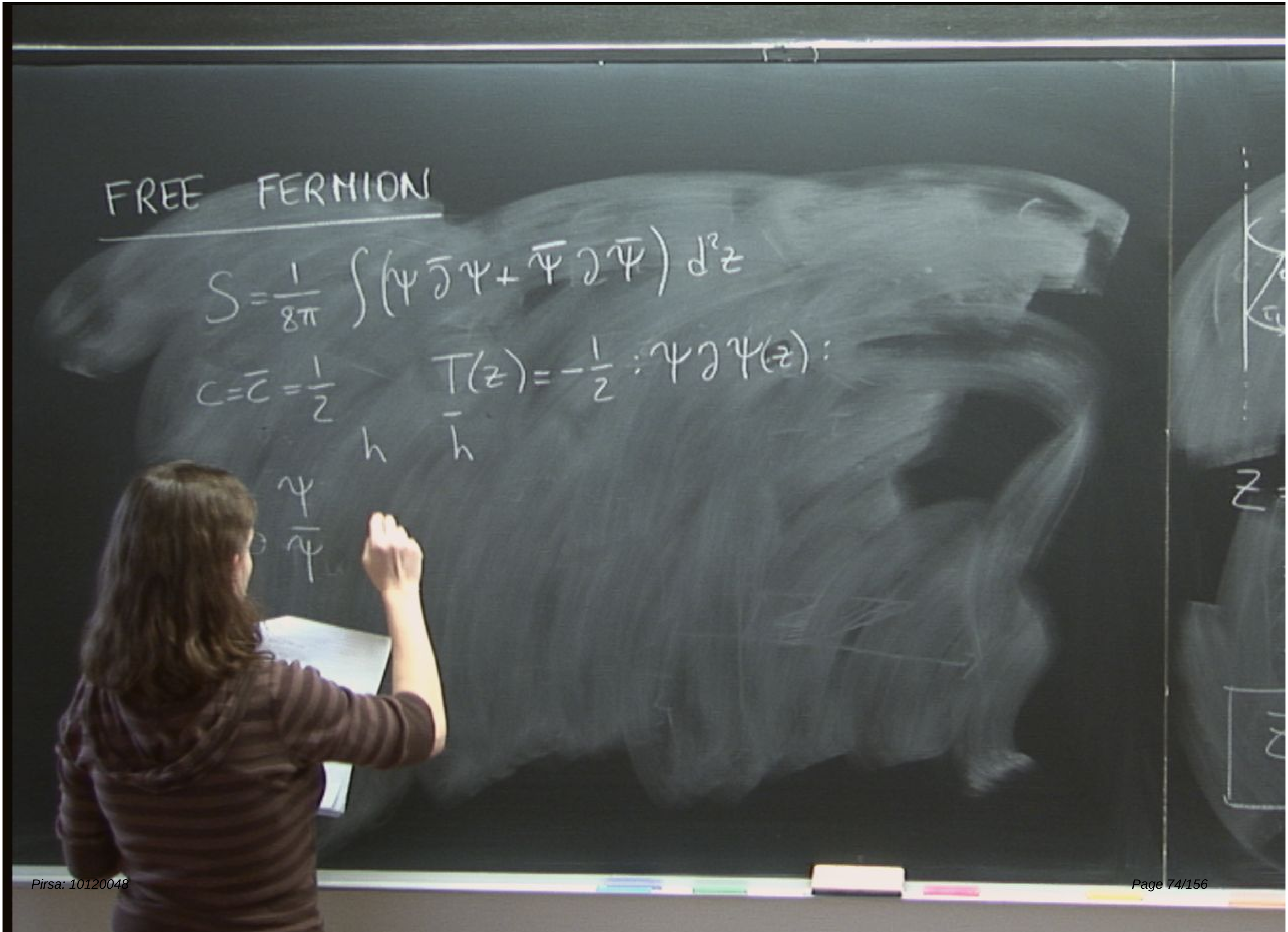

$\mathrm{Tr}_h\, q^{L_0} = \sum_{N\geq 0} d(N)\, q^{h+N}\, q^{-c/24}$
$\chi_h(\tau)$
CHARACTER
STATES
AT LEVE
$M_{h,\bar{h}}\ \chi_h(\tau)\ \bar{\chi}_{\bar{h}}(\bar{\tau})$
MODULAR INVARIANCE
$= |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2$

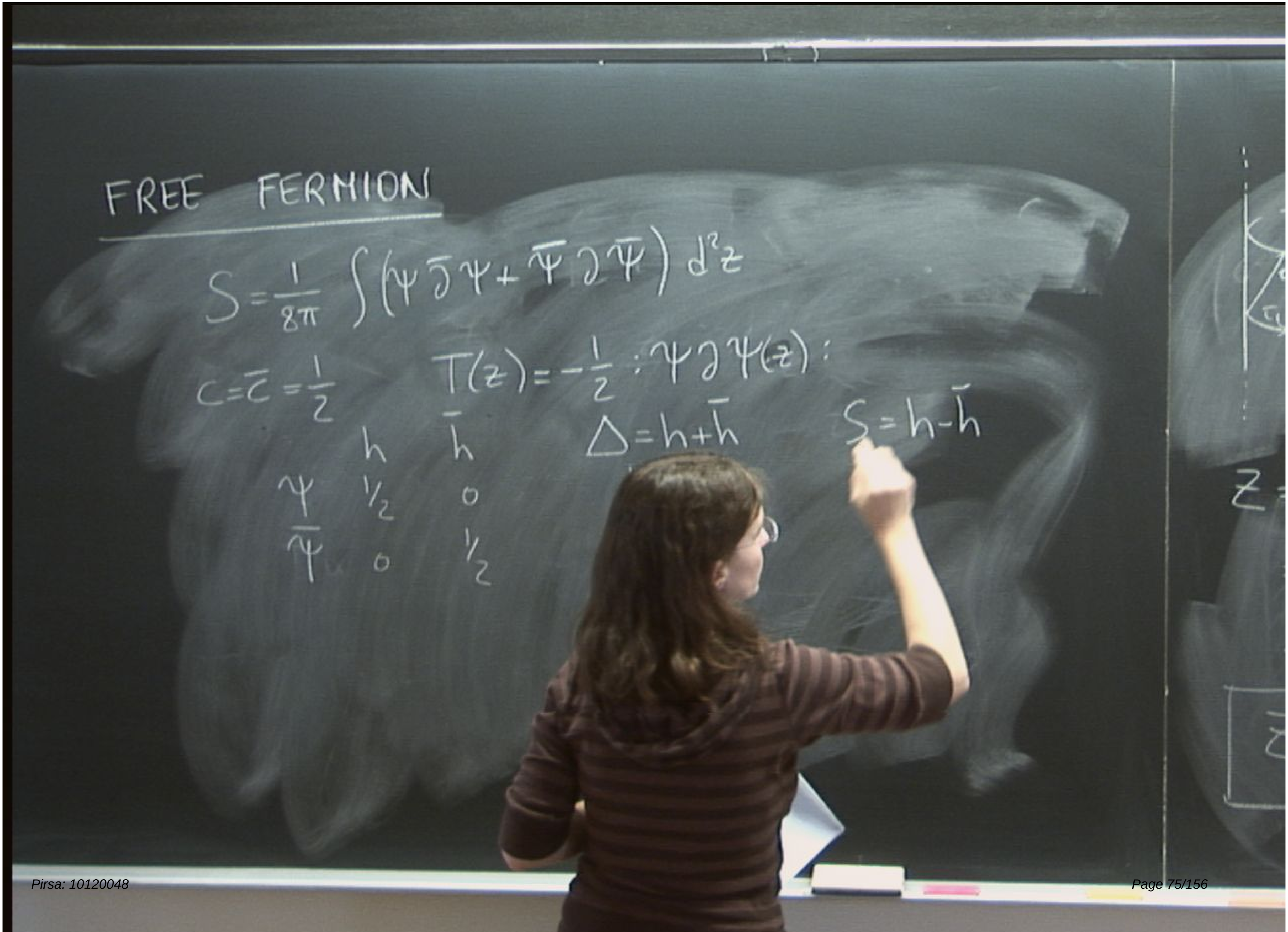

$\mathrm{Tr}_h\, q^{L_0} = \sum_{N \geq 0} d(N)\, q^{h+N}\, q^{-c/24}$
$\equiv$
$\chi_h(\tau)$
CHARACTER
STATES
AT LEVEL N
$Z(\tau) = \sum_{h,\bar{h}} M_{h,\bar{h}}\, \chi_h(\tau)\, \bar{\chi}_{\bar{h}}(\bar{\tau})$
MODULAR INVARIANCE
$Z_{ISING} = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2$
$\mathbb{1}$ ε σ

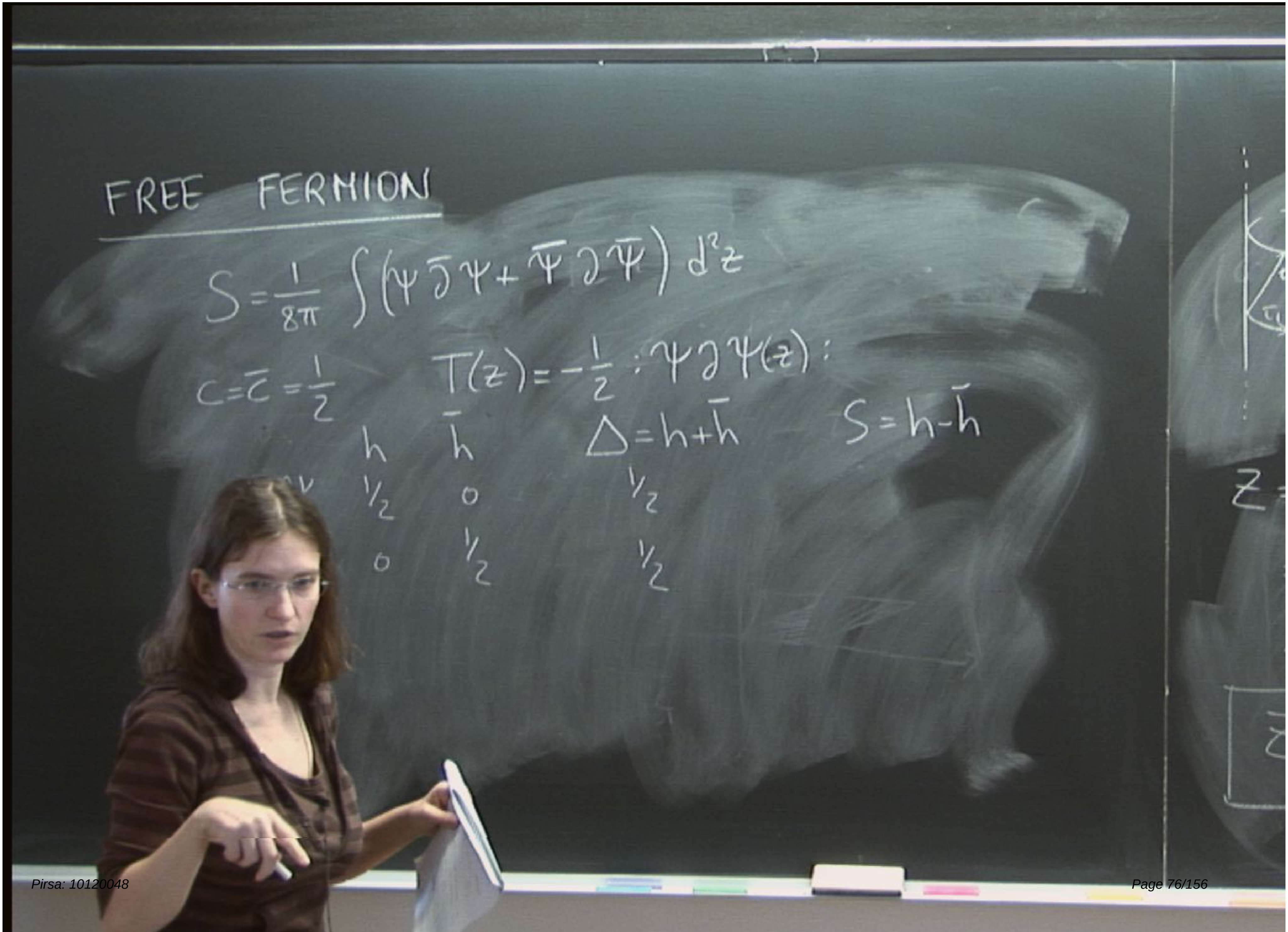

FREE FERMION

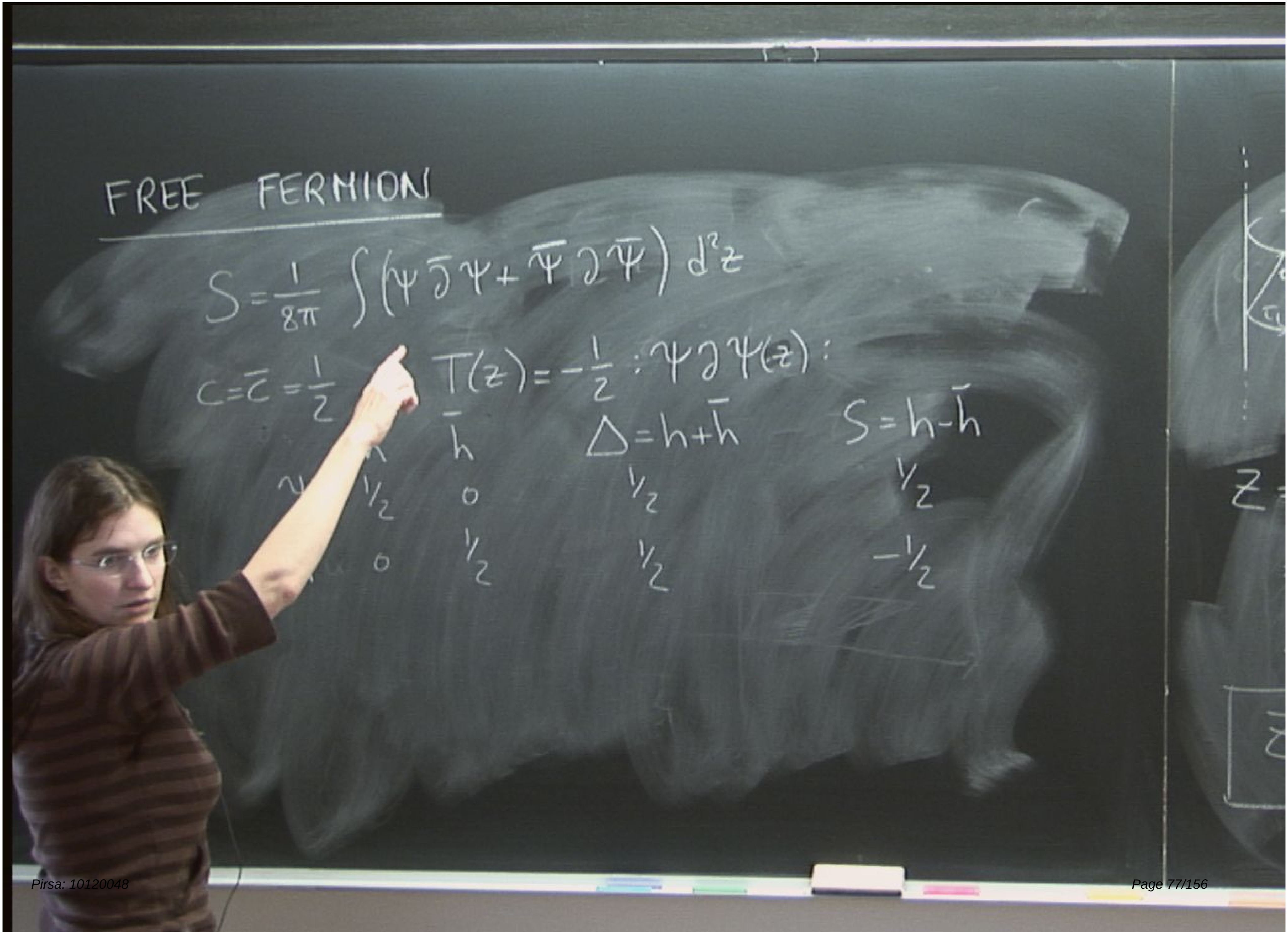

FREE FERMION
$S=\frac{1}{8\pi}\int\left(\psi\bar{\partial}\psi+\bar{\psi}\partial\bar{\psi}\right)d^2z$

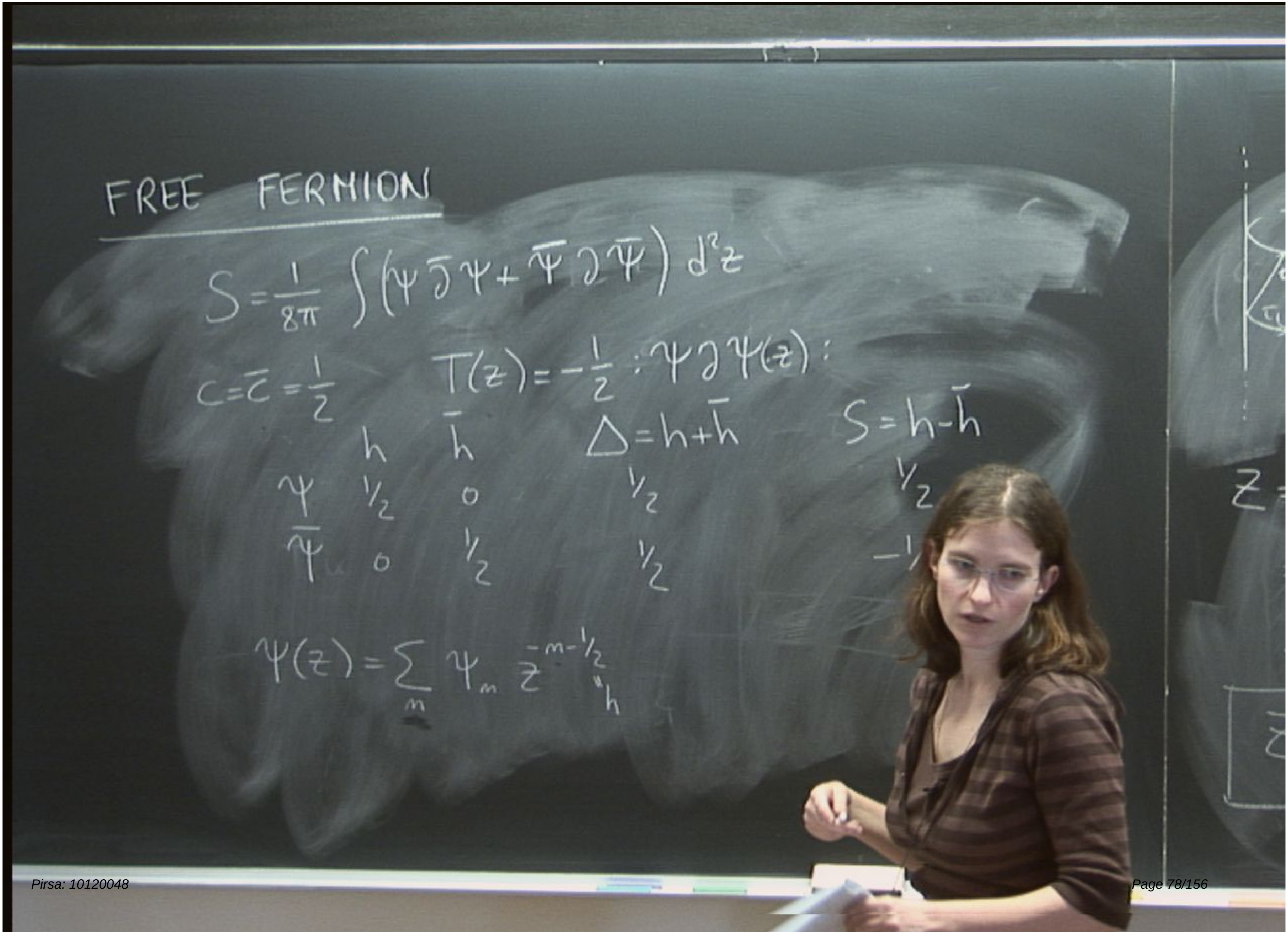

FREE FERMION
$S = \frac{1}{8\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) \, d^2 z$
$c = \bar{c} = \frac{1}{2}$
$T(z) = -\frac{1}{2} :\psi \partial \psi($

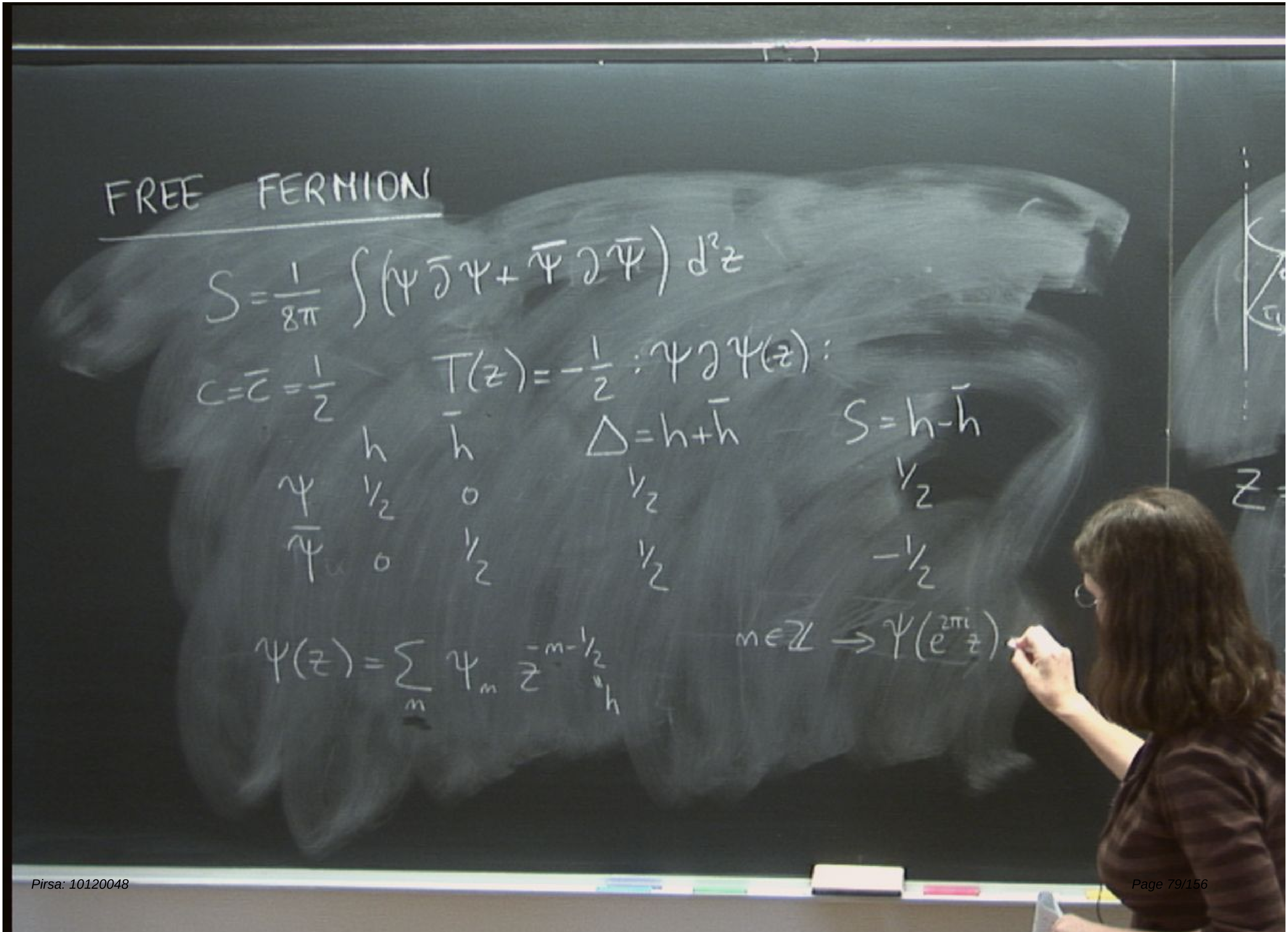

FREE FERMION
$S = \frac{1}{8\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) \, d^2 z$
$c = \bar{c} = \frac{1}{2}$
$T(z) = -\frac{1}{2} :\psi \partial \psi(z):$
h
$\bar{h}$
ψ
$\bar{\psi}$


FREE FERMION
$S = \frac{1}{8\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) \, d^2 z$
$c = \bar{c} = \frac{1}{2}$
$T(z) = -\frac{1}{2} :\psi \partial \psi(z):$
$\Delta = h + \bar{h}$
$S = h - \bar{h}$
h $\bar{h}$
ψ $1/2$ 0
$\bar{\psi}$ 0 $1/2$


FREE FERMION
S = 1/8π ∫(ψ ∂̄ ψ + ψ̄ ∂ ψ̄) d²z
c = c̄ = 1/2
T(z) = −1/2 :ψ ∂ ψ(z):
h h̄
Δ = h + h̄
S = h − h̄
1/2 0 1/2
0 1/2 1/2


FREE FERMION
S = 1/8π ∫(ψ ∂̄ ψ + ψ̄ ∂ ψ̄) d²z
c = c̄ = 1/2
T(z) = −1/2 :ψ ∂ψ(z):
h̄
Δ = h + h̄
S = h − h̄
1/2 0 1/2 1/2
0 1/2 1/2 −1/2


FREE FERMION
S = 1/8π ∫ (ψ ∂̄ ψ + ψ̄ ∂ ψ̄) d²z
c = c̄ = 1/2
T(z) = −1/2 :ψ∂ψ(z):
h h̄ Δ = h + h̄ S = h − h̄
ψ 1/2 0 1/2 1/2
ψ̄ 0 1/2 1/2
ψ(z) = Σ_m ψ_m z^(−m−1/2)


FREE FERMION
$S = \frac{1}{8\pi} \int (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) \, d^2z$
$c = \bar{c} = \frac{1}{2}$
$T(z) = -\frac{1}{2} :\psi \partial \psi(z):$
h $\bar{h}$ $\Delta = h + \bar{h}$ $S = h - \bar{h}$
ψ $1/2$ 0 $1/2$ $1/2$
$\bar{\psi}$ 0 $1/2$ $1/2$ $-1/2$
$\psi(z) = \sum_m \psi_m z^{-m-1/2}$
h
$m \in \mathbb{Z} \rightarrow \psi(e^{2\pi i} z)$
Z

FREE FERMION

$$S = \frac{1}{8\pi}\int \left(\psi\bar\partial\psi + \bar\psi\partial\bar\psi\right) d^2z$$

$$c = \bar c = \frac{1}{2} \qquad T(z) = -\frac{1}{2} :\psi\partial\psi(z):$$

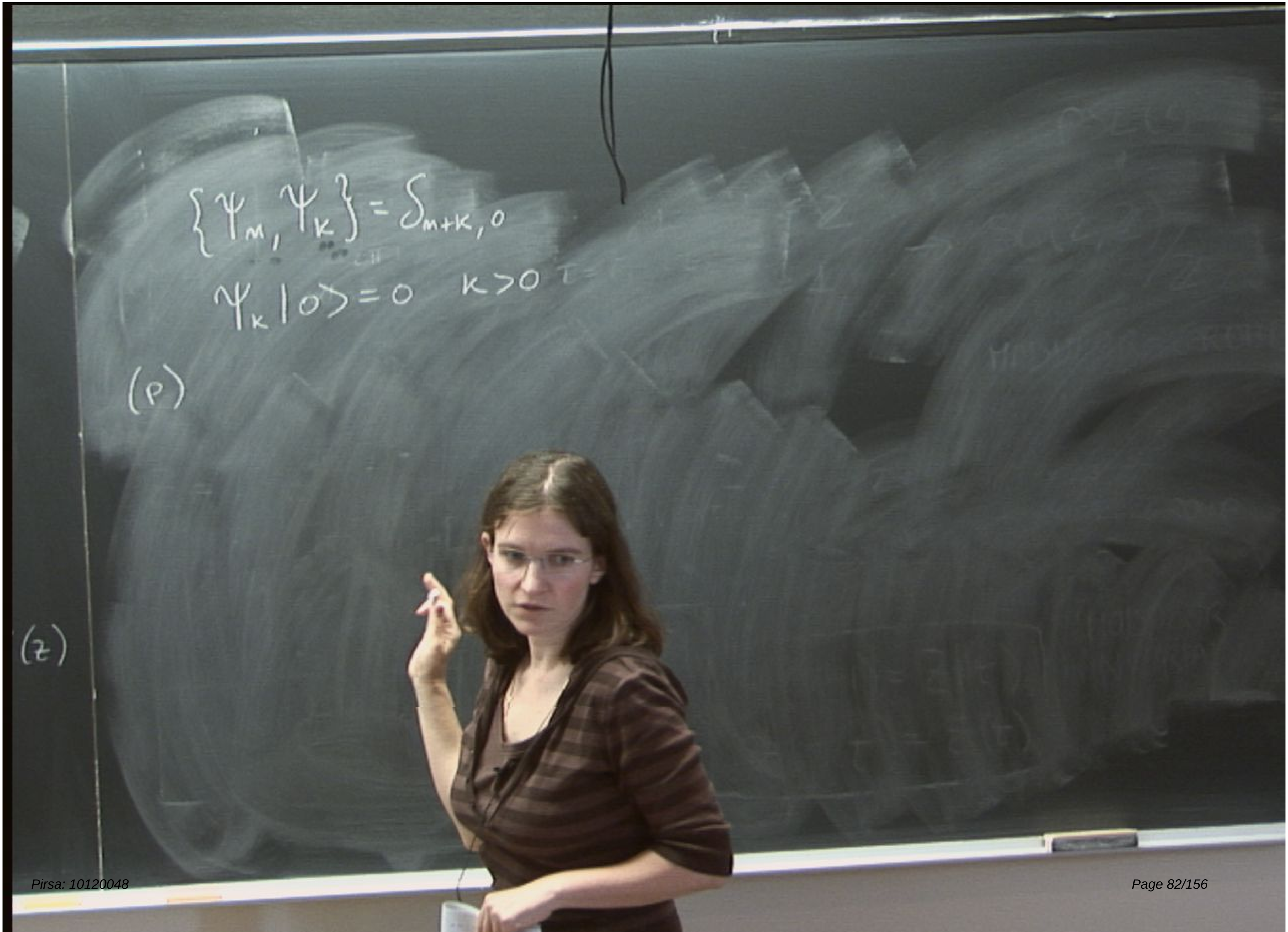
	h	$\bar h$	$\Delta = h + \bar h$	$S = h - \bar h$
ψ	$1/2$	0	$1/2$	$1/2$
$\bar\psi$	0	$1/2$	$1/2$	$-1/2$

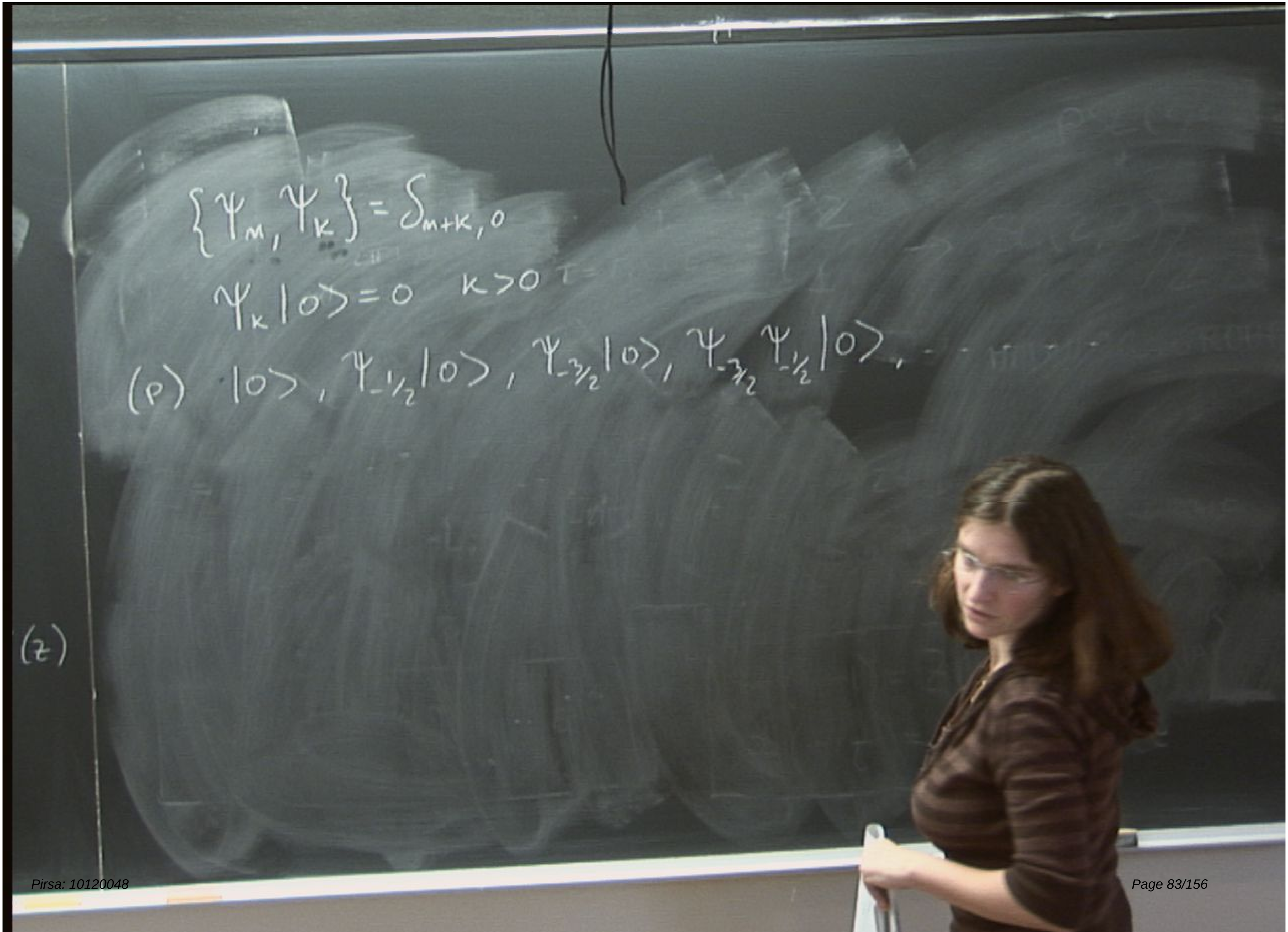
$$\psi(z) = \sum_m \psi_m z^{-m-1/2}$$

(A): $m \in \mathbb{Z} \rightarrow \psi(e^{2\pi i} z) = -\psi(z)$

(P): $m \in \mathbb{Z} + \frac{1}{2}$


$\{\Psi_m, \Psi_k\} = \delta_{m+k,0}$
(z)


$\{\psi_m, \psi_k\} = \delta_{m+k,0}$
$\psi_k |0\rangle = 0 \quad k>0$
(p)
(z)


$\{\Psi_m, \Psi_k\} = \delta_{m+k,0}$
$\Psi_k |0\rangle = 0 \quad k>0$
(P) $|0\rangle, \Psi_{-1/2}|0\rangle, \Psi_{-3/2}|0\rangle, \Psi_{-3/2}\Psi_{-1/2}|0\rangle, \dots$
(z)

