Title: Part I: Don't Shake That Solenoid Too Hard: Particle Production from Aharonov-Bohm

Date: Nov 30, 2010 02:00 PM

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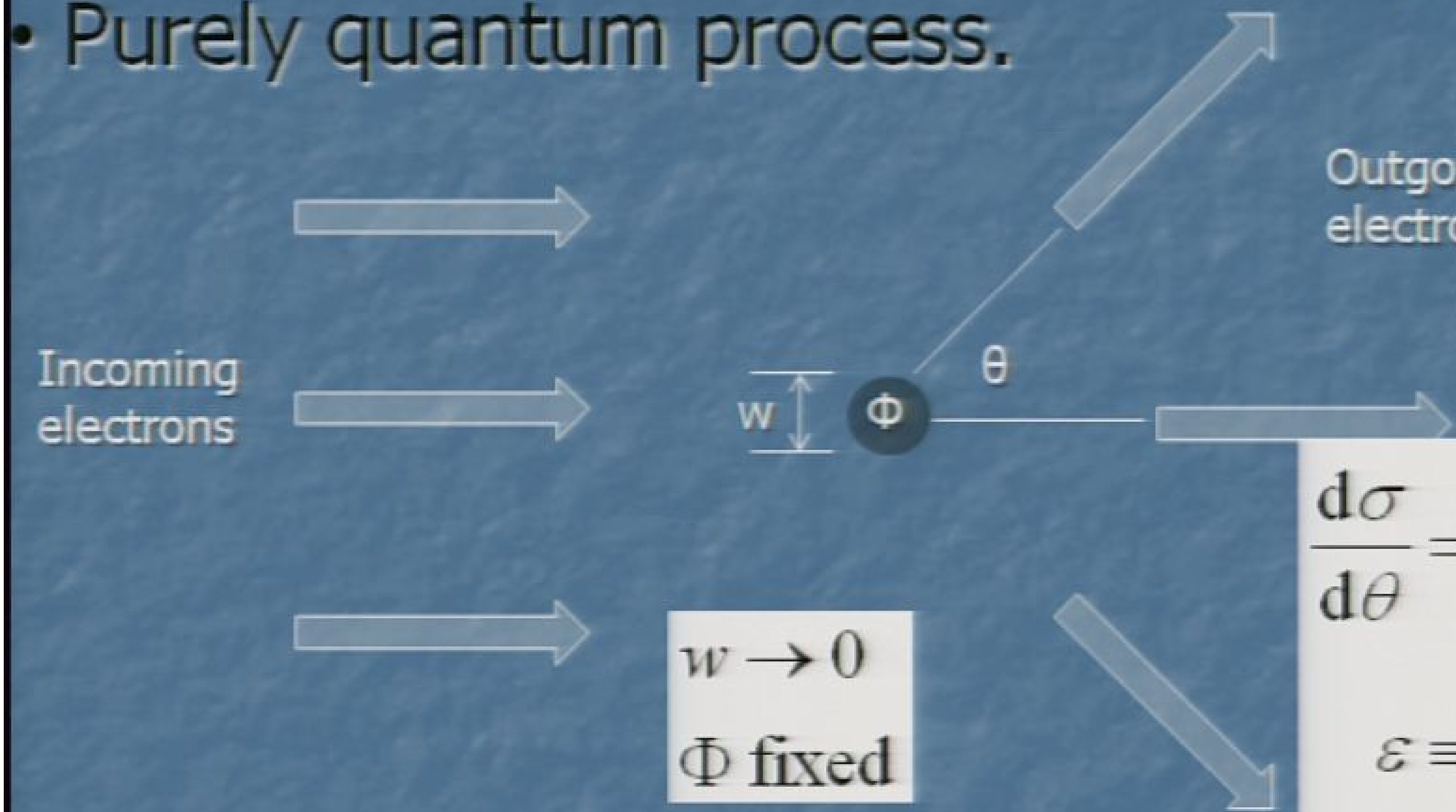
Abstract: Five decades ago, Aharonov and Bohm illustrated the indispensable role of the vector potential in quantum dynamics by showing (theoretically) that scattering electrons around a solenoid, no matter how thin, would give rise to a non-trivial cross section that had a periodic dependence on the product of charge and total magnetic flux. (This periodic dependence is due to the topological nature of the interaction.) We extend the Aharonov-Bohm analysis to the field theoretic domain: starting with the quantum vacuum (with zero particles) we compute explicitly the rate of production of electrically charged particle-antiparticle pairs induced by shaking a solenoid at some fixed frequency. (This body of work can be found in arXiv: 0911.0682 and 1003.0674.)

Part II: The N-Body Problem in General Relativity from Perturbative QFT

In the second portion of the talk, I will describe how one may use methods usually associated with perturbative quantum field theory to develop what is commonly known as the post-Newtonian program in General Relativity -- the weak field, non-relativistic, gravitational dynamics of compact astrophysical objects. The 2 body aspect of the problem is a large industry by now, driven by the need to model the gravitational waves expected from compact astrophysical binaries. I will discuss my efforts to generalize these calculations to the N-body case. (This work can be found in arXiv: 0812.0012.)

Aharonov and Bohm (1959)

- Purely quantum process.



$$w \to 0$$
$$\Phi \text{ fixed}$$

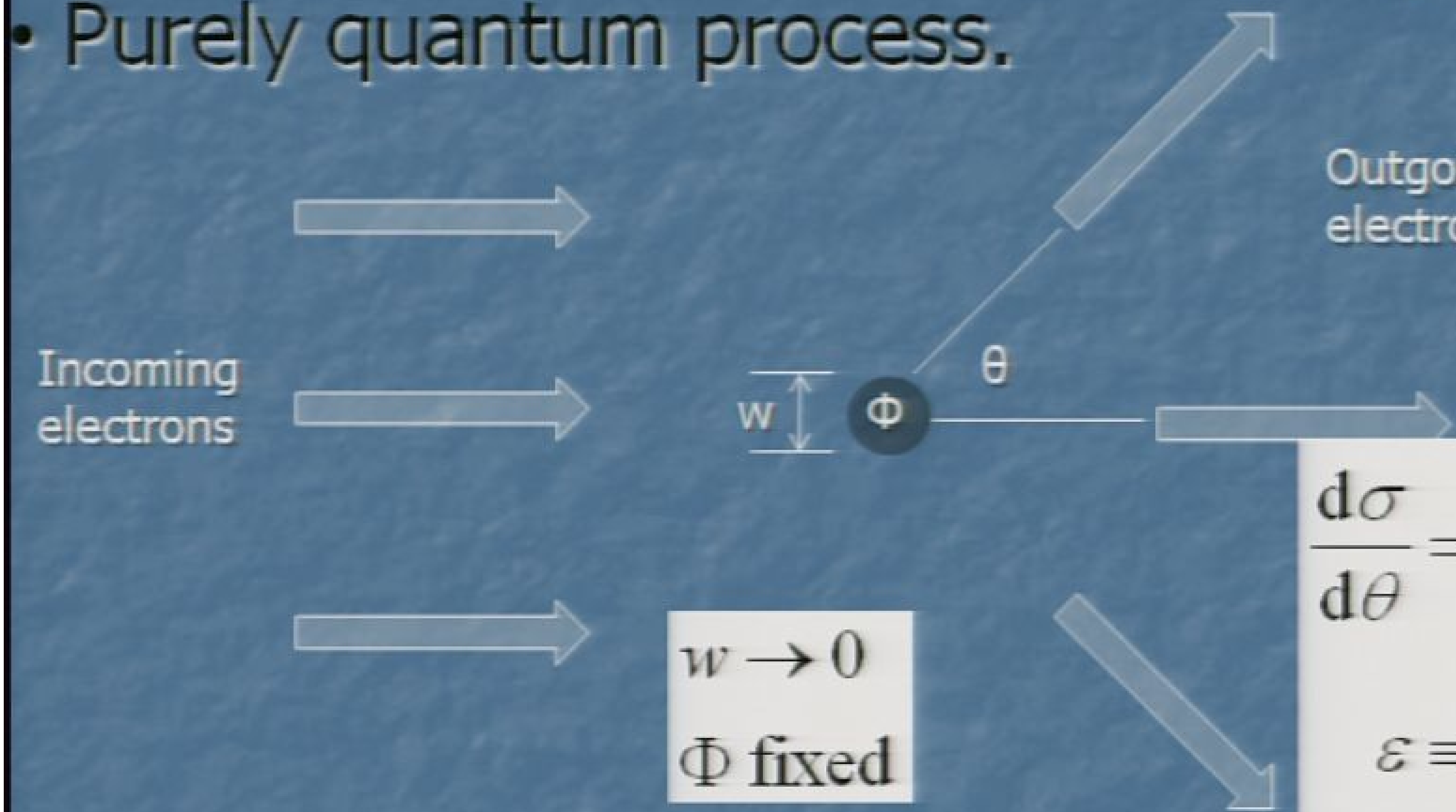
$$\frac{d\sigma}{d\theta} = \frac{\sin^2[\pi\varepsilon]}{2\pi k \cos^2[\theta/2]}$$

$$\varepsilon \equiv \frac{e\Phi}{2\pi}$$

- Maxwell tensor is zero almost everywhere in thin solenoid limit – no classical dynamics.
- Non-zero cross section (AB 1959; Alford, Wilczek 1989)

Aharonov and Bohm (1959)

- Purely quantum process.



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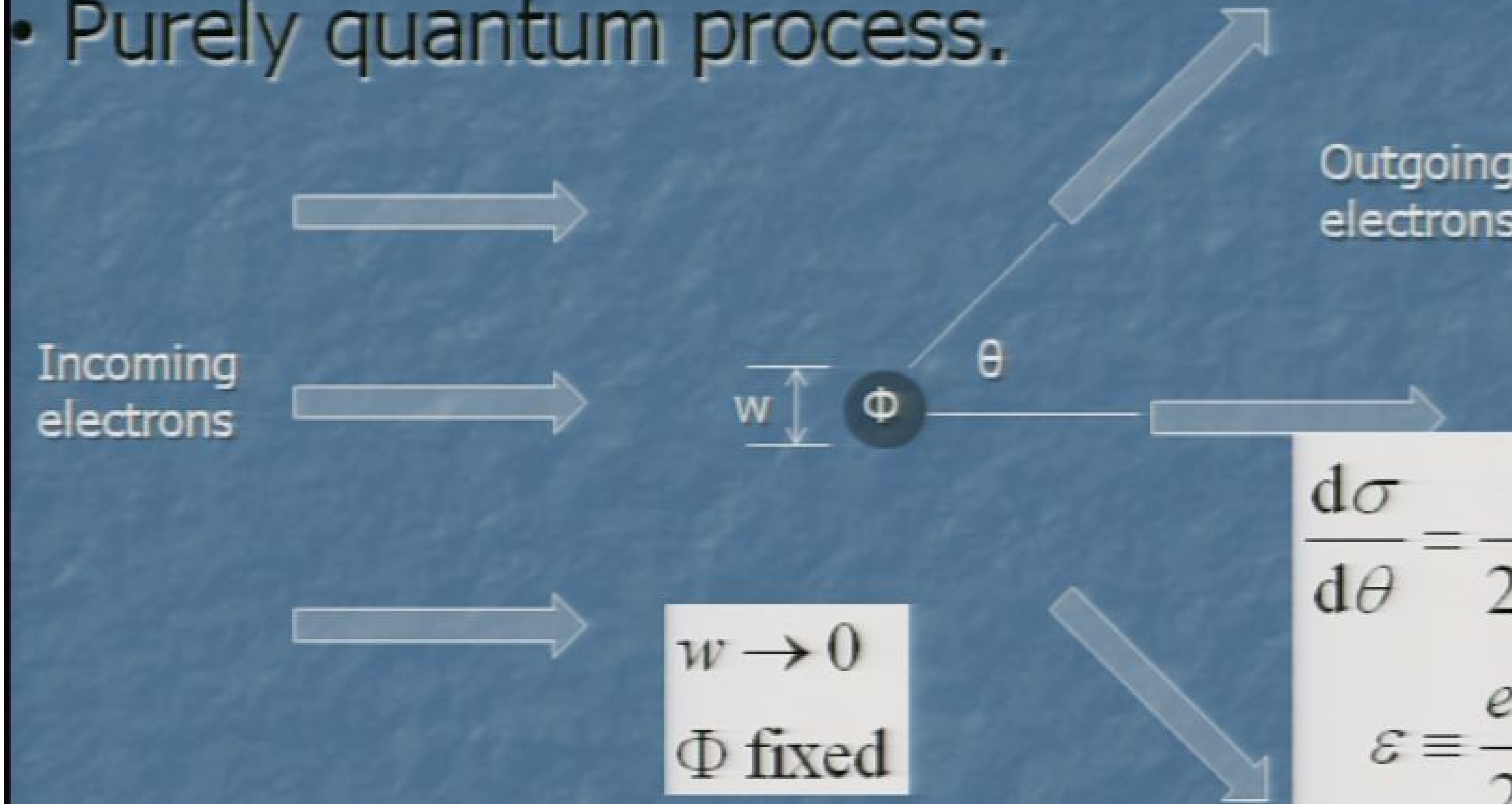
- Maxwell tensor is zero almost everywhere in thin solenoid limit – no classical dynamics.
- Quantum dyanmics: A_μ is non-zero outside solenoid.

Aharonov and Bohm (1959)

- Purely quantum process.



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$$\varepsilon \equiv \frac{e\Phi}{2\pi}$$

$w \to 0$

Φ fixed

- Maxwell tensor is zero almost everywhere in thin solenoid limit – no classical dynamics.
- Periodic dependence on eΦ – topological interaction.

Aharonov-Bohm Interaction

- Purely quantum process.
- Topological aspect:

AB QM

$$\left\langle t, \vec{x} \middle| t', \vec{x}' \right\rangle = \int_{\vec{q}[t']=\vec{x}'}^{\vec{q}[t]=\vec{x}} \exp[iS_0[\vec{q}, \dot{\vec{q}}] + ie\int A_i \mathrm{d}x^i] D\vec{q}$$

- QM: Amp. for paths that cannot be deformed into each other will differ by exp[i(integer)eΦ]

Aharonov-Bohm Interaction

- Purely quantum process.
- Topological aspect:

AB

QM

$$\Delta M = \exp[ie\Phi]$$

$$\left\langle t, \vec{x} \middle| t', \vec{x}' \right\rangle = \int_{\vec{q}[t']=\vec{x}'}^{\vec{q}[t]=\vec{x}} \exp[iS_0[\vec{q}, \dot{\vec{q}}] + ie\int A_i \mathrm{d}x^i] D\vec{q}$$

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Aharonov-Bohm Interaction

- Purely quantum process.
- Topological aspect:

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$$\Delta M = \exp[i2e\Phi]$$

$$\left\langle t,\vec{x}\middle|t',\vec{x}'\right\rangle = \int_{\vec{q}[t']=\vec{x}'}^{\vec{q}[t]=\vec{x}} \exp[iS_0[\vec{q},\dot{\vec{q}}] + ie\int A_i \mathrm{d}x^i] D\vec{q}$$

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Aharonov-Bohm Interaction

- Purely quantum process.
- Topological aspect:

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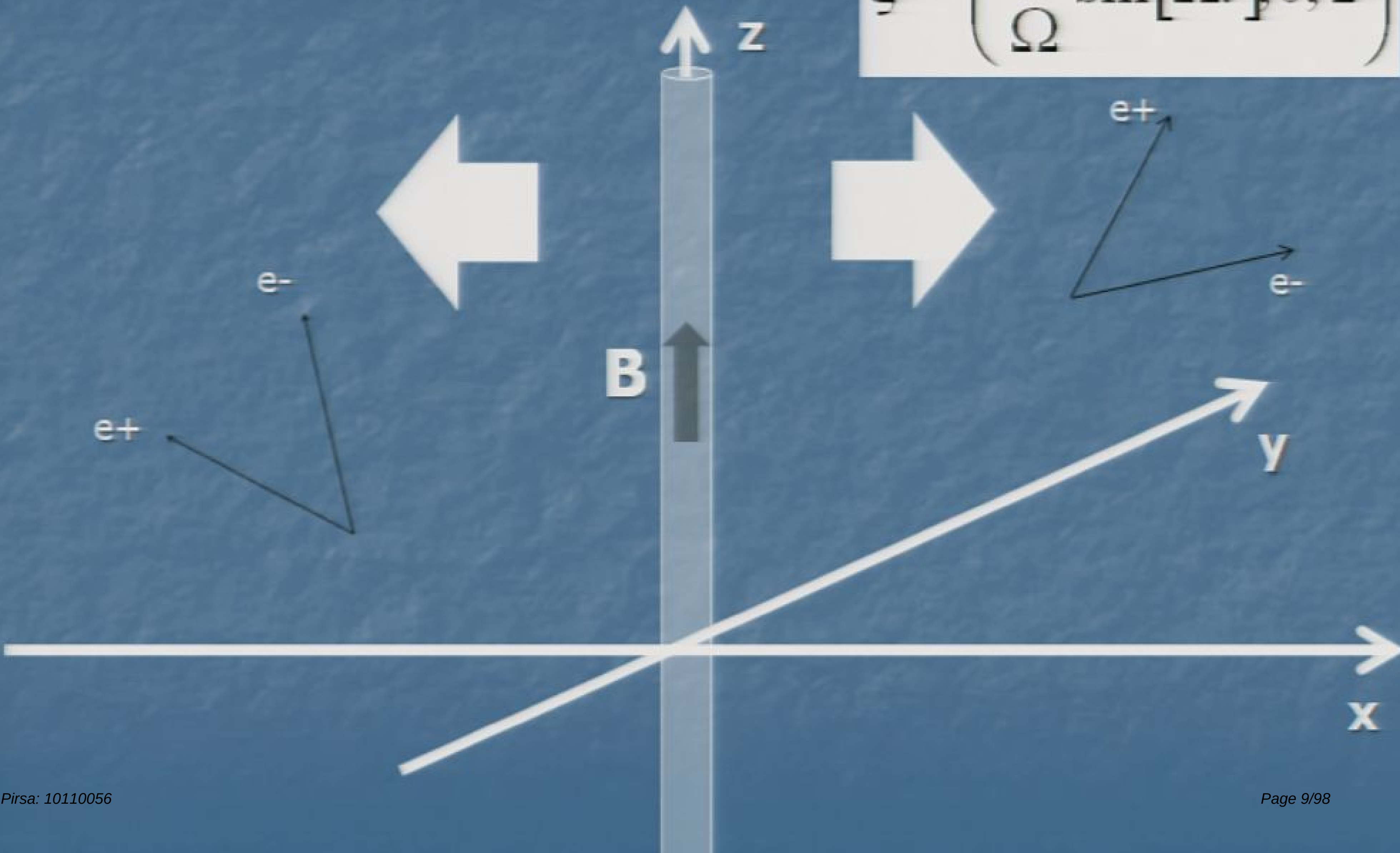
$$\langle t,\vec{x}|t',\vec{x}'\rangle = \int_{\vec{q}[t']=\vec{x}'}^{\vec{q}[t]=\vec{x}} \exp[iS_0[\vec{q},\dot{\vec{q}}] + ie\int A_i \mathrm{d}x^i] D\vec{q}$$

- Expect: Pair production rate to have periodic dependence on AB phase eΦ.

Setup

$$\vec{\xi} = \left(\frac{v_0}{\Omega}\sin[\Omega t], 0, z\right)$$



Setup

- Effective theory of magnetic flux tube: Alford and Wilczek (1989).
- Bosonic or fermionic quantum electrodynamics (QED)

$$S = S_{\_\mathrm{QED}} + S_{\otimes}$$

$$S_{\otimes} = -\frac{\Phi}{2} \iint_{\mathrm{worldsheet}} \widetilde{F}_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$$

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

$$\Phi \equiv \text{Magnetic flux}$$

Bosonic QED

$$S_{\varphi\mathrm{QED}} = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + | D_{\mu}\varphi |^2 - m^2 | \varphi |^2 \right)$$

$$D_{\mu} \equiv \partial_{\mu} + ieA_{\mu}$$

Fermionic QED

$$S_{\psi\mathrm{QED}} = \int \mathrm{d}^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\gamma^{\mu} D_{\mu} - m \right) \psi \right)$$

$$D_{\mu} \equiv \partial_{\mu} + ieA_{\mu}$$

Why are particles produced?

- The gauge potential A_μ around a moving solenoid is time-dependent.

$$A_\mu = \left(0,0,-\Phi\Theta[x-\xi^x(t)]\delta[y],0\right)$$

- Hamiltonian of QFT is explicitly time-dependent: $H_i = \int d^3x\, A_\mu J^\mu$
- Zero particle state (in the Heisenberg picture) at different times not the same vector – i.e. particle creation occurs.

