Title: Orthogonality catastrophe 40 years later: fidelity approach, criticality and boundary CFT

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Abstract: More than forty years ago Nobel laureate P.W. Anderson studied the overlap between two nearby ground states. The result that the overlap tends to zero in the thermodynamics limit was catastrophic for those times. More recently the study of the overlap between ground states, i.e. the fidelity, led to the formulation of the so called fidelity approach to (quantum) phase transition (QPT). This new approach to QPT does not rely on the identification of order parameters or symmetry pattern; rathers it embodies the theory of phase transitions with an operational meaning in terms of measurements. Nowadays orthogonality of ground states is much less surprising. I will provide the general scaling behavior of the fidelity at regular and at critical points of the phase diagrams, Anderson's result being a particular case. These results are useful to many areas of theoretical physics. A related quantity extensively studied here, the fidelity susceptibility, is well known in various other contexts under different names. In metrology it is called quantum Fisher information; in the theory of adiabatic computation it represents the figure of merit for efficient computation; in yet another context it is known as (real part of) the Berry geometric tensor.

Orthogonality catastrophe
40 years later

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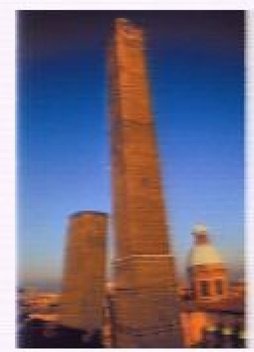

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December 10, 2010

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Approaches to criticality

Standard

$$f=\frac{1}{\beta L^d}\log tr\left(e^{-\beta H}\right)$$

$$e=\frac{\langle\Psi|H|\Psi\rangle}{L^d}$$

$$H=H_0+\lambda V$$

$$\chi_{VV}=\frac{\partial e}{\partial\lambda^2}=\frac{1}{L^d}\sum_{n\neq 0}\frac{|\langle n|V|0\rangle|^2}{E_n-E_0}$$

Fidelity

$$F=|\langle\Psi|\Psi'\rangle|$$

Why the fidelity?

How do we distinguish states? $|\Psi_P\rangle$ $|\Psi_Q\rangle$

by measuring! $A \to \{a_i, |i\rangle\}$

Observable

$$p_i = |\langle i|\Psi_P\rangle|^2 \quad q_i = p_i + \delta p_i$$

distinguishing experiments

Fisher (infinitesimal) distance

$$d_F(\{p_i\},\{q_i\}) = \sum_i \frac{(\delta p_i)^2}{p_i}$$

distance in probability space

Problem: each preparation $|\psi_P\rangle$ defines infinitely many probability distribution p_i, one for each observable.

One has to maximize over all possible experiments!

Surprise!

$$\max_{Exps} d_F(P,Q) =$$ Projective Hilbert space distance!

$$D_{FS}(P,Q) \equiv \cos^{-1}\frac{|\langle Q|P\rangle|}{\|Q\|\cdot\|P\|} \equiv \cos^{-1}(F)$$

← *Q-fidelity*!

(Wootters 1981)

Statistical distance and geometrical one collapse:

Hilbert space geometry ***is*** *(quantum) information geometry.....*

Question: What about non pure preparations ?!? ρ_P

Answer: Bures metric & Uhlmann fidelity!
(Braunstein Caves 1994)

$$d(\rho_0,\rho_1) = \cos^{-1}(F)$$

$$F(\rho_0,\rho_1) = tr\sqrt{\rho_1^{1/2}\rho_0\rho_1^{1/2}}$$

Remark: In the classical case one has commuting objects, simplification ....