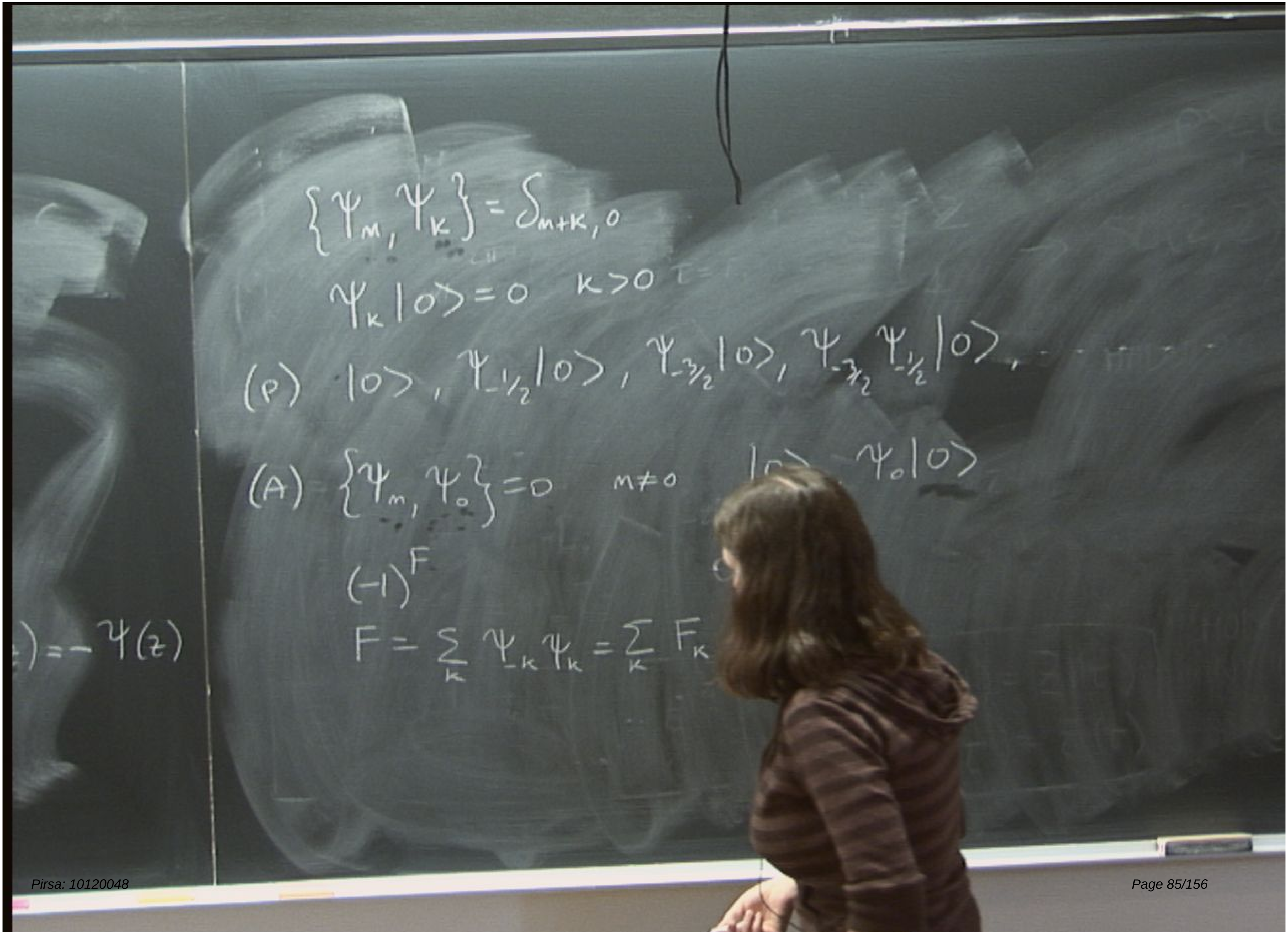
$$\{\psi_m, \psi_k\} = \delta_{m+k,0}$$

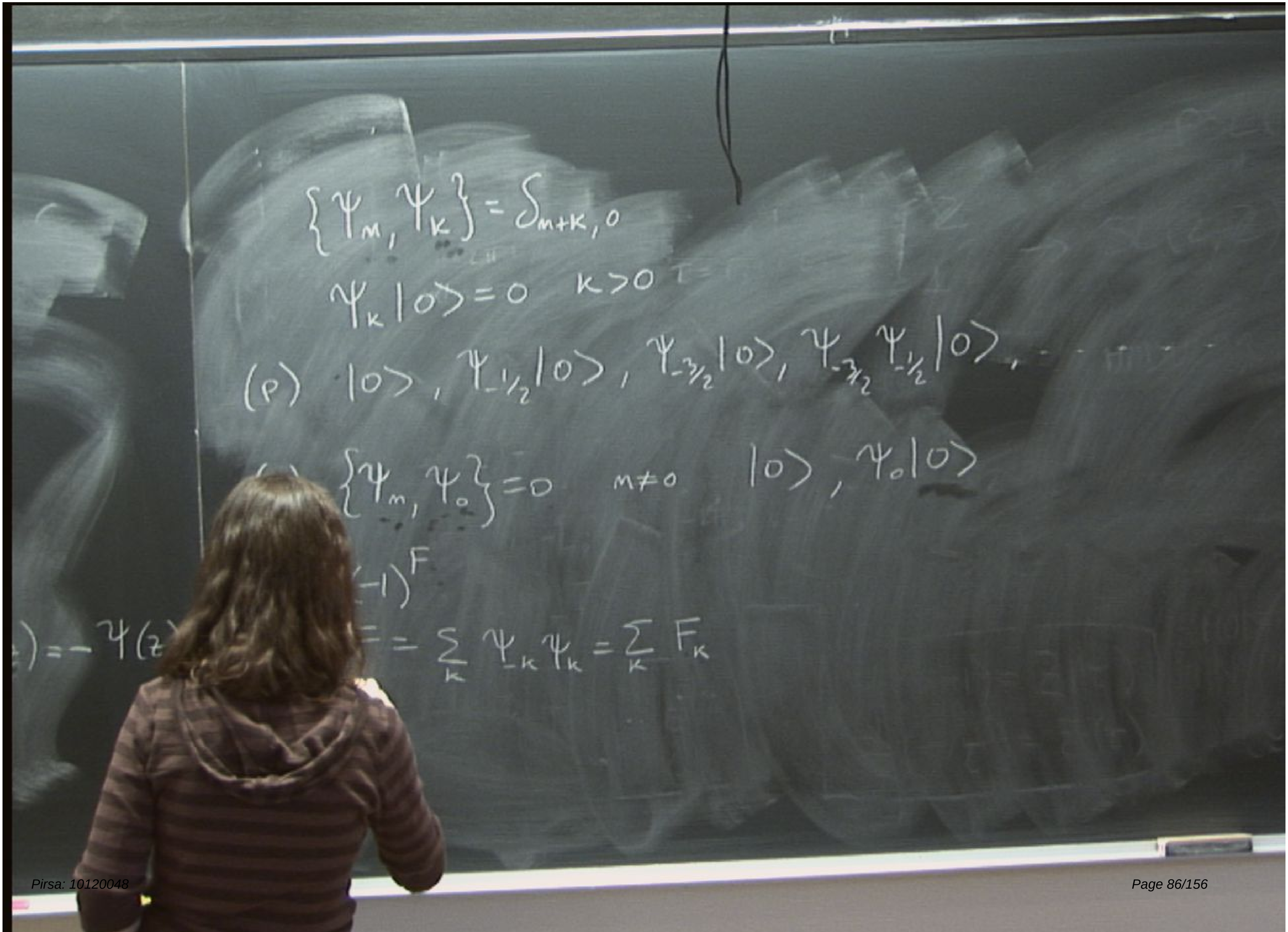
$$\psi_k |0\rangle = 0 \quad k>0$$

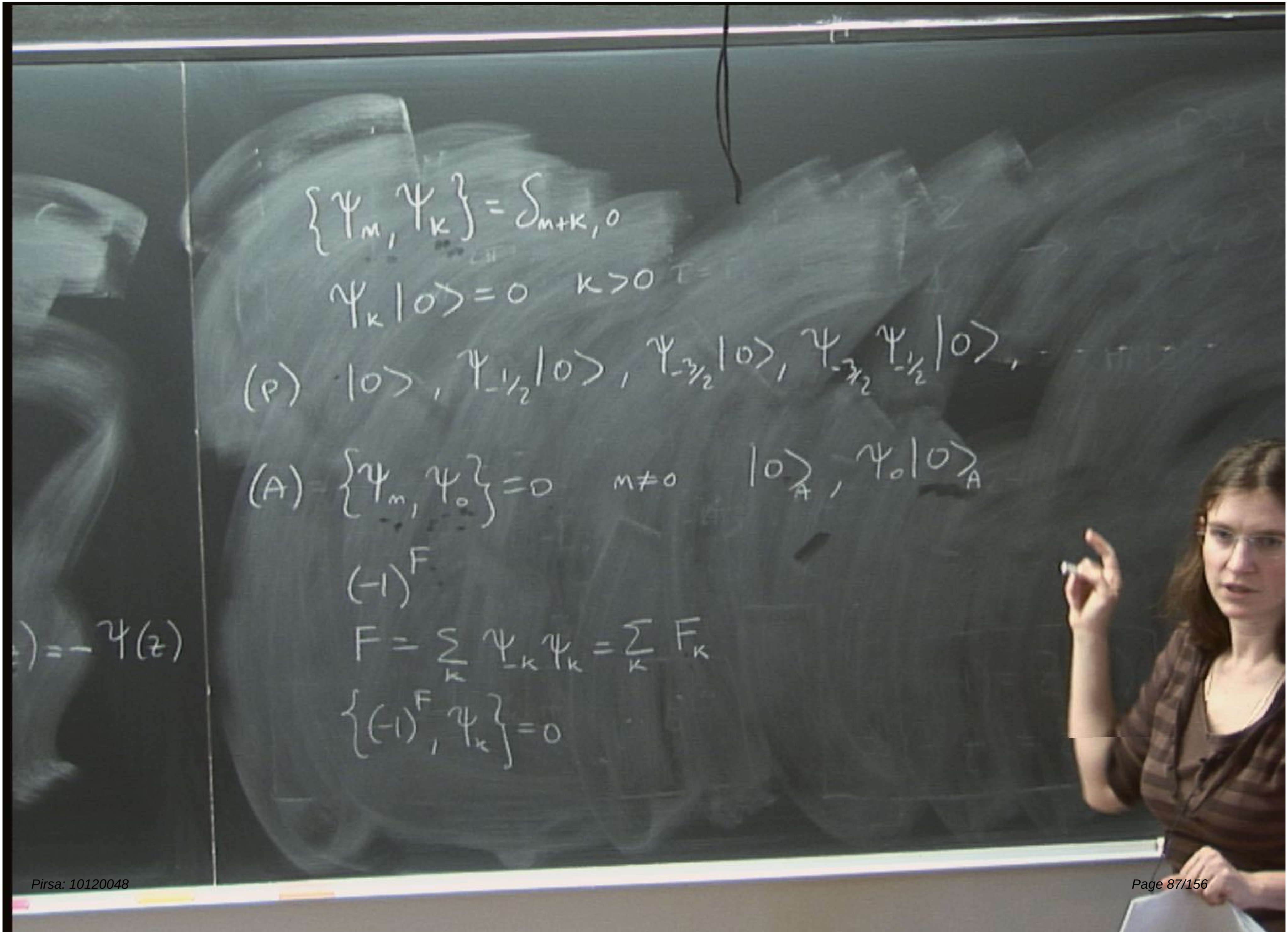
(P) $|0\rangle, \psi_{-1/2}|0\rangle, \psi_{-3/2}|0\rangle, \psi_{-3/2}\psi_{-1/2}|0\rangle, \ldots$

(A) $\{\psi_m, \psi_0\} = 0 \quad m \neq 0$

(z)


$\{\psi_m, \psi_k\} = \delta_{m+k,0}$
$\psi_k |0\rangle = 0 \quad k>0$
(P) $|0\rangle, \psi_{-1/2}|0\rangle, \psi_{-3/2}|0\rangle, \psi_{-3/2}\psi_{-1/2}|0\rangle, \ldots$
(A) $\{\psi_m, \psi_0\} = 0 \quad m \neq 0 \qquad |0\rangle \quad \psi_0|0\rangle$
$(-1)^F$
$F = \sum_k \psi_{-k}\psi_k = \sum_k F_k$
$) = -\psi(z)$


$\{\psi_m, \psi_k\} = \delta_{m+k,0}$
$\psi_k|0\rangle = 0 \quad k>0$
(P) $|0\rangle, \psi_{-1/2}|0\rangle, \psi_{-3/2}|0\rangle, \psi_{-3/2}\psi_{-1/2}|0\rangle, \ldots$
$\{\psi_m, \psi_0\} = 0 \quad m \neq 0 \qquad |0\rangle, \psi_0|0\rangle$
$(-1)^F$
$) = -\psi(z)$
$= \sum_k \psi_{-k}\psi_k = \sum_k F_k$


$\{\psi_m, \psi_k\} = \delta_{m+k,0}$
$\psi_k|0\rangle = 0 \quad k>0$
(P) $|0\rangle, \psi_{-1/2}|0\rangle, \psi_{-3/2}|0\rangle, \psi_{-3/2}\psi_{-1/2}|0\rangle, \ldots$
(A) $\{\psi_m, \psi_0\} = 0 \quad m \neq 0 \qquad |0\rangle_A, \psi_0|0\rangle_A$
$(-1)^F$
$F = \sum_k \psi_{-k}\psi_k = \sum_k F_k$
$\{(-1)^F, \psi_k\} = 0$
$) = -\psi(z)$

$$\{\psi_m, \psi_k\} = \delta_{m+k,0}$$

$$\psi_k |0\rangle = 0 \quad k>0$$

(P) $|0\rangle, \ \psi_{-1/2}|0\rangle, \ \psi_{-3/2}|0\rangle, \ \psi_{-3/2}\psi_{-1/2}|0\rangle, \ldots$

(A) $\{\psi_m, \psi_0\} = 0 \quad m \neq 0 \qquad |0\rangle_A, \ \psi_0|0\rangle_A$

$(-1)^F$

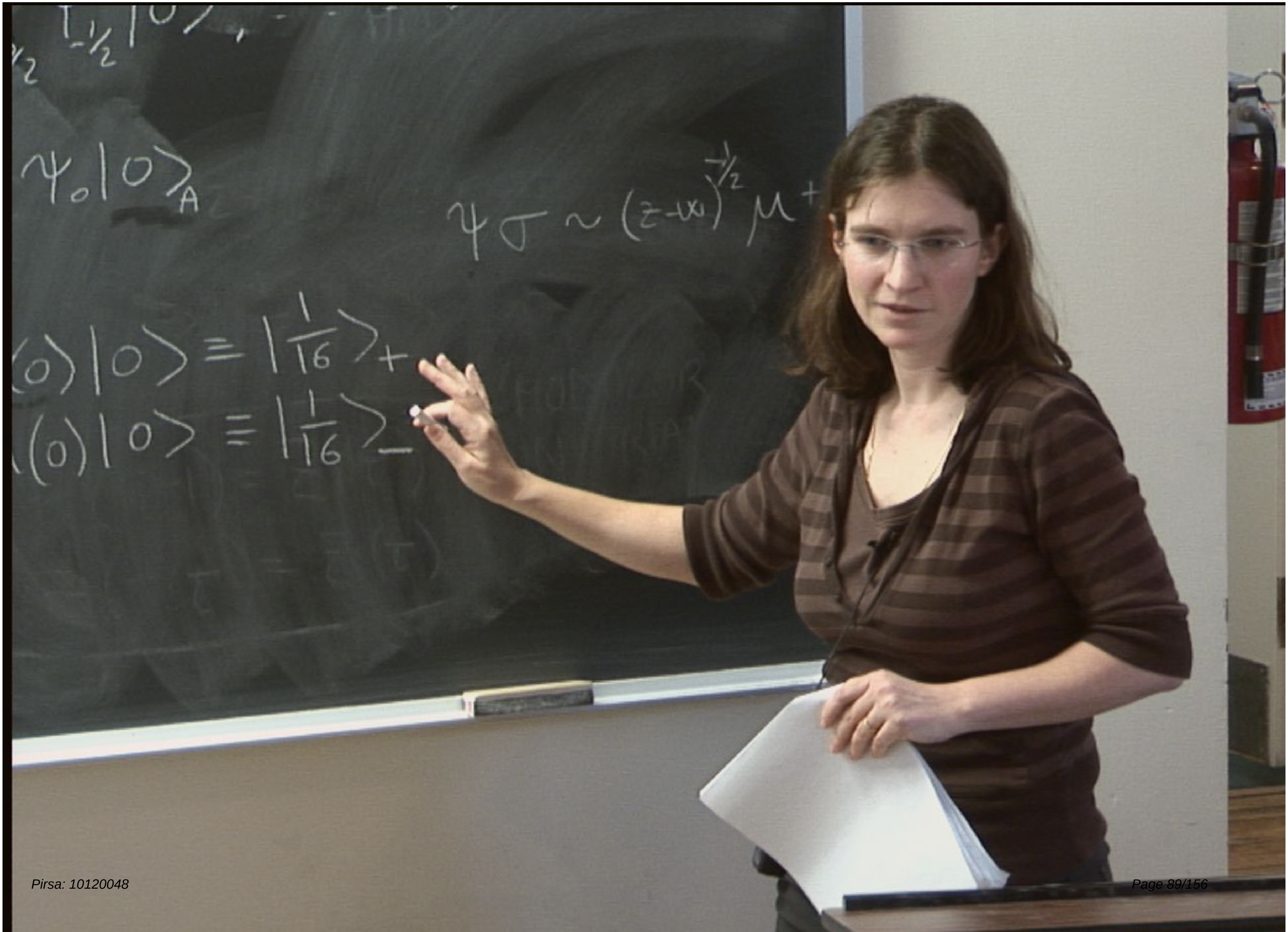
$$F = \sum_k \psi_{-k}\psi_k = \sum_k F_k$$

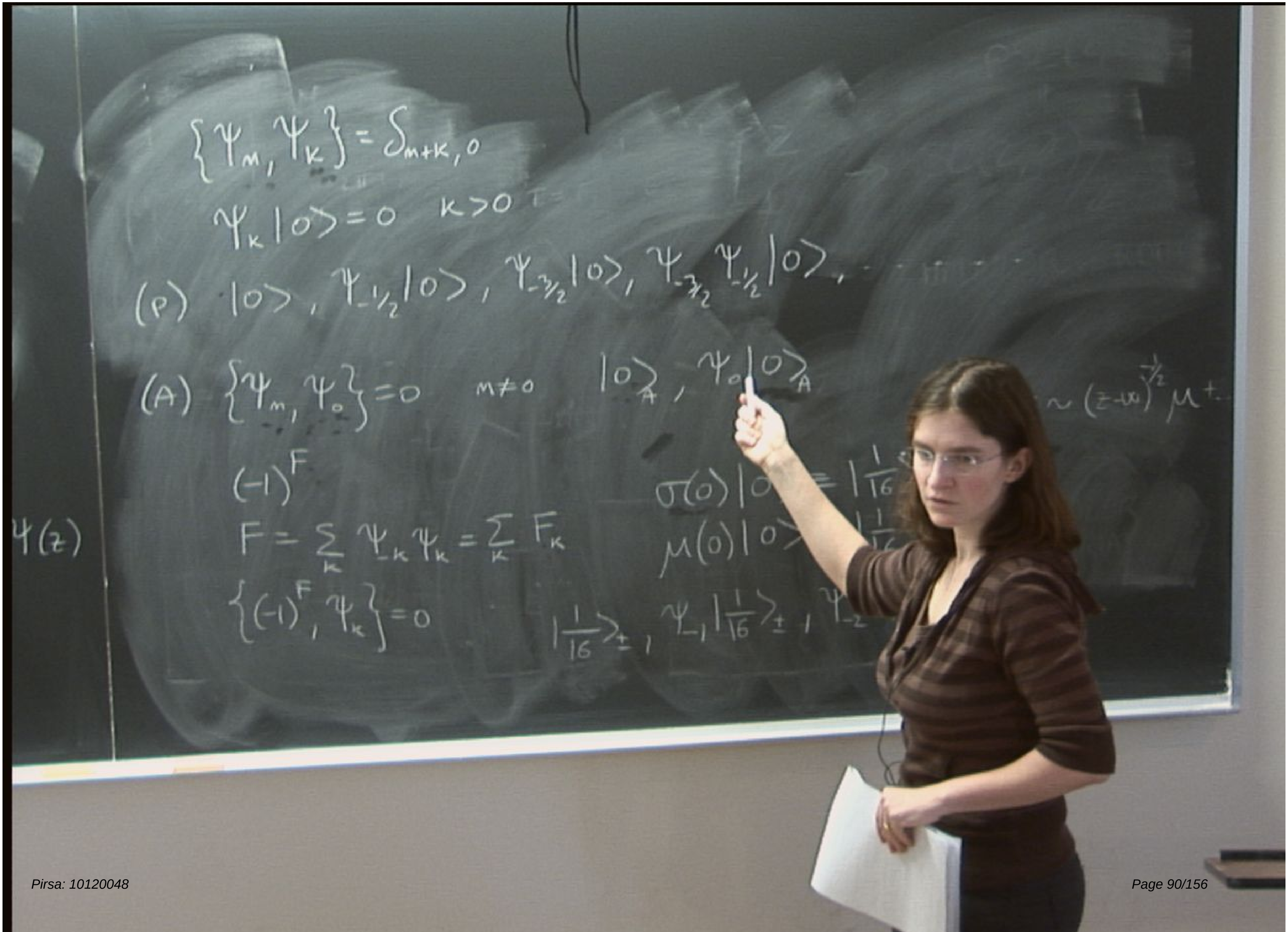
$$\{(-1)^F, \psi_k\} = 0$$

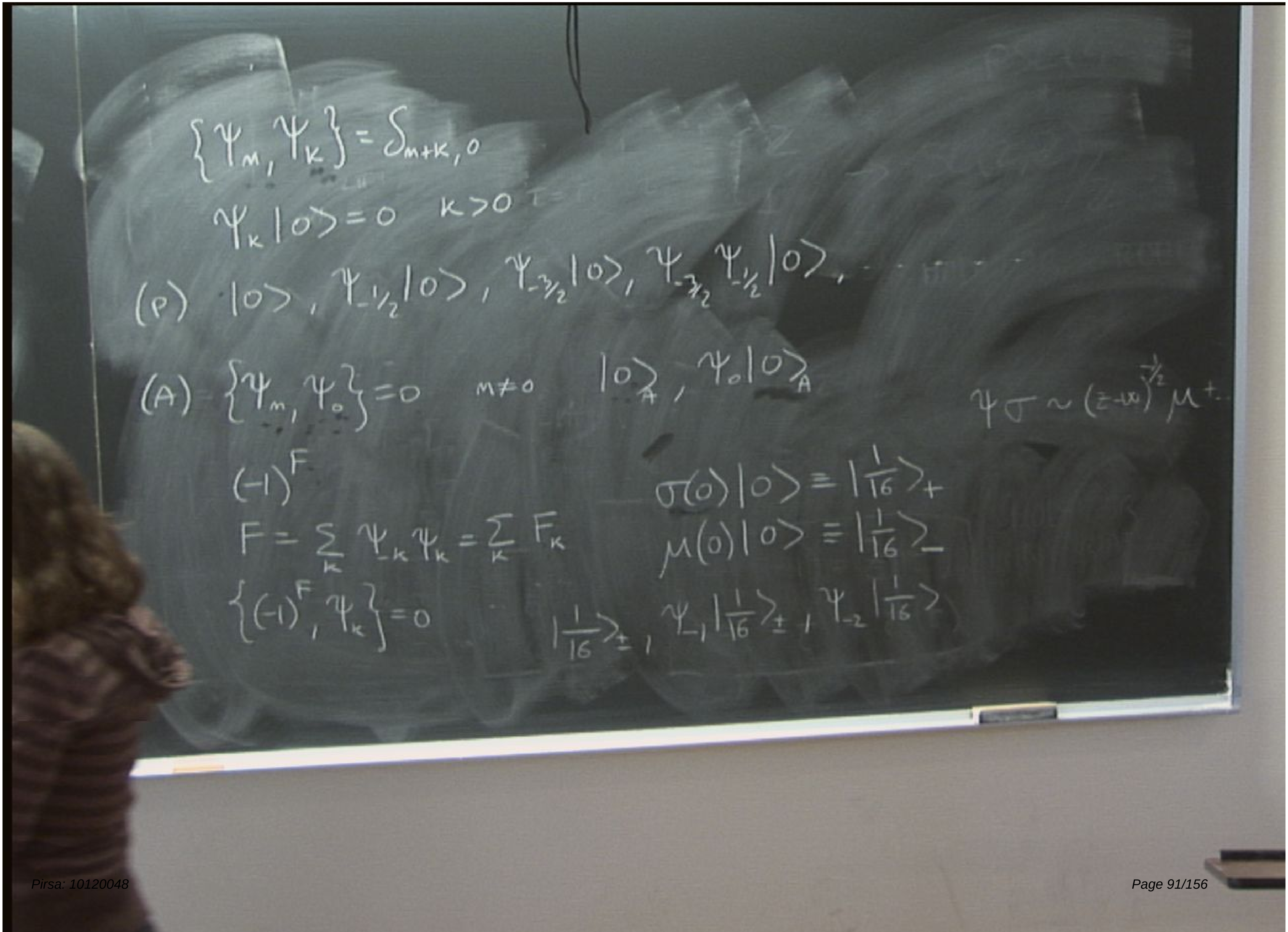
$$\sigma(0)|0\rangle \equiv |\tfrac{1}{16}\rangle_+$$

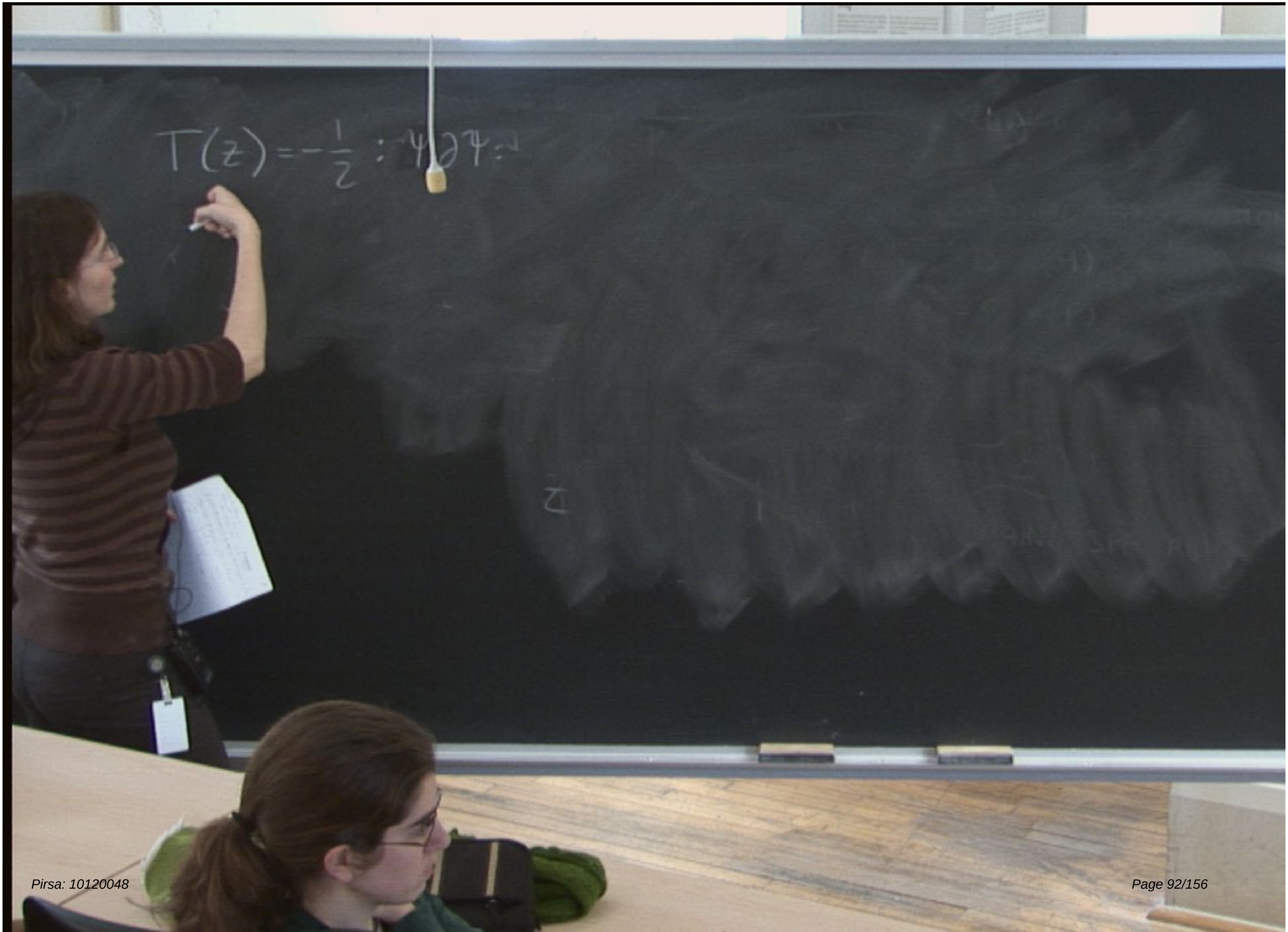
$$\mu(0)|0\rangle \equiv |\tfrac{1}{16}\rangle_-$$

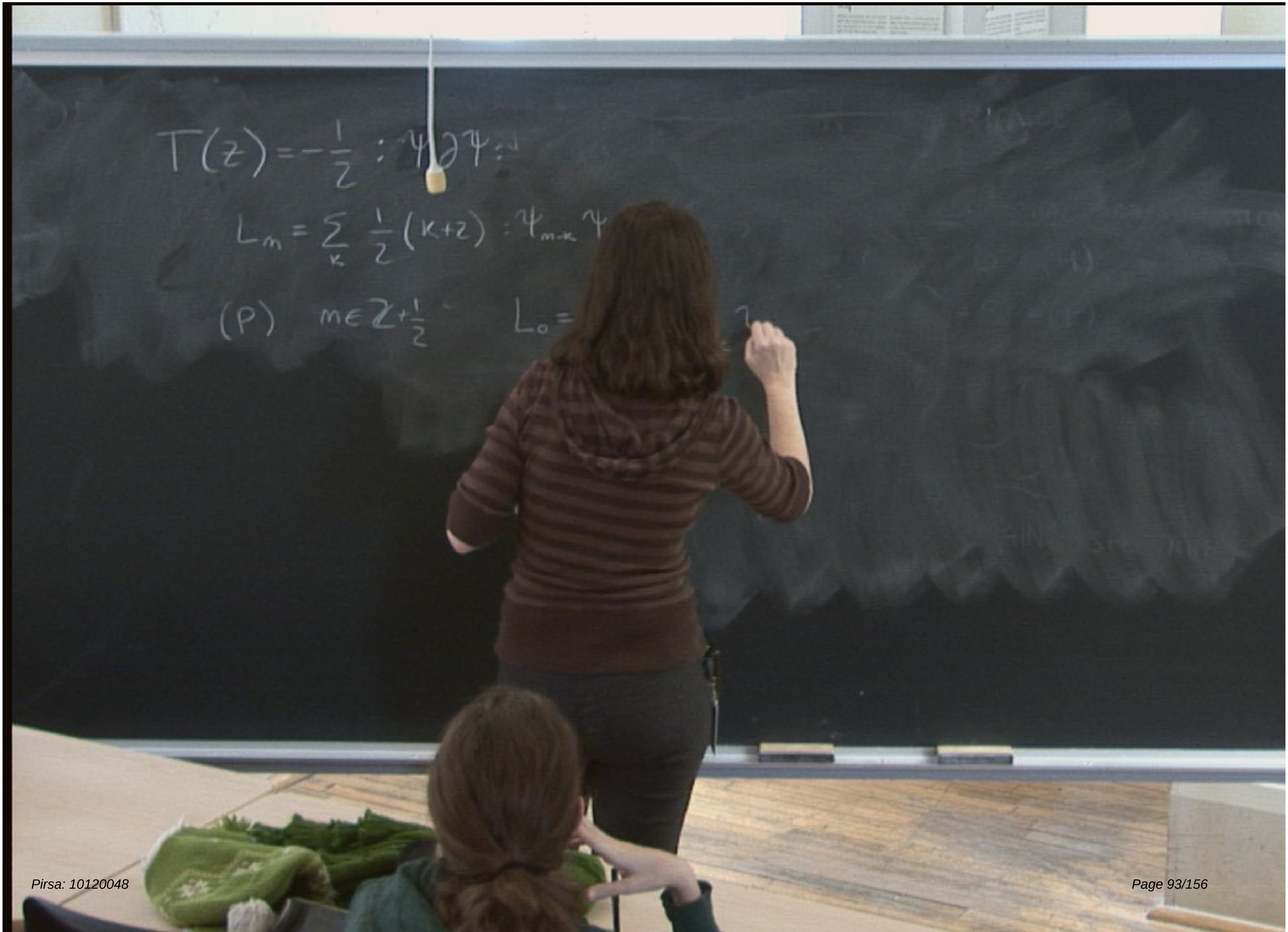
$\ldots) = -\psi(z)$

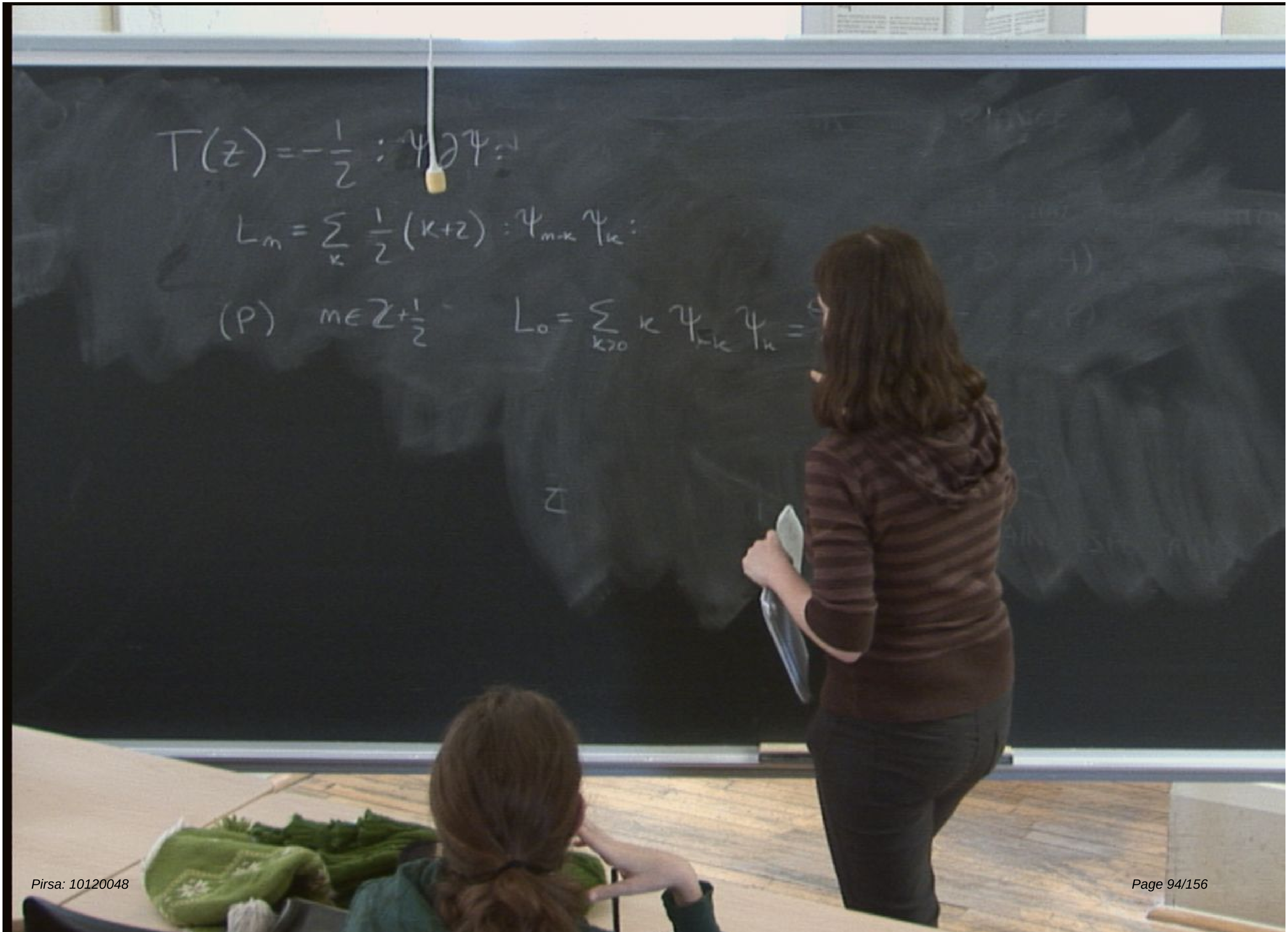


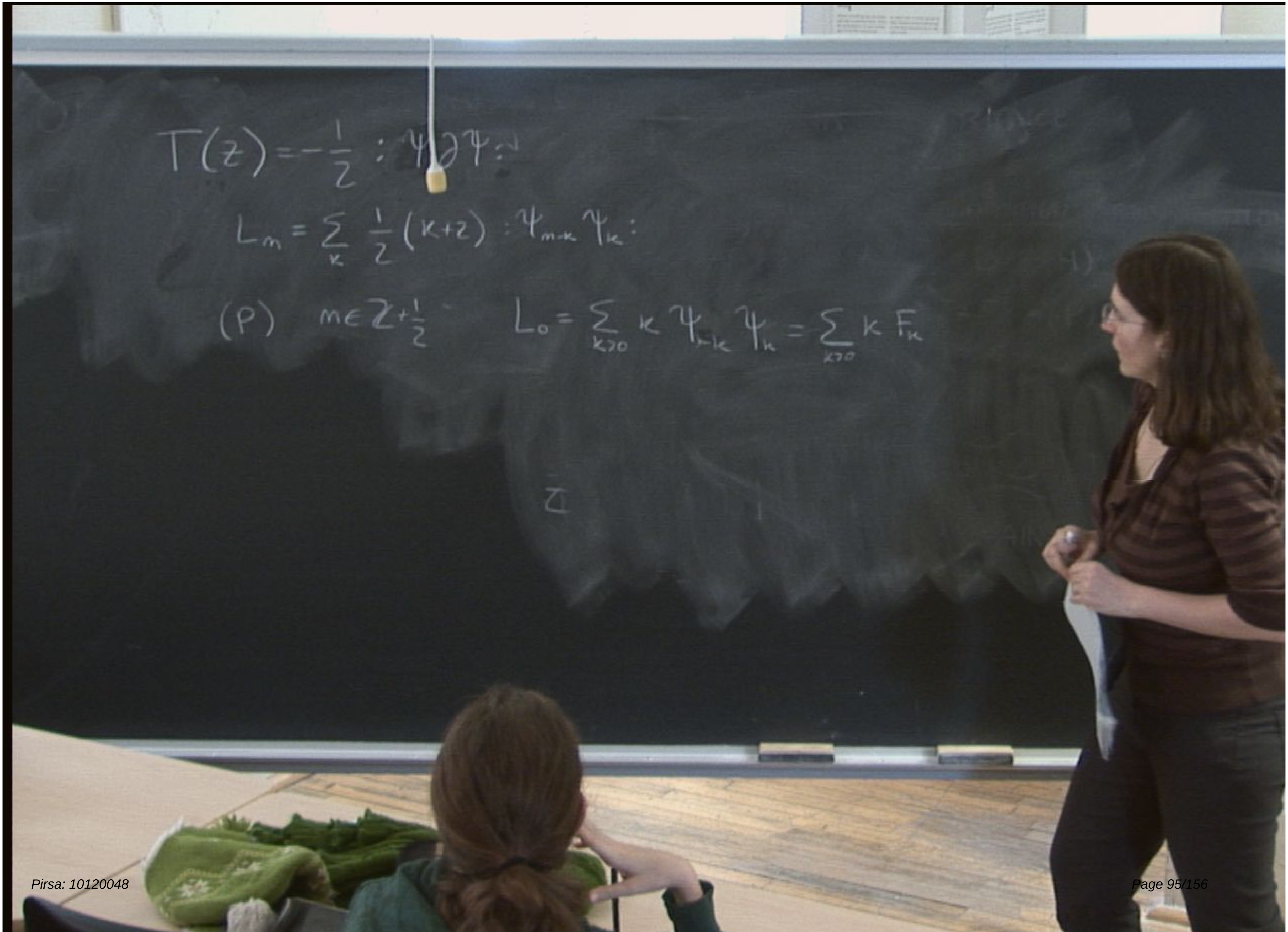

{ψm, ψk} = δm+k,0
ψk |0> = 0   k > 0
(P) |0>, ψ-1/2 |0>, ψ-3/2 |0>, ψ-3/2 ψ-1/2 |0>, ...
(A) {ψm, ψ0} = 0   m ≠ 0 |0>A, ψ0 |0>A
(-1)F
F = Σk ψ-k ψk = Σk Fk
{(-1)F, ψk} = 0
ψ(z)
σ(0)|0
μ(0)|0>
|1/16>±, ψ-1 |1/16>±, ψ-2