Moving frames scheme

Adiabatic approximation

- Mode expansion: Solve the mode functions for the stationary solenoid problem and shift them by ξ, location of moving solenoid.

$$\varphi = \sum_k \begin{pmatrix} a_k(t) f_k[t, \vec{x} - \vec{\xi}(t)] \\ + b_k^{H.C.}(t) h_k[t, \vec{x} - \vec{\xi}(t)] \end{pmatrix}, \quad H f_k = E_k f_k$$

$$\psi = \sum_k \begin{pmatrix} a_k(t) u_k[t, \vec{x} - \vec{\xi}(t)] \\ + b_k^{H.C.}(t) v_k[t, \vec{x} - \vec{\xi}(t)] \end{pmatrix}, \quad H u_k = E_k u_k$$

Moving frames scheme

- Compute 0 particle to 2 particle transition amplitude

$$\left\langle \varphi^* \varphi \,\mathrm{or}\, e^+ e^-, t = \infty \middle| 0, t = -\infty \right\rangle$$

$$= \left\langle 0, t = -\infty \middle| a^{H.C.}{}_{(t=-\infty)} b^{H.C.}{}_{(t=-\infty)} U(+\infty, -\infty) \middle| 0, t = -\infty \right\rangle$$

$$\approx -i \left\langle 0, t = -\infty \middle| a^{H.C.}{}_{(t=-\infty)} b^{H.C.}{}_{(t=-\infty)} \int_{-\infty}^{\infty} H(\tau) d\tau \middle| 0, t = -\infty \right\rangle$$

Moving frames φ results

Rate of pair production of $\varphi^* \varphi$ per unit length

$$= \int_0^\infty dk \int_{-\infty}^\infty dk_z \Theta\left[\Omega - k_0 - \sqrt{k_z^2 + m^2}\right]$$

$$\times \frac{v_0^2 \sin^2[\pi\kappa] k_c^2 k}{8\pi^2 \Omega^2 k_0} \left(\left(\frac{k}{k_c}\right)^{2\kappa} + \left(\frac{k_c}{k}\right)^{2\kappa - 2} \right)$$

$$k_c \equiv \sqrt{\Omega^2 + k^2 - 2\Omega k_0}, \kappa \equiv \frac{e\Phi}{2\pi} \bmod 1$$

- Rate carries periodic dependence on eΦ
- Non-relativistic

Moving frames ψ results

Rate of pair production of e^+e^- per unit length

$$= \int_0^\infty dk\,k \int_0^\infty dk'\,k' \int_{-\infty}^{\infty} dk_z \int_{-\infty}^{\infty} dk_z'\,\delta\left[k_0 + k_0' - \Omega\right]\delta\left[k_z + k_z'\right]$$

$$\times \frac{v_0^2 \sin^2\left[\pi\kappa\right]}{8\pi^2\Omega^2 k_0 k_0'}\left(m^2 + k_z^2 + k_0 k_0'\right)\left(\left(\frac{k}{k'}\right)^{2\kappa} + \left(\frac{k'}{k}\right)^{2\kappa}\right)$$

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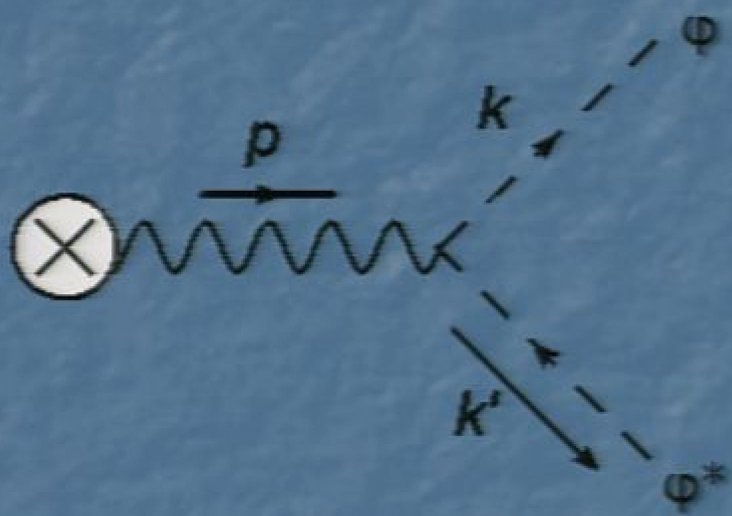
- Rate carries periodic dependence on eΦ
- Non-relativistic

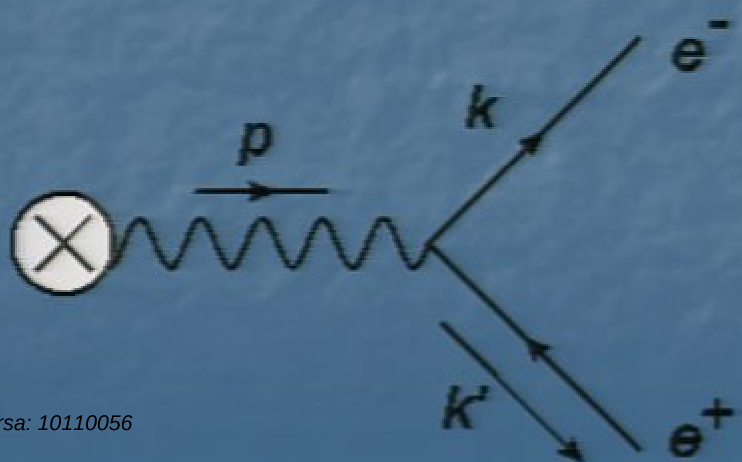
Relativistic
eΦ << 1

$$S = S_{\_\mathrm{QED}} + S_{\otimes}$$

$$S_{\otimes} = -\frac{\Phi}{2} \iint_{\text{world sheet}} \widetilde{F}_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}$$



$$S_{\varphi\mathrm{QED}} = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_{\mu}\varphi|^2 - m^2 |\varphi|^2 \right)$$



$$S_{\psi\mathrm{QED}} = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\left(i\gamma^{\mu} D_{\mu} - m \right)\psi \right)$$

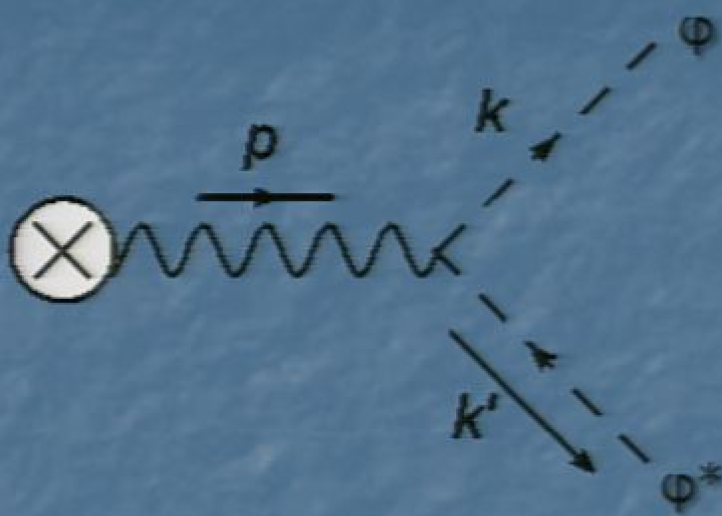
Small AB phase: eΦ << 1

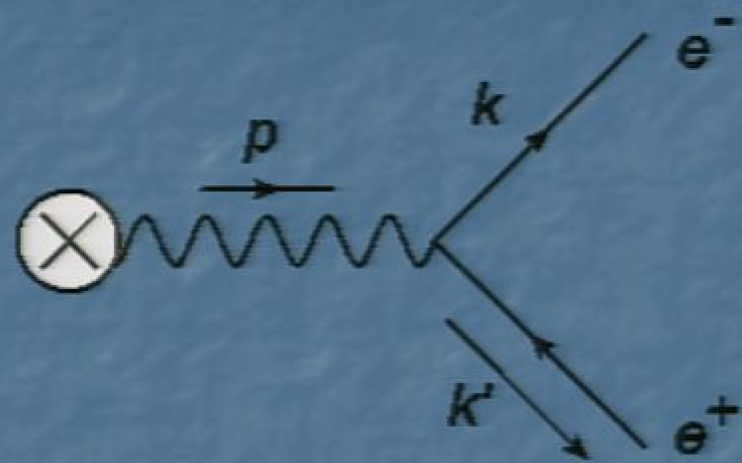
$$M[0\to 2]=\frac{e\Phi}{p^{\delta}p_{\delta}}\varepsilon_{\mu\nu\alpha\beta}p^{\mu}J^{\nu}\widetilde{S}^{\alpha\beta}=\frac{e\Phi}{p_0}\vec{J}\cdot\left(\vec{I}_{+}\times\vec{I}_{-}\right)$$

$$\widetilde{S}^{\alpha\beta}=\frac{1}{2}\iint_{\text{worldsheet}}\exp[ip_{\mu}x^{\mu}]\mathrm{d}x^{\alpha}\wedge\mathrm{d}x^{\beta}=\frac{1}{2}I_{+}^{[\alpha}I_{-}^{\beta]}$$

- Valid for any flux tube trajectory.

Small AB phase: eΦ << 1





$$M[0 \to 2] = \frac{e\Phi}{p^{\delta} p_{\delta}} \varepsilon_{\mu\nu\alpha\beta} p^{\mu} J^{\nu} \tilde{S}^{\alpha\beta} = \frac{e\Phi}{p_0} \vec{J} \cdot \left(\vec{I}_+ \times \vec{I}_- \right)$$

$\vec{I}_+ \times \vec{I}_-$ (Moving solenoid)

$$= \hat{y}(2\pi)^2 \sum_{\ell=-\infty}^{+\infty} \delta[p_z]\delta[p_0 - \ell\Omega](-)^{\ell} J_{\ell}\left[\frac{p_x v_0}{\Omega}\right]\frac{p_0}{p_x}$$

Small AB phase: eΦ << 1

$$M[0 \to 2] = \frac{e\Phi}{p^{\delta} p_{\delta}} \varepsilon_{\mu\nu\alpha\beta} p^{\mu} J^{\nu} \tilde{S}^{\alpha\beta} = \frac{e\Phi}{p_0} \vec{J} \cdot \left(\vec{I}_{+} \times \vec{I}_{-} \right)$$

$\vec{I}_{+} \times \vec{I}$

$= \hat{y}(2$

- Spins of e⁺e⁻ anti-correlated along direction determined by their momenta and $I_+ \times I_-$.

Small AB phase results

Rate of pair production of $\varphi^* \varphi$
per unit length per unit phase space

$$=\left(\frac{e\Phi}{4\pi^2}\right)^2 \sum_{\ell=1}^{\infty} \frac{(k_y - k_y')^2}{k_0 k_0' p_x^2}$$

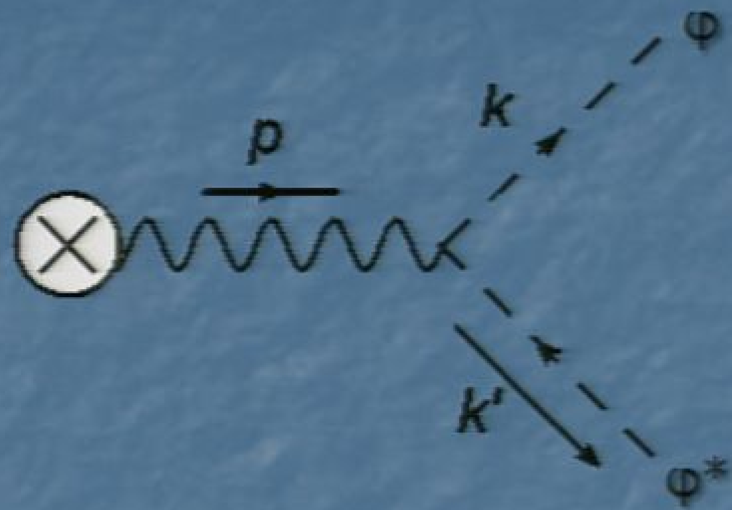
$$\times J_\ell^2\left[\frac{p_x v_0}{\Omega}\right]\delta[k_z + k_z']\delta[k_0 + k_0' - \ell\Omega]$$

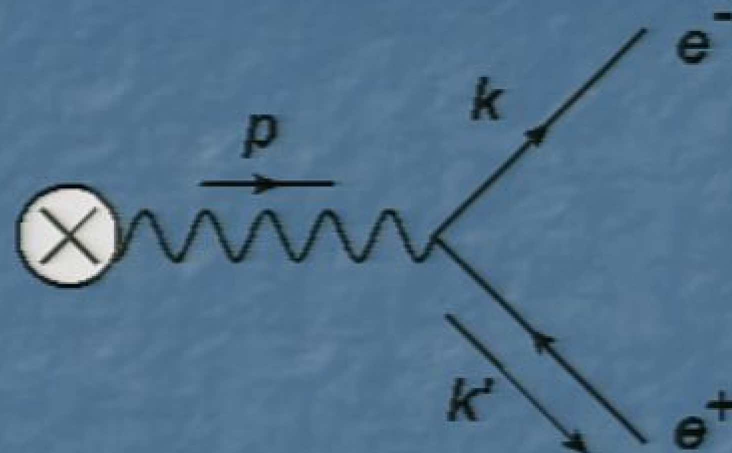
Rate of pair production of e^+e^-
per unit length per unit phase space

$$=\left(\frac{e\Phi}{4\pi^2}\right)^2 \sum_{\ell=1}^{\infty} \frac{(\ell\Omega)^2 - (k_x + k_x')^2 - (k_y - k_y')^2}{2k_0 k_0' p_x^2}$$

$$\times J_\ell^2\left[\frac{p_x v_0}{\Omega}\right]\delta[k_z + k_z']\delta[k_0 + k_0' - \ell\Omega]$$

Small AB phase: eΦ << 1





$$M[0 \to 2] = \frac{e\Phi}{p^\delta p_\delta}\varepsilon_{\mu\nu\alpha\beta}p^\mu J^\nu \tilde{S}^{\alpha\beta} = \frac{e\Phi}{p_0}\vec{J}\cdot\left(\vec{I}_+ \times \vec{I}_-\right)$$

$\vec{I}_+ \times \vec{I}$

$= \hat{y}(2$

- Spins of e^+e^- anti-correlated along direction determined by their momenta and $I_+ \times I_-$.

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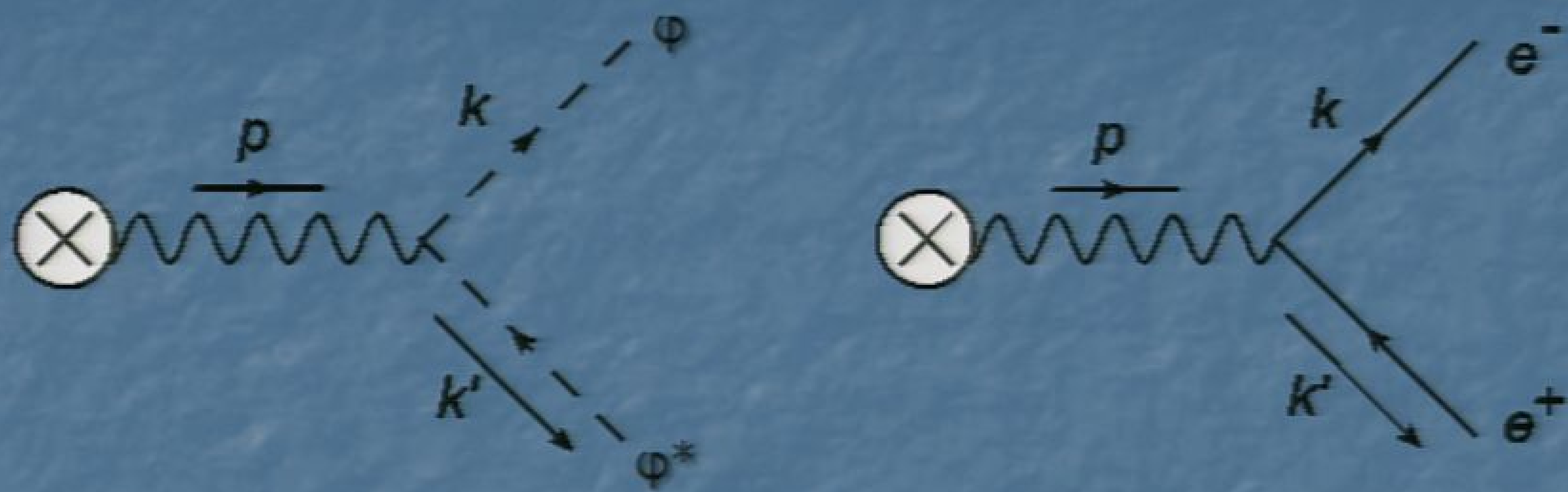
Small AB phase: eΦ << 1



$$M[0 \to 2] = \frac{e\Phi}{p^{\delta} p_{\delta}} \varepsilon_{\mu\nu\alpha\beta} p^{\mu} J^{\nu} \widetilde{S}^{\alpha\beta} = \frac{e\Phi}{p_0} \boxed{\vec{J} \cdot \left(\vec{I}_{+} \times \vec{I}_{-} \right)}$$

$\vec{I}_{+} \times \vec{I}$

$= \hat{y}(2$

- Spins of $e^{+}e^{-}$ anti-correlated along direction determined by their momenta and $I_{+} \times I_{-}$.

Small AB phase results

Rate of pair production of $\varphi^* \varphi$

per unit length per unit phase space

$$= \left(\frac{e\Phi}{4\pi^2} \right)^2 \sum_{\ell=1}^{\infty} \frac{(k_y - k_y^{'})^2}{k_0 k_0^{'} p_x^2}$$

$$\times J_\ell^2 \left[\frac{p_x v_0}{\Omega} \right] \delta[k_z + k_z^{'}] \delta[k_0 + k_0^{'} - \ell\Omega]$$

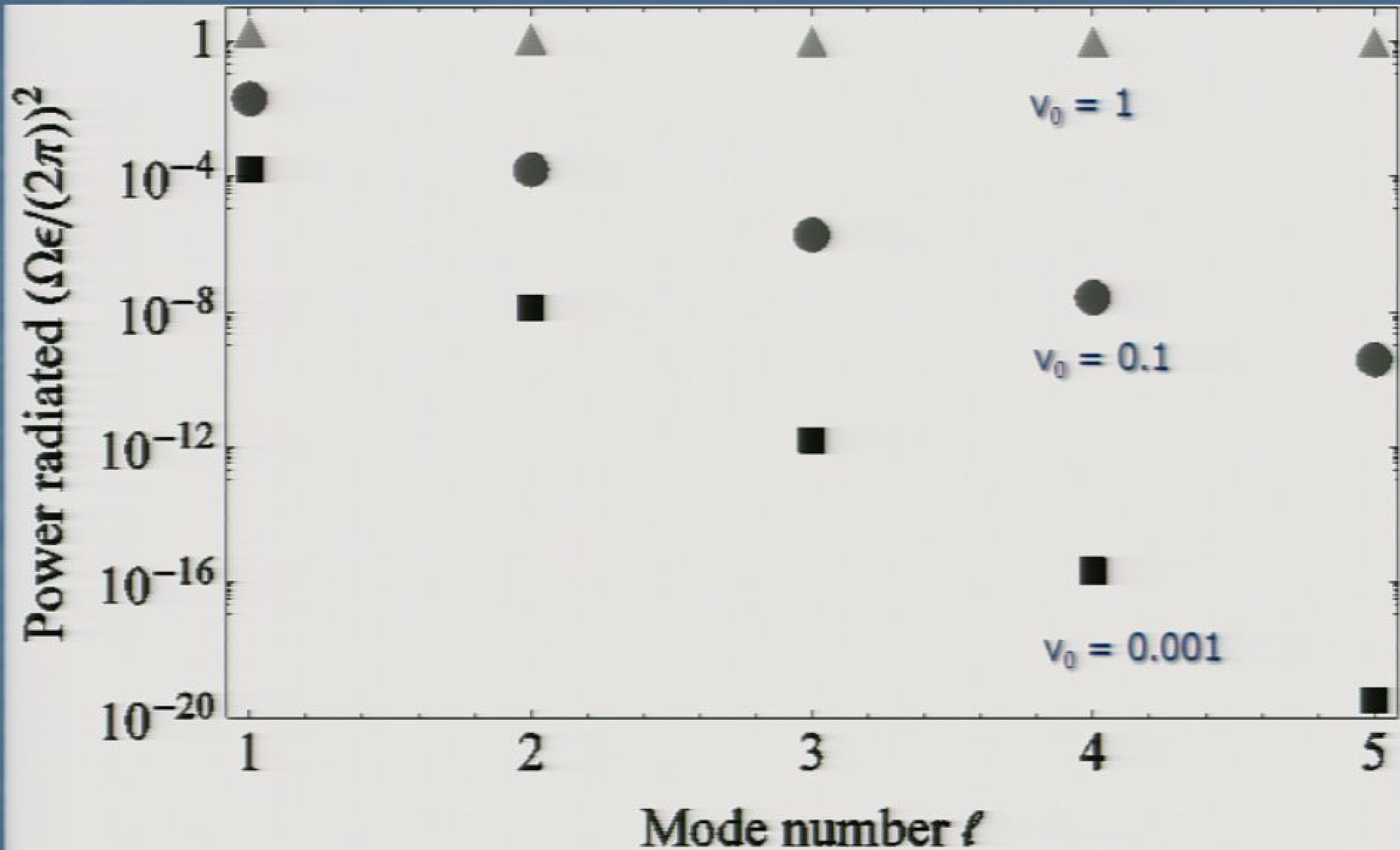
Rate of pair production of e^+e^-

per unit length per unit phase space

$$= \left(\frac{e\Phi}{4\pi^2} \right)^2 \sum_{\ell=1}^{\infty} \frac{(\ell\Omega)^2 - (k_x + k_x^{'})^2 - (k_y - k_y^{'})^2}{2 k_0 k_0^{'} p_x^2}$$

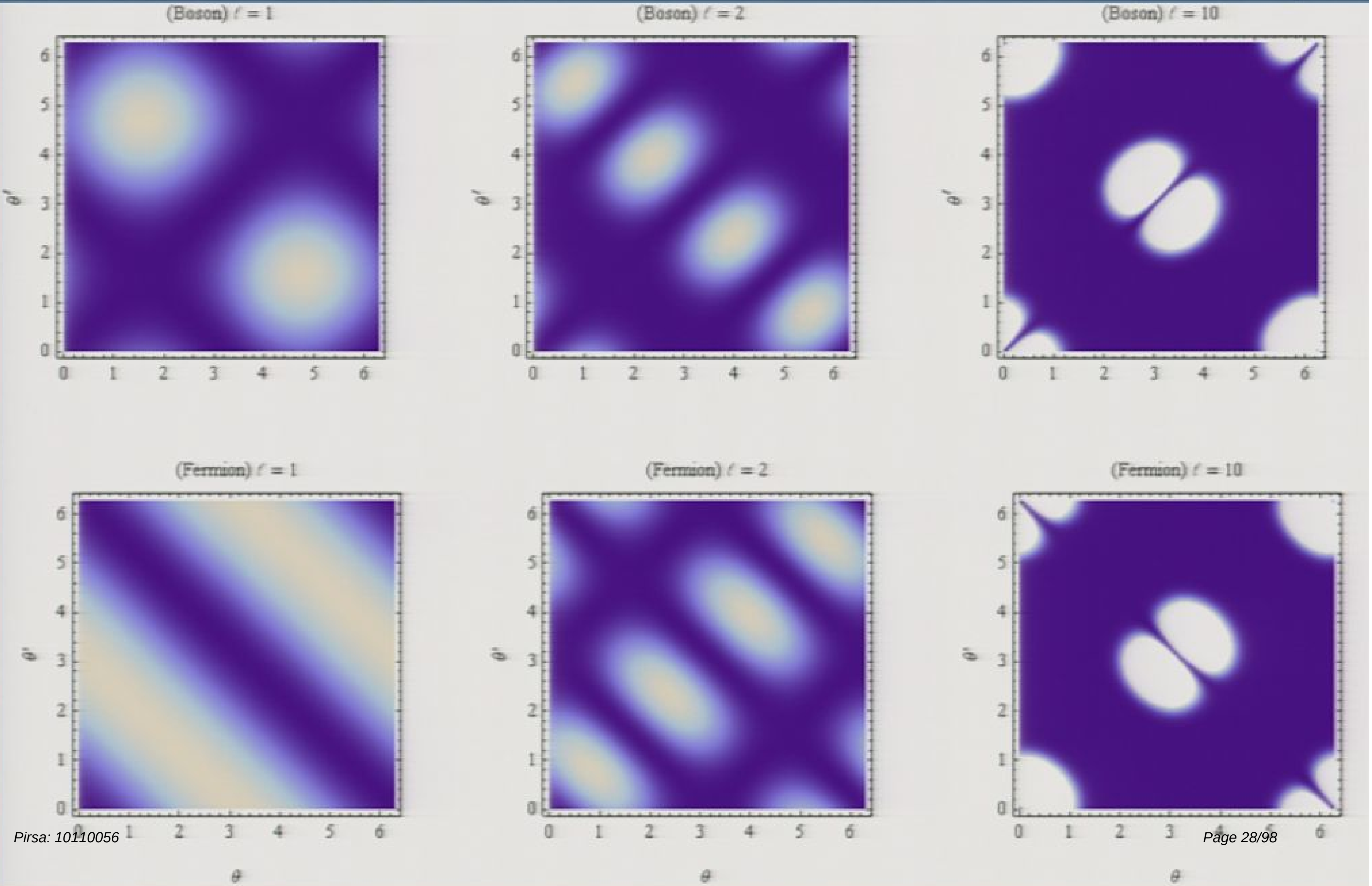
$$\times J_\ell^2 \left[\frac{p_x v_0}{\Omega} \right] \delta[k_z + k_z^{'}] \delta[k_0 + k_0^{'} - \ell\Omega]$$

eΦ<<1: Total Power for Ω>>m



- Similar plot for bosons.

