Title: Why do spinning black hole binaries "bob" and "kick"?

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Abstract: Numerical simulations of binary black holes with spin have revealed some surprising behavior: for antialigned spins in the orbital plane, 1) one sees an up-and-down "bobbing" of the entire orbital plane at the orbital frequency and 2) the merged black hole receives an enormous kick that depends on the phase at merger. Natural questions are: What causes the bobbing? Can the kick be viewed as a post-merger continuation of the bobbing?

We show that this type of bobbing is in fact ubiquitous in relativistic mechanics, occurring independently of the type of force holding two spinning bodies in orbit. The cause can be identified as a spin correction to the naive center of mass of a body; the effect is analogous to Thomas precession and is ``purely kinematical" in the same way. Since a kick requires the release of field momentum, it is instead very dependent on the type of force holding bodies in orbit. In a mechanical analog (spinning balls connected by a string), there is bobbing but can be no kick. In an electromagnetic analog, one should be able to tune the kick independently of the bobbing. In the gravitational case the spin parameter happens to control both bobbing and kick, making separate tuning impossible and giving the appearance of causation to two essentially unrelated phenomena.

Our answers are therefore: the bobbing is caused by a purely kinematical effect of spin, and the kick cannot be viewed its post-merger continuation.

Why do Spinning Black Hole Binaries Bob and Kick?

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with A. Harte and R. Wald
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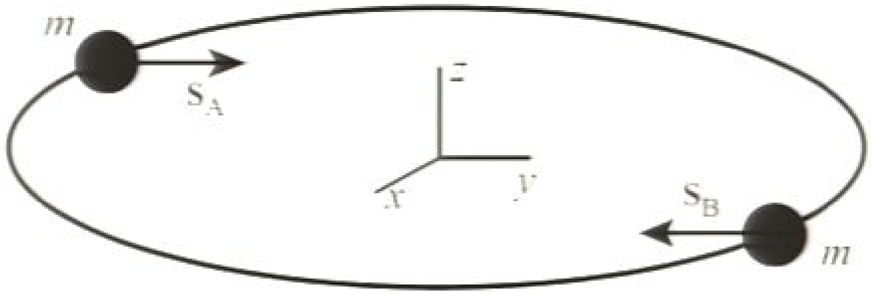
Perimeter Institute Strong Gravity Seminar
November 11, 2010

$$F = m\,a$$

(for suitable F, m, a)

Put two spinning black holes in the following orbital configuration,



Credit: Keppel, Nichols, Chen, Thorne

and observe...

(Campanelli, Lousto, Zlochower, Merritt; Gonzalez, Sperhake, Brugmann, Hannam, Husa)

1. **Bobbing.** The whole orbital plane bobs up and down in phase with the orbit.

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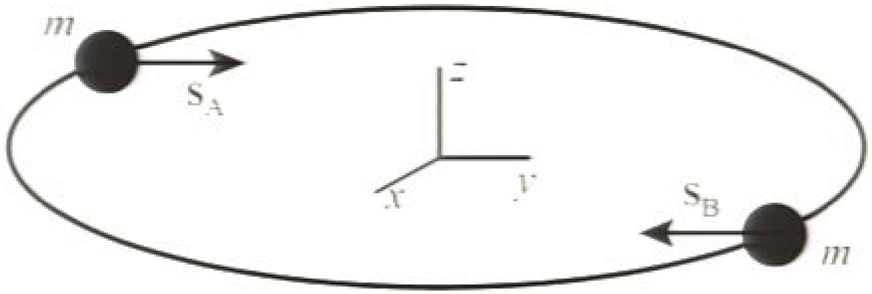

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What is going on??

What you might think

1. The bobbing is some weird GR effect, which is enabled by exchange of momentum between the bodies and the gravitational field

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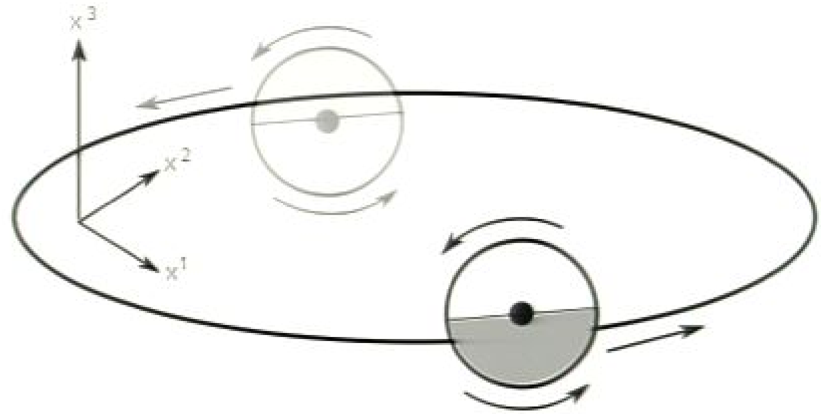
What I will show/argue today

show: 1a. Bobbing is totally normal. Two bodies connected by a string will bob just as do binary black holes, due to a "kinematical effect".

how/argue: 1b. While there is momentum exchange, it has nothing to do with the bobbing, because of something called hidden momentum.

argue: 2. The kick has little to do with the bobbing. The appearance of causation in the gravitational case has to do with a quirk of that system.

Outline of Talk

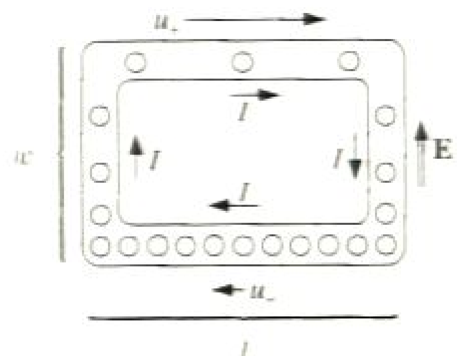


1. Kinematical bobbing

    a. Failure of lab-frame intuition
    b. Simple derivation of bobbing

2. Electromagnetic Binary

    a. Hidden momentum prevents dynamical bobbing
    b. Momentum Budget



3. Gravitational Correspondence

4. Kicks and the appearance of causation

Center of mass

Freshman physics center of mass: $$Z^i = \frac{1}{\int \rho \, d^3x} \int \rho \, x^i d^3x$$

