Title: Turning pictures into calculations: the duotensor framework

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Abstract: A picture can be used to represent an experiment. In this talk we will consider such pictures and show how to turn them into pictures representing calculations (in the style of Penrose's diagrammatic tensor notation). In particular, we will consider circuits described probabilistically. A circuit represents an experiment where we act on various systems with boxes, these boxes being connected by the passage of systems between them. We will make two assumptions concerning such circuits. These two assumptions allow us to set up the duotensor framework (a duotensor is like a tensor except that each position is associated with two possible bases). We will see that quantum theory can be formulated in this framework. Each of the usual objects of
quantum theory (states, measurements, transformations) are special cases of duotensors. The framework is motivated by the objective of providing a formulation of quantum theory which is local in the sense that, in doing a calculation pertaining to a particular region of spacetime, we need only use mathematical objects that pertain to this same region. This is, I argue, a prerequisite in a theory of quantum gravity.
Reference for this talk: http://arxiv.org/abs/1005.5164

Turning pictures into calculations: the duotensor framework[1]

The operation-duotensor tango

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Perimeter Institute, Waterloo, Ontario, Canada

- Prelude. Formalism locality
- Part I. Operational descriptions
- Part II. Probabilities. Objective.
- Part III. The simple tango. The advanced tango.
- Summary.

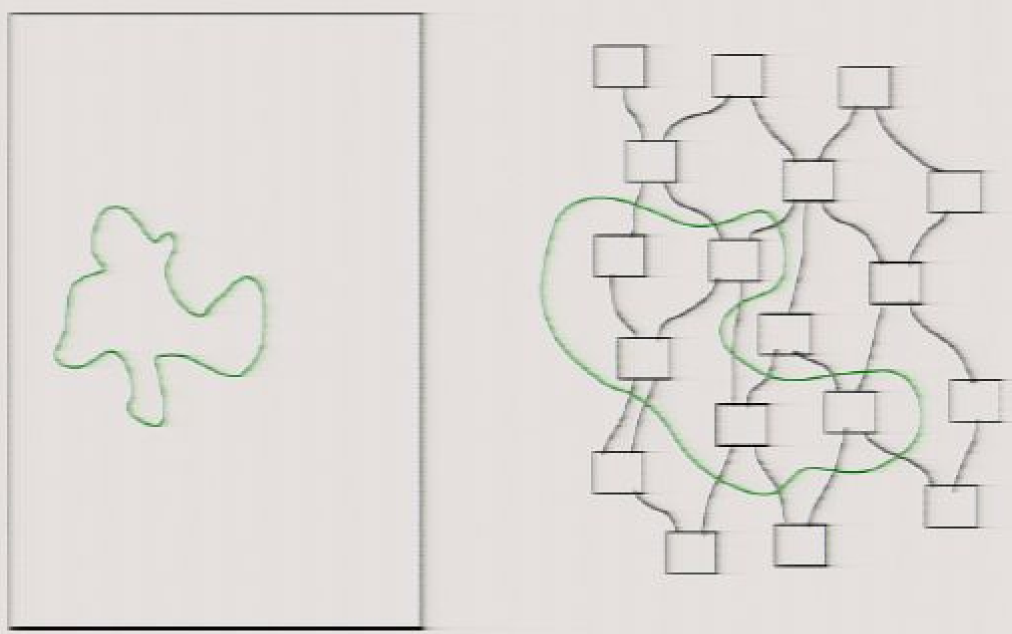
relude. Formalism-locality

Motivated by considerations from Quantum Gravity, we wish to have the following property

> **Formalism locality:** *A formalism for a physical theory is said to have the property of "formalism locality" if we can do calculations pertaining to any region of spacetime employing only mathematical objects associated with that region.*

Note that this is a property of the way a theory is formulated.

arbitrary region of space time ⇔ arbitrary fragment of a circuit



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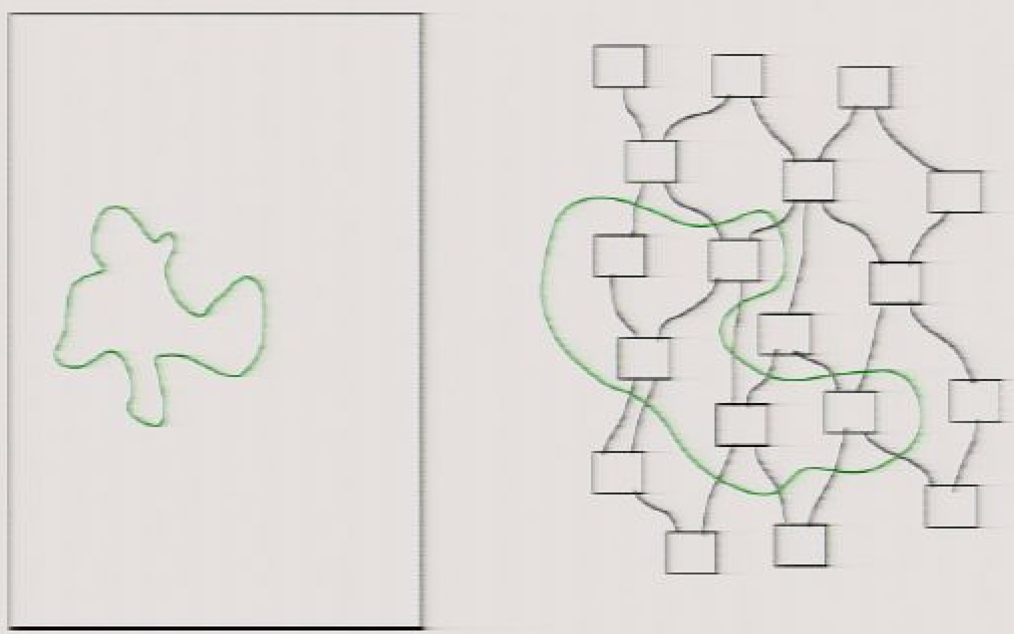

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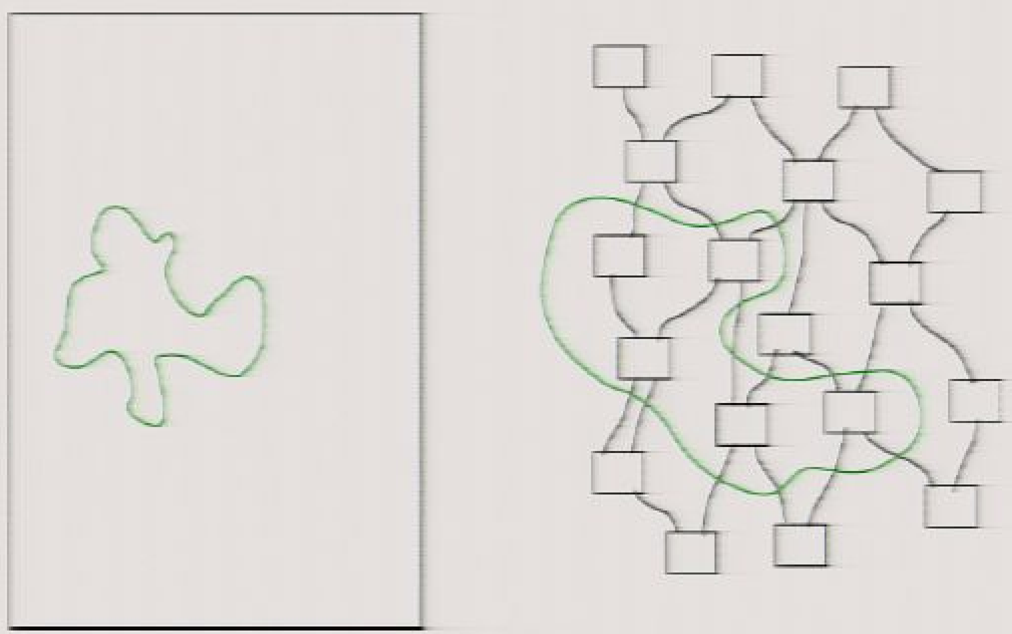
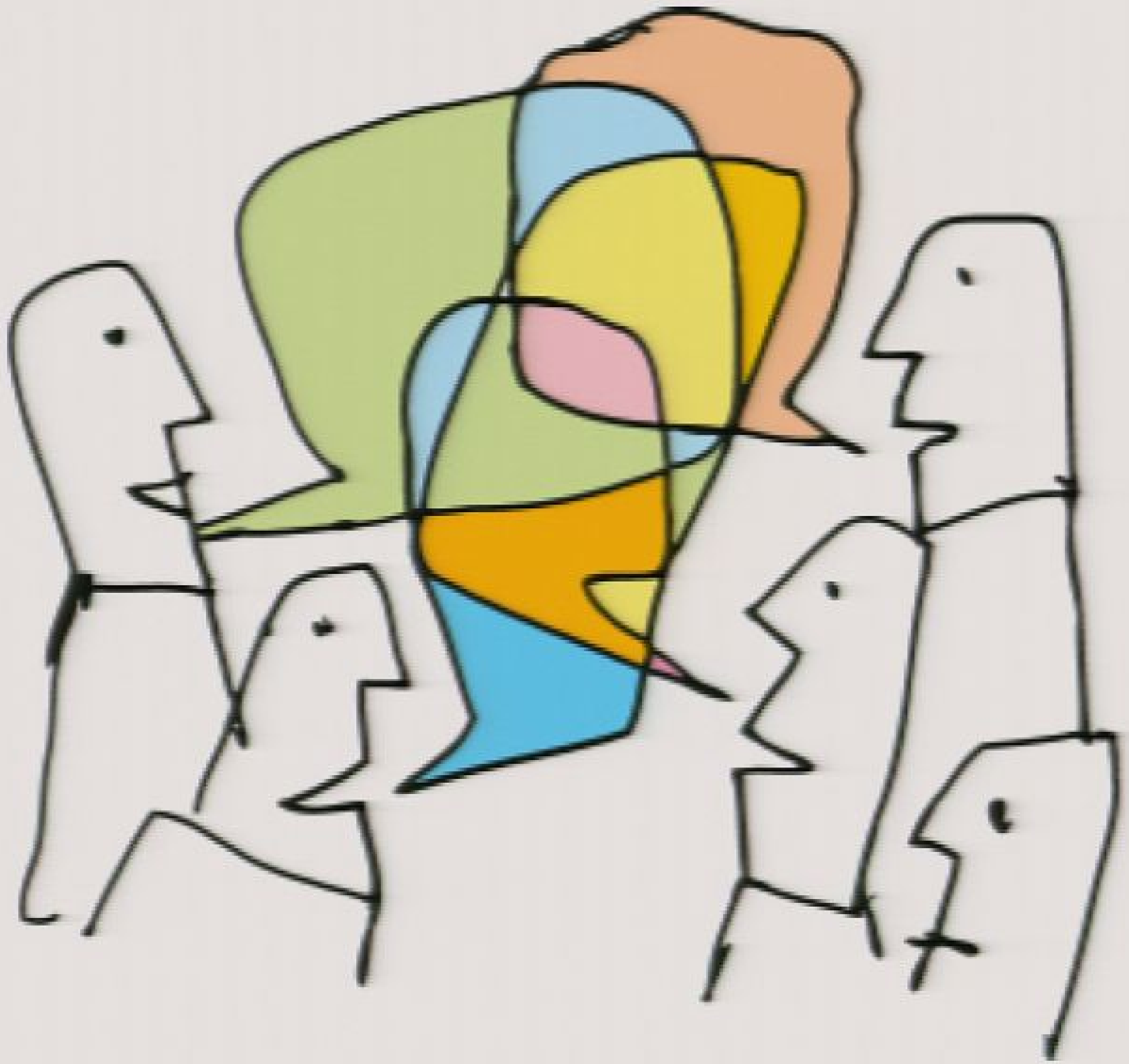


Incomplete list of related work:

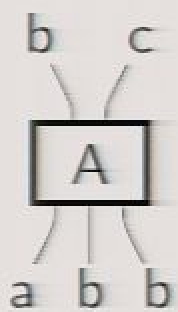
- S. Abramsky and B. Coecke, work on quantum picturalism (categories).
- L. Hardy, "Reasonable axioms for quantum theory", quant-ph/0101012.
- L. Hardy, "Foliable operational structures for general probabilistic theories", arXiv:0912.4740 (2009).
- G. Chiribella, G. M. D'Ariano, P. Perinotti, "Probabilistic theories with purification", arXiv:0908.1583 (2009)
- Causal set work by R. Sorkin
- Quantum causal histories approach of F. Markopoulou and related work by Blute, Ivanov, and Panangaden
- Time symmetric quantum theory work by Y. Aharanov and collaborators
- R. Oeckl, work on General boundary quantum field theory
- ...

art I. Operational descriptions.

Language used when theorists and experimentalists talk to each other.



perations - take 1



$\Longleftrightarrow$

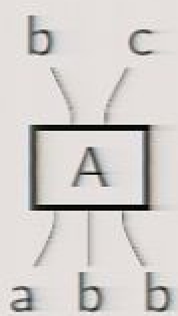
$$A^{b_4 c_5}_{a_1 b_2 b_3}$$

An operation, A, corresponds to one use of an apparatus and has the following features.

- *Inputs and outputs.* Come in various types, a, b, . . .
- *A setting*, s(A).
- *An outcome*, x_A.

If the outcome is x_A then we say operation A "happened".

perations - take 2 (with coarse graining)



$\Longleftrightarrow$ $A_{a_1 b_2 b_3}^{b_4 c_5}$

An operation, A, corresponds to one use of an apparatus and has the following features.

