

Title: Mathematical Physics (PHYS 624) - Lecture 14

Date: Dec 09, 2010 09:00 AM

URL: <http://pirsa.org/10120006>

Abstract:

## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^\nu$$



## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^\nu \sum \underline{\text{series}} \quad (\text{as } x \rightarrow a)$$

## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^v \sum \underline{\text{series}} \quad (\text{as } x \rightarrow a)$$

$$-y'' + (x^2 + c)y = 0$$

## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^v \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

$$-y'' + (x^2 + c)y = 0$$

$y \sim ? \text{ as } x \rightarrow \infty$

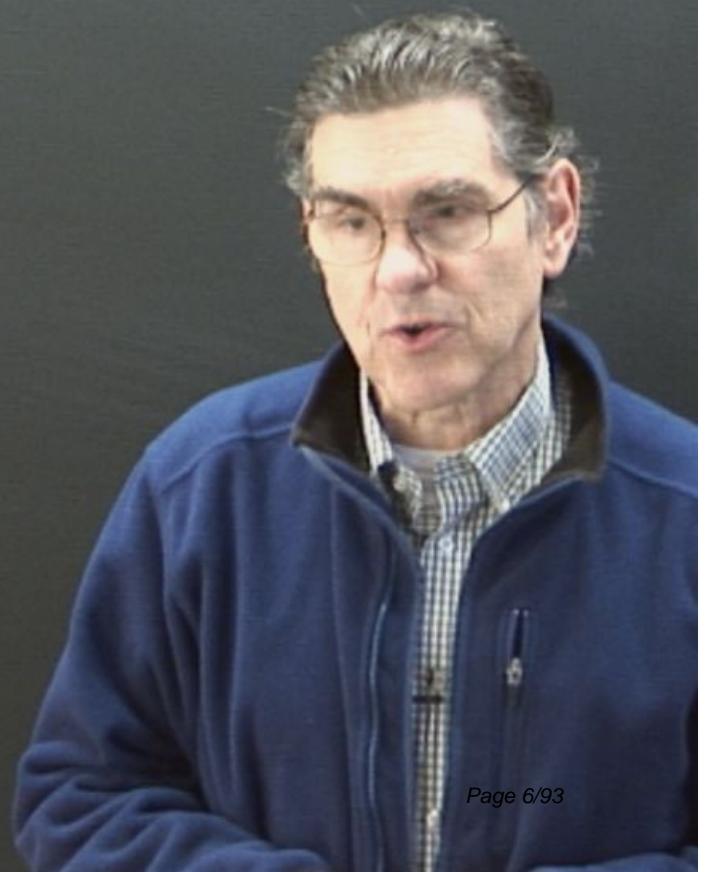


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$$y(x) \sim e^{\text{blah}(x)} x^v \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

$$-y'' + (x^2 + c)y = 0 \quad y'' = Q(x)y$$

$y \sim ? \text{ as } x \rightarrow \infty$

## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^{\nu} \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

$$-y'' + (x^2 + c)y = 0 \\ y \sim ? \text{ as } x \rightarrow \infty$$

$$y'' = Q(x)y \\ y \sim e^{\pm \int \sqrt{Q}}$$



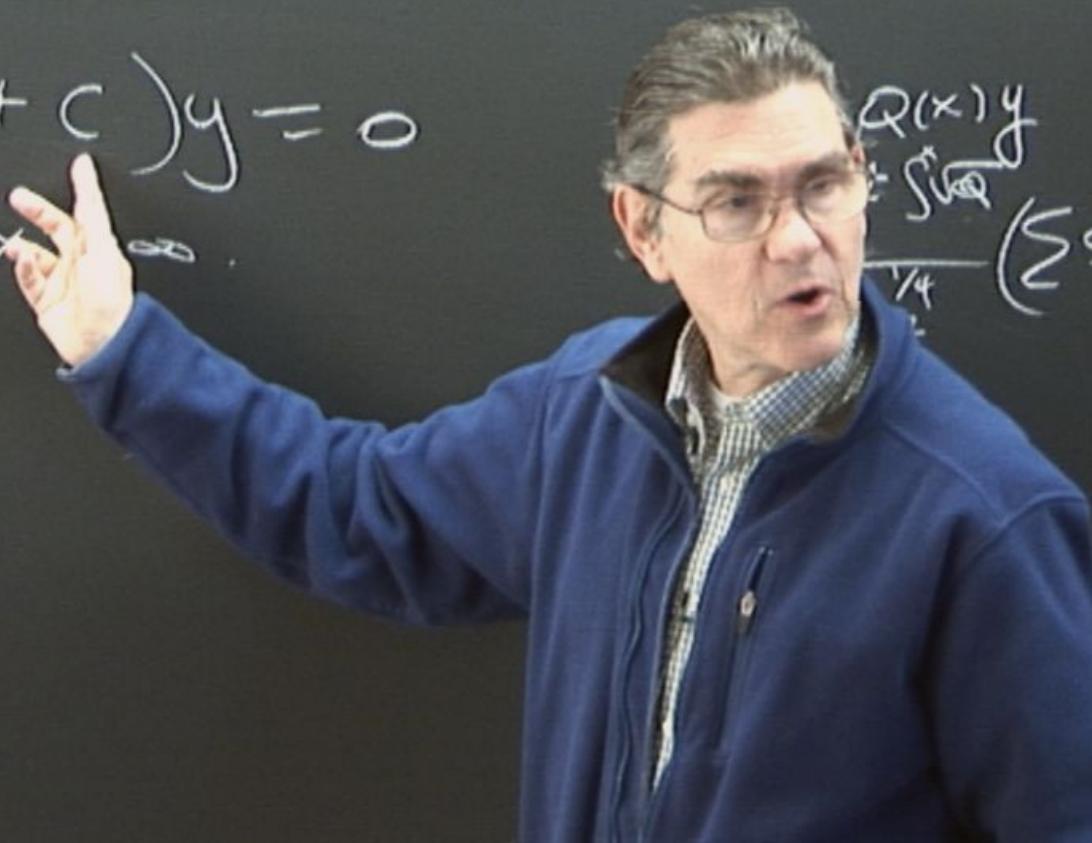
## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^v \sum \underline{\text{series}} \quad (\text{as } x \rightarrow a)$$

$$-y'' + (x^2 + c)y = 0$$

$y \sim ? \text{ as } x \rightarrow \infty$

$$\frac{Q(x)y}{\sqrt{x}} = \sqrt{\frac{Q(x)}{x}} \left( \sum \text{series} \right)$$



## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^{\nu} \sum \text{series} \quad (\text{as } x \rightarrow \infty)$$

$$-y'' + (x^2 + c)y = 0 \\ y \sim ? \text{ as } x \rightarrow \infty.$$

$$\begin{aligned} y'' &= Q(x)y \\ y &\sim e^{\pm \sqrt{Q}} \left( \sum \text{series} \right) \\ y &\sim e^{\pm \frac{x}{\sqrt{Q}}} \end{aligned}$$

## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^{\nu} \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

$$-y'' + (x^2 + c)y = 0 \\ y \sim ? \text{ as } x \rightarrow \infty.$$

$$\begin{aligned} y'' &= Q(x)y \\ y &\sim \frac{e^{\pm \sqrt{Q}}}{\sqrt{Q}} (\sum \text{series}) \\ y &\sim \frac{e^{\pm \sqrt{Q}}}{\sqrt{Q}} \sum_{n=0}^{\infty} \frac{a_n}{x^{2n}} \end{aligned}$$

## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^{\nu} \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

$$-y'' + (x^2 + c)y = 0 \\ y \sim ? \text{ as } x \rightarrow \infty$$

$$y'' = Q(x)y \\ y \sim e^{\pm \sqrt{Q}x} \left( \sum \text{series} \right)$$

$$y \sim \frac{e^{\pm \sqrt{Q}x}}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{a_n}{x^{2n}} \\ (\text{as } x \rightarrow \infty)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(xy')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$



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$x=\infty$  is an ISP

$$y(x) \sim \underline{e^{\frac{2\sqrt{x}}{\sqrt{x}}}} \text{ series}$$

(as  $x \rightarrow 0$ )

$$y'' = Q(x)y$$

$$y \sim e^{\pm \int Q(x) dx} (\sum \text{series})$$

$$y \sim e^{\pm \sqrt{x}} \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$

(as  $x \rightarrow \infty$ )

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)}$$

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$$y \sim e^{\pm \frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{a_n}{x^{2n}} \quad (\text{as } x \rightarrow 0)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)}$$

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$$y(x) \sim e^{\frac{2\sqrt{x}}{x^{1/4}}} \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$

(as  $x \rightarrow 0$ )

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$$y \sim e^{\pm \int \sqrt{Q}} (\sum \text{series})$$

$$y \sim e^{\pm \int \frac{x^2}{\sqrt{Q}}} \sum_{n=0}^{\infty} \frac{a_n}{x^{2n}} \quad (\text{as } x \rightarrow \infty)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)}$$

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$$y \sim e^{\pm \int Q(x) dx} (\sum \text{series})$$

$$y \sim e^{\pm \int \frac{x^2}{Q(x)} dx} \sum_{n=0}^{\infty} \frac{a_n}{x^{2n}} \quad (\text{as } x \rightarrow \infty)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)}$$

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$$y(x) \sim e^{\frac{2\sqrt{x}}{x^{1/4}}} \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \quad \leftarrow$$

$$\text{I. } \sum_{n=0}^{10}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

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$$y(x) \sim e^{2\sqrt{x}} \cdot \frac{1}{x^{1/4}} \cdot \frac{1}{n=0} \cdot \frac{1}{x^n} \quad \leftarrow$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \quad \leftarrow$$

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$$\text{I. } \sum_{n=0}^{10} \frac{x^n}{(n!)^2} \\ y(x)$$