The relativistic generalization? $$Z^i = \frac{1}{\int T_{00} \, d^3x} \int T_{00} \, x^i d^3x$$

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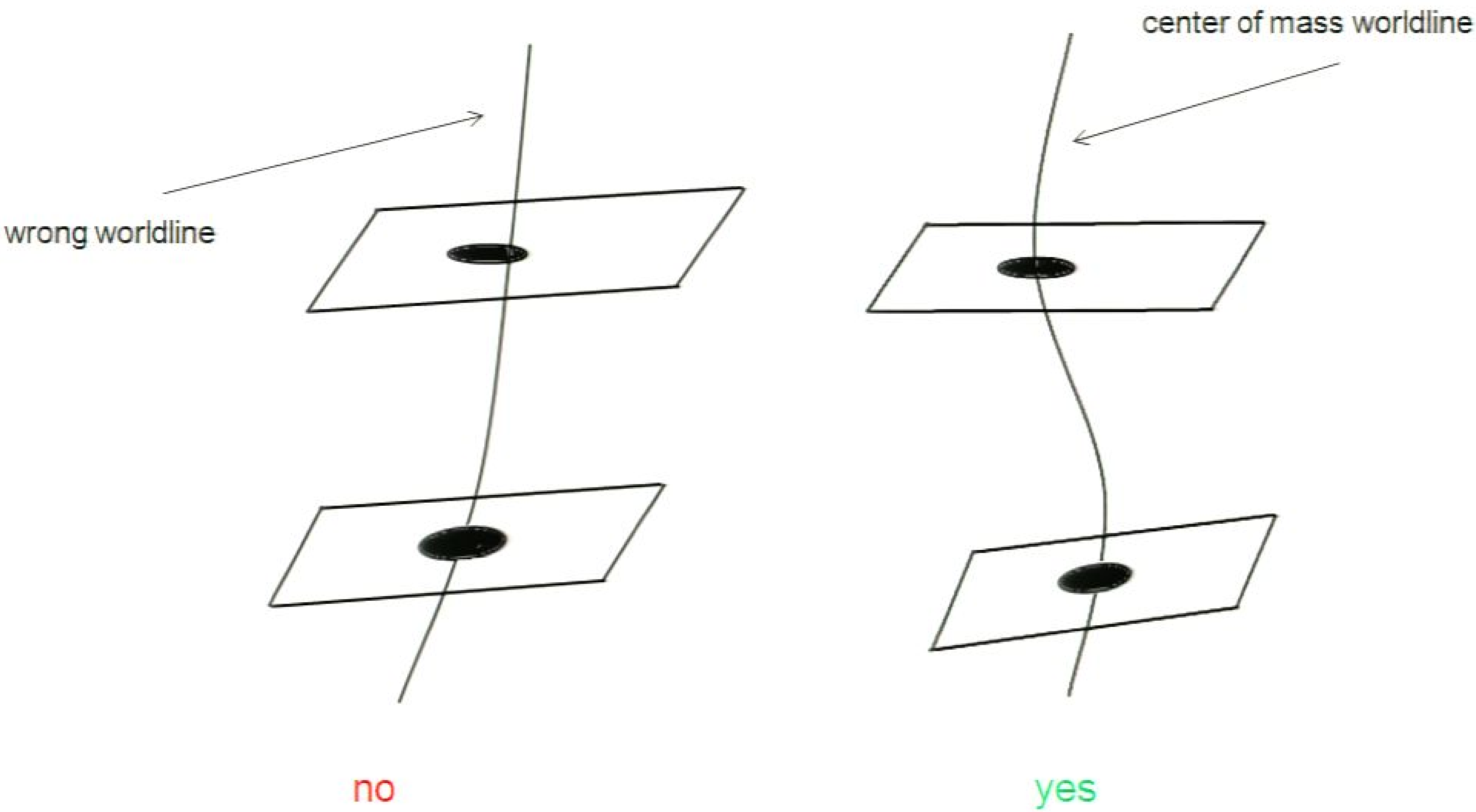
Instead define center of mass implicitly in the rest frame:

Demand $\int T_{\hat{0}\hat{0}} \, \hat{x}^i d^3\hat{x} = 0$ where hats mean rest (Fermi) frame

> The center of mass worldline is the (unique) worldline for which the center of energy always vanishes in the instantaneous rest frame of that worldline.

(for actual definition as well as uniqueness and other properties, see Dixon 1974)