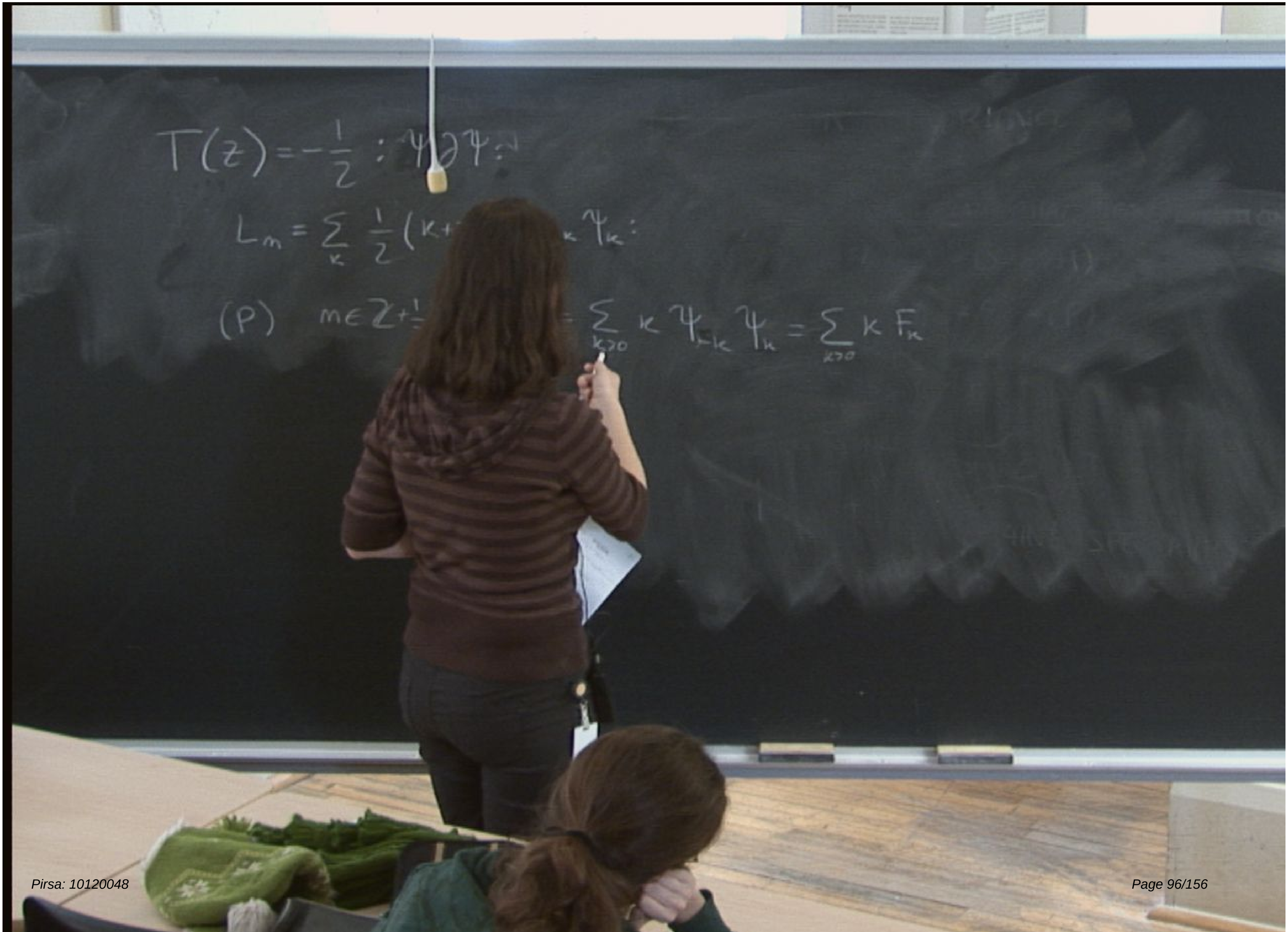

$\{\psi_m, \psi_k\} = \delta_{m+k,0}$
$\psi_k |0\rangle = 0 \quad k > 0$
(P) $|0\rangle, \psi_{-1/2}|0\rangle, \psi_{-3/2}|0\rangle, \psi_{-3/2}\psi_{-1/2}|0\rangle, \ldots$
(A) $\{\psi_m, \psi_0\} = 0 \quad m \neq 0 \quad |0\rangle_A, \psi_0|0\rangle_A$
$\psi\sigma \sim (z-w)^{-1/2}\mu^{+} \ldots$
$(-1)^F$
$F = \sum_k \psi_{-k}\psi_k = \sum_k F_k$
$\{(-1)^F, \psi_k\} = 0$
$\sigma(0)|0\rangle \equiv |\tfrac{1}{16}\rangle_+$
$\mu(0)|0\rangle \equiv |\tfrac{1}{16}\rangle_-$
$|\tfrac{1}{16}\rangle_\pm, \psi_{-1}|\tfrac{1}{16}\rangle_\pm, \psi_{-2}|\tfrac{1}{16}\rangle$

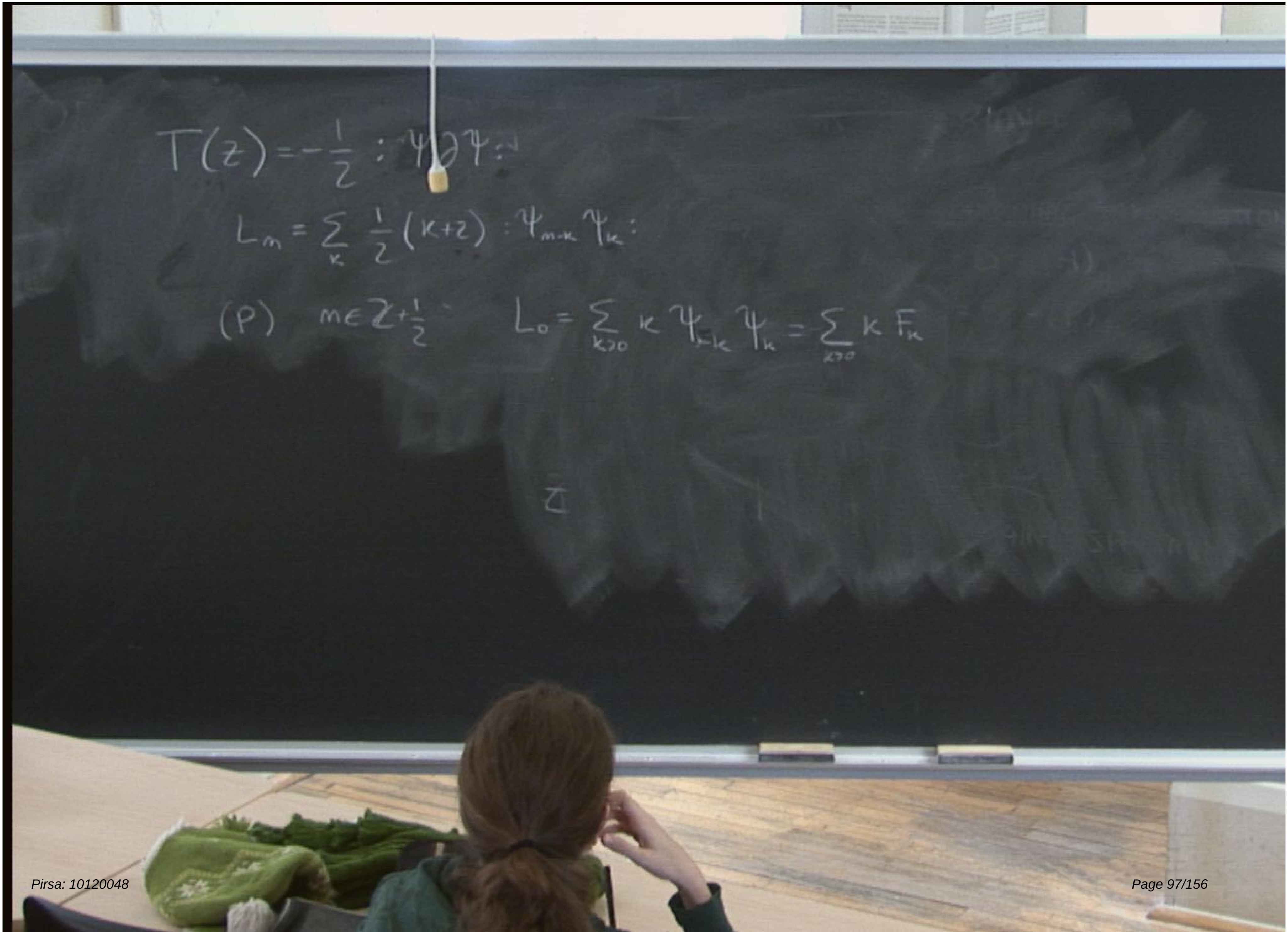

T(z) = -1/2 :ψ∂ψ:
z

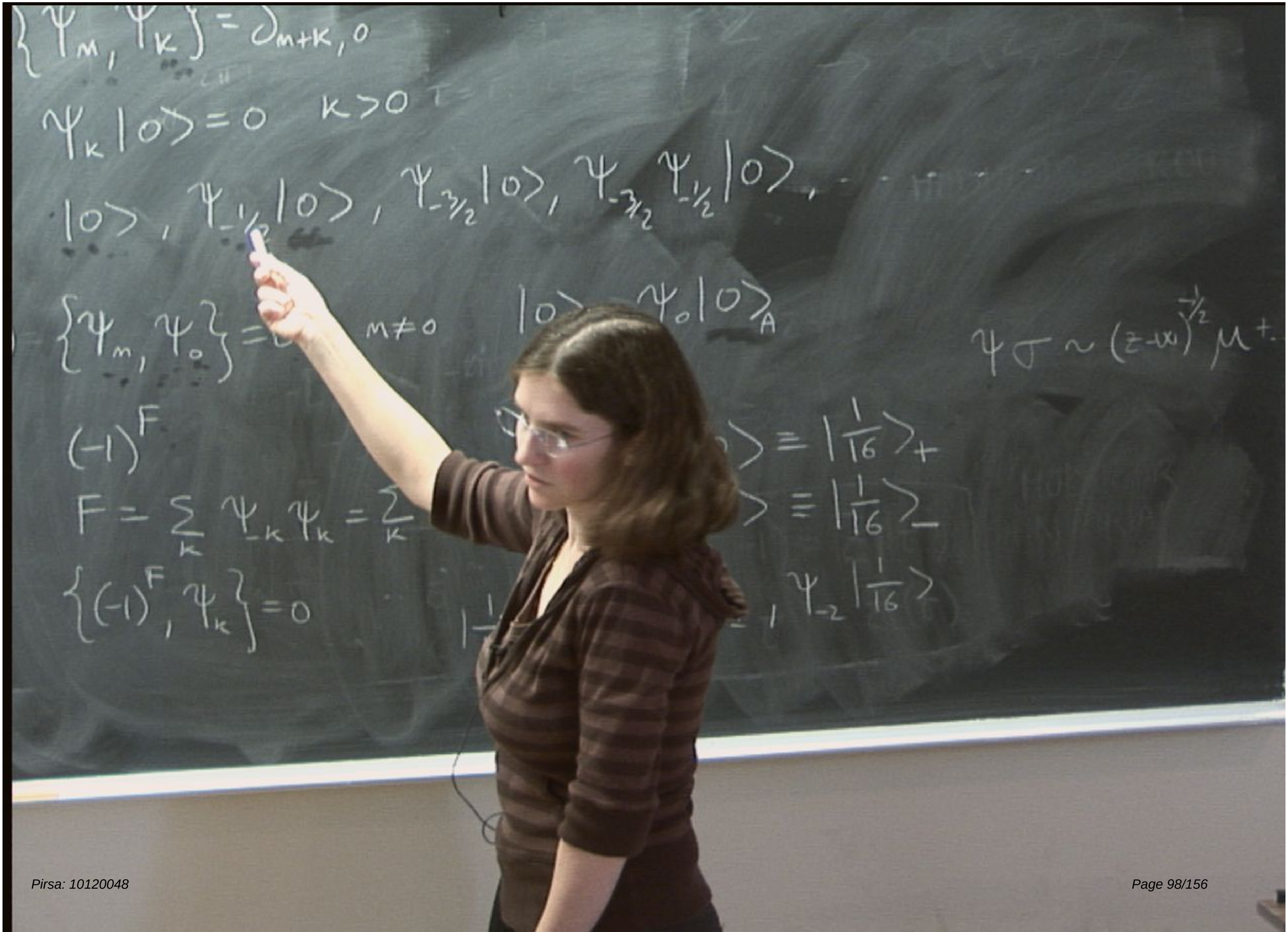

T(z) = -1/2 :ψ∂ψ:
L_m = Σ_k 1/2 (k+2) :ψ_{m-k} ψ
(P) m ∈ Z + 1/2
L_0 =

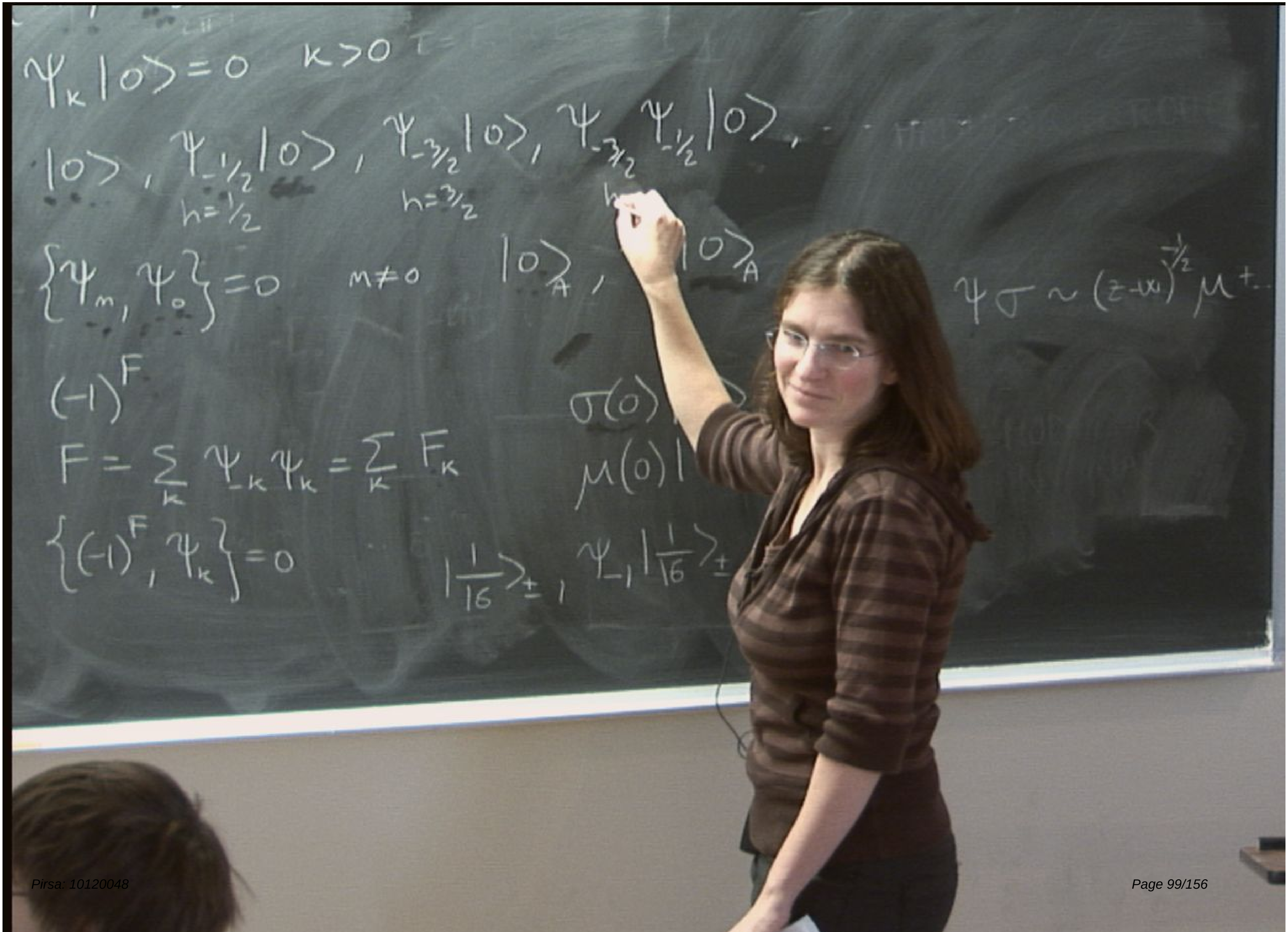

T(z) = -1/2 :ψ∂ψ:
L_m = Σ_k 1/2 (k+z) :ψ_{m-k} ψ_k:
(P) m∈Z+1/2
L_0 = Σ_{k>0} k ψ_{-k} ψ_k =

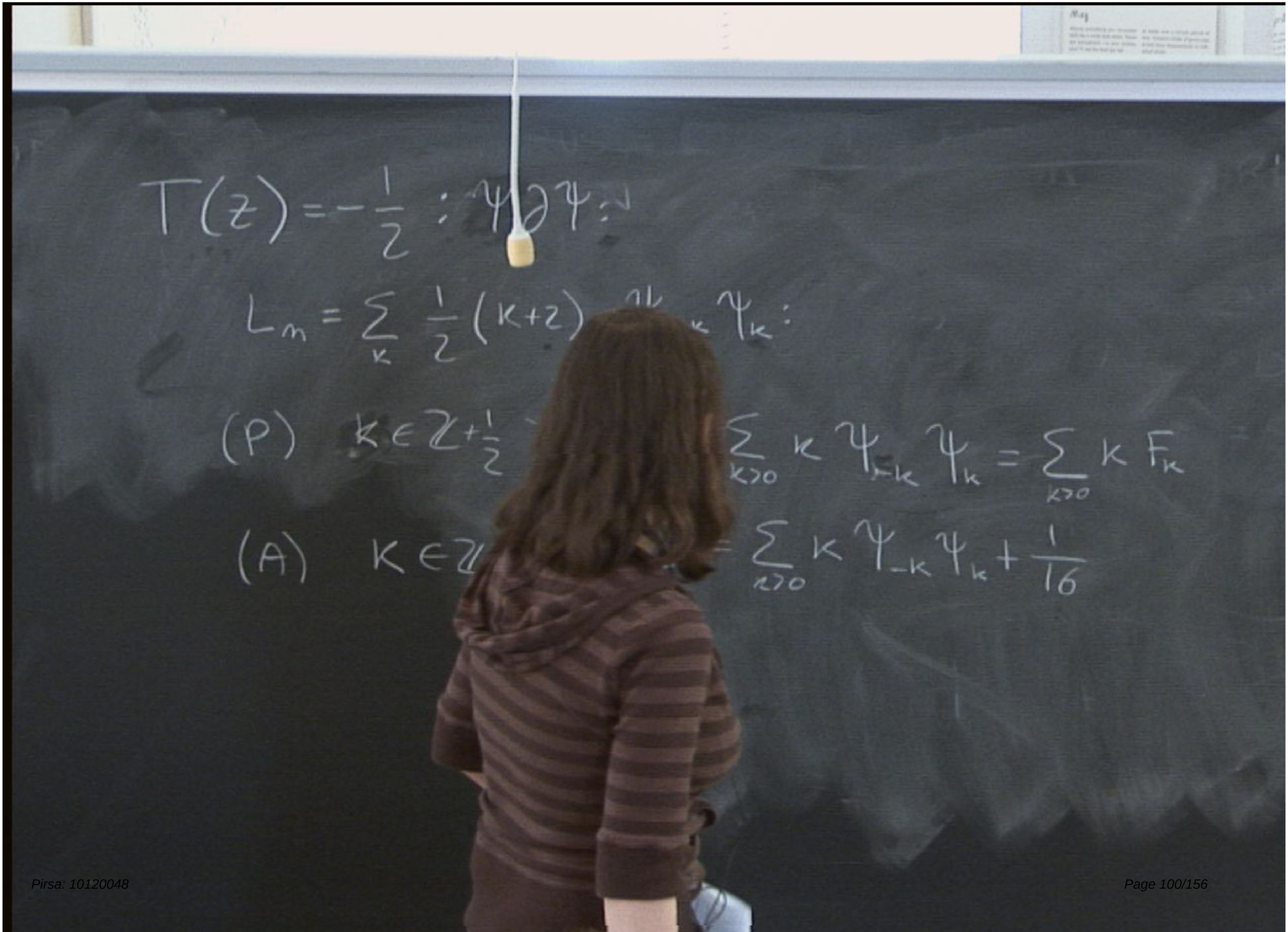

T(z) = -1/2 :ψ∂ψ:
L_m = Σ_k 1/2 (k+2) :ψ_{m-k} ψ_k:
(P) m ∈ Z + 1/2
L_0 = Σ_{k>0} k ψ_{-k} ψ_k = Σ_{k>0} k F_k
Z

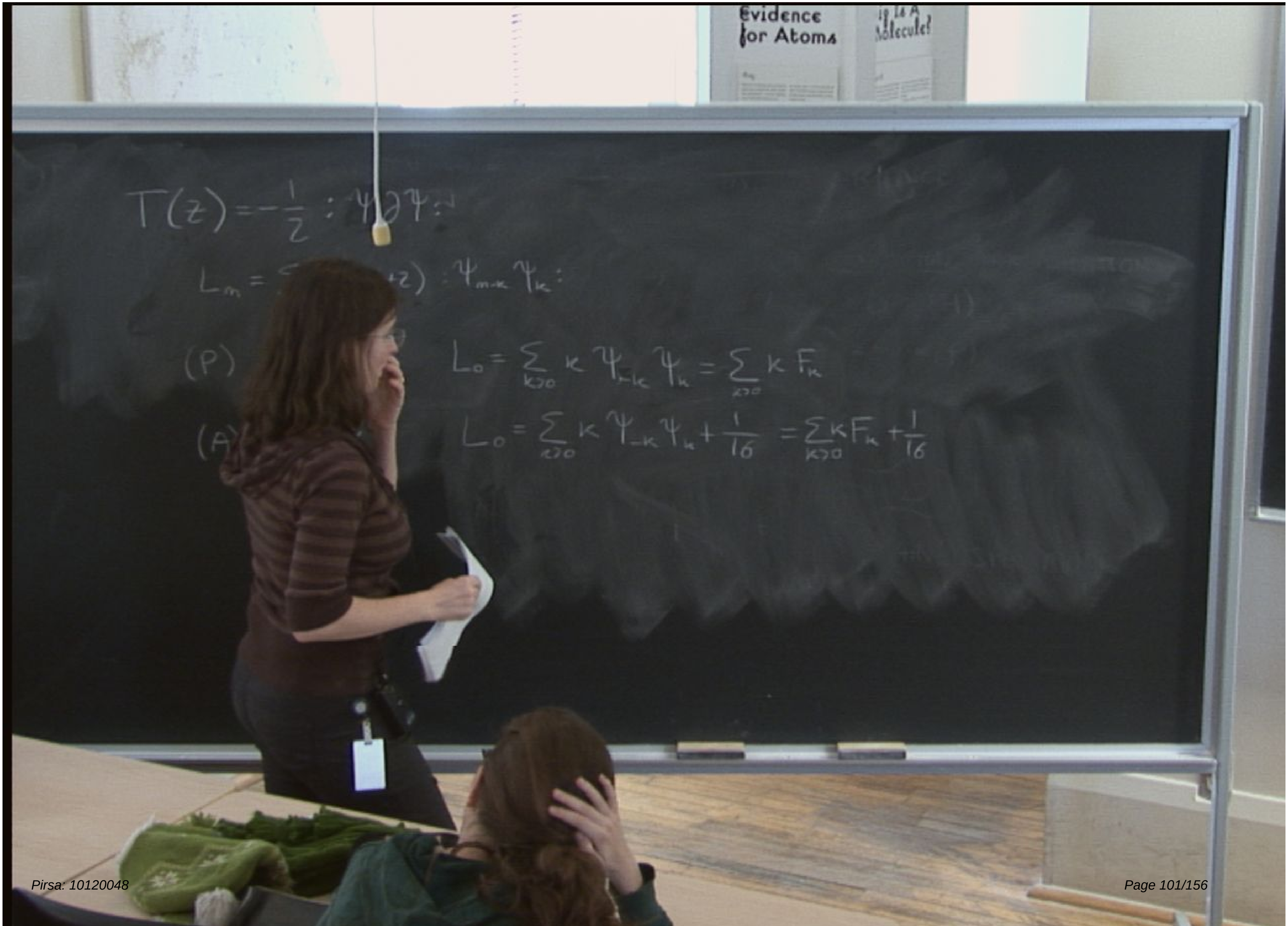

T(z) = -1/2 :ψ∂ψ:
(P) m∈Z+1/2
= Σ k>0 k ψ-k ψk = Σ k>0 k Fk

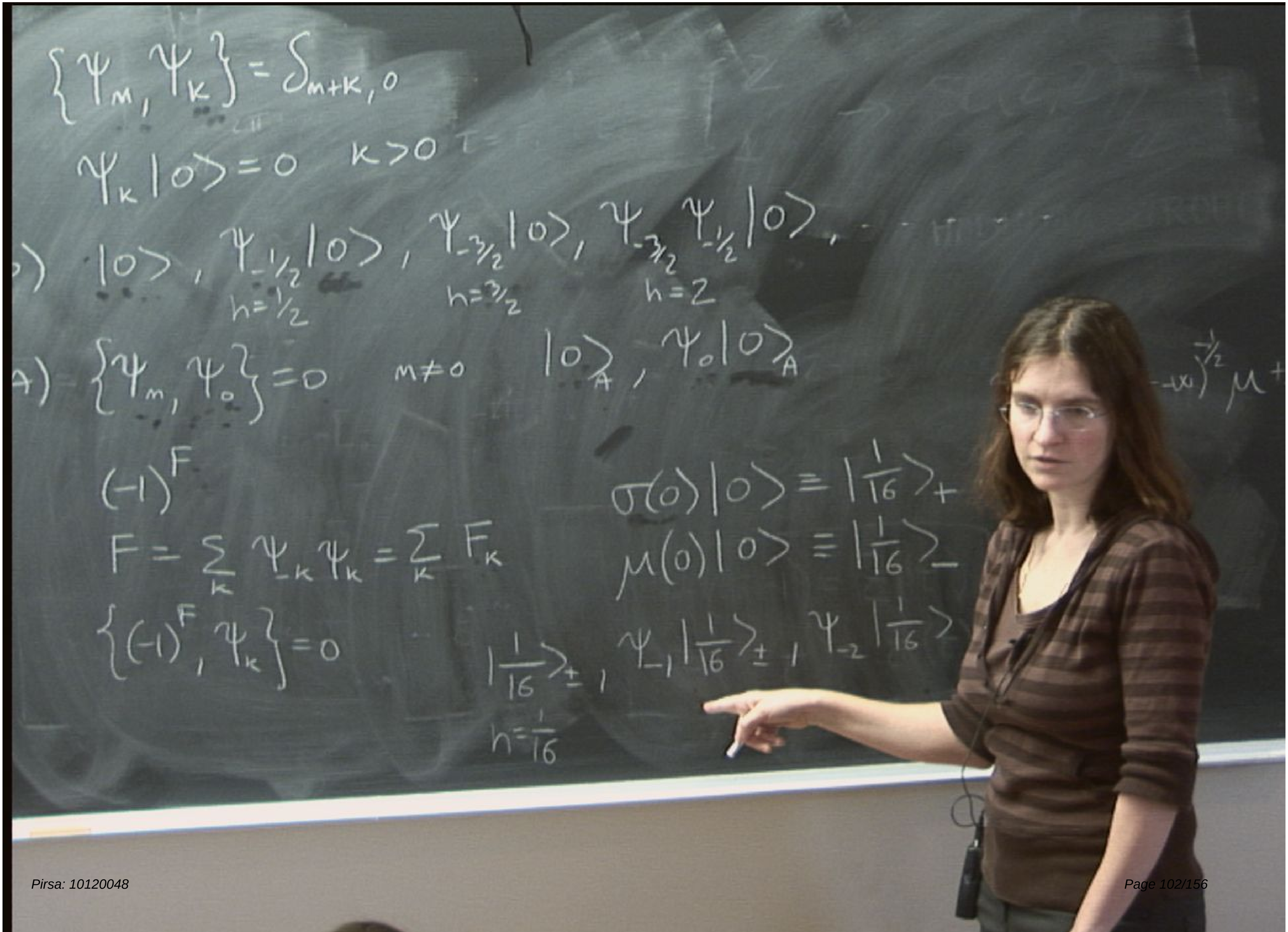

$T(z) = -\frac{1}{2} :\psi \partial \psi:$
$L_m = \sum_k \frac{1}{2}(k+2) :\psi_{m-k} \psi_k:$
(P) $m \in \mathbb{Z} + \frac{1}{2}$
$L_0 = \sum_{k>0} k \psi_{-k} \psi_k = \sum_{k>0} k F_k$


{Ψm, Ψk} = δm+k,0
Ψk |0> = 0   k > 0
|0>, Ψ-1/2 |0>, Ψ-3/2 |0>, Ψ-3/2 Ψ-1/2 |0>, ...
{Ψm, Ψ0} = 0   m ≠ 0
|0>   Ψ0 |0>A
Ψσ ~ (z-w)^-1/2 μ+
(-1)^F
F = Σk Ψ-k Ψk = Σk
{(-1)^F, Ψk} = 0
> ≡ |1/16>+
> ≡ |1/16>-
Ψ-2 |1/16>


ψ_k |0> = 0   k > 0
|0> , ψ_{-1/2} |0> , ψ_{-3/2} |0> , ψ_{-3/2} ψ_{-1/2} |0> , ...
h = 1/2   h = 3/2
{ψ_m , ψ_0} = 0   m ≠ 0
|0>_A , |0>_A
ψ σ ~ (z-w)^{-1/2} μ + ...
(-1)^F
σ(0)
F = Σ_k ψ_{-k} ψ_k = Σ_k F_k
μ(0)|
{(-1)^F , ψ_k} = 0
|1/16>_± , ψ_{-1} |1/16>_±


$T(z) = -\frac{1}{2} :\psi \partial \psi:$
$L_n = \sum_k \frac{1}{2}(k+2)$
$\psi_k :$
(P) $k \in \mathbb{Z} + \frac{1}{2}$
$\sum_{k>0} k \, \psi_{-k} \psi_k = \sum_{k>0} k \, F_k$
(A) $k \in \mathbb{Z}$
$= \sum_{k>0} k \, \psi_{-k} \psi_k + \frac{1}{16}$


Evidence for Atoms
T(z) = -1/2 :ψ∂ψ:
(P)
L_0 = Σ_{k>0} k ψ_{-k} ψ_k = Σ_{k>0} k F_k
(A)
L_0 = Σ_{k>0} k ψ_{-k} ψ_k + 1/16 = Σ_{k>0} k F_k + 1/16