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$x = \infty$  is an ISP

$$y(x) \sim e^{2\sqrt{x}} \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$

I.  $\frac{\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}}{y(x)}$

II  $\frac{e^{\frac{x^{1/4}}{2\sqrt{x}}}}{y(x)}$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \quad \leftarrow$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

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$x = \infty$  is an ISP

$$y(x) \sim \frac{e^{2\sqrt{x}}}{x^{1/4}} \sum_{n=0}^{\infty} \frac{u_n}{x^n} \quad \leftarrow$$

$$\text{I. } \sum_{n=0}^{10} \frac{x^n}{(n!)^2} \overline{y(x)}$$

$$\text{II} \quad \frac{e^{2\sqrt{x}}}{x^{1/4}} \overline{y(x)}$$

$$f(x) \sim x \text{ as } x \rightarrow \infty$$

$$\int f(x) \sim \frac{x^2}{2}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \quad \leftarrow$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

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$$y(x) \sim \underbrace{e^{\frac{x}{2\sqrt{x}}}}_{x} \sum_{n=0}^{\infty} \frac{u_n}{x^n} \quad \leftarrow$$

$$\text{I. } \sum_{n=0}^{10} \frac{x^n}{(n!)^2} \quad ?$$

$$\text{II} \quad \frac{e^{\frac{x}{2\sqrt{x}}}}{y(x)} \quad ?$$

$$f(x) \sim x \text{ as } x \rightarrow \infty$$

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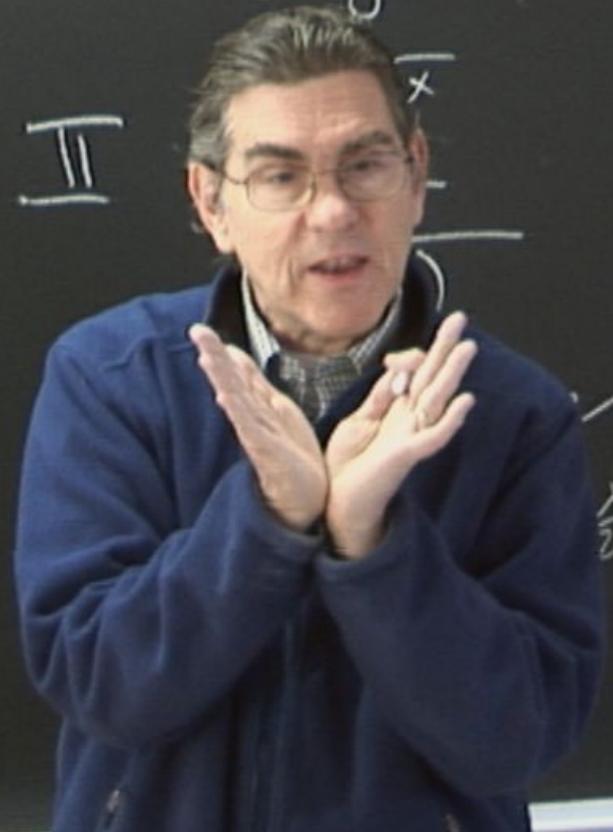
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II

?



$$x \text{ as } x \rightarrow \infty$$

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$$y(x) \sim \underbrace{e^{\frac{2\sqrt{x}}{x^{1/4}}}}_{n \geq 0} \sum_{n \geq 0} \frac{a_n}{x^n}$$

$$\text{I. } \sum_{n=0}^{10} \frac{x^n}{(n!)^2} \quad ?$$

$$\text{II} \quad \frac{e^{\frac{2\sqrt{x}}{x^{1/4}}}}{y(x)} \quad ?$$

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## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^v \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

$$\frac{-y'' + (x^2 + c)y = 0}{y \sim ? \text{ as } x \rightarrow \infty}$$

$$y \sim \frac{e^{x^2/2}}{\sqrt{x}} \{ \dots \} \quad |x| \rightarrow \infty$$

$$y'' = Q(x)y$$
$$y \sim e^{\pm \int \sqrt{Q}} \left( \sum \text{series} \right)$$

$$y \sim \frac{e^{\pm \int \frac{a_n}{x^n}}}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{a_n}{x^n} \quad (\text{as } x \rightarrow \infty)$$



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$$y(x) \sim e^{\text{blah}(x)} x^v \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

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$$y'' = Q(x)y$$
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$$y \sim \frac{e^{\pm \int \frac{1}{\sqrt{x}}}}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$
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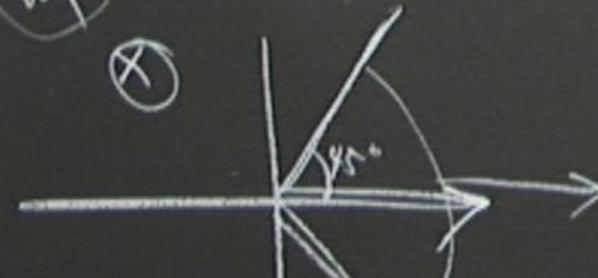
## Fun with asymptotic Series

$$y(x) \sim e^{\text{blah}(x)} x^v \sum \underline{\text{series}} \quad (\text{as } x \rightarrow \infty)$$

$$\underline{y'' + (x^2 + c)y = 0}$$

$y \sim ?$  as  $x \rightarrow \infty$

$$y \sim \boxed{e^{x^2/2}} \sum \dots \quad |x| \rightarrow \infty$$



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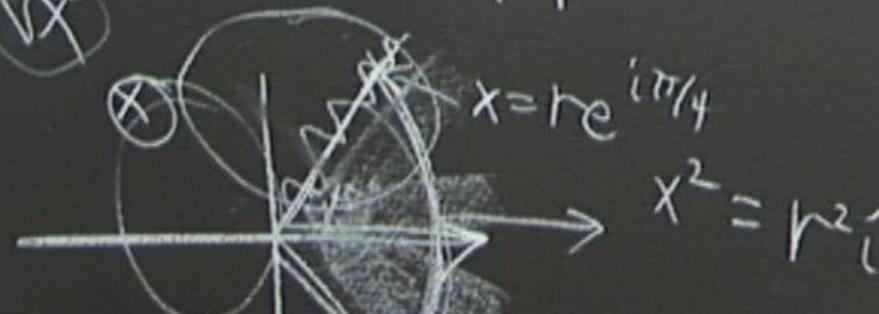
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$$y'' = Q(x) y$$
$$y \sim e^{\pm \int \sqrt{Q} dx} \left( \sum \text{series} \right)$$

$$y \sim \frac{e^{\pm \int \frac{Q}{2} dx}}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{a_n}{x^n} \quad (\text{as } x \rightarrow \infty)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(xy')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{((n-1)!)^2}$$

$x = \infty$  is an ISP

$$y(x) \sim e^{\frac{2\sqrt{x}}{x^{1/4}}} \sum_{n=0}^{\infty}$$

Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (v + \frac{1}{2})y$$



$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

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### Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (\nu + \frac{1}{2})y$$

Parab cyl fns  $D_\nu(x)$

$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{(series)}$$

$$D_\nu(x)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

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$x = \infty$  is an ISP

$$y(x) \sim e^{-\frac{x}{2\sqrt{x}}} \frac{1}{x^{1/4}}$$

### Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (\nu + \frac{1}{2})y$$

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$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{ (series)}$$

$$D_\nu(x) \sim e^{-\frac{x^2}{4}}$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

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$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{ (series)}$$

$$D_\nu(x) \sim e^{-\frac{x^2}{4}} \sum \dots \quad (x \rightarrow +\infty)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

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$x = \infty$  is an ISP

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### Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (\nu + \frac{1}{2})y$$

Parab cyl fn's  $D_\nu(x)$

$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{ (series)}$$

$$D_\nu(x) \sim e^{-\frac{x^2}{4}} \sum \dots \quad (x \rightarrow +\infty)$$

$$D_\nu(x) \sim e^{-\frac{x^2}{4x}} + e^{\frac{x^2}{4x}} \quad (x \rightarrow -\infty)$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(xy')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{((n-1)!)^2}$$

$x=\infty$  is an ISP

$$y(x) \sim \frac{e^{\frac{2\sqrt{x}}{x}}}{x} \sum_{n=0}^{\infty}$$

### Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (\nu + \frac{1}{2})y$$

Parab cyl fn's  $D_\nu(x)$

$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{ (series)}$$

$$D_\nu(x) \sim \frac{e^{-\frac{x^2}{4}}}{\sqrt{x}} \sum \dots \quad (x \rightarrow +\infty)$$

$$D_\nu(x) \sim c_1 e^{-\frac{x^2}{4x}} + c_2 e^{\frac{x^2}{4x}} \left( \frac{e^{\frac{x^2}{4x}}}{\Gamma(-\nu)} \right) \quad (x \rightarrow -\infty)$$

$$\Gamma(-\nu) = 0$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

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$x = \infty$  is an ISP

$$y(x) \sim \frac{e^{\frac{2\sqrt{x}}{x}}}{x^{\frac{1}{4}}} \quad n \geq 0$$

### Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (\nu + \frac{1}{2})y$$

Parab cyl fn's  $D_\nu(x)$

$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{ (series)}$$

$$D_\nu(x) \sim \frac{e^{-\frac{x^2}{4}}}{\sqrt{x}} \sum \dots \quad (x \rightarrow +\infty)$$

$$D_\nu(x) \sim c_1 e^{-\frac{x^2}{4x}} + \frac{e^{\frac{x^2}{4x}}}{\sqrt{x}} \left( \frac{e^{\frac{i\nu\pi}{2}}}{1-i\nu} \right) \quad (x \rightarrow -\infty)$$

$$\Gamma(-\nu) = \infty$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(xy')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{((n-1)!)^2}$$

$x=\infty$  is an ISP

$$y(x) \sim \frac{e^{\frac{x}{\sqrt{x}}}}{x^{\frac{1}{\sqrt{x}}}} \quad n \neq 0$$

### Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (\nu + \frac{1}{2})y$$

Parab cyl fns  $D_\nu(x)$

$$x \rightarrow +\infty \quad y(x) \sim \frac{e^{\pm \frac{x^2}{4}}}{\sqrt{x}} \text{ (series)}$$

$$D_\nu(x) \sim \frac{e^{-\frac{x^2}{4}}}{\sqrt{x}} \sum \dots \quad (x \rightarrow +\infty)$$

$$D_\nu(x) \sim c_1 e^{-\frac{x^2}{4}} + c_2 e^{\frac{x^2}{4}} \frac{e^{\nu \pi i}}{\Gamma(-\nu)} \quad (x \rightarrow -\infty)$$

$$\Gamma(-\nu) = \infty \quad \nu = 0, 1, 2, \dots$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(xy')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{((n-1)!)^2}$$

$x=\infty$  is an I.S.P

$$y(x) \sim e^{\frac{2\sqrt{x}}{x}} \sum_{n=0}^{\infty}$$

Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4} y = \frac{E}{(V+\frac{1}{2})} y$$

Parab cyl fns  $D_V(x)$

$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{ (series)}$$

$$D_V(x) \sim \frac{e^{-\frac{x^2}{4}}}{\sqrt{x}} \sum \dots \quad (x \rightarrow +\infty)$$

$$D_V(x) \sim C_1 e^{-\frac{x^2}{4x}} + C_2 e^{\frac{x^2}{4x}} \left( \frac{e^{\frac{2\pi i}{\lambda} x}}{\Gamma(-V)} \right) \quad (x \rightarrow -\infty)$$

$$\Gamma(-V) = \infty \quad V=0, 1, 2, \dots$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(xy')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{((n-1)!)^2}$$

$x=\infty$  is an I.S.P

$$y(x) \sim \frac{e^{\frac{x}{2\sqrt{x}}}}{x^{1/4}}$$

Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = \frac{E}{(V+\frac{1}{2})}y$$

Parab cyl fns  $D_V(x)$

$$x \rightarrow +\infty \quad y(x) \sim \frac{e^{\pm \frac{x^2}{4}}}{\sqrt{x}} \text{ (series)}$$

$$D_V(x) \sim \frac{e^{-\frac{x^2}{4}}}{\sqrt{x}} \sum \dots \quad (x \rightarrow +\infty)$$

$$D_V(x) \sim e^{-\frac{x^2}{4x}} + \frac{e^{+\frac{x^2}{4}}}{\sqrt{x}} \frac{e^{\pm i\pi}}{1-y} \quad (x \rightarrow -\infty)$$

$$\Gamma(-V)=0 \quad V=0, 1, 2, \dots$$

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

$$xy'(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!}$$

$$(xy')' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!^2}$$

$x=\infty$  is an ISP

$$y(x) \sim e^{\frac{2\sqrt{x}}{x}} \sum_{n=0}^{\infty}$$

Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = \frac{E}{(v+\frac{1}{2})}y$$

Parab cyl fns  $D_V(x)$

$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \sum_{n=0}^{\infty} \dots \quad (\text{series})$$

$$D_V(x) \sim e^{-\frac{x^2}{4}} \sum_{n=0}^{\infty} \dots \quad (x \rightarrow +\infty)$$

$$D_V(x) \sim e^{-\frac{x^2}{4x}} + e^{\frac{x^2}{4x}} \sum_{n=0}^{\infty} \frac{(-V)^n}{\sqrt{x}} \quad (x \rightarrow -\infty)$$

$$\Gamma(-V) = \infty \quad V=0, 1, 2, \dots$$

Convergent

$$\sum_{n=0}^{\infty} a_n x^n$$

Convergent

$$\sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \frac{1}{n!}$$

for Atoms

Molecules

asympt (div.)

$$\sum a_n x^n$$

$$a_n = n!$$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

means

for Atoms

Molecules

Convergent

$$\sum_{n=0}^{\infty} a_n x^n$$

$$a_n = \frac{1}{n!}$$

asympt (div.)

$$\sum a_n x^n$$

$$a_n = n!$$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

$$\text{means } \left[ f(x) - \sum_{n=0}^N a_n x^n \right]$$

# for Atoms

# Molecules

a symP (div.)

$$\sum a_n x^n$$

$$a_n = n!$$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

means  $\left[ f(x) - \sum_{n=0}^N a_n x^n \right] \sim a_{N+1} x^{N+1} \text{ as } x \rightarrow 0$

for ALL N

# for Atoms

# Molecules

$$\text{asympt} \quad (\text{div.})$$

$$\sum a_n x^n$$

$$a_n = n!$$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

means  $\left[ f(x) - \sum_{n=0}^N a_n x^n \right] \sim a_{N+1} x^{N+1} \text{ as } x \rightarrow 0$  for ALL N

$$\lim_{x \rightarrow 0} \frac{f(x) - \sum_{n=0}^N a_n x^n}{x^{N+1}} = a_{N+1}$$

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Is It A  
Molecule?

Convergent

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \quad a_n$$

$$S_N = \sum_{n=0}^N a_n x^n$$

does limit as  $N \rightarrow \infty$  of  $S_N$   
exist?

asympt (div.)

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \quad a_n$$

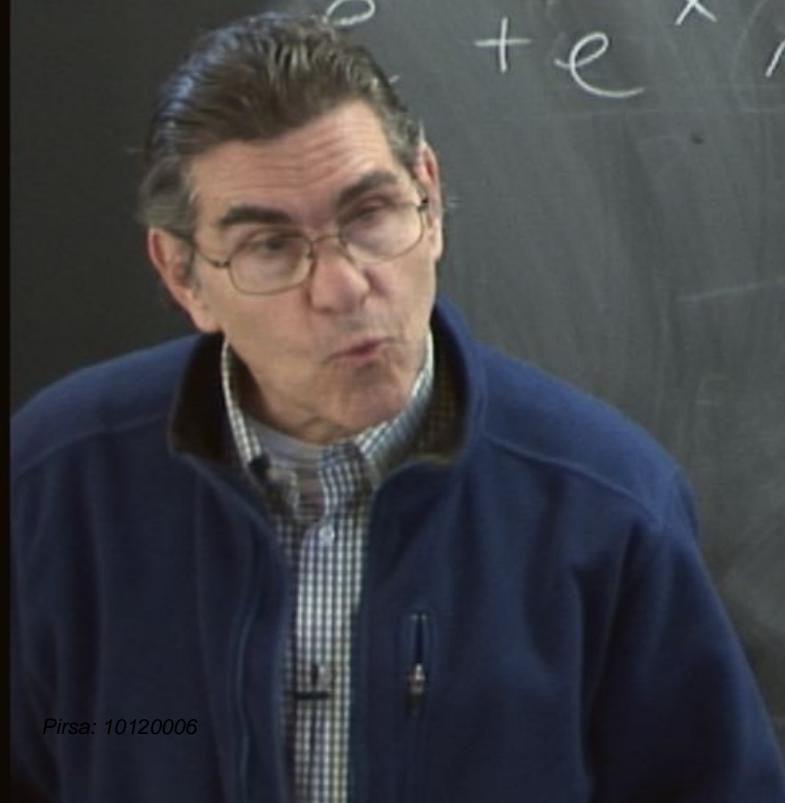
$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

$$\text{means } \left[ f(x) - \sum_{n=0}^N a_n x^n \right] \sim a_{N+1} x^{N+1} \text{ as } x \rightarrow 0 \quad \text{for ALL } N$$

$$\lim_{x \rightarrow 0} \frac{f(x) - \sum_{n=0}^N a_n x^n}{x^{N+1}} = a_{N+1}$$

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$

$$e^x + e^{-x} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$



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$$e^x + e^{-x} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ as } x \rightarrow 0^+$$

OPT. ASYMP APPROX

Parabolic Cylinder Eqn

$$-\gamma'' + \frac{x^2}{4}y = \left(\nu + \frac{1}{2}\right)y$$

Parab cyl fns  $D_\nu(x)$

$$x \rightarrow +\infty \quad y(x) \sim e^{\pm \frac{x^2}{4}} \text{ (series)}$$

$$D_\nu(x) \sim e^{-\frac{x^2}{4}} \sum \dots \quad (x \rightarrow +\infty)$$

$$D_\nu(x) \sim e^{-\frac{x^2}{4}} + e^{+\frac{x^2}{4}} \frac{\Gamma(\nu+1)}{\sqrt{x}} \quad (x \rightarrow -\infty)$$

$$\Gamma(-\nu) = \infty \quad \nu = 0, 1, 2, \dots$$

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it a  
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Convergent

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \quad a_n$$

$$S_N = \sum_{n=0}^N a_n x^n$$

does limit as  $N \rightarrow \infty$  of  $S_N$   
exist?

a.simp (div.)