Cartoon for CM Definition



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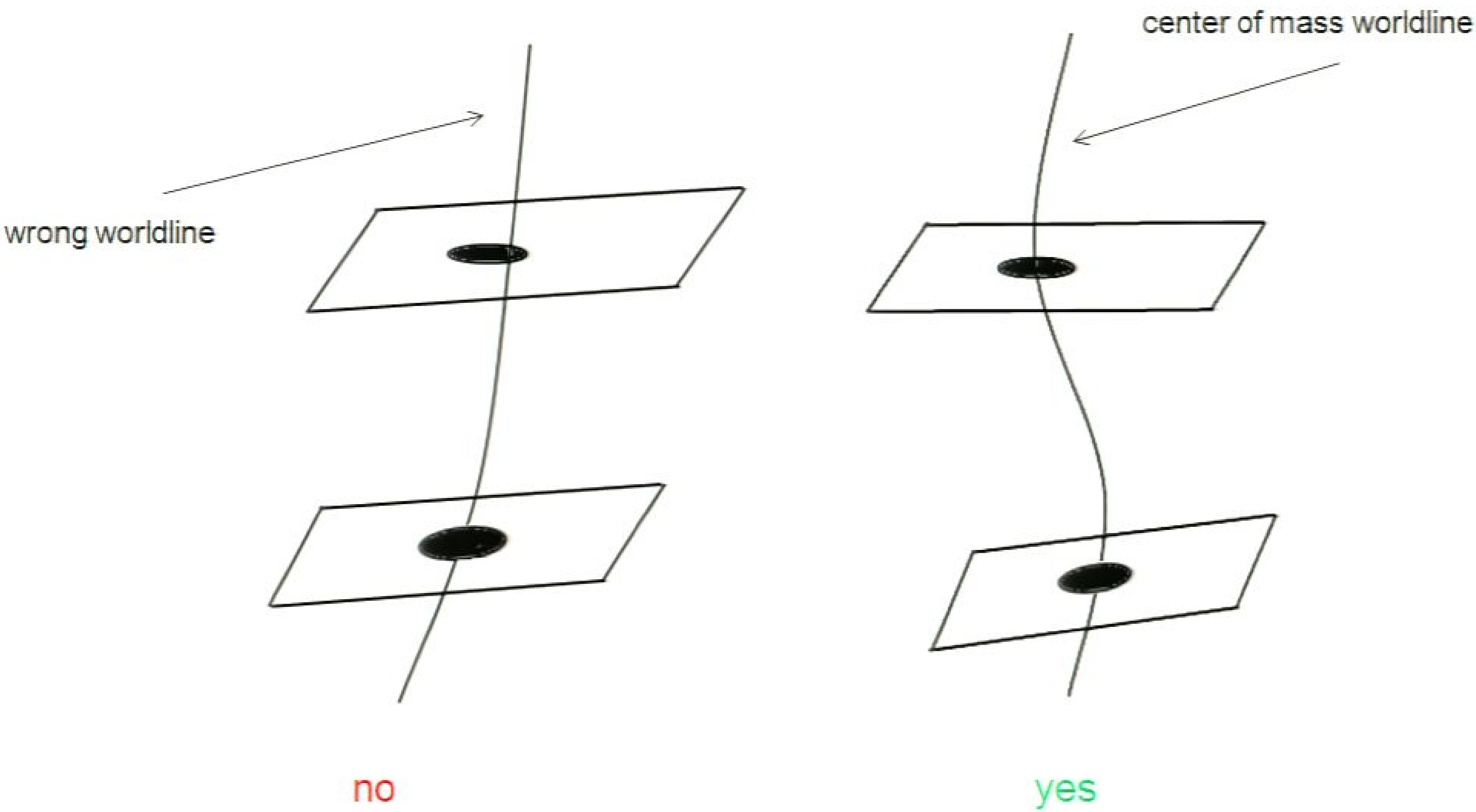

Lab-Frame Center of Energy

The lab-frame center of energy mixes up information about the particle with information about the lab. But it is nevertheless useful because it appears in two important places…

1. Conservation laws

2. Your intuition

So, we are going to use this concept to understand bobbing. The key point is its relationship to the true center of mass,

$$\vec{Z}_L = \vec{Z}_{CM} + \vec{v} \times \vec{S} + O(v^2)$$

This is easy to understand. There is more lab-frame kinetic energy on one side of a spinning, moving body than on the other.

Killing Vector Trivia

Symmetry	Conserved Quantity
Time translation	Energy
Space translation	Momentum
Rotation	Angular Momentum
Boost	???

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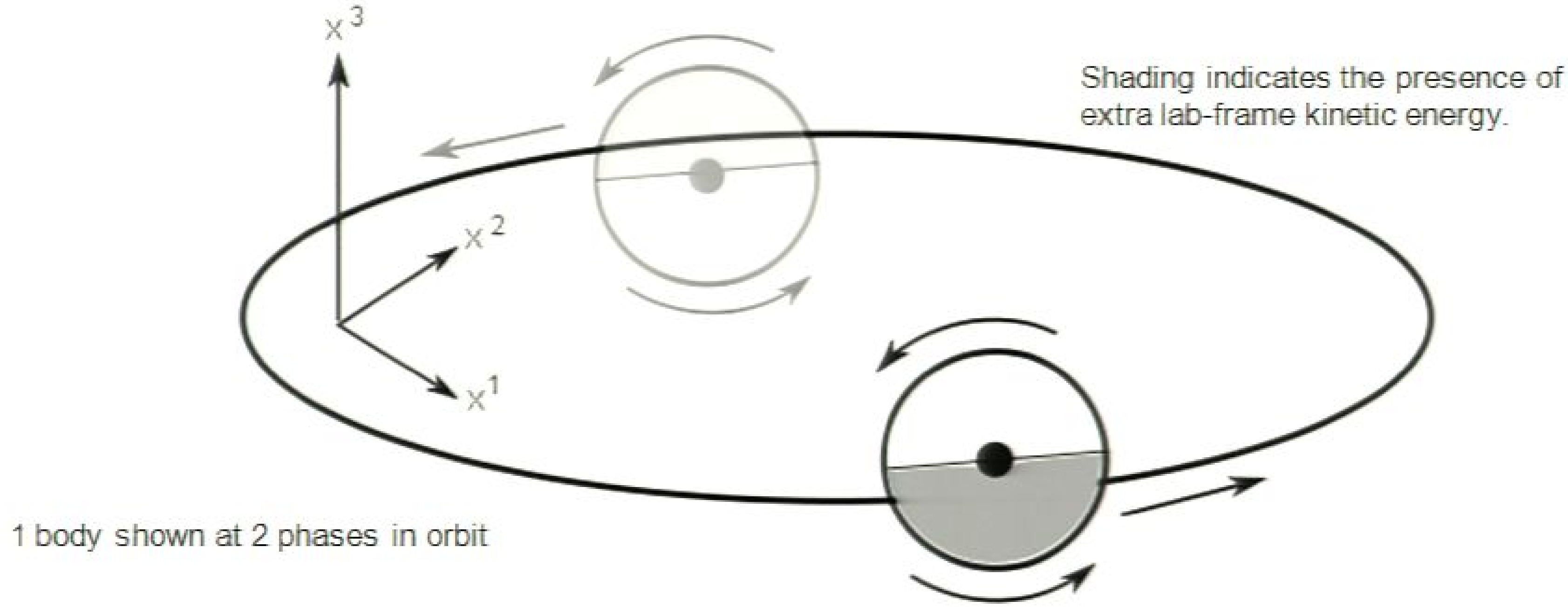
Answer: lab frame center of energy minus t times momentum

$$(\xi_I)^\alpha = (x^I, t\ \delta^\alpha{}_I) \qquad \int T_{0\beta}(\xi_I)^\beta d^3x = \int T_{00}x^I d^3x + t\int T_{0I}$$

conserved: $\int T_{00}x^i d^3x - tP^i$

In an adapted frame the lab frame center of energy is conserved.