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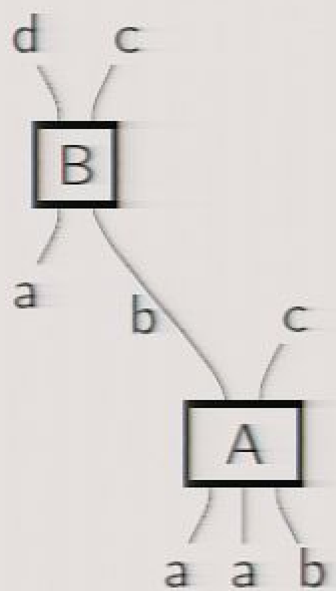
- *An outcome set*, o(A).

If $x_A \in o(A)$ then we say operation A "happened".

/ires

Outputs can be connected to inputs by wires.



$$\Longleftrightarrow \quad A^{b_4 c_5}_{a_1 a_2 b_3} B^{d_6 c_7}_{a_5 b_4}$$

Wiring rules.

- *One wire:* At most one wire can be connected to any given input or output.
- *Type matching:* Wires can connect inputs and outputs of the same type.
- *No closed loops.*

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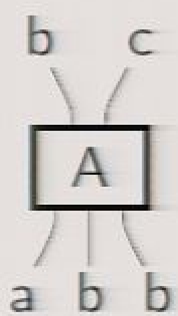

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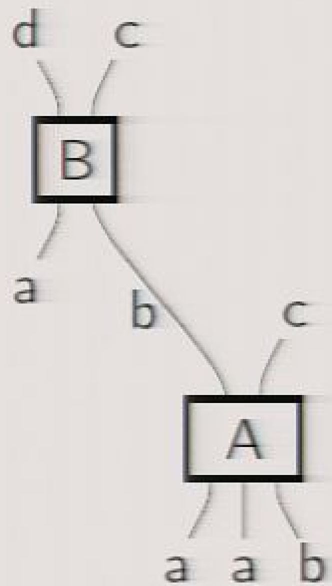

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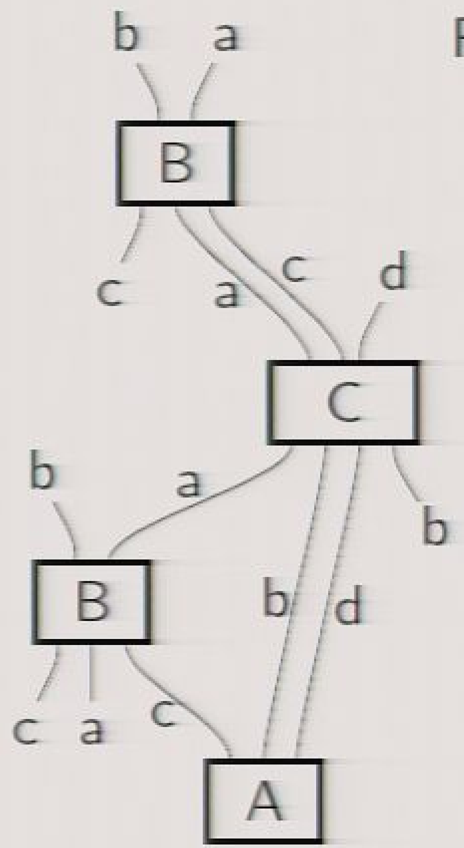


Fragment E

$$\Longleftrightarrow \quad A^{c_1 b_2 d_3} B^{b_6 a_7}_{c_4 a_5 c_1} C^{a_9 c_{10} d_{11}}_{a_7 b_2 d_3 b_8} B^{b_{13} a_{14}}_{c_{12} a_9 c_{10}}$$

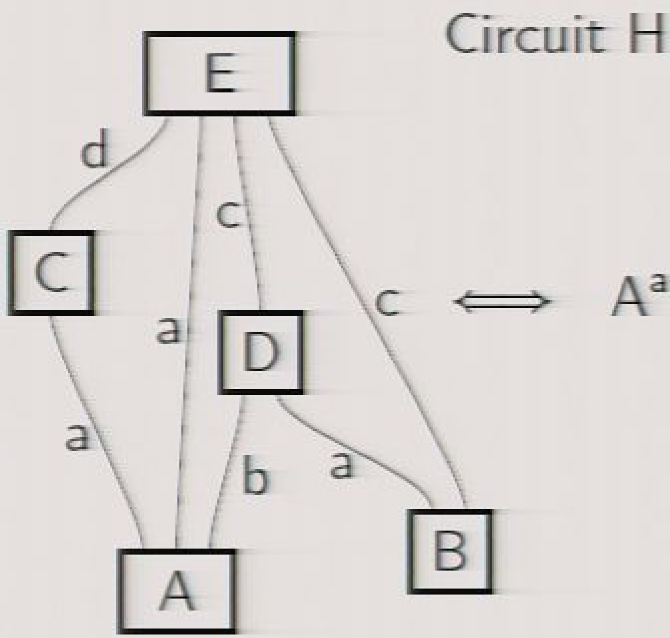
Fragments have

- *A setting*, s(E)
- *An outcome set*, o(E)
- *A wiring*, w(E)

Use notation sw(E) for settings and wiring

ircuits

Circuits have no open inputs or outputs.



$$\Longleftrightarrow \quad A^{a_1 a_2 b_3} B^{a_4 c_7} C^{d_5}_{a_1} D^{c_6}_{b_3 a_4} E_{d_5 a_2 c_6 c_7}$$

Circuits are special cases of fragments.

art II. Probabilities.

robabilities - notation

We write

$$\mathrm{Prob}(A|B)$$

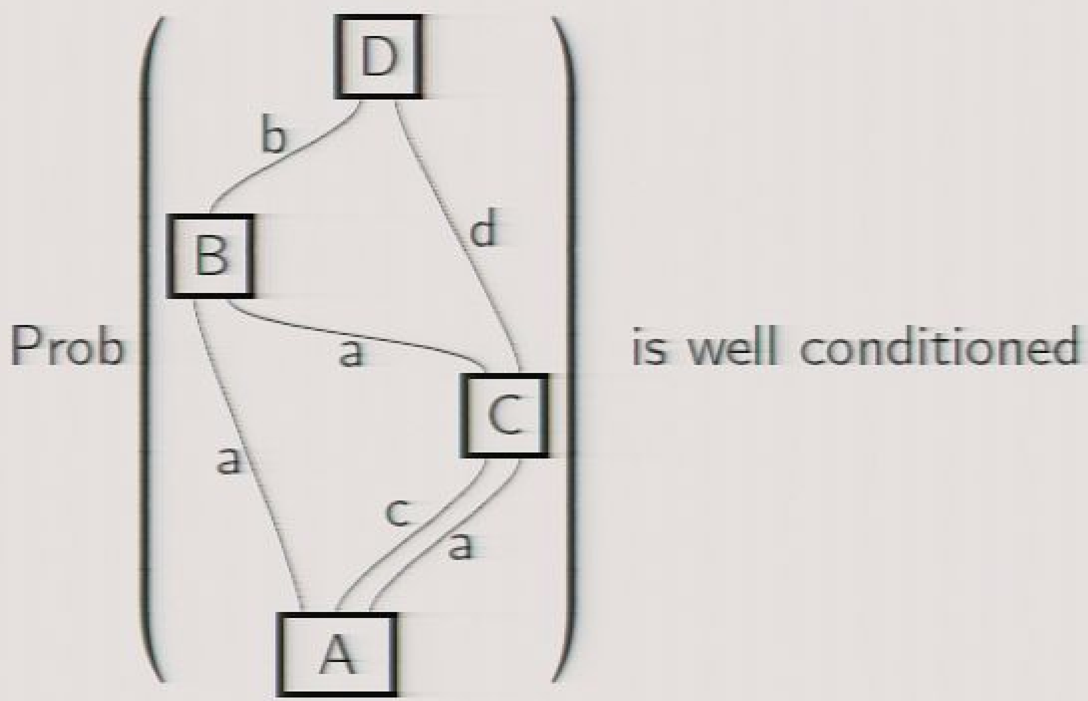
as shorthand notation for

$$\mathrm{Prob}\big(x_A \in o(A) | sw(AB), x_B \in o(B)\big)$$

We will always take A, B, C, ... to be non-overlapping in such expressions

ssumption 1

Assumption 1 *The probability, Prob(A), for any circuit, A (this has no open inputs or outputs), is well conditioned - it is determined by the operations and the wiring of the circuit alone and is independent of settings and outcomes elsewhere.*

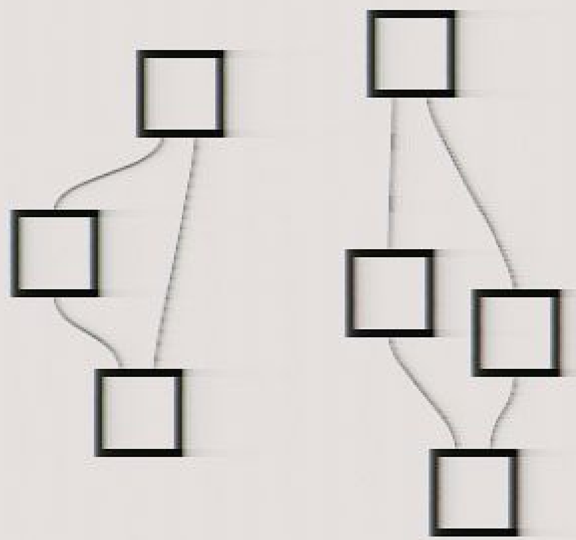


robabilities factorize over circuits

It follows from Assumption 1 that

$$\mathrm{Prob}(AB) = \mathrm{Prob}(A)\mathrm{Prob}(B)$$

for circuits A and B



(Chiribella, D'Ariano, and Perinotti take this factorization as an assumption.)

he $p(\cdot)$ function

We define the function $p(\cdot)$ as follows

$$p(\alpha \mathsf{A} + \beta \mathsf{B} + \dots) := \alpha \mathrm{Prob}(\mathsf{A}) + \beta \mathrm{Prob}(\mathsf{B}) + \dots$$

for circuits A, B, ... and real numbers α, β, ... (these can be negative).

quivalence relations

Equivalence: *We write*

$$expression_1 \equiv expression_2$$

if

$$p(expression_1 \mathsf{E}) \equiv p(expression_2 \mathsf{E})$$

for any fragment E *that makes the contents of the argument on both sides of this equation into a linear sum of circuits.*

Equivalence is a weaker notion than equality.

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xample of equivalence

Have

$$\alpha A^{a_1} + \beta B^{a_1} \equiv \gamma C^{a_1} + \delta D^{a_1}$$

if

$$p([\alpha A^{a_1} + \beta B^{a_1}]E_{a_1}) = p([\gamma C^{a_1} + \delta D^{a_1}]E_{a_1}) \quad \text{for all} \quad E_{a_1}$$

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nother example of equivalence

In general, we have

$$\mathsf{A} \equiv \mathsf{Prob}(\mathsf{A}) \quad \text{for any circuit } \mathsf{A}$$

Proof: For any circuit E

$$p(\mathsf{AE}) = p(\mathsf{A})p(\mathsf{E}) = p(\mathsf{Prob}(\mathsf{A})\mathsf{E})$$

quivalence relations

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wo general types of equivalence

1. Each expression is a real number plus a linear combination of circuits:
$$\alpha + \beta A + \gamma B + \cdots \equiv \delta + \epsilon C + \zeta D + \ldots$$
where A, B, ..., C, D, ..., are all circuits.
2. Each expression is a linear combination of fragments
$$\alpha A + \beta B + \cdots \equiv \gamma C + \delta D + \ldots$$
where A, B, ..., C, D, ..., are all fragments having the same causal structure.

art III. The tango

The simple tango

The advanced tango

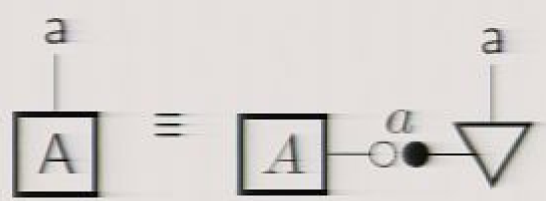
iducial preparations

Fiducial preparations

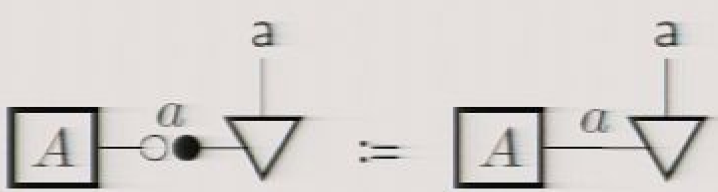
$\Longleftrightarrow \quad {}_{a_1}X^{a_1}$ where $a_1 = 1$ to K_a

For any preparation A^{a_1}

$$A^{a_1} \equiv {}^{a_1}A \; {}_{a_1}X^{a_1} \quad \Longleftrightarrow$$



We define



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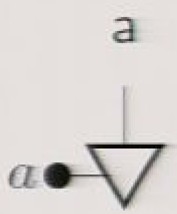
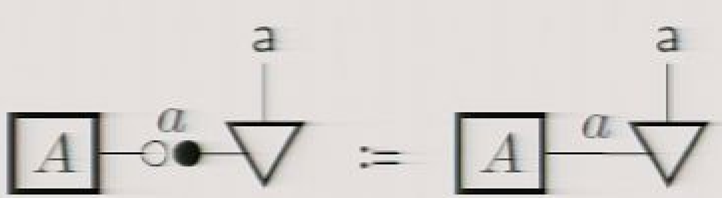


$\Longleftrightarrow \quad {}_{a_1}\mathsf{X}^{\mathsf{a}_1}$ where $a_1 = 1$ to K_{a}