$e\Phi << 1,\ \Omega >> m,\ v_0 \sim 1,\ k_z = 0,\ k_{xy} = k'_{xy}$



The N-Body Problem in General Relativity from Perturbative (Quantum) Field Theory

Y.-Z.Chu, Phys. Rev. D 79: 044031, 2009
arXiv: 0812.0012 [gr-qc]

Yi-Zen Chu

@ University of Toronto, High Energy Physics Seminar

Monday, 29 November 2010

n-Body Problem in GR

- System of $n \geq 2$ gravitationally bound compact objects:
  - Planets, neutron stars, black holes, etc.
- What is their effective gravitational interaction?

n-Body Problem in GR

- Compact objects ≈ point particles
- n-body problem: Dynamics for the coordinates of the point particles
- Assume non-relativistic motion
  - GR corrections to Newtonian gravity: an expansion in $(v/c)^2$

$$L_{\text{eff}} = \sum_{a=1}^{n} \frac{1}{2} M_a \vec{v}_a^{\,2} + \frac{1}{2} \sum_{a \neq b}^{n} \frac{G_N M_a M_b}{|\vec{x}_a - \vec{x}_b|} + \ldots ?$$

Nomenclature: $O[(v/c)^{2Q}] = Q$ PN

n-Body Problem in GR

- Note that General Relativity is non-linear.
  - Superposition does not hold
  - 2 body lagrangian is not sufficient to obtain n-body lagrangian

$$L_{\text{eff}} = \sum_{a=1}^{n} \frac{1}{2} M_a \vec{v}_a^{\,2} + \frac{1}{2} \sum_{a \neq b}^{n} \frac{G_N M_a M_b}{|\vec{x}_a - \vec{x}_b|} + \ldots ?$$

Nomenclature: $O[(v/c)^{2Q}] = Q$ PN

n-Body Problem in GR

- n-body problem known up to $O[(v/c)^2]$:
  - Einstein-Infeld-Hoffman lagrangian
  - Eqns of motion used regularly to calculate solar system dynamics, etc.
    - Precession of Mercury's perihelion begins at this order
- $O[(v/c)^4]$ only known partially.
  - Damour, Schafer (1985, 1987)
  - **Compute using field theory? (Goldberger, Rothstein, 2004)**

Motivation I

- Solar system probes of GR beginning to go beyond $O[(v/c)^2]$:
  - New lunar laser ranging observatory APOLLO; Mars and/or Mercury laser ranging missions?
  - ASTROD, LATOR, GTDM, BEACON, etc.
  - See e.g. Turyshev (2008)

Motivation I

- n-body L_{eff} gives not only dynamics but also geometry.
  - Add a test particle, M->0: it moves along geodesic in the spacetime metric generated by the rest of the n masses
  - Metric can be read off its L_{eff}

$$L_{\text{eff}}^{(\text{test})} = -M_\varepsilon \sqrt{g_{\mu\nu} \frac{dz^\mu}{dt} \frac{dz^\nu}{dt}}$$

$$\boxed{g_{\mu\nu} \equiv \eta_{\mu\nu} + \delta g_{\mu\nu}}$$

$$= -M_\varepsilon + \frac{M_\varepsilon}{2}\left(\frac{d\vec{z}}{dt}\right)^2 - M_\varepsilon\left(\frac{1}{2}\delta g_{00} + \delta g_{0i}\frac{dz^i}{dt} + \frac{1}{2}\delta g_{ij}\frac{dz^i}{dt}\frac{dz^j}{dt} + \dots\right)$$

Motivation II

- Gravitational wave observatories may need the 2 body L_{eff} beyond $O[(v/c)^7]$:
  - LIGO, VIRGO, etc. can track gravitational waves (GWs) from compact binaries over $O[10^4]$ orbital cycles.
  - GW detection: Raw data integrated against theoretical templates to search for correlations.
  - Construction of accurate templates requires 2 body dynamics.
  - Currently, 2 body dynamics known up to $O[(v/c)^7]$, i.e. 3.5 PN
  - See e.g. Blanchet (2006).

Why (Quantum) Field Theory

- Starting at 3 PN, $O[(v/c)^6]$, GR computations of 2 body L_{eff} start to give divergences – due to the point particle approximation – that were eventually handled by dimensional regularization.
- Perturbation theory beyond $O[(v/c)^7]$ requires systematic, efficient methods.

QFT Offers:

- Renormalization & regularization
- Computational algorithm – Feynman diagrams with appropriate dimensional analysis.

Dynamics: Action

- GR: Einstein-Hilbert
- n point particles: any scalar functional of geometric tensors, *d*-velocities, etc. integrated along world line

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}^{(d/2)-1}}, \qquad M_{pl} = (32\pi G_N)^{-1/2}$$

$$S = S_{GR} + S_{pp}$$

$$S_{GR} = -2M_{pl}^{d-2}\int d^d x\sqrt{g}R$$

$$S_{pp} = -\sum_{a=1}^{n} M_a \int ds_a \begin{pmatrix} 1 + c_1 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + c_2\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 \\ + c_3 R_{\mu\nu\alpha\beta}R^{\mu\sigma\alpha\rho}u_\sigma u_\rho u^\nu u^\beta + \ldots \end{pmatrix}$$

$$ds_a = dt_a\sqrt{g_{\mu\nu}\frac{dx_a^\mu}{dt_a}\frac{dx_a^\nu}{dt_a}}, \qquad u^\mu = \frac{dx_a^\mu}{ds_a}$$

Dynamics: Action

- GR: Einstein-Hilbert
- n point particles: any scalar functional of geometric tensors, *d*-veloc

- Point particle approximation gives us computational control.
- Infinite series of actions truncated based on desired accuracy of theoretical prediction.

$$g_{\mu\nu} = \eta_{\mu\nu} +$$

$$S = S_{GR} +$$

$$S_{GR} = -2M$$

$$S_{pp} = -\sum_{a=1}^{n} M_a \int ds_a \begin{pmatrix} 1 + c_1 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + c_2 \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}\right)^2 \\ + c_3 R_{\mu\nu\alpha\beta} R^{\mu\sigma\alpha\rho} u_\sigma u_\rho u^\nu u^\beta + \ldots \end{pmatrix}$$

$$ds_a = dt_a \sqrt{g_{\mu\nu} \frac{dx_a^\mu}{dt_a} \frac{dx_a^\nu}{dt_a}}, \qquad u^\mu = \frac{dx_a^\mu}{ds_a}$$

Dynamics: Action

- GR: Einstein-Hilbert
- n point particles: any scalar functional of geometric tensors, *d*-velocities, etc. integrated along world line

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}^{(d/2)-1}}, \qquad M_{pl} = (32\pi G_N)^{-1/2}$$

- $-M\int ds$ describes structureless point particle

$$S = S_{GR} + S_{pp}$$

$$S_{GR} = -2M_{pl}^{d-2}\int d^d x\sqrt{g}R$$

$$S_{pp} = -\sum_{a=1}^{n} M_a \int ds_a \left(\begin{array}{l} 1 + c_1 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + c_2\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 \\ + c_3 R_{\mu\nu\alpha\beta}R^{\mu\sigma\alpha\rho}u_\sigma u_\rho u^\nu u^\beta + \ldots \end{array} \right)$$

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$$S = S_{GR} + S_{pp}$$

$$S_{GR} = -2M_{pl}^{d-2}\int d^d x$$

- Non-minimal terms encode information on the non-trivial structure of individual objects.

$$S_{pp} = -\sum_{a=1}^{n} M_a \int ds_a \begin{pmatrix} 1 + c_1 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + c_2\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 \\ + c_3 R_{\mu\nu\alpha\beta}R^{\mu\sigma\alpha\rho}u_\sigma u_\rho u^\nu u^\beta + \ldots \end{pmatrix}$$

$$ds_a = dt_a \sqrt{g_{\mu\nu}\frac{dx_a^\mu}{dt_a}\frac{dx_a^\nu}{dt_a}}, \qquad u^\mu = \frac{dx_a^\mu}{ds_a}$$

Dynamics: Action

- GR: Einstein-Hilbert
- n point particles: any scalar functional of geometric tensors, *d*-veloci

- Coefficients $\{c_x\}$ have to be tuned to match physical observables from full description of objects.
- E.g. n non-rotating black holes.

$$g_{\mu\nu} = \eta_{\mu\nu} + \ldots$$

$$S = S_{GR} + \ldots$$

$$S_{GR} = -2M_{pl}^{d-\ldots}\ldots$$

$$S_{pp} = -\sum_{a=1}^{n} M_a \int ds_a \begin{pmatrix} 1 + c_1 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + c_2\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 \\ + c_3 R_{\mu\nu\alpha\beta}R^{\mu\sigma\alpha\rho}u_\sigma u_\rho u^\nu u^\beta + \ldots \end{pmatrix}$$

$$ds_a = dt_a\sqrt{g_{\mu\nu}\frac{dx_a^\mu}{dt_a}\frac{dx_a^\nu}{dt_a}}, \qquad u^\mu = \frac{dx_a^\mu}{ds_a}$$

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- n point particles: any scalar functional of geometric tensors, *d*-velocities, etc. integrated along world line

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}^{(d/2)-1}},$$

$$S = S_{GR} + S_{pp}$$

$$S_{GR} = -2M_{pl}^{d-2}\int d^d x\sqrt{g}R$$

$$S_{pp} = -\sum_{a=1}^{n} M_a \int ds_a \left(1 + \cancel{c_1 R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + c_2\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2} \\ \cancel{+ c_3 R_{\mu\nu\alpha\beta}R^{\mu\sigma\alpha\rho}u_\sigma u_\rho u^\nu u^\beta} + \ldots \right)$$

$$ds_a = dt_a\sqrt{g_{\mu\nu}\frac{dx_a^\mu}{dt_a}\frac{dx_a^\nu}{dt_a}}, \qquad u^\mu = \frac{dx_a^\mu}{ds_a}$$

- For non-rotating compact objects, up to $O[(v/c)^8]$, only minimal terms $-M_a\int ds_a$ needed

Perturbation Theory

- Expand GR and point particle action in powers of graviton fields $h_{\mu\nu}$ ...

$$\exp\left[i\int dt\, L_{\rm eff}\right] = \left(\prod_{\mu<\nu=0}^{d-1}\int Dh_{\mu\nu}\,\exp\left[i\left(S+S_{\rm gf}\right)\right]\right)_{\rm classical}$$

$$= \exp\left[\sum \text{Fully connected tree diagrams}\right]$$

$$L_{\rm eff} = L_{\rm eff}\left[\{x_a, \dot{x}_a, \ddot{x}_a, \ldots\}\right]$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_{pl}^{(d/2)-1}}$$

$$S = -2M_{pl}^{d-2}\int d^d x\sqrt{g}R$$

$$-\sum_{a=1}^{n} M_a \int dt_a \sqrt{1-\dot{\vec{x}}_a^2 + M_{pl}^{1-d/2}\left(h_{00} + 2h_{0i}\dot{x}_a^i + h_{ij}\dot{x}_a^i\dot{x}_a^j\right)}$$

Perturbation Theory

- Expand GR and point particle action in powers of graviton fields $h_{\mu\nu}$ ...

$$\exp\left[i\int dt\, L_{\text{eff}}\right] = \left(\prod_{\mu<\nu=0}^{d-1}\int Dh_{\mu\nu}\exp\left[i\left(S+S_{\text{gf}}\right)\right]\right)_{\text{classical}}$$

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- ∞ terms just from Einstein-Hilbert and $-M_a\int ds_a$.

Dimensional Analysis

- … but some dimensional analysis before computation makes perturbation theory much more systematic
- The scales in the n-body problem
  - **r** – typical separation between n bodies.
  - **v** – typical speed of point particles
  - **r/v** – typical time scale of n-body system

$$\frac{d}{dx^0} \sim \delta[x^0 - x'^0] \sim \frac{v}{r}$$

$$\int d^d x \sim r^d v^{-1}$$

Dimensional Analysis

- Lowest order effective action

$$S_0 = \int dt \left(\sum_{a=1}^{n} \frac{1}{2} M_a \dot{\vec{x}}_a^2 + \kappa \sum_{a \neq b}^{n} \frac{G_N^{(d/2)-1} M_a M_b}{|\vec{x}_a - \vec{x}_b|^{d-3}} \right)$$

- Schematically, conservative part of effective action is a series:

$$S_{\text{eff}} \sim S_0 + S_2 v^2 + S_4 v^4 + \ldots, \qquad S_n \sim S_0$$

$$S_0 \sim \int dt\, M v^2 \sim \int dt\, G_N^{(d/2)-1} M^2 r^{3-d} \sim M v r$$

- Virial theorem

$$\frac{G_N^{(d/2)-1} M}{r^{d-3}} \sim v^2$$

Dimensional Analysis

- Look at Re[Graviton propagator], non-relativistic limit:

$$\mathrm{Re}\left\langle 0\middle| T\left\{h_{\mu\nu}[x^0,\vec{x}]h_{\alpha\beta}[x'^0,\vec{x}']\right\}\middle|0\right\rangle_{v<<c}$$

$$\approx -\frac{iP_{\mu\nu;\alpha\beta}}{8\pi^{(d-1)/2}}\frac{\Gamma[(d-3)/2]}{|\vec{x}-\vec{x}'|^{d-3}}\delta[x^0-x'^0]$$

$$P_{\mu\nu;\alpha\beta}\equiv\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\frac{2}{d-2}\eta_{\mu\nu}\eta_{\alpha\beta}$$

Dimensional Analysis

- Look at Re[Graviton propagator], non-relativistic limit:

$$\mathrm{Re}\left\langle 0\left|T\left\{h_{\mu\nu}[x^0,\vec{x}]h_{\alpha\beta}[x'^0,\vec{x}']\right\}\right|0\right\rangle_{v<<c}$$

$$\approx -\frac{iP_{\mu\nu;\alpha\beta}}{8\pi^{(d-1)/2}}\frac{\Gamma[(d-3)/2]}{|\vec{x}-\vec{x}'|^{d-3}}\delta[x^0-x'^0]$$

$$\boxed{\begin{aligned} h_{\mu\nu} &\sim r^{1-d/2}v^{1/2} \\ \partial_i h_{\mu\nu} &\sim r^{-d/2}v^{1/2} \\ \partial_0 h_{\mu\nu} &\sim r^{-d/2}v^{3/2} \end{aligned}}$$

Dimensional Analysis

- n-graviton piece of $-M_a\int ds_a$ with χ powers of velocities scales as $S_0^{1-n/2}v^{2n-2+\chi}$
- n-graviton piece of Einstein-Hilbert action with ψ time derivatives scales as $S_0^{1-n/2}v^{2n-4+\psi}$
- With $n_{(w)}$ world line terms $-M_a\int ds_a$,
- With $n_{(v)}$ Einstein-Hilbert action terms,
- With N total gravitons,
- Every Feynman diagram scales as $S_0^{n_{(w)}+n_{(v)}-N/2}v^{2(n_{(w)}-2+\lambda)}$

Dimensional Analysis

- n-graviton piece of $-M_a \int ds_a$ with χ powers of velocities scales as $S_0^{1-n/2} v^{2n-2+\chi}$
- n-graviton piece of Einstein-Hilbert action with ψ time derivatives scales as $S_0^{1-n/2} v^{2n-4+\psi}$
- With $n_{(w)}$ world line terms $-M_a \int ds_a$,
- With $n_{(v)}$ Einstein-Hilbert action terms,
- With N total gravitons,

- Know exactly which terms in action & diagrams are necessary.

$$\underbrace{S_0^{n_{(w)}+n_{(v)}-N/2}}_{=1 \text{ for classical problem}} \; v^{\underbrace{2\left(n_{(w)}-2+\lambda\right)}_{\text{Q PN}}}$$

Superposition

- Every Feynman diagram scales as

$$S_0\, v^{2\left(n_{(w)}-2+\lambda\right)}, \qquad \lambda > 0$$

$$\left(n_{(w)} - 2 + \lambda\right)\mathrm{PN}$$

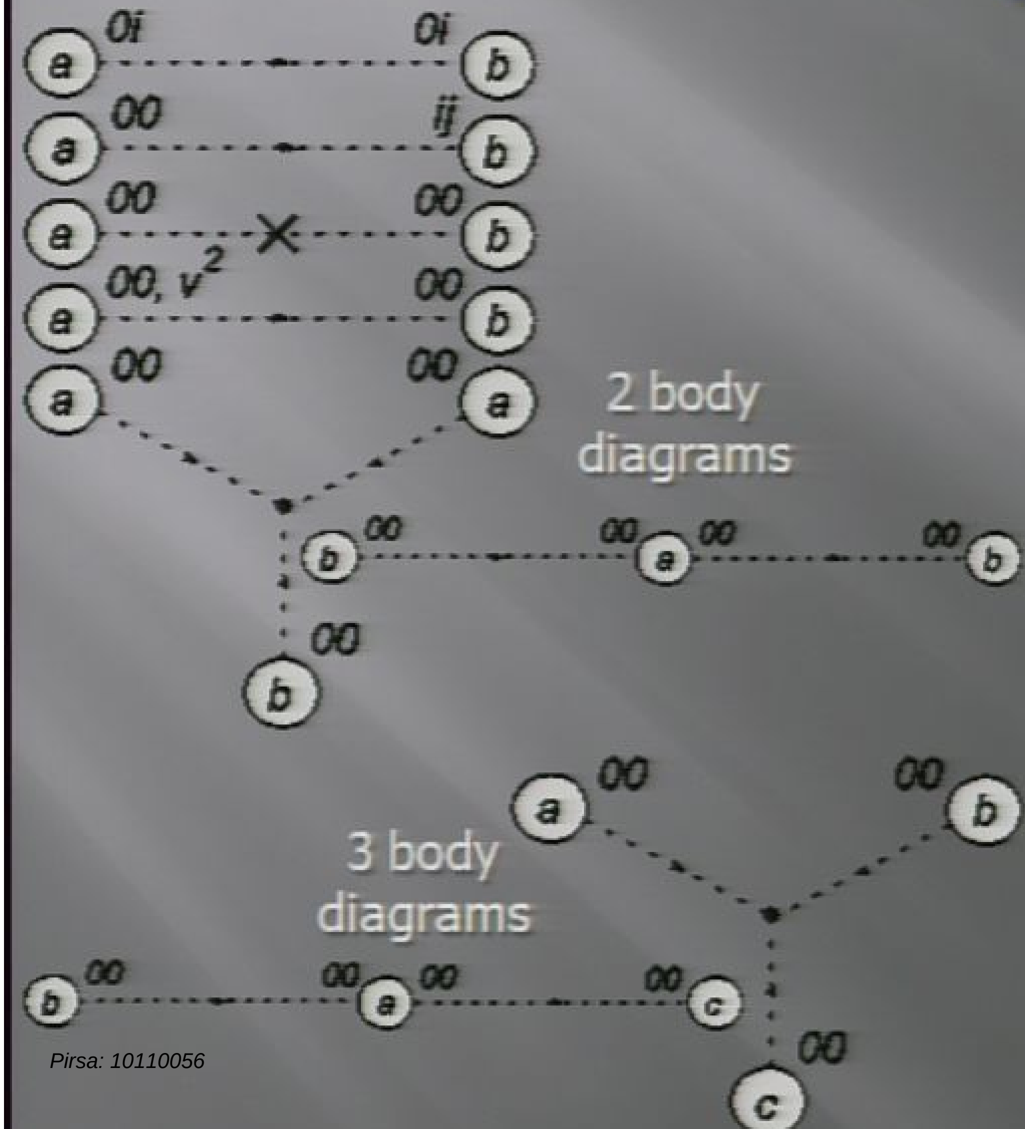
- Limited form of superposition holds
  - At Q PN, i.e. $O[(v/c)^{2Q}]$, max number of distinct point particles in a given diagram is Q+2
  - 1 PN, $O[(v/c)^2]$: 3 body problem
  - 2 PN, $O[(v/c)^4]$: 4 body problem
  - 3 PN, $O[(v/c)^6]$: 5 body problem

