Title: Bringing order through disorder: Localization in the toric code

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Abstract: Anderson localization emerges in quantum systems when randomised parameters cause the exponential suppression of motion. In this talk we will consider the localization phenomenon in the toric code, demonstrating its ability to sustain quantum information in a fault tolerant way. We show that an external magnetic field induces quantum walks of anyons, causing logical information to be destroyed in a time linear with the system size when even a single pair of anyons is present. However, by taking into account the disorder inherent in any physical realisation of the code, it is found that localization allows the memory to be stable in the presence of a finite anyon density. Enhancements to this effect are also considered using random lattices, and similar problems for anyons transported by thermal errors are considered.

Motivation

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- The toric code is a **quantum memory**, protected by a **gap**
- Realistically, this is always subjected to stray **'magnetic' fields**
- The effects of these on the gap and topological order have been well studied
- Here we study the **dynamic effects** on excited states
- **Quantum walks** are induced, propagating errors
- We find the quantum memory is **destroyed in linear time**
- Can **disorder** be used to protect the stored information through **localization**?

Overview

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- Toric code
  - Qubits, errors and anyons
  - Hamiltonian and protection

- Magnetic fields
  - Effects on the toric code
  - Quantum walks
  - Decoherence of quantum memory

- Disorder and localization
  - Random graphs
  - Anderson localization
  - Error suppression

The toric code

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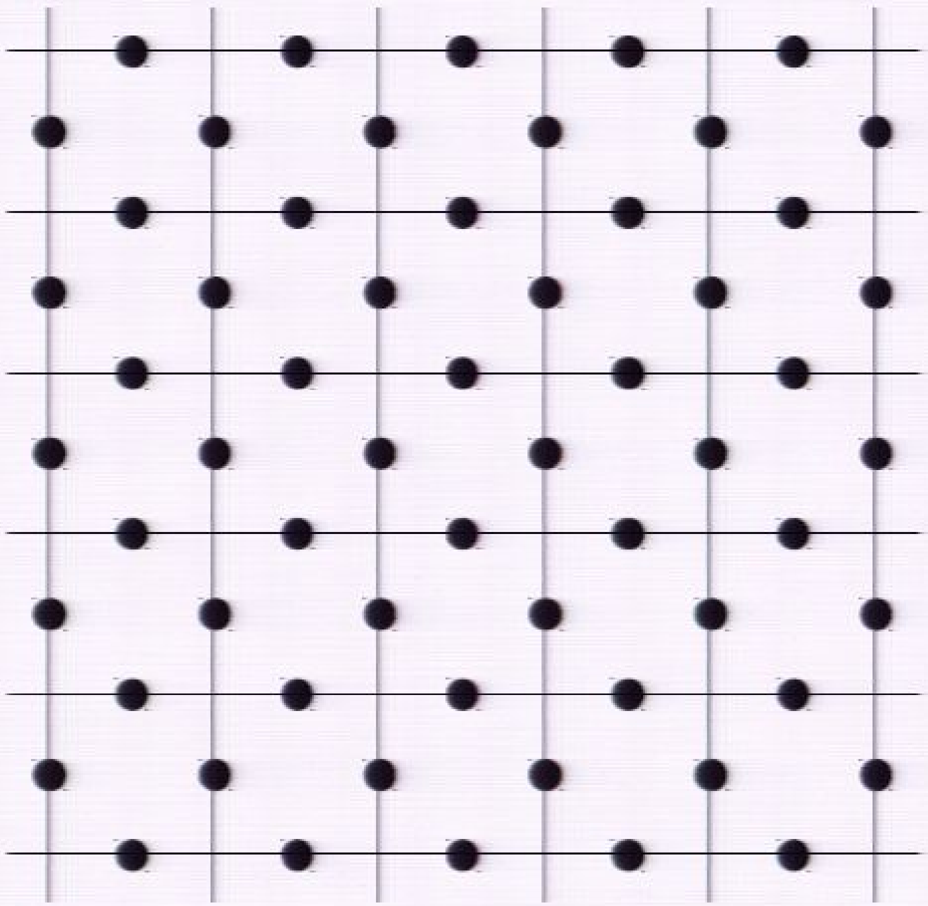
- Proposed by Kitaev
- Stabilizer code
- Defined on 2D spin lattice
- Spin-1/2 on edges
- Lattice wrapped around torus
  (other surfaces may also be used)



The toric code

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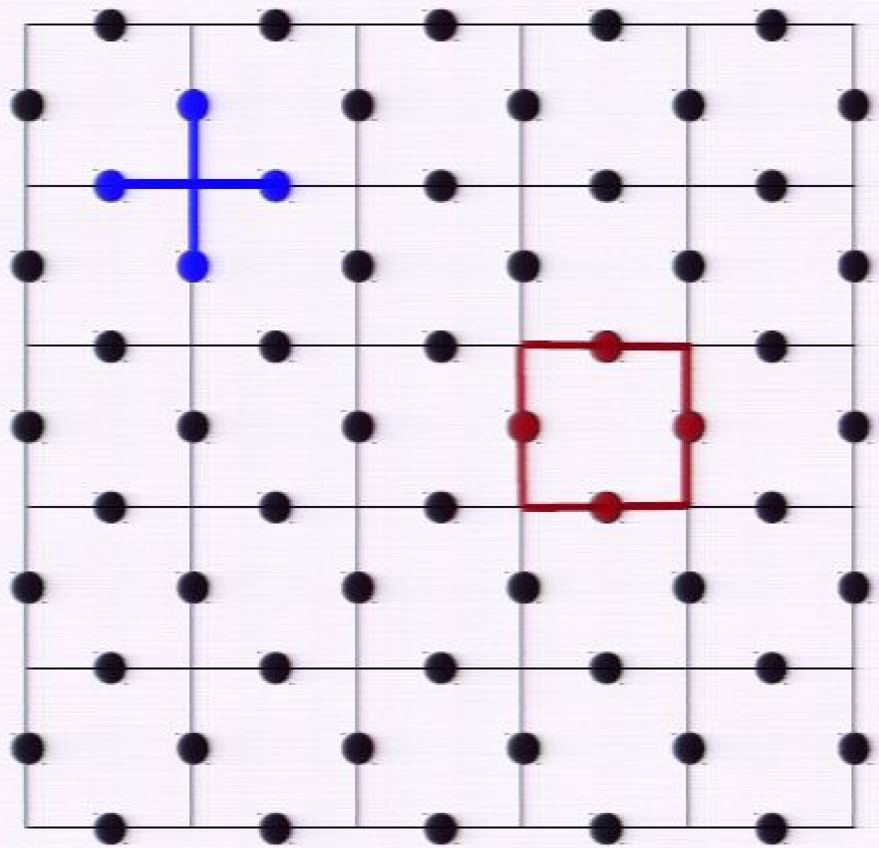
- Stabilizers defined on spins around each plaquette and vertex

$$A_v = \sigma_i^x \sigma_j^x \sigma_k^x \sigma_l^x,$$

$$B_p = \sigma_i^z \sigma_j^z \sigma_k^z \sigma_l^z$$



- Quantum information stored in stabilizer space

$$A_v |\psi\rangle = |\psi\rangle \ \forall \ v$$

$$B_v |\psi\rangle = |\psi\rangle \ \forall \ p$$

- Four dimensional Hilbert space: two logical qubits

The toric code

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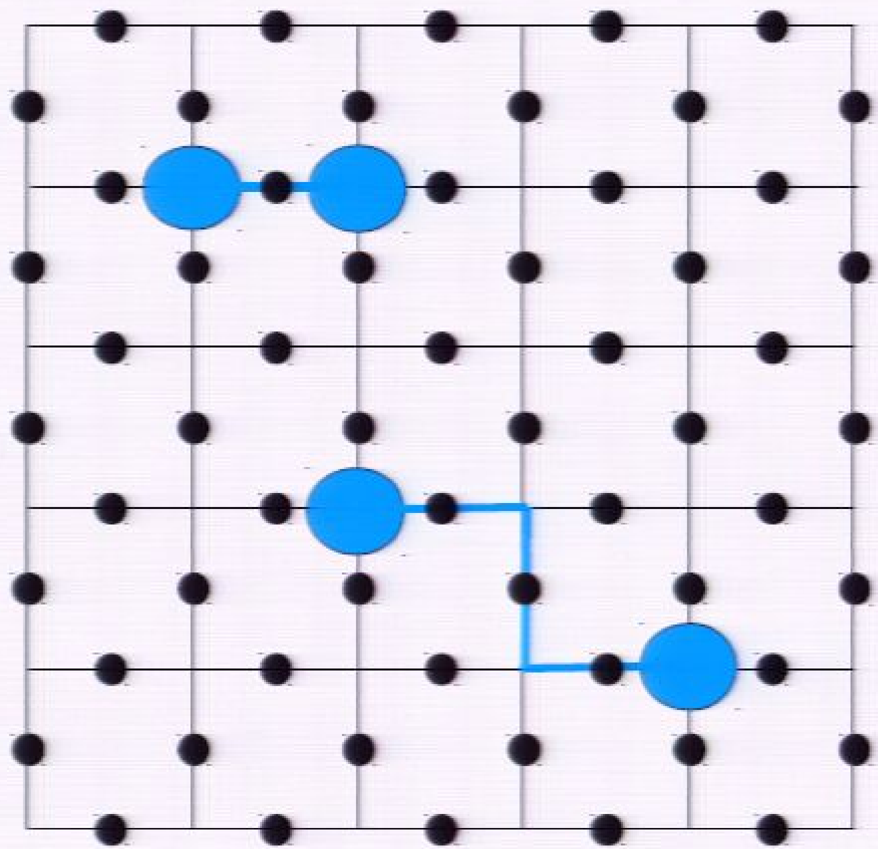
- Local errors move state out of stabilizer space



- Stabilizers can be measured to determine whether such errors have occurred
- Best means to correct errors can then be determined and performed
- Single spin errors affect pairs of neighbouring stabilizers
- Can be interpreted as pair creation of quasiparticles
- $A_v|\psi\rangle = -|\psi\rangle$ implies an e anyon on v
- Created and moved by σ_i^z errors

The toric code

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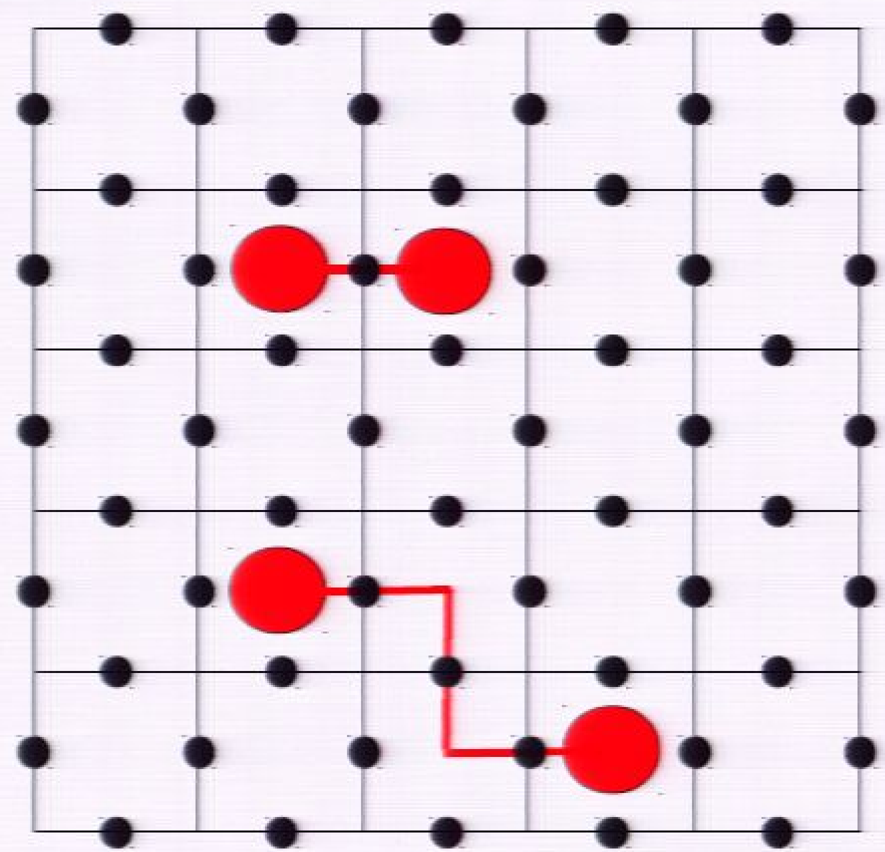
- $B_p|\psi\rangle = -|\psi\rangle$ implies an m anyon on p

- Created and moved by σ_i^x operations



The toric code

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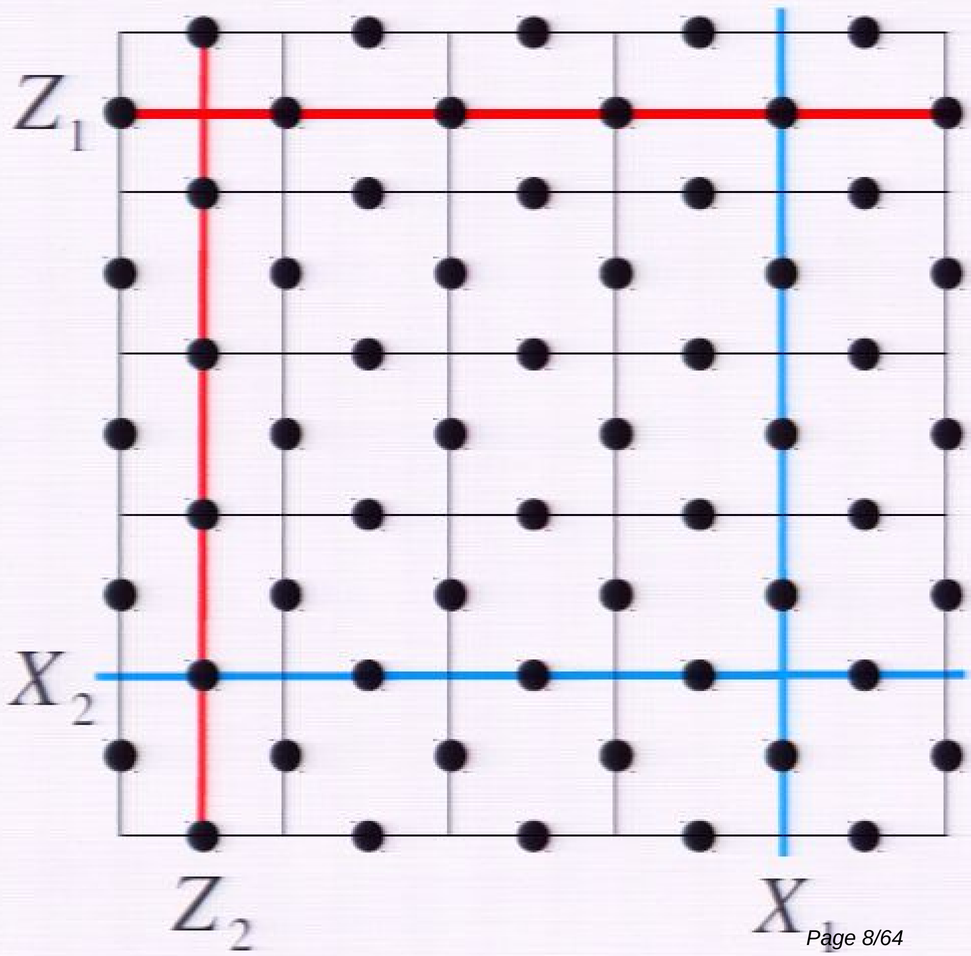
- Logical operations correspond to moving anyons around the torus in topologically non-trivial loops

- Trivial loops have no effect on logical qubits – equivalent to stabilizers

- Error correction attempts to annihilate anyons without creating non-trivial loops

- Error correction successful when density of anyons is less than a critical value

$$\rho_c \approx 0.31$$



The toric code

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- Quasilocal stabilizers mean Hamiltonian can be implemented

$$H_{TC} = -J\sum_v A_v - J\sum_p B_p$$

- Degenerate ground state corresponds to stabilizer space

- Anyon creation suppressed by energy gap

- Gap and topological order stable against local perturbations

- Quantum memory is vulnerable to dynamic effects (Kay, Pastawski)

Magnetic fields and the toric code

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- Consider the toric code Hamiltonian, perturbed with a magnetic field

$$H = -J\sum_v A_v - J\sum_p B_p + h\sum_i \sigma_i^z$$

- This can create, annihilate and transport e anyons

- Symmetry between e's and m's mean study of e's alone is sufficient

Magnetic fields and the toric code

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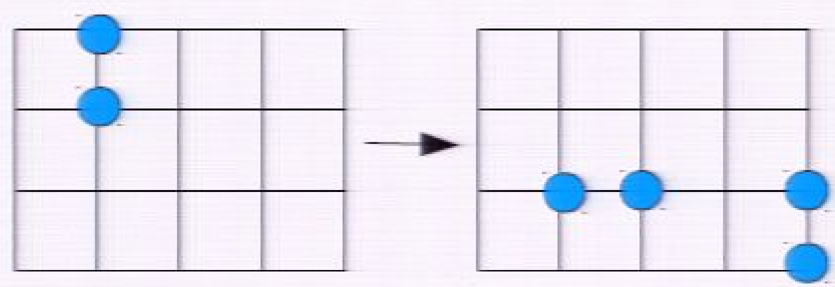
- The magnetic field term can be written

$$\sum_i \sigma_i^z = T + C \qquad T = \sum_n P_n \left(\sum_i \sigma_i^z\right) P_n$$