Bobbing Cartoon



$$\vec{Z}_L = \vec{Z}_{CM} + \vec{v} \times \vec{S} + O(v^2)$$

The x^3 component of the lab frame center of energy is conserved, so the center of mass must bob!

More careful approach

A body subject to an external force density will be described by a stress-energy tensor satisfying

$$\partial_\mu T^{\mu\nu} = f^\nu$$

Integrate the time component,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int T^{0\nu}\mathrm{d}^3x = \int f^\nu \mathrm{d}^3x.$$

dp / dt= net force

Multiplying by x^i before integrating the time component,

$$\mathcal{E}\frac{\mathrm{d}z^i_{\mathrm{L}}}{\mathrm{d}t} = \int T^{0i}\mathrm{d}^3x + \int f^0(x^i - z^i_{\mathrm{L}})\mathrm{d}^3x.$$

m v = p + mysterious stuff

$$\mathcal{E} = \int T^{00}\mathrm{d}^3x$$

$$z^i_{\mathrm{L}} \equiv \frac{1}{\mathcal{E}}\int T^{00}x^i\mathrm{d}^3x.$$

Combine the two into a second-order equation…

$$\mathcal{E}\frac{\mathrm{d}^2 z^i_{\mathrm{L}}}{\mathrm{d}t^2} = \int f^i \mathrm{d}^3x + \frac{\mathrm{d}}{\mathrm{d}t}\int f^0(x^i - z^i_{\mathrm{L}})\mathrm{d}^3x \ - \frac{\mathrm{d}z^i_{\mathrm{L}}}{\mathrm{d}t}\int f^0 \mathrm{d}^3x.$$

m a = F + mysterious stuff

Now assume:

1. Non-relativistic motion. E = m and f^0 is small.
2. The force is centered about the center of mass.

And at long last F = m a:

$$\boxed{m\frac{\mathrm{d}^2 z^i_{\mathrm{L}}}{\mathrm{d}t^2} \simeq \int f^i \mathrm{d}^3x,}$$

But the acceleration refers to the lab frame center of energy.

Bobbing of Tetherballs

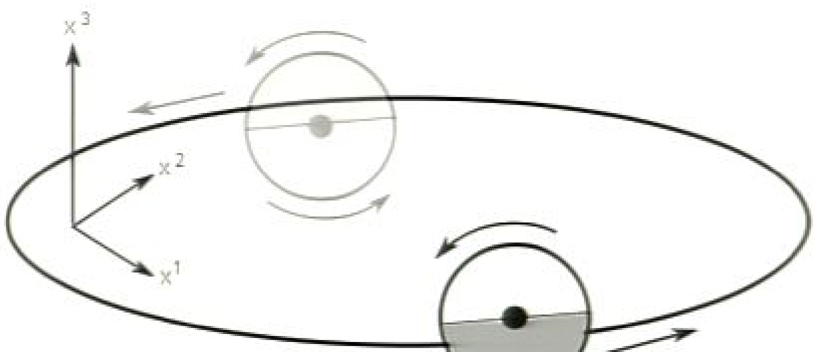
Attach two spinning bodies by a string and anti-align the spins parallel with the string. Set the bodies in a slow-motion orbit and the equation of motion is

$$m\frac{\mathrm{d}^2 z^i_{\mathrm{L}}}{\mathrm{d}t^2} \simeq \int f^i \mathrm{d}^3 x.$$

The x^3 force is zero for a string, so there is no x^3 acceleration of Z_L. But we have

$$\vec{Z}_L = \vec{Z}_{CM} + \vec{v} \times \vec{S} + O(v^2)$$

So as the bodies orbit and v x S changes, the center of mass must bob.



Most Careful Approach

$\nabla_b(T^{ab}_{\mathrm{A}} + T^{ab}_{\mathrm{B}} + T^{ab}_{\mathrm{T}}) = 0.$ (include the stress-energy of the tether)

1. In the lab frame, the motion of the balls and tether is non-relativistic and (55) holds. $|\vec{f}|R^4/M \lesssim O(|\vec{V}|) \ll 1.$
2. Each component of the lab frame momenta satisfies[10] $P^\mu_{\mathrm{T}} \ll P^\mu_{\mathrm{A}}$.
3. The motion of the system is symmetric with respect to rotations by π about the x^3-axis.

(make precise assumptions)

$$\frac{\mathrm{D}p^a}{\mathrm{d}s} = F^a$$
$$m\dot{z}^a = p^a - \epsilon^a{}_{bcd}F^b p^c S^d/m^2$$
$$\frac{\mathrm{D}S^a}{\mathrm{d}s} = p^a(F^b S_b)/m^2.$$

(employ results from extended body theory)

$$\boxed{(m_{\mathrm{A}}\dot{\vec{z}}_{\mathrm{A}} + m_{\mathrm{B}}\dot{\vec{z}}_{\mathrm{B}}) \cdot \vec{\zeta} = \left[\left(\frac{\vec{S}_{\mathrm{A}}}{m_{\mathrm{A}}} - \frac{\vec{S}_{\mathrm{B}}}{m_{\mathrm{B}}}\right) \times \vec{F}_{\mathrm{A}}\right] \cdot \vec{\zeta}.}$$

(final result)

Summary of Part I

Bobbing is ubiquitous! A kinematical effect makes all spinning bodies bob just as do binary black holes.

You're surprised by bobbing only because your intuition is built from equations satisfied by the lab frame center of energy.

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Questions for Parts II,III,IV

Can there be dynamical bobbing in addition to kinematical bobbing?

Is there a relationship to kicks?

Dynamical Bobbing and EM Analog

Since gravitational fields carry momentum, there is nothing stopping an exchange of momentum between the bodies and the field, which could give rise to “dynamical bobbing”.

But gravitational field momentum can’t be localized. You can use a pseudotensor formalism to build intuition (Keppel, Nichols, Chen, Thorne 2009), but the results depend on the choice of pseudotensor and gauge.

Since electromagnetic field momentum *can* be localized, we will study an electromagnetic analog: two bodies with mass, charge, spin, and magnetic dipole.

The kinematical bobbing (proportional to spin) will be there. But will there be any dynamical bobbing (proportional to magnetic dipole)?