Newtonian Gravity
n = 2 Body Problem

$$L_{\text{eff}}^{(0\,\text{PN})} = \sum_{a=1}^{n} \frac{1}{2} M_a \bar{v}_a^2 + \frac{1}{2} \sum_{\substack{a,b=1 \\ a\neq b}}^{n} \frac{2^{\frac{5d}{2}-8}\,\Gamma[(d-1)/2]}{\pi^{1/2}(d-2)} \frac{G_N^{\frac{d}{2}-1} M_a M_b}{R_{ab}^{d-3}}$$

$$R_{ab} \equiv |\bar{x}_a - \bar{x}_b|$$

O[(v/c)^2]: 1 PN
n = 3 Body Problem



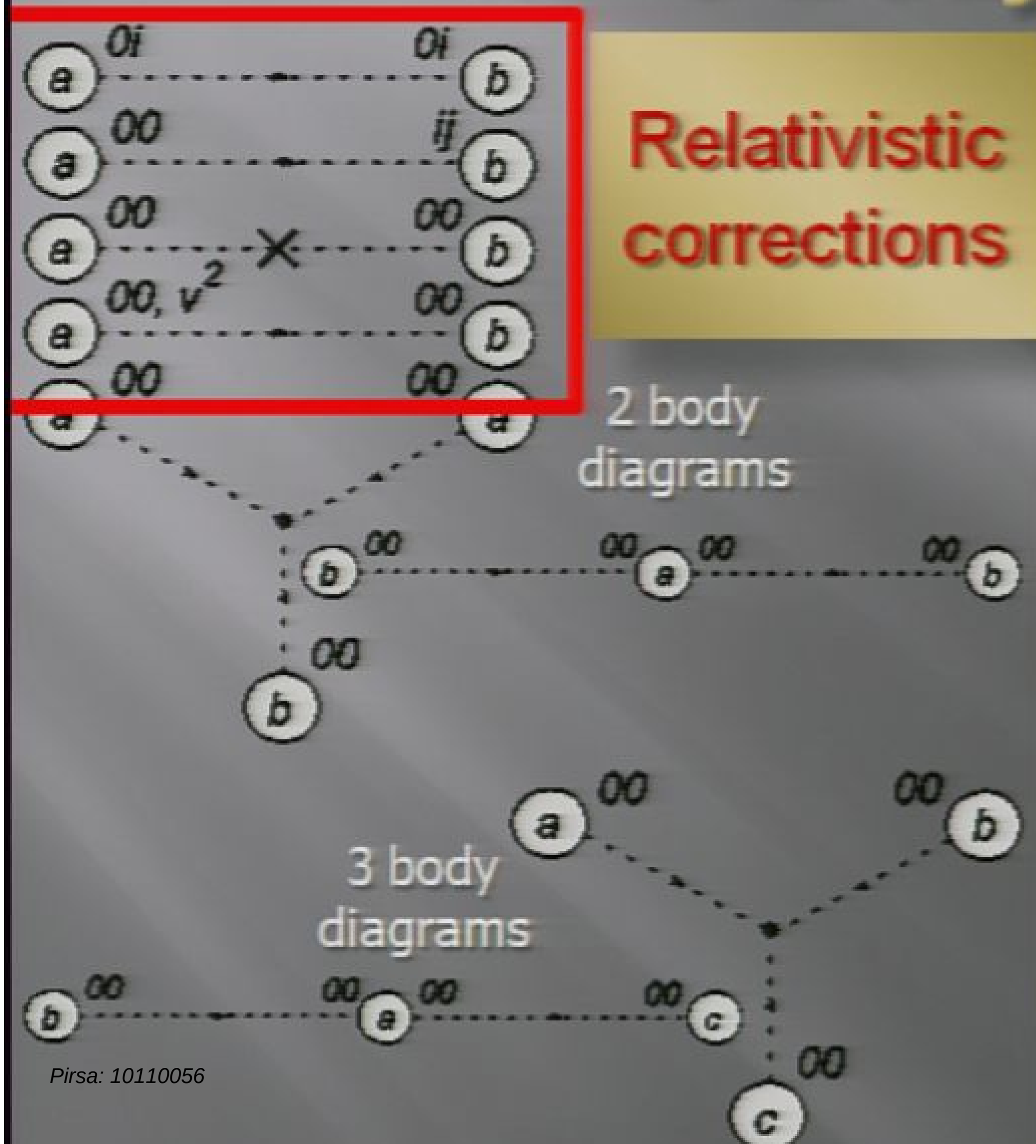
Einstein-Infeld-Hoffman
***d*-spacetime dimensions**

$$
\begin{aligned}
L_{\text{eff}}^{(1\text{ PN})} &= \sum_{a=1}^{n} \frac{1}{8} M_a \vec{v}_a^4 \\
&+ \frac{1}{2} \sum_{\substack{1 \le a,b \le n \\ a \neq b}} \frac{2^{\frac{5(d-4)}{2}} \Gamma\left[\frac{d-3}{2}\right]}{(d-2)\pi^{1/2}} \frac{G_{\text{N}}^{\frac{d}{2}-1} M_a M_b}{R_{ab}^{d-3}} \\
&\quad \times \left((d-3)^2 \frac{\vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_b}{R_{ab}^2} \right. \\
&\quad\quad \left. + (d-1)\left(\vec{v}_a^2 + \vec{v}_b^2\right) - (3d-5)\vec{v}_a \cdot \vec{v}_b \right) \\
&- \frac{1}{2} \sum_{\substack{1 \le a,b \le n \\ a \neq b}} \frac{2^{5d-17} \Gamma^2\left[\frac{d-1}{2}\right]}{(d-2)^2 \pi} \frac{G_{\text{N}}^{d-2} M_a M_b (M_a + M_b)}{R_{ab}^{2(d-3)}} \\
&- \frac{1}{3!} \sum_{\substack{1 \le a,b,c \le n \\ a,b,c \text{ distinct}}} \frac{2^{5d-16} \Gamma^2\left[\frac{d-1}{2}\right]}{(d-2)^2 \pi} G_{\text{N}}^{d-2} M_a M_b M_c \\
&\quad \times \left(R_{ab}^{3-d} R_{ac}^{3-d} + R_{ba}^{3-d} R_{bc}^{3-d} + R_{ca}^{3-d} R_{cb}^{3-d} \right)
\end{aligned}
$$

$R_{ab} \equiv |\vec{x}_a - \vec{x}_b|$

$\vec{R}_{ab} \equiv \vec{x}_a - \vec{x}_b$

$O[(v/c)^2]$: 1 PN
n = 3 Body Problem



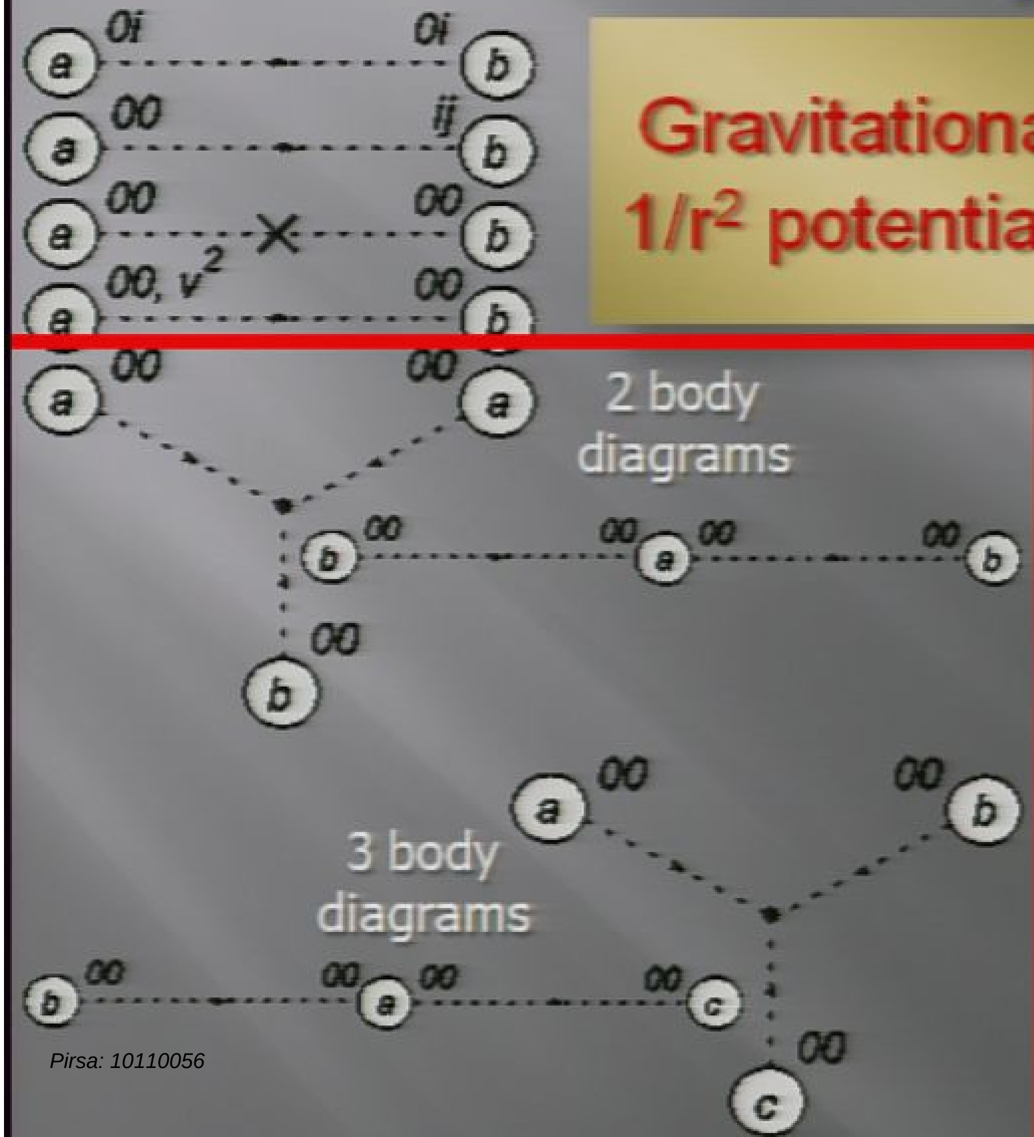
Einstein-Infeld-Hoffman
d-spacetime dimensions

$$
\begin{aligned}
L_{\text{eff}}^{(1\text{ PN})} &= \sum_{a=1}^{n} \frac{1}{8} M_a \vec{v}_a^{\,4} \\
&+ \frac{1}{2} \sum_{\substack{1 \le a,b \le n \\ a \neq b}} \frac{2^{\frac{5(d-4)}{2}} \Gamma\left[\frac{d-3}{2}\right]}{(d-2)\pi^{1/2}} \frac{G_N^{\frac{d}{2}-1} M_a M_b}{R_{ab}^{d-3}} \\
&\times \left((d-3)^2 \frac{\vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_b}{R_{ab}^2} + (d-1)\left(\vec{v}_a^{\,2} + \vec{v}_b^{\,2}\right) - (3d-5)\vec{v}_a \cdot \vec{v}_b \right) \\
&- \frac{1}{2} \sum_{\substack{1 \le a,b \le n \\ a \neq b}} \frac{2^{5d-17}\Gamma^2\left[\frac{d-1}{2}\right]}{(d-2)^2\pi} \frac{G_N^{d-2} M_a M_b (M_a + M_b)}{R_{ab}^{2(d-3)}} \\
&- \frac{1}{3!} \sum_{\substack{1 \le a,b,c \le n \\ a,b,c \text{ distinct}}} \frac{2^{5d-16}\Gamma^2\left[\frac{d-1}{2}\right]}{(d-2)^2\pi} G_N^{d-2} M_a M_b M_c \\
&\times \left(R_{ab}^{3-d} R_{ac}^{3-d} + R_{ba}^{3-d} R_{bc}^{3-d} + R_{ca}^{3-d} R_{cb}^{3-d} \right)
\end{aligned}
$$

$R_{ab} \equiv |\vec{x}_a - \vec{x}_b|$

$\vec{R}_{ab} \equiv \vec{x}_a - \vec{x}_b$

O[(v/c)^2]: 1 PN
n = 3 Body Problem



Gravitational 1/r^2 potentials

Einstein-Infeld-Hoffman
d-spacetime dimensions

$$
\begin{aligned}
L_{\text{eff}}^{(1\text{ PN})} &= \sum_{a=1}^{n} \frac{1}{8} M_a \vec{v}_a^{\,4} \\
&+ \frac{1}{2} \sum_{\substack{1 \leq a,b \leq n \\ a \neq b}} \frac{2^{\frac{5(d-4)}{2}} \Gamma\left[\frac{d-3}{2}\right]}{(d-2)\pi^{1/2}} \frac{G_{\text{N}}^{\frac{d}{2}-1} M_a M_b}{R_{ab}^{d-3}} \\
&\times \left((d-3)^2 \frac{\vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_b}{R_{ab}^2} \right. \\
&\left. + (d-1)\left(\vec{v}_a^{\,2} + \vec{v}_b^{\,2}\right) - (3d-5)\vec{v}_a \cdot \vec{v}_b \right) \\
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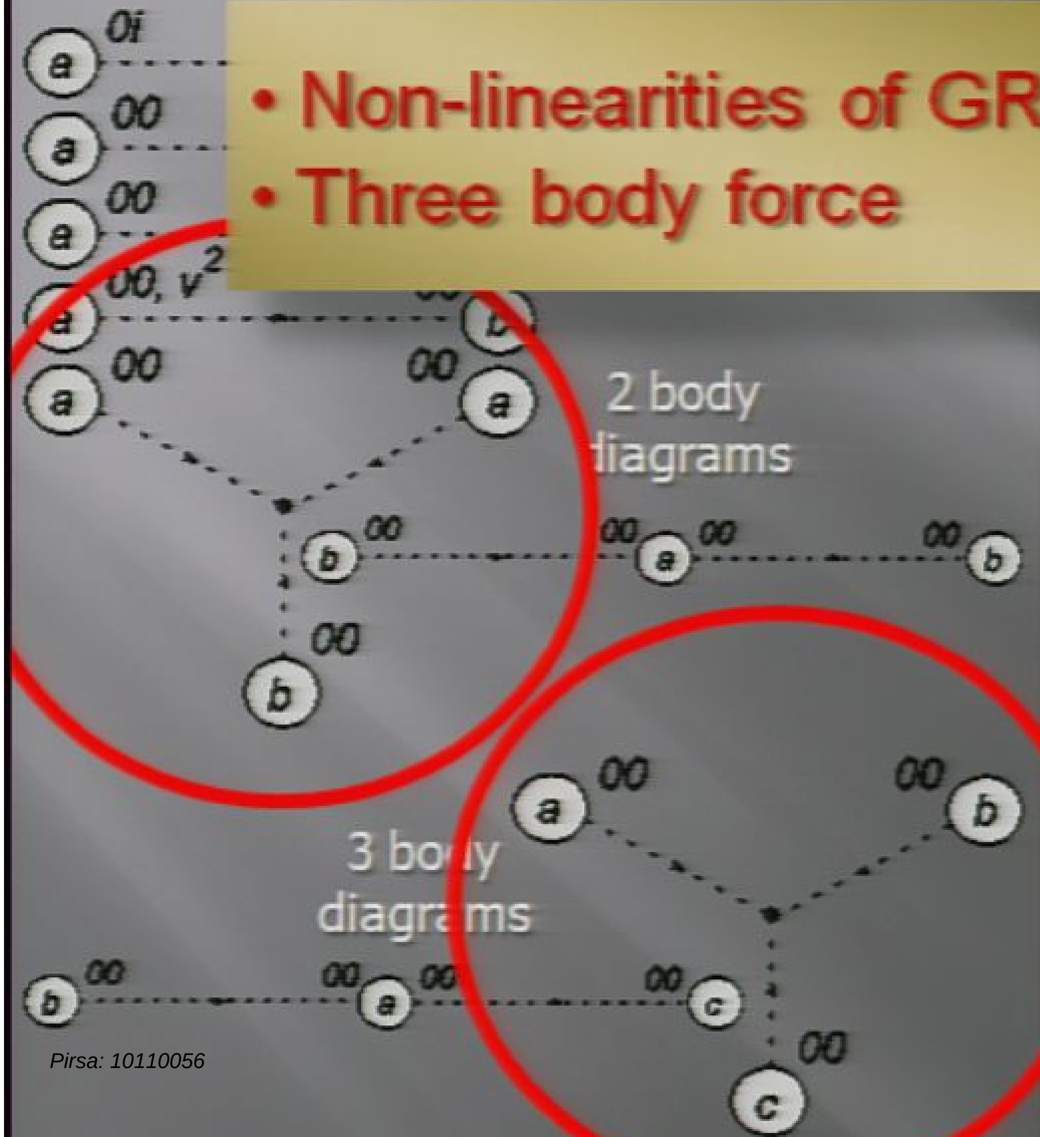
&\times \left(R_{ab}^{3-d} R_{ac}^{3-d} + R_{ba}^{3-d} R_{bc}^{3-d} + R_{ca}^{3-d} R_{cb}^{3-d} \right)
\end{aligned}
$$

$R_{ab} \equiv |\vec{x}_a - \vec{x}_b|$

$\vec{R}_{ab} \equiv \vec{x}_a - \vec{x}_b$

O[(v/c)^2]: 1 PN
n = 3 Body Problem

- Non-linearities of GR
- Three body force



Einstein-Infeld-Hoffman
d-spacetime dimensions

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\begin{aligned}
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&+ \frac{1}{2} \sum_{\substack{1\le a,b\le n \\ a\neq b}} \frac{2^{\frac{5(d-4)}{2}}\Gamma\left[\frac{d-3}{2}\right]}{(d-2)\pi^{1/2}} \frac{G_{\rm N}^{\frac{d}{2}-1} M_a M_b}{R_{ab}^{d-3}} \\
&\quad \times \left((d-3)^2 \frac{\vec{R}_{ab}\cdot\vec{v}_a \vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}^2} \right. \\
&\quad \left. + (d-1)\left(\vec{v}_a^{\,2}+\vec{v}_b^{\,2}\right) - (3d-5)\vec{v}_a\cdot\vec{v}_b \right) \\
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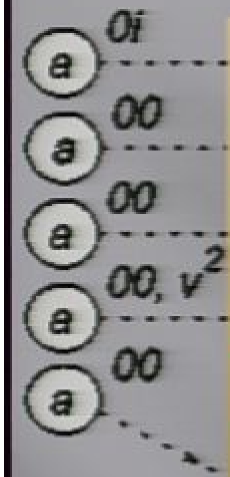
&- \frac{1}{3!} \sum_{\substack{1\le a,b,c\le n \\ a,b,c\ \text{distinct}}} \frac{2^{5d-16}\Gamma^2\left[\frac{d-1}{2}\right]}{(d-2)^2\pi} G_{\rm N}^{d-2} M_a M_b M_c \\
&\quad \times \left(R_{ab}^{3-d}R_{ac}^{3-d} + R_{ba}^{3-d}R_{bc}^{3-d} + R_{ca}^{3-d}R_{cb}^{3-d}\right)
\end{aligned}
$$

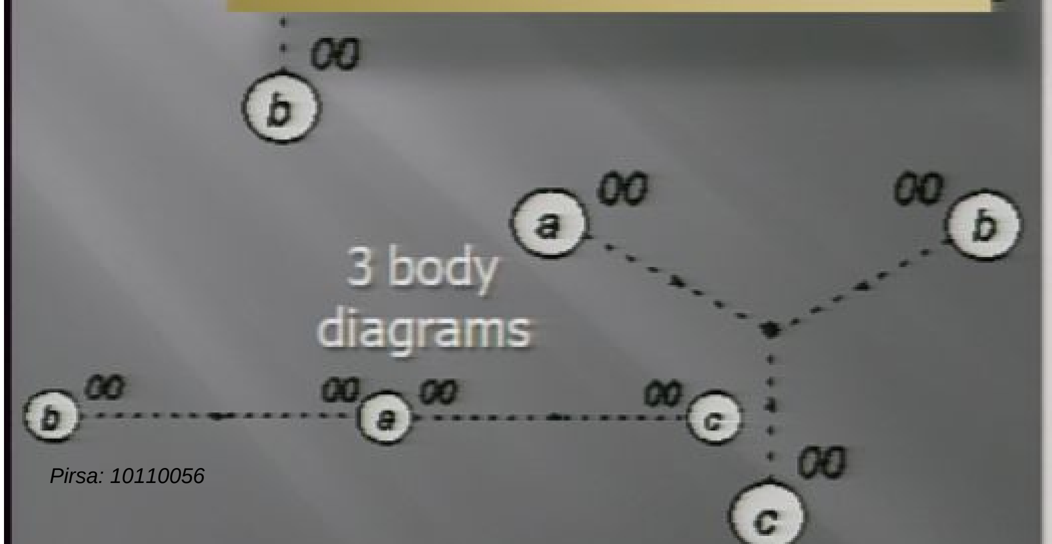
$R_{ab} \equiv |\vec{x}_a - \vec{x}_b|$

$\vec{R}_{ab} \equiv \vec{x}_a - \vec{x}_b$

$O[(v/c)^2]$: 1 PN
n = 3 Body Problem



- Time derivative of Runge-Lenz vector gives perihelion precession of planetary orbits.