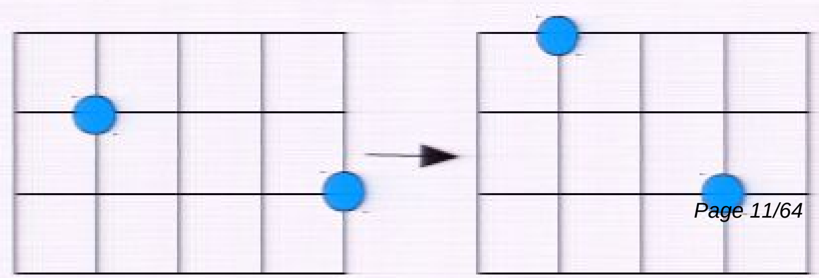
$$C = \sum_{n \neq m} P_n \left(\sum_i \sigma_i^z\right) P_m$$

- Here P_n is the projector onto the space of states with n vertex anyons

- C creates and annihilates vertex anyons



- T moves them



2D anyonic quantum walks

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- T commutes with the the toric Hamiltonian, but C does not. The action of C is penalized by the gap

- For $J \gg h$ the effects of C are suppressed, hence

$$H \approx -J \sum_v A_v - J \sum_p B_p + hT$$

- This is the Hamiltonian for a continuous time quantum walk

- Quantum analogue of classical random walk

- Coherence allows complex behaviour to manifest

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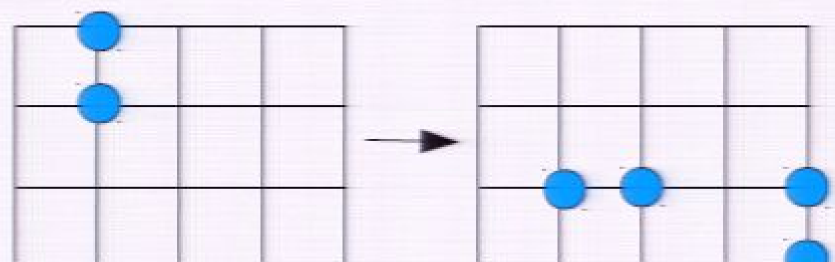

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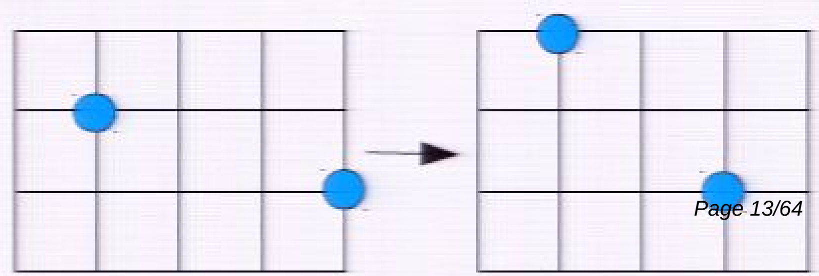

2D anyonic quantum walks

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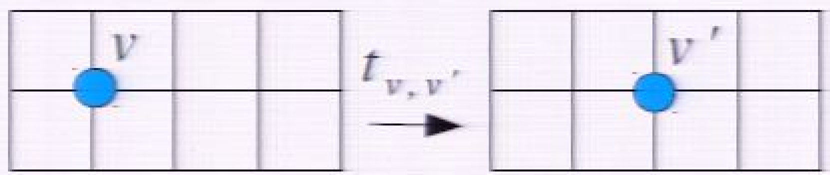
- To study the walks, we simplify the Hamiltonian

$$H=\sum_{v,v'} M_{v,v'} t_{v,v'} \qquad M_{v,v'}=J\,\delta_{v,v'}+h\,\delta_{\langle v,v'\rangle}$$



- M acts on a space of single walker position states.

- It contains all the properties of the walk

- This contains no trace of anyonic statistics

- Both e and m anyons are bosonic w.r.t. themselves

2D anyonic quantum walks

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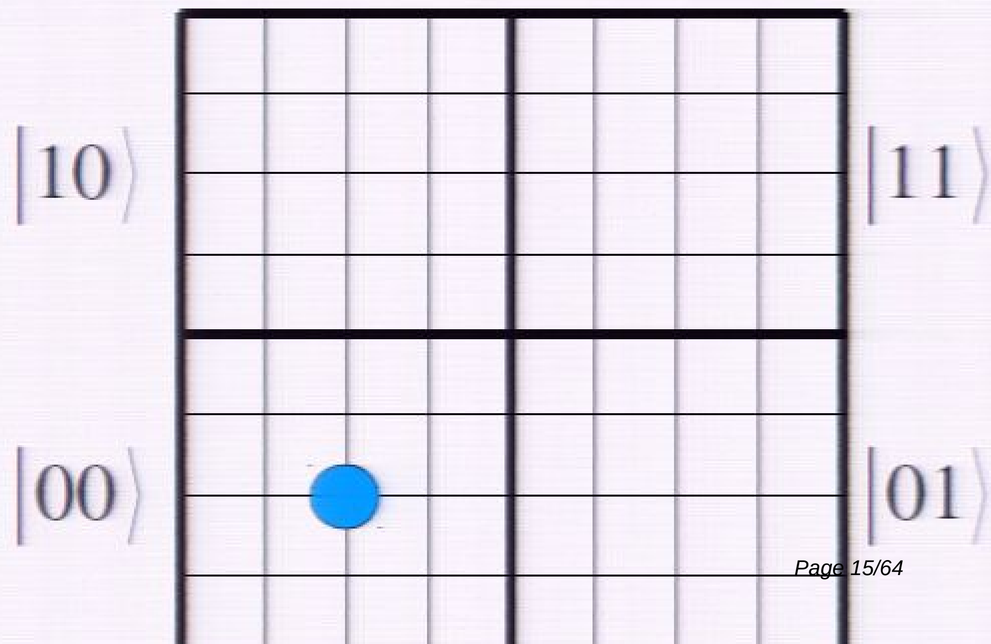
- M is Hamiltonian for single anyon walk
- Single anyon walks can be used to determine its properties
- Degenerate ground state can be represented by extended lattice
- Each quadrant corresponds to original lattice, but with a different value for the logical qubits
- Consider an initial state of $|00\rangle$
- Anyon moving beyond its quadrant implies logical error



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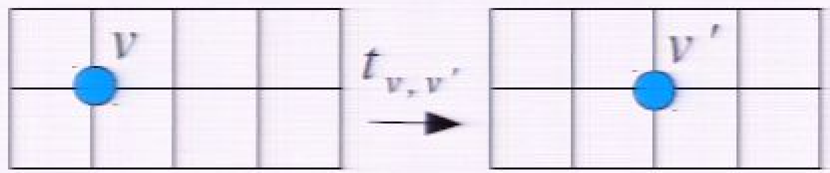

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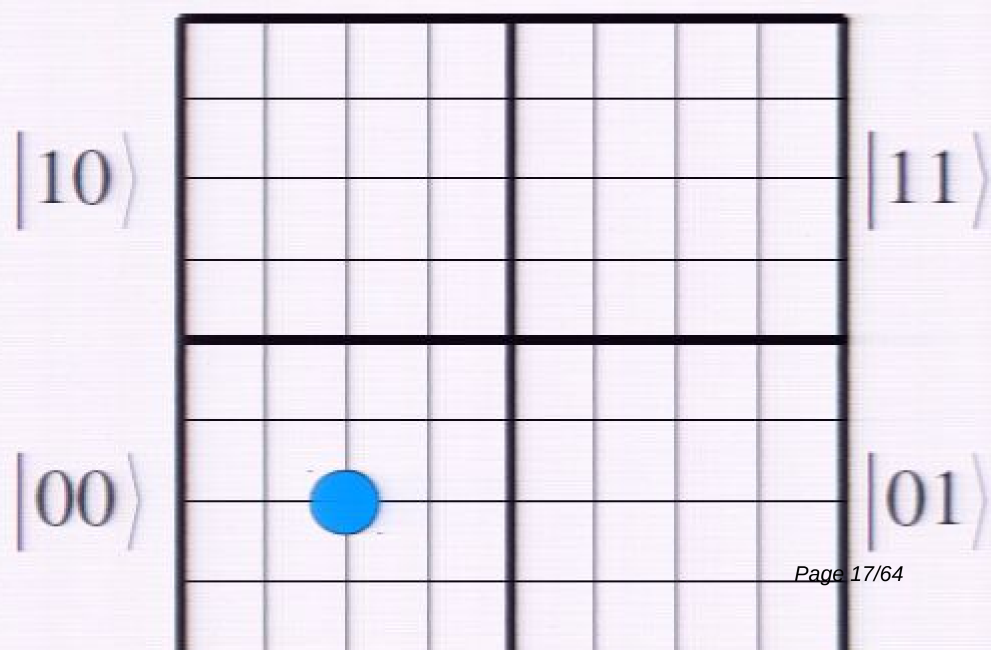
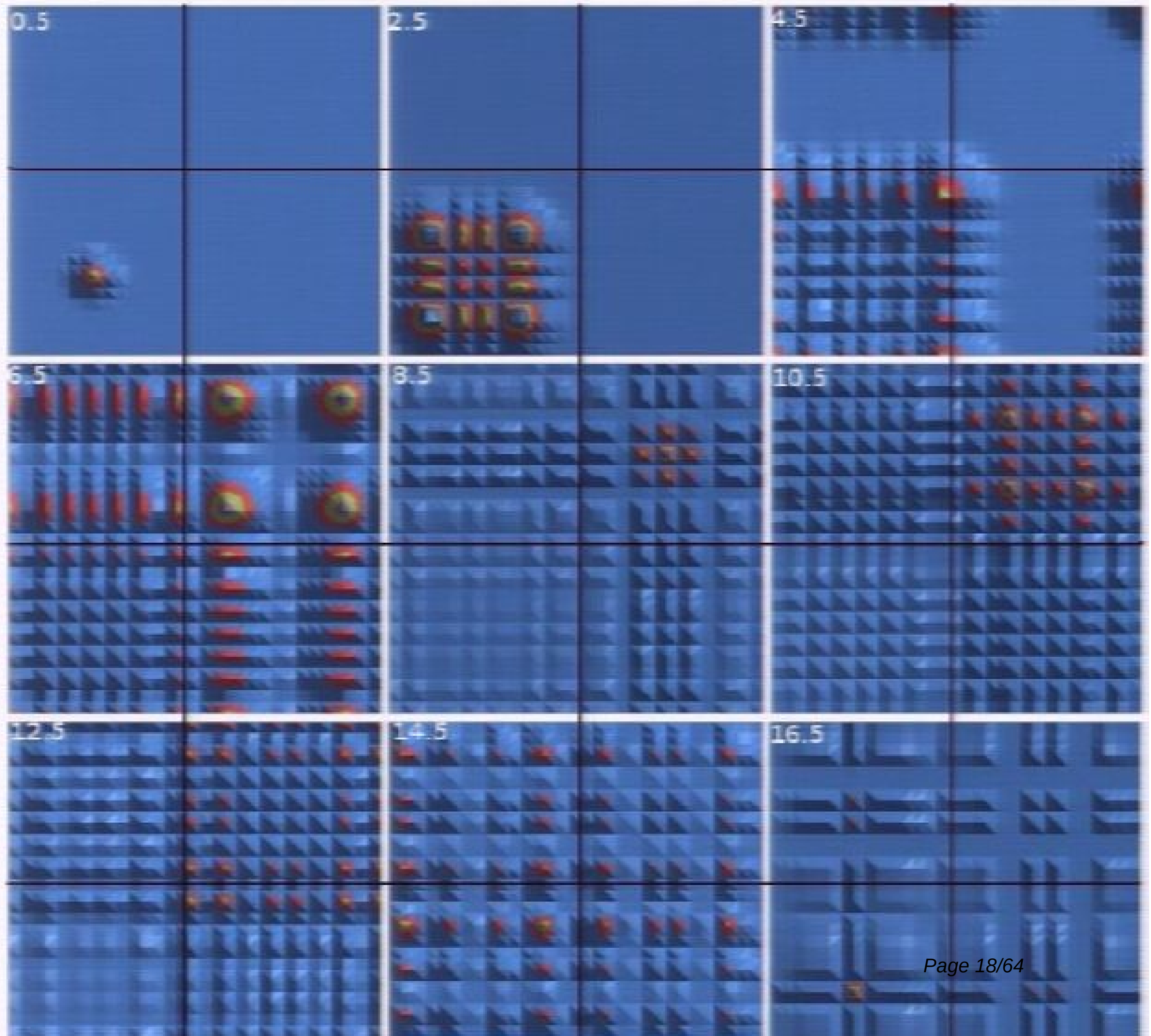


2D anyonic quantum walks

UNIVERSITY OF LEEDS

- Quantum walk of a single anyon on 16x16 square lattice
- Distribution spreads quickly
- Logical errors occur quickly
- Error on both qubits after time ht=L/2
- Any anyons therefore cause uncorrectable errors in linear time
- **Critical density** of anyons becomes **zero** in the presence of a **magnetic field**



Disorder and Localization

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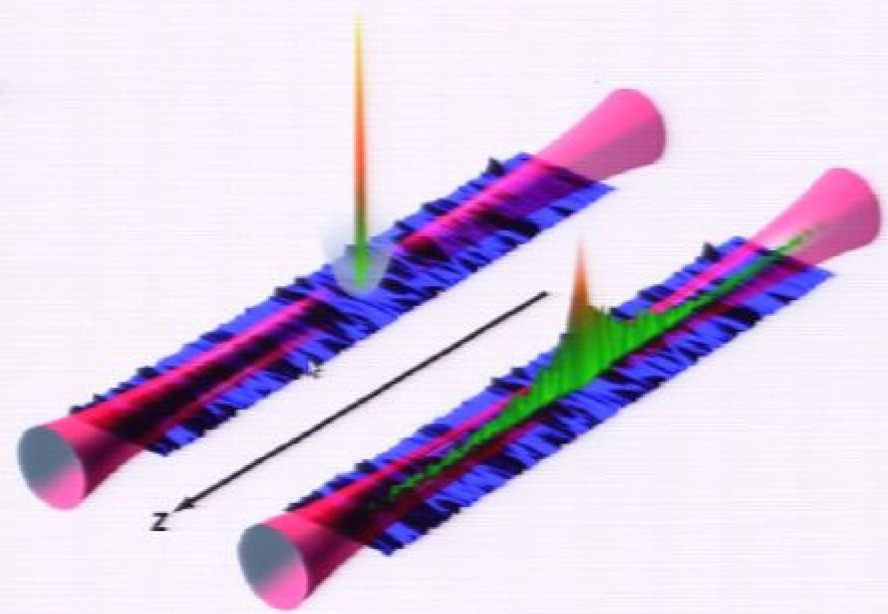
- **Disorder** slows quantum walks
- This should give the memory a **longer lifetime**
- **Anderson localization** may also be induced
- **Universal effect** in wave propagation
- Random interference **exponentially suppresses** motion
- Could this help regain finite critical anyon density?