{ψ_m , ψ_k} = δ_{m+k,0}
ψ_k |0> = 0   k > 0
|0> , ψ_{-1/2}|0> , ψ_{-3/2}|0> , ψ_{-3/2} ψ_{-1/2}|0> , . . .
h=1/2   h=3/2   h=2
{ψ_m , ψ_0} = 0   m ≠ 0 |0>_A , ψ_0|0>_A
(-1)^F
F = Σ_k ψ_{-k} ψ_k = Σ_k F_k
{(-1)^F , ψ_k} = 0
σ(0)|0> ≡ |1/16>_+
μ(0)|0> ≡ |1/16>_−
|1/16>_± , ψ_{-1}|1/16>_± , ψ_{-2}|1/16>
h = 1/16

$$\{\psi_m, \psi_k\} = \delta_{m+k,0}$$

$$\psi_k |0\rangle = 0 \quad k > 0$$

$$|0\rangle, \quad \psi_{-1/2}|0\rangle, \quad \psi_{-3/2}|0\rangle, \quad \psi_{-3/2}\psi_{-1/2}|0\rangle, \ldots$$

$$h = 1/2 \qquad h = 3/2 \qquad h = 2$$

$$\text{A)} \quad \{\psi_m, \psi_0\} = 0 \quad m \neq 0 \qquad |0\rangle_A, \quad \psi_0|0\rangle_A$$

$$\psi\sigma \sim (z-w)^{-1/2} \mu^{+}$$

$$(-1)^F$$

$$\sigma(0)|0\rangle \equiv |\tfrac{1}{16}\rangle_+$$

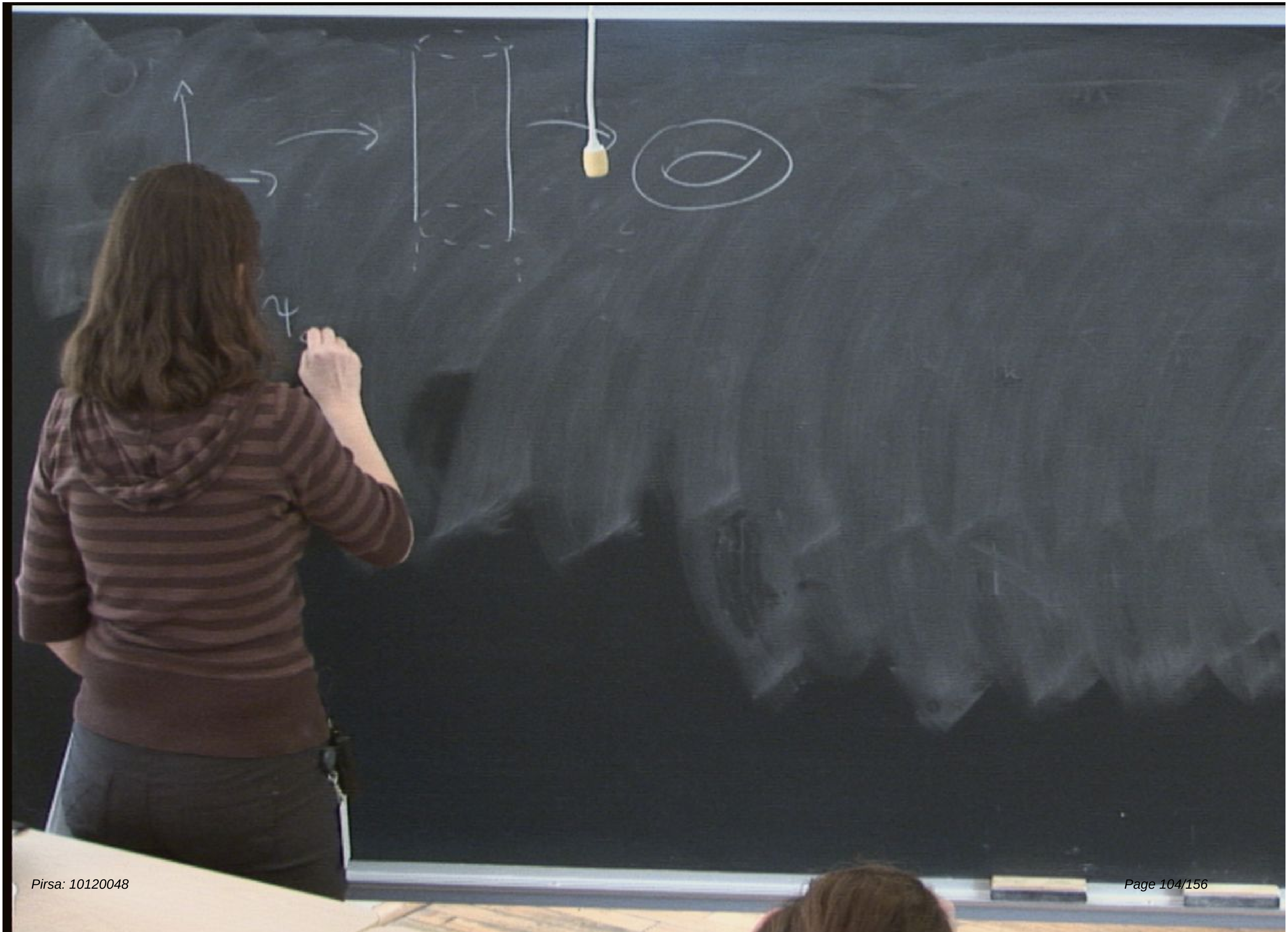
$$F = \sum_k \psi_{-k}\psi_k = \sum_k F_k$$

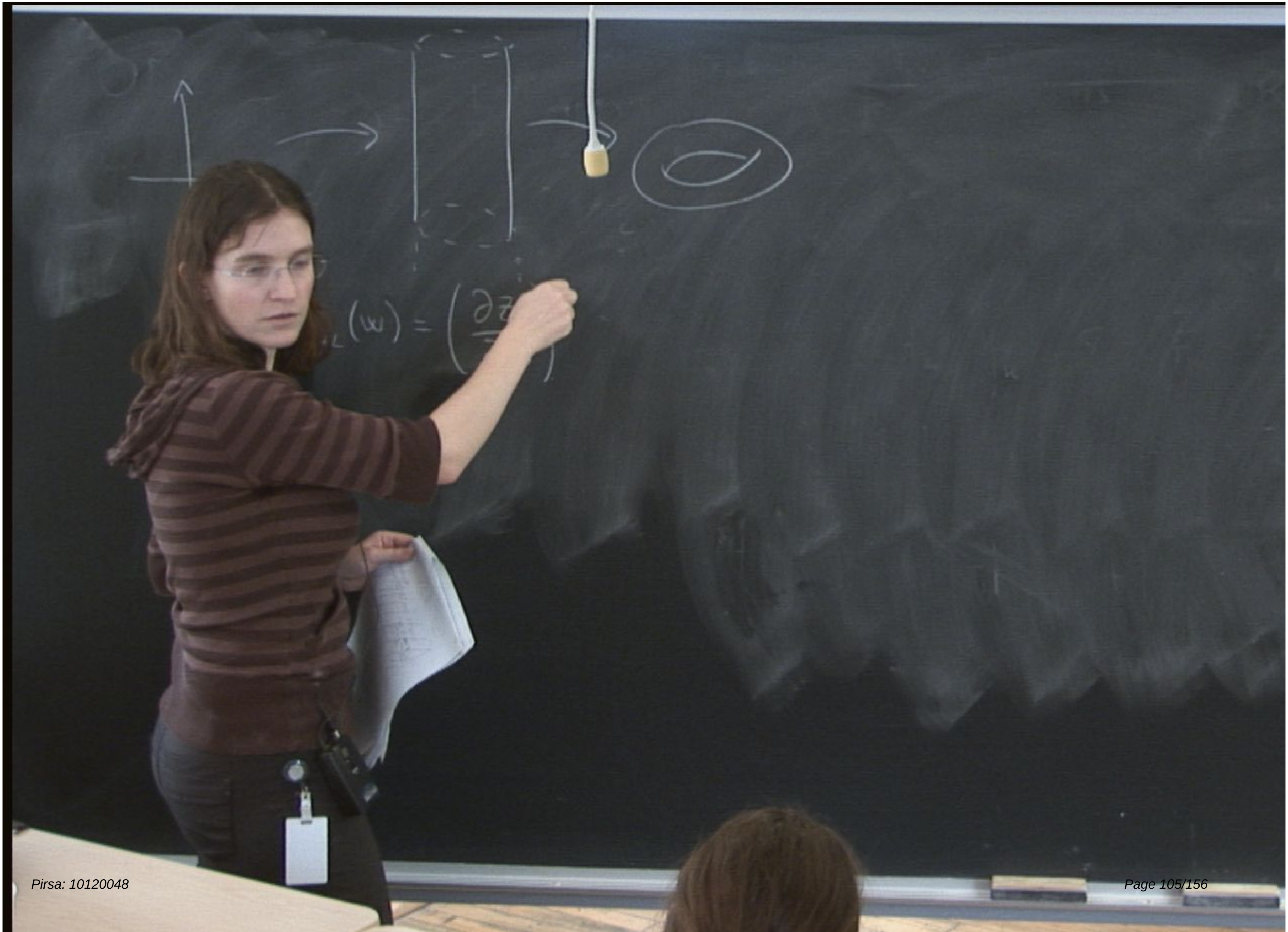
$$\mu(0)|0\rangle \equiv |\tfrac{1}{16}\rangle_-$$

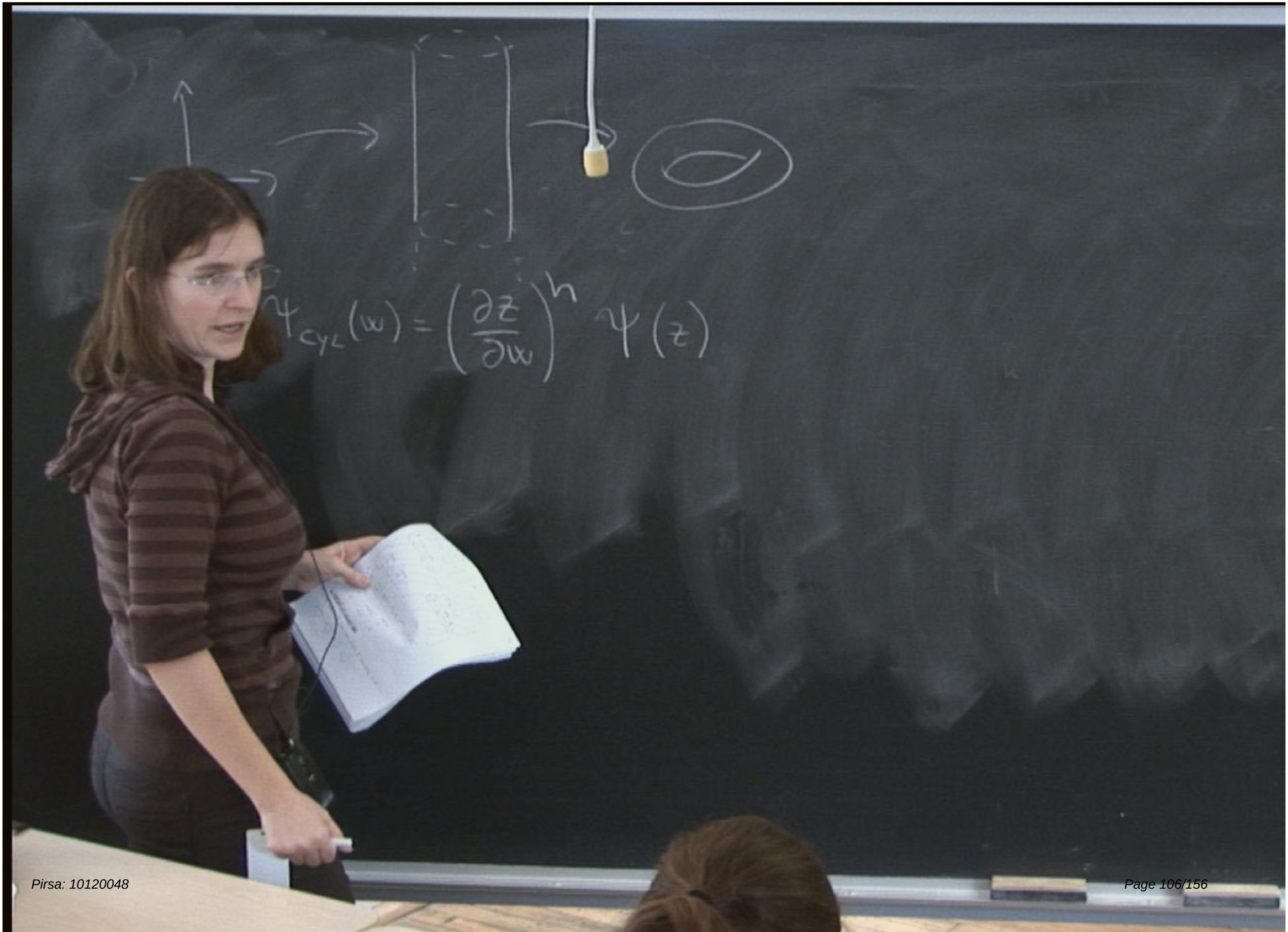
$$\{(-1)^F, \psi_k\} = 0$$

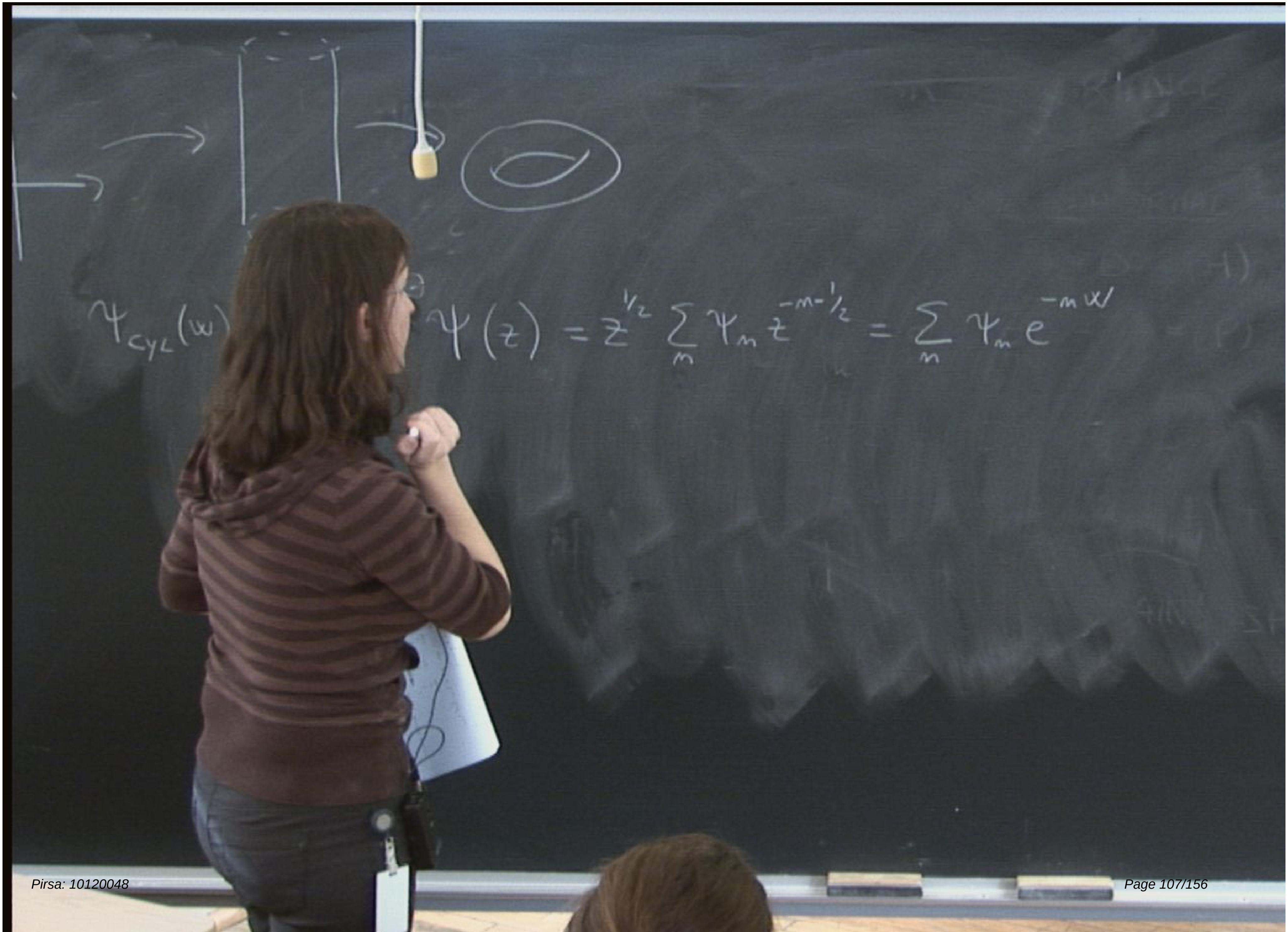
$$|\tfrac{1}{16}\rangle_\pm, \quad \psi_{-1}|\tfrac{1}{16}\rangle_\pm, \quad \psi_{-2}|\tfrac{1}{16}\rangle, \ldots$$

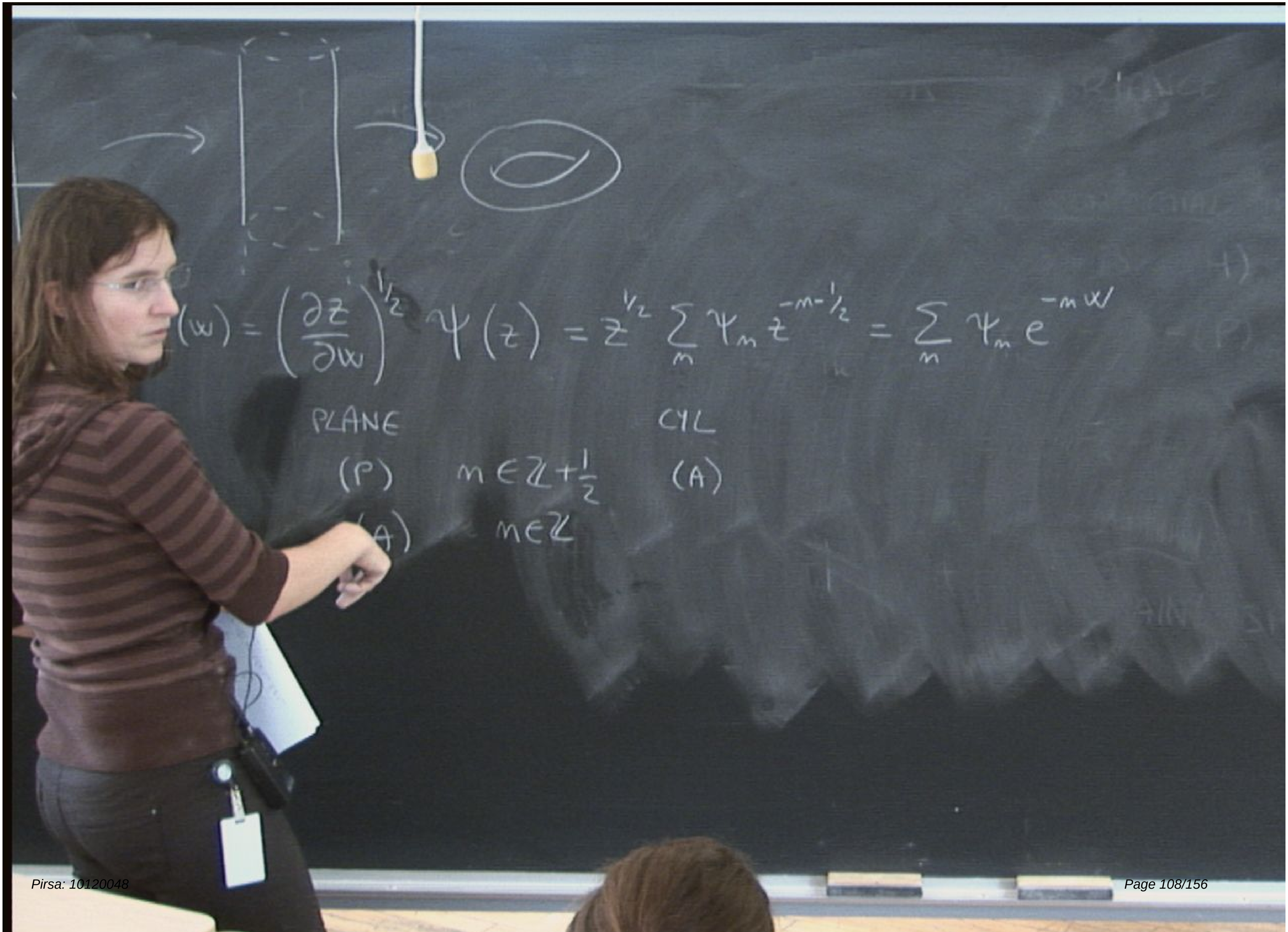
$$h = \tfrac{1}{16} \qquad 1 + \tfrac{1}{16} \qquad 2 + \tfrac{1}{16}$$

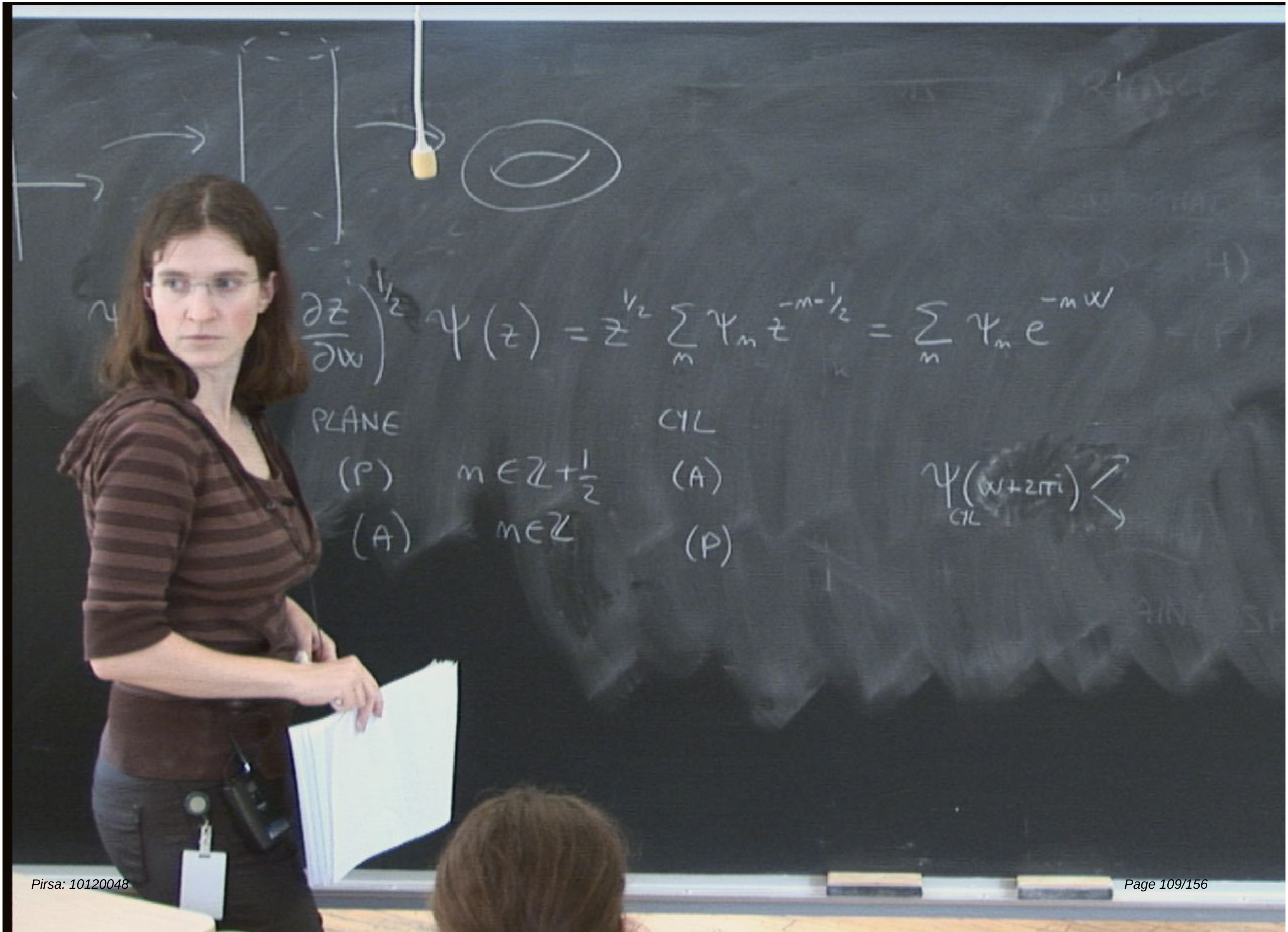


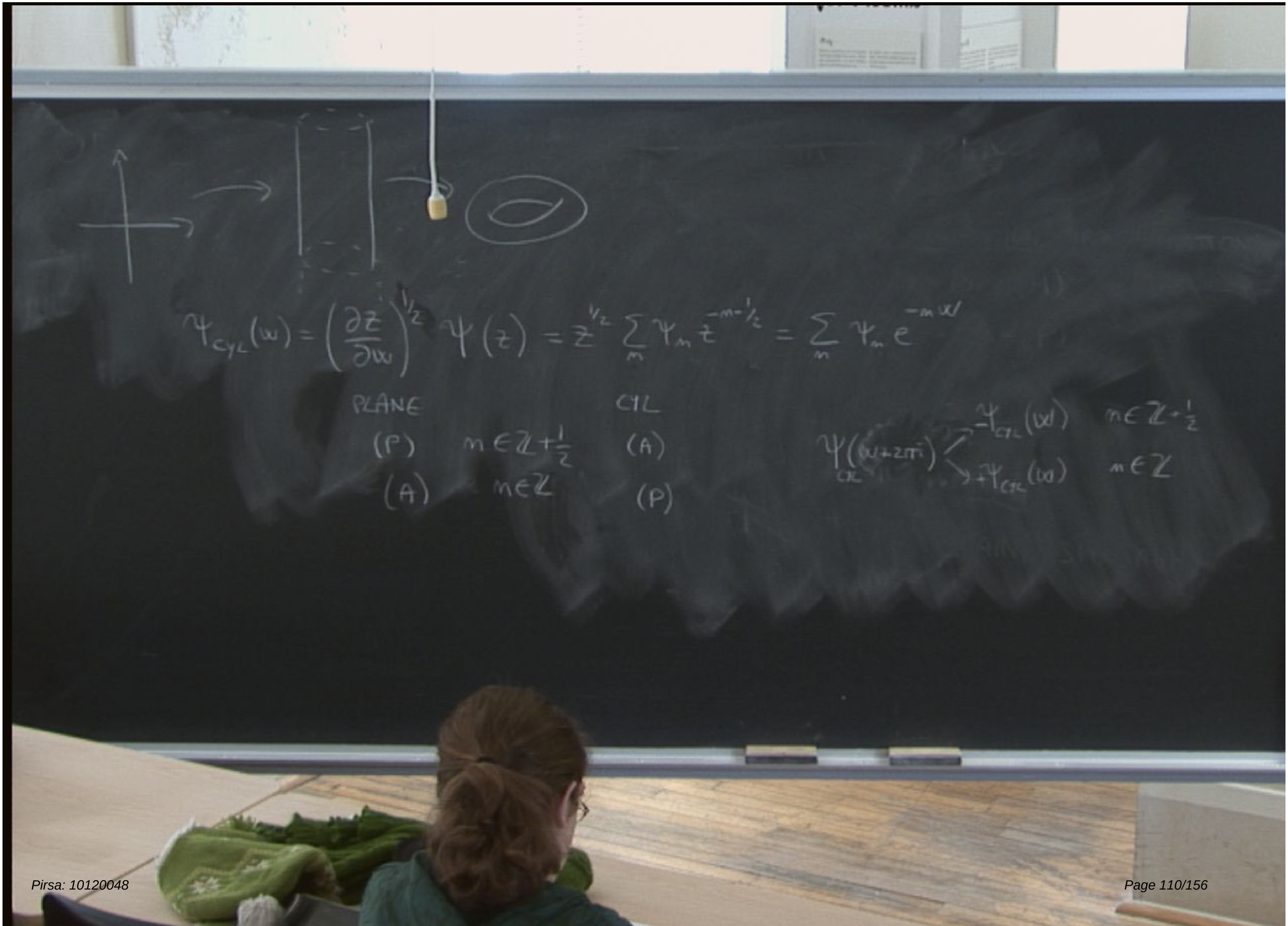


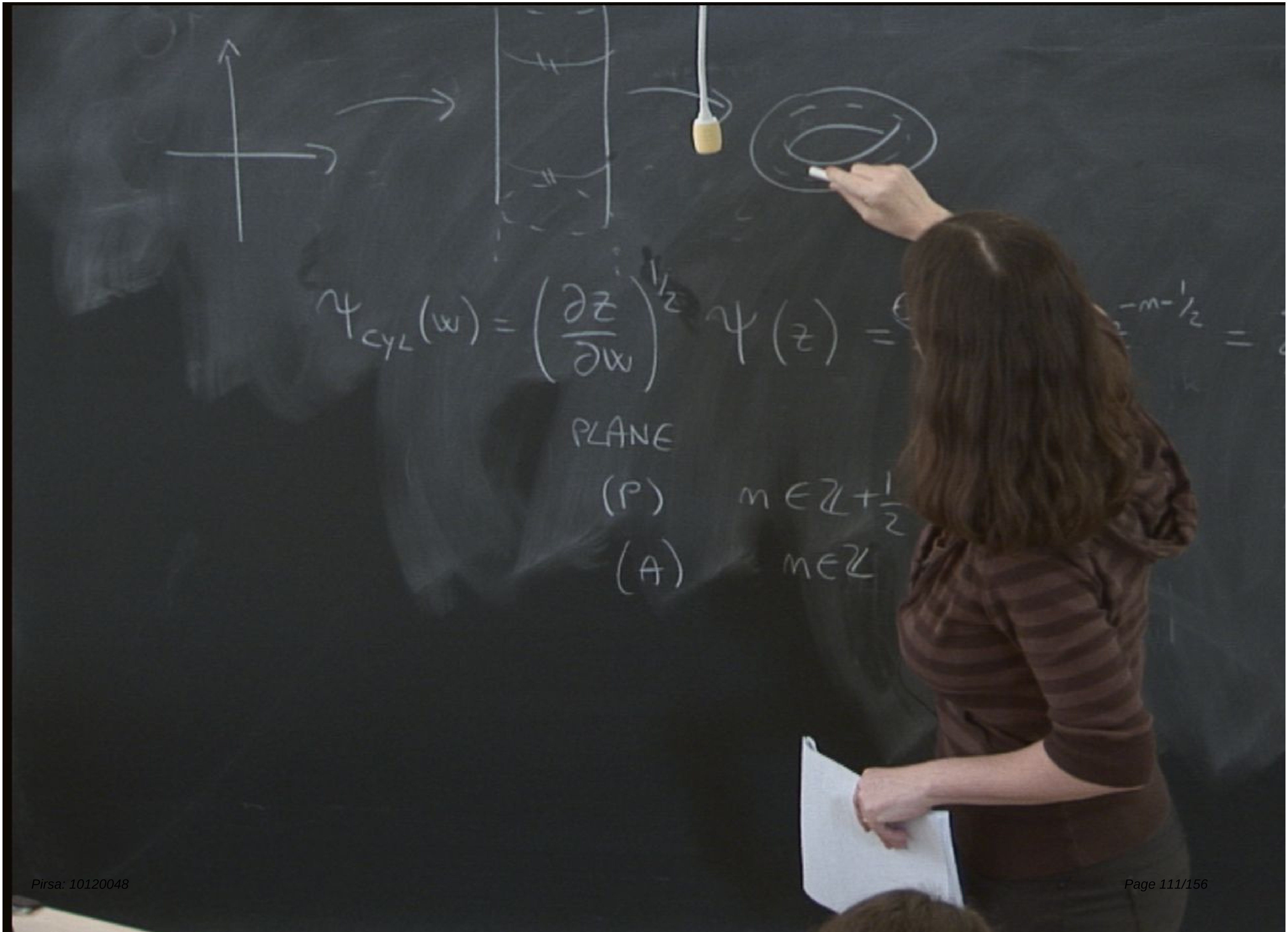

$\psi_{cyl}(w) = \left(\frac{\partial z}{\partial w}\right)^h \psi(z)$

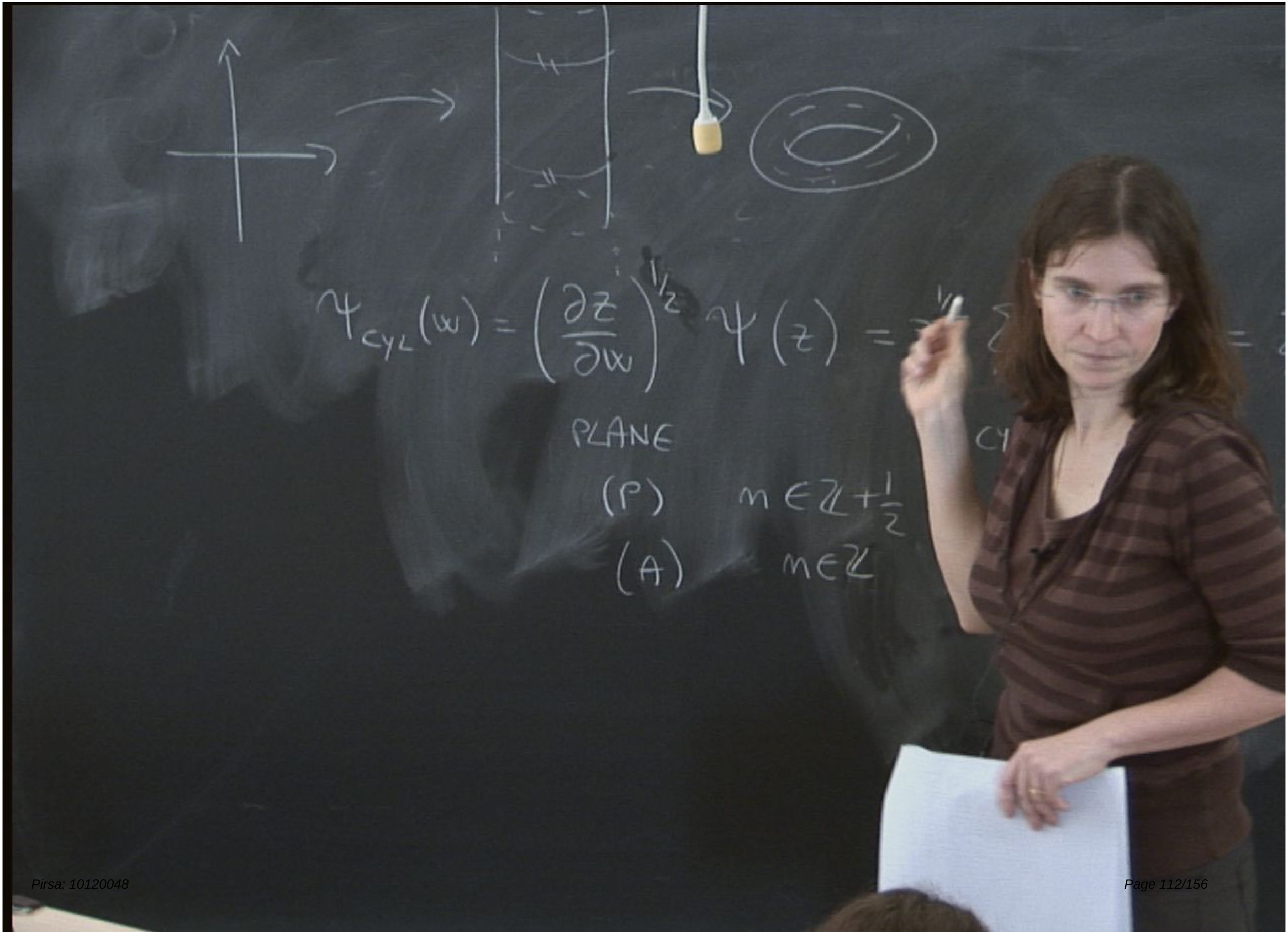

$\psi_{cyl}(w)$
$\psi(z) = z^{1/2} \sum_m \psi_m z^{-m-1/2} = \sum_m \psi_m e^{-mw}$

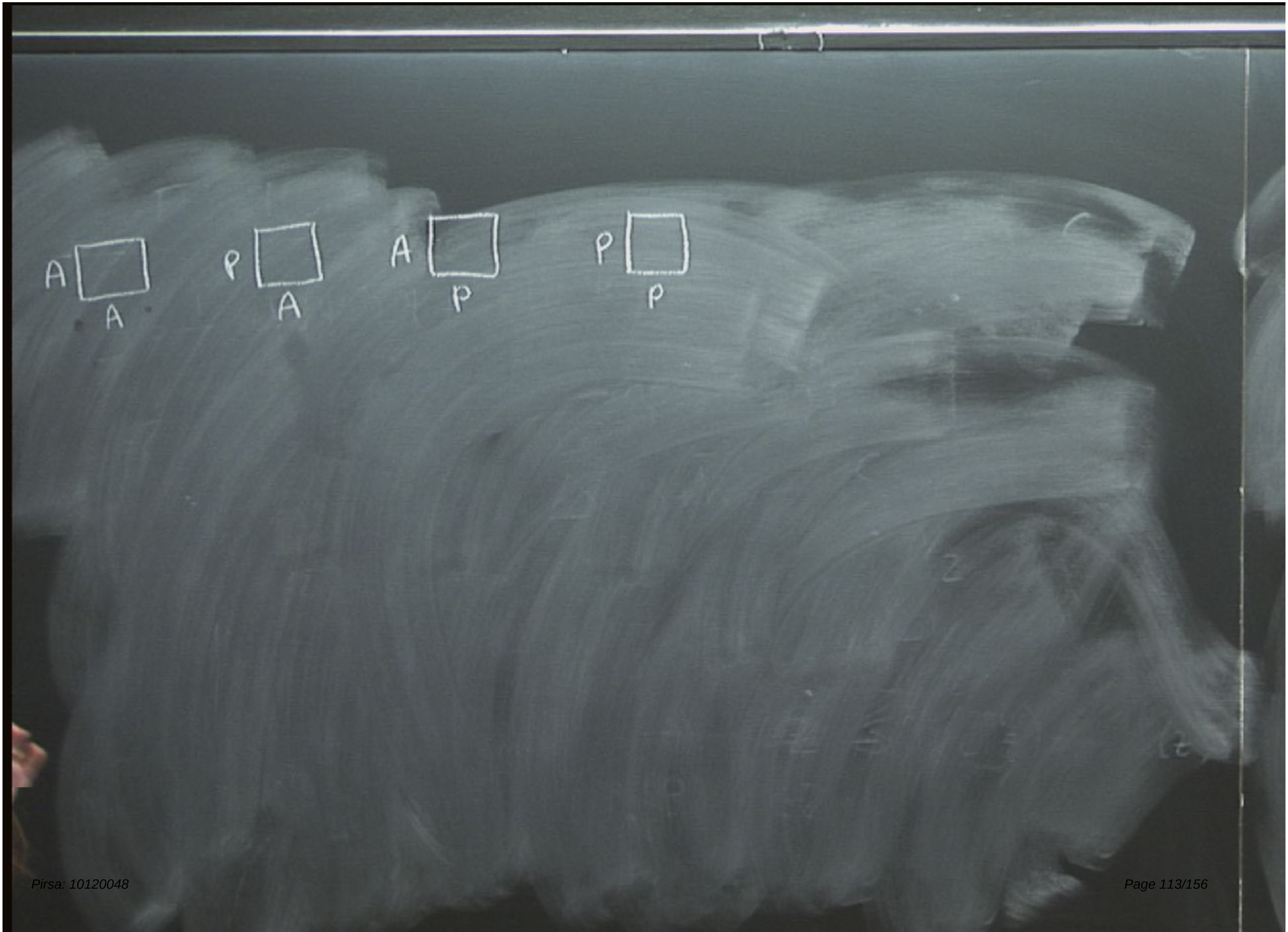

(w) = (∂z/∂w)^1/2 ψ(z) = z^1/2 Σ_m ψ_m z^-m-1/2 = Σ_m ψ_m e^-mw
PLANE
CYL
(P) m ∈ ℤ + 1/2 (A)
(A) m ∈ ℤ