For any preparation $\mathsf{A}^{\mathsf{a}_1}$

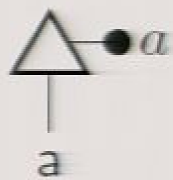
$\mathsf{A}^{\mathsf{a}_1} \equiv {}^{a_1}A \;\; {}_{a_1}\mathsf{X}^{\mathsf{a}_1} \quad \Longleftrightarrow$

We define



iducial effects

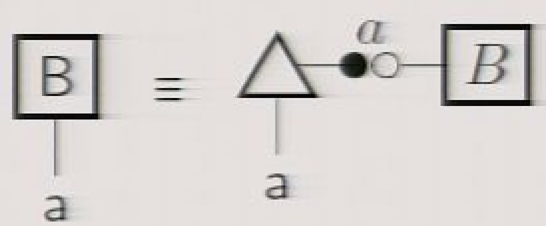
Fiducial effects



$\Longleftrightarrow \quad X^{a_1}_{\mathsf{a}_1}$ where $a_1 = 1$ to K_{a}

For any effect for a system of type a

$B_{\mathsf{a}_1} \equiv B_{a_1} X^{a_1}_{\mathsf{a}_1} \quad \Longleftrightarrow$



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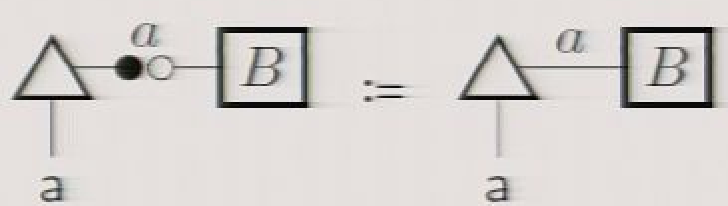
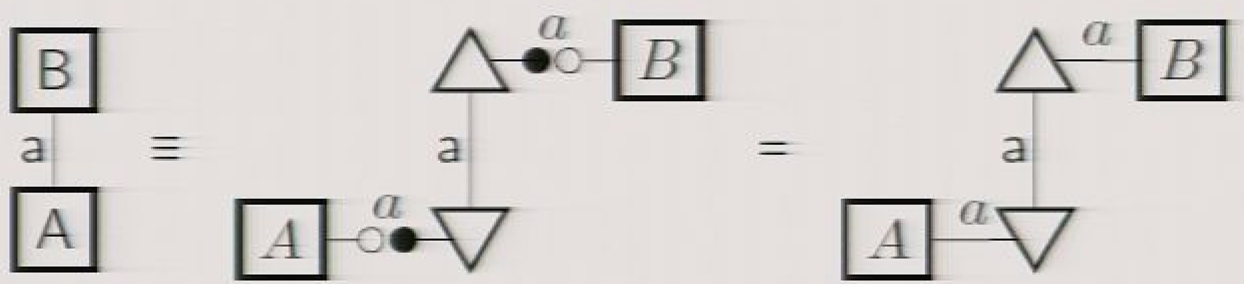
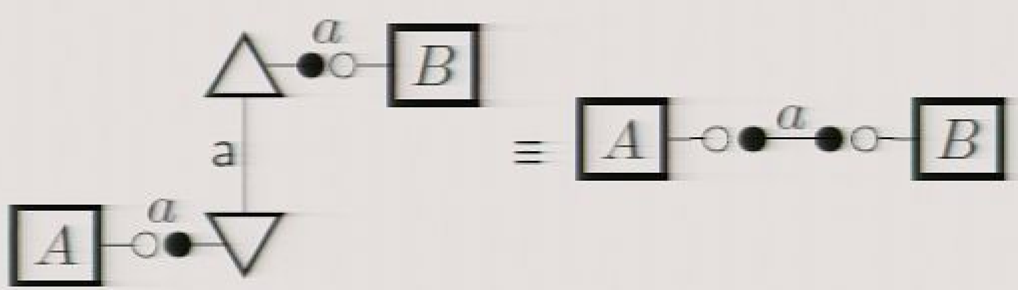


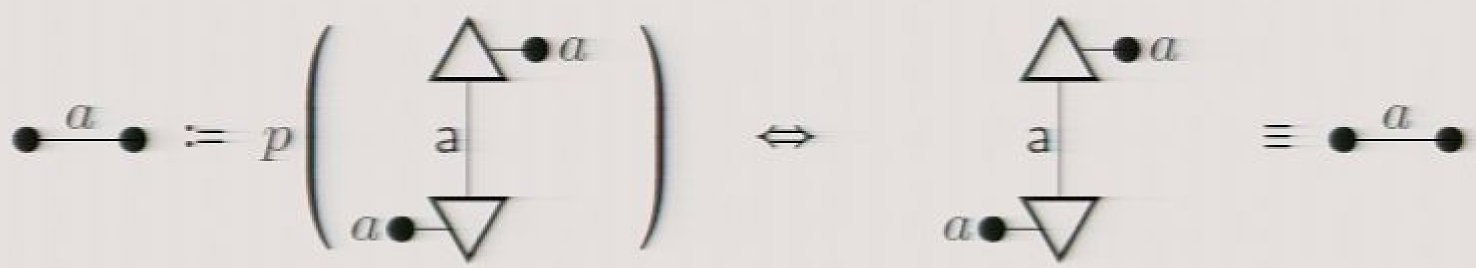
he simple tango



Using the linearity of the $p(\cdot)$ function we have

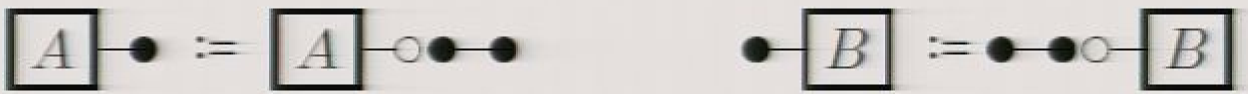


where we define *the hopping metric*

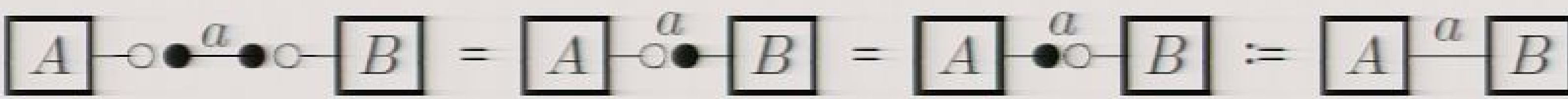


lack and white dots

We define

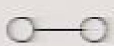
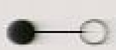


Hence

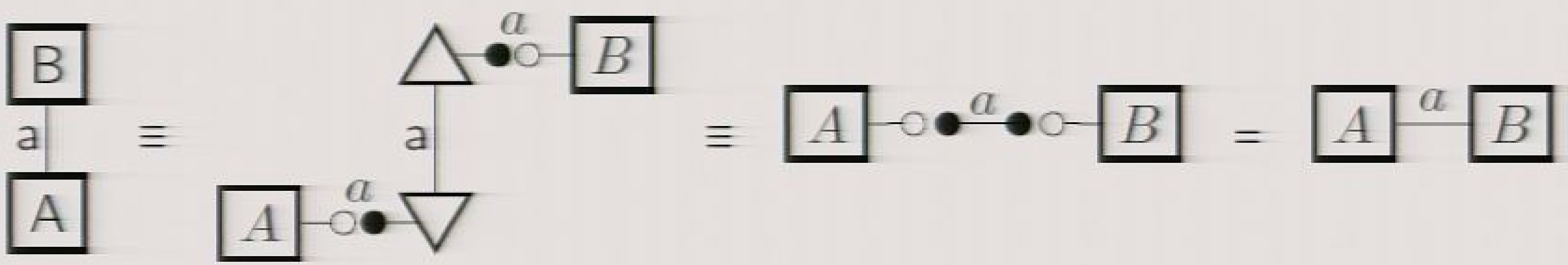


We have

Hence, we can insert and delete pairs of black and white dots as we like. Consistency requires

-  to be the inverse of 
-  to be equal to the identity
-  to be equal to the identity

he steps of the simple tango



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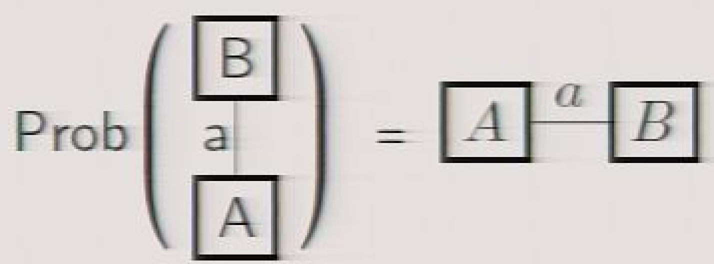

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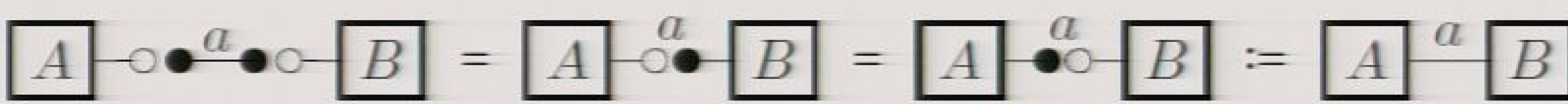

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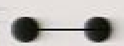
-  to be the inverse of 
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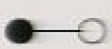
-  to be equal to the identity

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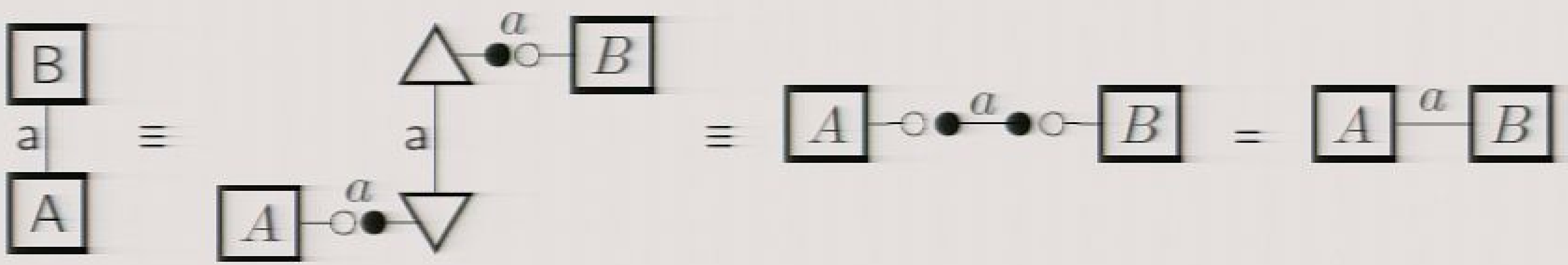

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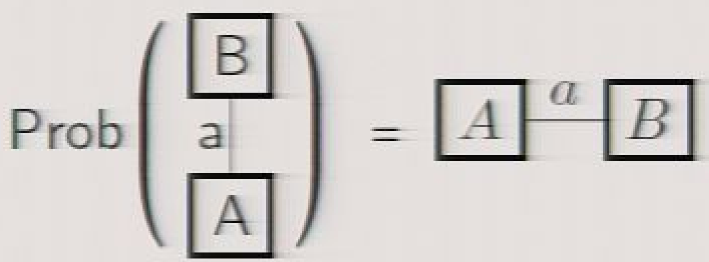
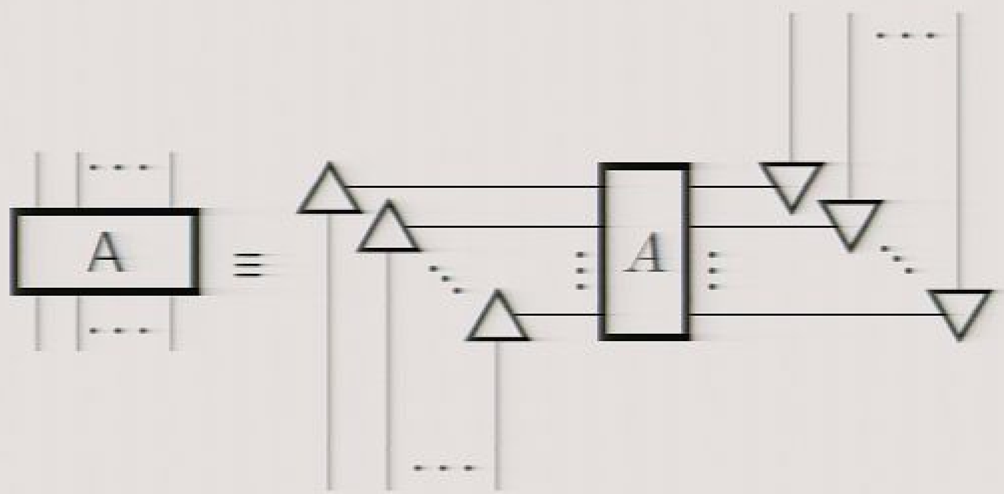


ssumption 2

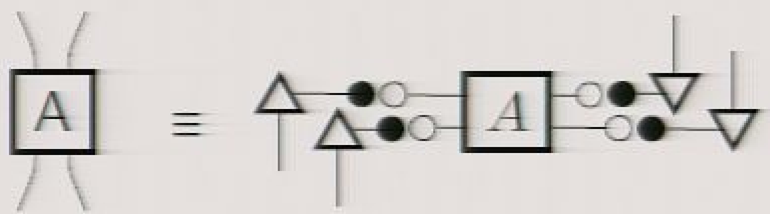
Assumption 2: Operations are fully decomposable. We assume that any operation can be written as



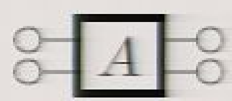
In words we will say that any operation is equivalent to a linear combination of operations each of which consists of an effect for each input and a preparation for each output.

uotensor with all white dots

Inserting black and white dots (with black next to the fiducial elements)



Therefore



(with all white dots) provides the weights in the sum over fiducial elements.
This is an example of a duotensor.