Idea:

Distance → **Metric**

$$d_{FS}(\lambda, \lambda+\delta\lambda) = \arccos(F)$$
$$= \sqrt{G\,\delta\lambda^2} \equiv ds$$
$$ds^2 = G_{\mu,\nu}\,\delta\lambda^\mu\,\delta\lambda^\nu$$

defines a metric in the space of states

$$G_{\mu,\nu} = \sum_{n>0} \frac{\langle n|\partial_\mu H|0\rangle\langle 0|\partial_\nu H|n\rangle}{\left[E_n(\lambda) - E_0(\lambda)\right]^2}$$

$$F = 1 - \frac{1}{2} G_{\mu,\nu}\,\delta\lambda^\mu\,\delta\lambda^\nu + O\left[\delta\lambda^3\right]$$

Expectation:

At (Quantum) Critical Points $F \rightarrow$ **sharp drop**
induced metric $G \rightarrow$ **sharp increase**

Theorem: **At regular (gapped) points**
$G = O(L^d)$ **(extensive)**

Result (scaling hyp.): **At critical points**
$G = O(L^{d+q})$

Comparison

$$H = H_0 + \lambda V$$

$$\chi_{VV} = \frac{1}{L^d} \sum_{n \neq 0} \frac{|\langle n|V|0\rangle|^2}{E_n - E_0}$$

$$g = \frac{1}{L^d} \sum_{n \neq 0} \frac{|\langle n|V|0\rangle|^2}{(E_n - E_0)^2}$$

quasi-critical region

$$\chi_{VV}(\lambda \approx \lambda_c, L) \sim L^{(\zeta + d - 2\Delta_v)} \qquad \xi \gg L \qquad g(\lambda \approx \lambda_c, L) \sim L^{(2\zeta + d - 2\Delta_v)}$$

off-critical region (TDL)

$$\chi_{VV}(\lambda) \sim |\lambda - \lambda_c|^{-\nu(\zeta + d - 2\Delta_v)} \qquad \xi \ll L \qquad g(\lambda) \sim |\lambda - \lambda_c|^{-\nu(2\zeta + d - 2\Delta_v)}$$

$$|\langle \Psi | \Psi' \rangle| = e^{-L^{d+q} g \delta\lambda^2 / 2 + O(\delta\lambda^3)}$$

$q = 0$ **Regular point**

$q = 2\zeta + d - 2\Delta_v$ **Critical point**

More divergent

Fidelity good for crossover

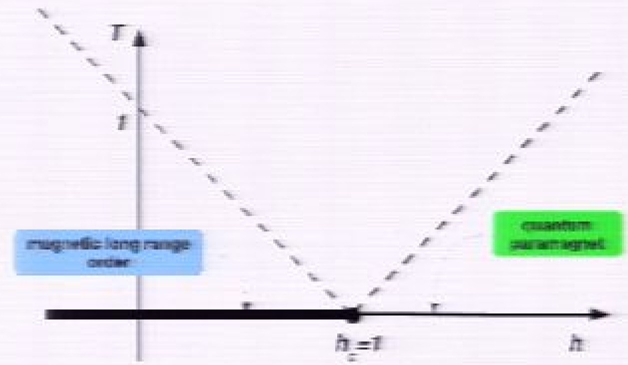
Zanardi, LVC, Giorda, PRA (2007)