Einstein-Infeld-Hoffman
d-spacetime dimensions

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\begin{aligned}
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&= \sum_{a=1}^{n} \frac{1}{8} M_a \vec{v}_a^4 \\
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&\quad \times \left((d-3)^2 \frac{\vec{R}_{ab}\cdot\vec{v}_a \vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}^2} \right. \\
&\quad\quad \left. + (d-1)\left(\vec{v}_a^2 + \vec{v}_b^2\right) - (3d-5)\vec{v}_a\cdot\vec{v}_b \right) \\
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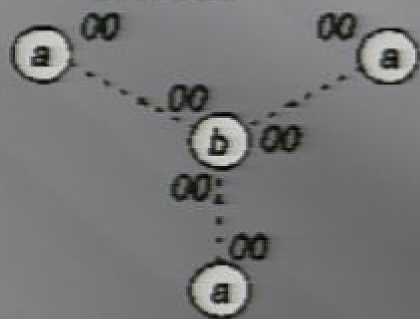
&\quad \times \left(R_{ab}^{3-d}R_{ac}^{3-d} + R_{ba}^{3-d}R_{bc}^{3-d} + R_{ca}^{3-d}R_{cb}^{3-d}\right)
\end{aligned}
$$

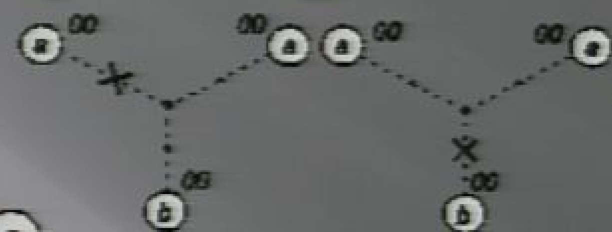
$$R_{ab} \equiv |\vec{x}_a - \vec{x}_b|$$

$$\vec{R}_{ab} \equiv \vec{x}_a - \vec{x}_b$$

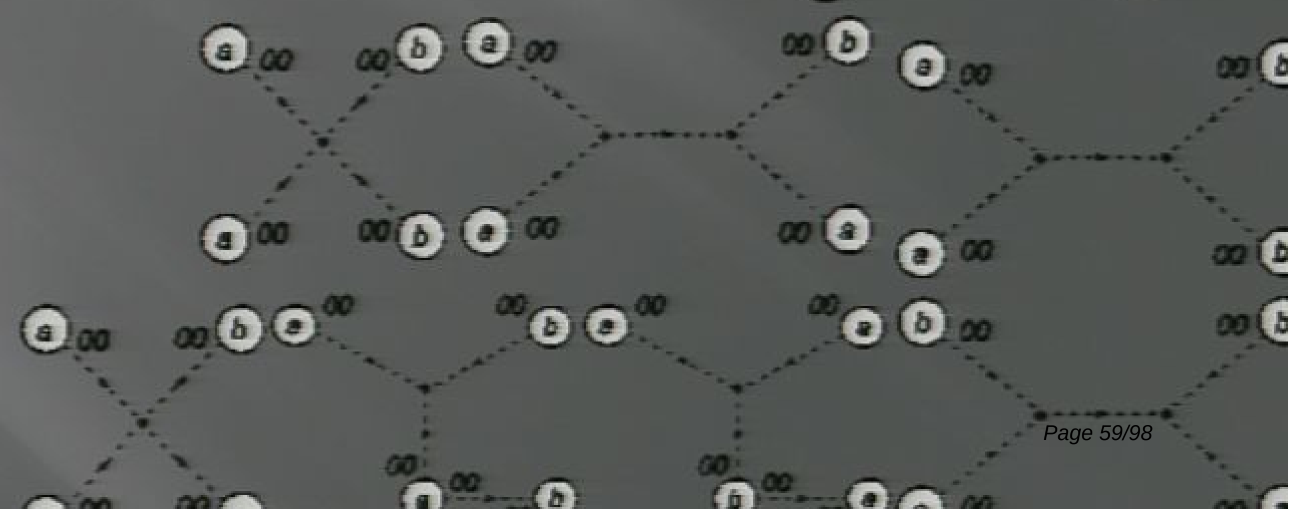
$O[(v/c)^4]$: 2 PN
n = 4 Body Problem
2 body diagrams

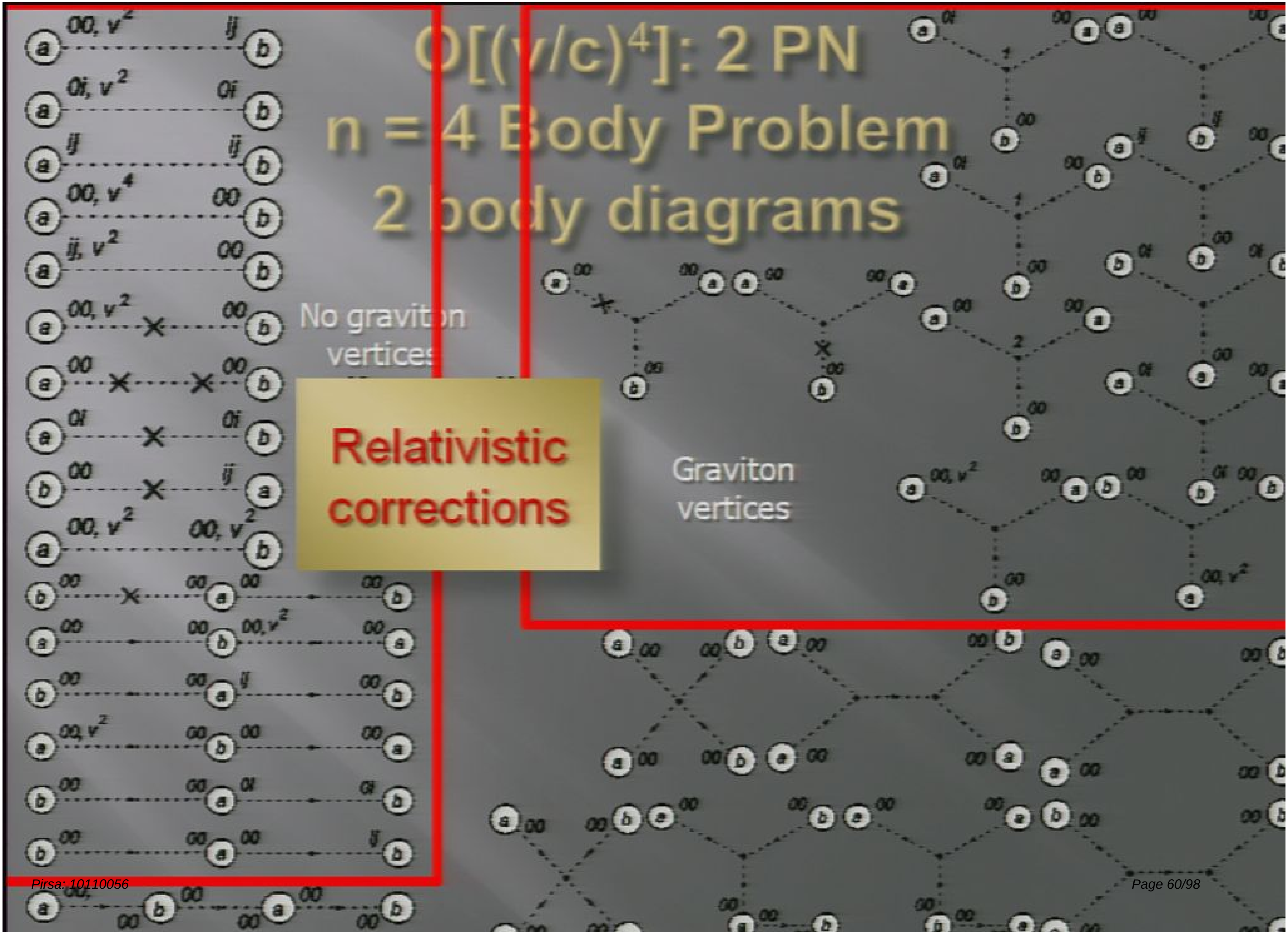
No graviton vertices

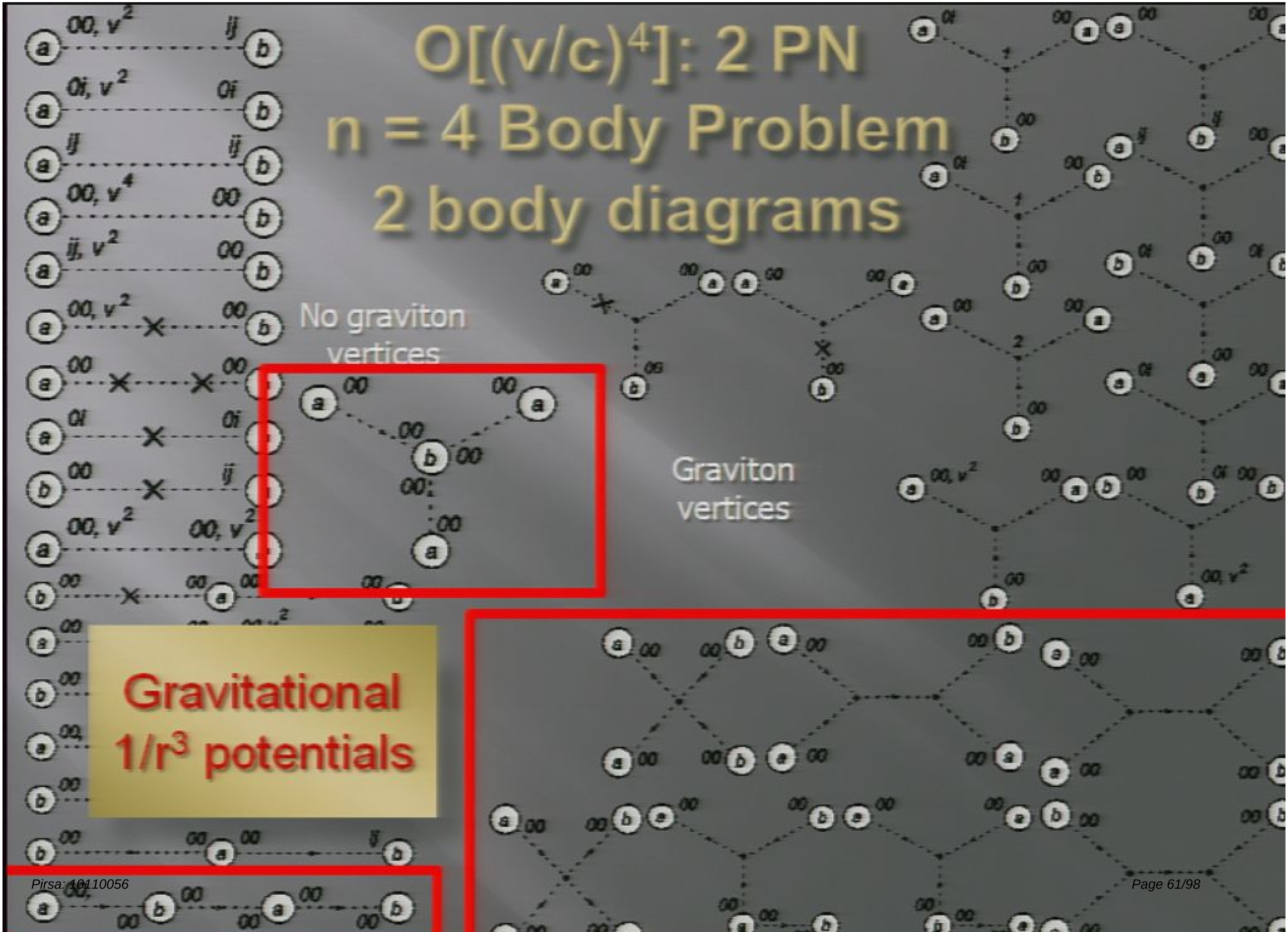


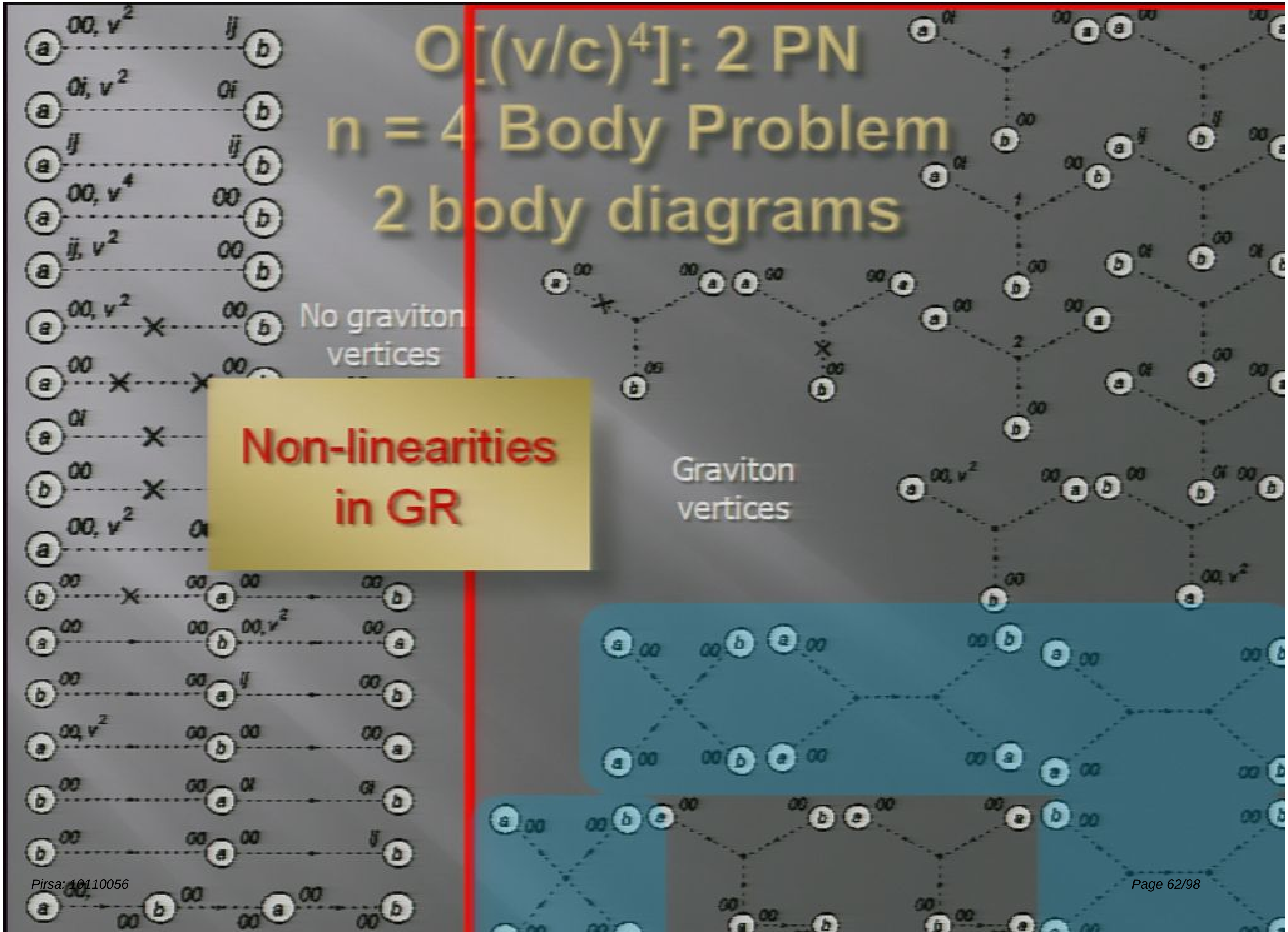


Graviton vertices

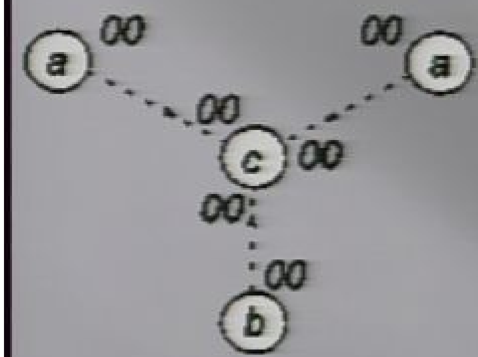



O[(v/c)^4]: 2 PN
n = 4 Body Problem
2 body diagrams
No graviton vertices
Relativistic corrections
Graviton vertices

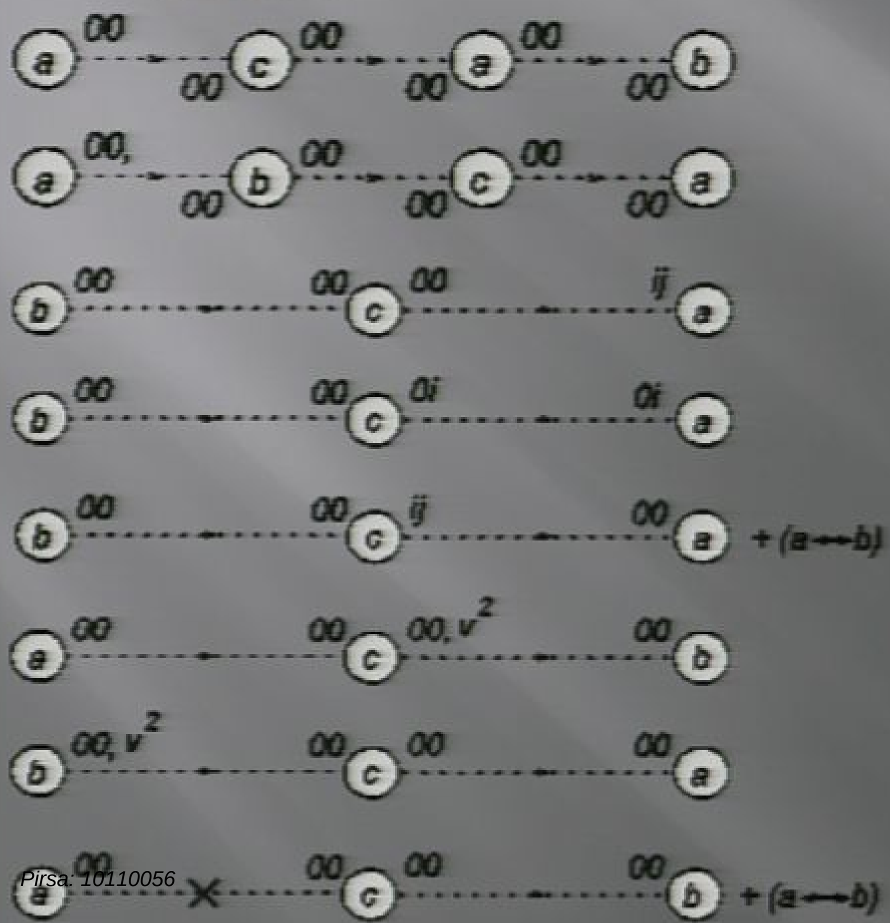

O[(v/c)^4]: 2 PN
n = 4 Body Problem
2 body diagrams
No graviton vertices
Graviton vertices
Gravitational
1/r^3 potentials


O[(v/c)⁴]: 2 PN
n = 4 Body Problem
2 body diagrams
No graviton vertices
Non-linearities in GR
Graviton vertices

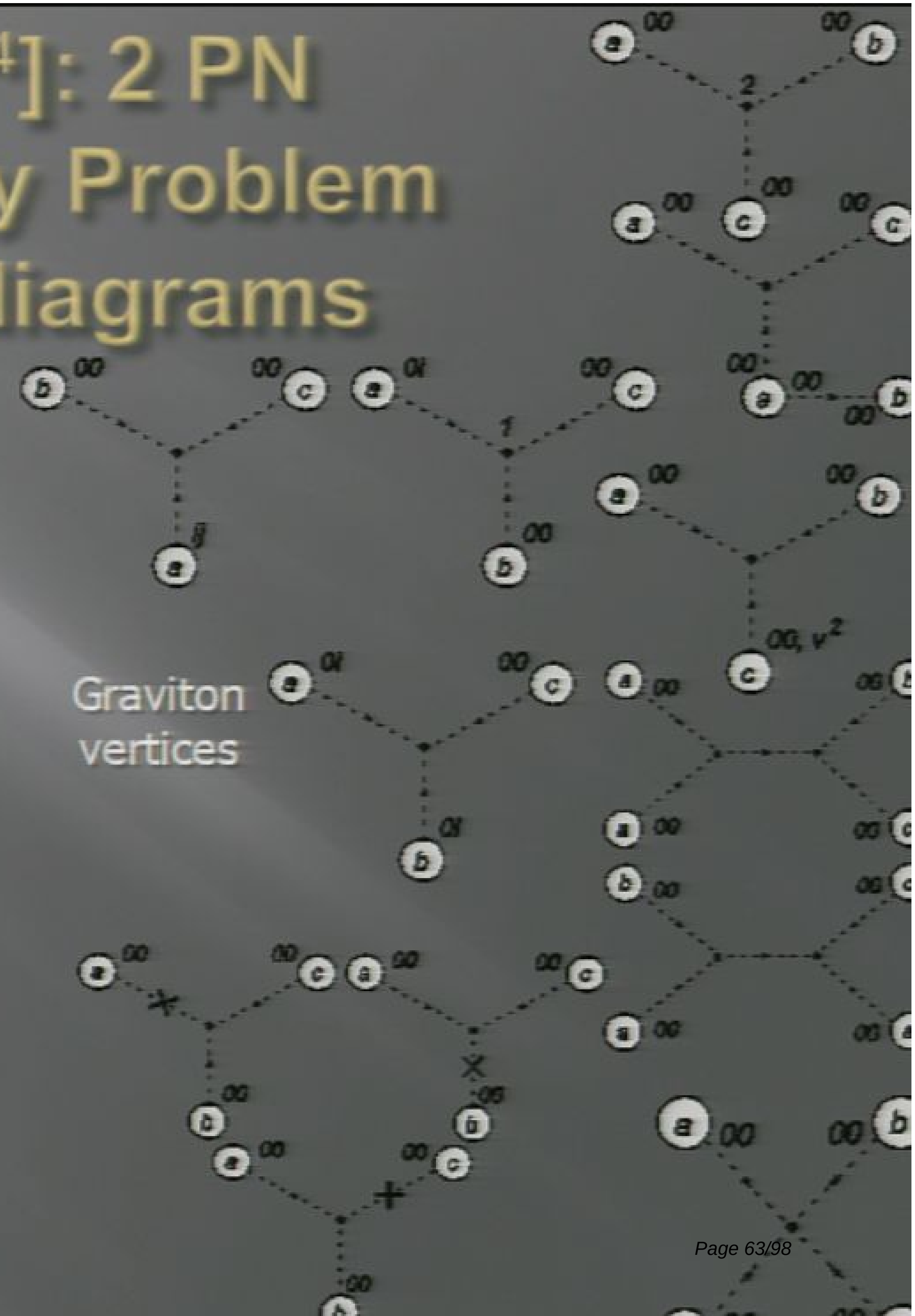
$O[(v/c)^4]$: 2 PN
n = 4 Body Problem
3 body diagrams