Electromagnetic Binary

A small body moving in an external field satisfies (SEG,Harte,Wald 2009; Harte 2009)

$$m\vec{a} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) + \frac{2}{3}q^2\frac{d\vec{a}^{(0)}}{dt} + p_i\vec{\nabla}E^i + \mu_i\vec{\nabla}B^i$$
$$+\frac{d}{dt}\left(\vec{S}\times\vec{a} - \vec{\mu}\times\vec{E} - \vec{p}\times\vec{B}\right) .$$

Compute two-body EOM at 1.5 "post-Coulombic order,"

$$\vec{v}\sim\lambda^{.5},\ m\sim q\sim\lambda,\ \vec{S}\sim\vec{\mu}\sim\lambda^2 \qquad \vec{p}=0$$

The terms proportional to spin and dipole are

$$\vec{r}\equiv\vec{z}_A-\vec{z}_B$$
$$\vec{v}\equiv\dot{\vec{z}}_{\rm A}-\dot{\vec{z}}_{\rm B}$$

$$\begin{aligned} m_{\rm A}\vec{a}^{\mu,S}_{\rm A} &= \frac{q_{\rm A}}{r^3}\left[3(\vec{v}\times\hat{r})(\vec{\mu}_{\rm B}\cdot\hat{r}) - \vec{v}\times\vec{\mu}_{\rm B}\right] \\ &+ \frac{q_{\rm B}}{r^3}\left[3((\vec{\mu}_{\rm A}\times\vec{v})\cdot\hat{r})\hat{r} + 3(\hat{r}\cdot\vec{v})(\vec{\mu}_{\rm A}\times\hat{r}) - 2\vec{\mu}_{\rm A}\times\vec{v}\right] \\ &+ \frac{q_{\rm A}q_{\rm B}}{m_{\rm A}r^3}\left[(\vec{S}_{\rm A}\times\vec{v} - 3(\vec{v}\cdot\hat{r})(\vec{S}_{\rm A}\times\hat{r})\right], \qquad (119)\end{aligned}$$

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Lorentz force

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dipole forces

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To focus on bobbing, dot into the instantaneous orbital plane,

$$a_{\rm A}^{\perp} \equiv \vec{a}_{\rm A} \cdot \frac{\hat{r} \times \vec{v}}{|\hat{r} \times \vec{v}|}.$$

For one body,

dynamical bobbing

$$m_{\rm A} a_{\rm A}^{\perp} = \frac{q_{\rm A}\vec{\mu}_{\rm B} + q_{\rm B}\vec{\mu}_{\rm A}}{r^3|\hat{r} \times \vec{v}|} \cdot \left[\hat{r}(-2v^2 + 3(\hat{r}\cdot\vec{v})^2) - \vec{v}(\hat{r}\cdot\vec{v})\right]$$
$$+ \frac{q_{\rm A}q_{\rm B}\vec{S}_{\rm A}/m_{\rm A}}{r^3|\hat{r} \times \vec{v}|} \cdot \left[\hat{r}(v^2 - 3(\hat{r}\cdot\vec{v})^2) + 2\vec{v}(\hat{r}\cdot\vec{v})\right].$$

kinematical bobbing

But if we consider the *net* bobbing of the whole orbital plane,

$$m_{\rm A} a_{\rm A}^{\perp} + m_{\rm B} a_{\rm B}^{\perp} = \frac{q_{\rm A}q_{\rm B}}{r^3|\hat{r} \times \vec{v}|} \left(\frac{\vec{S}_{\rm A}}{m_{\rm A}} - \frac{\vec{S}_{\rm B}}{m_{\rm B}}\right) \cdot \left[\hat{r}\left(v^2 - 3(\vec{v}\cdot\hat{r})^2\right) + 2\vec{v}(\vec{v}\cdot\hat{r})\right].$$

No net dynamical bobbing! Mysteriously, Newton's 3rd law is preserved for the EM forces.

Hidden Mechanical Momentum

The total momentum of a stationary system is zero.

Proof: $\partial_0 \overset{0}{\cancel{T^{00}_{\text{tot}}}} + \partial_i T^{i0}_{\text{tot}} = 0.$ => $0 = \int \partial_i T^{i0}_{\text{tot}} x^j d^3x \overset{\text{I.B.P.}}{=} \int T^{j0}_{\text{tot}} d^3x$

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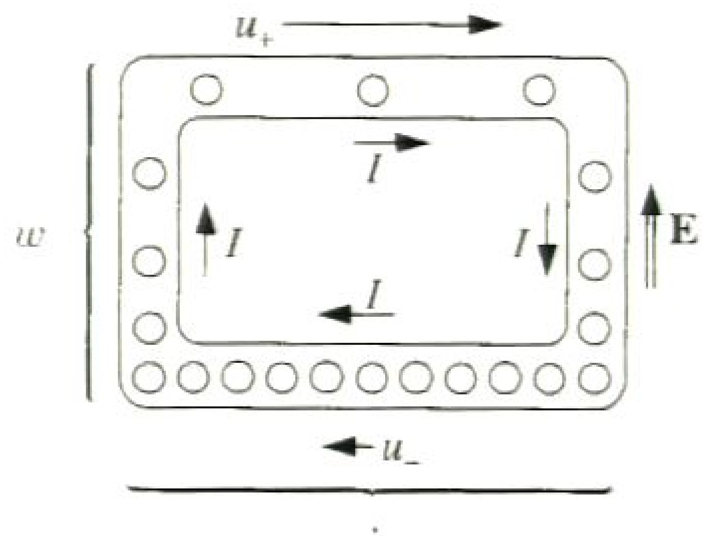
But consider a magnetic dipole immersed in a constant electric field. An easy calculation shows that there is electromagnetic field momentum,

$$\vec{P}_{EM} = \frac{1}{4\pi}\int \vec{E}\times\vec{B} = -\vec{\mu}\times\vec{E}$$

This means that there must be an equal and opposite amount *of mechanical* momentum stored in the magnetic dipole!

There is a hidden mechanical momentum of $\vec{\mu}\times\vec{E}$ stored in the stationary dipole.