Eigenvector of $g_{\mu,\nu}$ → direction of maximal distinguishability

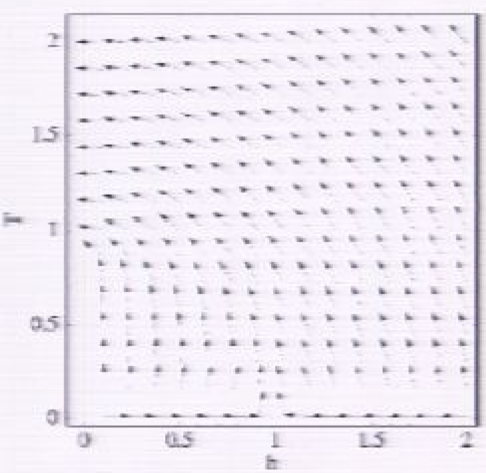
Consider (h,T) phase diagram

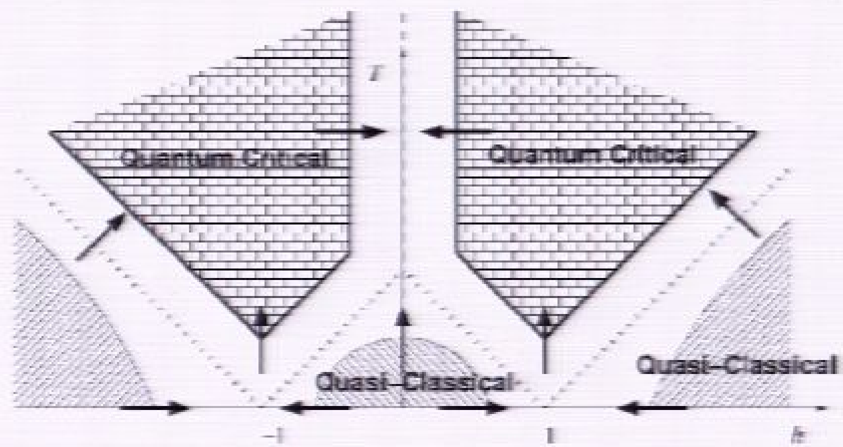
$$H_{Q-Ising} = -\sum_i \left[\sigma_i^x \sigma_{i-1}^x + h\sigma_i^z\right]$$

Sachdev



But in reality:





See also A. Kahn, P. Pieri PRA (2009) for BEC-BCS crossover

Fidelity enters in a multitude of topics

Fermi Edge Singularities in X-Ray absorpion
(also Loschmidt echo: Q-chaos)

Mahan, Nozieres, De Dominicis
Levitov, ...

Estimation theory: Quantum metrology
best measurement to *estimate* a given parameter

Braunstein, Caves, Lloyd,
Maccone, LCV, ...

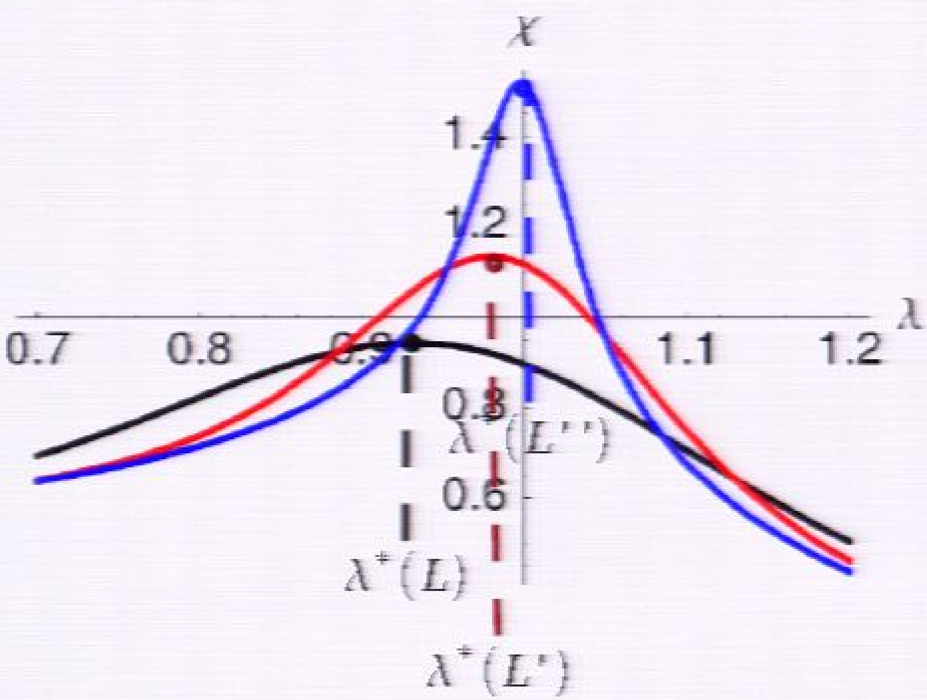
Adiabatic theorem and adiabatic quantum computation

Lidar, Rezakhani,
Polkovnikov,...