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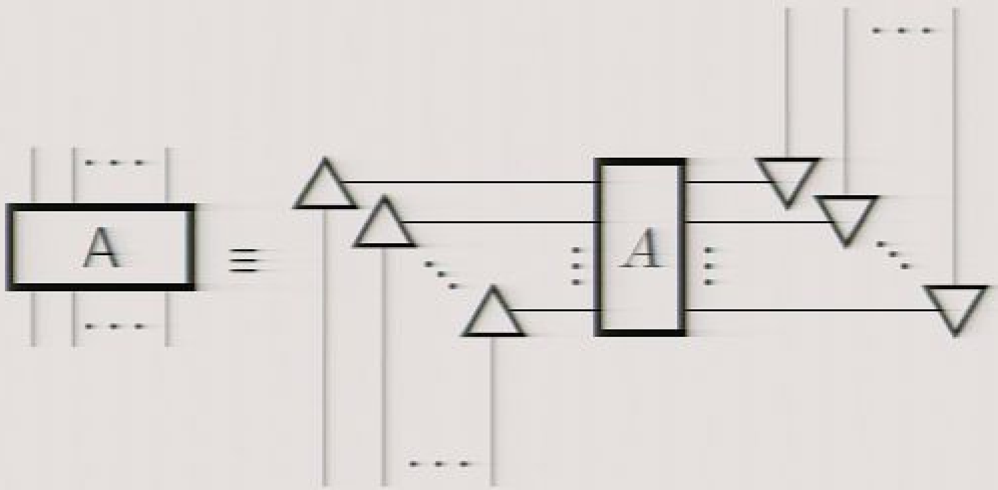

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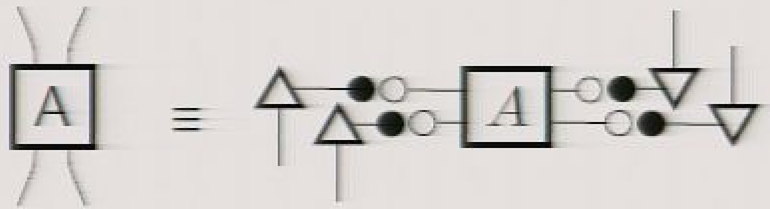

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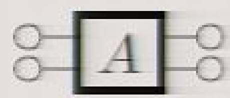
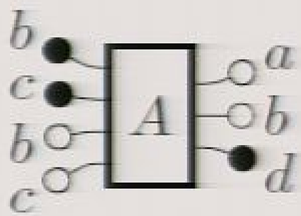


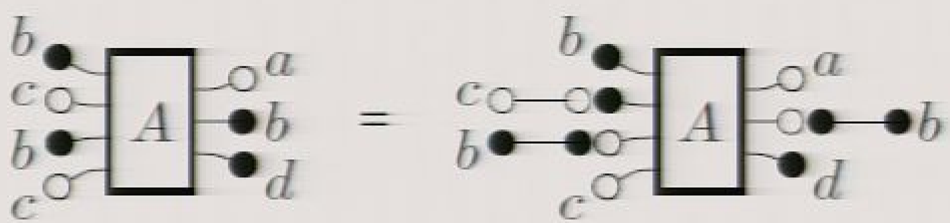
(with all white dots) provides the weights in the sum over fiducial elements.
This is an example of a duotensor.

/hat are duotensors?

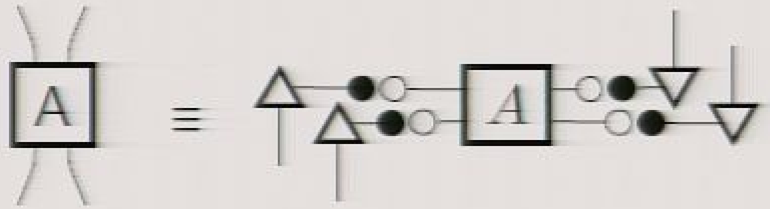
- Like tensors except that each index is associated with two bases.
- They transform like tensors but with respect to two bases.
-  corresponds to ${}^{a_1 b_2}_{b_3 c_4}A^{d_5}_{b_6 c_7}$
- Have map

$$ {}^{\circ}_{\bullet}A^{\bullet}_{\circ} $$

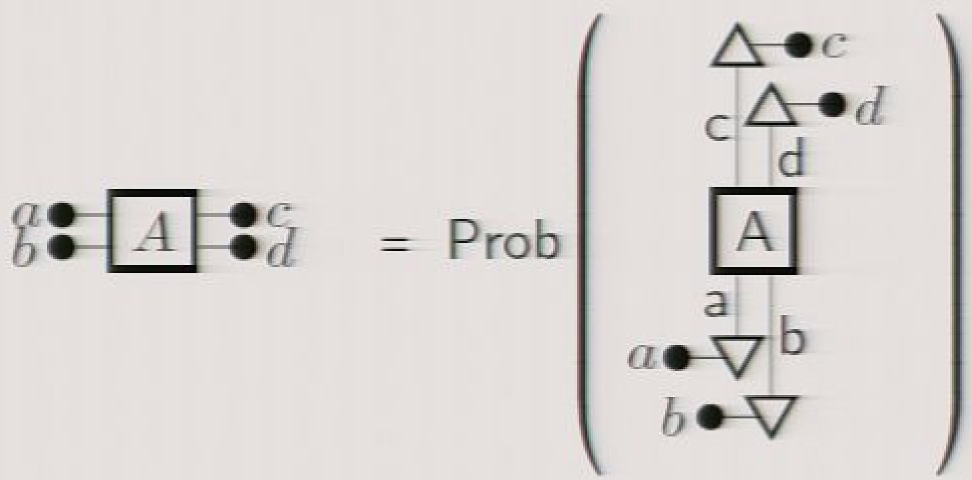
- Can change colours of dots using ●—● and ○—○



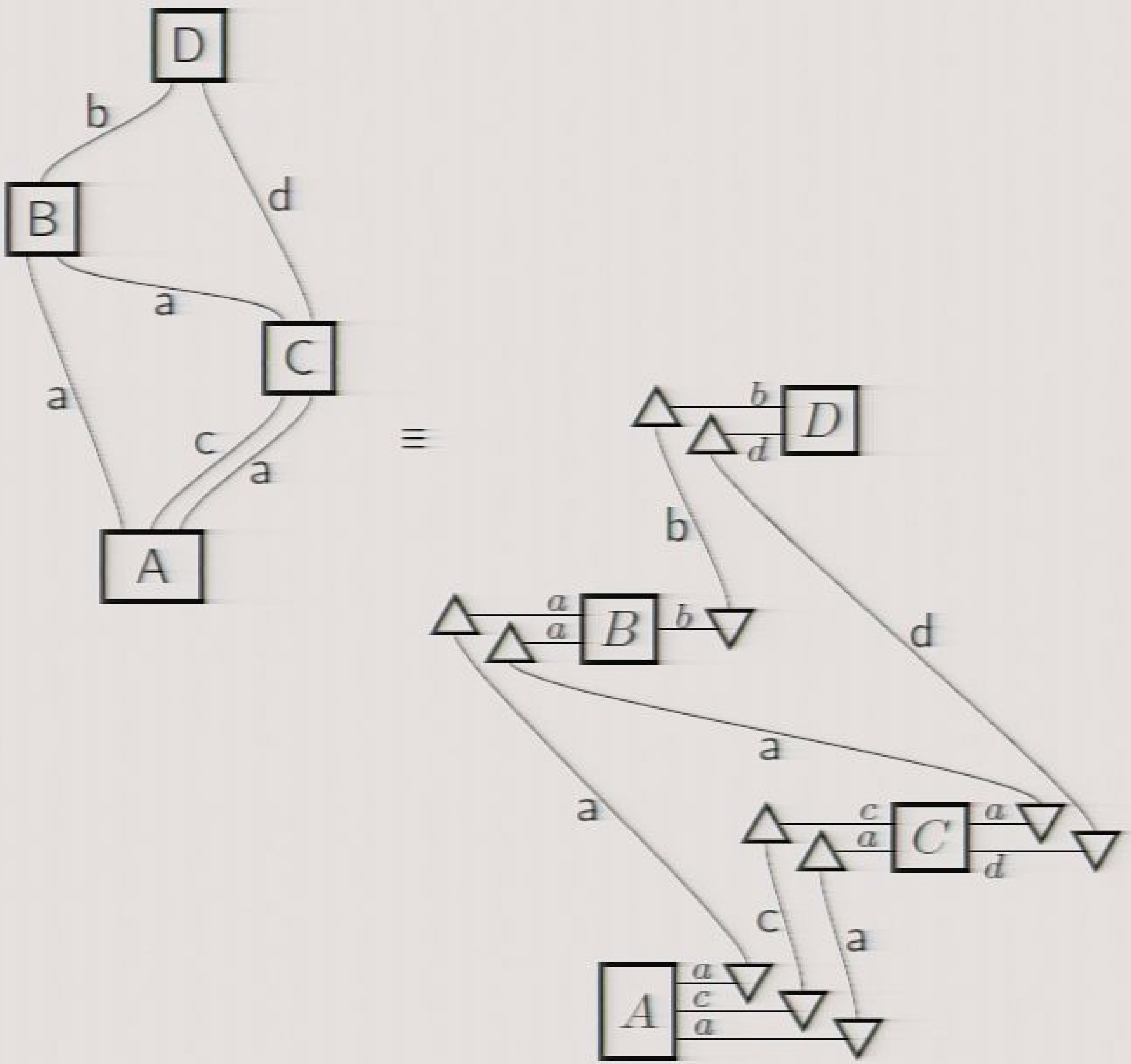
All white dots gives coefficients in sum over fiducials



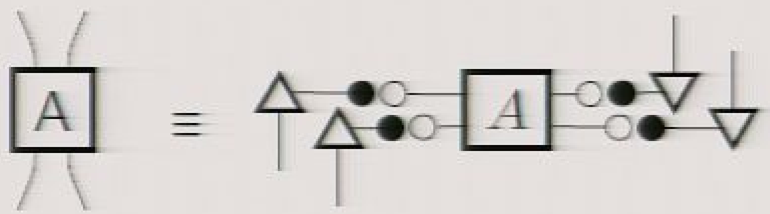
All black dots gives fiducial probabilities