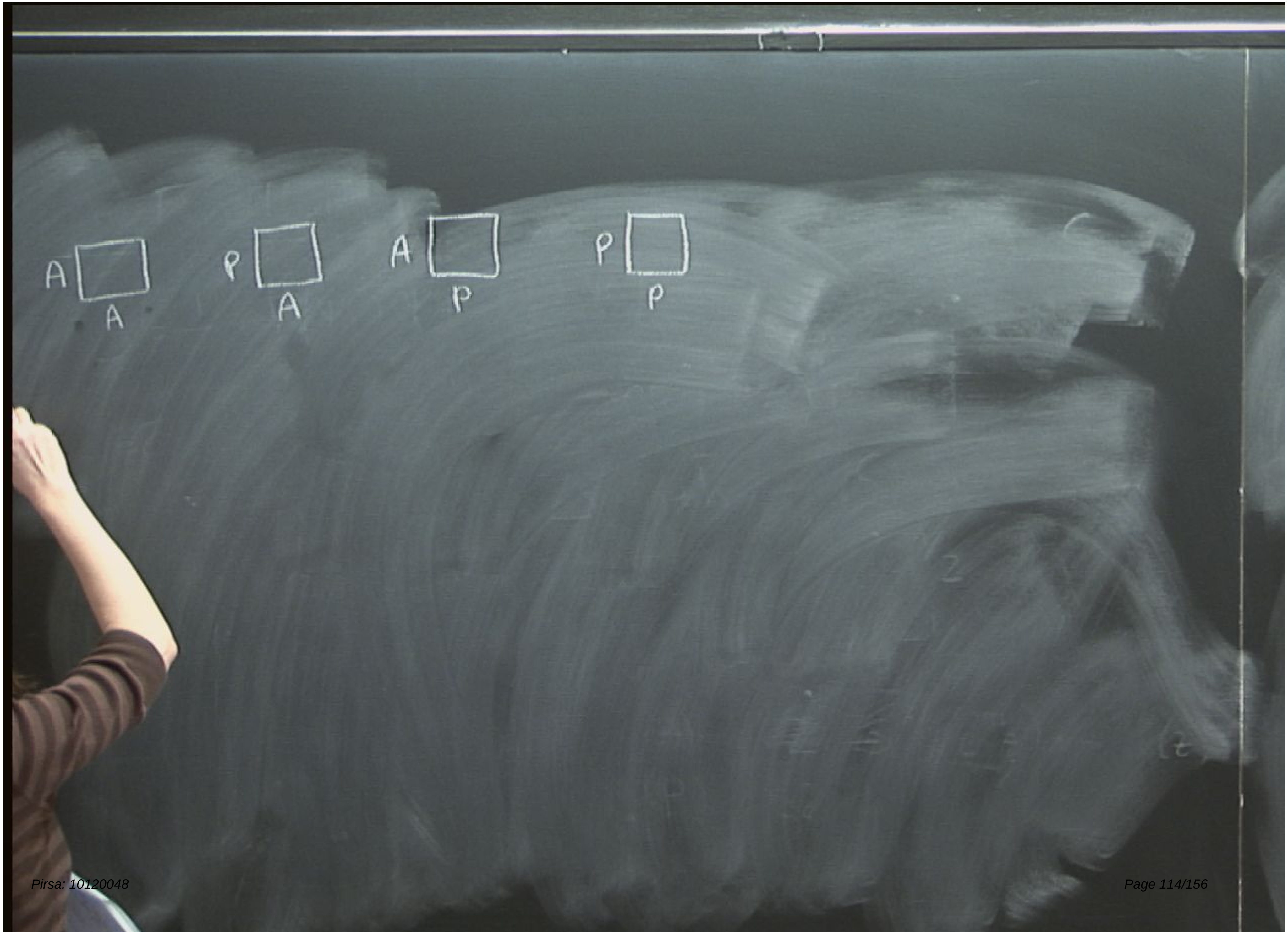

= z^1/2 ∑_m ψ_m z^-m-1/2 = ∑_m ψ_m e^-mw
PLANE
CYL
(P)
m ∈ ℤ + 1/2
(A)
(A)
m ∈ ℤ
(P)
ψ_CYL(w+2πi)

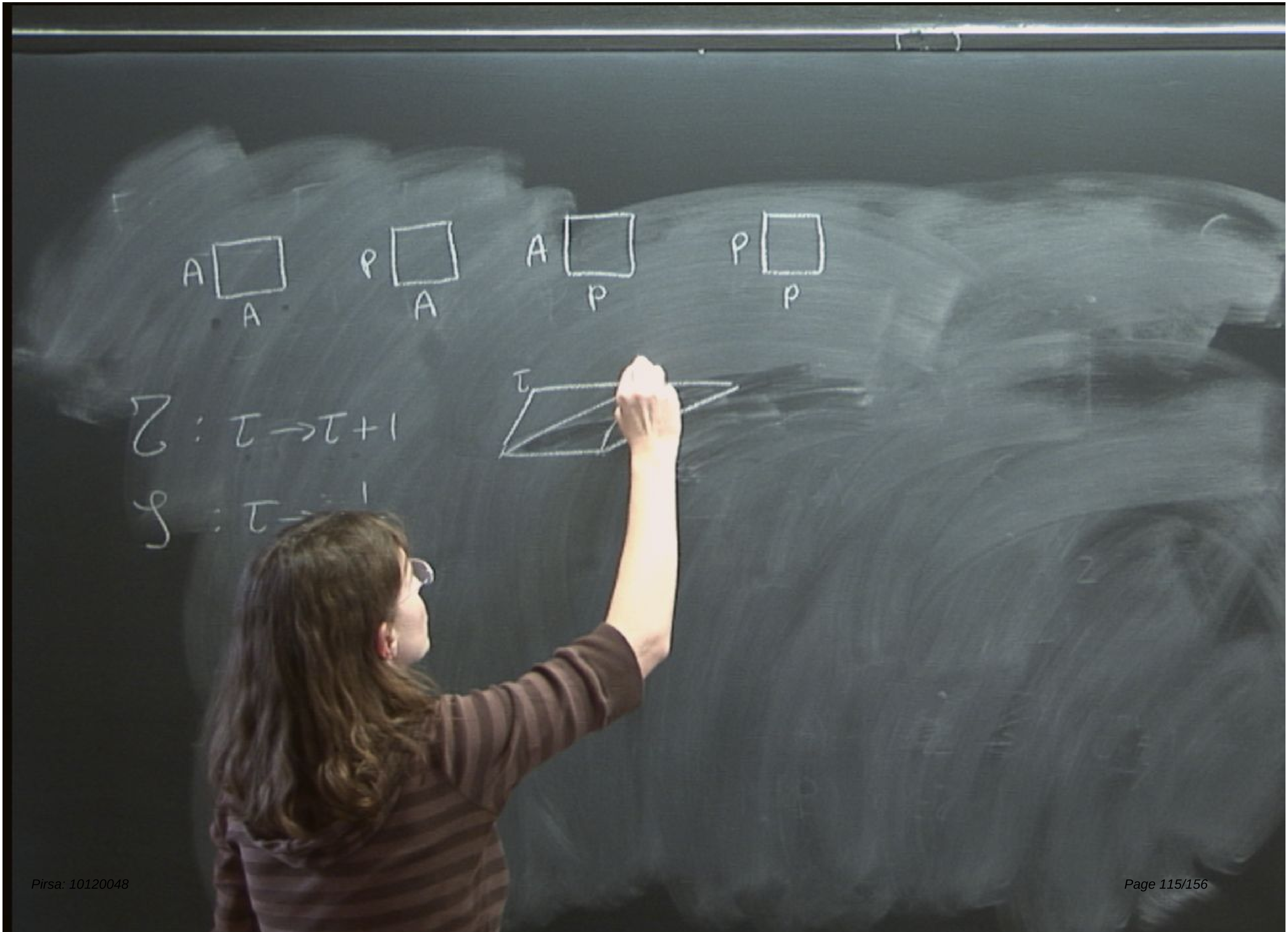

$\psi_{CYL}(w) = \left(\frac{\partial z}{\partial w}\right)^{1/2} \psi(z) = z^{1/2} \sum_m \psi_m z^{-m-1/2} = \sum_m \psi_m e^{-mw}$
PLANE
CYL
(P) $m \in \mathbb{Z}+\frac{1}{2}$ (A)
(A) $m \in \mathbb{Z}$ (P)
$\psi_{CYL}(w+2\pi i)$
$-\psi_{CYL}(w)$ $m \in \mathbb{Z}+\frac{1}{2}$
$+\psi_{CYL}(w)$ $m \in \mathbb{Z}$

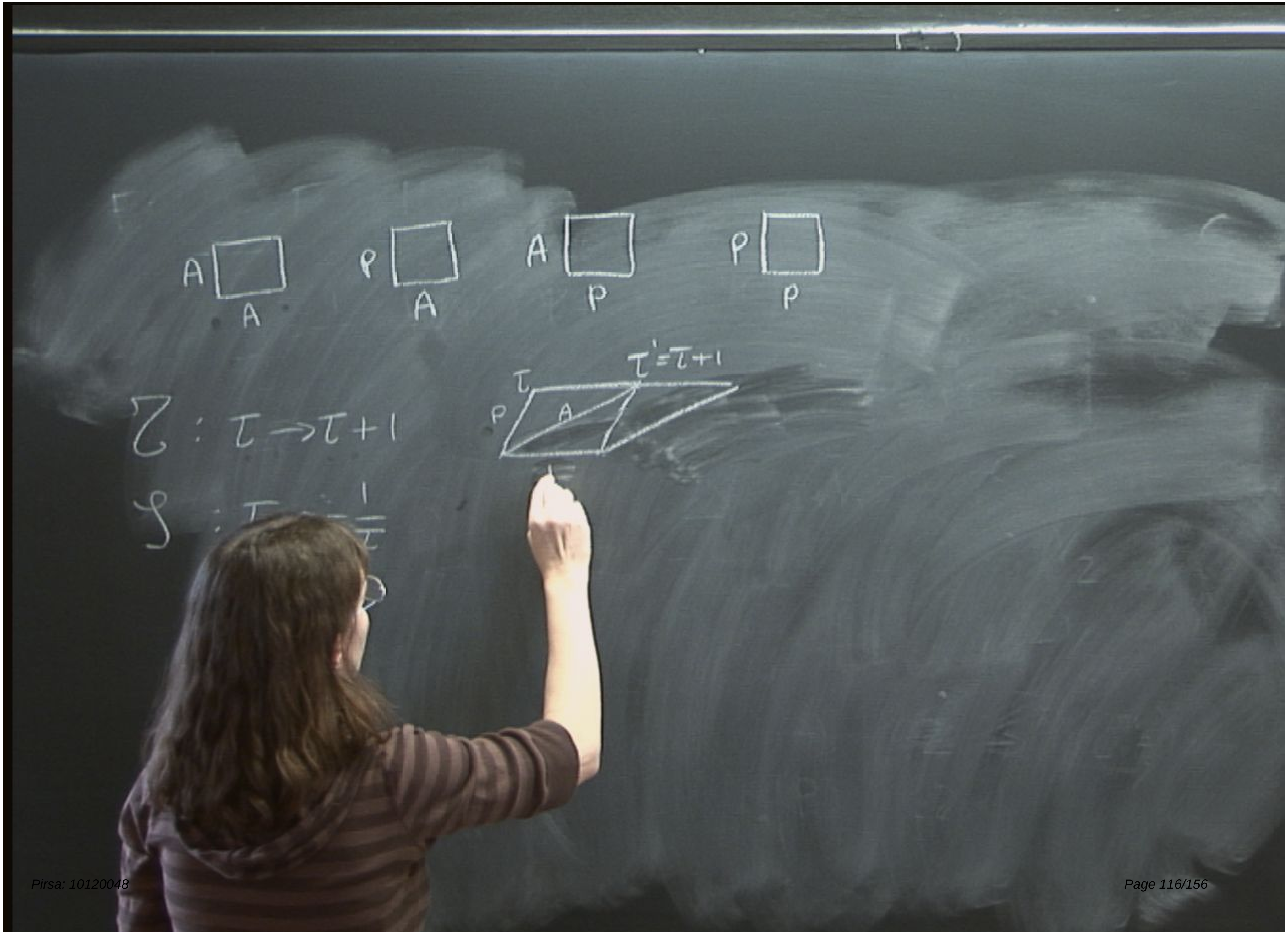

ψcyl(w) = (∂z/∂w)1/2 ψ(z) =
PLANE
(P) m ∈ ℤ + 1/2
(A) m ∈ ℤ

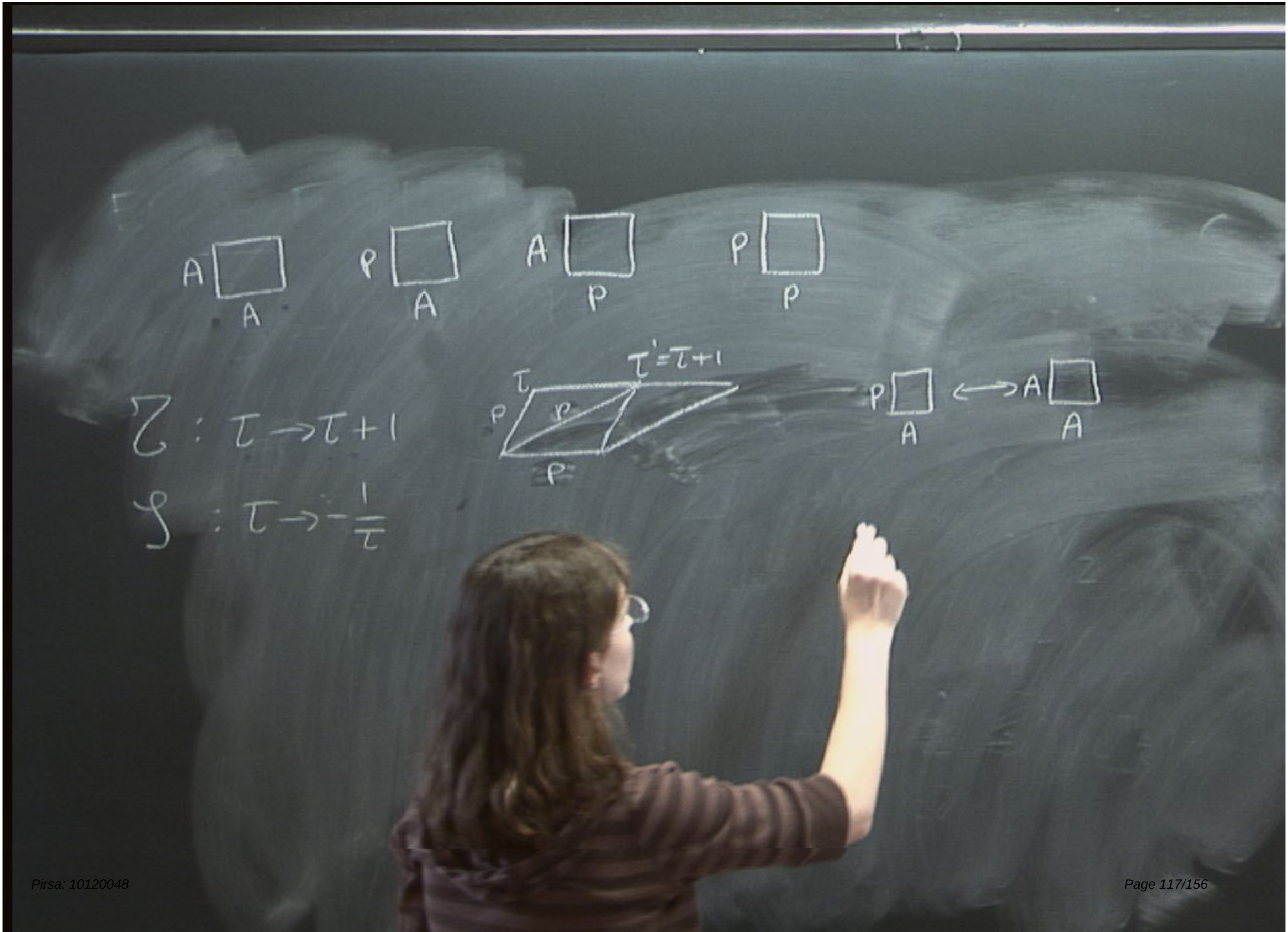

ψ_cyl(w) = (∂z/∂w)^1/2 ψ(z) =
PLANE
(P) m ∈ ℤ + 1/2
(A) m ∈ ℤ

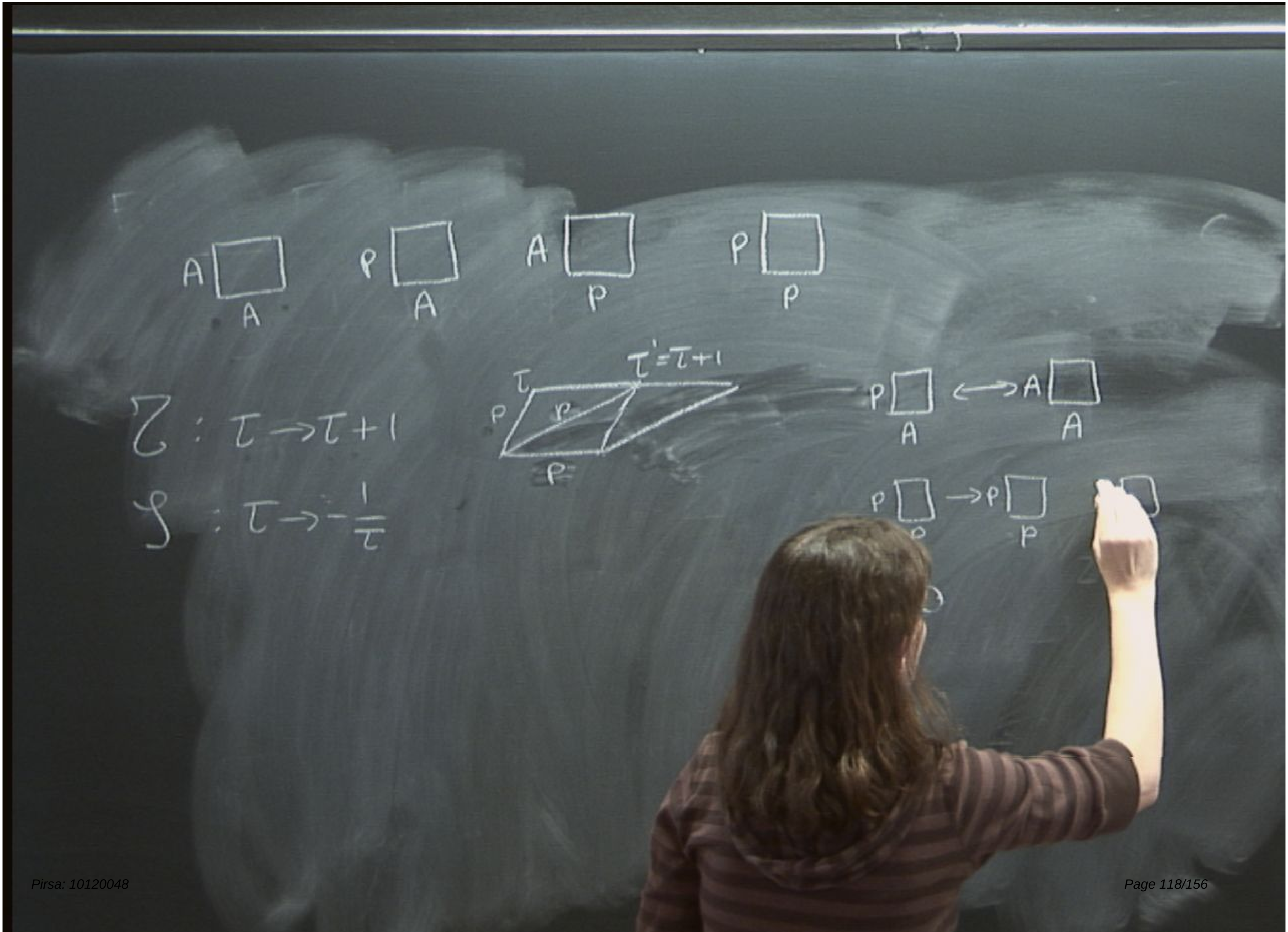

A A
P A
A P
P P

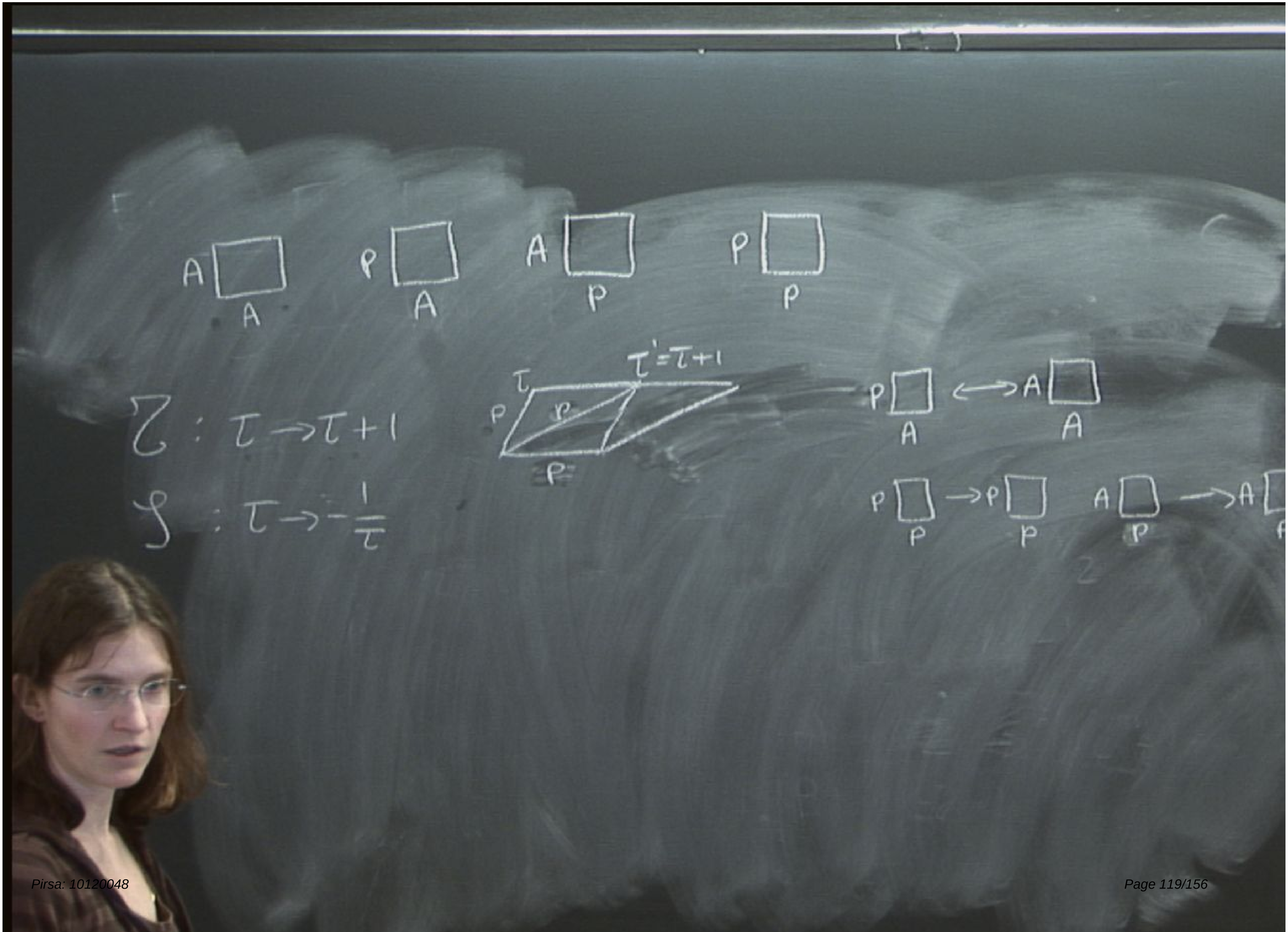

A A
P A
A P
P P


A A
P A
A P
P P
τ
τ : τ → τ+1
σ : τ → ...

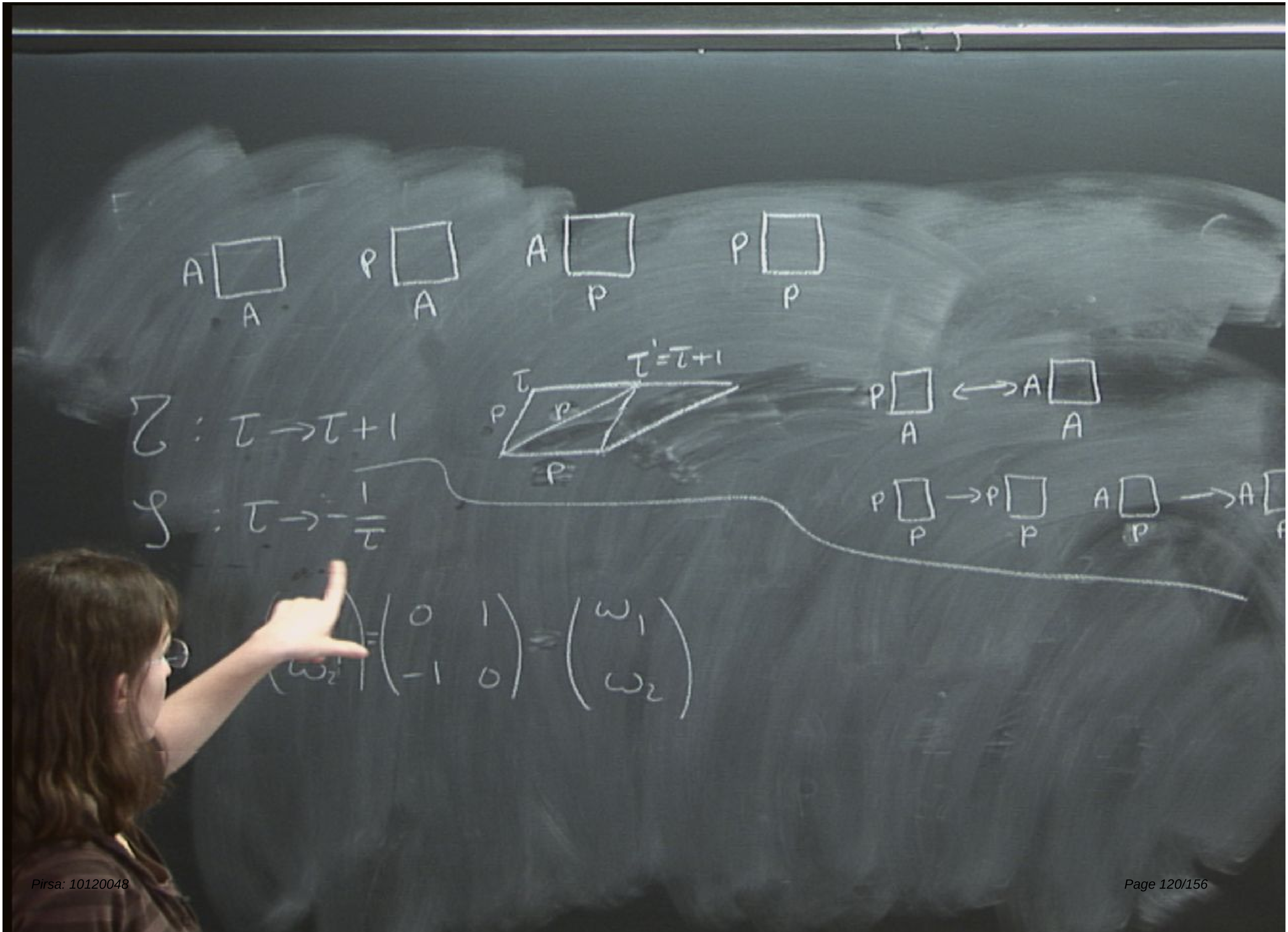

A
A
P
A
A
P
P
P
τ : τ → τ+1
τ' = τ+1
τ
P
A

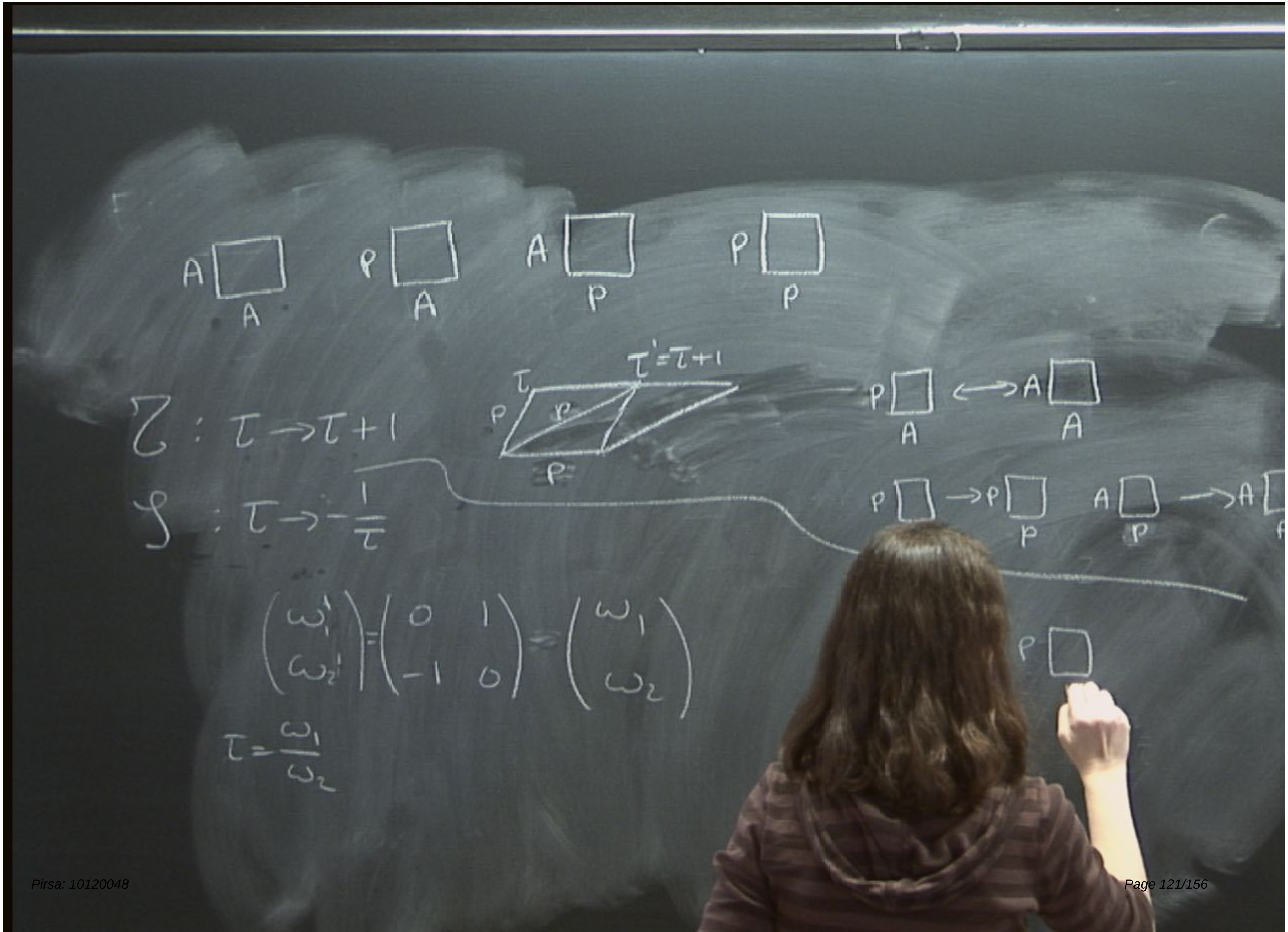

A A
P A
A P
P P
τ' = τ + 1
P A ⟷ A A
τ : τ → τ + 1
S : τ → −1/τ

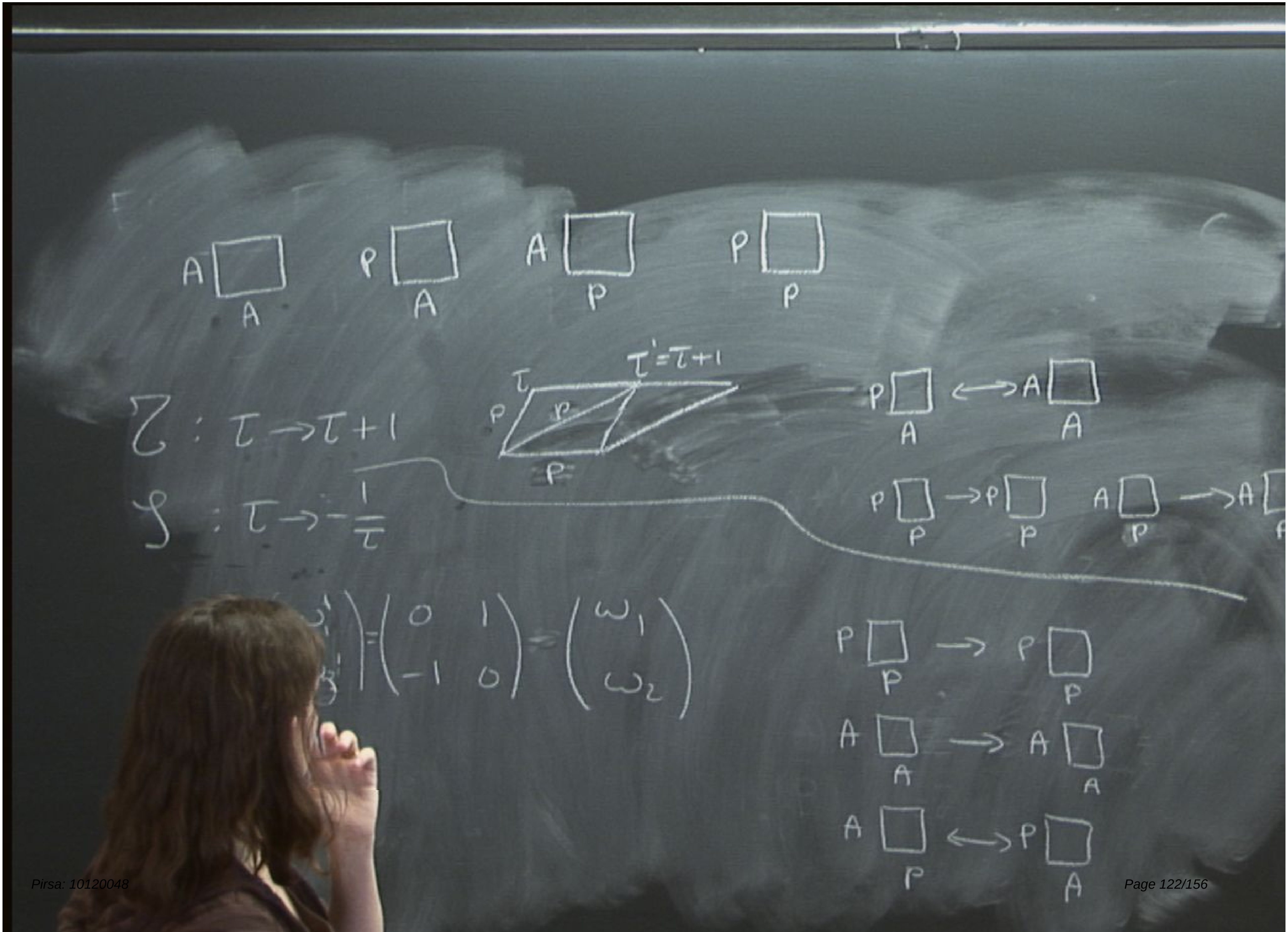

A
A
P
A
A
P
P
P
τ' = τ + 1
τ
T : τ → τ + 1
S : τ → −1/τ
P ↔ A
A
A
P → P
P

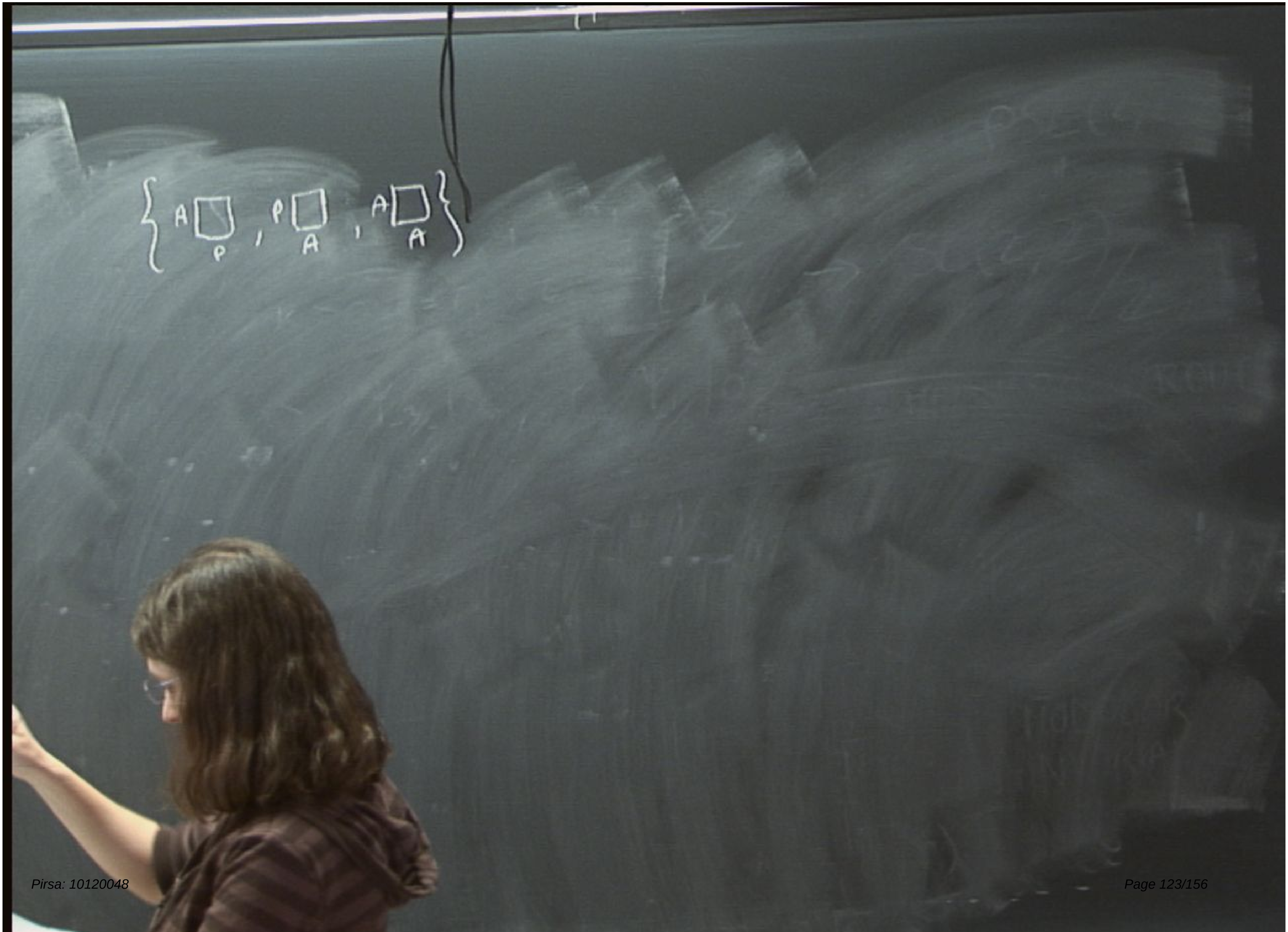


A A
P A
A P
P P
$T: \tau \to \tau + 1$
$S: \tau \to -\frac{1}{\tau}$
$\tau' = \tau + 1$
P
P
P
P A ⟷ A A
P P → P P
A P → A