No graviton vertices

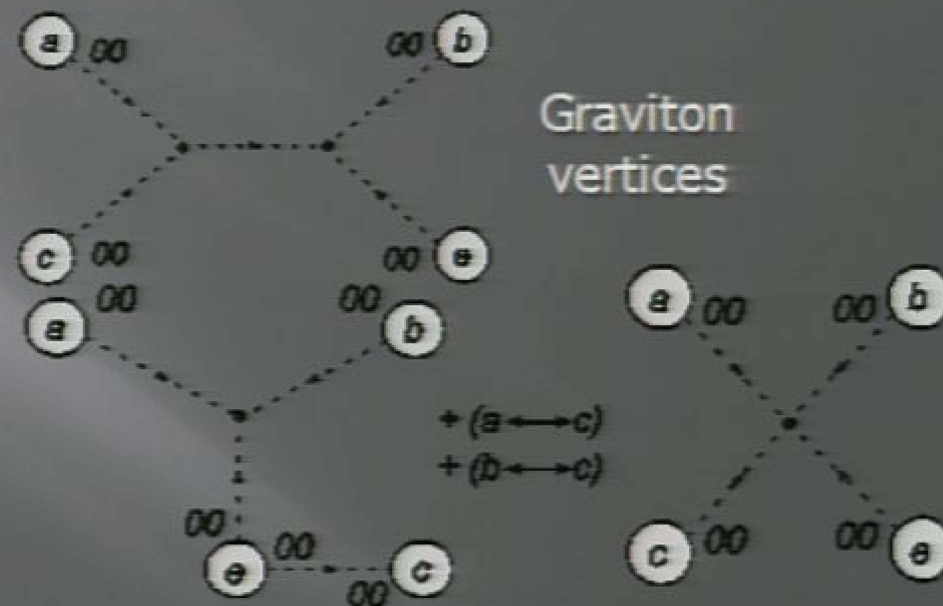


Graviton vertices



$O[(v/c)^4]$: 2 PN
4 Body Diagrams

No graviton vertices



Graviton vertices

$O[(v/c)^4]$: 2 PN
n = 4 Body Problem

$$L_{\text{eff}}^{(2\text{ PN})} \overset{d=4}{=} L_4^{2\text{ Body}} + L_4^{3\text{ Body}} + L_4^{4\text{ Body}}$$

$$\begin{aligned}
L_4^{2\text{ Body}} = \frac{1}{2}\sum_{\substack{1\le a,b\le n\\ a\ne b}} \Bigg\{ & \frac{M_a}{16}\vec{v}_a^{\,6} + \frac{M_b}{16}\vec{v}_b^{\,6} \\
& + \frac{G_N M_a M_b}{R_{ab}}\Bigg(\vec{R}_{ab}\cdot\vec{v}_a\left(\frac{7}{4}\dot{\vec{v}}_a\cdot\vec{v}_b - \frac{3}{2}\vec{v}_b\cdot\dot{\vec{v}}_b\right) + \vec{R}_{ba}\cdot\vec{v}_b\left(\frac{7}{4}\dot{\vec{v}}_b\cdot\vec{v}_a - \frac{3}{2}\vec{v}_a\cdot\dot{\vec{v}}_a\right) \\
& \quad - \frac{1}{8}\left(\frac{\vec{R}_{ab}\cdot\vec{v}_a}{R_{ab}}\right)^2\left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b + \vec{v}_b^{\,2}\right) - \frac{1}{8}\left(\frac{\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}}\right)^2\left(\vec{R}_{ab}\cdot\dot{\vec{v}}_a + \vec{v}_a^{\,2}\right) \\
& \quad + \frac{3}{4}\left(\vec{v}_a^{\,2} + \vec{v}_b^{\,2} - 2\,\vec{v}_a\cdot\vec{v}_b\right)\frac{\vec{R}_{ab}\cdot\vec{v}_a}{R_{ab}}\frac{\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}} + \frac{3}{8}\frac{(\vec{R}_{ab}\cdot\vec{v}_a)^2(\vec{R}_{ba}\cdot\vec{v}_b)^2}{R_{ab}^4} \\
& \quad + \frac{1}{8}\left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b\,\vec{v}_a^{\,2} + \vec{R}_{ab}\cdot\dot{\vec{v}}_a\,\vec{v}_b^{\,2}\right) + \frac{1}{8}\vec{R}_{ab}\cdot\dot{\vec{v}}_a\,\vec{R}_{ba}\cdot\dot{\vec{v}}_b \\
& \quad + \frac{15}{8}\dot{\vec{v}}_a\cdot\dot{\vec{v}}_b\,R_{ab}^2 + \frac{7}{8}(\vec{v}_a^{\,4} + \vec{v}_b^{\,4}) + \frac{1}{4}(\vec{v}_a\cdot\vec{v}_b)^2 + \frac{3}{8}\vec{v}_a^{\,2}\vec{v}_b^{\,2} - \frac{5}{4}(\vec{v}_a^{\,2} + \vec{v}_b^{\,2})\vec{v}_a\cdot\vec{v}_b\Bigg) \\
& + \frac{G_N^2 M_a M_b}{R_{ab}^2}\Bigg(-\frac{3}{2}\frac{M_a(\vec{R}_{ba}\cdot\vec{v}_b)^2 + M_b(\vec{R}_{ab}\cdot\vec{v}_a)^2}{R_{ab}^2} - 2(M_a + M_b)\frac{\vec{R}_{ab}\cdot\vec{v}_a\,\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}^2} \\
& \quad - (2M_a + M_b)\vec{R}_{ab}\cdot\dot{\vec{v}}_a - (2M_b + M_a)\vec{R}_{ba}\cdot\dot{\vec{v}}_b \\
& \quad + \vec{v}_a^{\,2}\left(2M_a + \frac{11}{4}M_b\right) + \vec{v}_b^{\,2}\left(2M_b + \frac{11}{4}M_a\right) - \frac{9}{2}\vec{v}_a\cdot\vec{v}_b(M_a + M_b)\Bigg) \\
& - \frac{G_N^3 M_a M_b}{R_{ab}^3}\left(M_a M_b + \frac{3}{2}(M_a^2 + M_b^2)\right)\Bigg\}
\end{aligned}$$

$$\begin{aligned}
L_4^{4\text{ Body}} = \frac{1}{4!}\sum_{\substack{1\le a,b,c,e\le n\\ a,b,c,e\text{ distinct}}} & G_N^3 M_a M_b M_c M_e \\
& \times\Bigg\{\frac{I_{22}[a,b,c,e]}{8\pi^2} - 3\left(\frac{1}{R_{ab}R_{ac}R_{ae}} + \frac{1}{R_{ba}R_{bc}R_{be}} + \frac{1}{R_{ca}R_{cb}R_{ce}} + \frac{1}{R_{ea}R_{eb}R_{ec}}\right) \\
& \quad + \Bigg[\frac{1}{R_{ab} + R_{ac} + R_{bc}}\left(\frac{R_{bc}}{R_{ab}R_{ac}R_{ae}} + \frac{2R_{ac}^2}{R_{ab}R_{be}^2} - \frac{2R_{ab}}{R_{ac}^2}\right) \\
& \qquad + 23\text{ other permutations of }\{a,b,c,e\}\Bigg]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
L_4^{3\text{ Body}} & \\
= \frac{1}{3!}\sum_{\substack{1\le a,b,c\le n\\ a,b,c\text{ distinct}}} & \Bigg\{G_N^2 M_a M_b M_c \\
& \times\Bigg(\frac{1}{R_{ab}R_{ac}}\left(\frac{9}{2}\vec{v}_a^{\,2} + 8\vec{v}_b\cdot\vec{v}_c\right) + \frac{1}{R_{ab}R_{bc}}\left(\frac{9}{2}\vec{v}_b^{\,2} + 8\vec{v}_a\cdot\vec{v}_c\right) + \frac{1}{R_{ac}R_{bc}}\left(\frac{9}{2}\vec{v}_c^{\,2} + 8\vec{v}_a\cdot\vec{v}_b\right) \\
& - \frac{8}{(R_{ab} + R_{ac} + R_{bc})^2}\left(\frac{\vec{R}_{ba}\cdot\vec{v}_b\vec{R}_{ca}\cdot\vec{v}_c}{R_{ab}R_{ac}} + \frac{\vec{R}_{ab}\cdot\vec{v}_a\vec{R}_{cb}\cdot\vec{v}_c}{R_{ab}R_{bc}} + \frac{\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{bc}\cdot\vec{v}_b}{R_{ac}R_{bc}}\right) \\
& + \frac{4}{R_{ab} + R_{ac} + R_{bc}}\left(\frac{\vec{v}_a^{\,2}}{R_{bc}} + \frac{\vec{v}_b^{\,2}}{R_{ac}} + \frac{\vec{v}_c^{\,2}}{R_{ab}}\right)\Bigg) \\
& + \Bigg[\frac{1}{2R_{ab}R_{ac}^2}\left(\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{ba}\cdot\vec{v}_b + \vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{ca}\cdot\vec{v}_c + \vec{R}_{ba}\cdot\vec{v}_b\vec{R}_{ca}\cdot\vec{v}_c + 2(\vec{R}_{ca}\cdot\vec{v}_c)^2\right) \\
& - \frac{1}{R_{ab}R_{ac}}\left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b + \frac{7}{2}\vec{v}_a\cdot\vec{v}_b + \frac{5}{2}\vec{v}_b^{\,2}\right) \\
& + \frac{1}{(R_{ab} + R_{ac} + R_{bc})^2}\Bigg(\frac{1}{R_{ab}^2}\left(4\vec{R}_{ab}\cdot\vec{v}_c\vec{R}_{ba}\cdot\vec{v}_b + 8\vec{R}_{ab}\cdot\vec{v}_a\vec{R}_{ab}\cdot\vec{v}_c - 2(\vec{R}_{ab}\cdot\vec{v}_a)^2 - 2(\vec{R}_{ab}\cdot\vec{v}_c)^2\right) \\
& \qquad + \frac{1}{R_{ab}R_{ac}}\left(4\vec{R}_{ac}\cdot\vec{v}_b\vec{R}_{ba}\cdot\vec{v}_b - 8\vec{R}_{ac}\cdot\vec{v}_b\vec{R}_{ba}\cdot\vec{v}_c + 12\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{ba}\cdot\vec{v}_c - 12\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{ba}\cdot\vec{v}_b\right)\Bigg) \\
& + \frac{1}{R_{ab} + R_{ac} + R_{bc}}\Bigg(\frac{1}{R_{ab}^3}\left(8\vec{R}_{ab}\cdot\vec{v}_a\vec{R}_{ab}\cdot\vec{v}_c + 4\vec{R}_{ab}\cdot\vec{v}_c\vec{R}_{ba}\cdot\vec{v}_b - 2(\vec{R}_{ab}\cdot\vec{v}_c)^2 - 2(\vec{R}_{ab}\cdot\vec{v}_a)^2\right) \\
& \qquad + \frac{1}{R_{ab}}\left(2\vec{v}_a^{\,2} - 4\vec{v}_a\cdot\vec{v}_c - 2\vec{R}_{ab}\cdot\dot{\vec{v}}_a\right)\Bigg) + 5\text{ other permutations of }\{a,b,c\}\Bigg]\Bigg) \\
+ G_N^3 M_a M_b M_c & \Bigg(\Bigg[\frac{(M_a + M_c)R_{ab}^2}{R_{ac}^2 R_{bc}^2} + \frac{2M_b R_{ab}}{R_{ac}R_{bc}^2} - \frac{3M_a}{R_{ab}^3} \\
& - \frac{1}{R_{ab}R_{ac}^2}\left(M_a + \frac{3}{2}M_c\right) + 5\text{ other permutations of }\{a,b,c\}\Bigg] \\
& + \frac{1}{16\pi^2}\left(M_a I_{22}[a,a,b,c] + M_b I_{22}[b,b,a,c] + M_c I_{22}[c,c,a,b]\right) - 2\left(\frac{M_a}{R_{bc}^3} + \frac{M_b}{R_{ac}^3} + \frac{M_c}{R_{ab}^3}\right)\Bigg)\Bigg\}
\end{aligned}$$

O[(v/c)⁴]: 2 PN
n = 4 Body Problem

$$
\begin{aligned}
L_4^{2\,\mathrm{Body}} \equiv \frac{1}{2} \sum_{\substack{1\le a,b\le n \\ a\neq b}} \Bigg\{ & \frac{M_a}{16}\vec{v}_a^6 + \frac{M_b}{16}\vec{v}_b^6 \\
& + \frac{G_N M_a M_b}{R_{ab}}\Bigg(\vec{R}_{ab}\cdot\vec{v}_a\left(\frac{7}{4}\vec{v}_a\cdot\dot{\vec{v}}_b - \frac{3}{2}\vec{v}_b\cdot\dot{\vec{v}}_b\right) + \vec{R}_{ba}\cdot\vec{v}_b\left(\frac{7}{4}\vec{v}_b\cdot\dot{\vec{v}}_a - \frac{3}{2}\vec{v}_a\cdot\dot{\vec{v}}_a\right) \\
& \quad - \frac{1}{8}\left(\frac{\vec{R}_{ab}\cdot\vec{v}_a}{R_{ab}}\right)^2\left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b + \vec{v}_b^2\right) - \frac{1}{8}\left(\frac{\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}}\right)^2\left(\vec{R}_{ab}\cdot\dot{\vec{v}}_a + \vec{v}_a^2\right) \\
& \quad + \frac{3}{4}\left(\vec{v}_a^2 + \vec{v}_b^2 - 2\,\vec{v}_a\cdot\vec{v}_b\right)\frac{\vec{R}_{ab}\cdot\vec{v}_a}{R_{ab}}\frac{\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}} + \frac{3}{8}\frac{(\vec{R}_{ab}\cdot\vec{v}_a)^2(\vec{R}_{ba}\cdot\vec{v}_b)^2}{R_{ab}^4} \\
& \quad + \frac{1}{8}\left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b\,\vec{v}_a^2 + \vec{R}_{ab}\cdot\dot{\vec{v}}_a\,\vec{v}_b^2\right) + \frac{1}{8}\vec{R}_{ab}\cdot\dot{\vec{v}}_a\,\vec{R}_{ba}\cdot\dot{\vec{v}}_b \\
& \quad + \frac{15}{8}\dot{\vec{v}}_a\cdot\dot{\vec{v}}_b\,R_{ab}^2 + \frac{7}{8}(\vec{v}_a^4 + \vec{v}_b^4) + \frac{1}{4}(\vec{v}_a\cdot\vec{v}_b)^2 + \frac{3}{8}\vec{v}_a^2\vec{v}_b^2 - \frac{5}{4}(\vec{v}_a^2 + \vec{v}_b^2)\vec{v}_a\cdot\vec{v}_b\Bigg) \\
& + \frac{G_N^2 M_a M_b}{R_{ab}^2}\Bigg(-\frac{3}{2}\frac{M_a(\vec{R}_{ba}\cdot\vec{v}_b)^2 + M_b(\vec{R}_{ab}\cdot\vec{v}_a)^2}{R_{ab}^2} - 2(M_a+M_b)\frac{\vec{R}_{ab}\cdot\vec{v}_a\,\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}^2} \\
& \quad - (2M_a + M_b)\vec{R}_{ab}\cdot\dot{\vec{v}}_a - (2M_b + M_a)\vec{R}_{ba}\cdot\dot{\vec{v}}_b \\
& \quad + \vec{v}_a^2\left(2M_a + \frac{11}{4}M_b\right) + \vec{v}_b^2\left(2M_b + \frac{11}{4}M_a\right) - \frac{9}{2}\vec{v}_a\cdot\vec{v}_b\,(M_a + M_b)\Bigg) \\
& - \frac{G_N^3 M_a M_b}{R_{ab}^3}\left(M_a M_b + \frac{3}{2}(M_a^2 + M_b^2)\right)\Bigg\}
\end{aligned}
$$

$O[(v/c)^4]$; 2 PN
n = 4 Body Problem

$$
\begin{aligned}
L_4^{\text{3 Body}}
&\equiv \frac{1}{3!} \sum_{\substack{1 \le a,b,c \le n \\ a,b,c \text{ distinct}}} \Bigg\{ G_N^2 M_a M_b M_c \\
&\times \Bigg(\frac{1}{R_{ab} R_{ac}} \left(\frac{9}{2} \vec{v}_a^2 + 8 \vec{v}_b \cdot \vec{v}_c \right) + \frac{1}{R_{ab} R_{bc}} \left(\frac{9}{2} \vec{v}_b^2 + 8 \vec{v}_a \cdot \vec{v}_c \right) + \frac{1}{R_{ac} R_{bc}} \left(\frac{9}{2} \vec{v}_c^2 + 8 \vec{v}_a \cdot \vec{v}_b \right) \\
&- \frac{8}{(R_{ab} + R_{ac} + R_{bc})^2} \left(\frac{\vec{R}_{ba} \cdot \vec{v}_b \vec{R}_{ca} \cdot \vec{v}_c}{R_{ab} R_{ac}} + \frac{\vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{cb} \cdot \vec{v}_c}{R_{ab} R_{bc}} + \frac{\vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{bc} \cdot \vec{v}_b}{R_{ac} R_{bc}} \right) \\
&+ \frac{4}{R_{ab} + R_{ac} + R_{bc}} \left(\frac{\vec{v}_a^2}{R_{bc}} + \frac{\vec{v}_b^2}{R_{ac}} + \frac{\vec{v}_c^2}{R_{ab}} \right) \Bigg) \\
&+ \Bigg[\frac{1}{2 R_{ab} R_{ac}^3} \left(\vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_b + \vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ca} \cdot \vec{v}_c + \vec{R}_{ba} \cdot \vec{v}_b \vec{R}_{ca} \cdot \vec{v}_c + 2 (\vec{R}_{ca} \cdot \vec{v}_c)^2 \right) \\
&- \frac{1}{R_{ab} R_{ac}} \left(\vec{R}_{ba} \cdot \dot{\vec{v}}_b + \frac{7}{2} \vec{v}_a \cdot \vec{v}_b + \frac{5}{2} \vec{v}_b^2 \right) \\
&+ \frac{1}{(R_{ab} + R_{ac} + R_{bc})^2} \Bigg(\frac{1}{R_{ab}^2} \left(4 \vec{R}_{ab} \cdot \vec{v}_c \vec{R}_{ba} \cdot \vec{v}_b + 8 \vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{ab} \cdot \vec{v}_c - 2 (\vec{R}_{ab} \cdot \vec{v}_a)^2 - 2 (\vec{R}_{ab} \cdot \vec{v}_c)^2 \right) \\
&\qquad + \frac{1}{R_{ab} R_{ac}} \left(4 \vec{R}_{ac} \cdot \vec{v}_b \vec{R}_{ba} \cdot \vec{v}_b - 8 \vec{R}_{ac} \cdot \vec{v}_b \vec{R}_{ba} \cdot \vec{v}_c + 12 \vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_c - 12 \vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_b \right) \Bigg) \\
&+ \frac{1}{R_{ab} + R_{ac} + R_{bc}} \Bigg(\frac{1}{R_{ab}^3} \left(8 \vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{ab} \cdot \vec{v}_c + 4 \vec{R}_{ab} \cdot \vec{v}_c \vec{R}_{ba} \cdot \vec{v}_b - 2 (\vec{R}_{ab} \cdot \vec{v}_c)^2 - 2 (\vec{R}_{ab} \cdot \vec{v}_a)^2 \right) \\
&\qquad + \frac{1}{R_{ab}} \left(2 \vec{v}_a^2 - 4 \vec{v}_a \cdot \vec{v}_c - 2 \vec{R}_{ab} \cdot \dot{\vec{v}}_a \right) \Bigg) + 5 \text{ other permutations of } \{a, b, c\} \Bigg] \Bigg) \\
&+ G_N^3 M_a M_b M_c \Bigg(\Bigg[\frac{(M_a + M_c) R_{ab}^2}{R_{ac}^2 R_{bc}^3} + \frac{2 M_b R_{ab}}{R_{ac} R_{bc}^3} - \frac{3 M_a}{R_{ab}^3} \\
&- \frac{1}{R_{ab} R_{ac}^2} \left(M_a + \frac{3}{2} M_c \right) + 5 \text{ other permutations of } \{a, b, c\} \Bigg] \\
&+ \frac{1}{16 \pi^2} \left(M_a I_{22}[a, a, b, c] + M_b I_{22}[b, b, a, c] + M_c I_{22}[c, c, a, b] \right) - 2 \left(\frac{M_a}{R_{bc}^3} + \frac{M_b}{R_{ac}^3} + \frac{M_c}{R_{ab}^3} \right) \Bigg) \Bigg\}
\end{aligned}
$$

O[(v/c)^4]: 2 PN
n = 4 Body Problem

$$L_4^{4\ \mathrm{Body}} \equiv \frac{1}{4!} \sum_{\substack{1 \le a,b,c,e \le n \\ a,b,c,e\ \text{distinct}}} G_N^3 M_a M_b M_c M_e$$

$$\times \left\{ \frac{I_{22}[a,b,c,e]}{8\pi^2} - 3\left(\frac{1}{R_{ab}R_{ac}R_{ae}} + \frac{1}{R_{ba}R_{bc}R_{be}} + \frac{1}{R_{ca}R_{cb}R_{ce}} + \frac{1}{R_{ea}R_{eb}R_{ec}} \right) \right.$$

$$+ \left[\frac{1}{R_{ab}+R_{ac}+R_{bc}} \left(\frac{R_{bc}}{R_{ab}R_{ac}R_{ae}} + \frac{2R_{ae}^2}{R_{ab}R_{be}^3} - \frac{2R_{ab}}{R_{ae}^3} \right) \right.$$

$$\left. \left. + 23 \text{ other permutations of } \{a,b,c,e\} \right] \right\}$$

$$I_{22} \sim \int d^{3-2\varepsilon} y \int d^{3-2\varepsilon} z\, \delta^{ij} \delta^{mn} \partial_i |\bar{x}_a - \bar{y}|^{-1+2\varepsilon}\, \partial_m |\bar{x}_b - \bar{y}|^{-1+2\varepsilon}$$

$$\times |\bar{y} - \bar{z}|^{-1+2\varepsilon}\, \partial_j |\bar{x}_c - \bar{z}|^{-1+2\varepsilon}\, \partial_n |\bar{x}_e - \bar{z}|^{-1+2\varepsilon}$$