Griffiths shows us where the momentum is stored

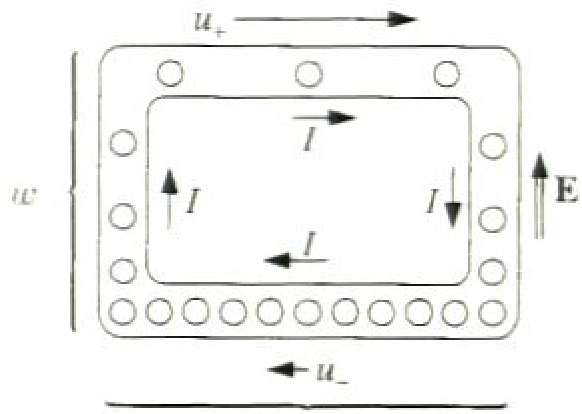


Solution: The momenta of the left and right segments cancel, so we need only consider the top and the bottom. Say there are N_+ charges in the top segment, going at speed u_+ to the right, and N_- charges in the lower segment, going at (slower) speed u_- to the left. The current ($I = \lambda u$) is the same in all four segments (or else charge would be piling up somewhere); in particular,

$$I = \frac{QN_+}{l} u_+ = \frac{QN_-}{l} u_-, \quad \text{so} \quad N_\pm u_\pm = \frac{Il}{Q},$$

where Q is the charge of each particle, and l is the length of the rectangle. *Classically*, the momentum of a single particle is $\mathbf{p} = M\mathbf{u}$ (where M is its mass), and the total momentum (to the right) is

$$p_{\text{classical}} = MN_+u_+ - MN_-u_- = M\frac{Il}{Q} - M\frac{Il}{Q} = 0.$$



as one would certainly expect (after all, the loop as a whole is not moving). But relativistically $\mathbf{p} = \gamma M \mathbf{u}$, and we get

$$p = \gamma_+ M N_+ u_+ - \gamma_- M N_- u_- = \frac{MIl}{Q}(\gamma_+ - \gamma_-),$$

which is *not* zero, because the particles in the upper segment are moving faster.

In fact, the gain in energy ($\gamma M c^2$), as a particle goes up the left segment, is equal to the work done by the electric force, QEw, where w is the height of the rectangle, so

$$\gamma_+ - \gamma_- = \frac{QEw}{Mc^2},$$

and hence

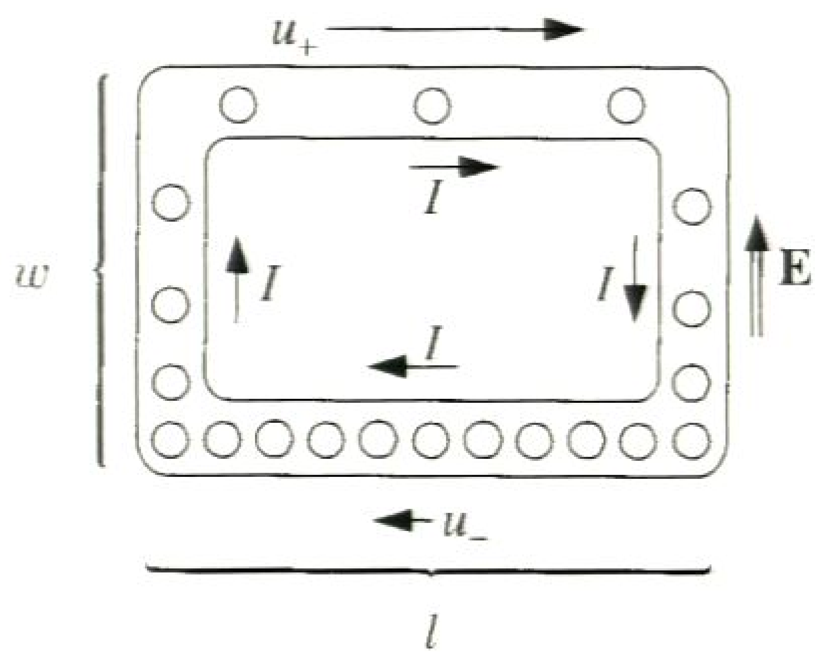
$$p = \frac{IlEw}{c^2}.$$

But Ilw is the magnetic dipole moment of the loop; as vectors, $\mathbf{m}$ points into the page and $\mathbf{p}$ is to the right, so

$$\mathbf{p} = \frac{1}{c^2}(\mathbf{m} \times \mathbf{E}).$$

In words, what is happening is:

1. The electric field makes the velocity of charge carriers uneven.
2. The number density adjusts to have no net *current* ~ m v
3. But the momentum is proportional to *[gamma]* m v.



Momentum in the EM Binary: bodies

Recall a formula for the momentum in a body…

$$\mathcal{E}\frac{\mathrm{d}z^i_{\mathrm{L}}}{\mathrm{d}t} = \int T^{0i}\mathrm{d}^3x + \int f^0(x^i - z^i_{\mathrm{L}})\mathrm{d}^3x.$$

m v = p + mysterious stuff

E-field approx. const

So the mechanical momentum is $\vec{P}_{\mathrm{body}} = m\vec{v}_L + \vec{\mu}\times\vec{E}$

Momentum in the EM Binary: bodies

Recall a formula for the momentum in a body…

$$\mathcal{E}\frac{\mathrm{d}z^i_{\mathrm{L}}}{\mathrm{d}t} = \int T^{0i}\mathrm{d}^3x + \int f^0(x^i - z^i_{\mathrm{L}})\mathrm{d}^3x.$$

m v = p + mysterious stuff

For the electromagnetic force $f^\mu = F^{\mu\nu}J_\nu$, the mysterious stuff is

$$f^0 = F^{0\nu}J_\nu = F^{0i}J_i = E^iJ_i$$

$$\int f^0(x^i - z^i)d^3x = \int E^jJ_j(x^i - z^i)d^3x = E_j\int J^j(x^i - z^i)d^3x = E_j\mu^{ij}$$

E-field approx const

So the mechanical momentum is $\vec{P}_{\mathrm{body}} = m\vec{v}_L + \vec{\mu}\times\vec{E}$

That is, the same hidden momentum formula shows in up the slow-motion case as a correction to lab-frame p=mv.