Random Lattices

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- We first consider **random lattices**

- These are designed such that
  - -Number of spins per plaquette and vertex remains small
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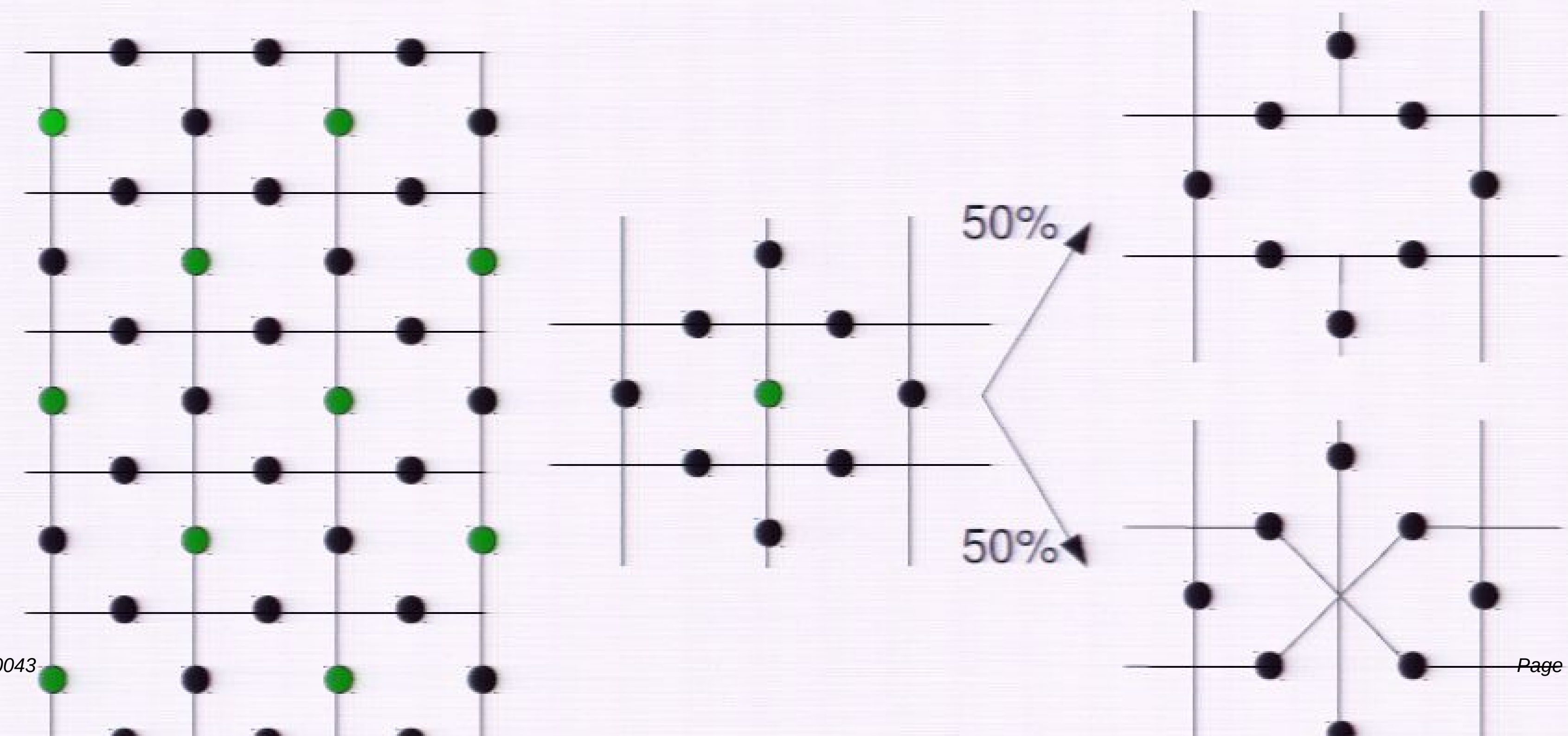

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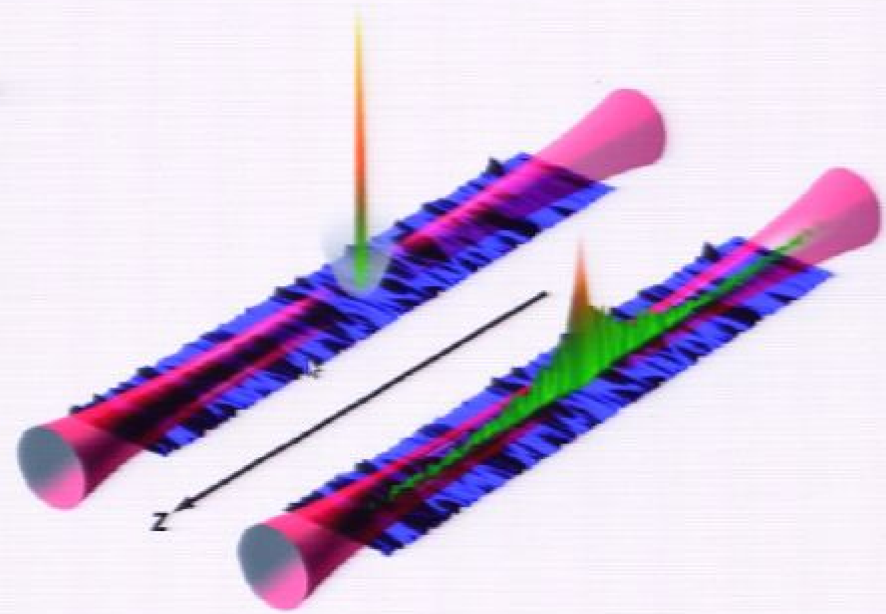

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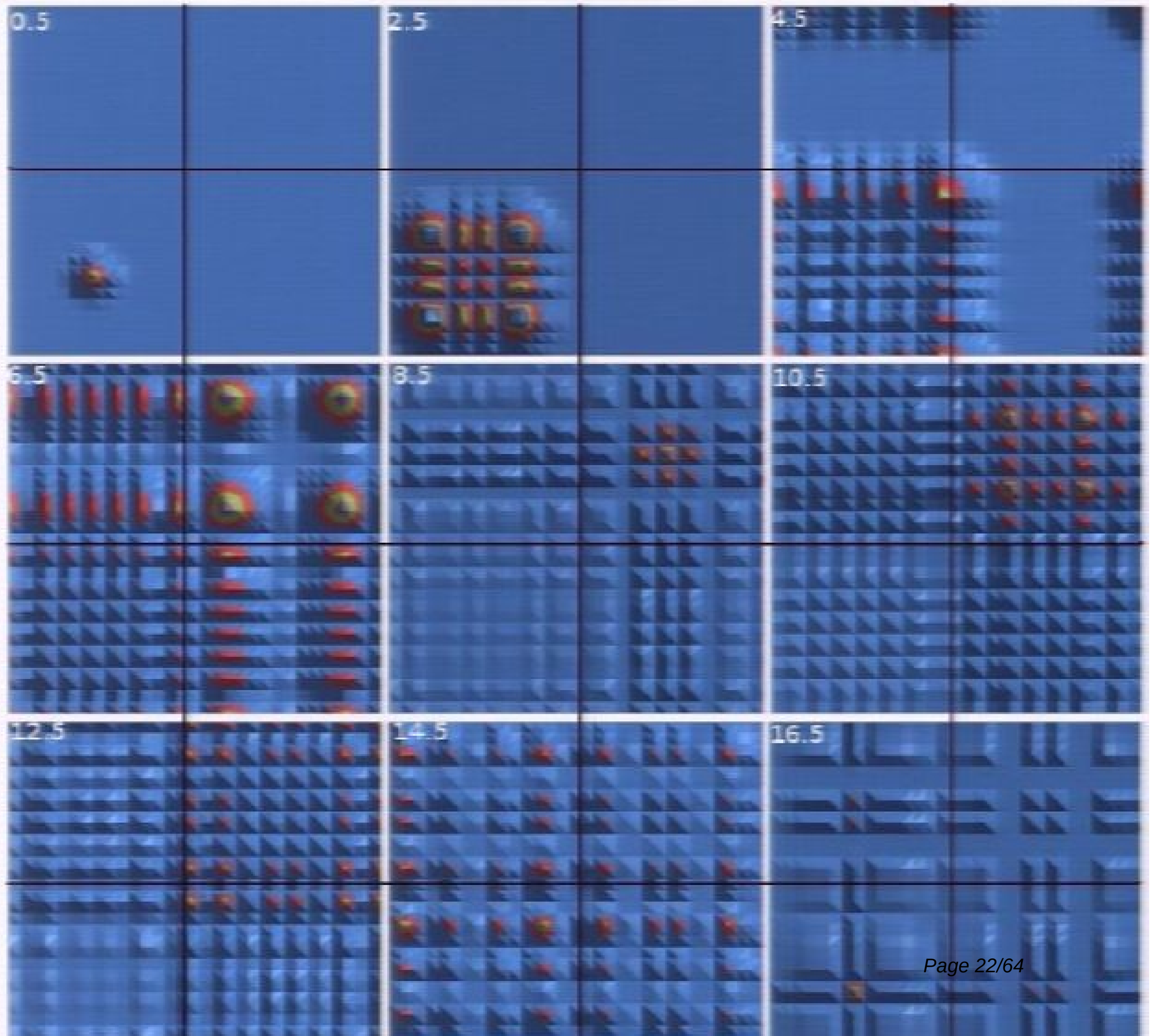

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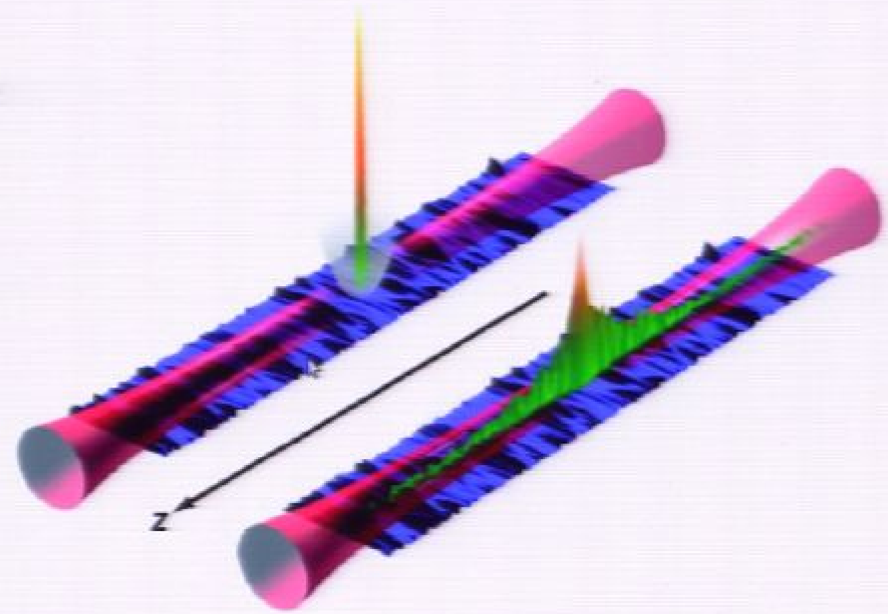

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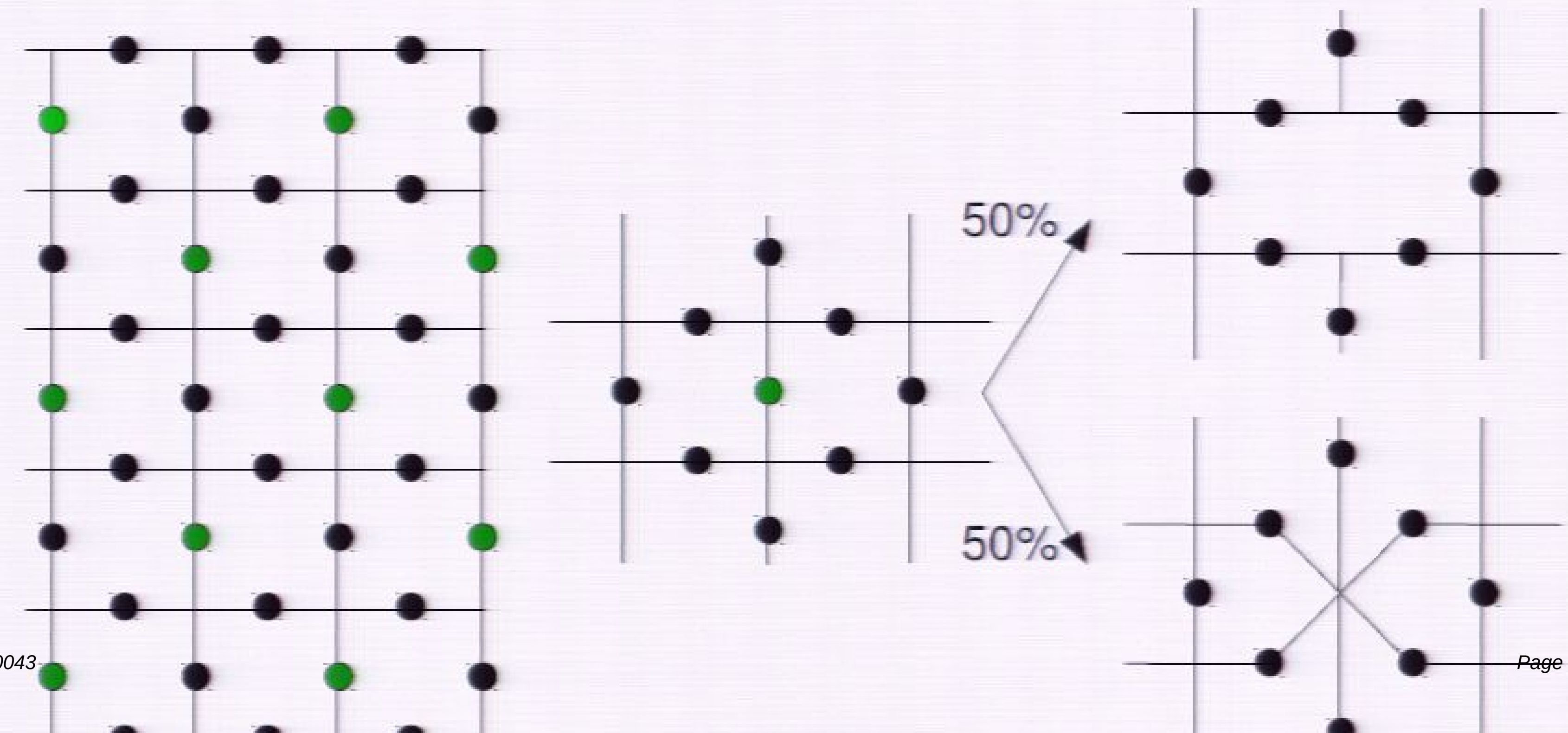
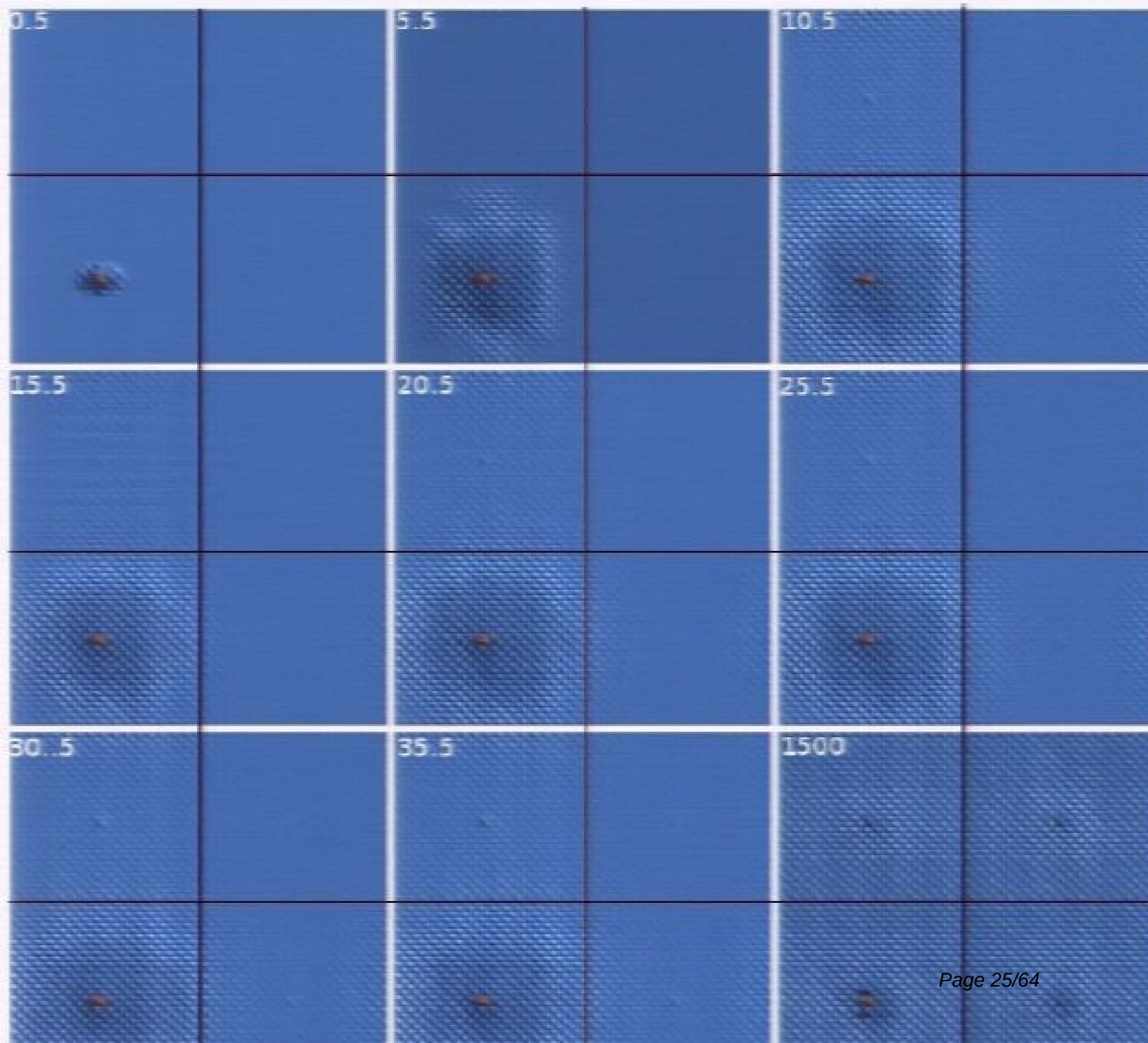