O[(v/c)^4]; 2 PN
n = 4 Body Problem

$$
\begin{aligned}
&L_4^{\text{3 Body}} \\
&\equiv \frac{1}{3!} \sum_{\substack{1\le a,b,c\le n \\ a,b,c \text{ distinct}}} \Bigg\{ G_N^2 M_a M_b M_c \\
&\quad \times \left(\frac{1}{R_{ab}R_{ac}} \left(\frac{9}{2}\vec{v}_a^2 + 8\vec{v}_b\cdot\vec{v}_c \right) + \frac{1}{R_{ab}R_{bc}} \left(\frac{9}{2}\vec{v}_b^2 + 8\vec{v}_a\cdot\vec{v}_c \right) + \frac{1}{R_{ac}R_{bc}} \left(\frac{9}{2}\vec{v}_c^2 + 8\vec{v}_a\cdot\vec{v}_b \right) \right. \\
&\quad - \frac{8}{(R_{ab}+R_{ac}+R_{bc})^2} \left(\frac{\vec{R}_{ba}\cdot\vec{v}_b \vec{R}_{ca}\cdot\vec{v}_c}{R_{ab}R_{ac}} + \frac{\vec{R}_{ab}\cdot\vec{v}_a \vec{R}_{cb}\cdot\vec{v}_c}{R_{ab}R_{bc}} + \frac{\vec{R}_{ac}\cdot\vec{v}_a \vec{R}_{bc}\cdot\vec{v}_b}{R_{ac}R_{bc}} \right) \\
&\quad \left. + \frac{4}{R_{ab}+R_{ac}+R_{bc}} \left(\frac{\vec{v}_a^2}{R_{bc}} + \frac{\vec{v}_b^2}{R_{ac}} + \frac{\vec{v}_c^2}{R_{ab}} \right) \right) \\
&\quad + \left[\frac{1}{2R_{ab}R_{ac}^3} \left(\vec{R}_{ac}\cdot\vec{v}_a \vec{R}_{ba}\cdot\vec{v}_b + \vec{R}_{ac}\cdot\vec{v}_a \vec{R}_{ca}\cdot\vec{v}_c + \vec{R}_{ba}\cdot\vec{v}_b \vec{R}_{ca}\cdot\vec{v}_c + 2(\vec{R}_{ca}\cdot\vec{v}_c)^2 \right) \right. \\
&\quad - \frac{1}{R_{ab}R_{ac}} \left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b + \frac{7}{2}\vec{v}_a\cdot\vec{v}_b + \frac{5}{2}\vec{v}_b^2 \right) \\
&\quad + \frac{1}{(R_{ab}+R_{ac}+R_{bc})^2} \left(\frac{1}{R_{ab}^2} \left(4\vec{R}_{ab}\cdot\vec{v}_c \vec{R}_{ba}\cdot\vec{v}_b + 8\vec{R}_{ab}\cdot\vec{v}_a \vec{R}_{ab}\cdot\vec{v}_c - 2(\vec{R}_{ab}\cdot\vec{v}_a)^2 - 2(\vec{R}_{ab}\cdot\vec{v}_c)^2 \right) \right. \\
&\qquad \left. + \frac{1}{R_{ab}R_{ac}} \left(4\vec{R}_{ac}\cdot\vec{v}_b \vec{R}_{ba}\cdot\vec{v}_b - 8\vec{R}_{ac}\cdot\vec{v}_b \vec{R}_{ba}\cdot\vec{v}_c + 12\vec{R}_{ac}\cdot\vec{v}_a \vec{R}_{ba}\cdot\vec{v}_c - 12\vec{R}_{ac}\cdot\vec{v}_a \vec{R}_{ba}\cdot\vec{v}_b \right) \right) \\
&\quad + \frac{1}{R_{ab}+R_{ac}+R_{bc}} \left(\frac{1}{R_{ab}^2} \left(8\vec{R}_{ab}\cdot\vec{v}_a \vec{R}_{ab}\cdot\vec{v}_c + 4\vec{R}_{ab}\cdot\vec{v}_c \vec{R}_{ba}\cdot\vec{v}_b - 2(\vec{R}_{ab}\cdot\vec{v}_c)^2 - 2(\vec{R}_{ab}\cdot\vec{v}_a)^2 \right) \right. \\
&\qquad \left.\left. + \frac{1}{R_{ab}} \left(2\vec{v}_a^2 - 4\vec{v}_a\cdot\vec{v}_c - 2\vec{R}_{ab}\cdot\dot{\vec{v}}_a \right) \right) + 5 \text{ other permutations of } \{a,b,c\} \right] \Bigg) \\
&+ G_N^3 M_a M_b M_c \left(\left[\frac{(M_a+M_c)R_{ab}^2}{R_{ac}^2 R_{bc}^3} + \frac{2M_b R_{ab}}{R_{ac}R_{bc}^3} - \frac{3M_a}{R_{ab}^3} \right. \right. \\
&\quad \left. - \frac{1}{R_{ab}R_{ac}^2} \left(M_a + \frac{3}{2}M_c \right) + 5 \text{ other permutations of } \{a,b,c\} \right] \\
&\quad \left. + \frac{1}{16\pi^2} \left(M_a I_{22}[a,a,b,c] + M_b I_{22}[b,b,a,c] + M_c I_{22}[c,c,a,b] \right) - 2 \left(\frac{M_a}{R_{bc}^3} + \frac{M_b}{R_{ac}^3} + \frac{M_c}{R_{ab}^3} \right) \right) \Bigg\}
\end{aligned}
$$

O[(v/c)^4]: 2 PN
n = 4 Body Problem

$$L_4^{4\,\text{Body}} \equiv \frac{1}{4!} \sum_{\substack{1\leq a,b,c,e\leq n \\ a,b,c,e \text{ distinct}}} G_N^3 M_a M_b M_c M_e$$

$$\times\left\{\frac{I_{22}[a,b,c,e]}{8\pi^2} - 3\left(\frac{1}{R_{ab}R_{ac}R_{ae}} + \frac{1}{R_{ba}R_{bc}R_{be}} + \frac{1}{R_{ca}R_{cb}R_{ce}} + \frac{1}{R_{ea}R_{eb}R_{ec}}\right)\right.$$

$$+\left[\frac{1}{R_{ab}+R_{ac}+R_{bc}}\left(\frac{R_{bc}}{R_{ab}R_{ac}R_{ae}} + \frac{2R_{ae}^2}{R_{ab}R_{be}^3} - \frac{2R_{ab}}{R_{ae}^3}\right)\right.$$

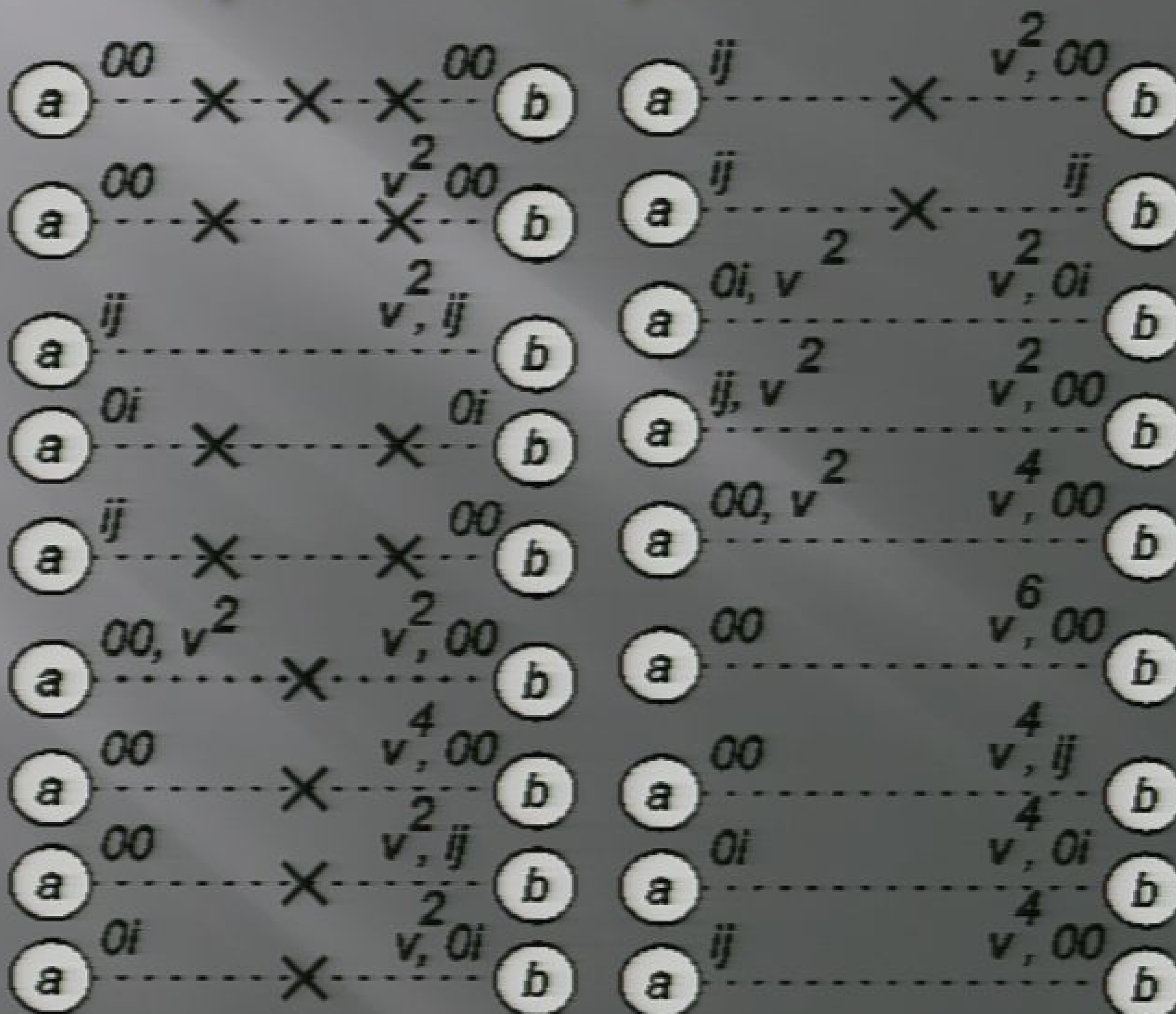
$$\left.\left. +23 \text{ other permutations of } \{a,b,c,e\}\right]\right\}$$

$$I_{22} \sim \int d^{3-2\varepsilon}y \int d^{3-2\varepsilon}z\, \delta^{ij}\delta^{mn}\partial_i |\bar{x}_a - \bar{y}|^{-1+2\varepsilon}\, \partial_m |\bar{x}_b - \bar{y}|^{-1+2\varepsilon}$$

$$\times |\bar{y} - \bar{z}|^{-1+2\varepsilon}\, \partial_j |\bar{x}_c - \bar{z}|^{-1+2\varepsilon}\, \partial_n |\bar{x}_e - \bar{z}|^{-1+2\varepsilon}$$

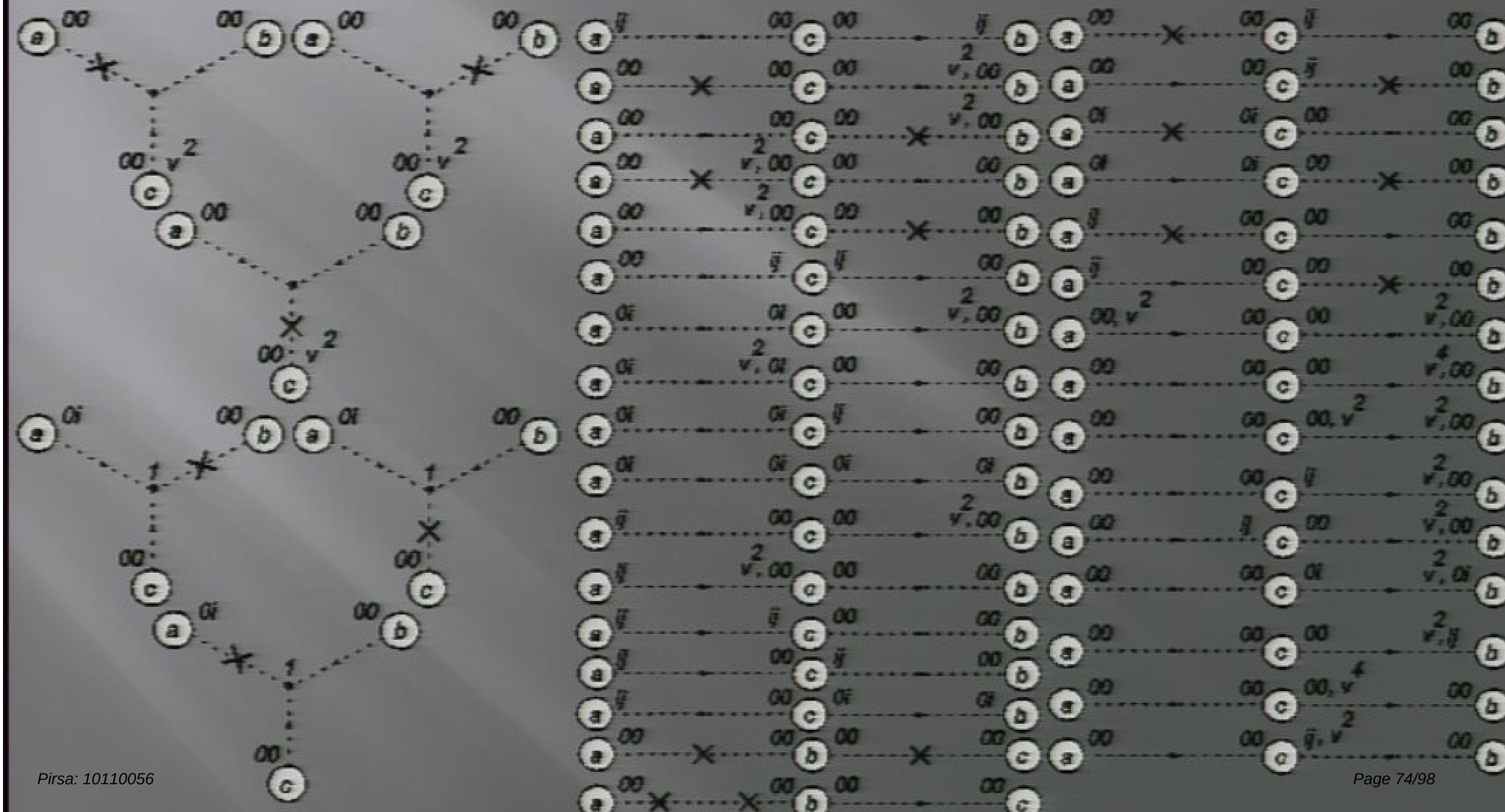
Third Order – $O[(v/c)^6]$ Beyond Newton

$O[(v/c)^6]$: 3 PN Feynman diagrams 2 (distinct) bodies

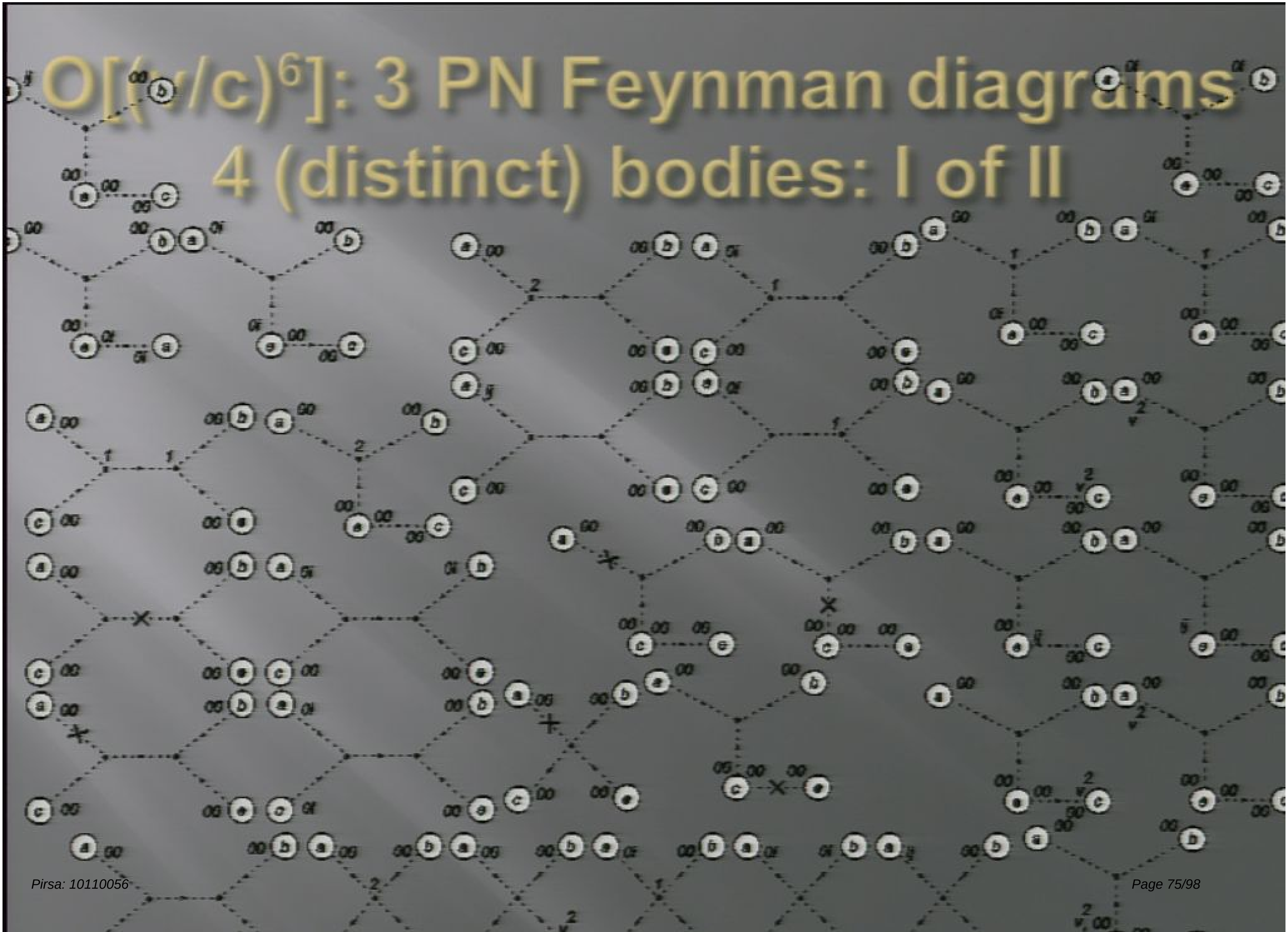


$O[(v/c)^6]$: 3 PN Feynman diagrams
3 (distinct) bodies: I or II

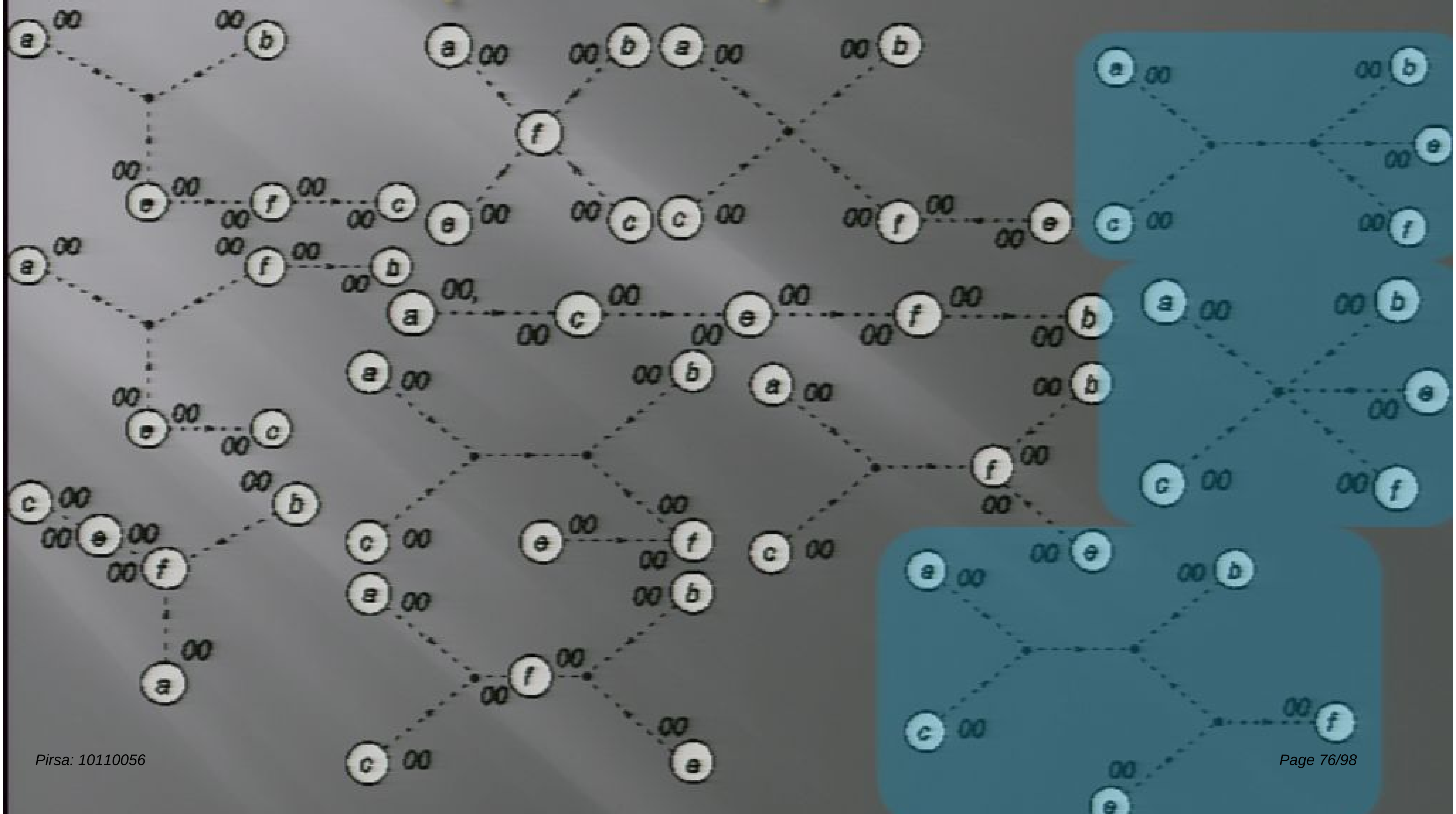
$O[(v/c)^6]$: 3 PN Feynman diagrams
3 (distinct) bodies: II of II



$O[(v/c)^6]$: 3 PN Feynman diagrams 4 (distinct) bodies: I of II



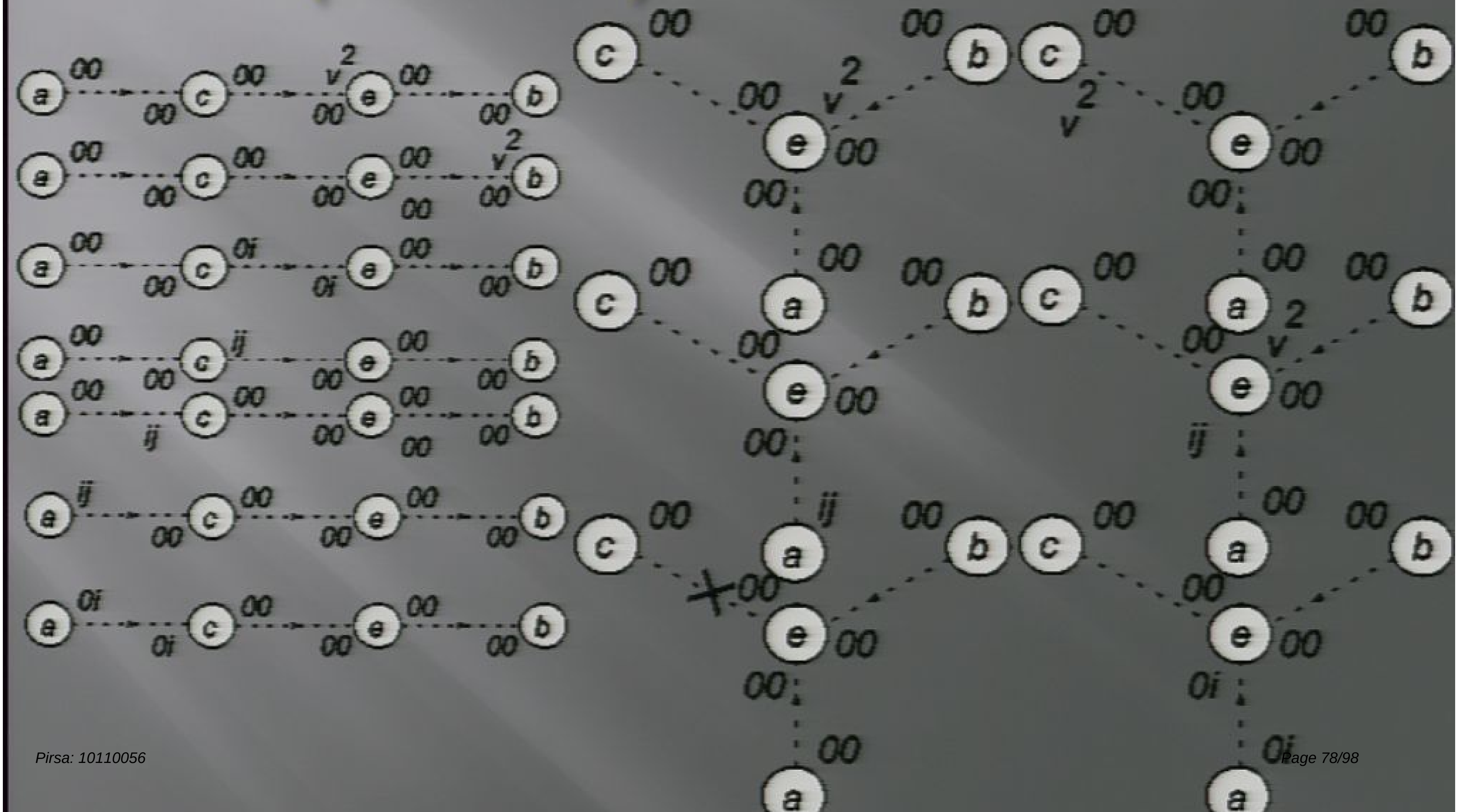
$O[(v/c)^6]$: 3 PN Feynman diagrams 5 (distinct) bodies