A A
P A
A P
P P
τ : τ → τ + 1
S : τ → − 1/τ
τ' = τ + 1
P ↔ A
P → P
A → A
ω1
ω2

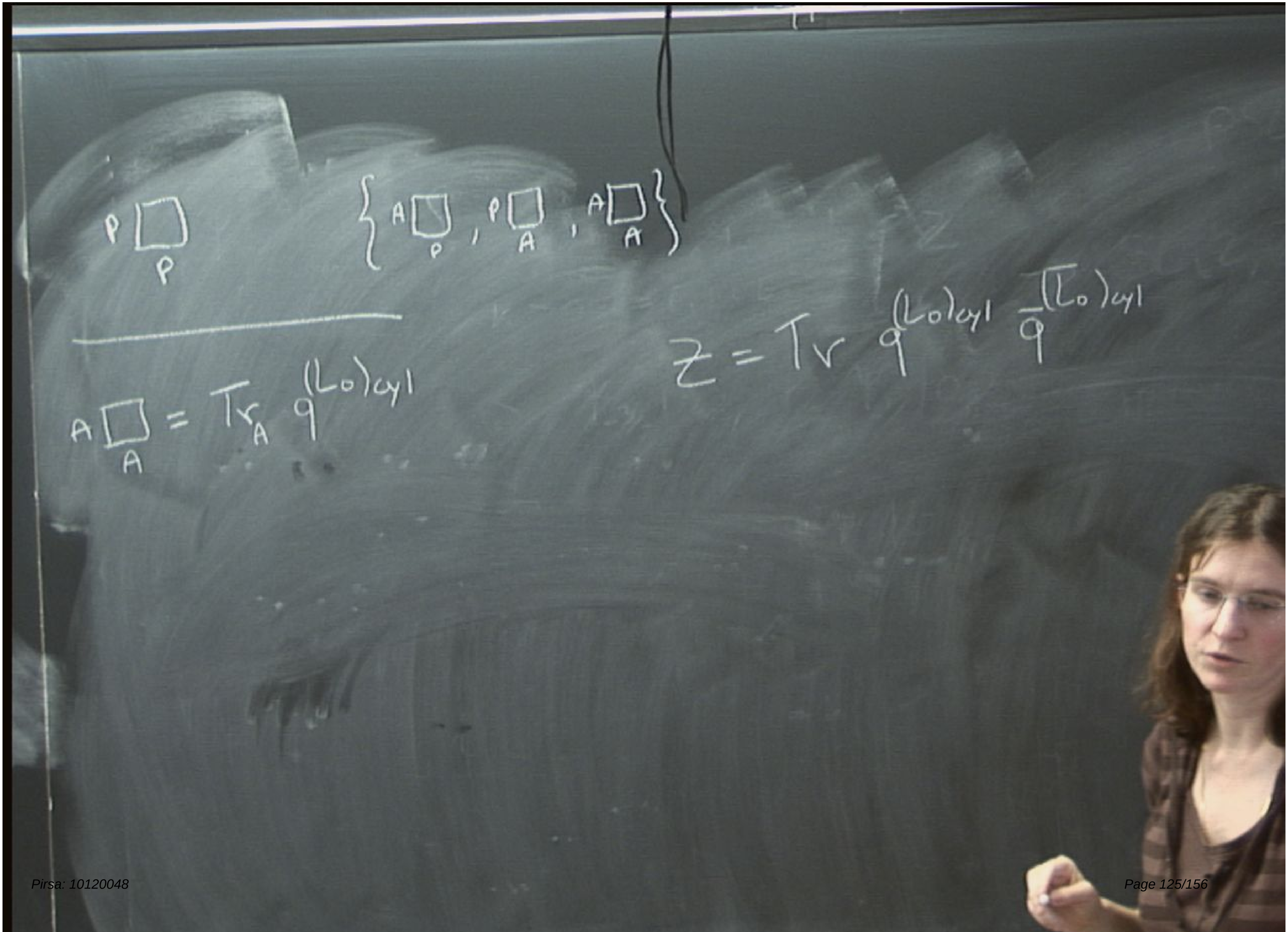

A A
P A
A P
P P
τ' = τ + 1
P A ⟷ A A
T : τ → τ + 1
S : τ → −1/τ
P → P P
A P → A
(ω'1 ω'2) = (0 1 −1 0) (ω1 ω2)
τ = ω1/ω2

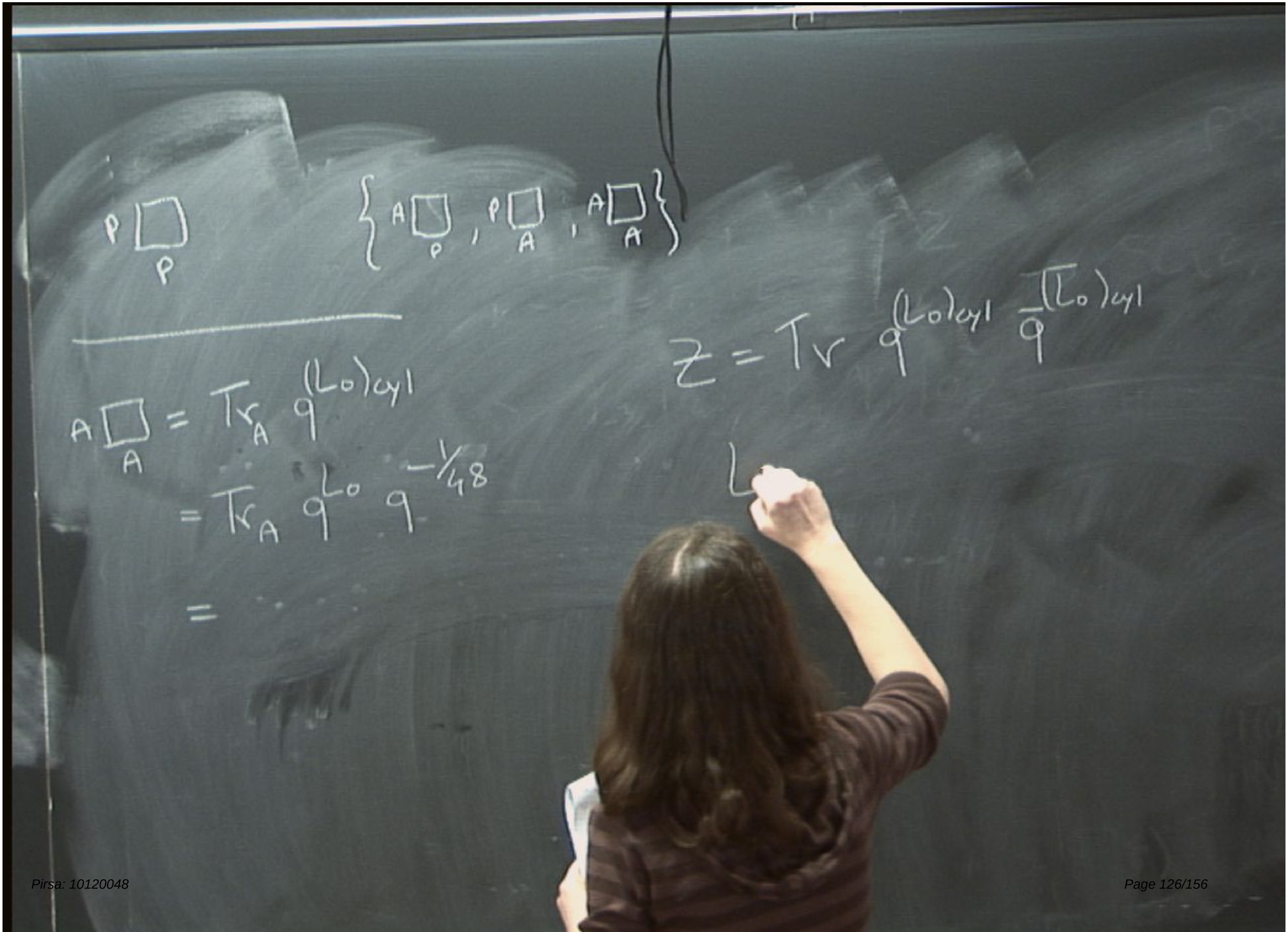

A A
P A
A P
P P
τ' = τ + 1
T : τ → τ + 1
S : τ → −1/τ
P A ⟷ A A
P P → P P
A P → A
P P → P P
A A → A A
A P ⟷ P A
(0 1; −1 0) (ω1; ω2)



$$ {}^{P}\square_{P} \qquad \left\{ {}^{A}\square_{P} \,,\ {}^{P}\square_{A} \,,\ {}^{A}\square_{A} \right\} $$

$$ {}^{A}\square_{A} = \mathrm{Tr}_{A}\, q^{(L_0)_{cyl}} $$


P □ P
{ A □ P , P □ A , A □ A }
Z = Tr q^(L₀)cyl q̄^(L̄₀)cyl
A □ A = Tr_A q^(L₀)cyl


$Z = \mathrm{Tr}\, q^{(L_0)_{cyl}}\, \bar{q}^{(L_0)_{cyl}}$
$A \square_A = \mathrm{Tr}_A\, q^{(L_0)_{cyl}}$
$= \mathrm{Tr}_A\, q^{L_0}\, q^{-1/48}$
=

$${}_P\Box_P \qquad \left\{ {}_A\Box_P \,,\ {}_P\Box_A \,,\ {}_A\Box_A \right\}$$

$$Z = \mathrm{Tr}\, q^{(L_0)_{cyl}}\, \bar{q}^{(\bar{L}_0)_{cyl}}$$

$${}_A\Box_A = \mathrm{Tr}_A\, q^{(L_0)_{cyl}}$$

$$= \mathrm{Tr}_A\, q^{L_0}\, q^{-1/48}$$

$$=$$

$$L_0 = \sum_{k>0} k F_k$$

$${}_P\square_P \qquad \left\{ {}_A\square_P ,\ {}_P\square_A ,\ {}_A\square_A \right\}$$

$$Z = \mathrm{Tr}\, q^{(L_0)_{cyl}}\, \bar{q}^{(\bar{L}_0)_{cyl}}$$

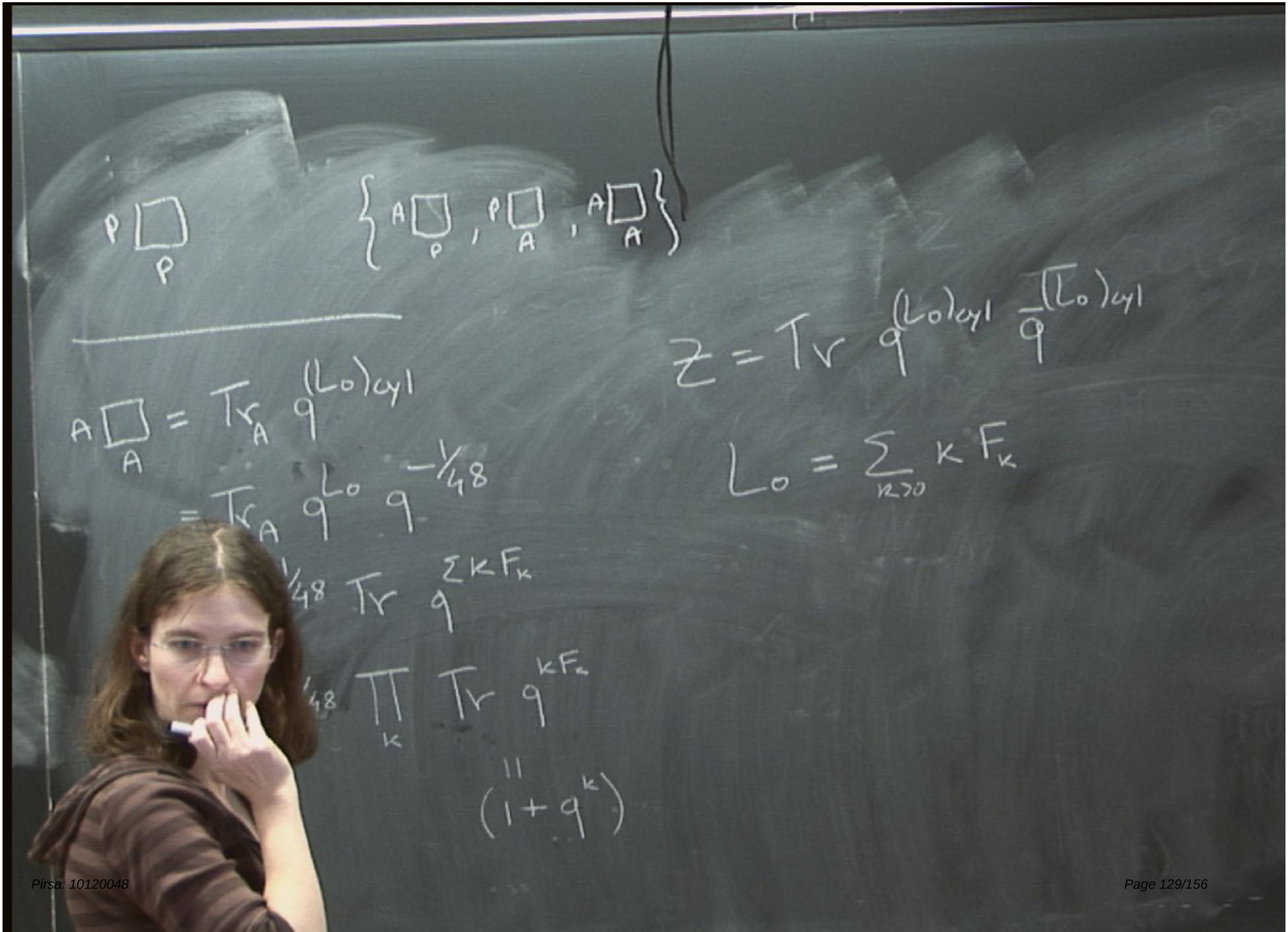
$${}_A\square_A = \mathrm{Tr}_A\, q^{(L_0)_{cyl}}$$

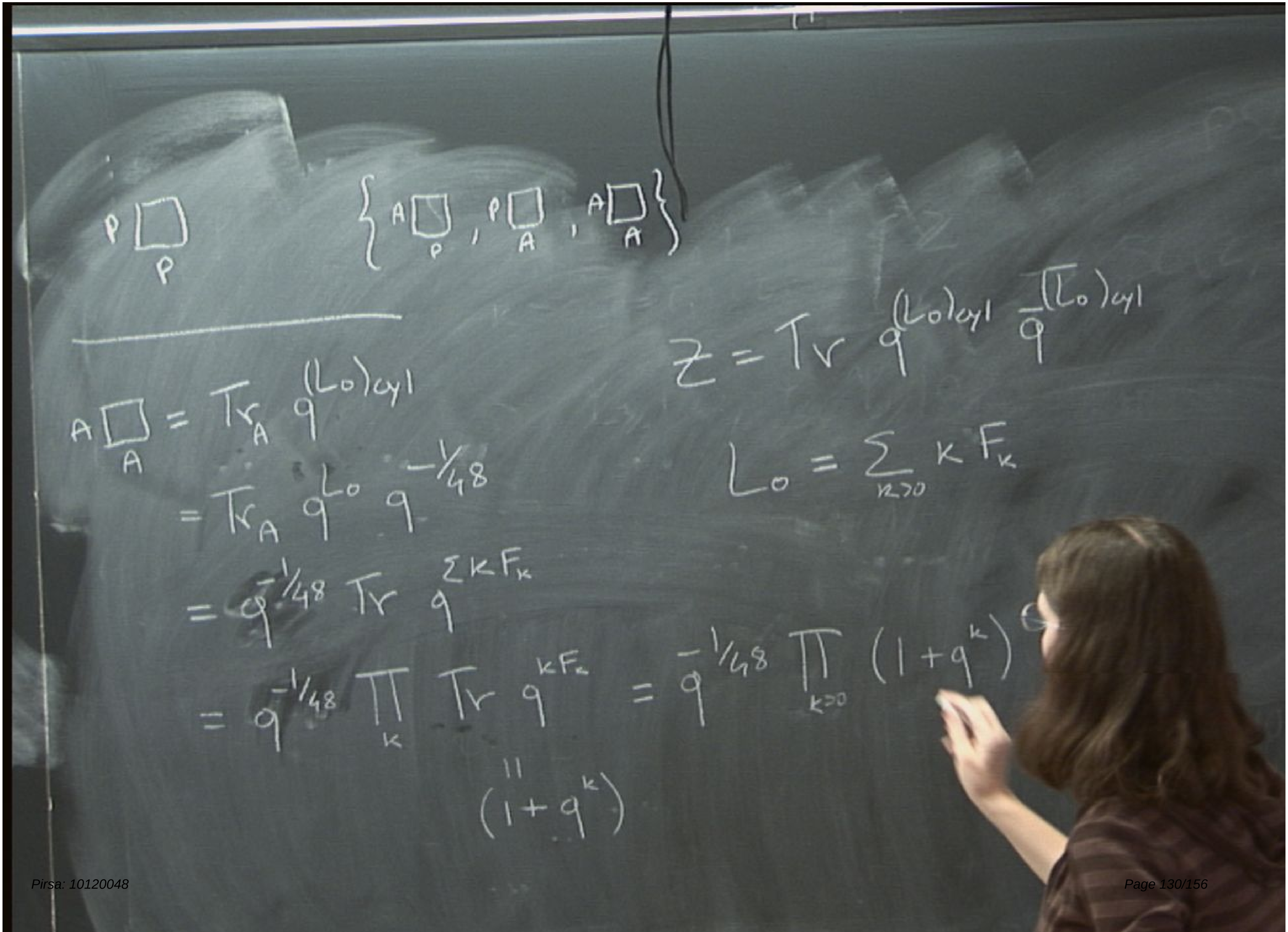
$$= \mathrm{Tr}_A\, q^{L_0}\, q^{-1/48}$$

$$= q^{-1/48}\, \mathrm{Tr}\, q^{\sum k F_k}$$

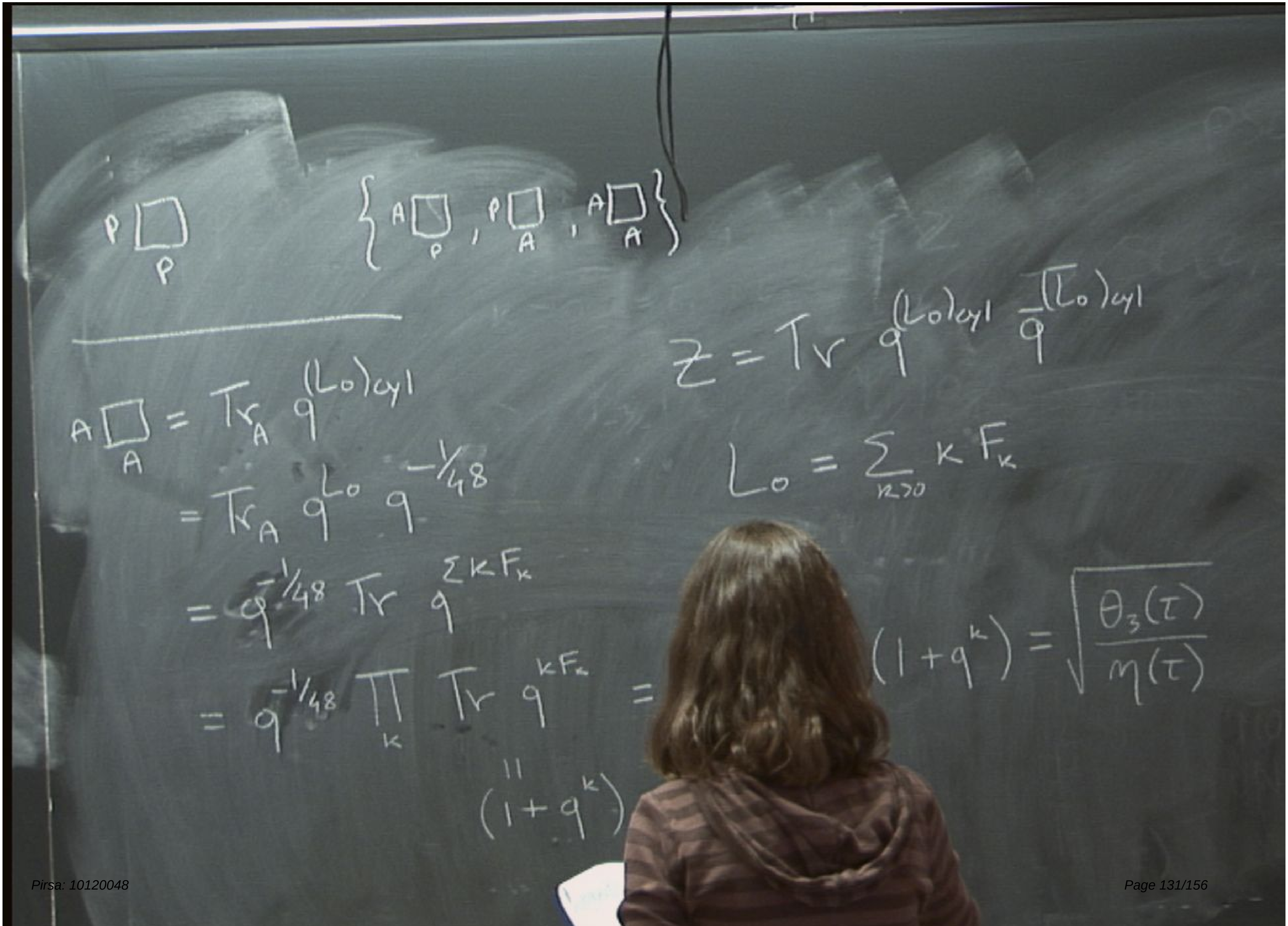
$$= q^{-1/48} \prod_k \mathrm{Tr}\, q^{k F_k}$$

$$L_0 = \sum_{k>0} k F_k$$


$Z = \mathrm{Tr}\, q^{(L_0)_{cyl}} \bar{q}^{(\bar{L}_0)_{cyl}}$
$L_0 = \sum_{k>0} k F_k$
$= \mathrm{Tr}_A\, q^{L_0}\, q^{-1/48}$
$\prod_k \mathrm{Tr}\, q^{k F_k}$
$(1+q^k)$



${}^{P}\square_{P}$
$\{ {}^{A}\square_{P}, {}^{P}\square_{A}, {}^{A}\square_{A} \}$
$Z = \mathrm{Tr}\, q^{(L_0)_{cyl}} \bar{q}^{(\bar{L}_0)_{cyl}}$
${}^{A}\square_{A} = \mathrm{Tr}_A\, q^{(L_0)_{cyl}}$
$= \mathrm{Tr}_A\, q^{L_0}\, q^{-1/48}$
$L_0 = \sum_{k>0} k F_k$
$= q^{-1/48}\, \mathrm{Tr}\, q^{\Sigma k F_k}$
$= q^{-1/48} \prod_k \mathrm{Tr}\, q^{k F_k} = q^{-1/48} \prod_{k>0} (1+q^k)$
$(1+q^k)$


P□P
{A□P, P□A, A□A}
Z = Tr q^(L0)cyl q̄^(L0)cyl
A□A = Tr_A q^(L0)cyl
= Tr_A q^L0 q^-1/48
= q^-1/48 Tr q^ΣkF_k
= q^-1/48 ∏_k Tr q^kF_k =
L0 = Σ_k>0 k F_k
(1+q^k) = √(θ3(τ)/η(τ))
(1+q^k)

$$ {}^{P}\Box_{P} \qquad \left\{ {}^{A}\Box_{P} ,\ {}^{P}\Box_{A} ,\ {}^{A}\Box_{A} \right\} $$

$$ {}^{A}\Box_{A} = \mathrm{Tr}_A\, q^{(L_0)_{cyl}} $$

$$ = \mathrm{Tr}_A\, q^{L_0}\, q^{-1/48} $$

$$ = q^{-1/48}\, \mathrm{Tr}\, q^{\Sigma k F_k} $$

$$ = q^{-1/48} \prod_k \mathrm{Tr}\, q^{k F_k} = q^{-1/48} \prod_{k>0} (1+q^k) = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} $$

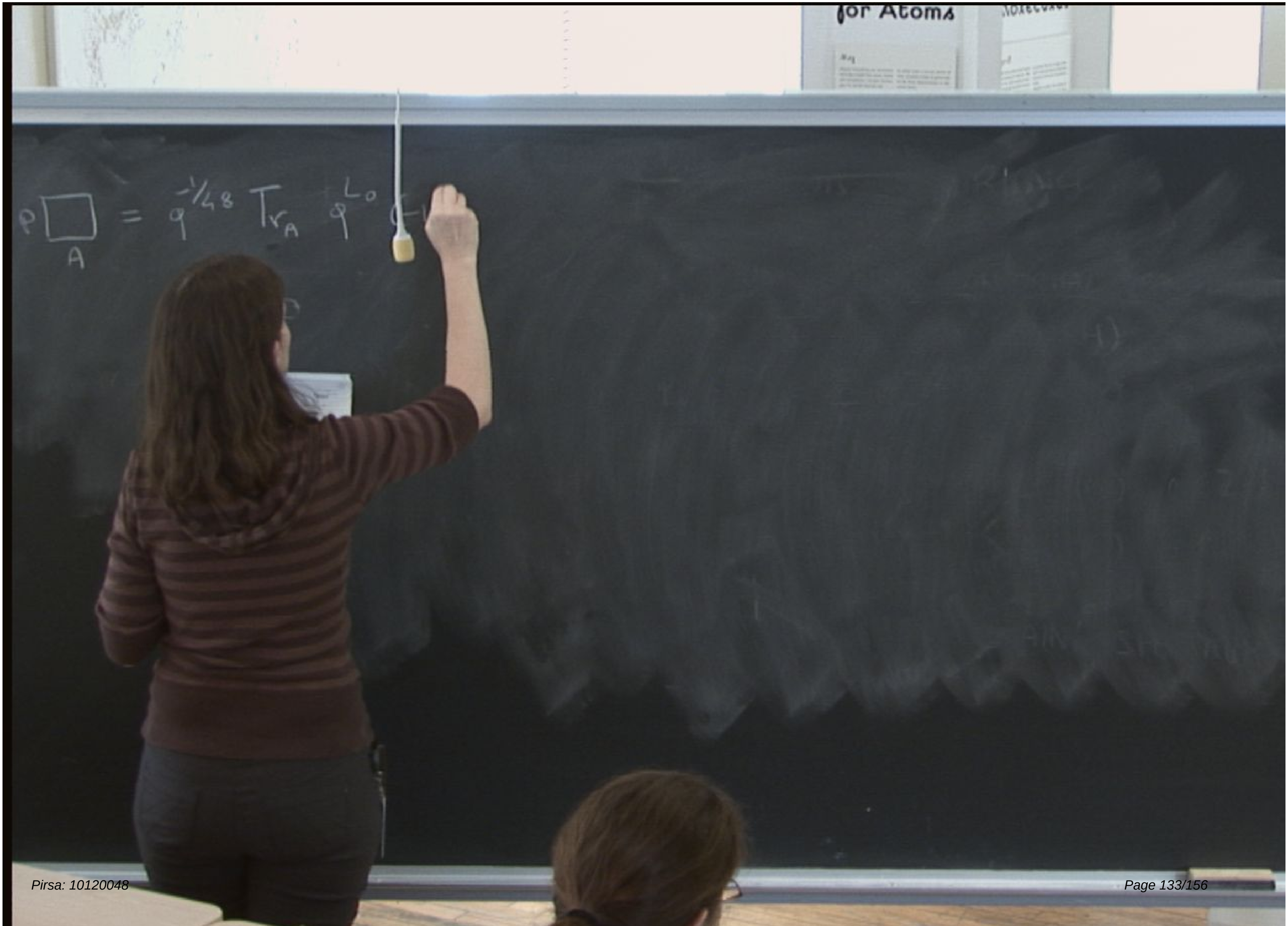
$$ \mathrm{Tr}\, q^{k F_k} = (1+q^k) $$

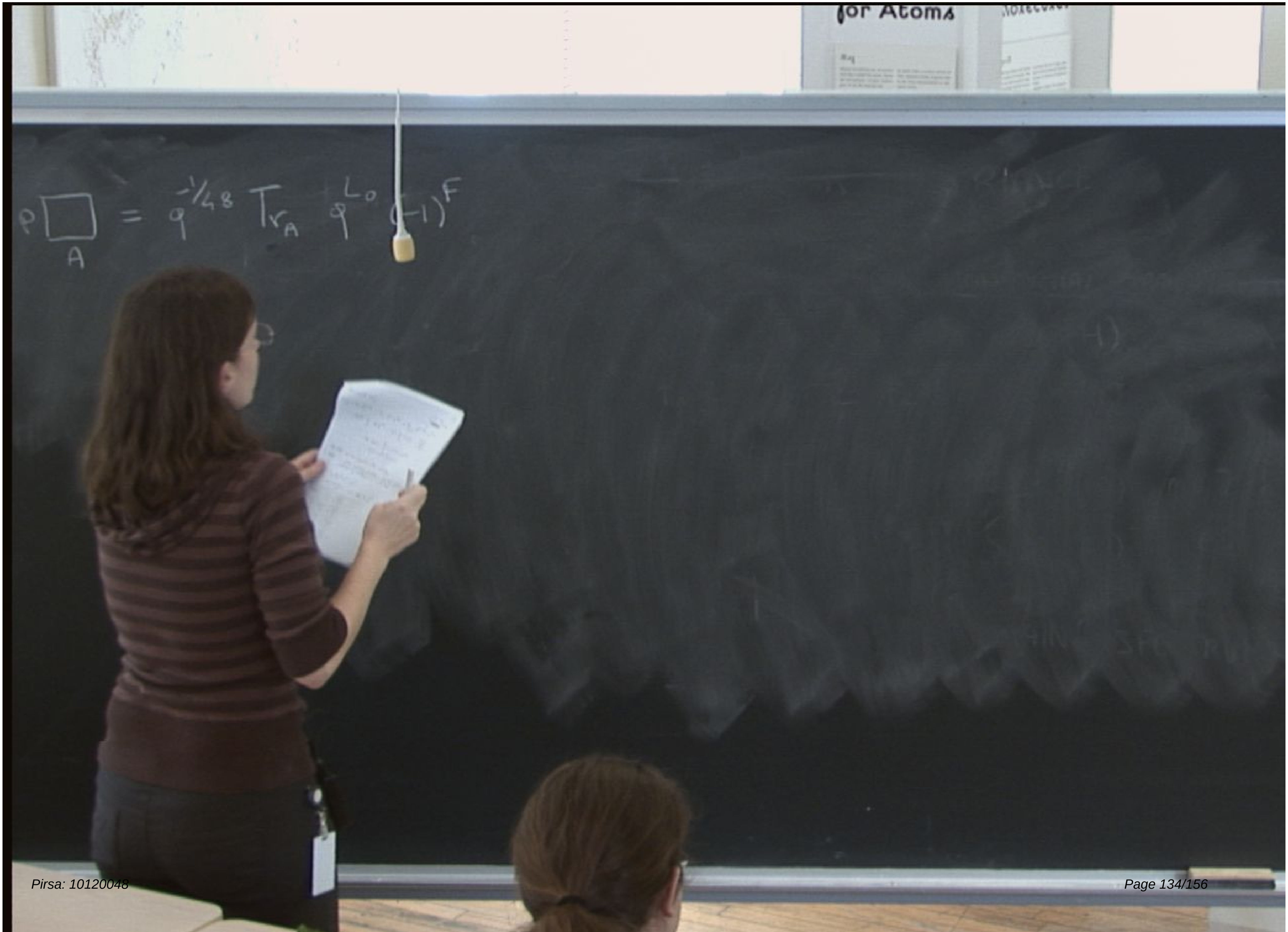
$$ Z = \mathrm{Tr}\, q^{(L_0)_{cyl}}\, \bar{q}^{(\bar{L}_0)_{cyl}} $$