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \quad a_n$$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

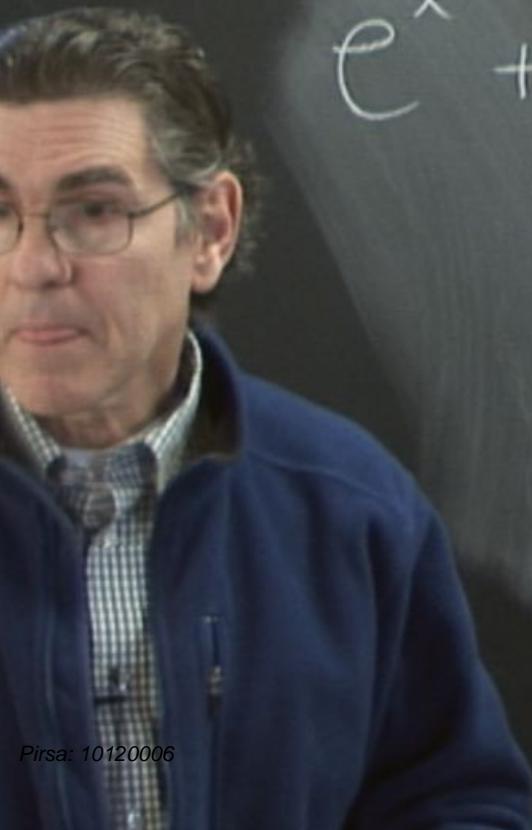
means  $\left[ f(x) - \sum_{n=0}^N a_n x^n \right] \sim a_{N+1} x^{N+1}$  as  $x \rightarrow 0$   
for ALL N

$$\lim_{x \rightarrow 0} \frac{f(x) - \sum_{n=0}^N a_n x^n}{x^{N+1}} = a_{N+1}$$

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$

$$e^x + e^{-x} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$

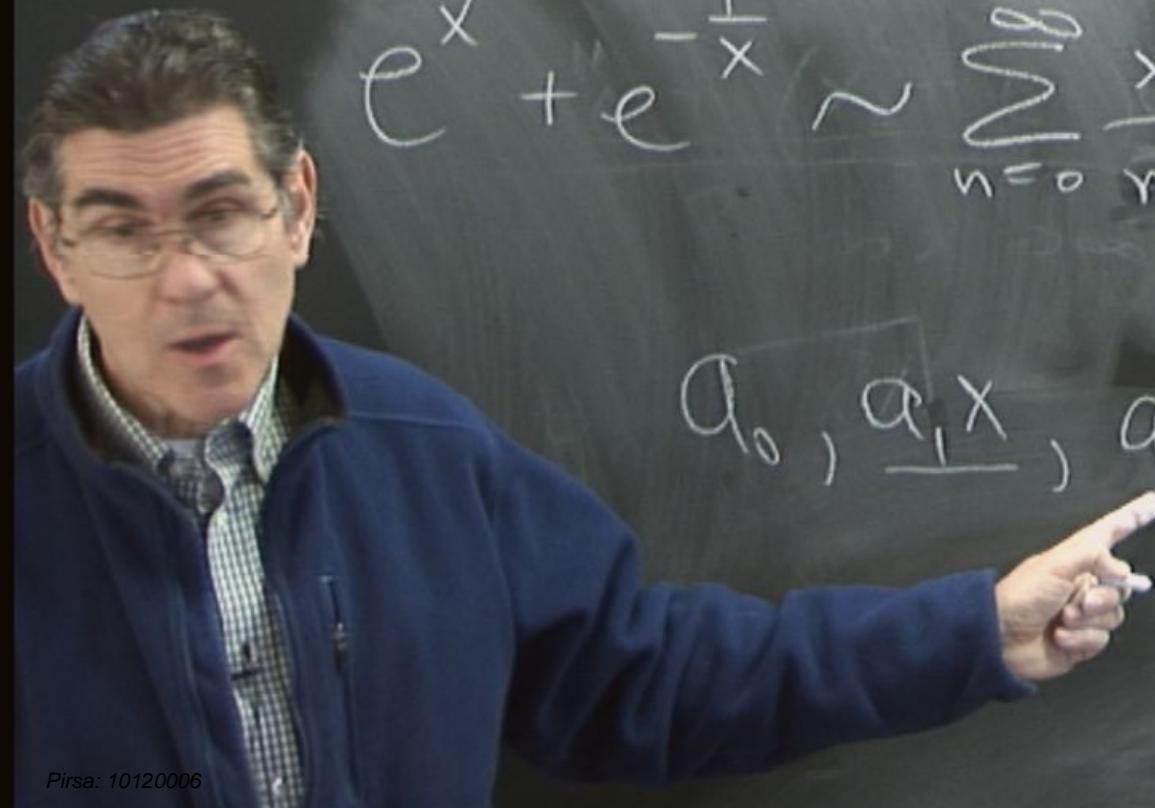
$$\alpha_0, \alpha_1 x$$



$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$

$$e^x + e^{-x} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$

$$a_0, \underline{a_1 x}, a_2 x^2$$



$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ as } x \rightarrow 0^+$$

$$e^x + e^{-\frac{1}{x}} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ as } x \rightarrow 0^+$$

$$a_0, \underline{a_1 x}, a_2 x^2$$



$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ as } x \rightarrow 0^+$$

$$e^x + e^{-x} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ as } x \rightarrow 0^+$$

$$a_0, a_1 x, a_2 x^2$$



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molecules

Convergent  
 $\left( \sum_{n=0}^{\infty} a_n x^n \right)$

$a_n$

$$S_N = \sum_{n=0}^N a_n x^n$$

does limit as  $N \rightarrow \infty$  of  $S_N$   
exist?

asymp (div.)  
 $\left( \sum_{n=0}^{\infty} a_n x^n \right)$   $a_n$   $x \text{ near } a$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

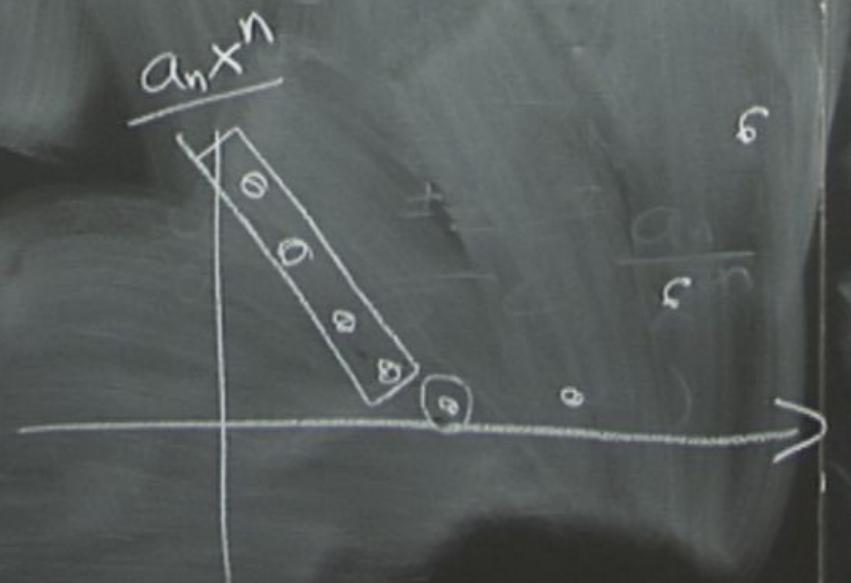
means  $\left[ f(x) - \sum_{n=0}^N a_n x^n \right] \sim \frac{a_{N+1} x^{N+1}}{N+1}$  as  $x \rightarrow 0$   
for ALL  $N$

$$\lim_{x \rightarrow 0} \frac{f(x) - \sum_{n=0}^N a_n x^n}{x^{N+1}} = a_{N+1}$$

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$

$$e^{-\frac{1}{x}} \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{as } x \rightarrow 0^+$$

$$\underline{a_1 x}, a_2 x^2$$



OPT. ASYMPT APPROX

Parabolic Cylinder Eqn

$$\varepsilon = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$$

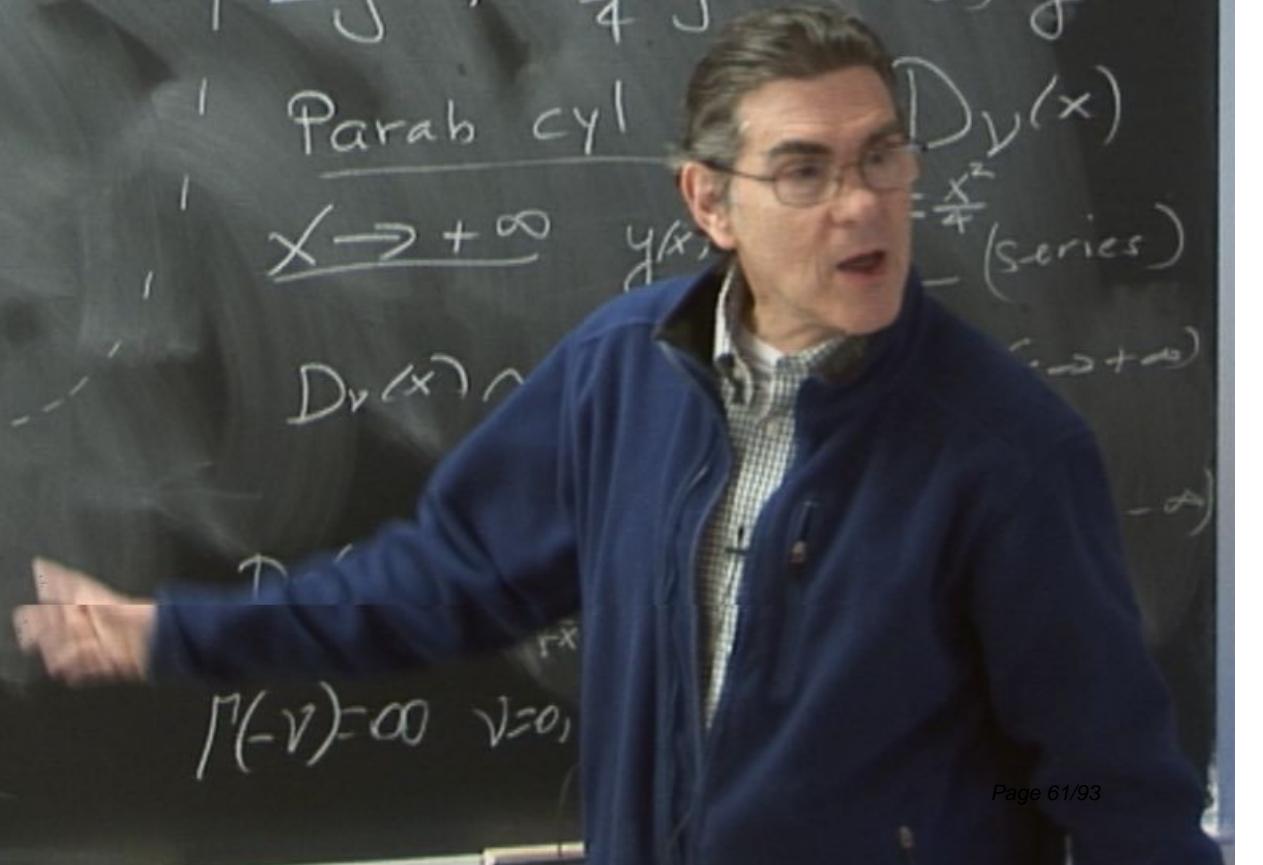
$$-y'' + \frac{x^2}{4}y = (v + \frac{1}{2})y$$