New universal quantities

LCV, Saluer, Zanardi

The search for critical point



$$\lambda^*(L) = \lambda_c + \frac{A}{L^{\theta}}$$

Shift exponent

Good method

⇕

Large Θ

Generally
(but not always)

$$\theta = \frac{1}{\nu}$$

$$\xi \sim |\lambda - \lambda_c|^{-\nu}$$

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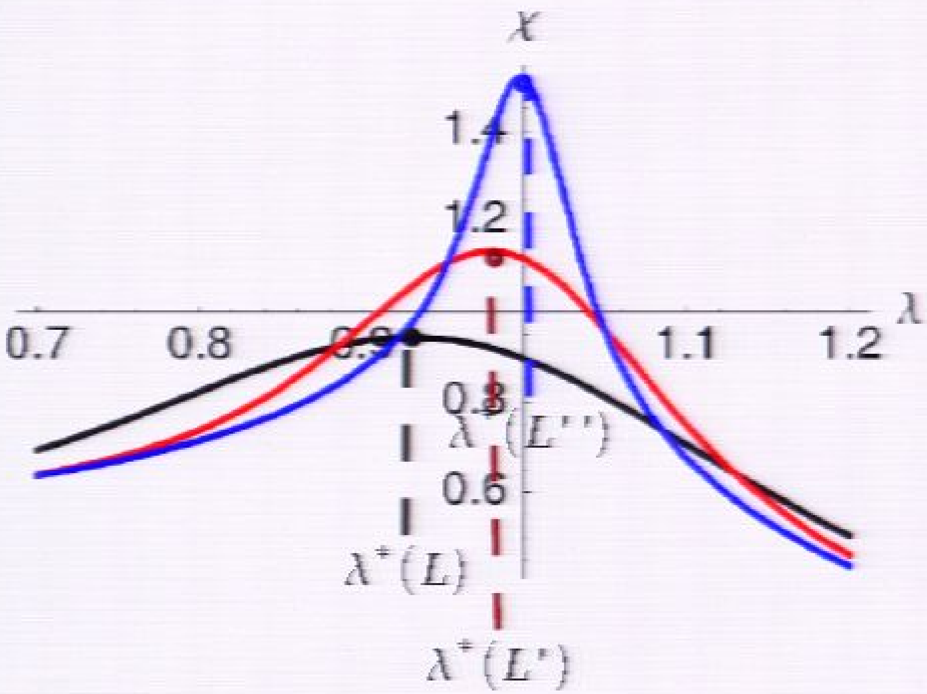

$$\lambda^*(L) = \lambda_c + \frac{A}{L^{\theta}}$$

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Efficiency of known methods

Name	Eqn	θ
Binder (4th order reduced) cumulant	$\partial_L B_L^{(4)}(\lambda^*)=0$	$1/\nu$
PRG	$\partial_L \vert L^{\zeta} \Delta_L(\lambda^*)\vert=0$	$1/\nu+c$
FSCM	$\partial_L \vert \langle V \rangle_L / L^d \vert=0$	$2/\nu$
Fidelity	$\partial_\lambda g_L(\lambda^*)=0$	$1/\nu$

Accelerated methods

Roncaglia, LVC, Degli Esposti Boschi, PRB (2008)

$$\tilde{\lambda}=\lambda-\lambda_c\,,\quad z=\tilde{\lambda}\,L^{1/\nu}$$

$$e_{sing}=L^{-(d+\zeta)}\left|\phi_0(z)+L^{-\omega}h(\tilde{\lambda})\phi_1(z)+\ldots\right|+F(\tilde{\lambda})L^{-(d+\zeta)}+\ldots$$