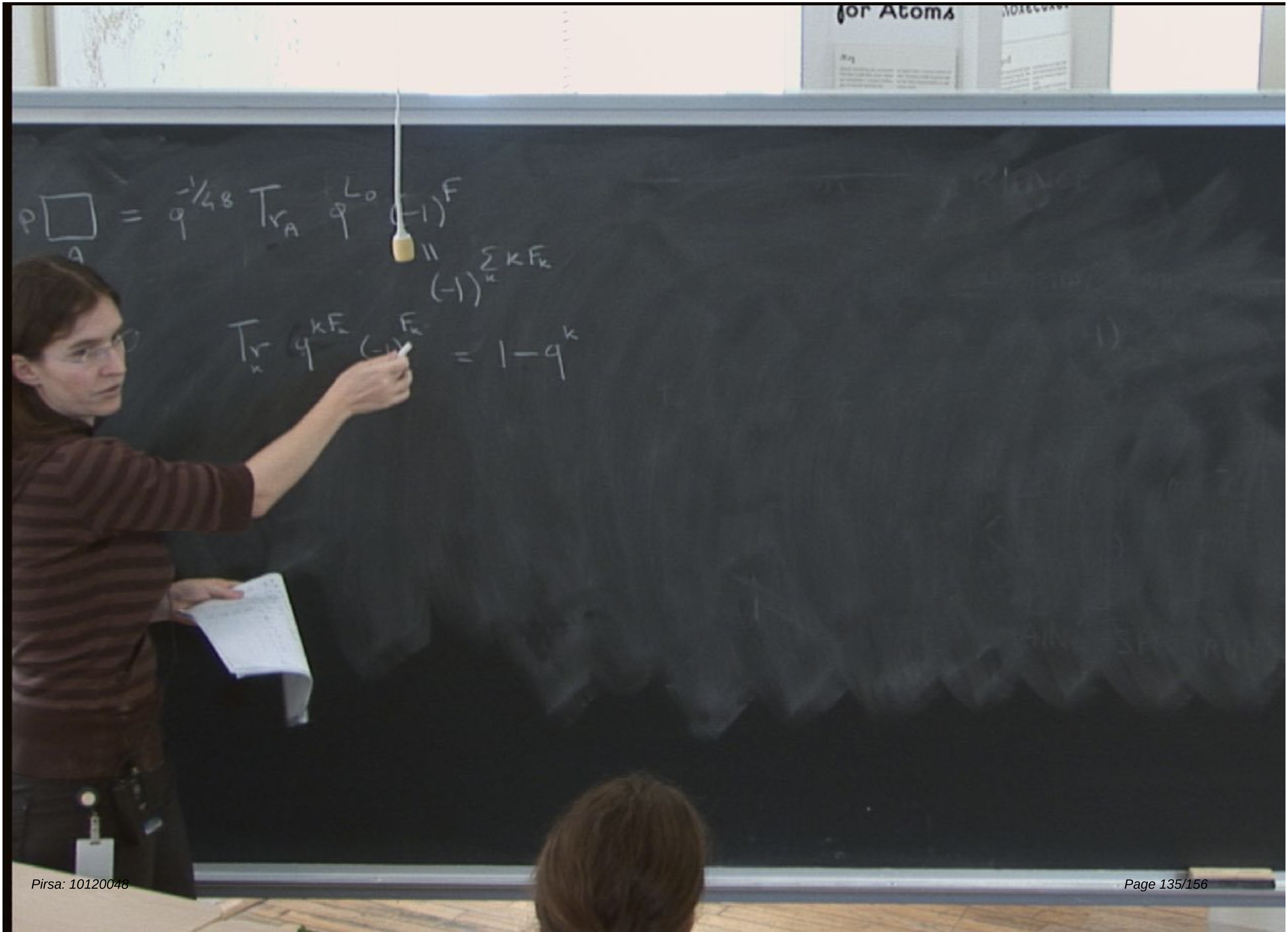
$$ L_0 = \sum_{k>0} k F_k $$

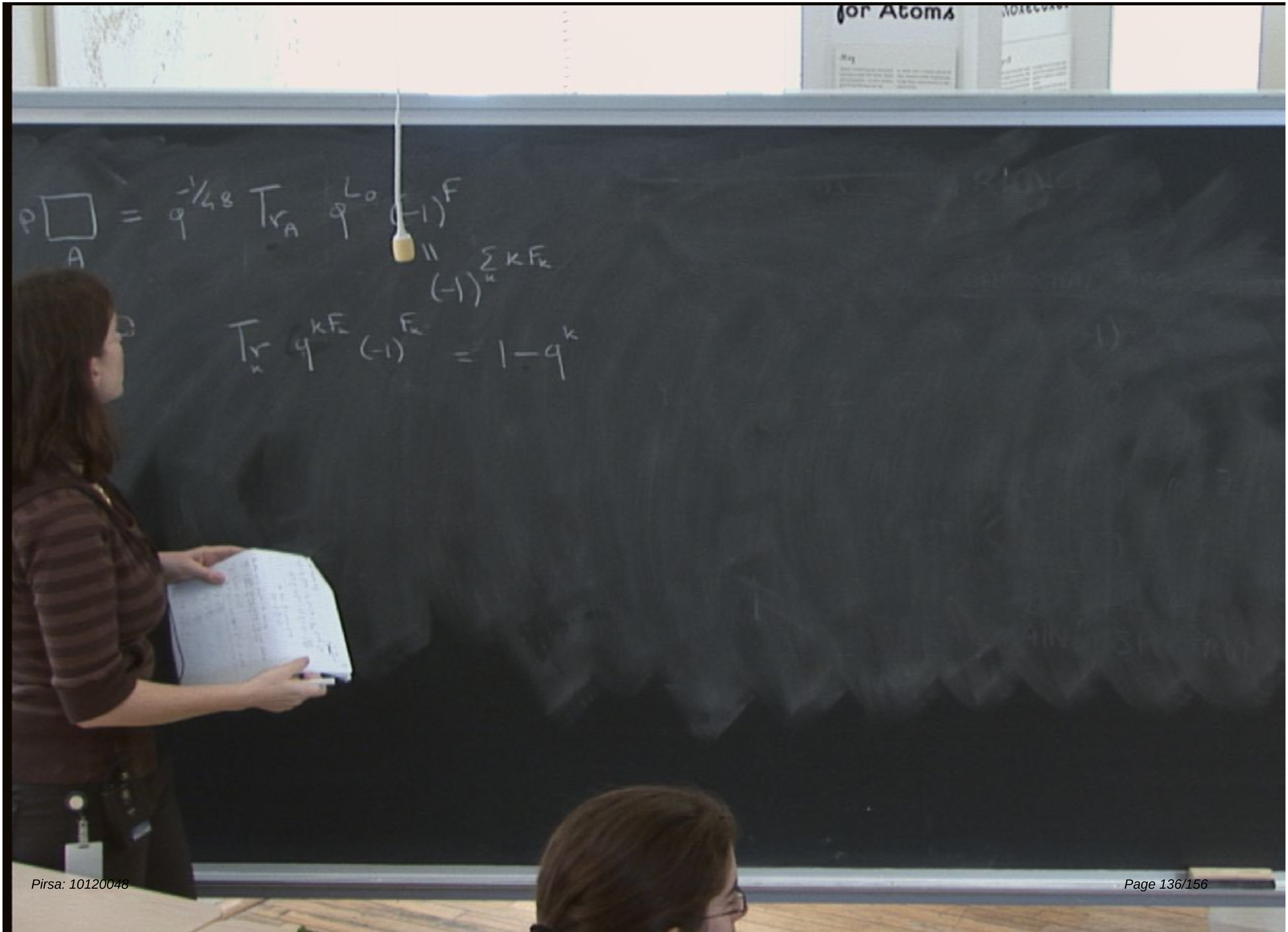
$$ \theta_3(\tau) = \prod_{m=1}^{\infty} (1-q^m)(1+q^{m-1/2} $$

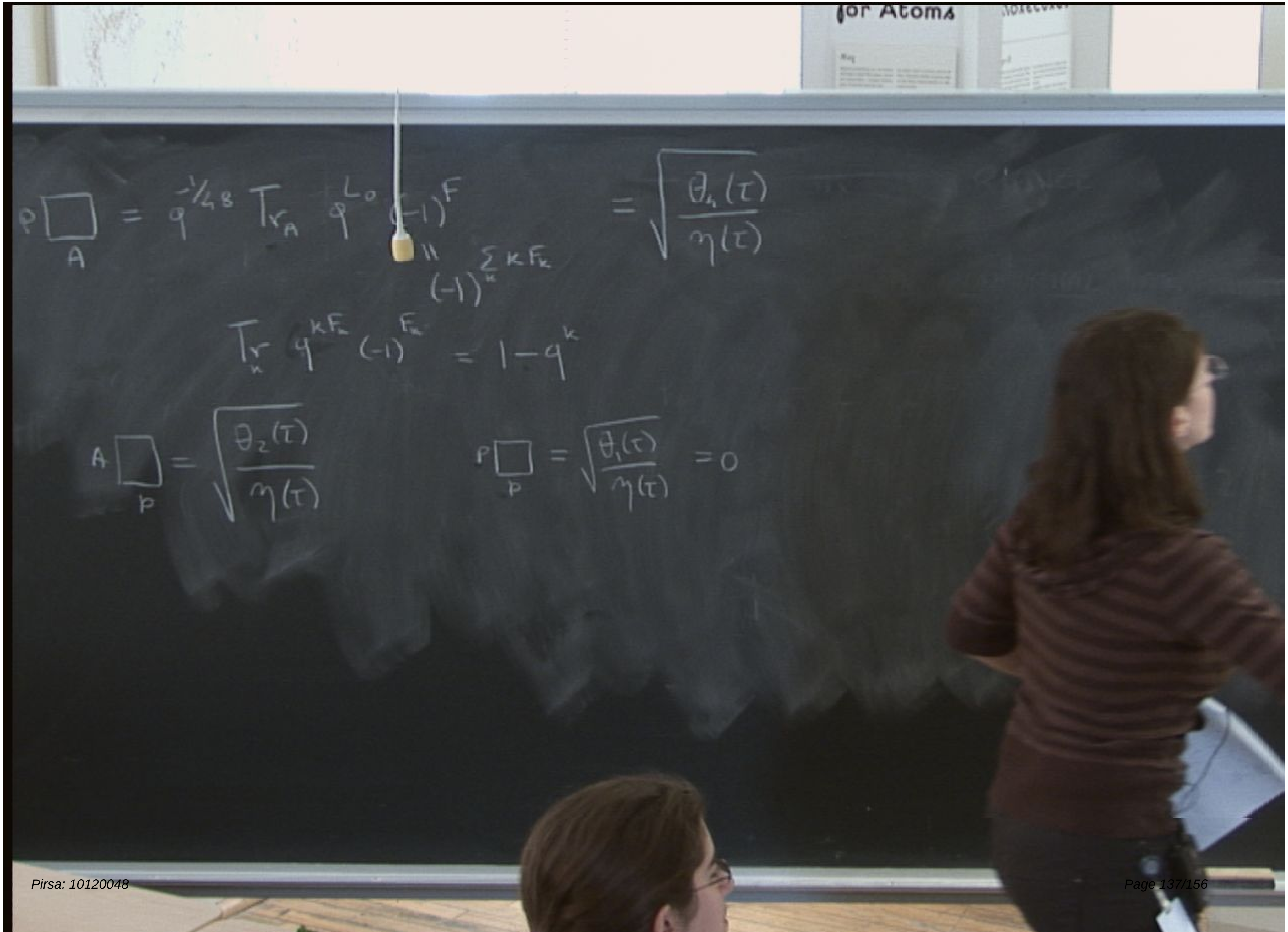
$$ \eta(\tau) = q^{1/24} \prod_{m=1}^{\infty} (1-q^m) $$

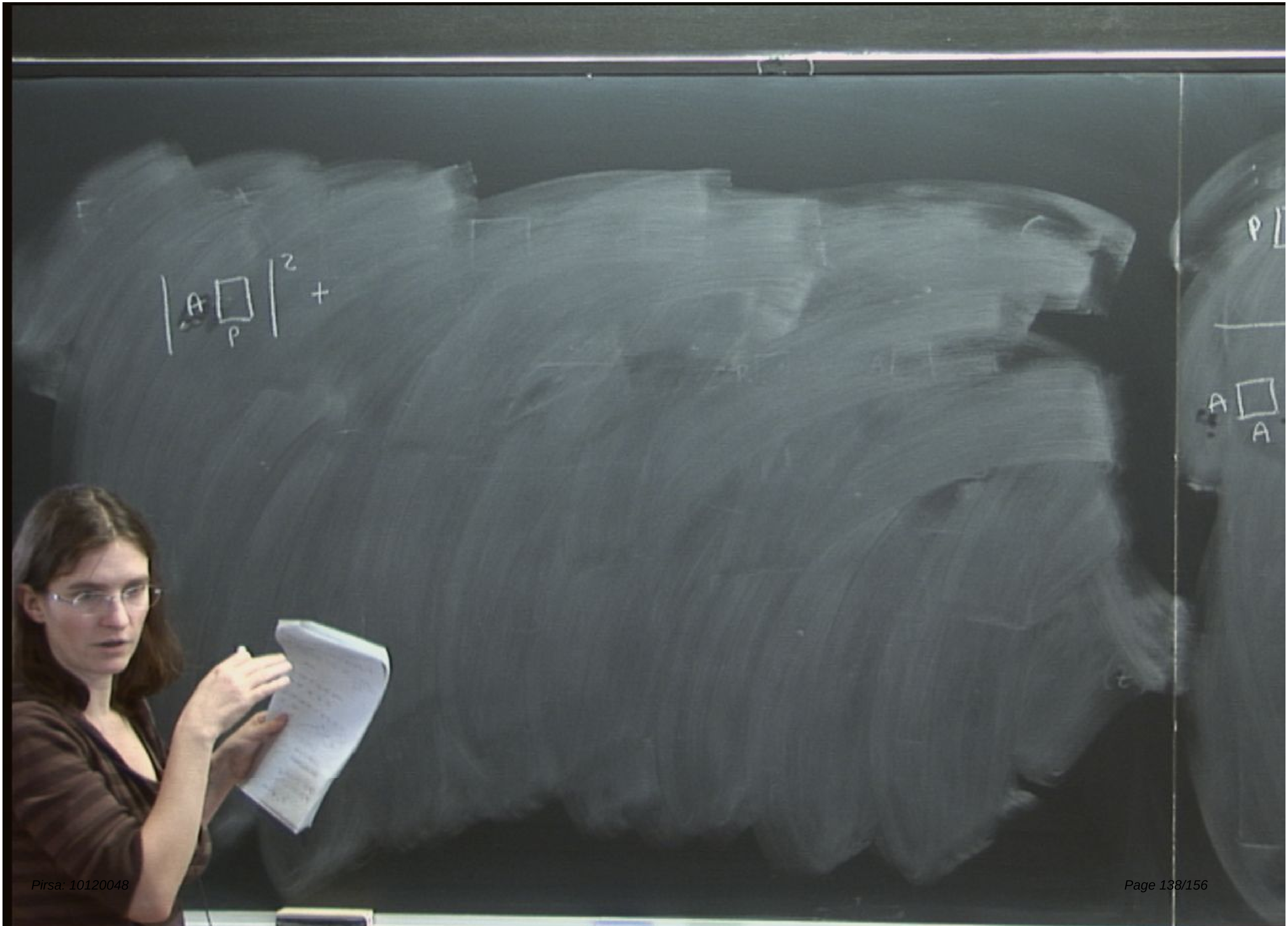

for Atoms

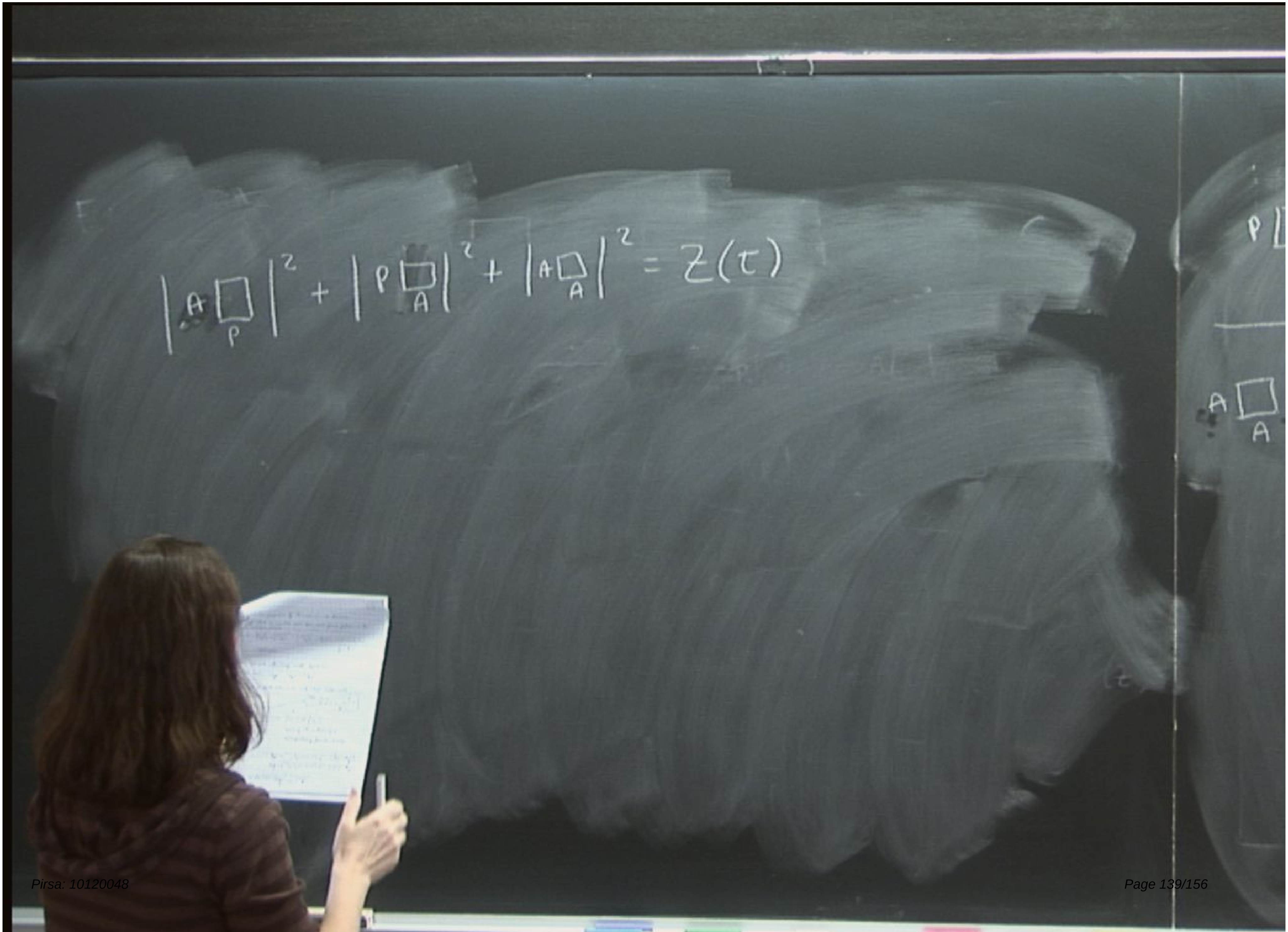

for Atoms
$\rho_{\square A} = q^{-1/48} \mathrm{Tr}_A \, q^{L_0} (-1)^F$

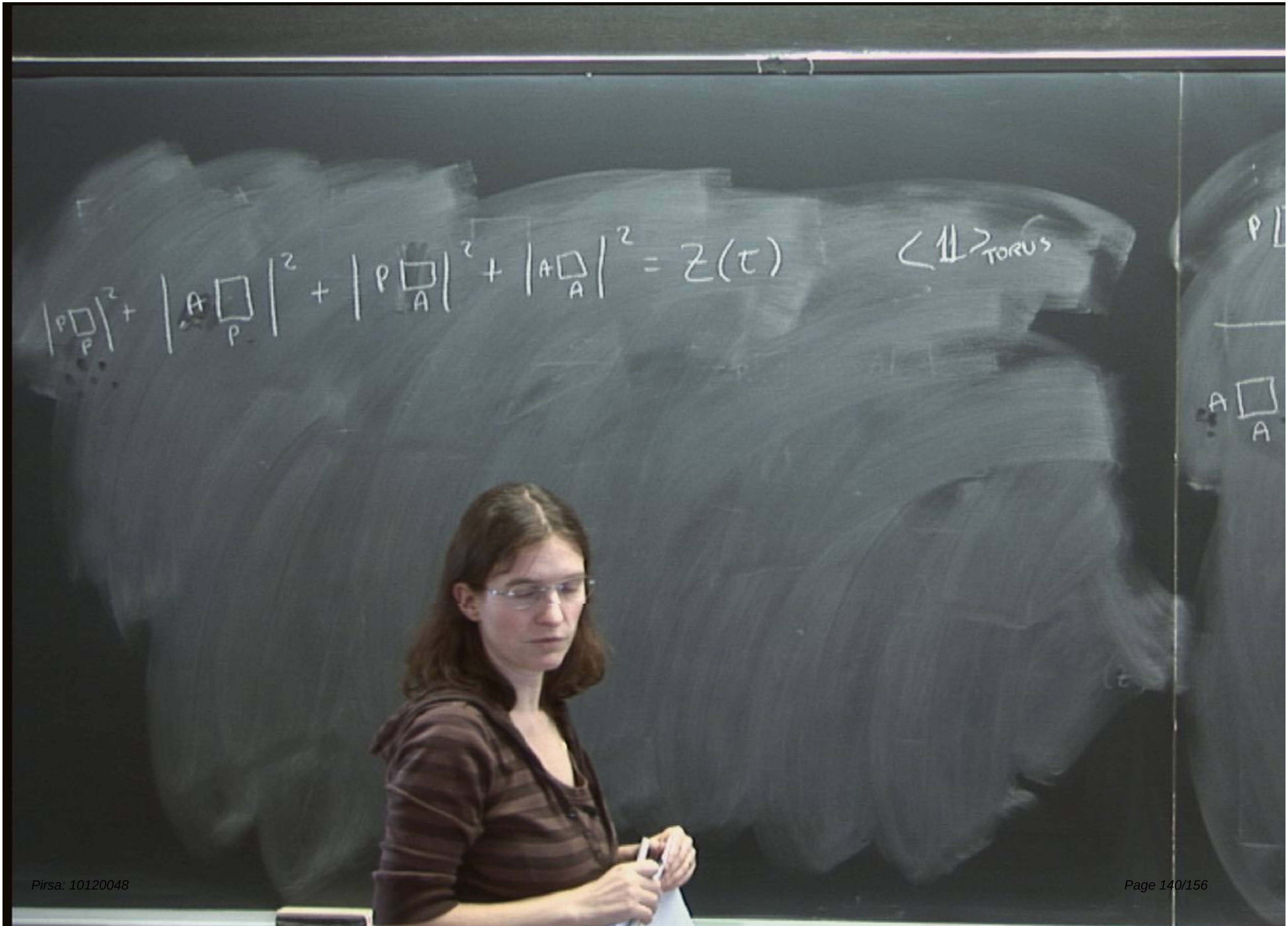

for Atoms
ρ□ = q^{-1/48} Tr_A q^{L_0} (-1)^F
A
= (-1)^{Σ_k k F_k}
Tr_k q^{kF_k} (-1)^{F_k} = 1 - q^k

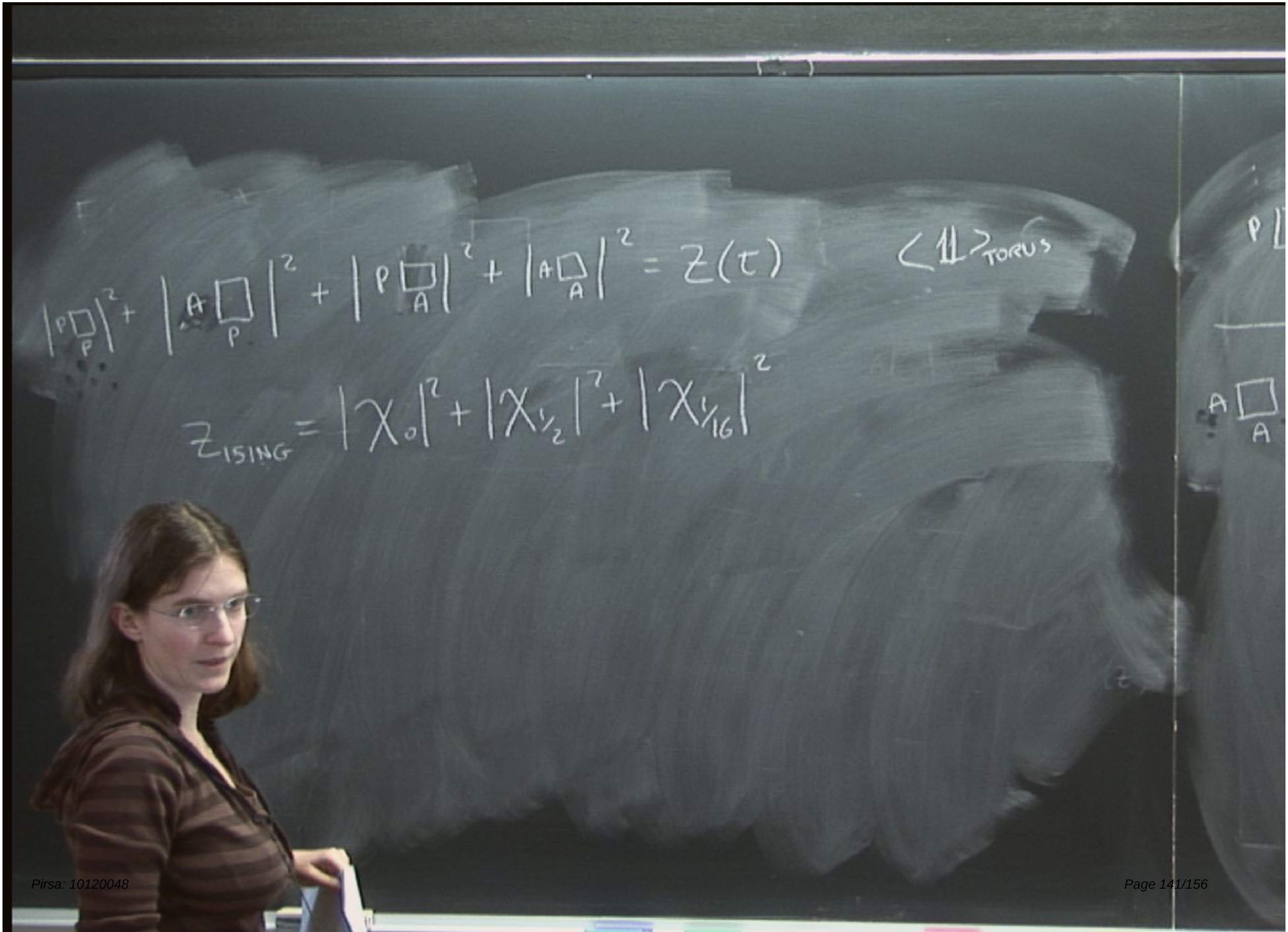

for Atoms
$\rho_{\square_A} = q^{-1/48} \, Tr_A \, q^{L_0} (-1)^F$
$\| \; (-1)^{\sum_k k F_k}$
$Tr_k \, q^{kF_k} (-1)^{F_k} = 1 - q^k$

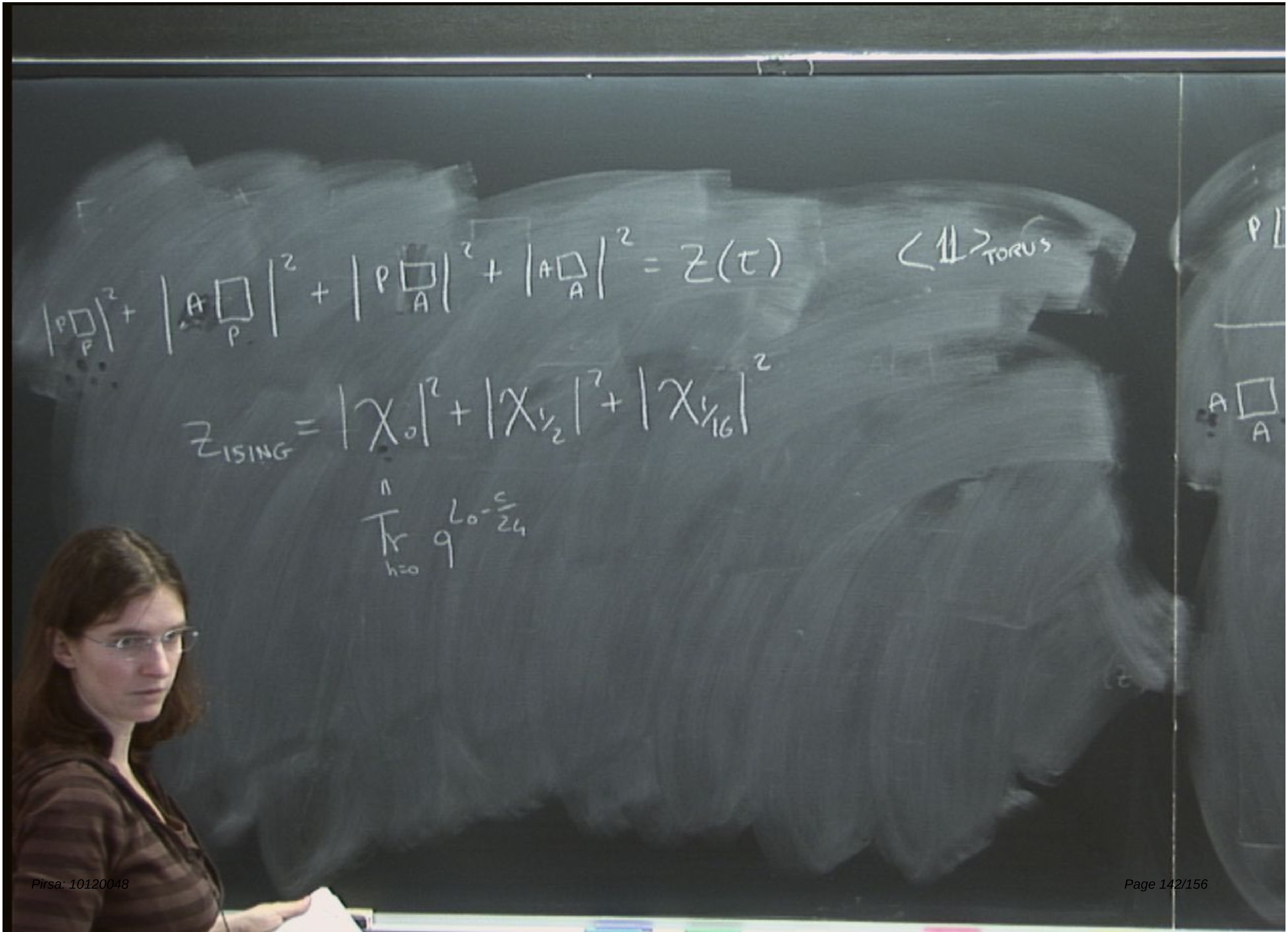

for Atoms
$P\square_A = q^{-1/48}\, Tr_A\, q^{L_0} (-1)^F = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}$
$(-1)^F = (-1)^{\sum_k k F_k}$
$Tr_k\, q^{kF_k} (-1)^{F_k} = 1 - q^k$
$A\square_P = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}$
$P\square_P = \sqrt{\frac{\theta_1(\tau)}{\eta(\tau)}} = 0$



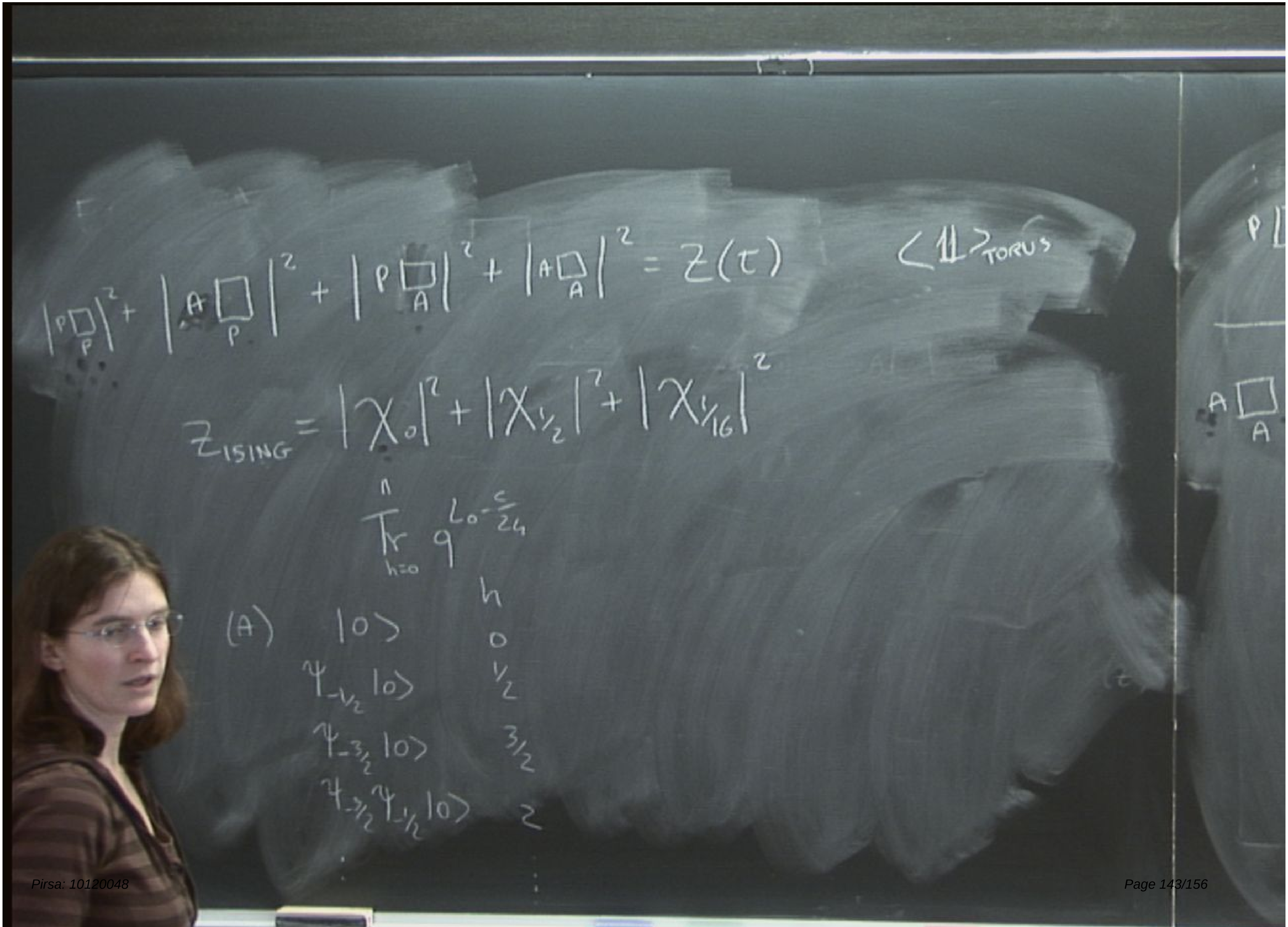

|A □ P|² + |P □ A|² + |A □ A|² = Z(τ)