Parab cyl  $D_v(x)$

$$x \rightarrow +\infty \quad y(x) = \frac{x^2}{4} \text{ (series)}$$

$$D_v(x) \sim$$

$$\Gamma(-v) = \infty \quad v=0,$$



OPT. ASYMPT APPROX

Parabolic Cylinder Eqn

$$\varepsilon = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$$

$$-y'' + \frac{x^2}{4}y = (v + \frac{1}{2})y$$

Parab cyl

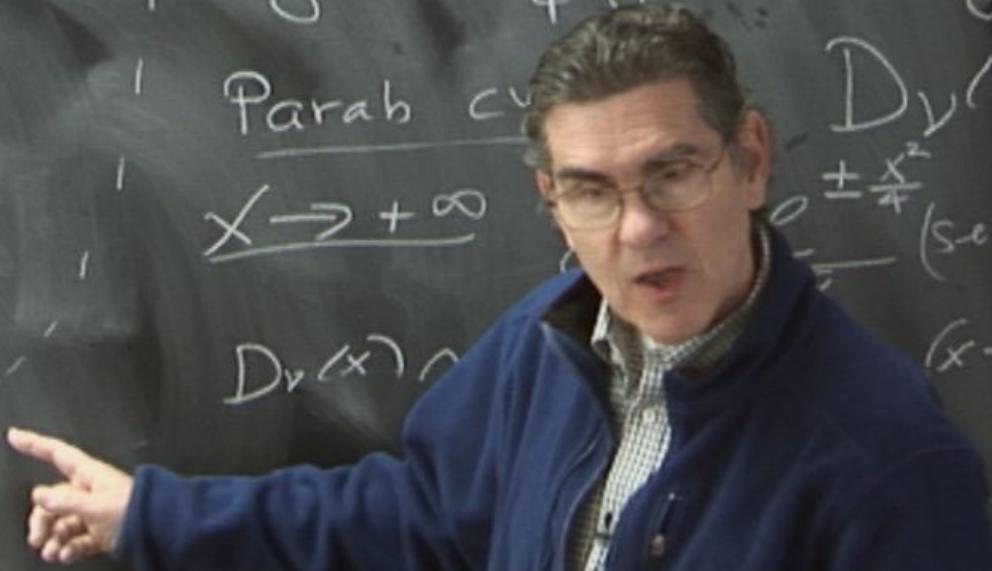
$$x \rightarrow +\infty$$

$$D_v(x) \sim$$

$$D_v(x)$$

$$\pm \frac{x^2}{4} \text{ (series)}$$

$$(x \rightarrow +\infty)$$



$$\Gamma(-v) = \infty \quad v =$$

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Molecule!

Convergent

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) a_n$$

$$S_N = \sum_{n=0}^N a_n x^n$$

does limit as  $N \rightarrow \infty$  of  $S_N$   
exist?

a sym? (div.)

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) a_n$$

$$a_n = n!$$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$$

$$\text{means } \left[ f(x) - \sum_{n=0}^N a_n x^n \right] \sim \frac{a_{N+1} x^{N+1}}{\text{for ALL } N}$$

$$\lim_{x \rightarrow 0} \frac{f(x) - \sum_{n=0}^N a_n x^n}{x^{N+1}} = a_{N+1}$$

ng behavior is given in

(3.5.8b)

$\rightarrow +\infty$ . (3.5.9b)

of the modified Bessel function  $K_v(x)$  decays exponentially as  $x \rightarrow +\infty$ ; (3.9b) with  $c_2 = (\pi/2)^{1/2}$ .

series as a formal asymptotic series

In last example we obtain the expansion to compute (3.5.9a), we obtain the

$x \rightarrow +\infty$ . (3.5.10)

ic. From the definition we see that if we terminate the series at the  $(N+2)$ th term in the  $x \rightarrow +\infty$ . If  $x$  is large, only an estimate of the numerical results from the series; typically the terms  $a_n$  diverge, they get the smallest term. We can ignore the smallest term. The next term, the estimation obtained in

the convergent power series to the accuracy; we sum. However, for a good approximation to the accuracy and if we sum according to our final accuracy, then to obtain the smallest term, we must take  $x$  closer

in (3.5.10) to evaluate partial sums truncated in to  $e^{-x}I_v(x)$  according to the exact number of terms.

Table 3.1 Asymptotic approximations to  $e^{-x}I_5(x)$  for five values of  $x$  using the series in (3.5.10)

Entries in the columns are the partial sums truncated after the  $x^{-k}$  term. Underlined partial sums are optimal asymptotic approximations. Notice that even when  $x = 7$  the leading term in the asymptotic expansion gives a very poor approximation while the optimal asymptotic truncation is very accurate. The number in parentheses is the power of 10 multiplying the entry.

$N$	$x$				
	3.0	4.0	5.0	6.0	7.0
0	2.30324 (-1)	1.99471 (-1)	1.78412 (-1)	1.62868 (-1)	1.50786 (-1)
2	1.08147 (0)	4.59816 (-1)	2.39128 (-1)	1.45372 (-1)	1.08004 (-1)
4	2.01953 (-1)	4.74361 (-2)	2.52641 (-2)	2.35810 (-2)	2.61284 (-2)
6	2.11127 (-2)	1.14538 (-2)	1.49262 (-2)	1.98392 (-2)	2.45412 (-2)
7	1.16597 (-2)	1.03611 (-2)	1.47212 (-2)	1.97870 (-2)	2.45248 (-2)
8	5.50542 (-3)	9.82749 (-3)	1.46411 (-2)	1.97700 (-2)	2.45202 (-2)
9	1.20401 (-4)	9.47732 (-3)	1.45991 (-2)	1.97626 (-2)	2.45184 (-2)
10	-5.73580 (-3)	9.19172 (-3)	1.45717 (-2)	1.97585 (-2)	2.45176 (-2)
11	-1.33001 (-2)	8.91505 (-3)	1.45504 (-2)	1.97559 (-2)	2.45172 (-2)
12	-2.45677 (-2)	8.60595 (-3)	1.45314 (-2)	1.97540 (-2)	2.45169 (-2)
13	-4.35276 (-2)	8.21586 (-3)	1.45122 (-2)	1.97523 (-2)	2.45167 (-2)
14	-7.90210 (-2)	7.66817 (-3)	1.44907 (-2)	1.97508 (-2)	2.45166 (-2)
15	-1.52078 (-1)	6.82267 (-3)	1.44641 (-2)	1.97492 (-2)	2.45164 (-2)
20	-1.31437 (1)	-3.61663 (-2)	1.39178 (-2)	1.97329 (-2)	2.45155 (-2)
35	-3.12759 (10)	-1.24079 (6)	-4.90286 (2)	-8.13340 (-1)	2.06197 (-2)
Exact value of $e^{-x}I_5(x)$					
4.54090 (-3)	9.24435 (-3)	1.45403 (-2)	1.97519 (-2)	2.45164 (-2)	
Relative error in optimal asymptotic approximation, %					
21.0	0.57	0.069	0.0024	0.000071	

Observe that as  $x$  increases the optimal number of terms increases and so does the accuracy of the corresponding partial sum. When  $x = 3$  the relative error is 21 percent, when  $x = 5$  it has improved to 0.07 percent, and when  $x = 7$  it is a whopping  $7 \times 10^{-5}$  percent. It is nice to know that the asymptotic series which was derived by considering a small neighborhood of infinity is dependable for values of  $x$  so far from  $\infty$ .

In Fig. 3.5 we plot the optimal asymptotic approximation [obtained by truncating the asymptotic series (3.5.10) according to our rule] to  $e^{-x}I_5(x)$  for  $x$  between 2.0 and 10.0. The graph shows that although  $e^{-x}I_5(x)$  is a continuous function the optimal approximation has discontinuities at the points  $x$  where the optimal number of terms in the truncated series changes by one. For  $x > 2.0$ , these points occur when two successive terms in the series are equal; these values of  $x$  are  $x = 2.63, 3.26, 3.88, 4.47, 5.05, 5.62, \dots$ . When  $2 \leq x \leq 2.63$ , we truncate after  $x^{-7}$ ; when  $2.63 \leq x \leq 3.26$ , we truncate after  $x^{-8}$ ; when  $3.26 \leq x \leq 3.88$ , we truncate after  $x^{-9}$ ; and so on. These crossover points are given explicitly by the formula  $x = [(2k+1)^2 - 100]/8(k+1)$  ( $k = 8, 9, 10, \dots$ ).

The optimal truncation of the asymptotic series gives a good numerical approximation to  $I_v(x)$  for all  $v$ . However, Fig. 3.6 shows that the smallest value of  $x$  at which the optimal approximation is accurate increases approximately linearly with  $v$ .