Random Lattices

UNIVERSITY OF LEEDS

- Quantum walk of a single anyon on a 32x32 random lattice
- Disorder leads the walk to slow significantly
- Logical errors take much longer to build up



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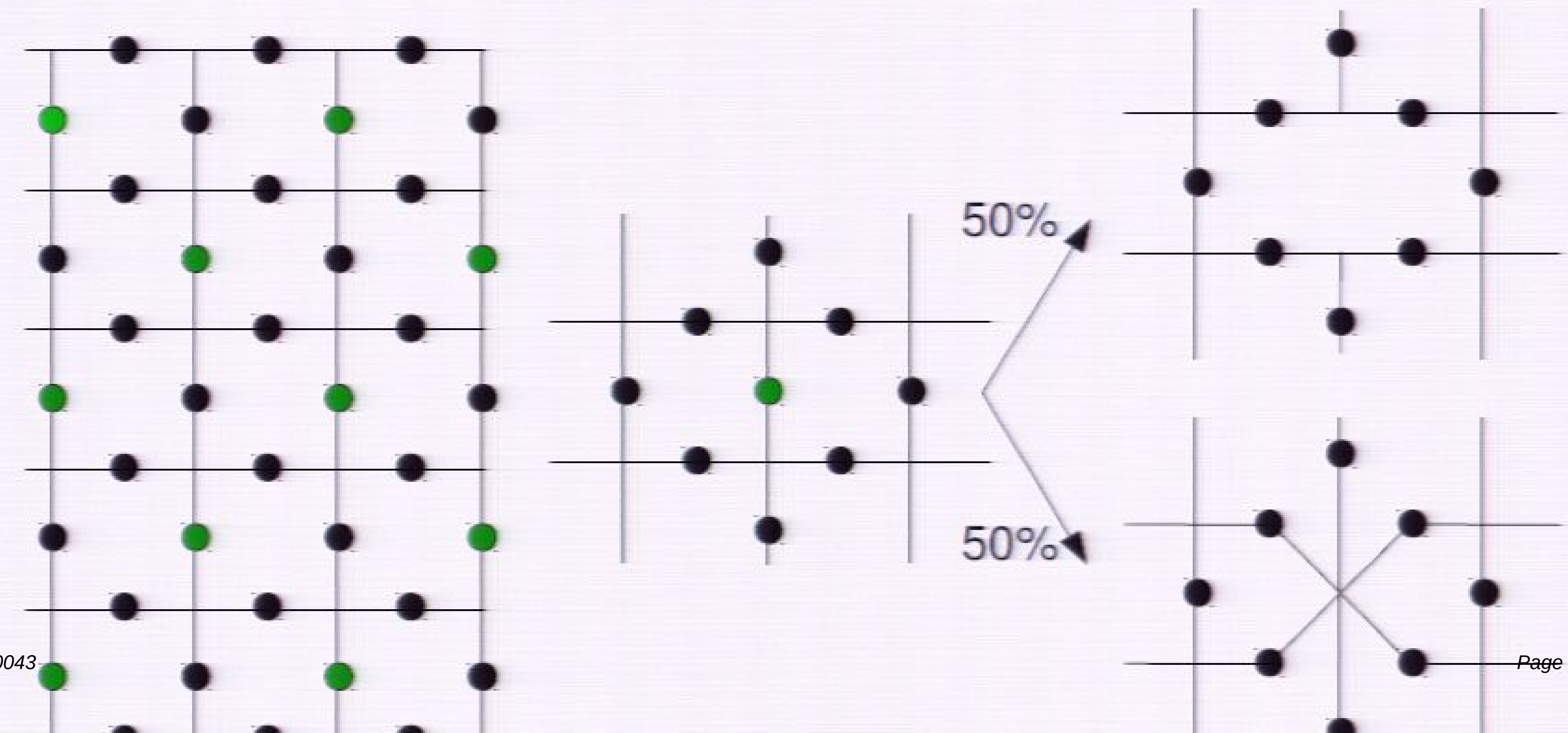

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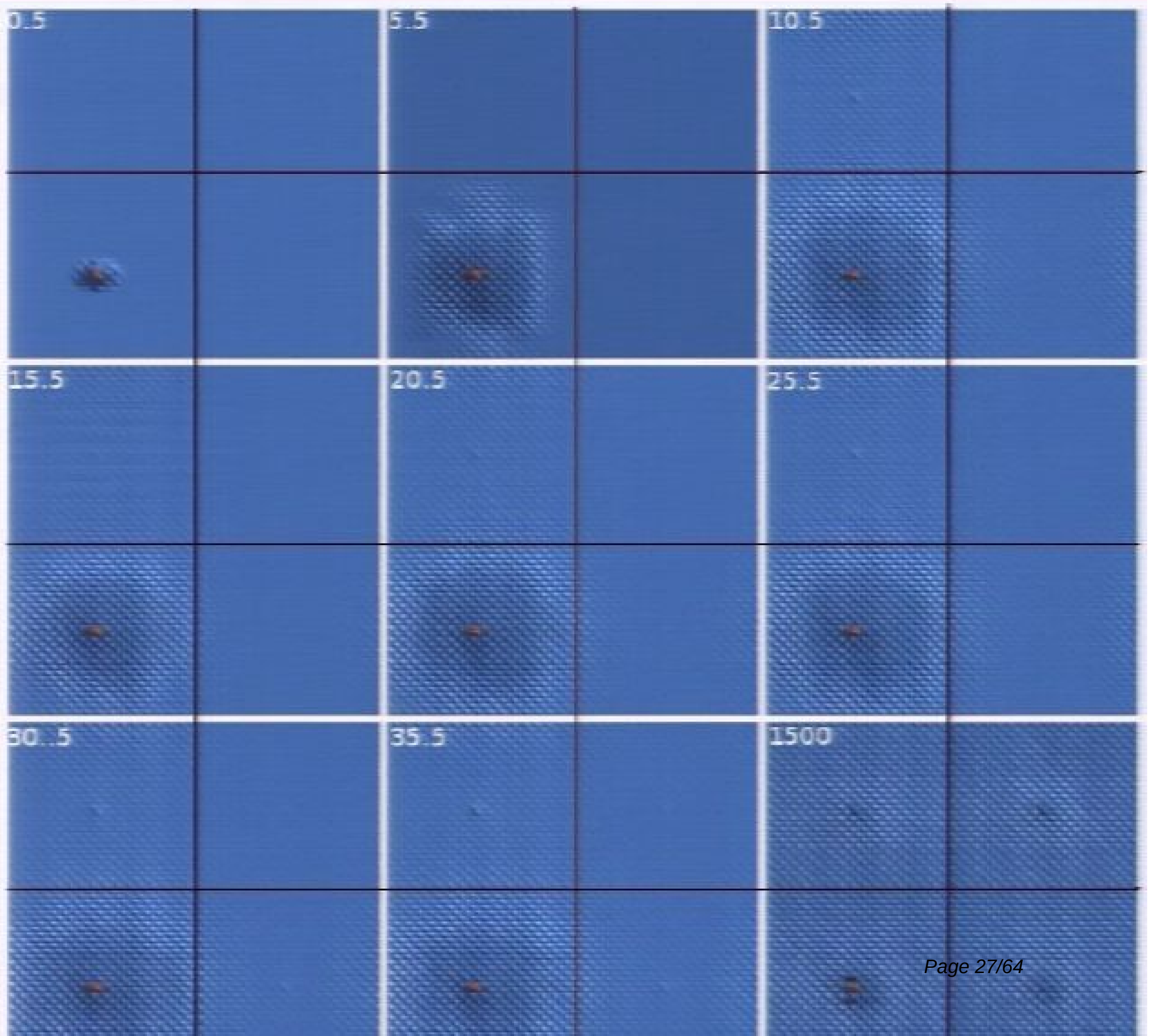

Random Lattices

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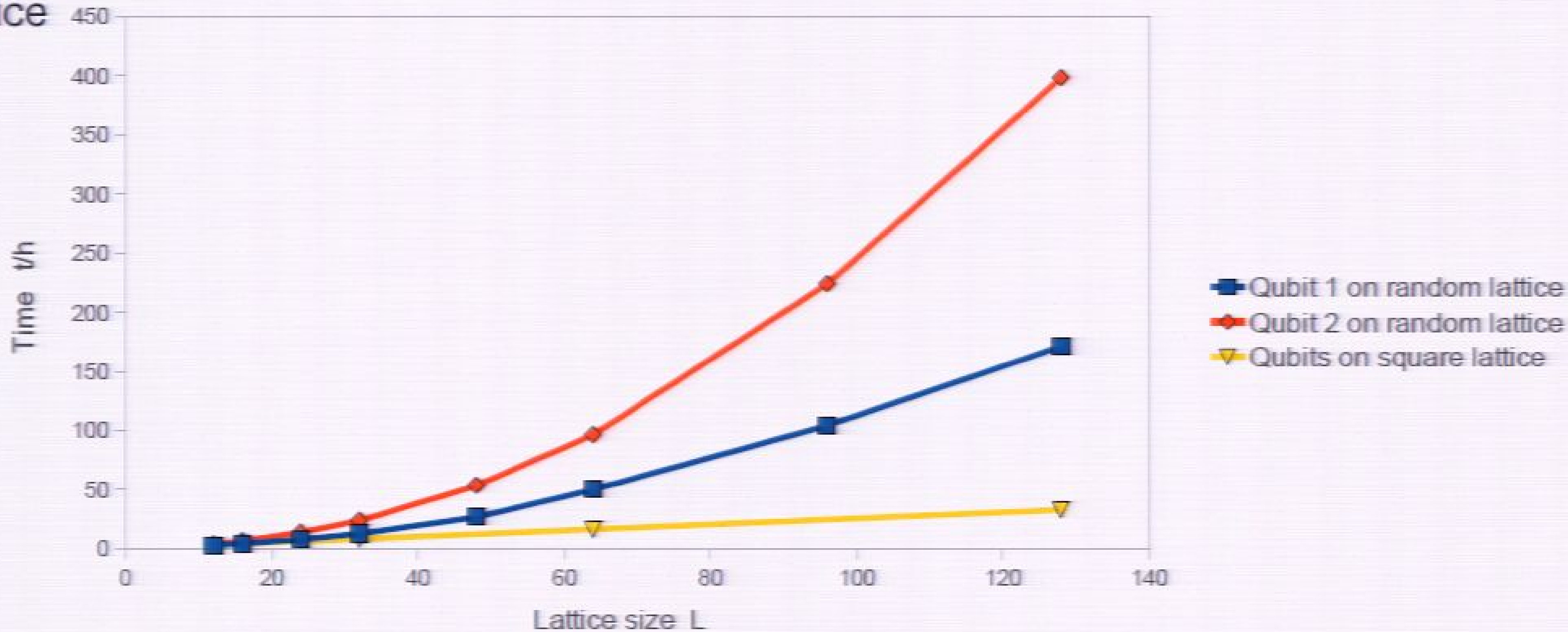
•The speed at which errors build up can be seen from the time taken until the error probability becomes p=0.1

•This increases linearly with L on the square lattice, but polynomially for the random lattice



•The memory still fails, and the critical density is still zero, but the lifetime is greatly increased by disorder

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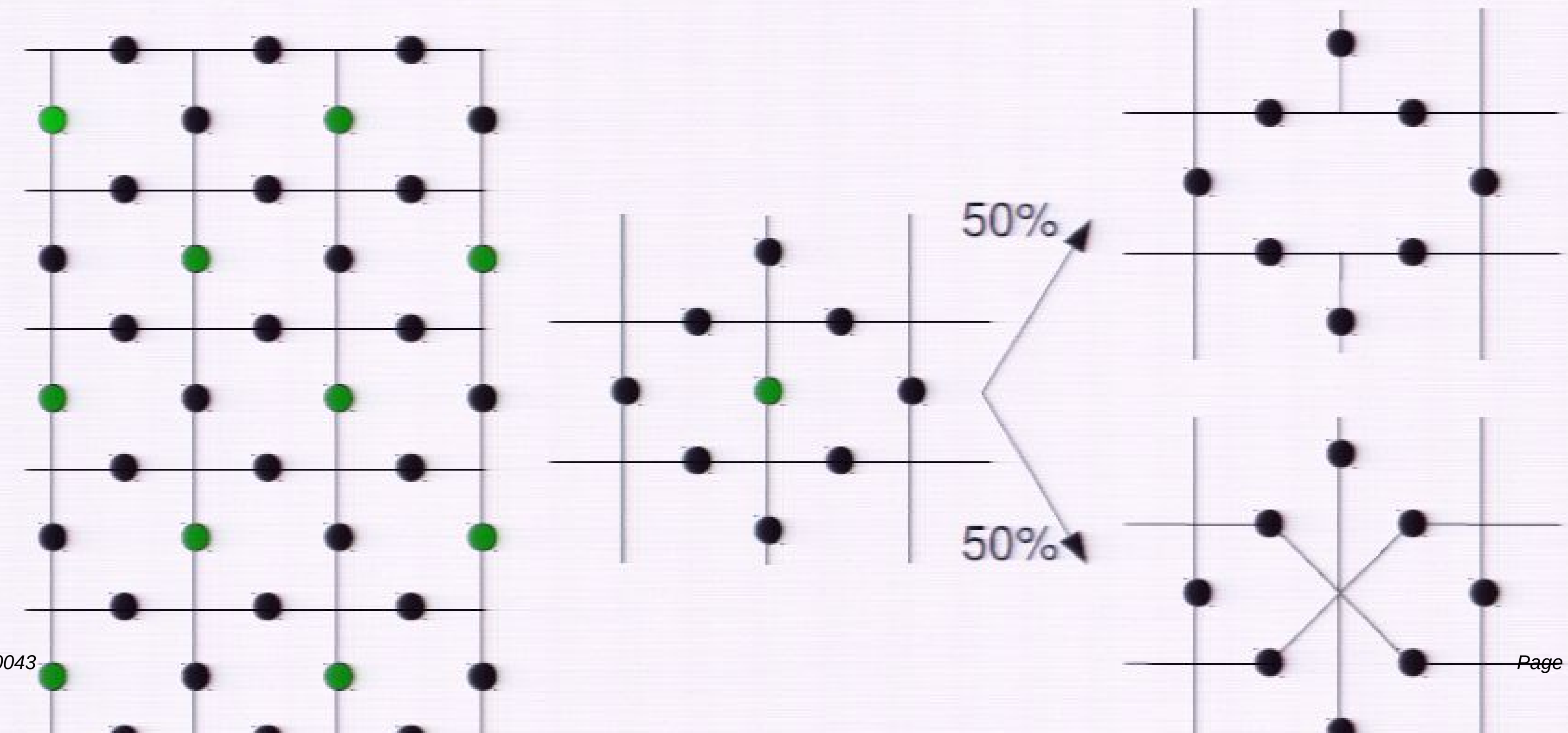

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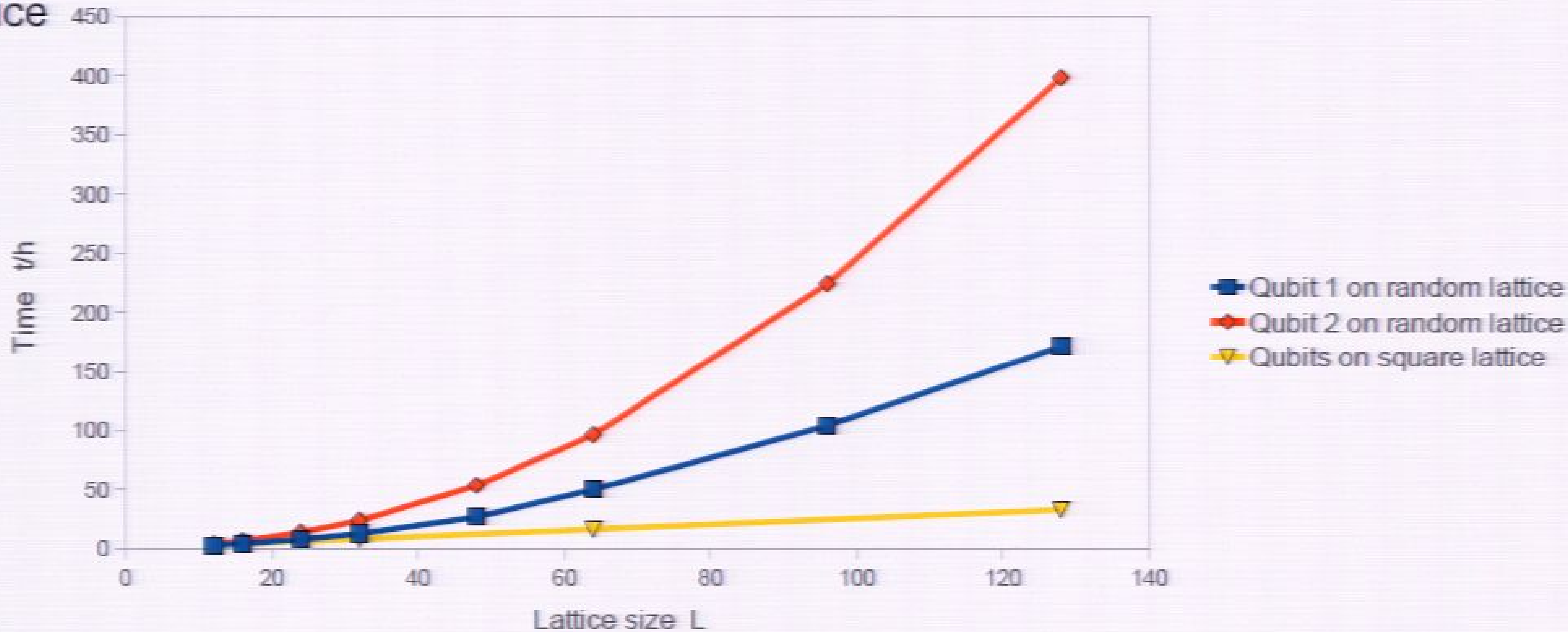