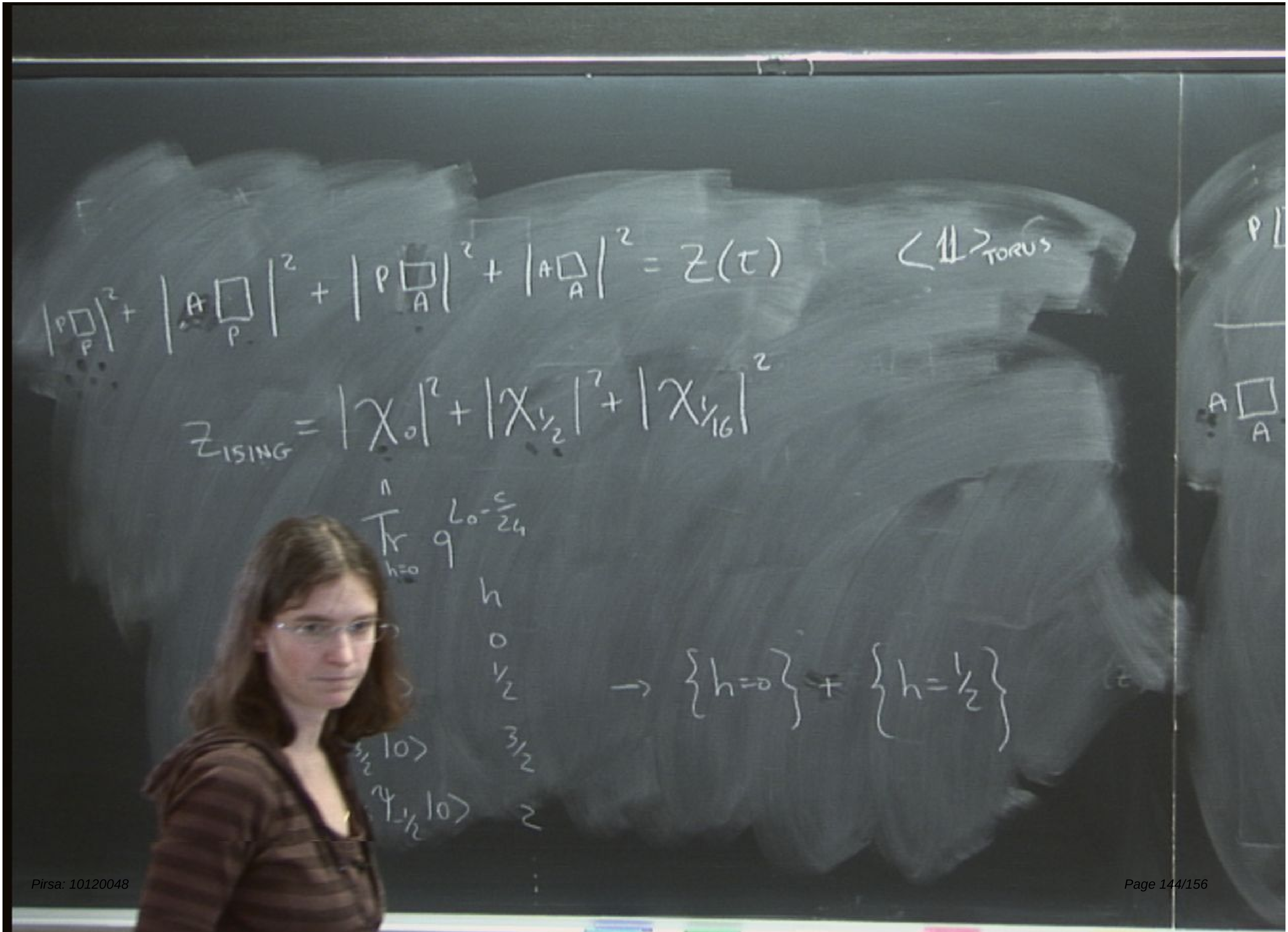

Z(τ)
TORUS

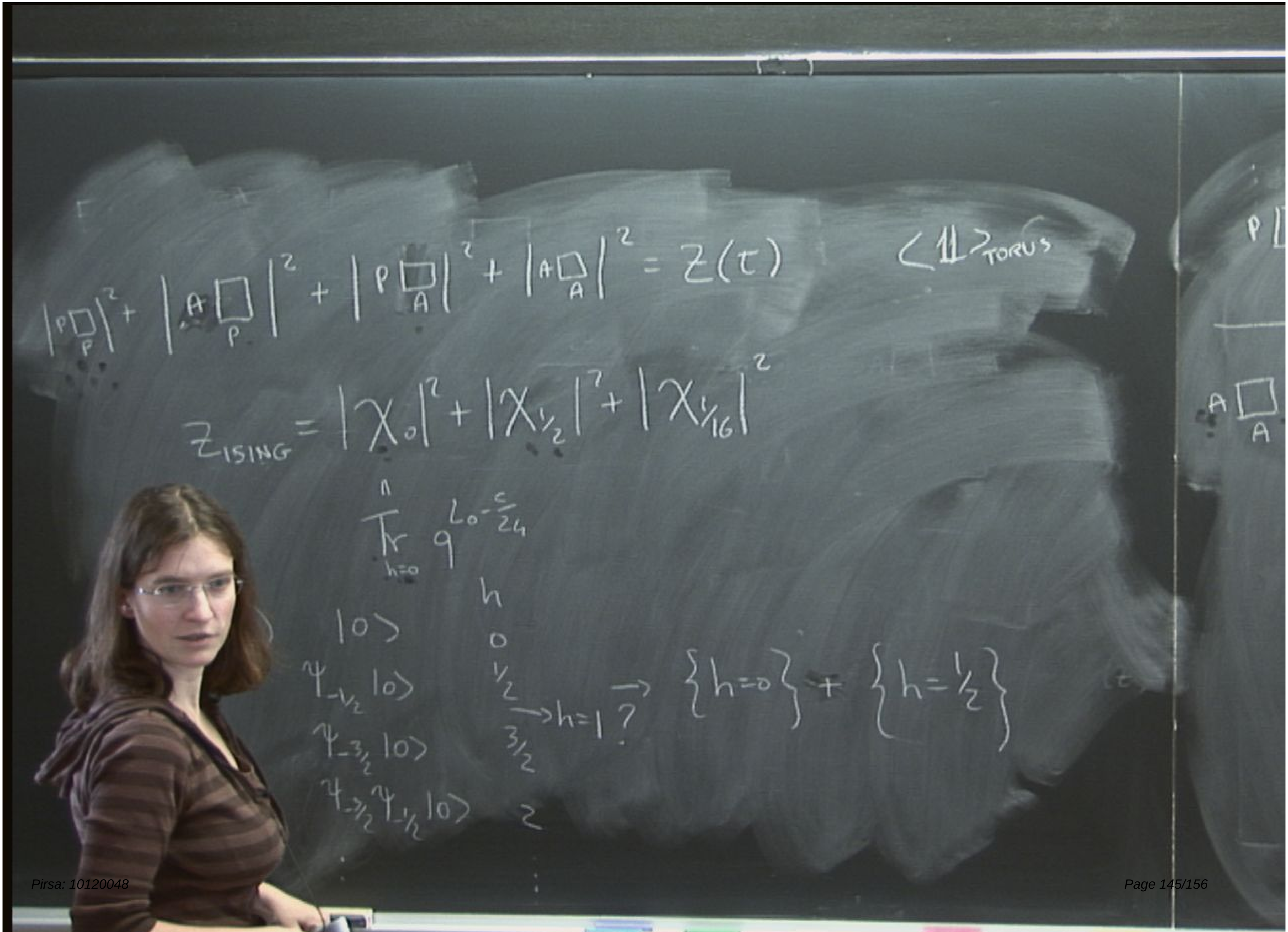

|P□ P|² + |A□ P|² + |P□ A|² + |A□ A|² = Z(τ)
⟨𝟙⟩TORUS
Z_ISING = |χ₀|² + |χ_½|² + |χ_1/16|²



|P□ P|² + |A□ P|² + |P□ A|² + |A□ A|² = Z(τ)
⟨11⟩ TORUS
Z ISING = |χ₀|² + |χ½|² + |χ1/16|²
Tr h=0 q^(L₀ − c/24)


|P□P|² + |A□P|² + |P□A|² + |A□A|² = Z(τ)
⟨𝟙⟩TORUS
Z_ISING = |χ_0|² + |χ_1/2|² + |χ_1/16|²
Tr_{h=0} q^{L_0 - c/24}
h
(A) |0⟩ 0
ψ_{-1/2} |0⟩ 1/2
ψ_{-3/2} |0⟩ 3/2
ψ_{-3/2} ψ_{-1/2} |0⟩ 2


|P□P|² + |A□P|² + |P□A|² + |A□A|² = Z(τ)
⟨𝟙⟩TORUS
ZISING = |χ0|² + |χ1/2|² + |χ1/16|²
Tr h=0 q^(L0 - c/24)
h
0
1/2
3/2 |0⟩
3/2
ψ-1/2 |0⟩
2
→ {h=0} + {h=1/2}


⟨11⟩ TORUS
$|P\square_P|^2 + |A\square_P|^2 + |P\square_A|^2 + |A\square_A|^2 = Z(\tau)$
$Z_{ISING} = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2$
$\hat{Tr}_{h=0}\, q^{L_0 - c/24}$
h
$|0\rangle$ 0
$\psi_{-1/2}|0\rangle$ 1/2
$\psi_{-3/2}|0\rangle$ 3/2
$\psi_{-3/2}\psi_{-1/2}|0\rangle$ 2
→ h=1 ?
→ {h=0} + {h=1/2}

$$\left|P\,\square_{P}\right|^2 + \left|A\,\square_{P}\right|^2 + \left|P\,\square_{A}\right|^2 + \left|A\,\square_{A}\right|^2 = Z(\tau) \qquad \langle 1\!\!1 \rangle_{\text{TORUS}}$$

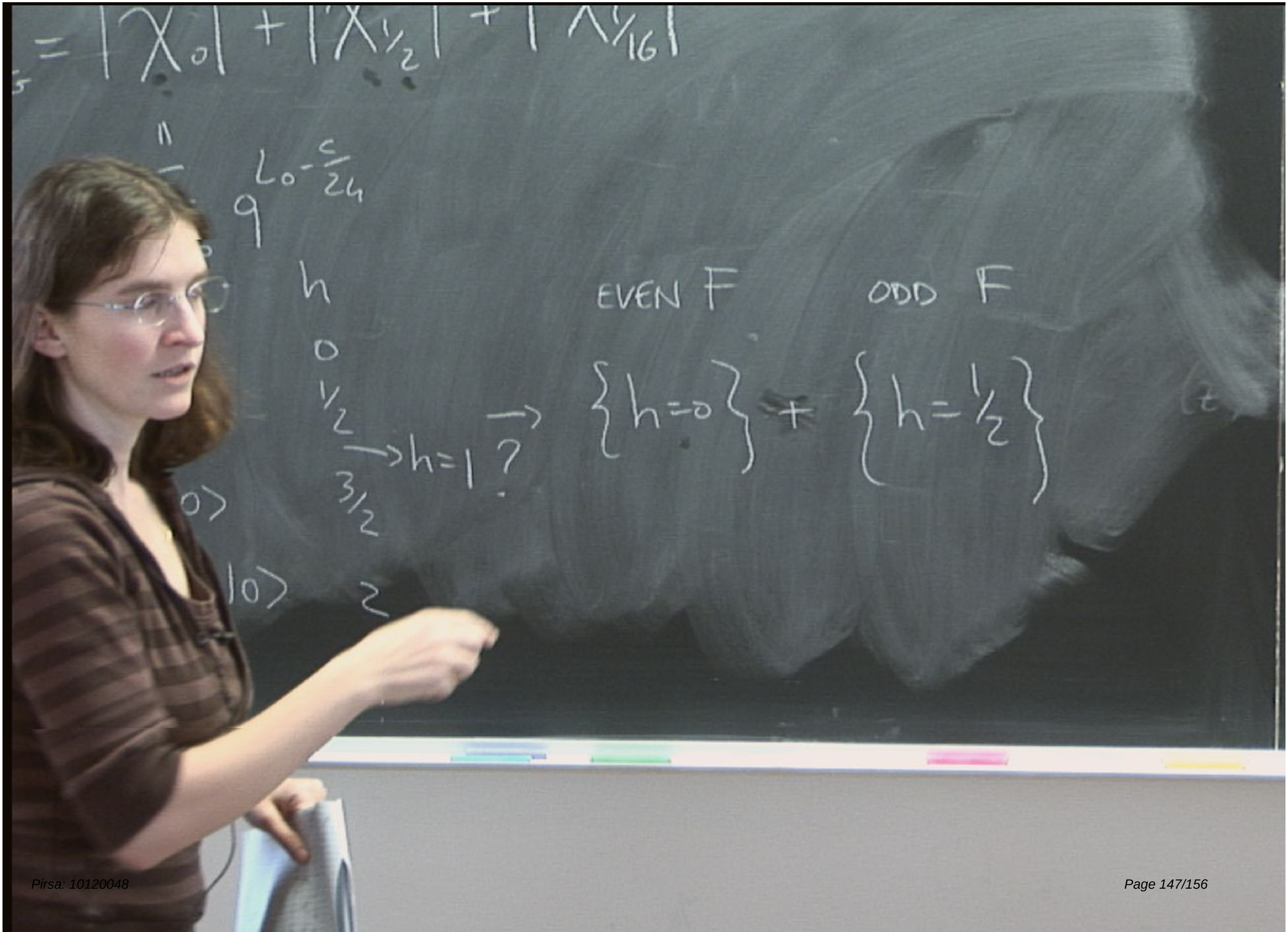
$$Z_{\text{ISING}} = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2$$

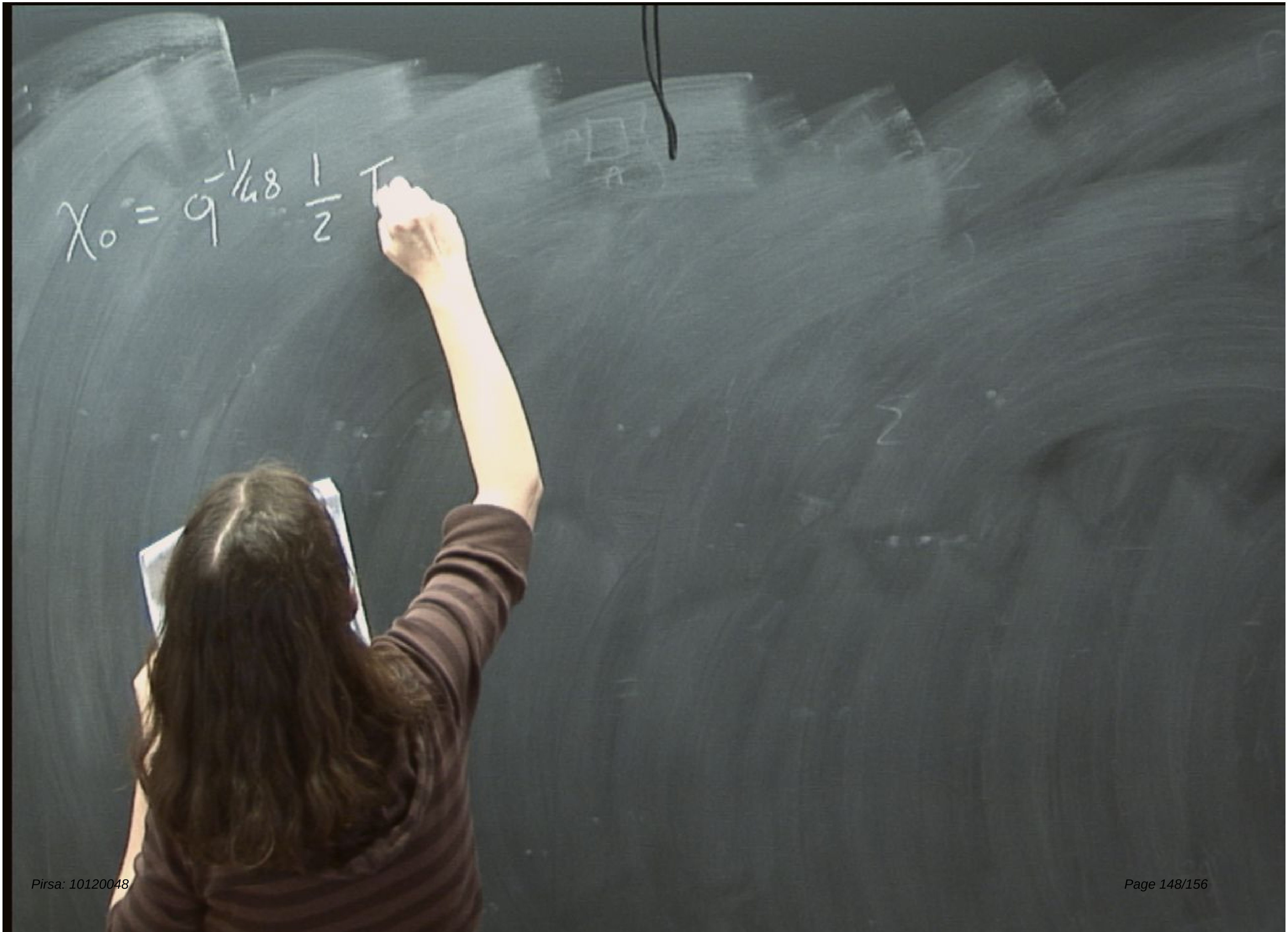
$$\widehat{\mathrm{Tr}}_{h=0}\; q^{L_0 - \frac{c}{24}}$$

(A)		h
	$\vert 0\rangle$	0
	$\psi_{-1/2}\vert 0\rangle$	$1/2$
	$\psi_{-3/2}\vert 0\rangle$	$3/2$
	$\psi_{-3/2}\psi_{-1/2}\vert 0\rangle$	2

$\rightarrow h=1\,?$

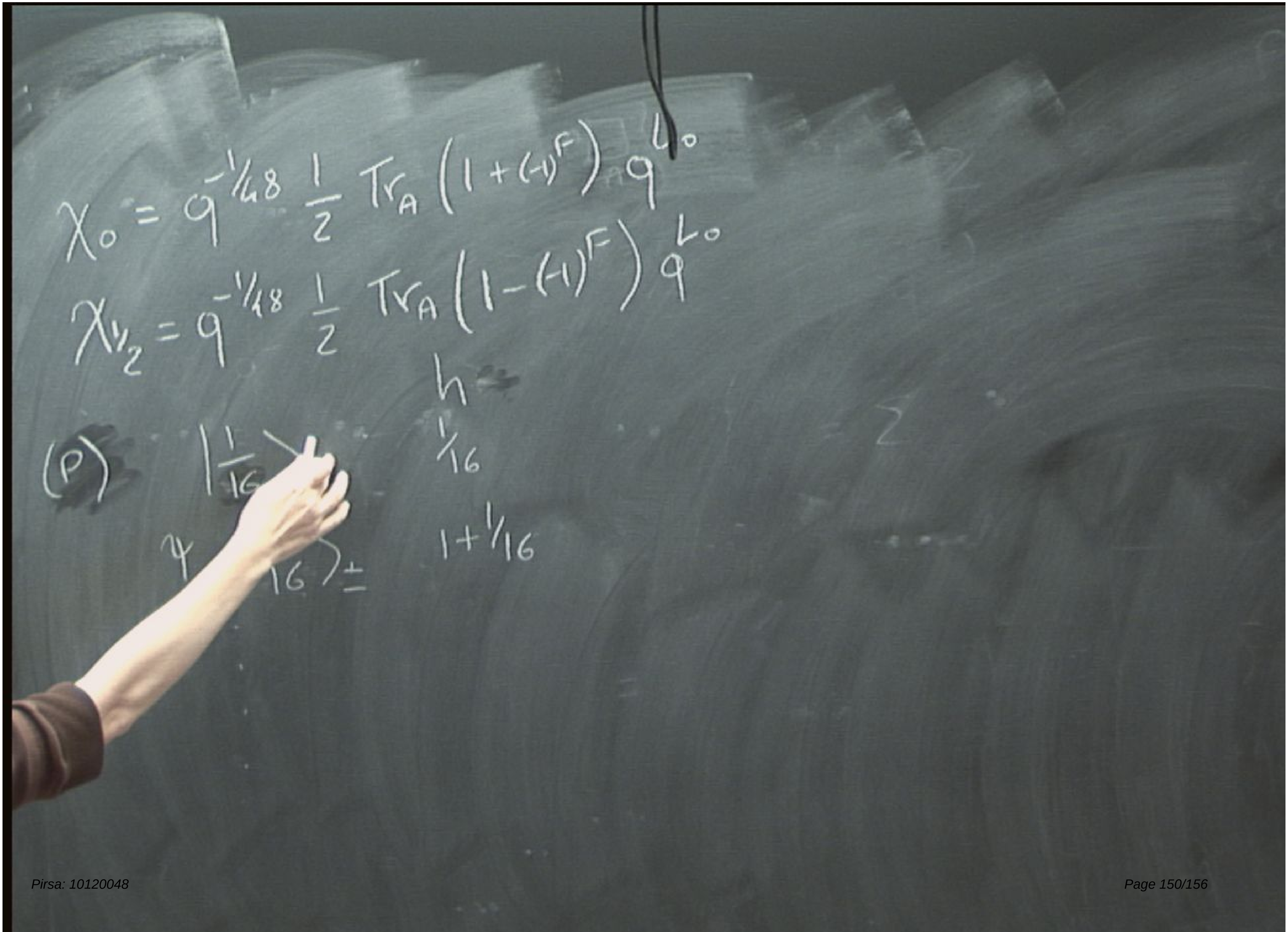
$\rightarrow \{h=0\} + \{h=\tfrac{1}{2}\}$

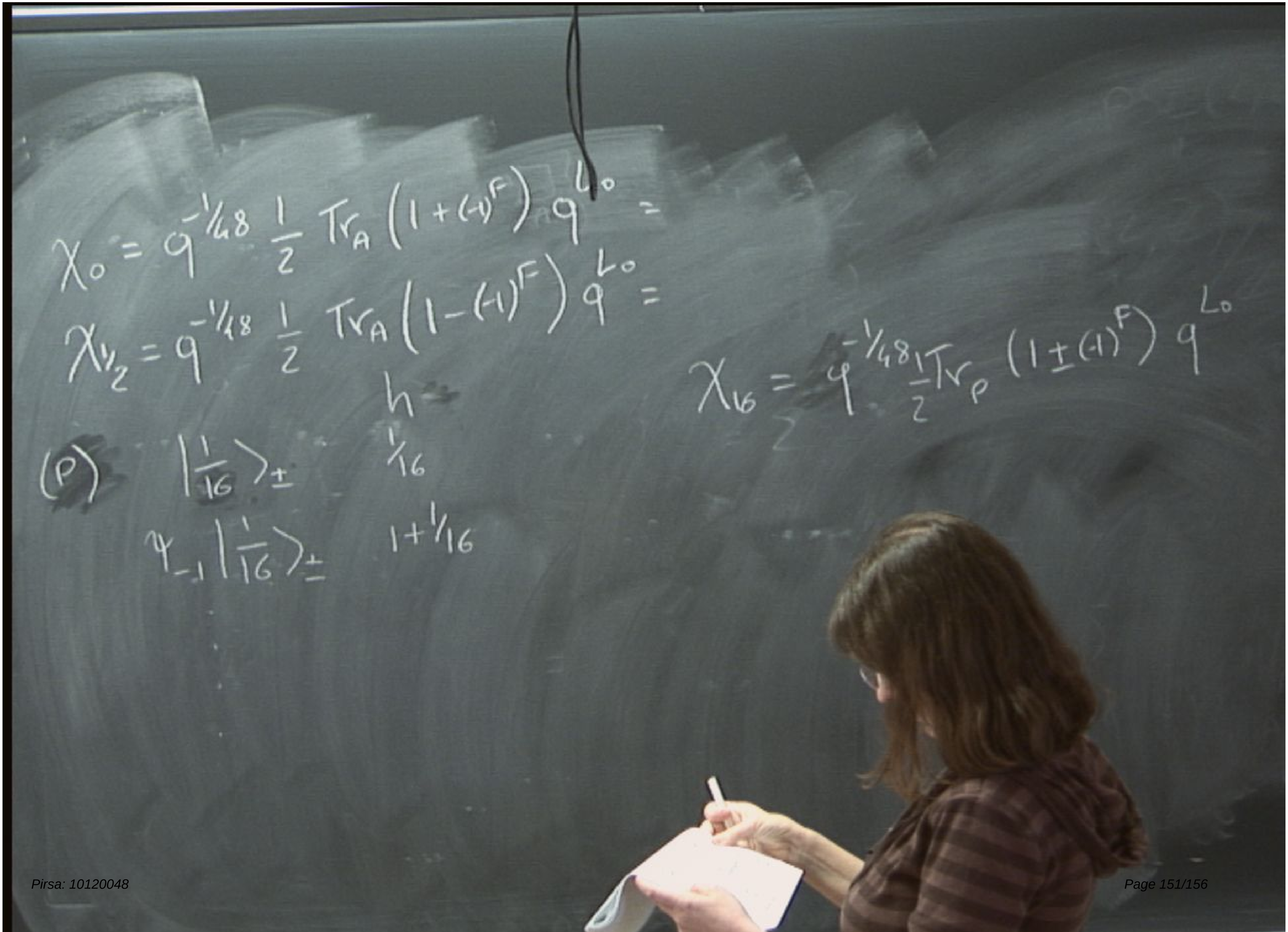

EVEN F
ODD F
h=1 ?
{h=0} + {h=1/2}

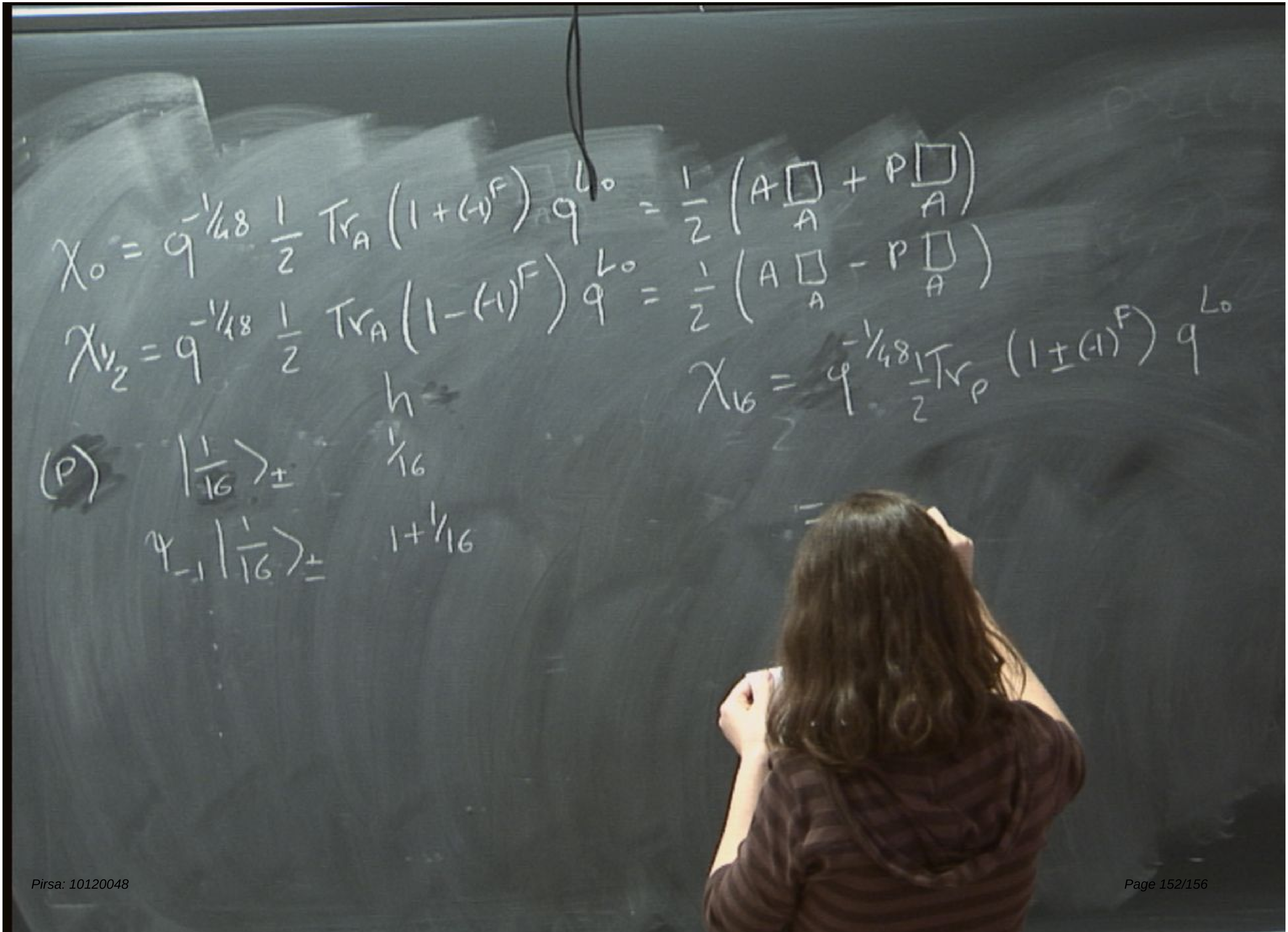

$x_0 = q^{-1/48} \frac{1}{2}$ T

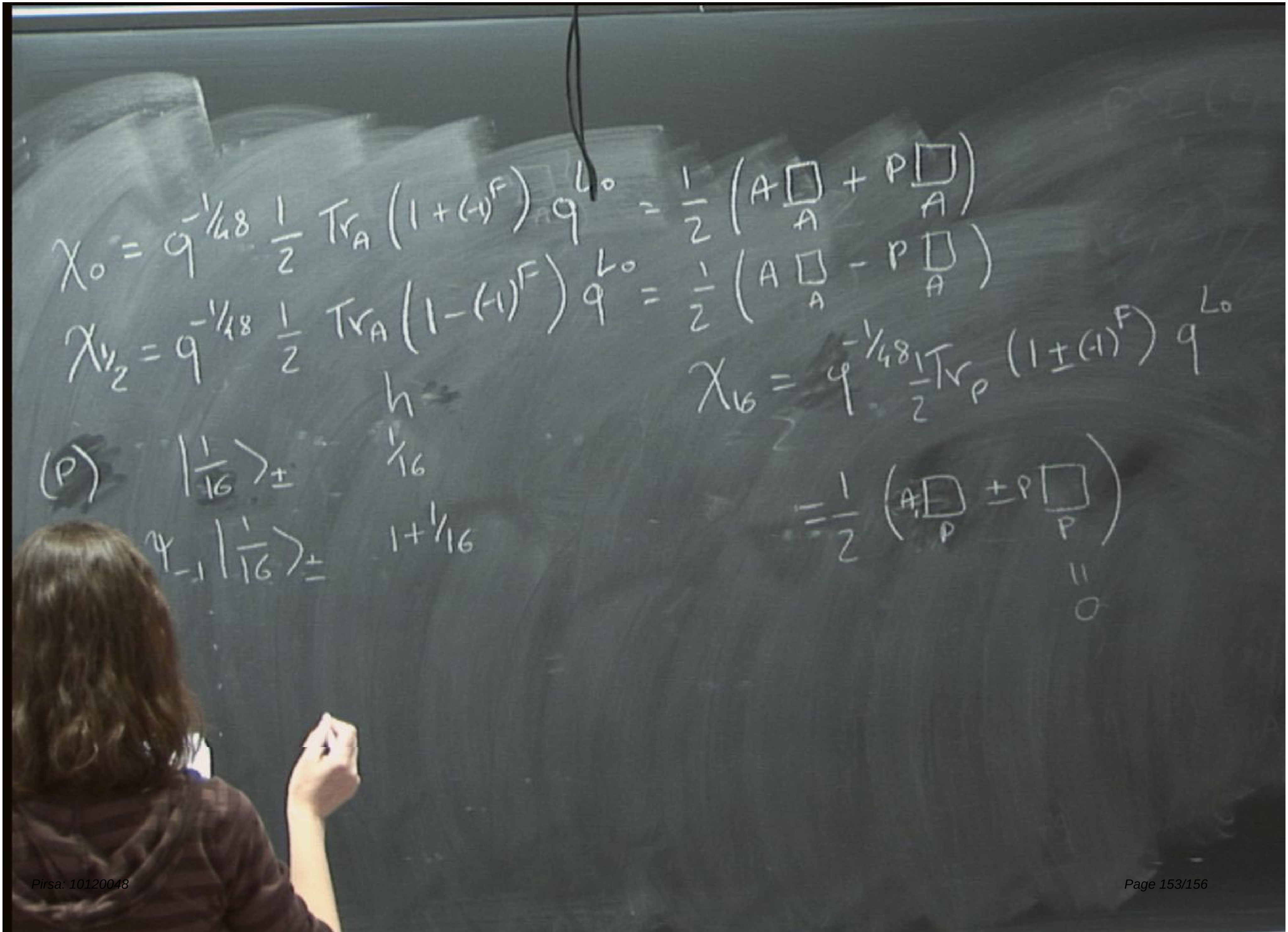
$$\chi_0 = q^{-1/48}\, \frac{1}{2}\, \mathrm{Tr}_A \left(1 + (-1)^F\right) q^{L_0}$$

$$\chi_{1/2} = q^{-1/48}\, \frac{1}{2}\, \mathrm{Tr}_A \left(1 - (-1)^F\right) q^{L_0}$$


χ_0 = q^{-1/48} 1/2 Tr_A (1 + (-1)^F) q^{L_0}
χ_{1/2} = q^{-1/48} 1/2 Tr_A (1 - (-1)^F) q^{L_0}
h
(P)
|1/16⟩
1/16
ψ
16⟩±
1+1/16


$\chi_0 = q^{-1/48} \frac{1}{2} \mathrm{Tr}_A (1 + (-1)^F) q^{L_0} =$
$\chi_{1/2} = q^{-1/48} \frac{1}{2} \mathrm{Tr}_A (1 - (-1)^F) q^{L_0} =$
$\chi_{16} = q^{-1/48} \frac{1}{2} \mathrm{Tr}_P (1 \pm (-1)^F) q^{L_0}$
h
(P) $|\frac{1}{16}\rangle_\pm$ $1/16$
$\psi_{-1} |\frac{1}{16}\rangle_\pm$ $1 + 1/16$


χ0 = q^(-1/48) 1/2 TrA (1 + (-1)^F) q^L0 = 1/2 (A□A + P□A)
χ1/2 = q^(-1/48) 1/2 TrA (1 - (-1)^F) q^L0 = 1/2 (A□A - P□A)
χ16 = q^(-1/48) 1/2 TrP (1 ± (-1)^F) q^L0
h
(P) |1/16>±   1/16
ψ-1 |1/16>±   1+1/16


$\chi_0 = q^{-1/48} \frac{1}{2} \mathrm{Tr}_A \left(1 + (-1)^F\right) q^{L_0} = \frac{1}{2}\left(A \square_A + P \square_A\right)$
$\chi_{1/2} = q^{-1/48} \frac{1}{2} \mathrm{Tr}_A \left(1 - (-1)^F\right) q^{L_0} = \frac{1}{2}\left(A \square_A - P \square_A\right)$
h
$(P) \quad |\frac{1}{16}\rangle_\pm \quad 1/16$
$\psi_{-1} |\frac{1}{16}\rangle_\pm \quad 1 + 1/16$
$\chi_{16} = q^{-1/48} \frac{1}{2} \mathrm{Tr}_P \left(1 \pm (-1)^F\right) q^{L_0}$
$= \frac{1}{2}\left(A \square_P \pm P \square_P\right)$
$P \square_P = 0$

$$\chi_0 = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_A\left(1+(-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\underset{A}{\Box} + P\underset{A}{\Box}\right)$$

$$\chi_{1/2} = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_A\left(1-(-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\underset{A}{\Box} - P\underset{A}{\Box}\right)$$

$$\chi_{16} = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_P\left(1\pm(-1)^F\right) q^{L_0}$$

$$= \frac{1}{2}\left(A\underset{P}{\Box} \pm P\underset{P}{\Box}\right)$$

		h
(P)	$\left\lvert\frac{1}{16}\right\rangle_\pm$	$1/16$
	$\psi_{-1}\left\lvert\frac{1}{16}\right\rangle_\pm$	$1+1/16$

$$Z_{\text{FERMIONS}} = \left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| = 2\left(|\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2\right)$$

$$\parallel$$

$$Z_{\text{ISING}}$$

$$\chi_0 = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_A\left(1+(-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\underset{A}{\square} + P\underset{A}{\square}\right)$$

$$\chi_{1/2} = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_A\left(1-(-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\underset{A}{\square} - P\underset{A}{\square}\right)$$

$$\chi_{16} = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_P\left(1\pm(-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\underset{P}{\square} \pm P\underset{P}{\square}\right)$$

		h
(P)	$\left|\frac{1}{16}\right\rangle_\pm$	$1/16$
	$\psi_{-1}\left|\frac{1}{16}\right\rangle_\pm$	$1+1/16$

$$Z_{\text{FERMIONS}} = \left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| = 2\left(|\chi_0|^2 + |\chi_{1/2}|^2 + |\chi$$

$\mathbb{1}$ $\quad \varepsilon(z,\bar{z})$

$\psi\bar{\psi}$

$$\chi_0 = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_A\left(1+(-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\,\underset{A}{\square} + P\,\underset{A}{\square}\right)$$

$$\chi_{1/2} = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_A\left(1-(-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\,\underset{A}{\square} - P\,\underset{A}{\square}\right)$$

$$\chi_{16} = q^{-1/48}\,\frac{1}{2}\,\mathrm{Tr}_P\left(1 \pm (-1)^F\right) q^{L_0} = \frac{1}{2}\left(A\,\underset{P}{\square} \pm P\,\underset{P}{\square}\right)$$

		h
(P)	$\left\lvert \frac{1}{16} \right\rangle_\pm$	$1/16$
	$\psi_{-1}\left\lvert \frac{1}{16} \right\rangle_\pm$	$1+1/16$

$$Z_{\text{FERMIONS}} = \left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| = 2\left(|\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2\right)$$

$\mathbb{1}$ $\qquad \varepsilon(z,\bar{z})$ $\qquad \sigma, \mu$

$\psi\bar{\psi}$