<i>N</i>	30	40	50	60	70
0	2.30129 (-1)	1.99471 (-1)	1.78417 (-1)	1.63988 (-1)	1.50786 (-1)
2	1.08147 (0)	4.99816 (-1)	2.49760 (-1)	1.46372 (-1)	1.00000 (-1)
4	2.901059 (-1)	4.74361 (-2)	2.52641 (-2)	1.35110 (-2)	2.61284 (-2)
6	2.11125 (-2)	1.14533 (-2)	1.49202 (-2)	1.98392 (-2)	2.46412 (-2)
7	1.16697 (-2)	1.03511 (-2)	1.47721 (-2)	1.97170 (-2)	2.46348 (-2)
8	5.99845 (-3)	9.16749 (-3)	1.46411 (-2)	1.97700 (-2)	2.46502 (-2)
9	1.33901 (-4)	1.47720 (-3)	1.45921 (-2)	1.97626 (-2)	2.46184 (-2)
10	-5.74590 (-2)	9.16102 (-3)	1.45717 (-2)	1.97585 (-2)	2.46176 (-2)
11	-1.33900 (-2)	1.91535 (-3)	1.45504 (-2)	1.97559 (-2)	2.46172 (-2)
12	-2.45457 (-2)	1.00295 (-3)	1.45310 (-2)	1.97540 (-2)	2.46169 (-2)
13	-4.35573 (-2)	8.23410 (-3)	1.45122 (-2)	1.97521 (-2)	2.46167 (-2)
14	-7.50010 (-2)	7.46017 (-3)	1.44937 (-2)	1.97508 (-2)	2.46164 (-2)
15	-1.22670 (-1)	6.72247 (-3)	1.44641 (-2)	1.97492 (-2)	2.46164 (-2)
20	-1.31437 (1)	-1.61463 (-2)	1.29178 (-2)	1.97329 (-2)	2.46153 (-2)
35	-3.12739 (0)	-1.20019 (1)	-4.90386 (2)	-8.13340 (-1)	2.06197 (-2)

Exact value of  $e^{-x} I_3(x)$ 

4.54090 (-2)	9.24435 (-2)	1.45403 (-2)	1.97519 (-2)	2.46164 (-2)
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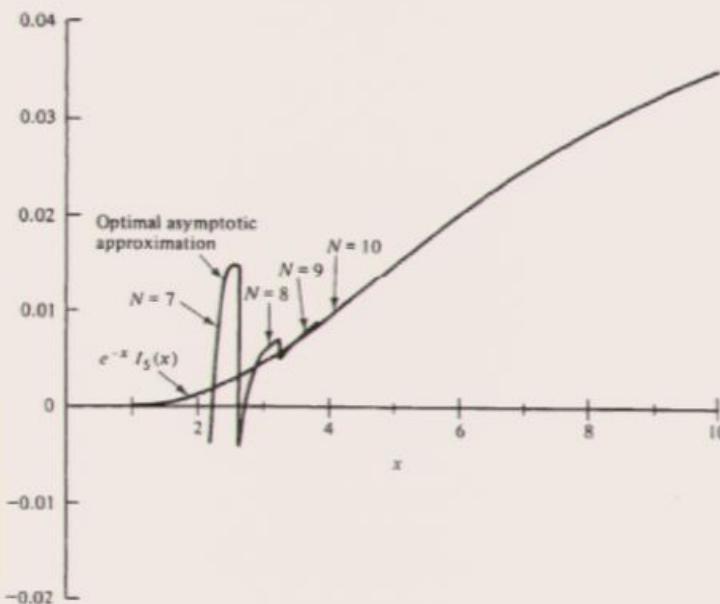
Relative error in optimal asymptotic approximation, %

N	2.0	4.0	5.0	6.0	7.0
0	2.53370 (-1)	1.99471 (-1)	1.70412 (-1)	1.62968 (-1)	1.59786 (-1)
2	1.89147 (0)	1.29110 (-1)	1.19123 (-1)	1.14572 (-1)	1.09904 (-1)
4	2.00490 (-1)	1.41163 (-2)	1.25341 (-2)	1.19310 (-2)	1.12123 (-2)
6	2.11127 (-2)	1.46630 (-2)	1.40962 (-2)	1.34592 (-2)	1.28412 (-2)
7	1.65977 (-2)	1.10351 (-2)	1.07212 (-2)	1.07770 (-2)	1.05348 (-2)
8	1.55342 (-2)	9.87149 (-2)	1.06411 (-2)	1.07700 (-2)	1.06200 (-2)
9	1.55901 (-2)	9.47740 (-2)	1.05911 (-2)	1.07636 (-2)	1.06156 (-2)
10	-5.72149 (-2)	9.11012 (-2)	1.05717 (-2)	1.07558 (-2)	1.06176 (-2)
11	-1.33001 (-2)	8.71565 (-2)	1.05904 (-2)	1.07539 (-2)	1.05172 (-2)
12	-2.25447 (-2)	8.30693 (-2)	1.06314 (-2)	1.07540 (-2)	1.05169 (-2)
13	-4.03975 (-2)	8.21413 (-2)	1.06123 (-2)	1.07513 (-2)	1.05167 (-2)
14	-7.80410 (-2)	8.20317 (-2)	1.06007 (-2)	1.07538 (-2)	1.05166 (-2)
15	-1.53373 (-1)	8.20318 (-2)	1.05941 (-2)	1.07502 (-2)	1.05164 (-2)
20	-1.31437 (0)	1.14106 (-2)	1.05978 (-2)	1.07529 (-2)	1.05155 (-2)
25	-1.27139 (0)	1.20774 (0)	1.05935 (2)	-0.11340 (-1)	1.06197 (-2)

Exact value of  $e^{-x} I_3(x)$

4.54000 (-2)	9.728435 (-1)	1.45603 (-2)	1.97519 (-2)	2.45164 (-2)
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Relative error in optimal asymptotic approximation, %



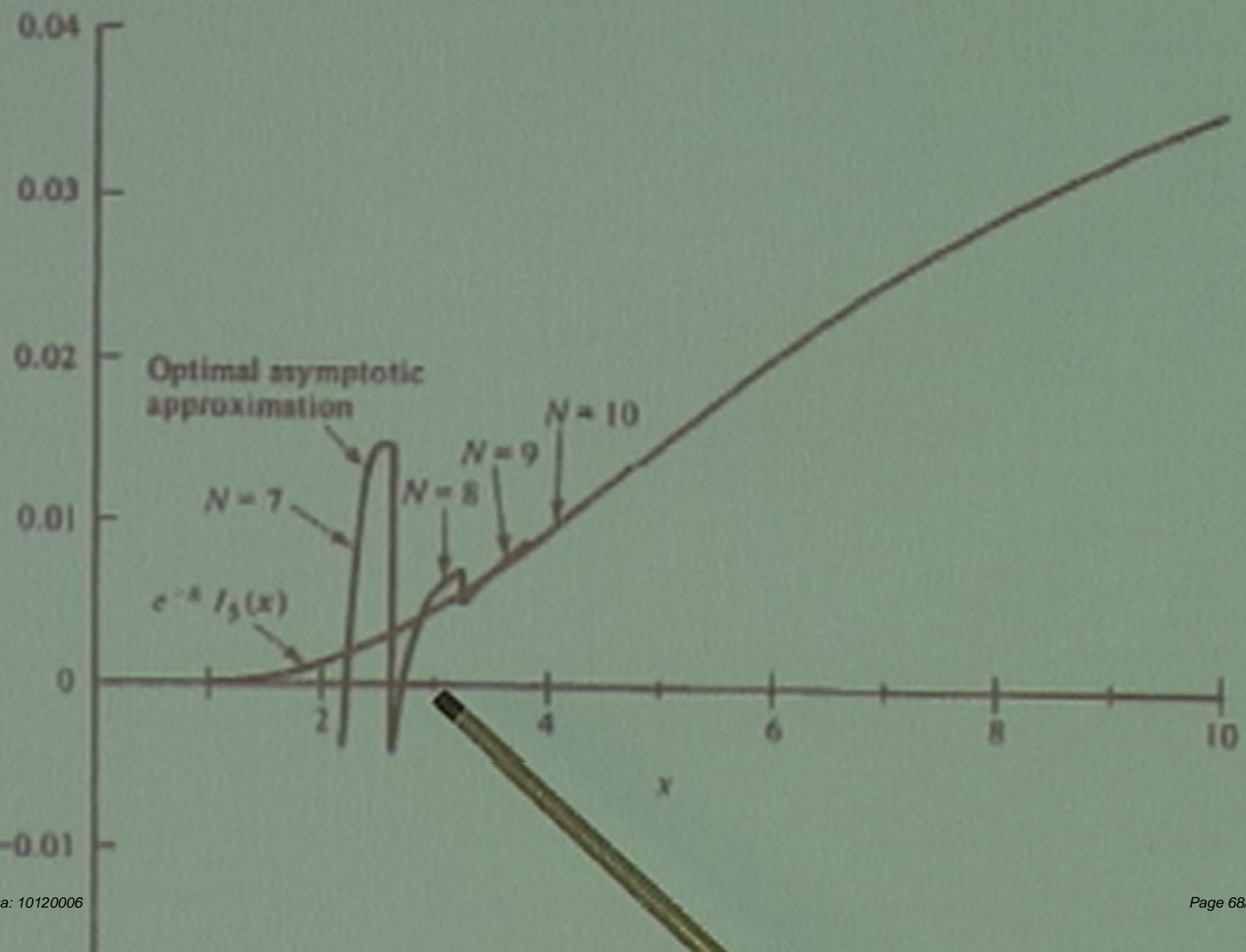
**Figure 3.5** A plot of the optimal asymptotic approximation to  $e^{-x} I_5(x)$  for  $2 \leq x \leq 10$ . For comparison, the exact numerical value of  $e^{-x} I_5(x)$  is also shown for  $0 \leq x \leq 10$ . These two curves are indistinguishable when  $x > 4$ . The discontinuities in the optimal asymptotic approximation occur when the optimal number of terms increases by one. Each segment of the optimal asymptotic approximation is labeled by a number  $N$  which is the highest power of  $l$  in the optimal truncation. [Note that we have chosen to plot  $e^{-x} I_5(x)$  instead of  $I_5(x)$  itself because  $I_5(x)$  rapidly runs off scale as  $x$  increases.]