Momentum in the EM Binary: Field

We can ignore the self-field stress-energy of each body because prior work shows that it just renormalizes the body's stress-energy moments. Thus we have only the "cross term" stress-energy, giving field momentum

$$\vec{P}_{\times} = \frac{1}{4\pi}\int (\vec{E}_{\mathrm{A}} \times \vec{B}_{\mathrm{B}} + \vec{E}_{\mathrm{B}} \times \vec{B}_{\mathrm{A}})\mathrm{d}^3x.$$

Use $\vec{E} \approx -\vec{\nabla}\Phi_{\text{Instantaneous Coulomb}}$, I.B.P, and Maxwell's equations,

$$\vec{P}_{\times} = \int \left(\Phi_{\mathrm{A}}(\vec{J}_{\mathrm{B}} + \frac{\partial \vec{E}_{\mathrm{B}}}{\partial t}) + \Phi_{\mathrm{B}}(\vec{J}_{\mathrm{A}} + \frac{\partial \vec{E}_{\mathrm{A}}}{\partial t})\right) \mathrm{d}^3x.$$

But now expand the local field of each body in a Taylor series,

$$\Phi_{\mathrm{A}}(\vec{x}) = \Phi_{\mathrm{A}}(\vec{z}_{\mathrm{B}}) - (\vec{x} - \vec{z}_{\mathrm{B}}) \cdot \vec{E}_{\mathrm{A}}(\vec{z}_{\mathrm{B}}) + \dots.$$

Of the non-small terms, those proportional to dipole are

$$\boxed{\vec{P}_{\times}^{\mathrm{dipole}} = -\vec{\mu}_{\mathrm{B}} \times \vec{E}_{\mathrm{A}}(\vec{z}_{\mathrm{B}}) - \vec{\mu}_{\mathrm{A}} \times \vec{E}_{\mathrm{B}}(\vec{z}_{\mathrm{A}}).}$$

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EM Binary Perpendicular Momentum Budget

$$P_\perp^{\text{bodies}} = \left(m\vec{v}_L^A + m\vec{v}_L^B + \mu^A \times \vec{E}^B|_{\vec{Z}^A} + \mu^B \times \vec{E}^A|_{\vec{Z}^B} \right)_\perp$$

$$P_\perp^{\text{EM field}} = \left(-\mu^A \times \vec{E}^B|_{\vec{Z}^A} - \mu^B \times \vec{E}^A|_{\vec{Z}^B} \right)_\perp$$

$$\boxed{P_\perp^{\text{total}} = \left(m\vec{v}_L^A + m\vec{v}_L^B \right)_\perp}$$

Conservation of momentum then implies no net dynamical bobbing. If an EM force pushes one body up, another EM force must always push the other body down. Newton's third law preserved!

Of course, there is still the kinematical bobbing. Rewriting in terms of the center of mass (which is shifted by $\vec{v} \times \vec{S}$),

$$P_\perp^{\text{total}} = m^A v_\perp^A + m^B v_\perp^B + (\vec{a}^A \times \vec{S}^A)_\perp + (\vec{a}^B \times \vec{S}^B)_\perp .$$

The final (curious) picture of the EM Binary

As the bodies orbit, they exchange perpendicular momentum with the EM field.

However, the momentum is always stored as hidden momentum, so no bobbing results!

Only the kinematical bobbing remains,

$$m_{\mathrm{A}} a_{\mathrm{A}}^{\perp} + m_{\mathrm{B}} a_{\mathrm{B}}^{\perp} = \frac{\mathrm{d}}{\mathrm{d}s}\left\{ \left[\left(\frac{\vec{S}_{\mathrm{A}}}{m_{\mathrm{A}}} - \frac{\vec{S}_{\mathrm{B}}}{m_{\mathrm{B}}} \right) \times (q_{\mathrm{A}} \vec{E}_{\mathrm{B}}) \right] \cdot \frac{\hat{r} \times \vec{v}}{|\hat{r} \times \vec{v}|} \right\}$$

Gravitational Correspondence

To what extent can we apply these results to gravity? There is a close correspondence between the motion of bodies in (weak-field) gravity and electromagnetism.

Of course, the leading-order (Coulomb) force is the same, but to what extent can we go beyond that?

We extended the analogy to include spin, dipole and even quadrupole effects. The basic recipe is to take the EM equation and

1. Leave all kinematical terms alone
2. For dynamical terms, send $q \rightarrow m$, $\mu \rightarrow S$, and make magnetic effects 4 times as strong.

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More Precisely…

Start with the EM equation of motion and:

Don't mess with kinematical terms; just change F_{EM} to F_{grav}

1. Identify all terms explicitly involving S_i. Replace qE_i by $-m\mathcal{E}_i$, and then leave as-is.

Make the obvious substitutions on the rest, while also making velocity effects twice as strong

2. To all other terms, send $m \to m$, $q \to m$, $\mu^i \to S^i$, $E_i \to \mathcal{E}_i$, $B_i \to 2\mathcal{B}_i$, $\dot{z}^i \to 2\dot{z}^i$, $\mathcal{Q}^{ij}_{\mathrm{TF}} \to \mathcal{J}^{ij}_{\mathrm{TF}}$, $\mathcal{Q}^{ijk}_{\mathrm{TF}} \to 2\mathcal{J}^{ijk}_{\mathrm{TF}}$. Multiply the overall result by "-1" (excluding the spin terms set aside in step one).

3. In the expression for acceleration in the gravitational case, add the term $2m\dot{z}^i(\dot{z}^j\mathcal{E}_j)$.

oops, it doesn't quite work at second order in v

This gives the equation of motion for gravity (at a level sufficient for our 1.5PC binary)

Applying the correspondence to the equations of motion for our EM binary,

EM:

$$m_A a_A^{\perp} = \frac{q_A \vec{\mu}_B + q_B \vec{\mu}_A}{r^3 |\hat{r} \times \vec{v}|} \cdot \overbrace{\left[\hat{r}(-2v^2 + 3(\hat{r}\cdot\vec{v})^2) - \vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{dynamical}}$$
$$+ \frac{q_A q_B \vec{S}_A / m_A}{r^3 |\hat{r} \times \vec{v}|} \cdot \overbrace{\left[\hat{r}(v^2 - 3(\hat{r}\cdot\vec{v})^2) + 2\vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{kinematical}}.$$

$$\downarrow$$

Gravity:

$$m_A a_A^{\perp} = \frac{-2(m_A \vec{S}_B + m_B \vec{S}_A)}{r^3 |\hat{r} \times \vec{v}|} \cdot \overbrace{\left[\hat{r}(-2v^2 + 3(\hat{r}\cdot\vec{v})^2) - \vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{dynamical}}$$
$$- \frac{m_B \vec{S}_A}{r^3 |\hat{r} \times \vec{v}|} \cdot \overbrace{\left[\hat{r}(v^2 - 3(\hat{r}\cdot\vec{v})^2) + 2\vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{kinematical}}.$$

These result agree with a direct post-Newtonian analysis (Caltech group). But a direct analysis can obscure the kinematical/dynamical split that is made manifest by using the analogy.