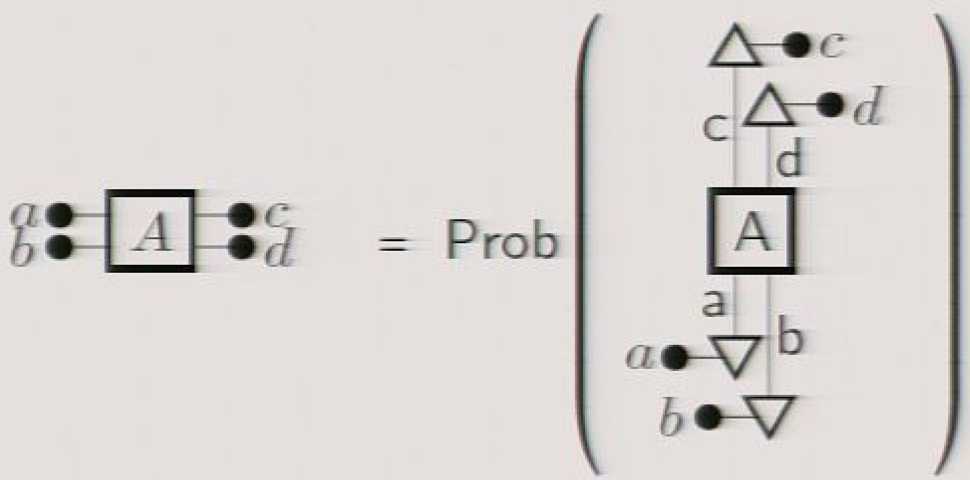
he advanced tango



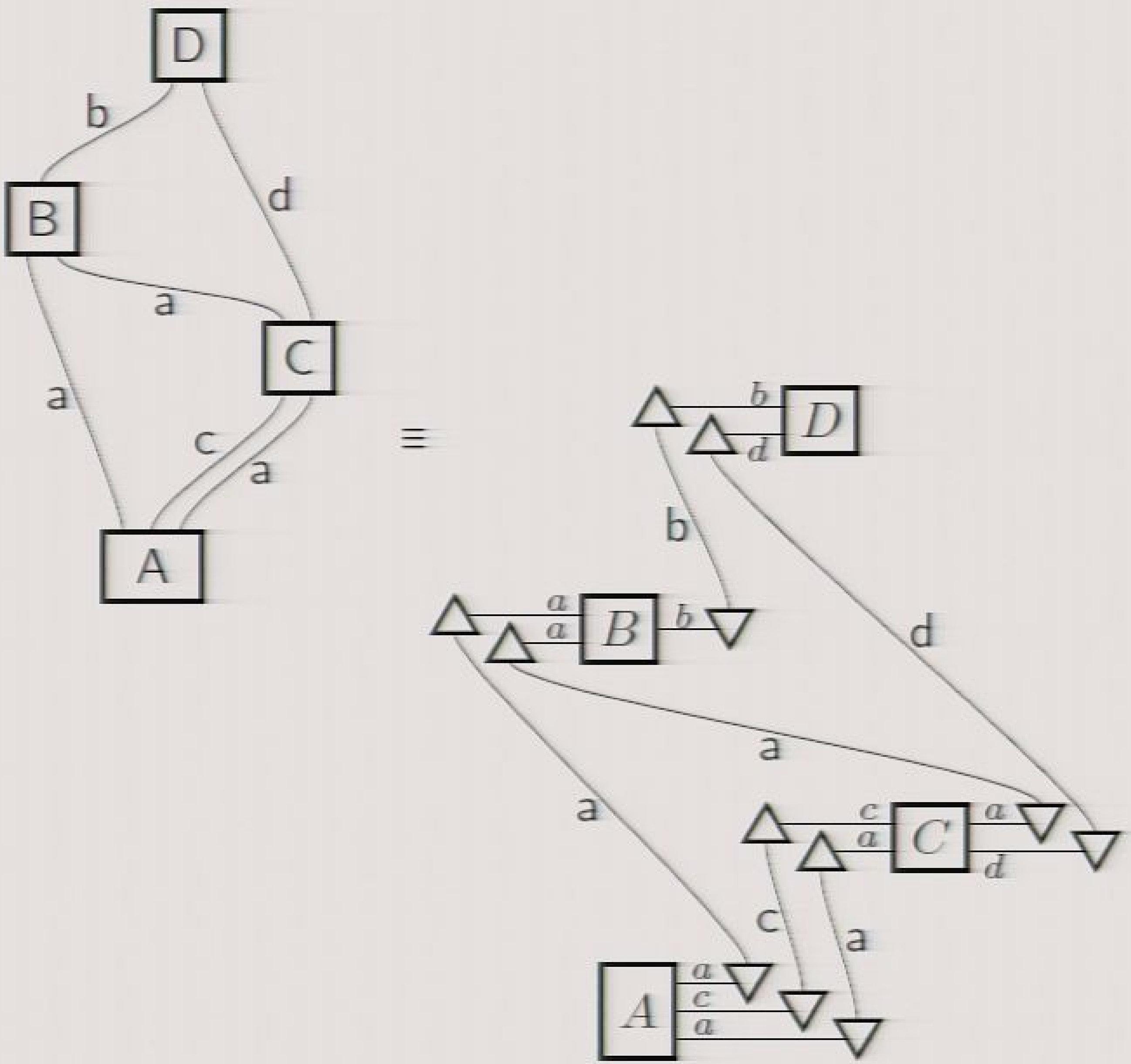
All white dots gives coefficients in sum over fiducials



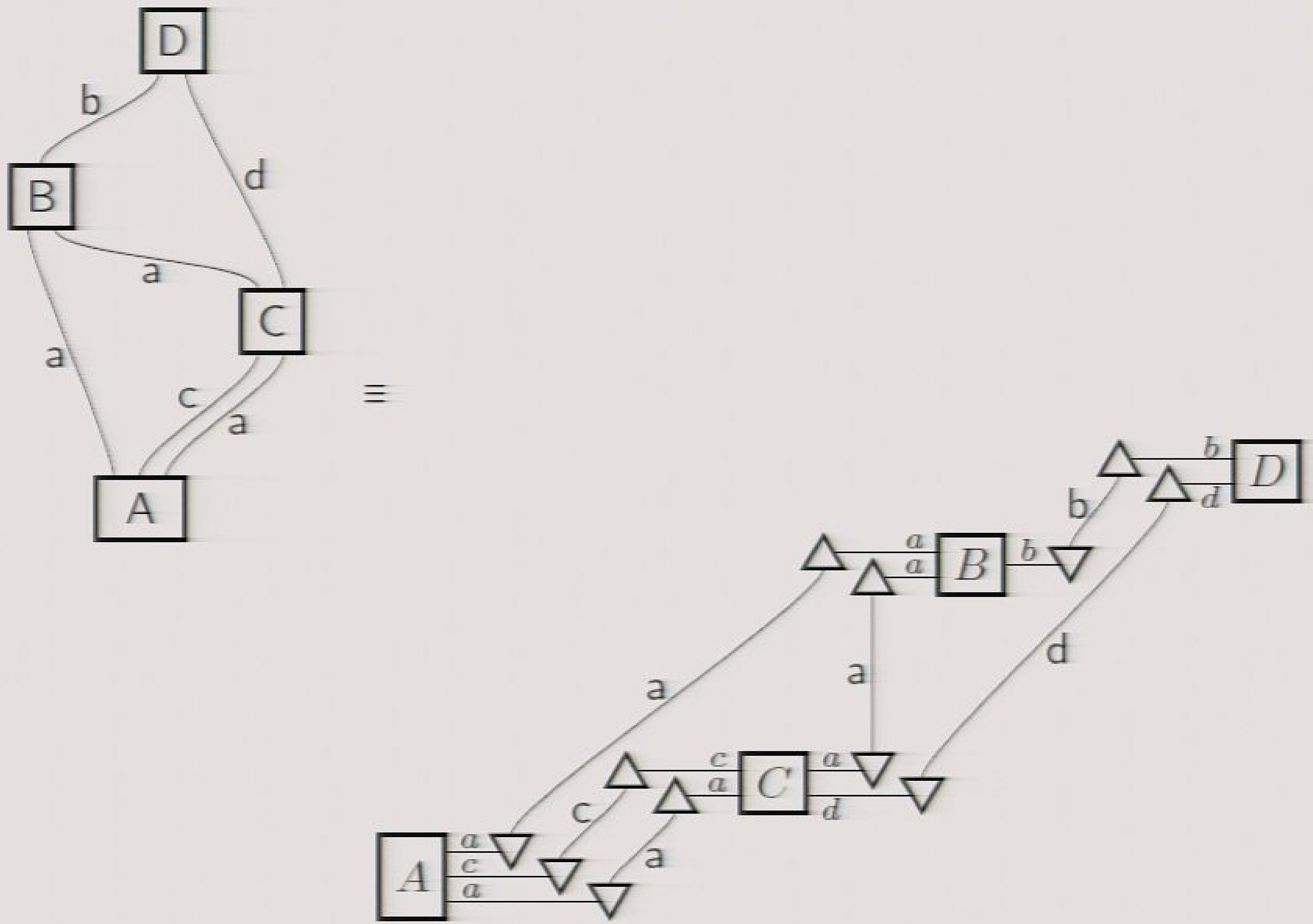
All black dots gives fiducial probabilities