Scaling (RG) terms — Off-Scaling

$$e_L=e_\infty-\frac{\pi c\,v(\lambda)}{6}\frac{1}{L^2}+o(L^{-2})$$

$$e_L\equiv\langle H\rangle/L^d\quad b_L\equiv\langle V\rangle/L^d$$

Name	Eqn	θ
HCM (Homogeneity condition method)	$(d+\zeta+1)\partial_L e_L+L\,\partial_L^2 e_L=0$	$2/\nu+\omega$
MHC (Modified homogeneity condition)	$\dfrac{\partial_L b_L}{\partial_L e_L}=\dfrac{\partial_L^2 b_L}{\partial_L^2 e_L}$	$2/\nu+\omega$

Accelerated methods

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$$\tilde{\lambda} = \lambda - \lambda_c , \quad z = \tilde{\lambda} L^{1/\nu}$$

$$e_{sing} = L^{-(d+\zeta)} [\phi_0(z) + L^{-\omega} h(\tilde{\lambda}) \phi_1(z) + \dots] + F(\tilde{\lambda}) L^{-(d+\zeta)} + \dots$$

Scaling (RG) terms — **Off-Scaling**

$$e_L = e_\infty - \frac{\pi c v(\lambda)}{6} \frac{1}{L^2} + o(L^{-2})$$

$$e_L \equiv \langle H \rangle / L^d \quad b_L \equiv \langle V \rangle / L^d$$

Name	Eqn	θ
HCM (Homogeneity condition method)	$(d+\zeta+1)\partial_L e_L + L \partial_L^2 e_L = 0$	$2/\nu + \omega$
MHC (Modified homogeneity condition)	$\frac{\partial_L b_L}{\partial_L e_L} = \frac{\partial_L^2 b_L}{\partial_L^2 e_L}$	$2/\nu + \omega$

Tests

XY model in transverse field

$$H_{XY}=-\sum_i\left[\frac{(1+\gamma)}{2}\sigma_i^x\sigma_{i-1}^x+\frac{(1-\gamma)}{2}\sigma_i^y\sigma_j^y+h\,\sigma_i^z\right]$$

$$\begin{aligned}e_L(h,\gamma)=&\,e_\infty(1,\gamma)-\frac{\pi|\gamma|}{6}L^{-2}-\frac{7\pi^3(4\gamma^2-3)}{1440|\gamma|}L^{-4}\\&+(h-1)\left[b_\infty(1,\gamma)-\frac{\pi}{12|\gamma|}L^{-2}-\frac{7\pi^3(3-2\gamma^2)}{2880|\gamma|^3}L^{-4}\right]\\&-(h-1)^2\frac{\log(L)+\gamma_E+\log(8|\gamma|/\pi)-1}{2\pi|\gamma|}+O\left[(h-1)^3\right]\end{aligned}$$