Despite these wonderful results in Table 3.1 and Figs. 3.5 and 3.6, you may be distressed about a rule for "summing" a divergent series which yields a maximal accuracy that cannot be surpassed. Maybe you are disappointed that the rest of the terms in the series must stand idle, unable to improve the optimal but relatively poor result for  $x = 3$  obtained by adding up the first nine terms. Why bother to compute the full asymptotic series if only nine terms are usable? In fact, there are sophisticated rules for "summing" divergent series which make use of all the information contained in the terms of the asymptotic series. In many cases these rules surpass the limited accuracy of the optimal truncations and give arbitrarily accurate approximations provided sufficiently many terms in the series are used. The existence of these more powerful rules, which are discussed in Chap. 8, vastly increases the value of asymptotic series.

**Example 4** *Behavior of parabolic cylinder functions for large  $x$ .* Let us examine the behavior of the solutions  $y(x)$  to the parabolic cylinder equation

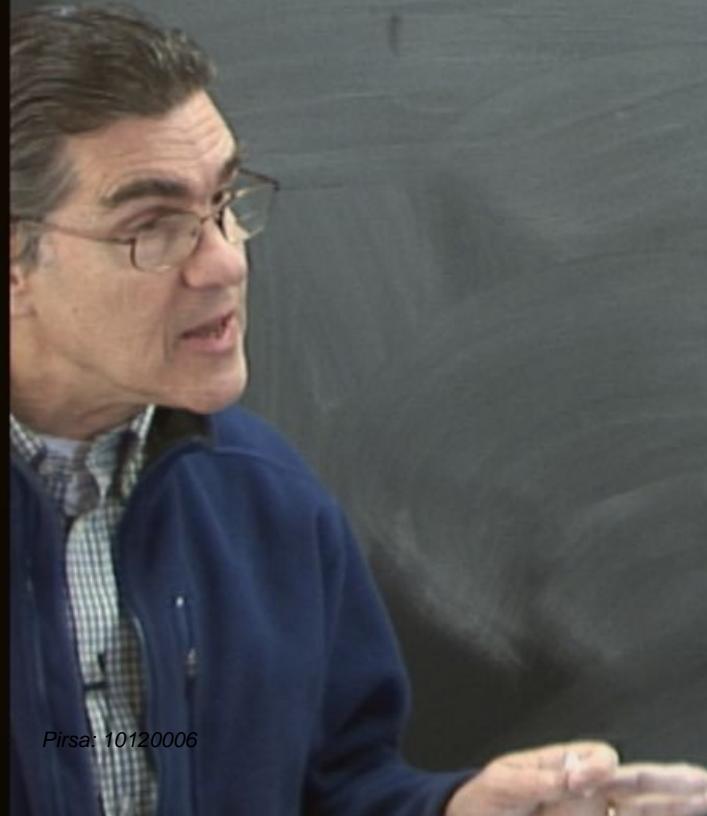
$$y'' + (v + \frac{1}{2} - \frac{1}{4}x^2)y = 0 \quad (3.5.11)$$

as  $x \rightarrow +\infty$ . In this equation  $v$  is a parameter.



$$G \sim a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4$$

$(g \rightarrow 0)$



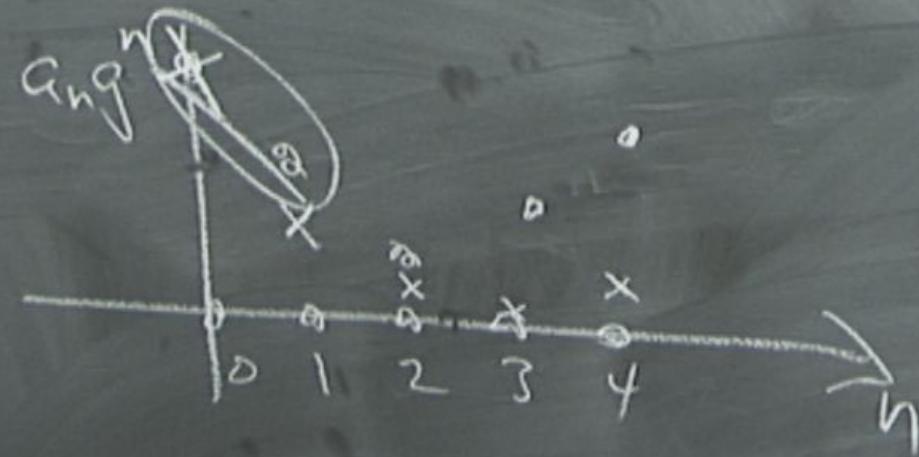
$$G \sim a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4$$

$(g \rightarrow 0)$

$g = 3$        $a_n g^n \phi$

G

$$G \sim a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4 \quad (g \rightarrow 0^+)$$

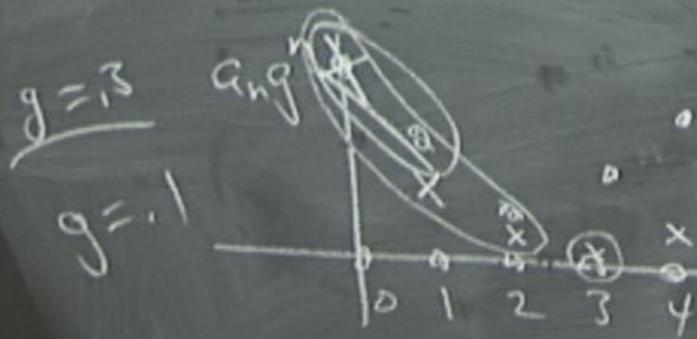


$$G \sim a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4$$

$(g \rightarrow 0^+)$

$\gamma$

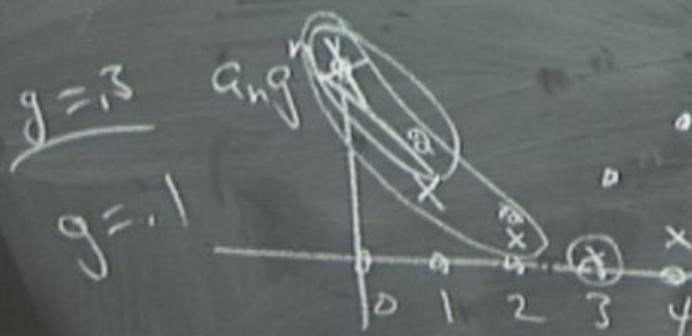
$$G \sim a_0 + a_1 \gamma + a_2 \gamma^2 + a_3 \gamma^3 + a_4 \gamma^4 \quad (\gamma \rightarrow 0^+)$$



$$\frac{dy''}{dx} + (V(x) + C e^{W(x)} + \phi U(x)) y = E y$$

COPT. AS

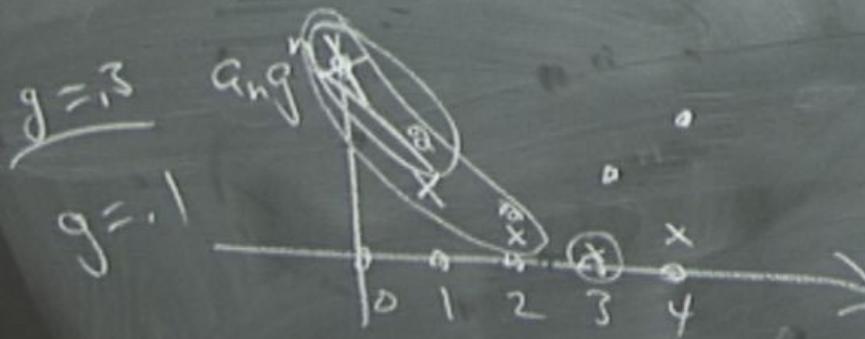
$$G \sim a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4 \quad (g \rightarrow 0^+)$$



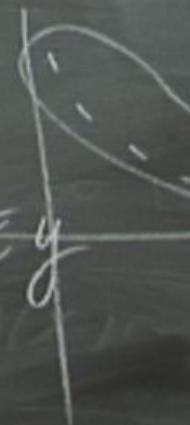
$$\frac{dy''}{dx} + (V(x) + C W(x) + \frac{C}{x} U(x)) y = E y$$

COPT. AS

$$G \sim a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4 \quad (g \rightarrow 0^+)$$



$$\frac{dy}{dx} + (V(x) + E W(x) + \Phi U(x)) y = E y$$



Stieltjes

$$w(x) \geq 0.$$

Stieljes

$$\underline{w(x) \geq 0} \quad (x \geq 0)$$

$$w(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$\int_0^\infty w(x) x^n$$

Stieljes

$$\underline{w(x) \geq 0} \quad (x \geq 0)$$

$$w(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$\int_0^\infty w(x) x^n$$

exists  
for all n

## Stieltjes

$$\underline{W(x) \geq 0}, \quad (x \geq 0)$$

$$W(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$a_n = \int_0^{\infty} x^n W(x) x^n \quad \begin{matrix} \text{exists} \\ \text{for all } n \end{matrix}$$