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•Note also that only ¾ of the spins are used

The toric code

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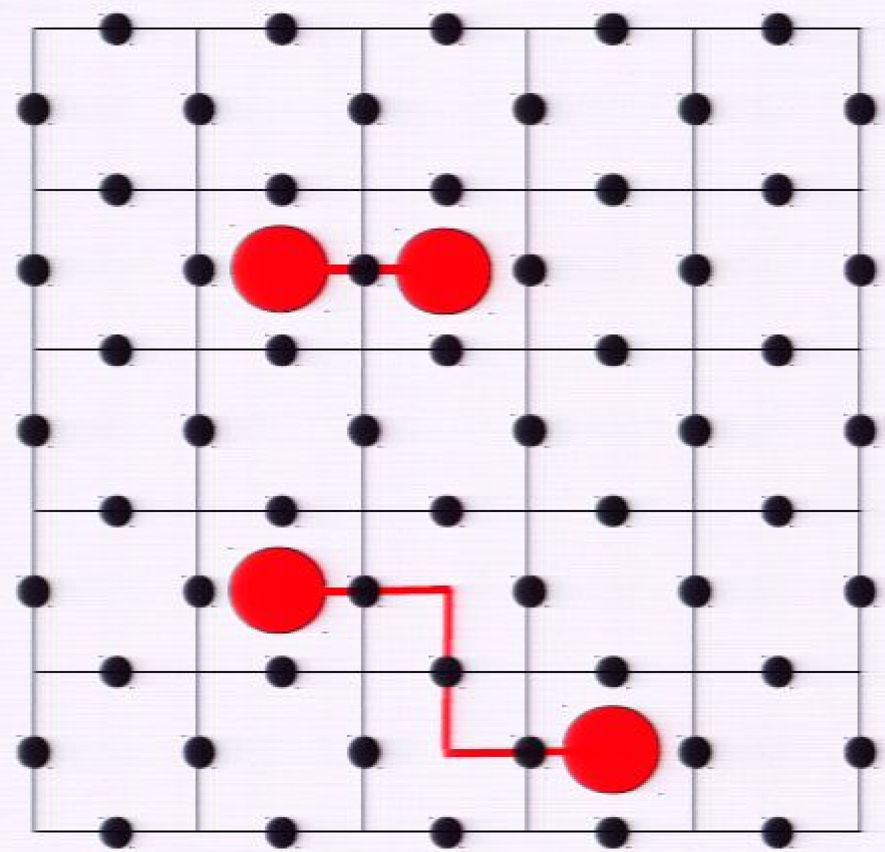
- Created and moved by σ_i^x operations



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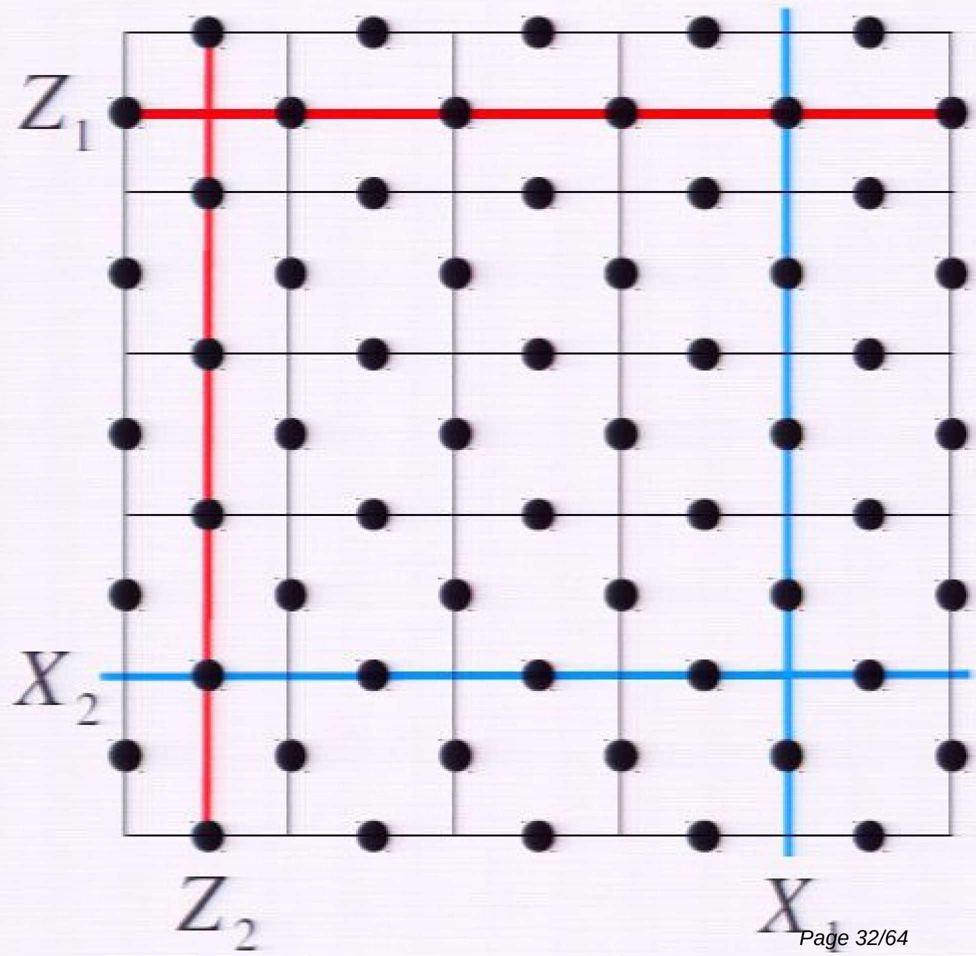
- Logical operations correspond to moving anyons around the torus in topologically non-trivial loops



- Trivial loops have no effect on logical qubits – equivalent to stabilizers

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$$\rho_c \approx 0.31$$

The toric code

Swipe your finger slower

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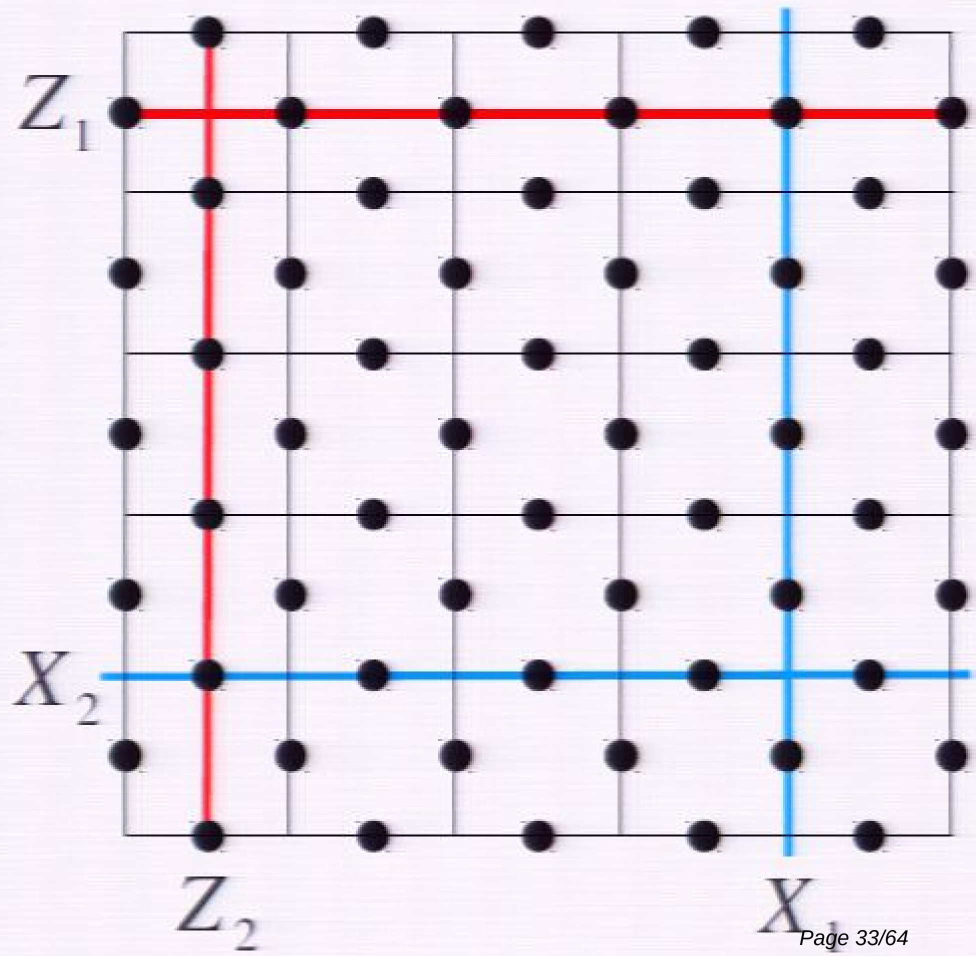

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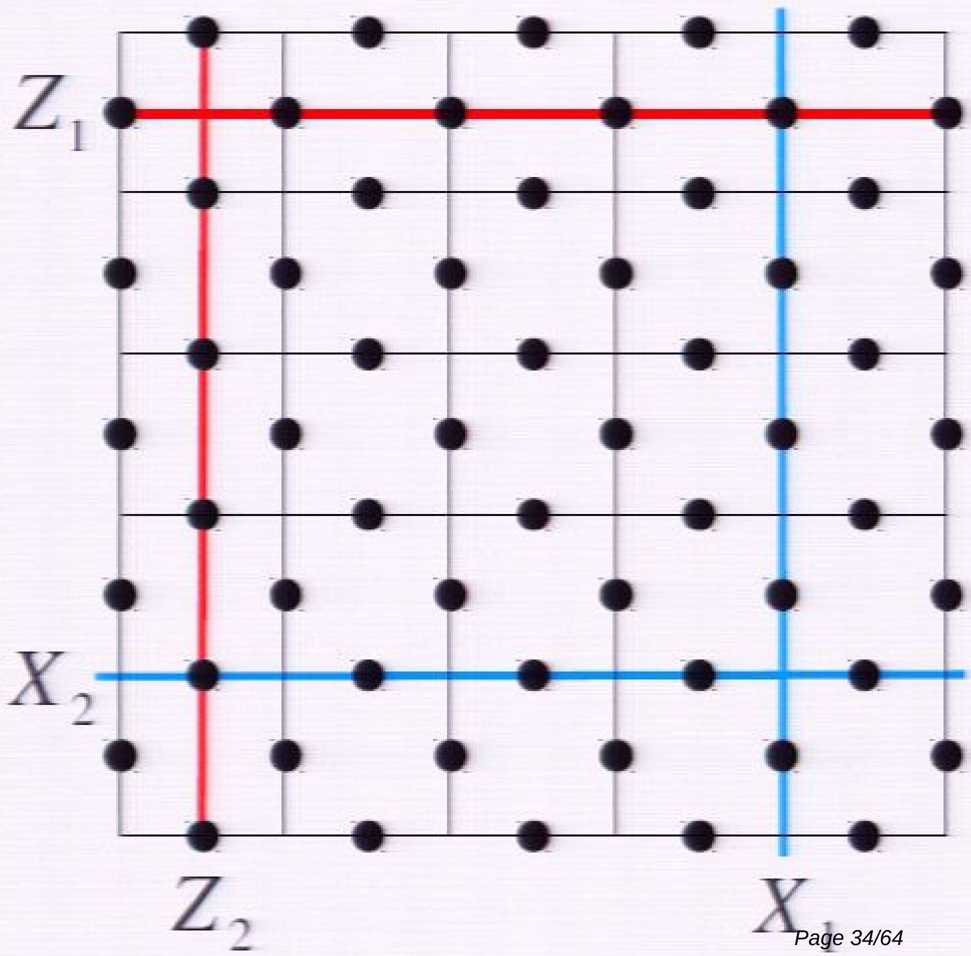

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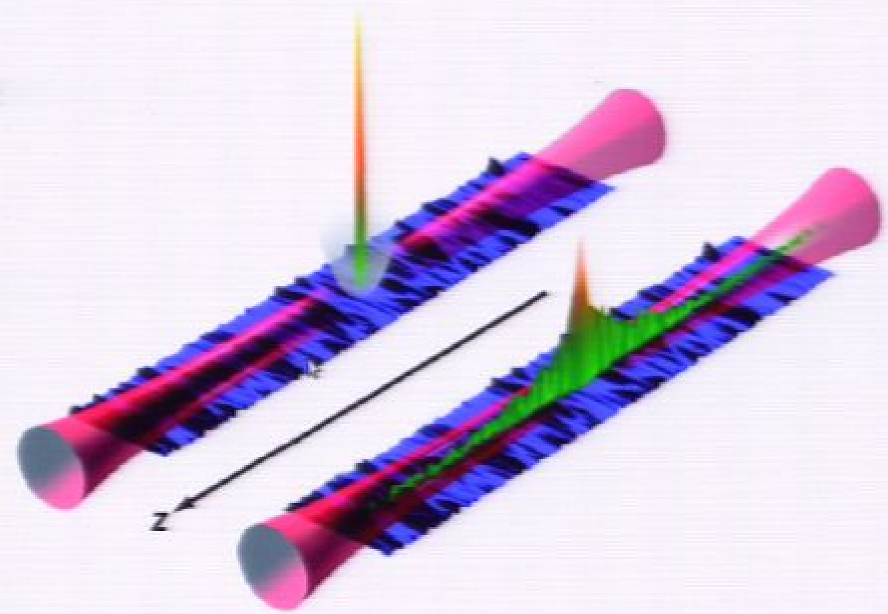

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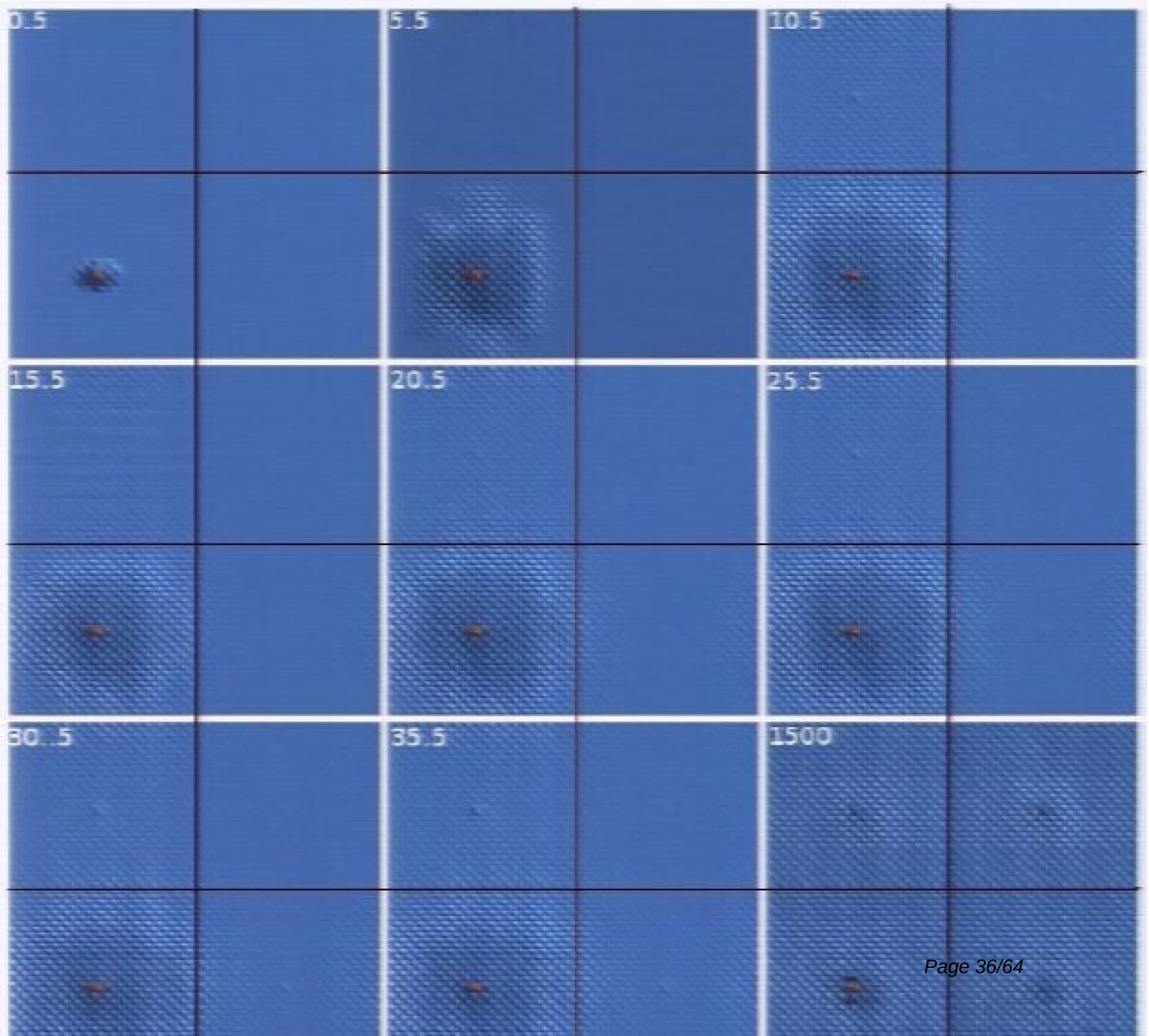

Disorder in Couplings

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- We also consider **disorder in the J couplings** of the toric code Hamiltonian

- This must be done with care. To suppress anyon creation, J must always be set as high as possible.