$$d=\zeta=\nu=1$$

$$\theta_{standard}=1$$

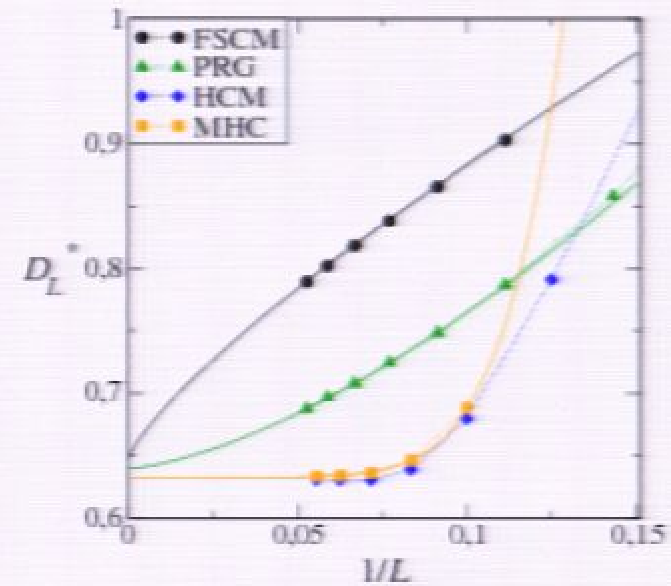
$$\theta_{PRG}=3$$

$$\theta_{HCM}=4$$

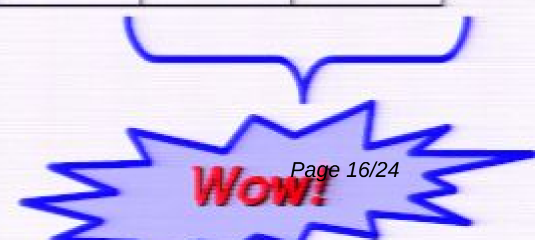
$$\theta_{MHC}=4$$

Anysotropic spin-1

$$H=J\sum_{\langle i,j\rangle}S_i^xS_j^x+S_i^yS_j^y+\lambda S_i^zS_j^z+D\left(S_i^z\right)^2$$



	FSCM	PRG	HCM	MHC
D_c	0.647	0.640	0.630	0.633
fitted θ	0.79	1.50	7.6	7.6



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New universal quantities (e.g. central charge, topological entropy)

LCV, Saluer, Zanardi,

		bulk	finite-size
specific heat	$C^{\sin}$	$t^{-\alpha}$	$L^{\alpha/\nu}$
magnetic susceptibility	χ	$t^{-\gamma}$	$L^{\gamma/\nu}$
correlation length	ξ	$t^{-\nu}$	L
free energy density	$f^{\sin}$	$t^{2-\alpha}$	L^{-d}
order parameter[a]	M	t^{β}	$L^{-\beta/\nu}$
latent heat[a]	ℓ_h	$t^{1-\alpha}$	$L^{(\alpha-1)/\nu}$

Universal quantities

Critical Point → Scale invariance → Universal Critical Exponents

Q: What about universal factors?

Energy of 1D Quantum Critical systems with PBC: I. Affleck, J. Cardy, PRL's 1986

$$E_L = e\,L - \frac{\pi c v}{6}\frac{1}{L} + o(L^{-1})$$

Von Neumann entropy of topologically ordered states of a region of boundary length *L*

$$S_L = s\,L - \gamma + o(L^0)$$

A. Hamma, R. Ionicioiu, P. Zanardi, PRA (2005)
A. Kitaev and J. Preskill, PRL (2006)
M. Levin and X. -G. Wen, PRL (2006)

} Universal Factors

Fidelity: universal factor

$$F = \left| \langle \Psi(\lambda), \Psi(\lambda') \rangle \right|$$

bulk term

boundary terms

$$\ln F = -f L^{d} - f_b L^{d-1} + \cdots + \ln g_0 + smaller \quad terms$$

"Residual distinguishability"

$$F = g_0 e^{-f L^{d} + \cdots}$$

Universal order one factor

&

Depends only on the boundary conditions!

Fidelity and Stat-Mech

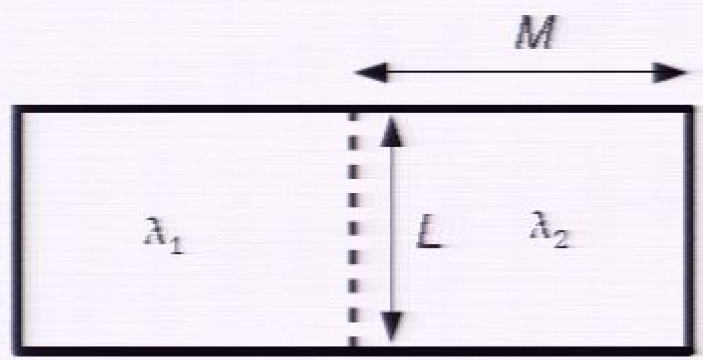
LCV, H. Saleur, P. Zanardi, PRB (2009)

d-dim Quantum Mechanics. Hamiltonian: $H(\lambda)$ ⟺ (*d*+1) Statistical Mechanics. Transfer matrix: $T(\lambda)$