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n a_n x^n$$

## Stieltjes

$$\underline{W(x) \geq 0}, \quad (x \geq 0)$$

$W(x) \rightarrow 0$  as  $x \rightarrow +\infty$

$$a_n = \int_0^{\infty} f(x) W(x) x^n dx \quad \text{exists for all } n$$

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n a_n x^n$$

$$a_n = \frac{1}{n+1}$$

all n

function

Parabolic Cylinder Eqn

$$-y'' + \frac{x^2}{4}y = (v + \frac{1}{2})y$$

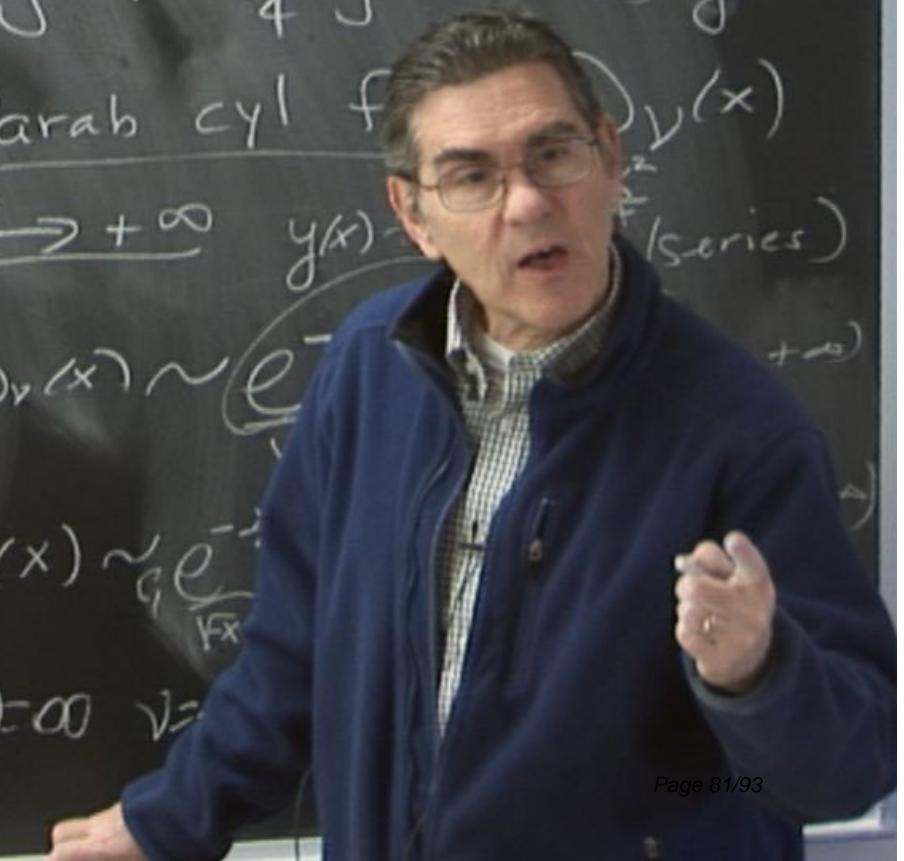
Parab cyl f  $y(x)$

$x \rightarrow +\infty$   $y(x)$   $\downarrow$   
 $\downarrow$  (series)

$$D_v(x) \sim e^{-vx}$$

$$, (x) \sim e^{-\frac{1}{4}x^2}$$

$$\Gamma(-v) = \infty \quad v =$$



## Parabolic Cylinder Eqn

$$E = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$$

$$-y'' + \frac{x^2}{4}y = \left(v + \frac{1}{2}\right)y$$

Parab

$$x \rightarrow +\infty$$

$$D_V(x)$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ (series)}$$

$$(x \rightarrow +\infty)$$

$$(\rightarrow -\infty)$$

$$a_n = \frac{1}{n+1}$$

all n

function

$$J(x) \sim$$

$J(-v) = \infty$

$$w(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

Parabolic Cylinder Eqn

$$E = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$$

$$-y'' + \frac{x^2}{4}y = \left(\nu + \frac{1}{2}\right)y$$

Parab cy

$$x \rightarrow +\infty$$

$D_y(x)$

$$e^{\pm \frac{x^2}{4}} \text{(series)}$$

$$(x \rightarrow +\infty)$$

$$(\rightarrow -\infty)$$

$$r(x) \sim$$

$$\Gamma(-\nu) = \infty$$

## Stieltjes

$$\underline{W(x) \geq 0}, \quad (x \geq 0)$$

$$W(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$\checkmark a_n = \int_0^{\infty} \underline{dx W(x)} x^n \quad \begin{matrix} \text{exists} \\ \text{for all } n \end{matrix}$$

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n \circled{a_n} x^n$$

② Stieltjes function

## Stieltjes

$$W(x) \geq 0, \quad (x \geq 0)$$

$\checkmark W(x) \rightarrow 0$  as  $x \rightarrow +\infty$

$$\checkmark a_n = \int_0^{\infty} \underline{W(x)} x^n \quad \begin{matrix} \text{exists} \\ \text{for all } n \end{matrix}$$

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n a_n x^n$$

② Stieltjes function

$$f(x) = \int_0^{\infty} \frac{dt}{1+xt} W(t)$$

② Stieltjes

$$f(x) = \int_0^{\infty} \frac{w(t)}{1+xt} dt$$

function

$$\sum_{n=0}^{\infty} (-1)^n a_n x^n$$
$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n x^n t^n$$



$I(-v)$

② Stieljes

$$\int_0^\infty \frac{dt}{1+xt} W(t)$$

function

$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n x^n t^n$$
$$\sum_{n=0}^{\infty} (-1)^n a_n x^n \quad (as x \rightarrow 0^+) \sim e^{-\frac{x}{t}}$$
$$\sim e^{-\frac{x}{t}}$$
$$P(-v) = \infty \quad v=0, 1, 2,$$

sts  
en

abol

$\frac{x}{4}$

n cyl

- $\infty$



From  
Grains of  
Pollen to  
Evidence  
for Atoms

Is  
it a  
molecule?

Convergent

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \quad a_n$$

$$S_N = \sum_{n=0}^N a_n x^n$$

does limit as  $N \rightarrow \infty$  of  $S_N$   
exist?

asympt (div.)

$$\left( \sum_{n=0}^{\infty} a_n x^n \right) \quad a_n \quad x \text{ near } 0$$

$a_n = n!$

$f(x) \sim \sum_{n=0}^{\infty} a_n x^n \text{ as } x \rightarrow 0$

*F. t. small.*

means  $\left[ f(x) - \sum_{n=0}^N a_n x^n \right] \sim \frac{a_{N+1} x^{N+1}}{N+1} \text{ as } x \rightarrow 0$

for ALL N

$\lim_{x \rightarrow 0} \frac{f(x) - \sum_{n=0}^N a_n x^n}{x^{N+1}} = a_{N+1}$

## Stieltjes

$$W(x) \geq 0 \quad (x \geq 0)$$

$$W(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$A_n = \int_0^{\infty} dx W(x) x^n$$

exists  
für alle n

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n A_n x^n$$

② Stieltjes function

$$f(x) = \int_0^x \frac{W(t)}{1+xt} dt \sim \sum_{n=0}^{\infty} (-1)^n$$

$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n$$

abulic Cylinder Eqn

$$\frac{x^2}{4} y = \left( \sqrt{1+\frac{1}{x^2}} \right)^3 y$$

n cyl fns  $D_y(x)$

$$\Rightarrow y(x) \sim e^{\pm \frac{x}{2}} \sqrt{x}$$

(series)

$$\Rightarrow \sim e^{\pm \frac{x}{2}} \sum_{n=0}^{\infty} (x^{2n+1})$$

$$\frac{e^{-\frac{x}{2}}}{\sqrt{x}} + \frac{e^{+\frac{x}{2}}}{\sqrt{x}} \sum_{n=0}^{\infty} (x^{2n+1}) \quad (x \rightarrow -)$$

Stieltjes

$$W(x) \geq 0, \quad (x \geq 0)$$

$$W(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$\checkmark n = \int_0^\infty dx W(x) x^n \quad \begin{matrix} \text{exists} \\ \text{for all } n \end{matrix}$$

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n \check{a}_n x^n$$

② Stieltjes function

$$f(x) = \int_0^\infty \frac{dt}{1+xt} \cdot \frac{W(t)}{t} \sim \sum_{n=0}^{\infty} (-1)^n a_n x^n$$
$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n x^n t^n$$

Stieltjes

$$W(x) \geq 0, \quad (x \geq 0)$$

$$W(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$\check{a}_n = \int_0^\infty dx W(x) x^n \quad \begin{matrix} \text{exists} \\ \text{for all } n \end{matrix}$$

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n \check{a}_n x^n$$

② Stieltjes

$$f(x) = \int_0^{\infty} \frac{dt}{1+xt} \sim \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad (\text{as } x \rightarrow 0)$$

$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n x^n t^n$$

$f(-t)$

Stieltjes

$$W(x) \geq 0, \quad (x \geq 0)$$

$$W(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$\int_0^{\infty} x W(x) x^n \quad \text{exists for all } n$$

① Stieltjes series

$$\sum_{n=0}^{\infty} (-1)^n a_n x^n$$

② Stieltjes

$$f(x) = \int_0^{\infty} \frac{dt \cdot W(t)}{1+xt} \sim \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad (\text{as } x \rightarrow 0)$$

$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n x^n t^n$$

$f(-t)$

abolic Cylinder Eqn

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x}$$

$$y = (V + \frac{x}{2})^{-1} y_0$$

n cyl fn's  $D_y(x)$

$$y(x) \sim e^{\pm \frac{x}{2}} \sqrt{x} \text{ (series)}$$

② Stieltjes function

$$f(x) = \int_0^\infty \frac{dt}{1+xt} \sim \sum_{n=0}^{\infty} (-1)^n a_n x^n \quad (\text{as } x \rightarrow 0^+) \sim \frac{e^{-\frac{x}{2}}}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \quad (x \rightarrow +\infty)$$
$$\frac{1}{1+xt} = \sum_{n=0}^{\infty} (-1)^n x^n t^n$$
$$\sim e^{-\frac{x}{2}} + e^{\frac{x}{2}} \frac{1}{\sqrt{x} \Gamma(1/2)} \quad (x \rightarrow -\infty)$$
$$\Gamma(-v) = \infty \quad v = 0, 1, 2, \dots$$