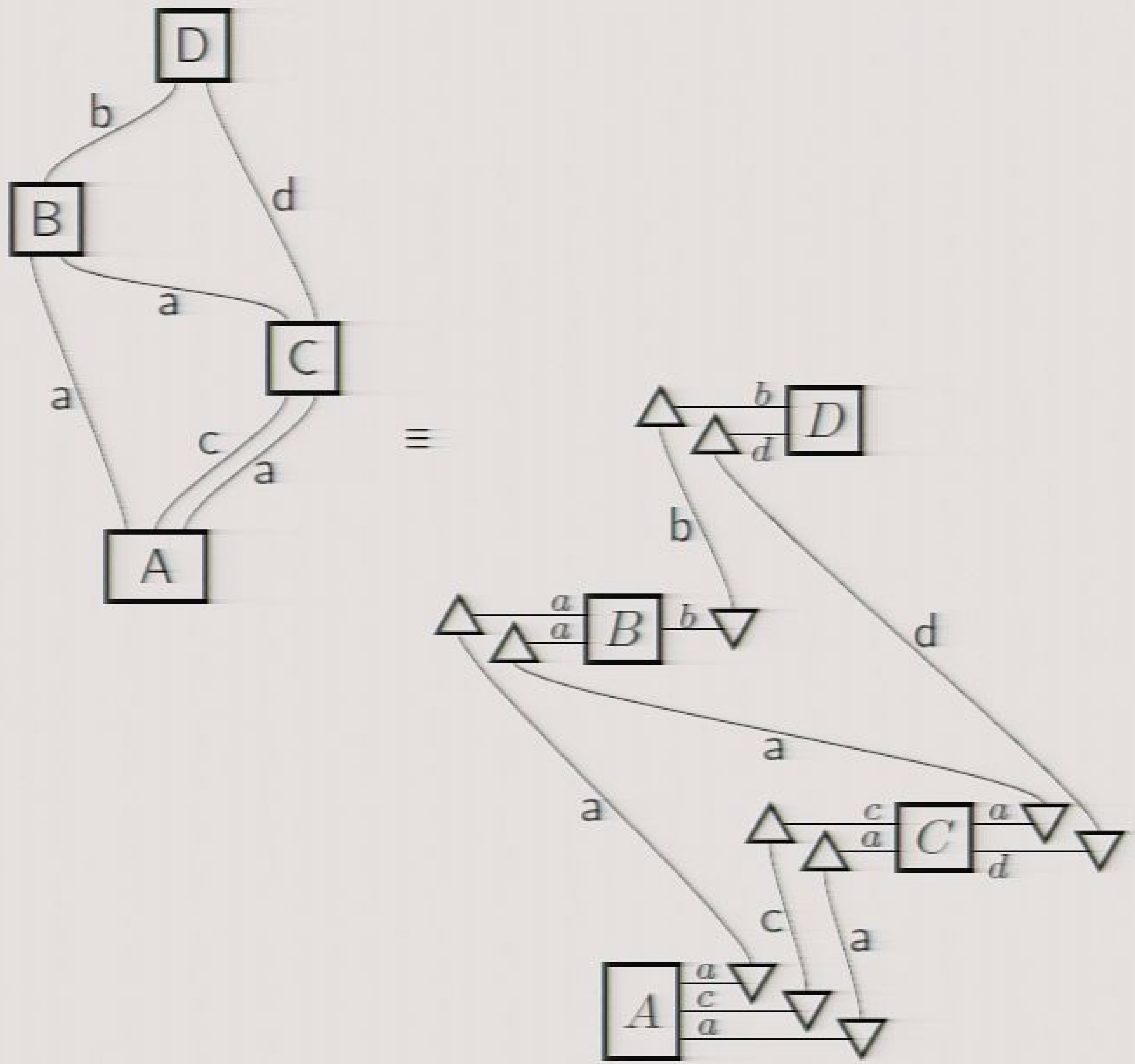
he advanced tango



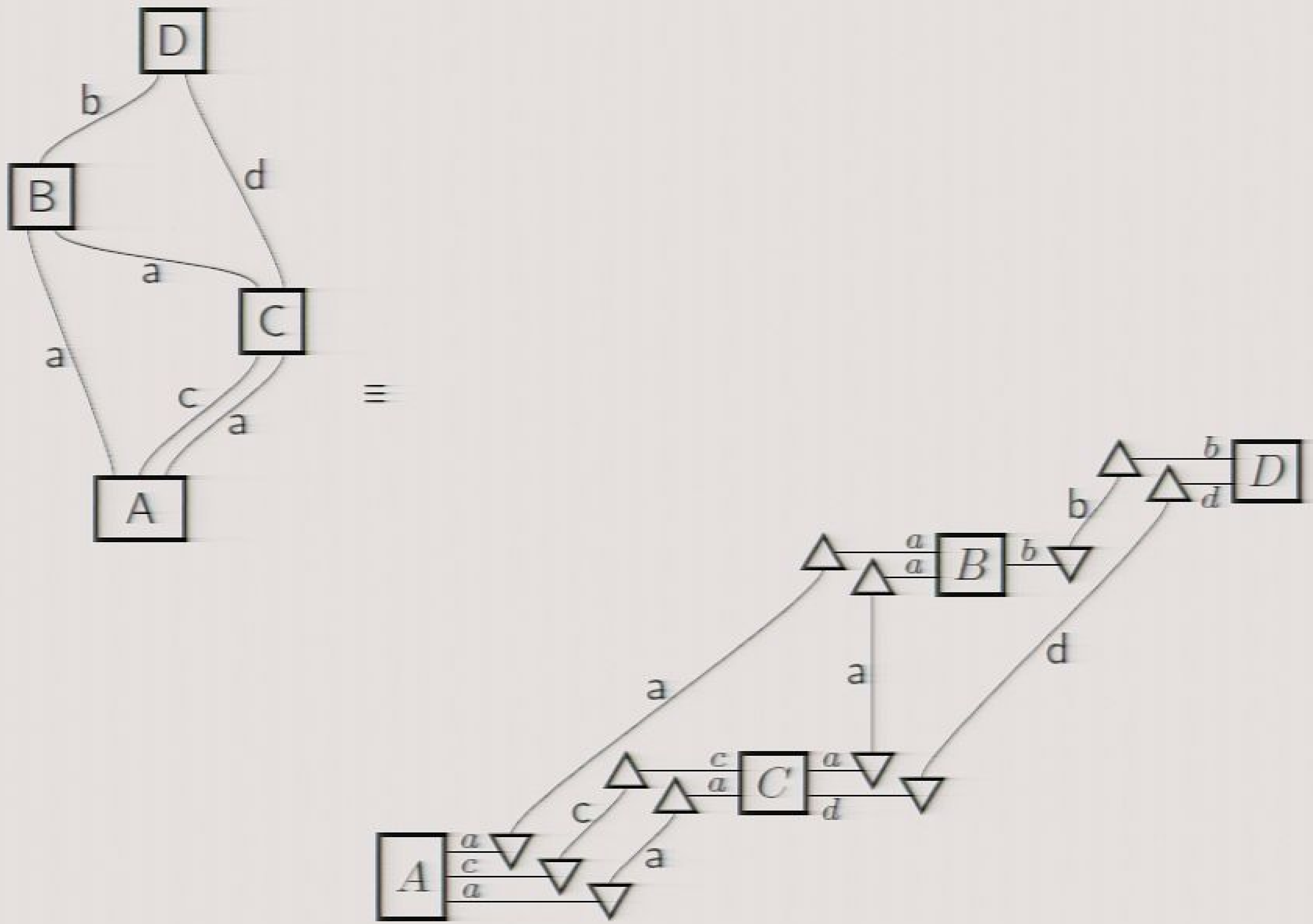
he advanced tango



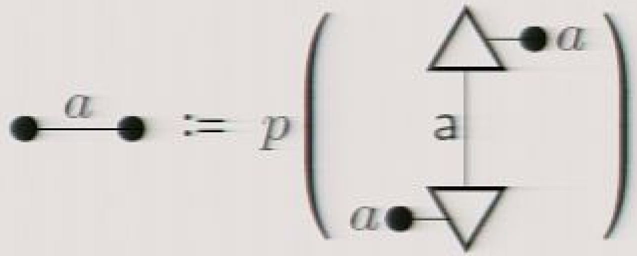
he advanced tango



he advanced tango

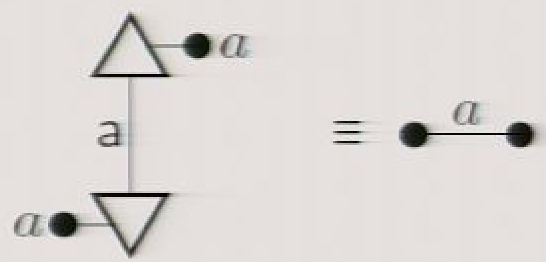


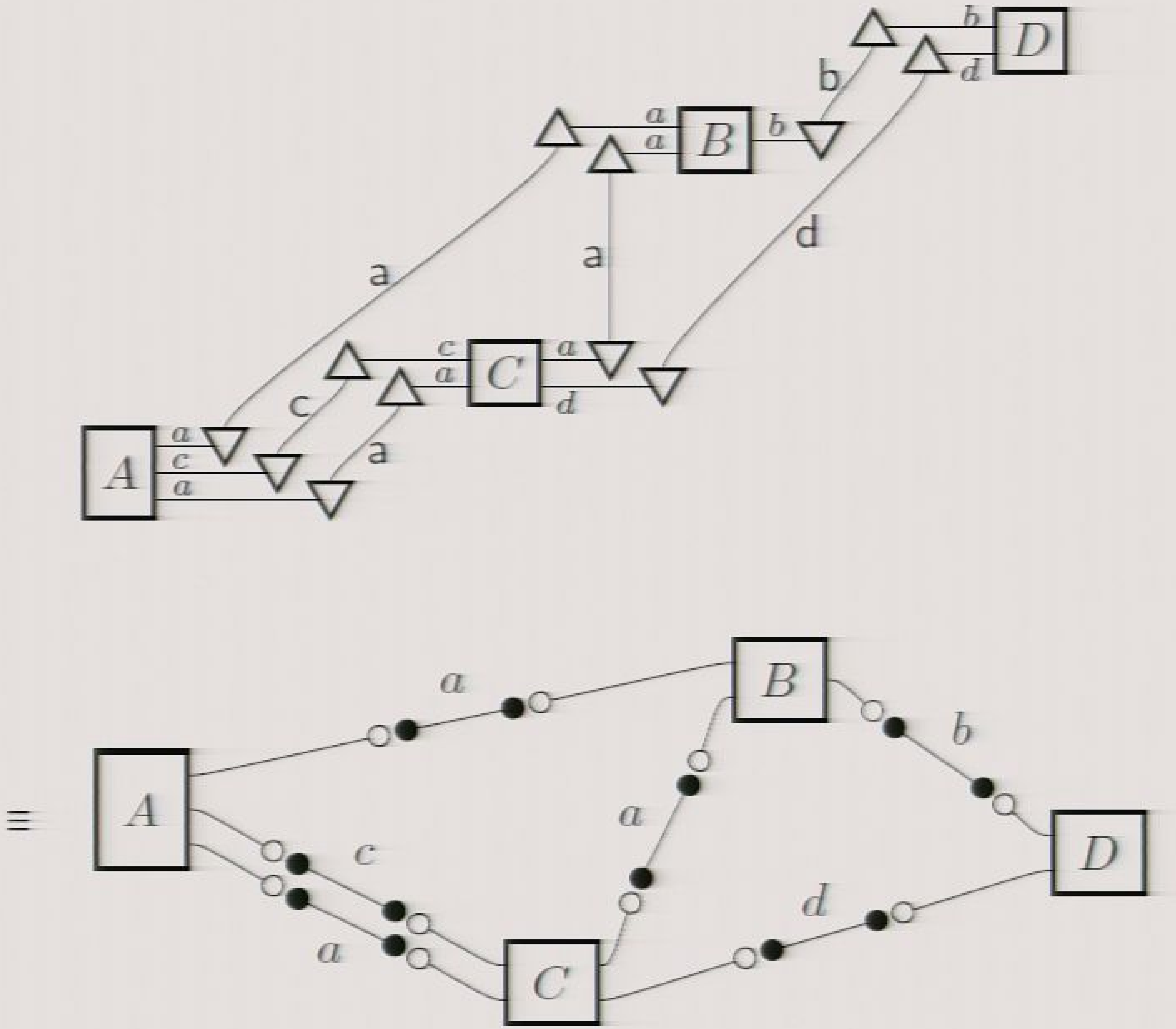
Recall



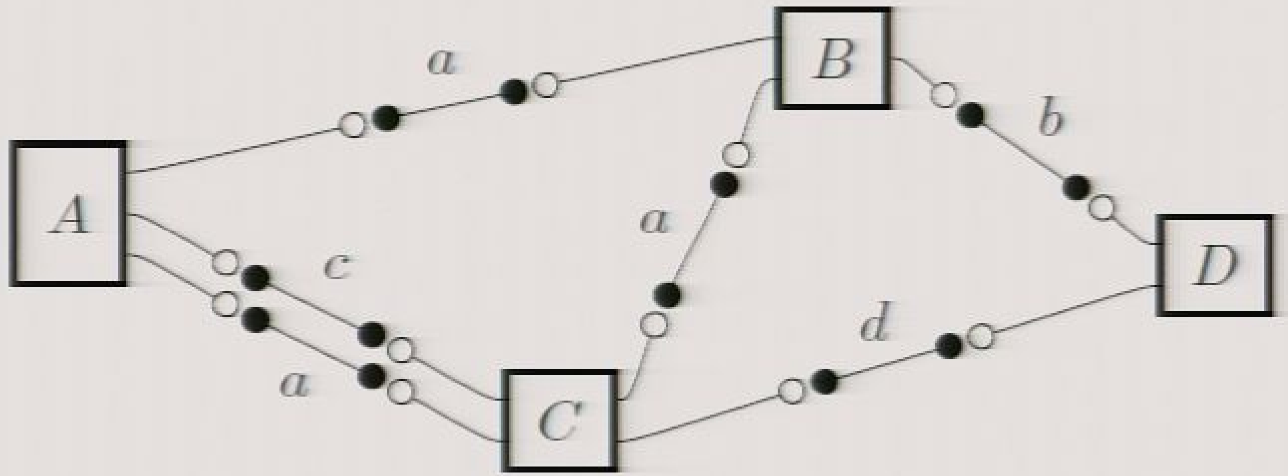
The hopping metric

This implies

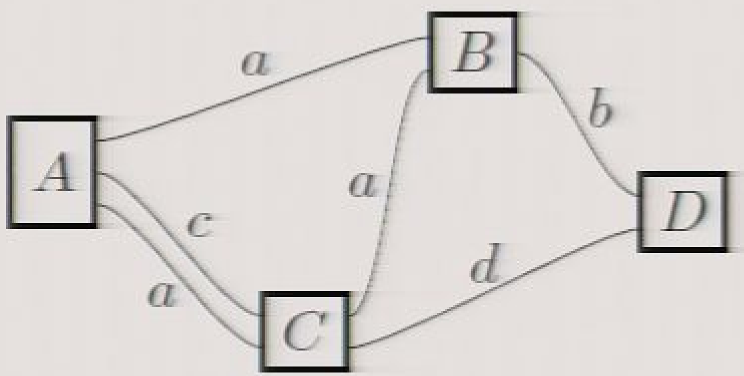




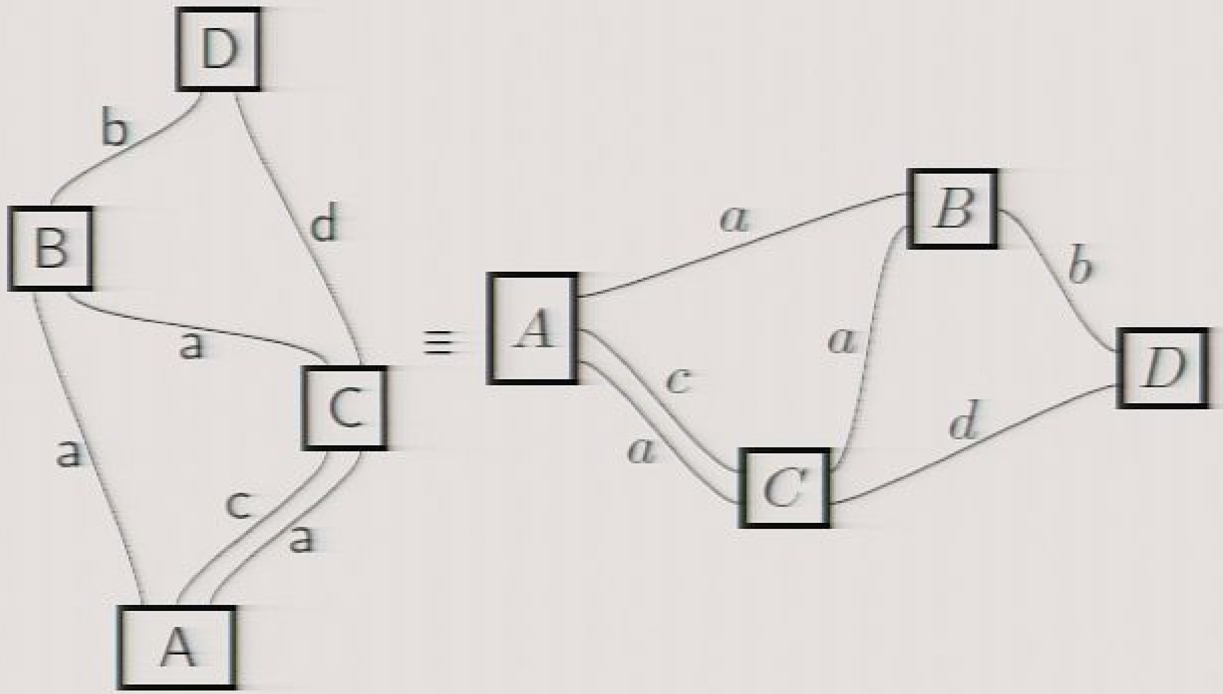
A
B
C
D
a
b
c
d
≡



equals

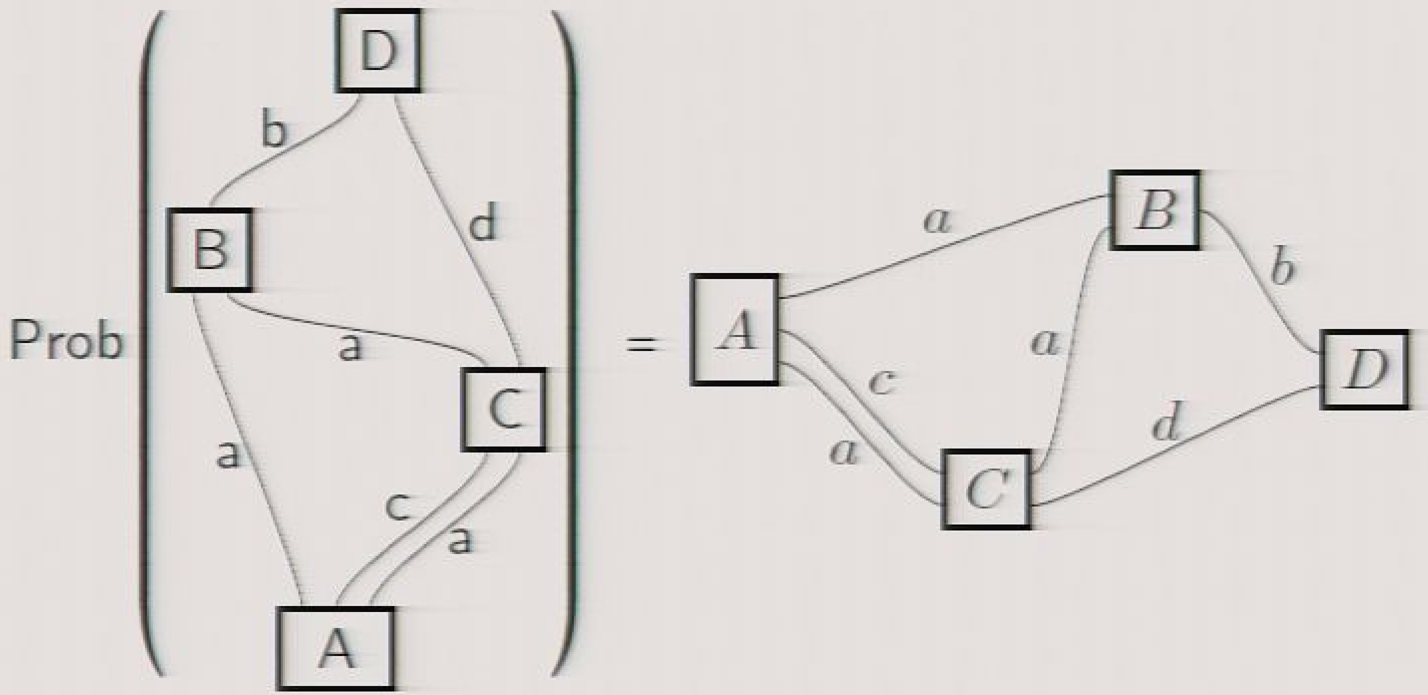


First and last step



Clearly works for any circuit.