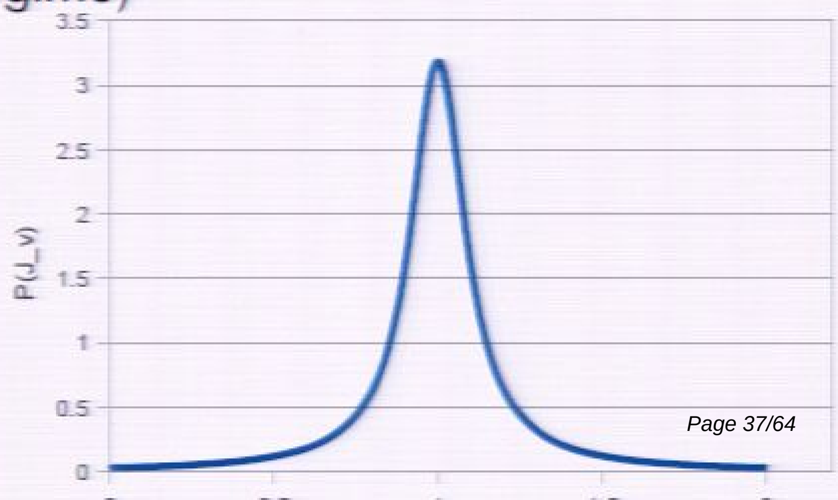
- Purposefully introducing disorder will mean lowering the J's on some vertices, and hence providing **nucleation points for anyons**

- This must be avoided, and so only the disorder inherent in any physical realization is considered (here we consider this parametric regime)

- Modelled as Cauchy distribution

$$M_{v,v'} = J_v \delta_{v,v'} + h \delta_{\langle v,v' \rangle}$$

$$P(J_v) = \frac{1}{\pi} \frac{\gamma}{(J_v - J)^2 + \gamma^2}$$



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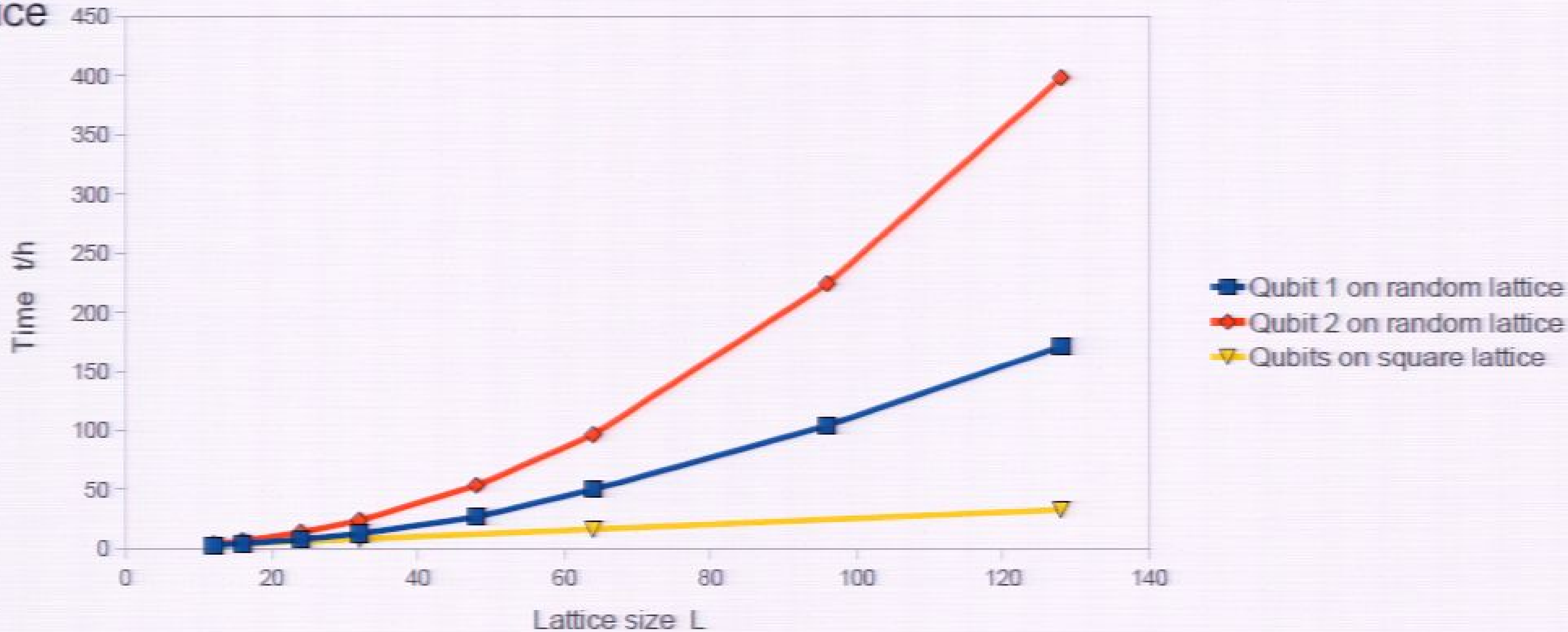

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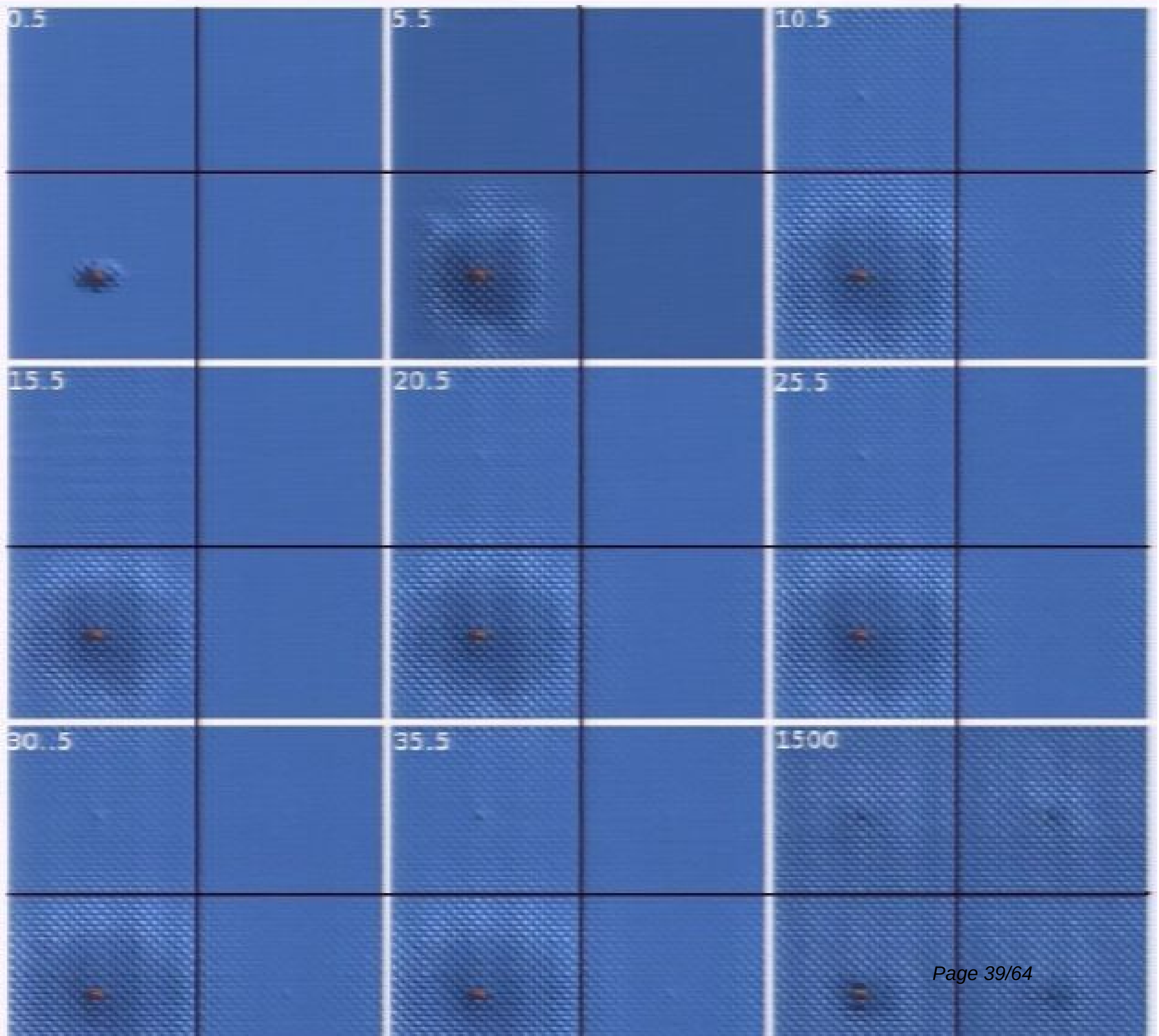

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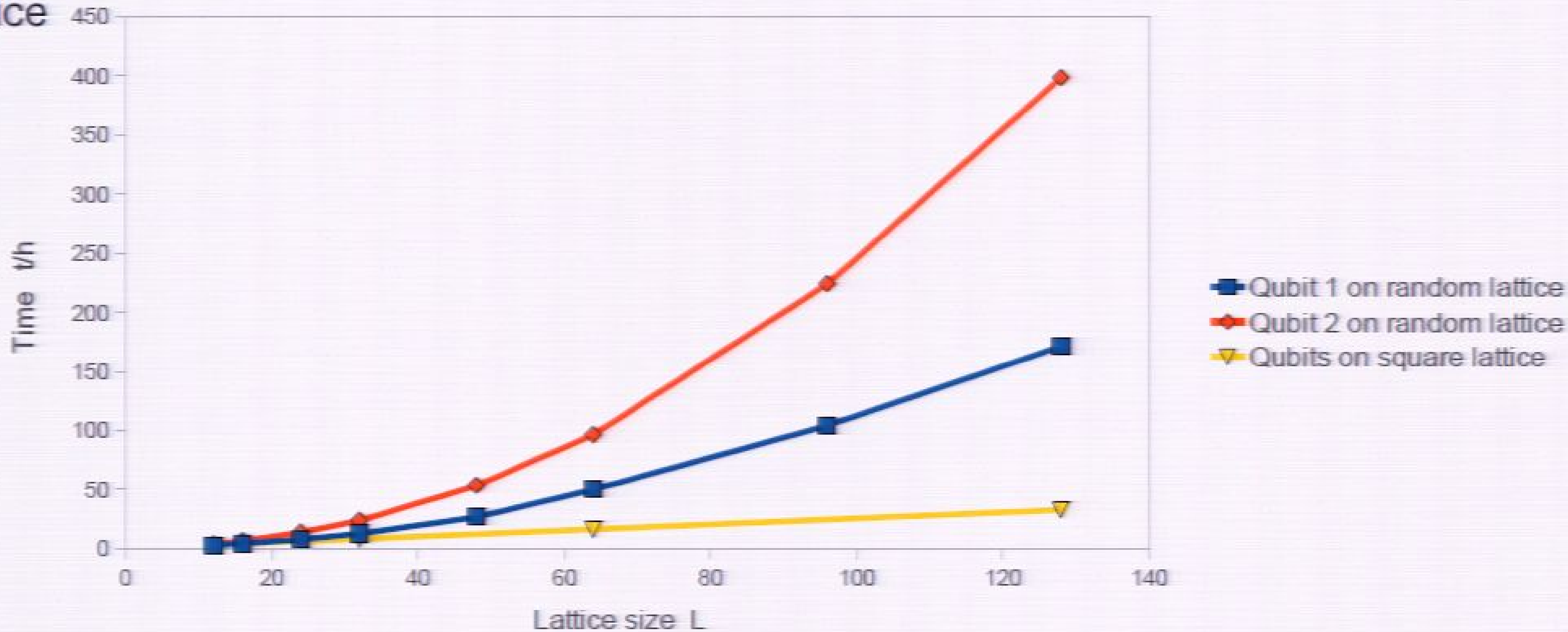

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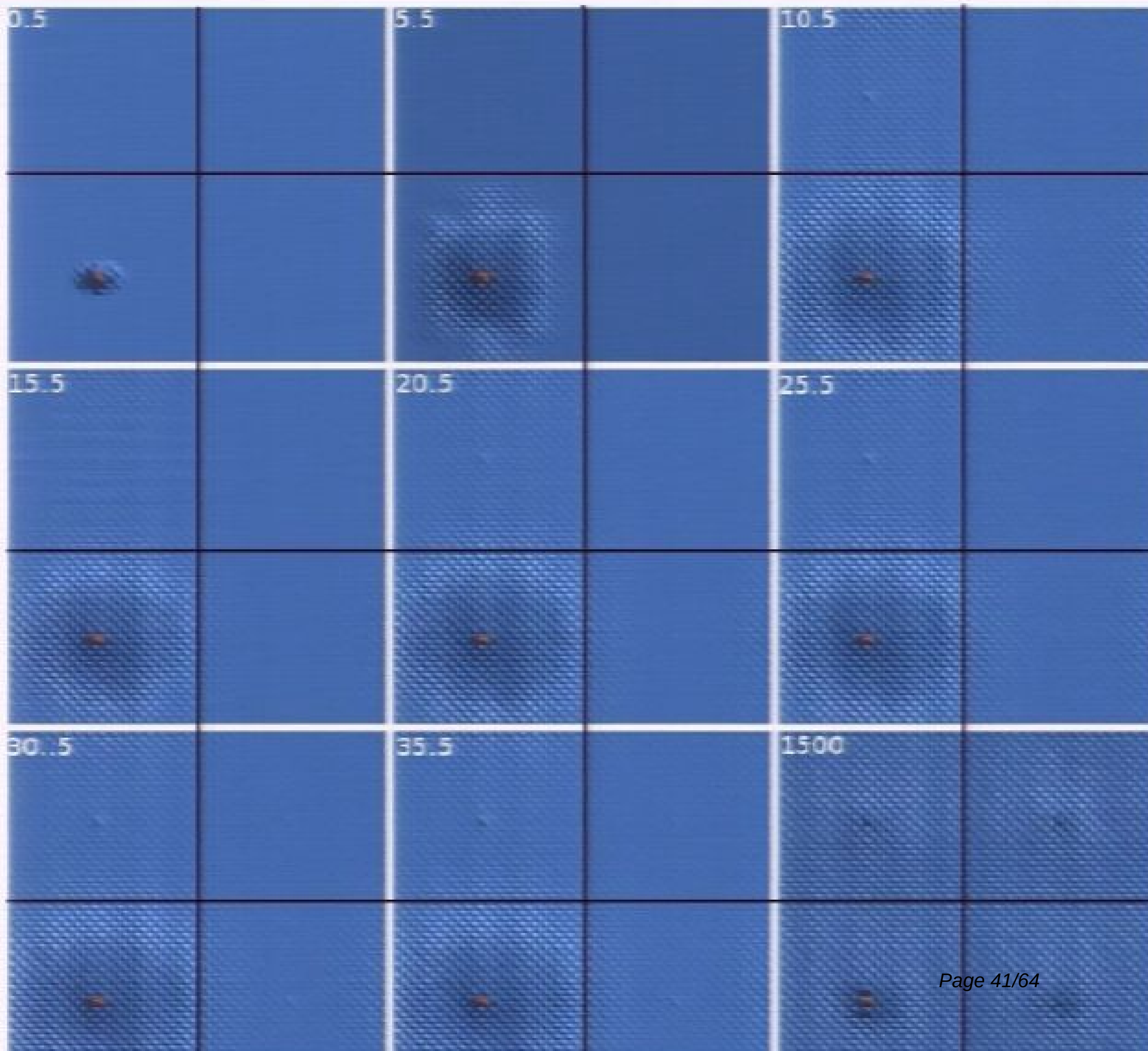