Considering the net bobbing, the result is of course the same for gravity: there is no net dynamical bobbing. One can rewrite as

$$m_{\mathrm{A}} a_{\mathrm{A}}^{\perp} + m_{\mathrm{B}} a_{\mathrm{B}}^{\perp} = \frac{\mathrm{d}}{\mathrm{d}s} \left\{ \left[\left(\frac{\vec{S}_{\mathrm{A}}}{m_{\mathrm{A}}} - \frac{\vec{S}_{\mathrm{B}}}{m_{\mathrm{B}}} \right) \times (-m_{\mathrm{A}} \vec{\mathcal{E}}_{\mathrm{B}}) \right] \cdot \frac{\hat{r} \times \vec{v}}{|\hat{r} \times \vec{v}|} \right\}.$$

Summary Parts I-III: Net Bobbing is Kinematical!

Tetherballs:

tether force

$$(m_A \dot{\vec{z}}_A + m_B \dot{\vec{z}}_B) \cdot \vec{\zeta} = \left[\left(\frac{\vec{S}_A}{m_A} - \frac{\vec{S}_B}{m_B} \right) \times \vec{F}_A \right] \cdot \vec{\zeta}.$$

Coulomb force

1.5PC Electromagnetism:

$$m_A a_A^{\perp} + m_B a_B^{\perp} = \frac{d}{ds} \left\{ \left[\left(\frac{\vec{S}_A}{m_A} - \frac{\vec{S}_B}{m_B} \right) \times (q_A \vec{E}_B) \right] \cdot \frac{\hat{r} \times \vec{v}}{|\hat{r} \times \vec{v}|} \right\}$$

1.5PN Gravity:

$$m_A a_A^{\perp} + m_B a_B^{\perp} = \frac{d}{ds} \left\{ \left[\left(\frac{\vec{S}_A}{m_A} - \frac{\vec{S}_B}{m_B} \right) \times (-m_A \vec{\mathcal{E}}_B) \right] \cdot \frac{\hat{r} \times \vec{v}}{|\hat{r} \times \vec{v}|} \right\}.$$

Newton force

In all cases the net bobbing is controlled by “S x F”, where F is the force causing orbit.

Applying the correspondence to the equations of motion for our EM binary,

EM:

$$m_{\mathrm{A}} a_{\mathrm{A}}^{\perp} = \frac{q_{\mathrm{A}}\vec{\mu}_{\mathrm{B}} + q_{\mathrm{B}}\vec{\mu}_{\mathrm{A}}}{r^3|\hat{r}\times\vec{v}|} \cdot \overbrace{\left[\hat{r}(-2v^2 + 3(\hat{r}\cdot\vec{v})^2) - \vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{dynamical}}$$

$$+ \frac{q_{\mathrm{A}} q_{\mathrm{B}} \vec{S}_{\mathrm{A}}/m_{\mathrm{A}}}{r^3|\hat{r}\times\vec{v}|} \cdot \overbrace{\left[\hat{r}(v^2 - 3(\hat{r}\cdot\vec{v})^2) + 2\vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{kinematical}}.$$

$$\Downarrow$$

Gravity:

$$m_{\mathrm{A}} a_{\mathrm{A}}^{\perp} = \frac{-2(m_{\mathrm{A}}\vec{S}_{\mathrm{B}} + m_{\mathrm{B}}\vec{S}_{\mathrm{A}})}{r^3|\hat{r}\times\vec{v}|} \cdot \overbrace{\left[\hat{r}(-2v^2 + 3(\hat{r}\cdot\vec{v})^2) - \vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{dynamical}}$$

$$- \frac{m_{\mathrm{B}}\vec{S}_{\mathrm{A}}}{r^3|\hat{r}\times\vec{v}|} \cdot \overbrace{\left[\hat{r}(v^2 - 3(\hat{r}\cdot\vec{v})^2) + 2\vec{v}(\hat{r}\cdot\vec{v})\right]}^{\text{kinematical}}.$$

These result agree with a direct post-Newtonian analysis (Caltech group). But a direct analysis can obscure the kinematical/dynamical split that is made manifest by using the analogy.

Part IV: Kicks

By conservation of momentum, a kick velocity requires the radiation of field momentum off to infinity.

This means that the tetherballs system cannot have a kick, so that bobbing and kicks are (in general) unrelated.

The EM binary could certainly have a kick, but our 1.5PC approximation is quasi-static and does not capture the radiation.

However, the momentum radiated during merger is likely related to the configuration of field momentum just before merger; for example, perhaps momentum that is "hanging around" before merger gets "released" during a rapid merger.

Since the bobbing is controlled by S and the kick is (presumably) controlled by μ, one should be able to tune the kick independently of the bobbing in the electromagnetic case.

For example, it should be possible to arrange to have the kick point in the opposite direction of the bobbing, or to have there be no kick at all.

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What about gravity?

In the gravitational analogy $\mu \rightarrow S$ and $S \rightarrow S$ and separate tuning is impossible!

Both effects are controlled by spin, explaining why they can appear related.

Summary of Findings

System	Net Bobbing	Kick
Tetherballs	kinematical	None
1.5PC EM Binary	kinematical	tunable
1.5PN Gravity	kinematical	tied to bobbing

Summary of Interpretations

At least for the bobbing and probably for the kick, there is no weird black hole physics or strange general relativity effect at play here.

Only a slight revision in freshman physics intuition is necessary to understand the basic physics.

The bobbing is kinematical and the kick is unrelated.

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