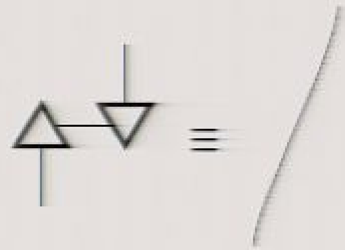
Hence



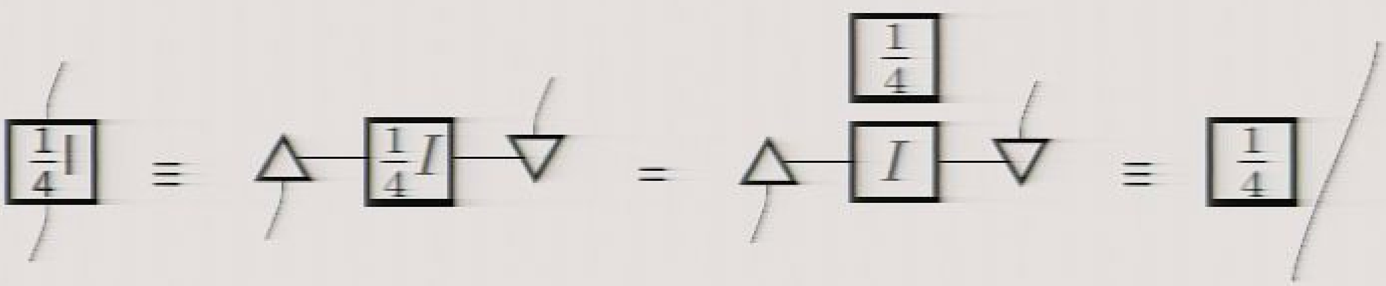
The diagram for the mathematical calculation looks the same as the diagram for the operational description.

eleportation tango (in honour of Oxford group)

First note

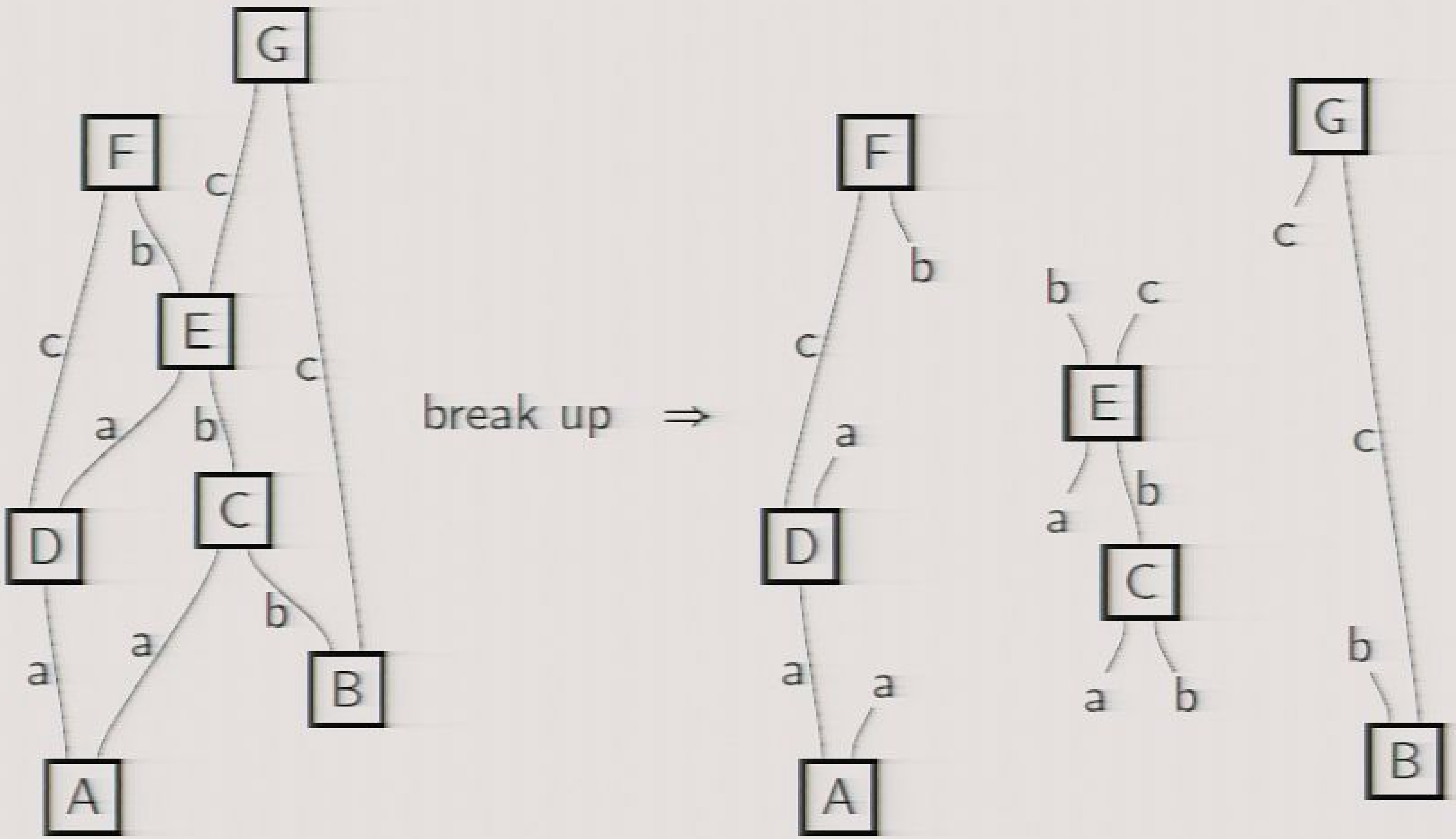


(easy to prove).
We have

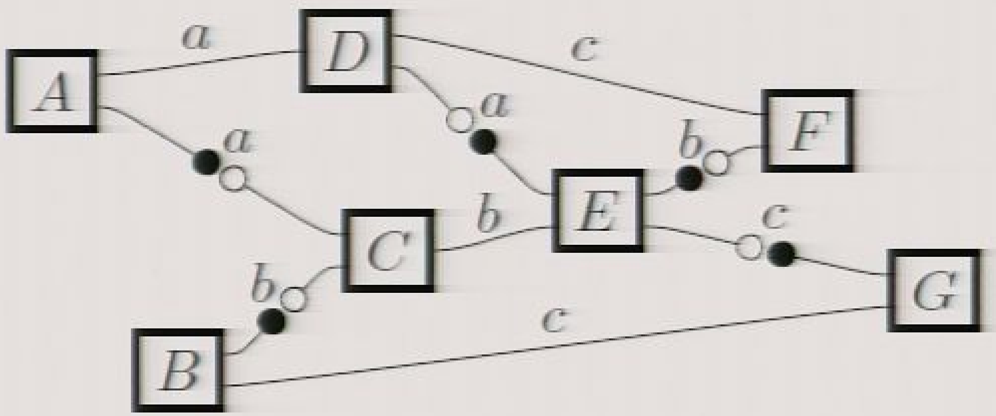


Maths and physics can inhabit the same diagram.

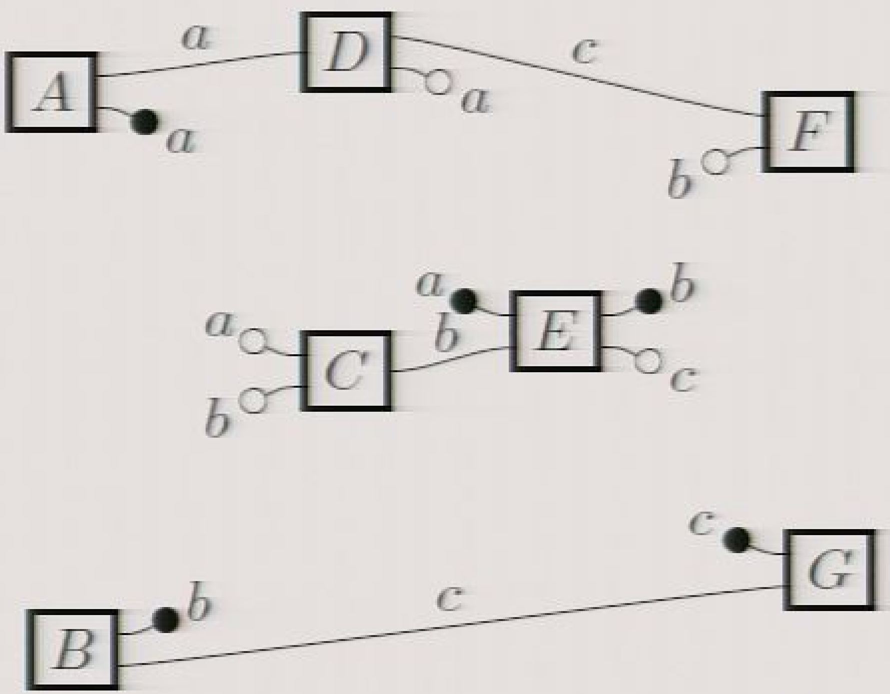
reaking circuits up into fragments



orresponding duotensor calculation



break up ⇒



Can break a circuit up into arbitrary fragments as we wish. Can do duotensor calculation for each fragment and then put them back together again.

May be interested in probabilities for one fragment alone ...

ramework has Formalism-locality

We simply quote the result:

The probability ratio

$$\frac{\text{Prob}(\text{E}[\text{i}])}{\text{Prob}(\text{E}[\text{j}])}$$

where E[i] and E[j] are two fragments corresponding to different outcome sets for the same experiment is

- **well conditioned** if and only if the corresponding duotensors, $E[i]$ and $E[j]$, are proportional, and
- **equal to** the constant of proportionality k in $E[i] = kE[j]$ (if well conditioned).

Hence we have formalism-locality.

ploading physical theories into framework

Physical theories can be uploaded into framework if the physical situation they pertain to can be described with operations and wires and Assumptions 1 and 2 are satisfied.

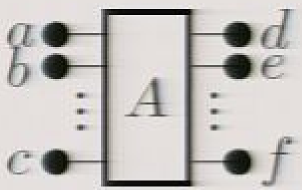
To upload a physical theory we need

1. A choice of fiducial effects and preparations for each system type.
2. An expression for the fiducial probabilities for each possible operation (these are the proabilities with fiducial preparations on the inputs and fiducial effects on the outputs). This gives us the duotensor with all black dots.
3. An expression for the hopping metric ●—● for each system type. The entries in this are the probabilities of the fiducial preparations followed by the fiducial effects. We can invert ●—● to get ○—○.

Can upload
CLASSICAL PROBABILITY THEORY
and
QUANTUM THEORY

ploading Quantum Theory

1. Have $\mathsf{X}^{\mathsf{a}_1}$ for $a_1 = 1$ to N_{a}^2, etc.
2. Fiducial probabilities given by[1]



$$= \text{Trace}\left[\hat{P}(\mathsf{X}^{d_4 e_5 \dots f_6}_{\mathsf{d}_4 \mathsf{e}_5 \dots \mathsf{f}_6}) \$ (\mathsf{A}^{\mathsf{d}_4 \mathsf{e}_5 \dots \mathsf{f}_6}_{\mathsf{a}_1 \mathsf{b}_2 \dots \mathsf{c}_3}) \hat{P}(\mathsf{X}^{\mathsf{a}_1 \mathsf{b}_2 \dots \mathsf{c}_3}_{a_1 b_2 \dots c_3})\right]$$

3. Hopping metric is given by

$$\bullet\!\!-\!\!\bullet = \text{Trace}\left(\hat{P}(\mathsf{X}^{a_1}_{\mathsf{a}_1}) \hat{P}({}_{a_1}\mathsf{X}^{\mathsf{a}_1})\right)$$