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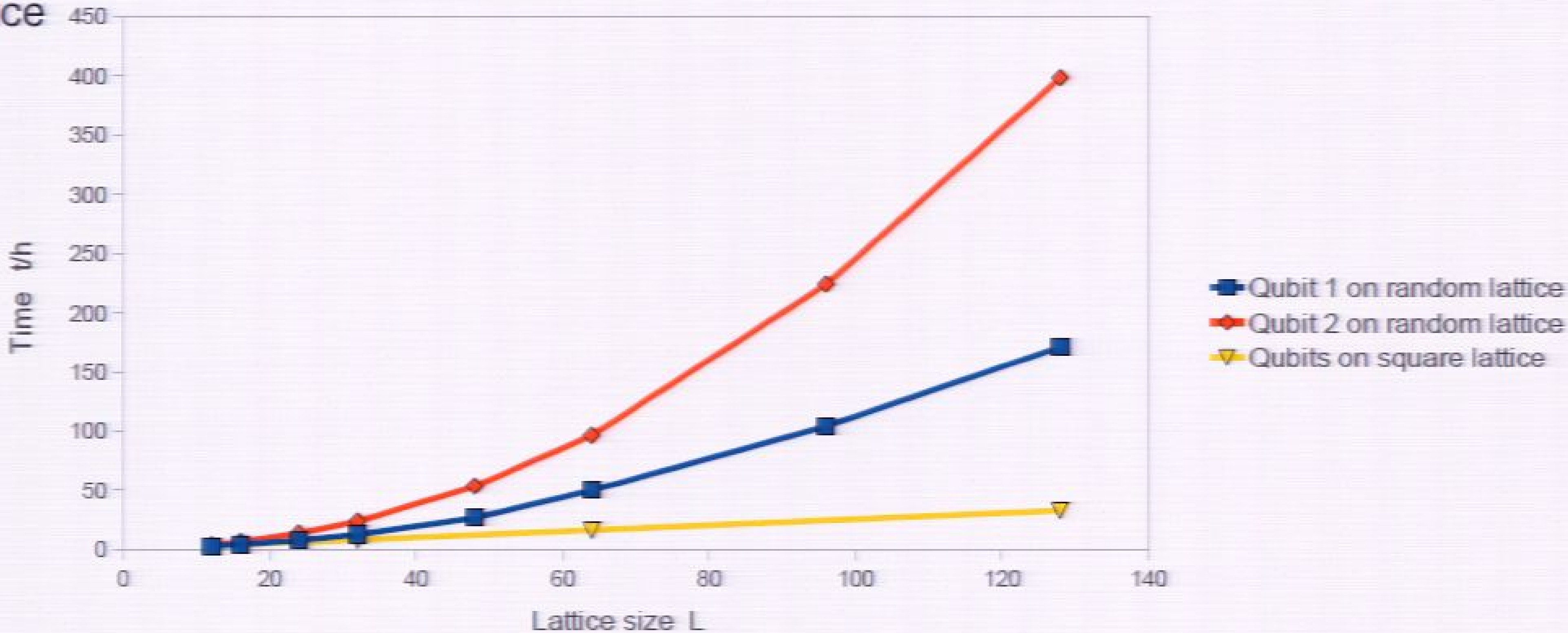

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The End

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- Thank you for your attention

Disorder and Error Suppression

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- Two walker Hamiltonian was diagonalized
- Probability distribution derived from each eigenstate
- Localization length of the eigenstate taken to be s.d. of distribution
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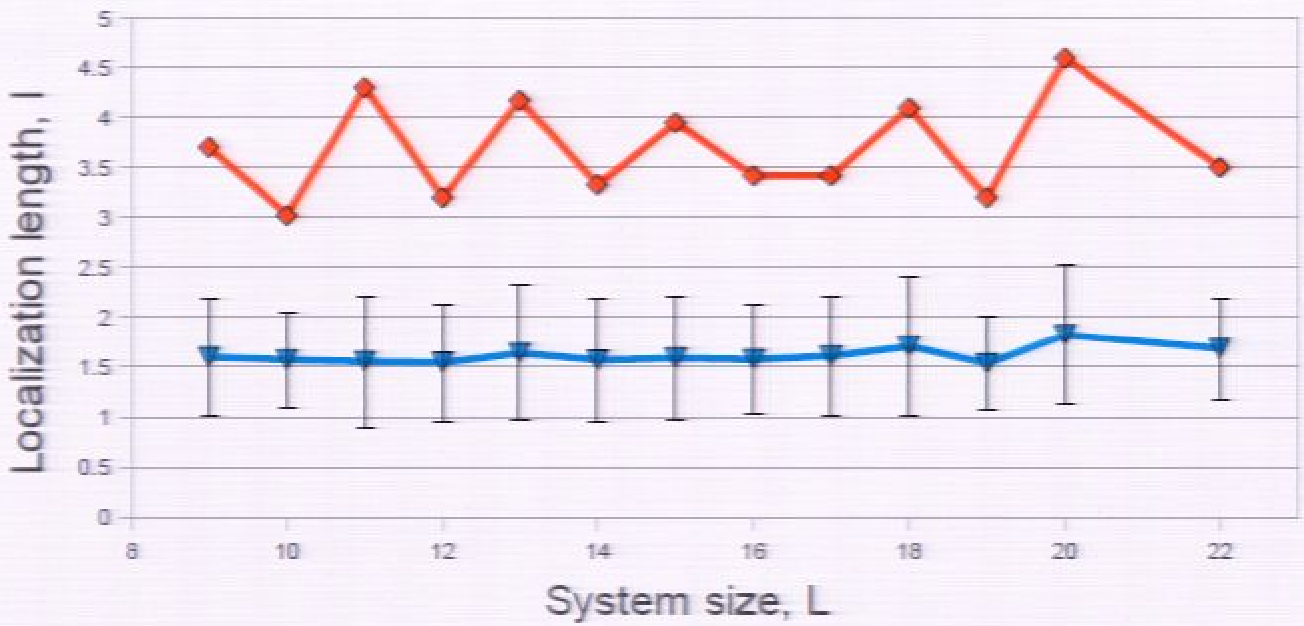

Disorder and Error Suppression

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•Localization lengths in 2D can be very large

•If too large, it may not be realistic to build codes big enough to benefit from localization

•Critical anyon density, though non-zero, will then become very small

•It's therefore important to determine the typical values of *l* for the toric code

•We consider the square lattice with

$$\gamma = J/10 \qquad h = J/100$$

•Specific value of J is unimportant

Disorder in Couplings

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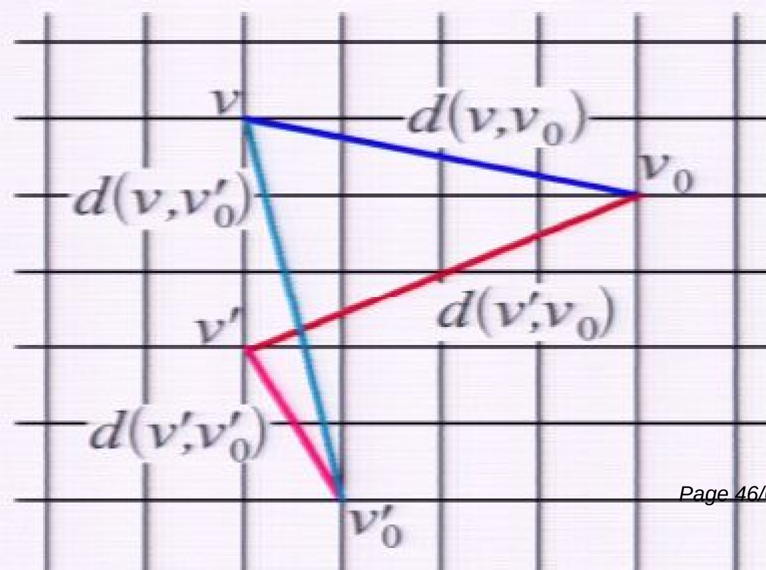
- Bound on eigenstates defined

$$\left|\left\langle v\, v' \middle| E_{v_0 v_0'}\right\rangle\right| < \exp\left[\frac{-d(v, v'; v_0, v_0')}{2 l_{v_0, v_0'}}\right]$$

$$d(v, v'; v_0, v_0') = min\left[d(v, v_0) + d(v', v_0'), d(v, v_0') + d(v', v_0)\right]$$

- Hamiltonian localization length l defined as maximum of all $l_{v_0, v_0'}$



Disorder in Couplings

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- We also consider **disorder in the J couplings** of the toric code Hamiltonian

- This must be done with care. To suppress anyon creation, J must always be set as high as possible.

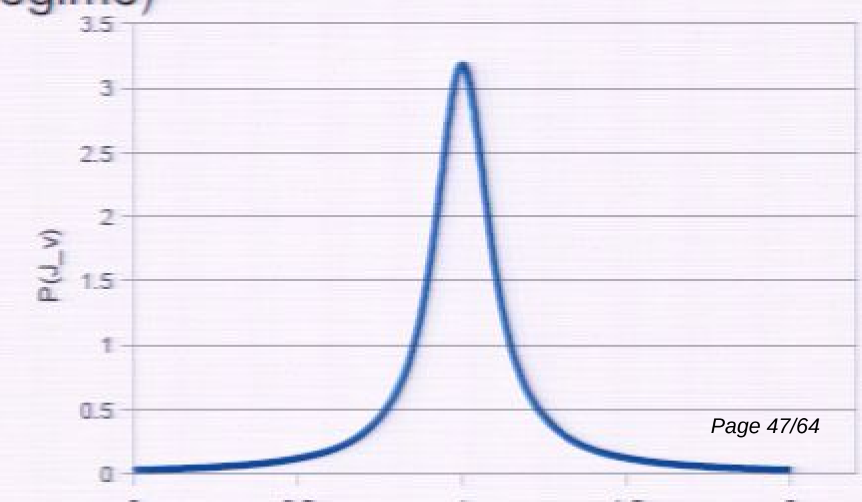
- Purposefully introducing disorder will mean lowering the J's on some vertices, and hence providing **nucleation points for anyons**

- This must be avoided, and so only the disorder inherent in any physical realization is considered (here we consider this parametric regime)

- Modelled as Cauchy distribution

$$M_{v,v'} = J_v \delta_{v,v'} + h \delta_{\langle v,v' \rangle}$$

$$P(J_v) = \frac{1}{\pi} \frac{\gamma}{(J_v - J)^2 + \gamma^2}$$



Disorder in Couplings

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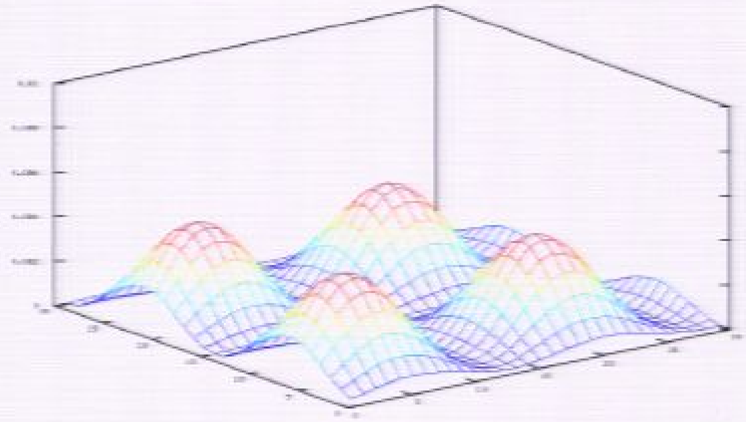
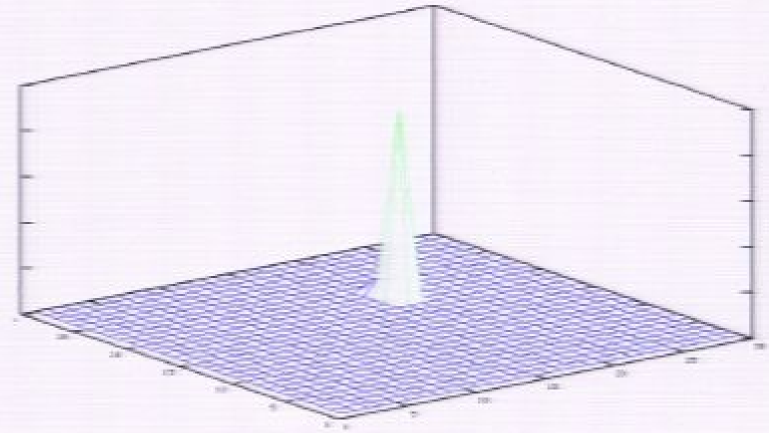
- Theory suggests that Anderson localization will occur for disorder in the J's

- Eigenstates of walker Hamiltonians become exponentially localized





- To truly consider whether localization occurs, should consider more than M

- Sparse configuration of anyon pairs, so should consider two walker Hamiltonian

Disorder in Couplings

UNIVERSITY OF LEEDS

- Bound on eigenstates defined

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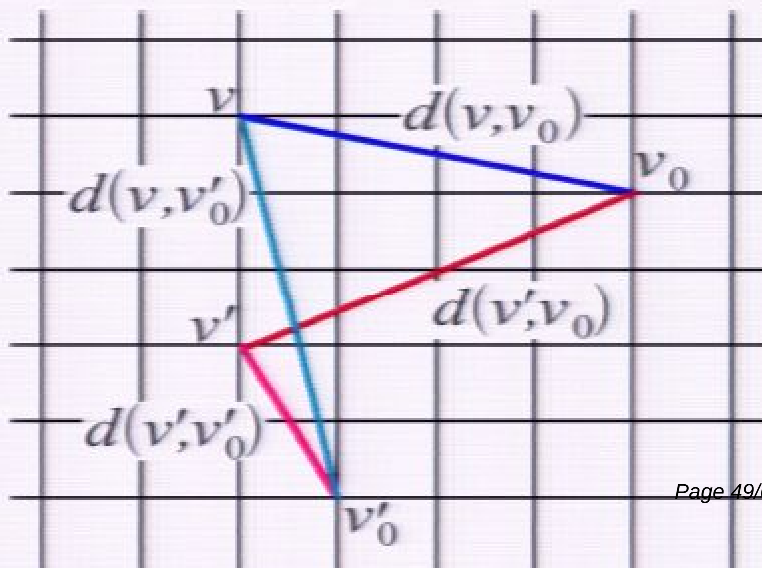

Disorder in Couplings

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- Localized eigenstates prevent the walkers moving freely

- Motion of the walker is exponentially suppressed

$$P_{v,v'}(t) < L^8 e^{-d/l}$$

- Anyons are bound to an area of radius ~*l* around their starting position at all times t.

- This allows a finite anyon density to be tolerable, even in the presence of the field

$$\rho_c \ll \frac{1}{l^2}$$

Disorder and Error Suppression

UNIVERSITY OF LEEDS

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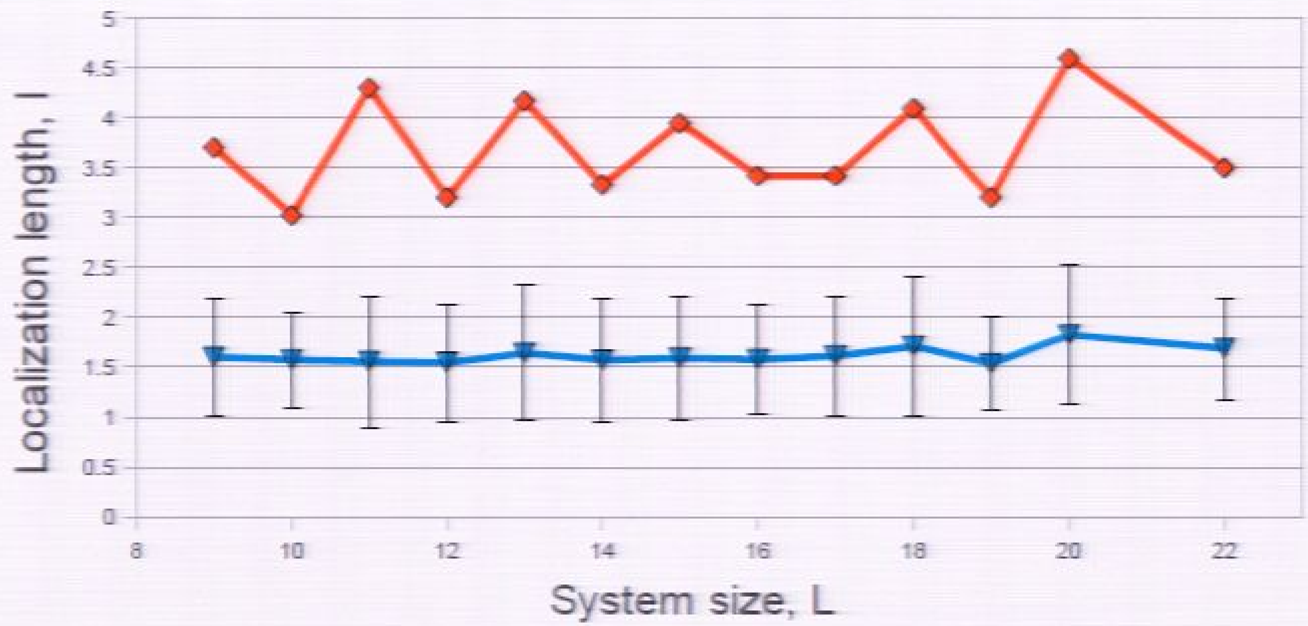