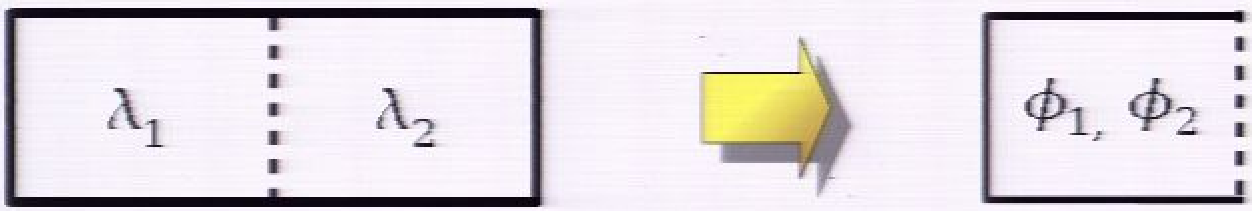
$$|\Psi(\lambda)\rangle = \lim_{M\to\infty} T(\lambda)^M |\phi_0\rangle / \sqrt{Z(\lambda)}$$

$$\langle \Psi(\lambda_1)|\Psi(\lambda_2)\rangle = \frac{Z(\lambda_1,\lambda_2)}{\sqrt{Z(\lambda_1)Z(\lambda_2)}}$$

$Z(\lambda_1,\lambda_2) =$



"Folding"



$= \langle \Omega_c | B \rangle$

Boundary state

From BCFT:
Affleck, Ludwig, Saleur, Oshikawa...

Universal order-one factor (boundary degeneracy)

$$\langle \Omega_c | B \rangle = g_0 \times \text{Bulk term}$$

⇒ Universal term in the fidelity g_0 depends only on the universality class and boundary conditions

Example: Luttinger liquid

consider (1+1) CFT:
example $c=1$ "free" Boson theory

$$S = \frac{\lambda}{2}\int dx^2\,(\nabla\phi)^2$$

direct calculation in the PBC case

M.-F. Yang (PRB 2007),
J. V. Fjaerestad JSTAT (2008)

$$F(\lambda_{1,}\lambda_2) = \frac{Z\left(\frac{\lambda_1+\lambda_2}{2}\right)}{\sqrt{Z(\lambda_1)Z(\lambda_2)}} = \prod_{k\neq 0}\sqrt{\frac{2\sqrt{\lambda_1\lambda_2}}{\lambda_1+\lambda_2}} = \left(\sqrt{\frac{2\sqrt{\lambda_1\lambda_2}}{\lambda_1+\lambda_2}}\right)^{L-1} = g_0 f^L$$

$$\Rightarrow \quad g_0 = \sqrt{\frac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}}}$$

agrees with BCFT result

I. Affleck, M. Oshikawa and H. Saleur
Nucl. Phys. B594 (2001)

Zero-order term: "experimental" check

(1+1) c=1 Universality class

XXZ 1D lattice quantum theory
(critical $|\Delta|<1$)