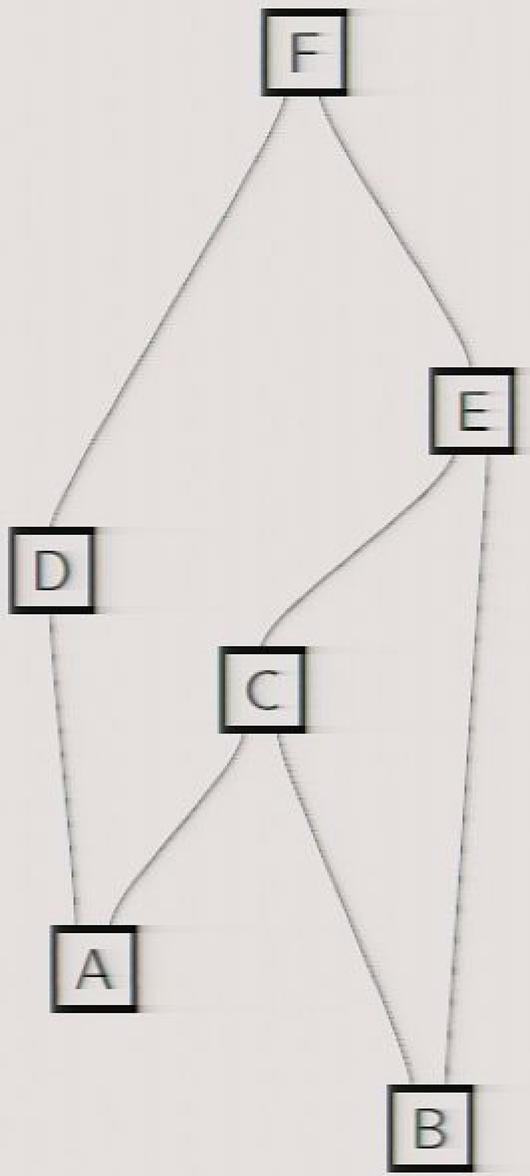
[1]where

$$\hat{P}(\mathsf{X}^{d_4 e_5 \dots f_6}_{\mathsf{d}_4 \mathsf{e}_5 \dots \mathsf{f}_6}) := \hat{P}(\mathsf{X}^{d_4}_{\mathsf{d}_4}) \otimes \hat{P}(\mathsf{X}^{e_5}_{\mathsf{e}_5}) \otimes \dots \otimes \hat{P}(\mathsf{X}^{f_6}_{\mathsf{f}_6})$$

$$\hat{P}(\mathsf{X}^{\mathsf{a}_1 \mathsf{b}_2 \dots \mathsf{c}_3}_{a_1 b_2 \dots c_3}) := \hat{P}({}_{a_1}\mathsf{X}^{\mathsf{a}_1}) \otimes \hat{P}({}_{b_2}\mathsf{X}^{\mathsf{b}_2}) \otimes \dots \otimes \hat{P}({}_{c_3}\mathsf{X}^{\mathsf{c}_3})$$

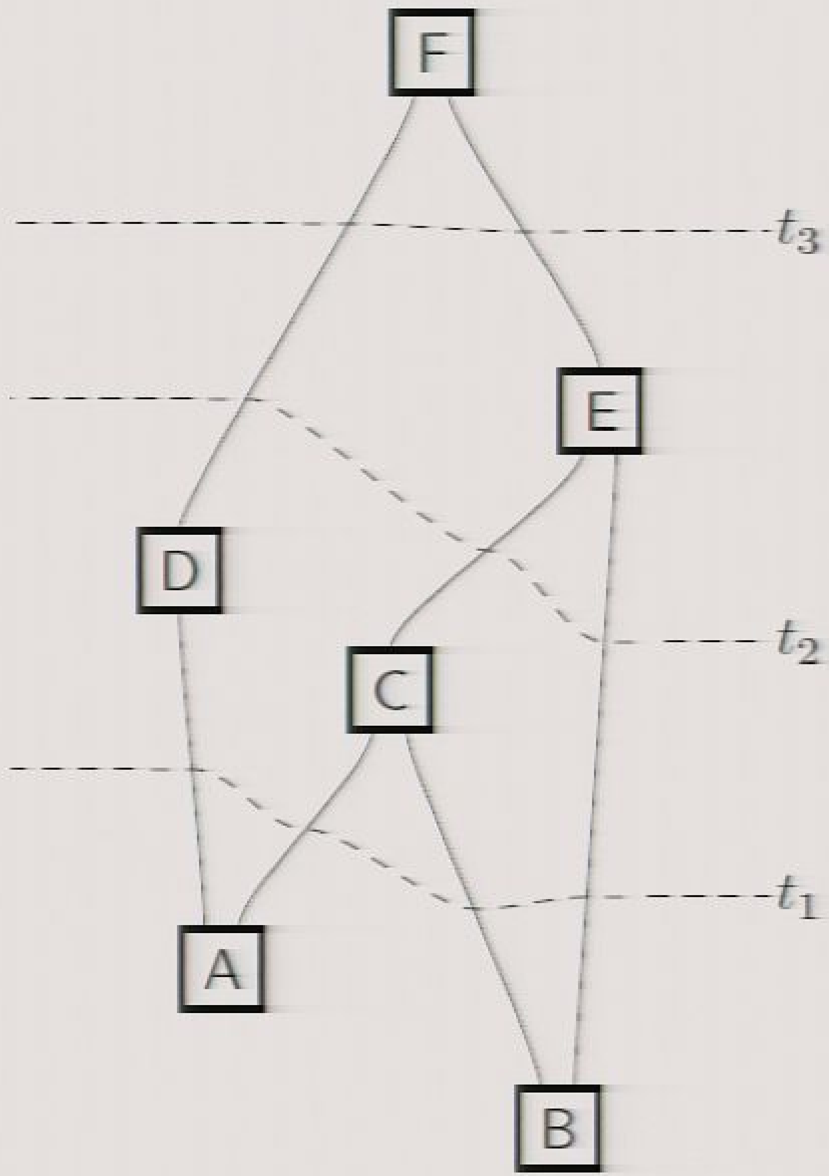
ow to foliate and why not to

Consider foliating the circuit

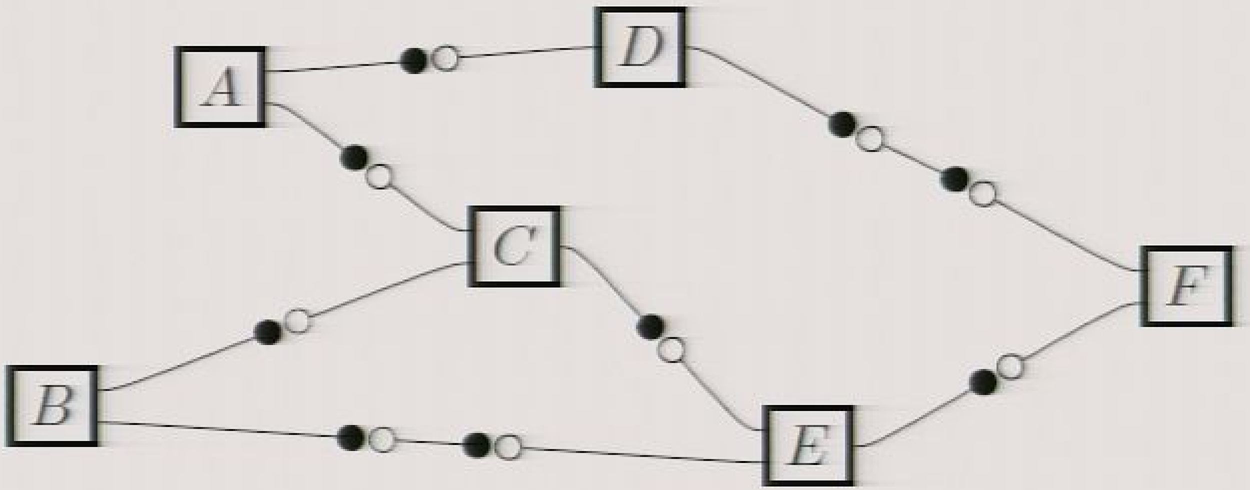


ow to foliate and why not to

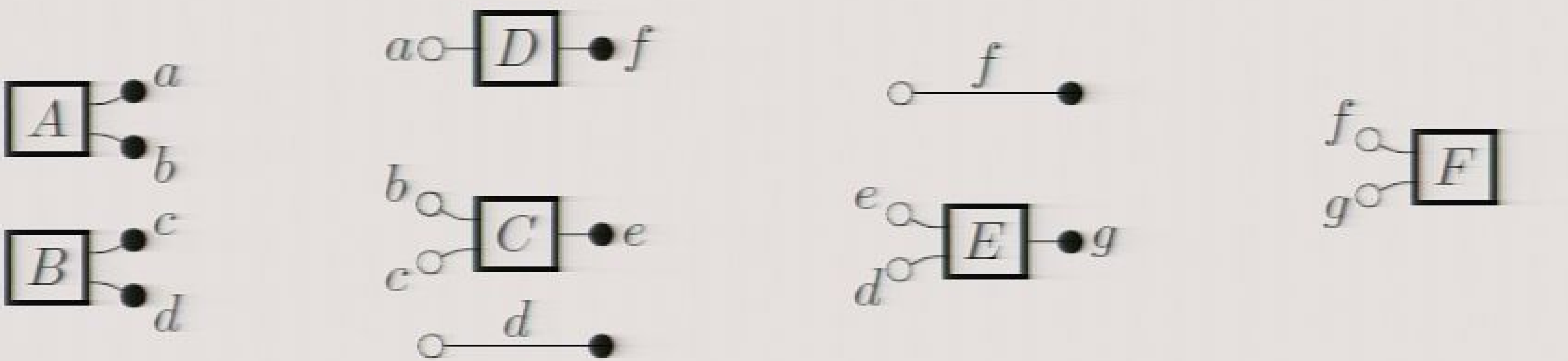
Consider foliating the circuit



Corresponds to

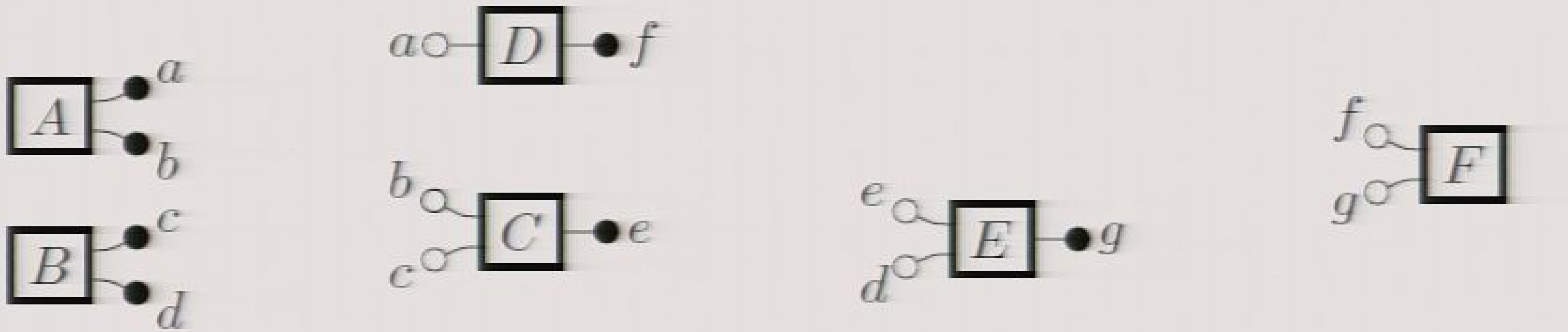


which we can break up into four duotensors

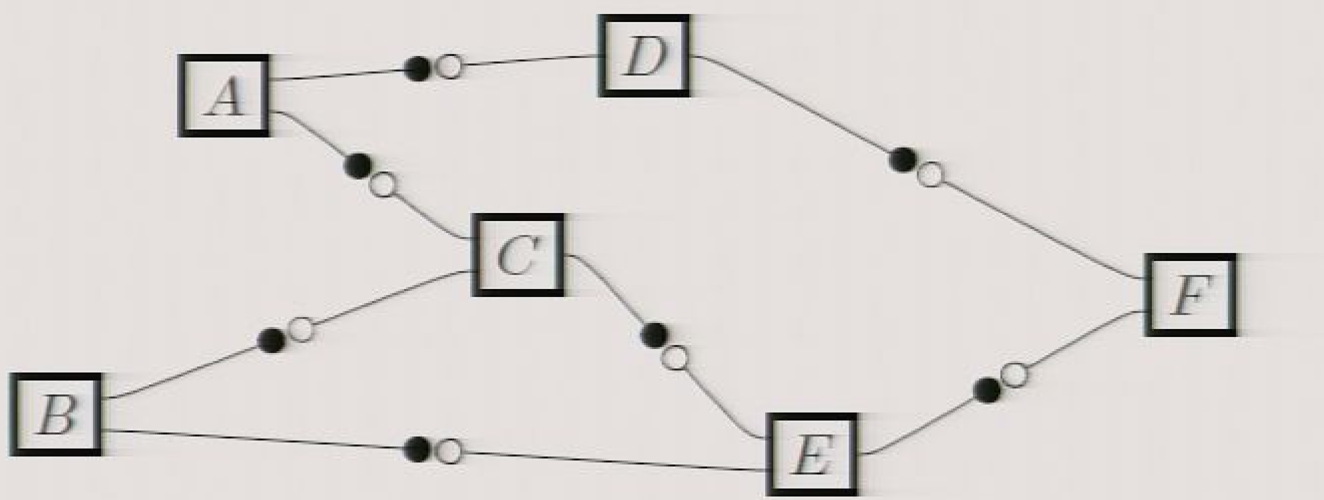


Have to pad calculation with identities.

Could simply drop the identities



This corresponds to



which does not result from a foliation.

Foliation can be done but, generically, results in unnecessary padding of calculation with identities.

ummary

Have two different worlds

- The world of physics (operational descriptions)
- The world of mathematics (duotensor calculations)

Have hybrid statements

- Assumption 1
- Assumption 2

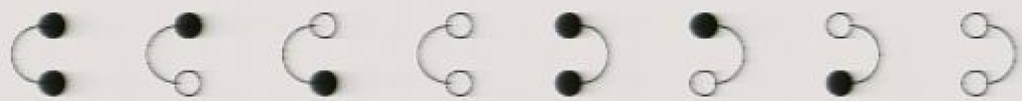
which allow these two worlds to tango.

Can have hybrid diagrams having physics and maths in the same diagram.

Classical Probability Theory and Quantum Theory fit naturally into this framework.

iscussion

- **Physics to mathematics correspondence principle.** For any physical theory, there exists a small number of simple hybrid statement that enable us to translate from the physical description to the corresponding mathematical calculation such that the mathematical calculation (in appropriate notation) looks the same as the physical description (in appropriate notation).
- Can we make use of



  May be related to the cups and caps of Abramsky, Coecke, ...
- Can we go beyond finite situation. Assumption 2 may be generalisable.
- Quantum Gravity?