Disorder and Error Suppression

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In order to probe larger system sizes, the one walker Hamiltonian was also considered

It is reasonable to suppose that the value of *l* here is of the same order as the two walker case

Over all samples, no value of *l* greater than *l=6* is found, hence

$$\rho_c \ll 10^{-2} \qquad \rho_c < 10^{-3}$$

This suggests the error threshold is similar to that in other schemes

Disorder and Error Suppression

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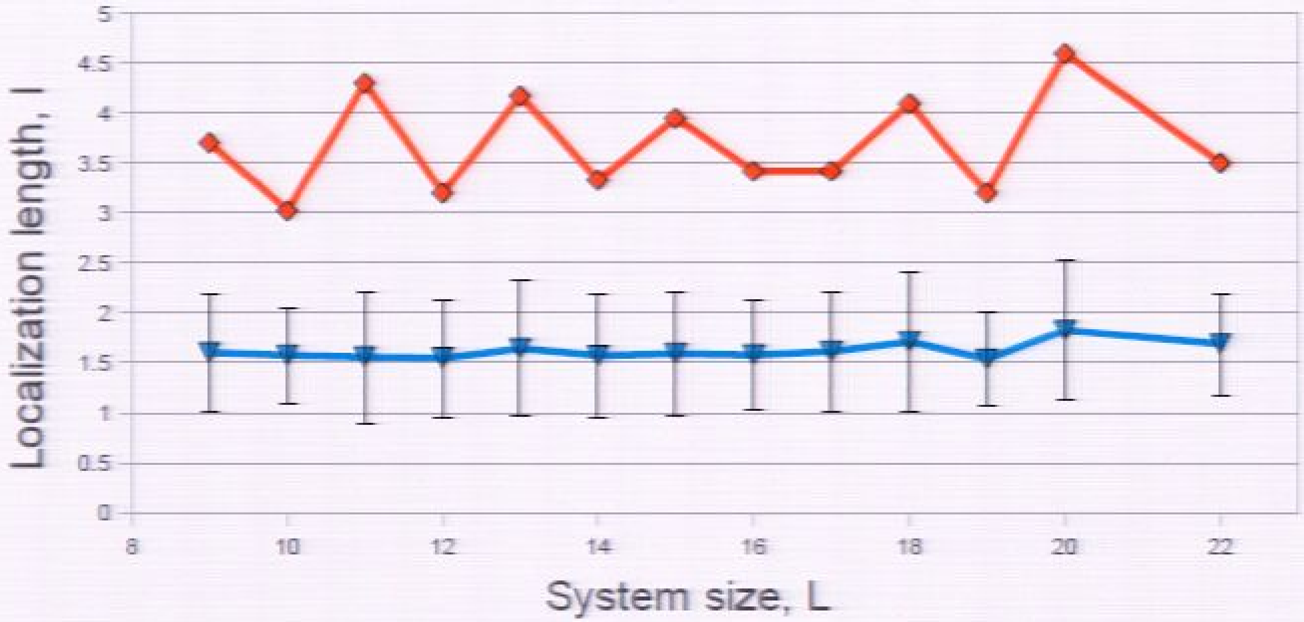

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Disorder and Error Suppression

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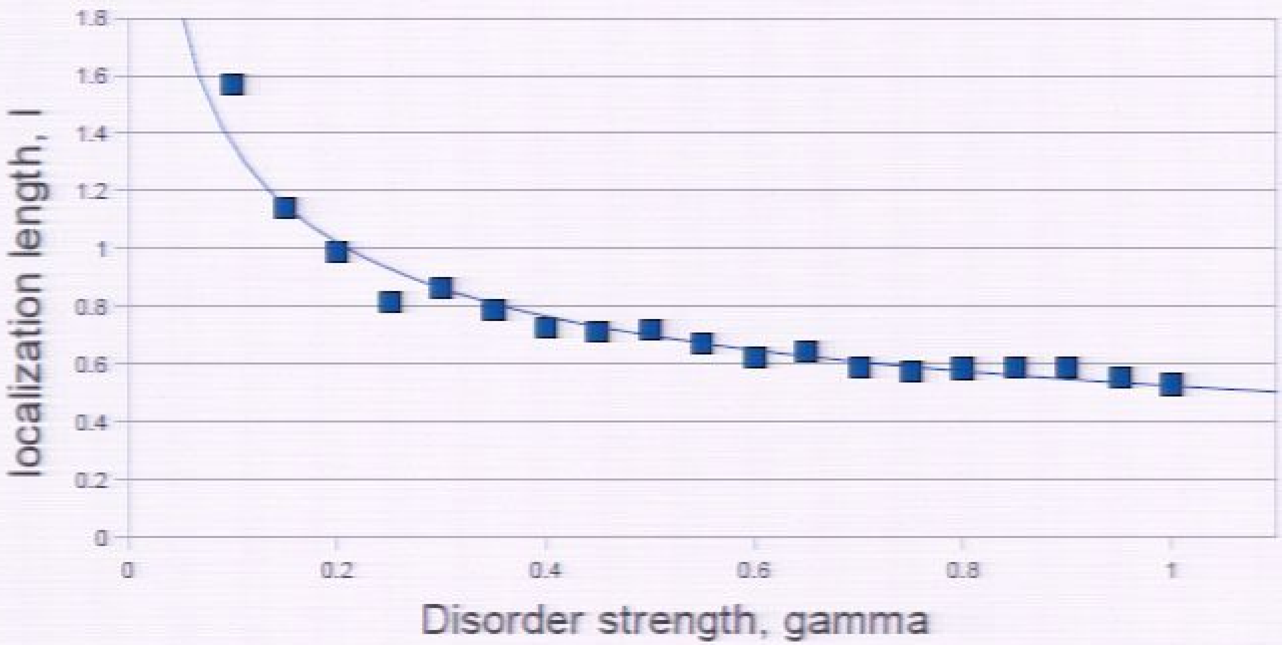
Other disorder strengths were also considered



$$l \sim \gamma^{-2}$$

Length is not strongly suppressed by increasing gamma

Must be careful if doing purposefully

Use in conjunction with random graphs may help

Conclusions

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- **Magnetic fields are fatal** for the toric code, inducing quantum walks

- This destroys memory in **linear time**, and sets the **critical anyon density** to **zero**

- Using a random graph slows these walks, increasing the life of the memory to **polynomial time**

- Disorder inherent in the J's will also cause **Anderson localization**, exponentially suppressing anyon motion and allowing the **critical anyon density** to be **finite**

- Reasonable parameters show that this will be relatively strong, giving good suppression of errors

- Trade off between value of l and suppression of errors should always be kept in mind for physical realizations, so that disorder in J's is not reduced too much

Thermal Errors

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- Thermal errors induce classical random walks of anyons
- Anderson localization not possible
- However, random graphs may still have effect
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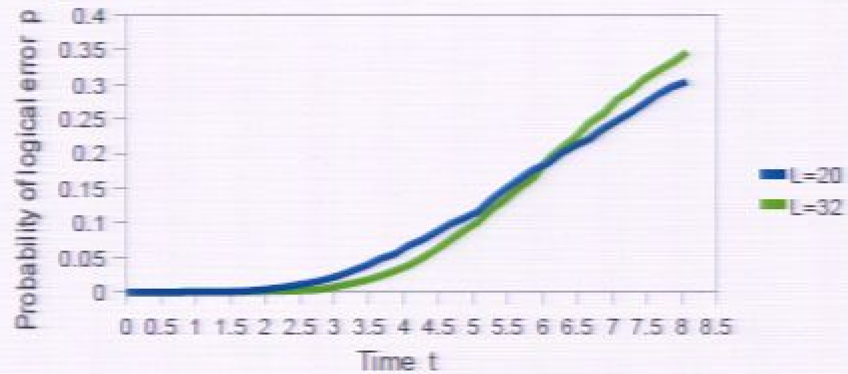

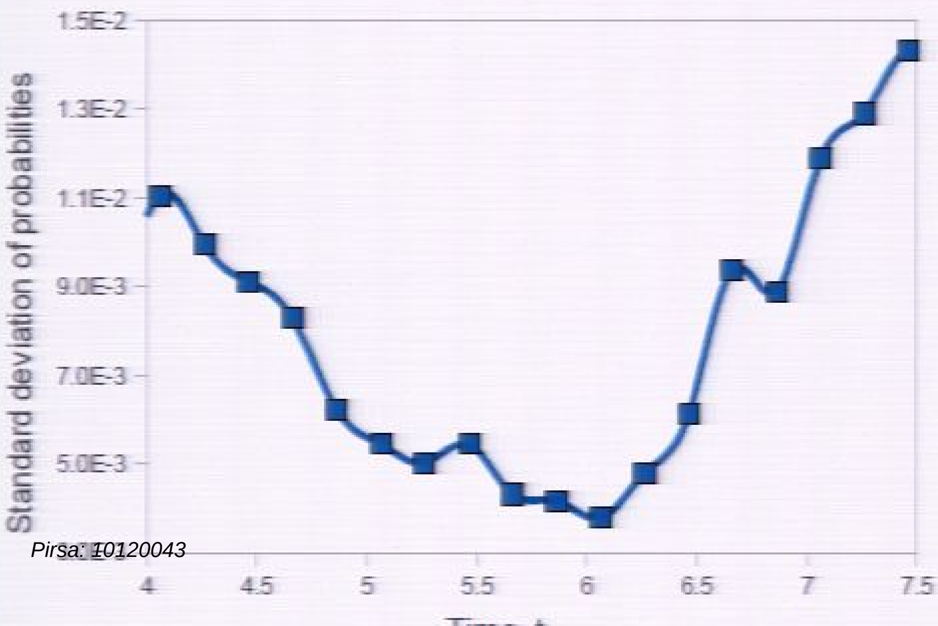

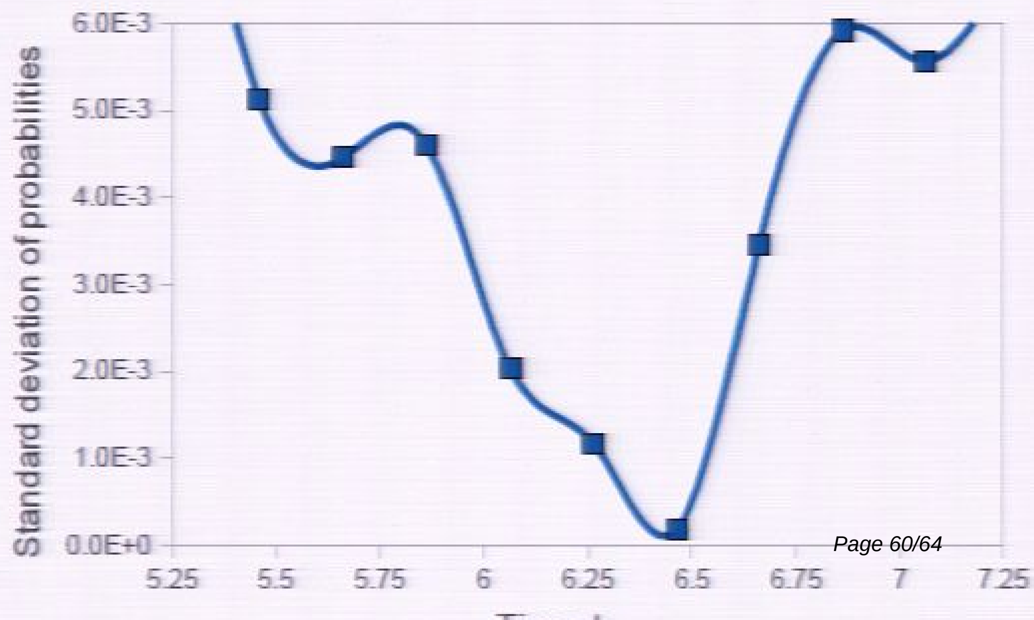

The End

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- Thank you for your attention

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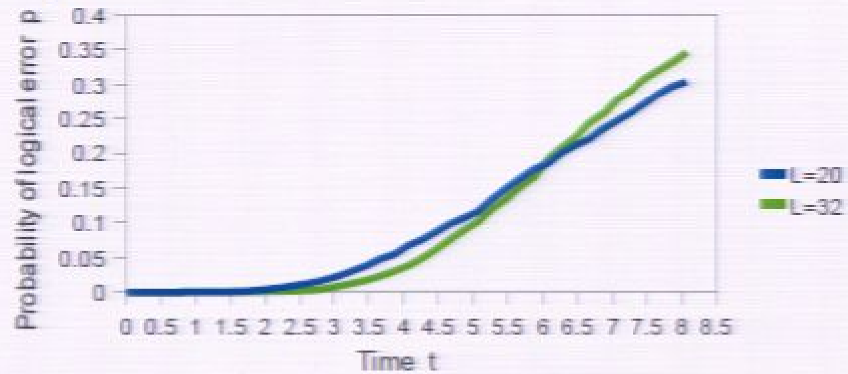

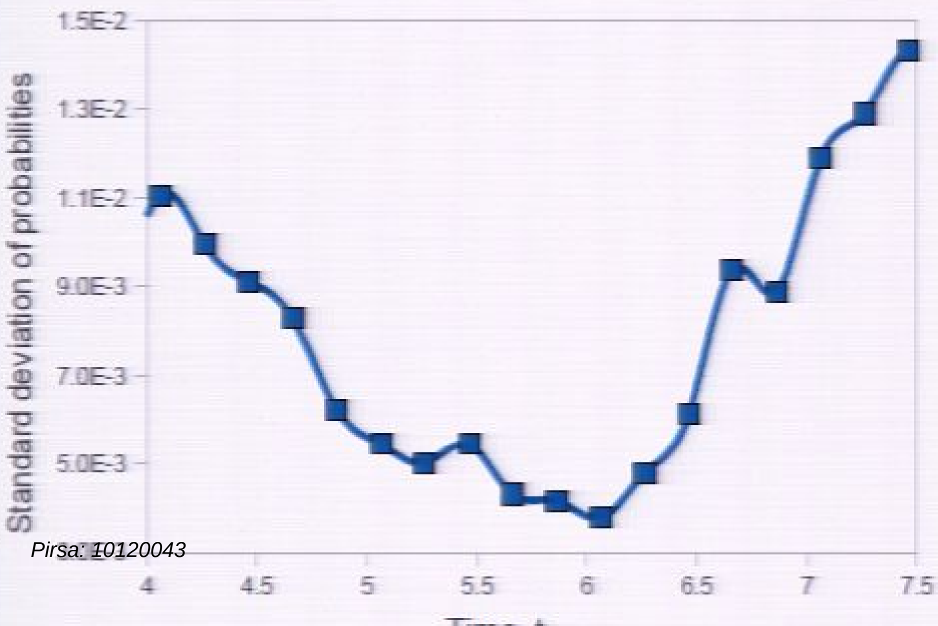

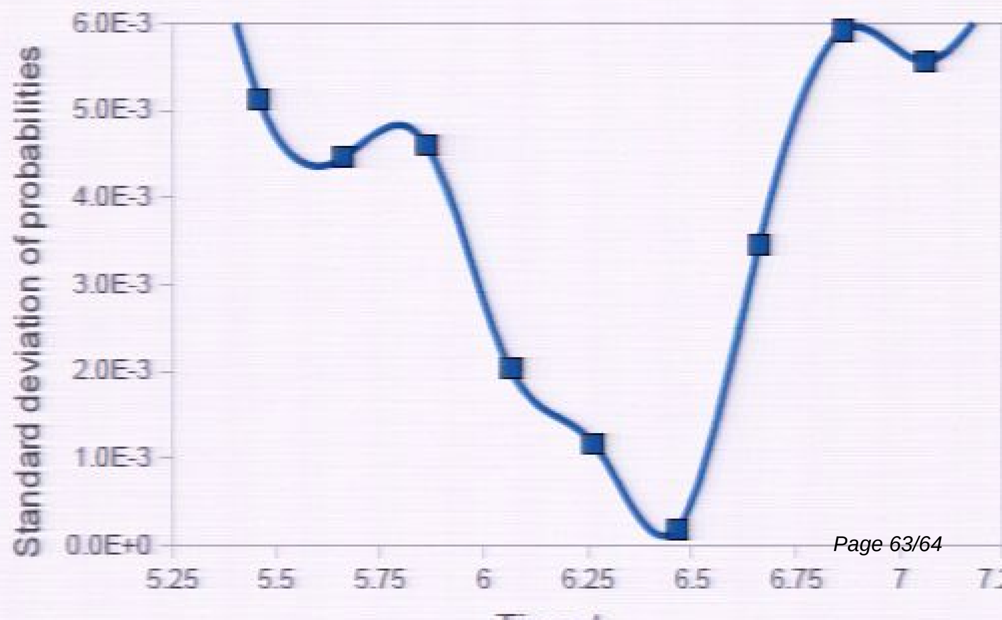

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