$$H=\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

(1+1) continuum field theory
c=1 CFT free boson λ

$$S = \frac{\lambda}{2}\int dx^2 (\nabla\phi)^2$$

calculation for **PBC**

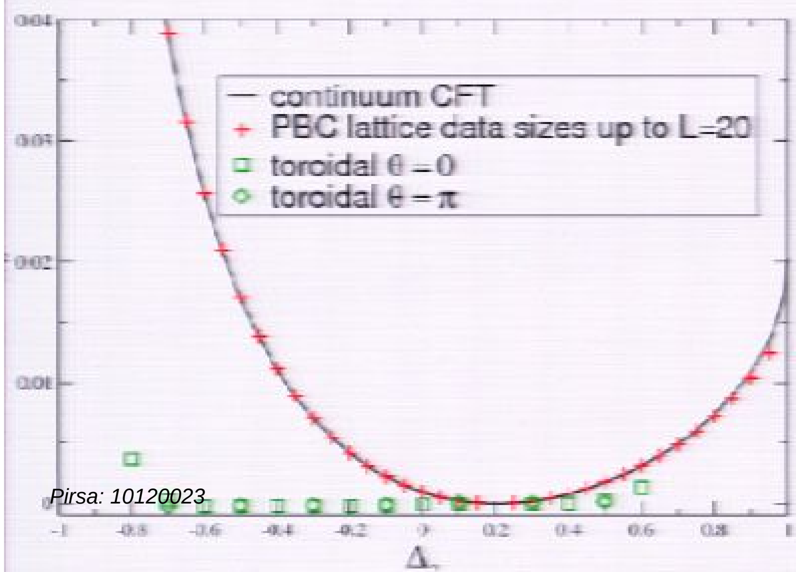
Bethe-Ansazt + Bosonization:

$$F(\Delta_1, \Delta_2) = |\langle \Psi(\Delta_1)|\Psi(\Delta_2)\rangle|$$

$$\lambda = \frac{\pi - \arccos(\Delta)}{2\pi^2}$$

$x_0 \equiv -\log(g_0)$ $\Delta_1 = 0.20$

$$g_0 = \sqrt{\frac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}}}$$



The **same** *g* appears also for excited states and/or free BCs!
instead **toroidal** BCs induce anti-periodic BCs in the CFT => $\ln g_0 = 0$!

toroidal BCs: $\sigma^{\pm}_{L+1} = e^{\pm i\theta}\sigma^{\pm}_1$, $\sigma^{z}_{L+1} = \sigma^{z}_1$

Generalization

$$H=\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

$$F[\Delta_1,\Delta_2]=\left|\langle\Psi(\Delta_1)|\Psi(\Delta_2)\rangle\right|$$

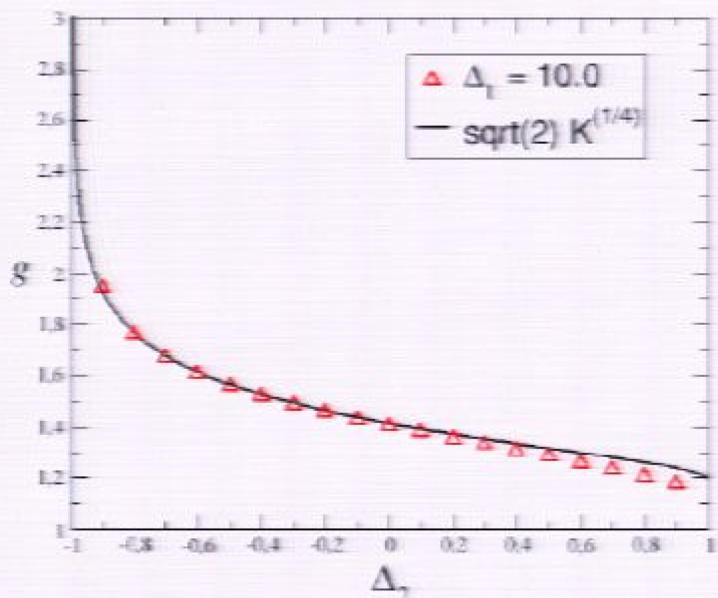
$$\Delta_1 \gg 1\,,\quad |\Psi(\Delta_1)\rangle \simeq \frac{1}{\sqrt{2}}\left[|Ne\acute{e}l\rangle + |ANe\acute{e}l\rangle\right]$$

Induces **Dirichlet** BCs on the field φ

$$g_D = K^{1/4}$$

Luttinger Liquid Parameter

$$g_{tot} = 2\times\frac{1}{\sqrt{2}}\times K^{1/4} = \sqrt{2}K^{1/4}$$



Summary

- Fidelity: geometric approach, operational meaning. Adiabatic quantum computation, Estimation theory, Equilibration...
- Want to spot the critical point?
  $\rightarrow$ Use accelerated methods
- Universal *factors: Topological Fidelity*

Thanks!