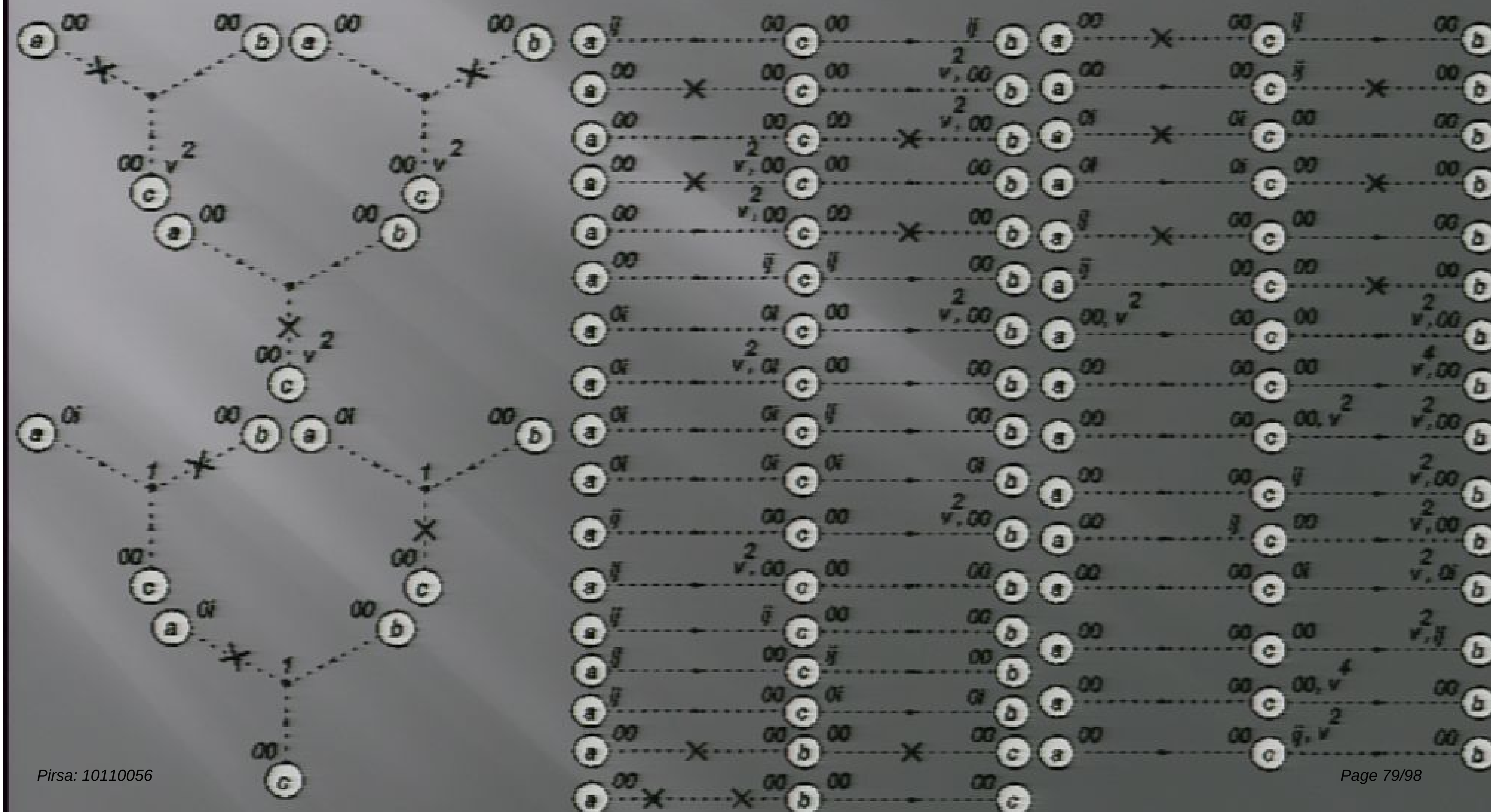
The road ahead

- N-body problem
  - Rotation, multipoles, tails, etc.
  - Gravitational radiation
- Higher PN computation:
  - Different field variables: ADM, Kol-Smolkin-Kaluza-Klein.
  - Different gravitational lagrangian: Bern-Grant.
  - Are there recursion relations for off-shell gravitational amplitudes?
  - Software development.

$O[(v/c)^6]$: 3 PN Feynman diagrams 4 (distinct) bodies: II of II



$O[(v/c)^6]$: 3 PN Feynman diagrams
3 (distinct) bodies: II of II



$O[(v/c)^4]$: 2 PN
n = 4 Body Problem

$$L_4^{4\text{ Body}} \equiv \frac{1}{4!} \sum_{\substack{1\le a,b,c,e\le n \\ a,b,c,e \text{ distinct}}} G_N^3 M_a M_b M_c M_e$$
$$\times \left\{ \frac{I_{22}[a,b,c,e]}{8\pi^2} - 3\left(\frac{1}{R_{ab}R_{ac}R_{ae}} + \frac{1}{R_{ba}R_{bc}R_{be}} + \frac{1}{R_{ca}R_{cb}R_{ce}} + \frac{1}{R_{ea}R_{eb}R_{ec}} \right) \right.$$
$$+ \left[\frac{1}{R_{ab}+R_{ac}+R_{bc}} \left(\frac{R_{bc}}{R_{ab}R_{ac}R_{ae}} + \frac{2R_{ae}^2}{R_{ab}R_{be}^3} - \frac{2R_{ab}}{R_{ae}^3} \right) \right.$$
$$\left. \left. +23 \text{ other permutations of } \{a,b,c,e\} \right] \right\}$$

$$I_{22} \sim \int d^{3-2\varepsilon} y \int d^{3-2\varepsilon} z\, \delta^{ij} \delta^{mn} \partial_i |\bar{x}_a - \bar{y}|^{-1+2\varepsilon} \partial_m |\bar{x}_b - \bar{y}|^{-1+2\varepsilon}$$
$$\times |\bar{y} - \bar{z}|^{-1+2\varepsilon} \partial_j |\bar{x}_c - \bar{z}|^{-1+2\varepsilon} \partial_n |\bar{x}_e - \bar{z}|^{-1+2\varepsilon}$$

$O[(v/c)^4]$: 2 PN
n = 4 Body Problem

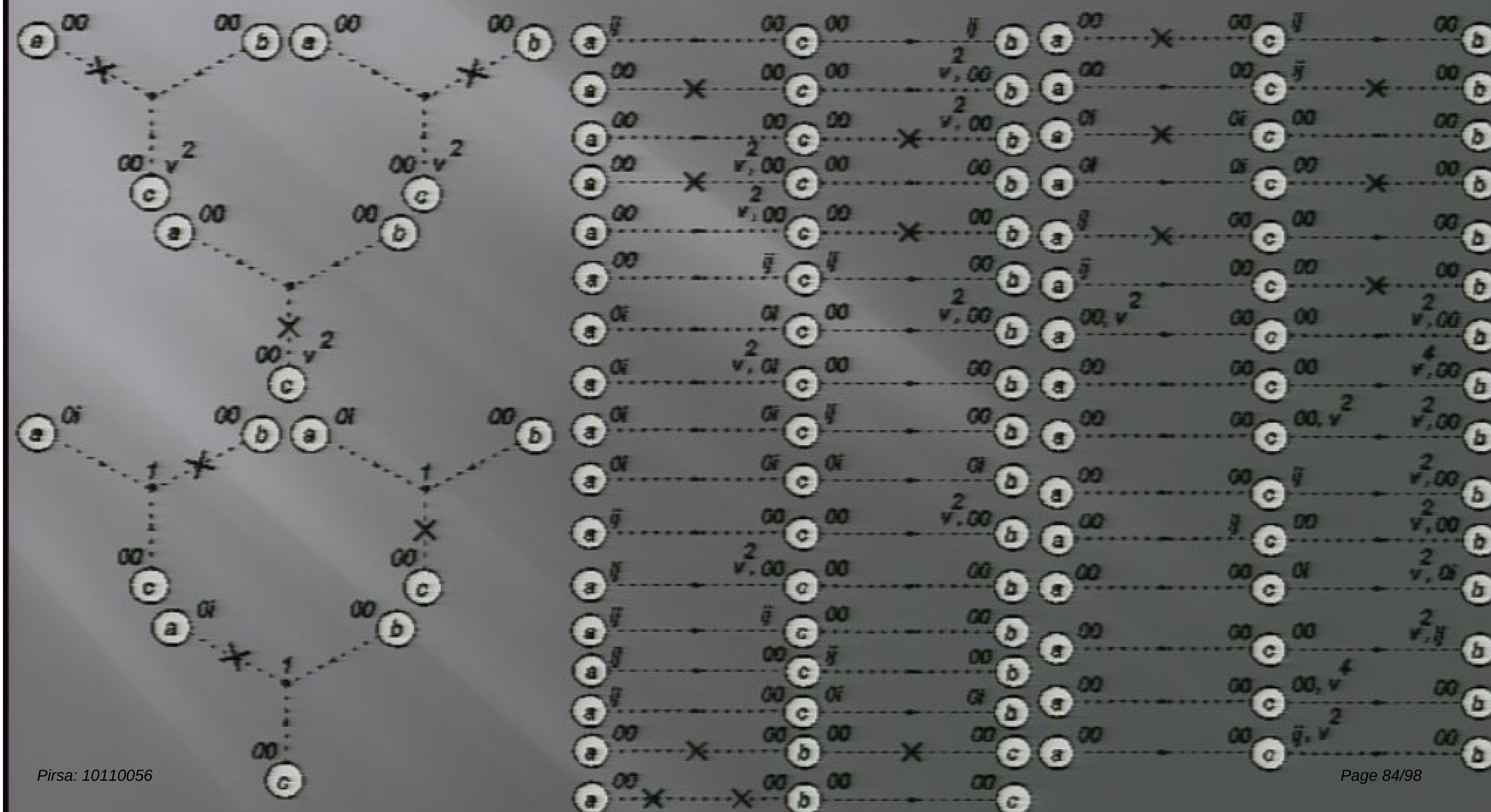
$$
\begin{aligned}
L_4^{2\text{ Body}} \equiv \frac{1}{2} \sum_{\substack{1\le a,b\le n \\ a\neq b}} \Bigg\{ & \frac{M_a}{16}\vec{v}_a^6 + \frac{M_b}{16}\vec{v}_b^6 \\
& + \frac{G_N M_a M_b}{R_{ab}} \Bigg(\vec{R}_{ab}\cdot\vec{v}_a \left(\frac{7}{4}\vec{v}_a\cdot\dot{\vec{v}}_b - \frac{3}{2}\vec{v}_b\cdot\dot{\vec{v}}_b \right) + \vec{R}_{ba}\cdot\vec{v}_b \left(\frac{7}{4}\vec{v}_b\cdot\dot{\vec{v}}_a - \frac{3}{2}\vec{v}_a\cdot\dot{\vec{v}}_a \right) \\
& \quad - \frac{1}{8}\left(\frac{\vec{R}_{ab}\cdot\vec{v}_a}{R_{ab}}\right)^2 \left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b + \vec{v}_b^2\right) - \frac{1}{8}\left(\frac{\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}}\right)^2 \left(\vec{R}_{ab}\cdot\dot{\vec{v}}_a + \vec{v}_a^2\right) \\
& \quad + \frac{3}{4}\left(\vec{v}_a^2 + \vec{v}_b^2 - 2\,\vec{v}_a\cdot\vec{v}_b\right) \frac{\vec{R}_{ab}\cdot\vec{v}_a}{R_{ab}} \frac{\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}} + \frac{3}{8}\frac{(\vec{R}_{ab}\cdot\vec{v}_a)^2(\vec{R}_{ba}\cdot\vec{v}_b)^2}{R_{ab}^4} \\
& \quad + \frac{1}{8}\left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b\,\vec{v}_a^2 + \vec{R}_{ab}\cdot\dot{\vec{v}}_a\,\vec{v}_b^2\right) + \frac{1}{8}\vec{R}_{ab}\cdot\dot{\vec{v}}_a\,\vec{R}_{ba}\cdot\dot{\vec{v}}_b \\
& \quad + \frac{15}{8}\dot{\vec{v}}_a\cdot\dot{\vec{v}}_b\,R_{ab}^2 + \frac{7}{8}(\vec{v}_a^4 + \vec{v}_b^4) + \frac{1}{4}(\vec{v}_a\cdot\vec{v}_b)^2 + \frac{3}{8}\vec{v}_a^2\vec{v}_b^2 - \frac{5}{4}(\vec{v}_a^2 + \vec{v}_b^2)\vec{v}_a\cdot\vec{v}_b \Bigg) \\
& + \frac{G_N^2 M_a M_b}{R_{ab}^2} \Bigg(-\frac{3}{2}\frac{M_a(\vec{R}_{ba}\cdot\vec{v}_b)^2 + M_b(\vec{R}_{ab}\cdot\vec{v}_a)^2}{R_{ab}^2} - 2(M_a + M_b)\frac{\vec{R}_{ab}\cdot\vec{v}_a\,\vec{R}_{ba}\cdot\vec{v}_b}{R_{ab}^2} \\
& \quad - (2M_a + M_b)\vec{R}_{ab}\cdot\dot{\vec{v}}_a - (2M_b + M_a)\vec{R}_{ba}\cdot\dot{\vec{v}}_b \\
& \quad + \vec{v}_a^2\left(2M_a + \frac{11}{4}M_b\right) + \vec{v}_b^2\left(2M_b + \frac{11}{4}M_a\right) - \frac{9}{2}\vec{v}_a\cdot\vec{v}_b\,(M_a + M_b) \Bigg) \\
& - \frac{G_N^3 M_a M_b}{R_{ab}^3}\left(M_a M_b + \frac{3}{2}(M_a^2 + M_b^2)\right) \Bigg\}
\end{aligned}
$$

$O[(v/c)^4]$: 2 PN
$n = 4$ Body Problem

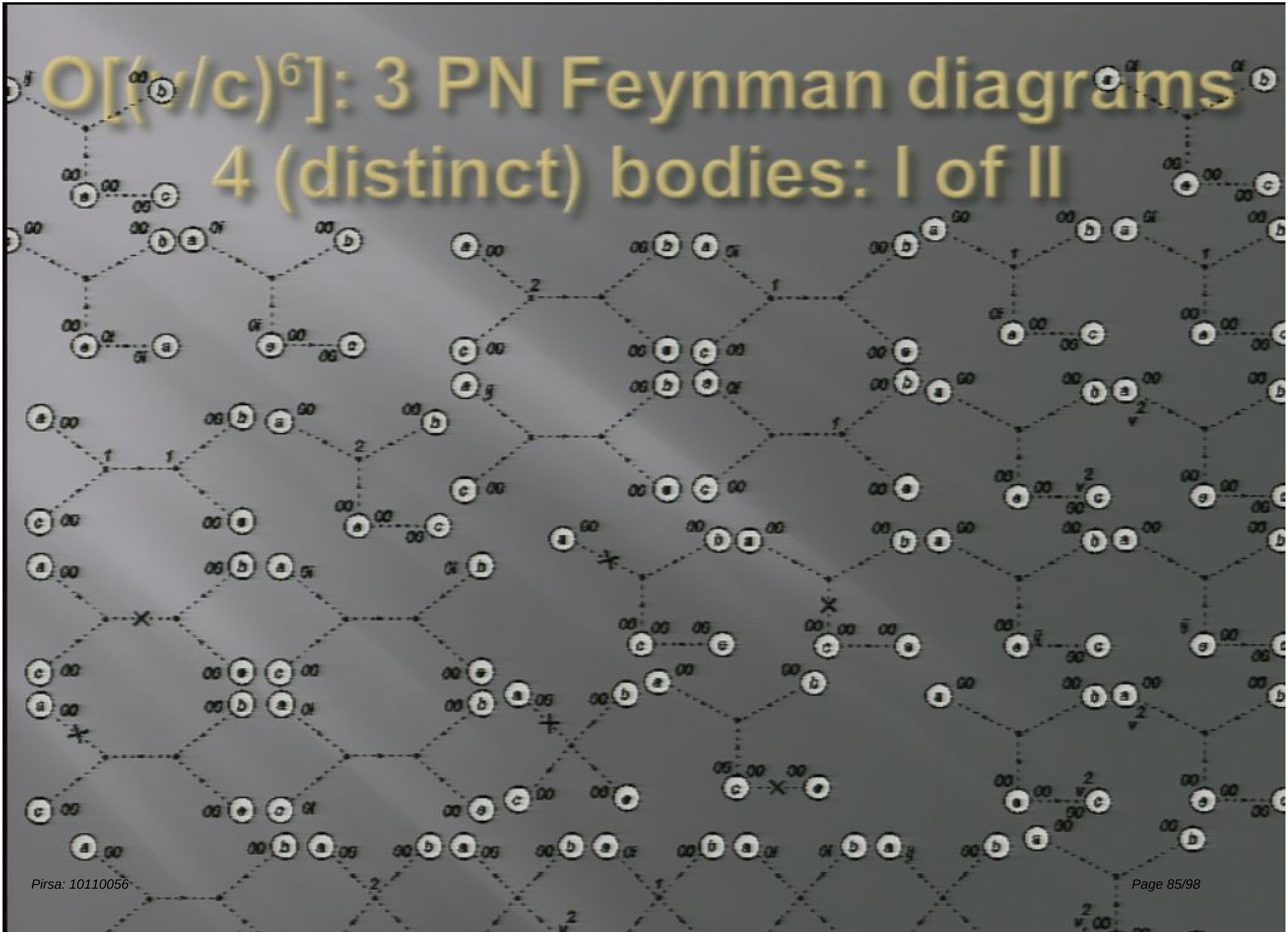
$$
\begin{aligned}
&L_4^{\text{3 Body}} \\
&\equiv \frac{1}{3!} \sum_{\substack{1 \le a,b,c \le n \\ a,b,c \text{ distinct}}} \Bigg\{ G_N^2 M_a M_b M_c \\
&\quad \times \Bigg(\frac{1}{R_{ab} R_{ac}} \left(\frac{9}{2} \vec{v}_a^2 + 8 \vec{v}_b \cdot \vec{v}_c \right) + \frac{1}{R_{ab} R_{bc}} \left(\frac{9}{2} \vec{v}_b^2 + 8 \vec{v}_a \cdot \vec{v}_c \right) + \frac{1}{R_{ac} R_{bc}} \left(\frac{9}{2} \vec{v}_c^2 + 8 \vec{v}_a \cdot \vec{v}_b \right) \\
&\quad - \frac{8}{(R_{ab} + R_{ac} + R_{bc})^2} \left(\frac{\vec{R}_{ba} \cdot \vec{v}_b \vec{R}_{ca} \cdot \vec{v}_c}{R_{ab} R_{ac}} + \frac{\vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{cb} \cdot \vec{v}_c}{R_{ab} R_{bc}} + \frac{\vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{bc} \cdot \vec{v}_b}{R_{ac} R_{bc}} \right) \\
&\quad + \frac{4}{R_{ab} + R_{ac} + R_{bc}} \left(\frac{\vec{v}_a^2}{R_{bc}} + \frac{\vec{v}_b^2}{R_{ac}} + \frac{\vec{v}_c^2}{R_{ab}} \right) \Bigg) \\
&\quad + \Bigg[\frac{1}{2 R_{ab} R_{ac}^3} \left(\vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_b + \vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ca} \cdot \vec{v}_c + \vec{R}_{ba} \cdot \vec{v}_b \vec{R}_{ca} \cdot \vec{v}_c + 2 (\vec{R}_{ca} \cdot \vec{v}_c)^2 \right) \\
&\quad - \frac{1}{R_{ab} R_{ac}} \left(\vec{R}_{ba} \cdot \dot{\vec{v}}_b + \frac{7}{2} \vec{v}_a \cdot \vec{v}_b + \frac{5}{2} \vec{v}_b^2 \right) \\
&\quad + \frac{1}{(R_{ab} + R_{ac} + R_{bc})^2} \Bigg(\frac{1}{R_{ab}^2} \left(4 \vec{R}_{ab} \cdot \vec{v}_c \vec{R}_{ba} \cdot \vec{v}_b + 8 \vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{ab} \cdot \vec{v}_c - 2 (\vec{R}_{ab} \cdot \vec{v}_a)^2 - 2 (\vec{R}_{ab} \cdot \vec{v}_c)^2 \right) \\
&\qquad + \frac{1}{R_{ab} R_{ac}} \left(4 \vec{R}_{ac} \cdot \vec{v}_b \vec{R}_{ba} \cdot \vec{v}_b - 8 \vec{R}_{ac} \cdot \vec{v}_b \vec{R}_{ba} \cdot \vec{v}_c + 12 \vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_c - 12 \vec{R}_{ac} \cdot \vec{v}_a \vec{R}_{ba} \cdot \vec{v}_b \right) \Bigg) \\
&\quad + \frac{1}{R_{ab} + R_{ac} + R_{bc}} \Bigg(\frac{1}{R_{ab}^3} \left(8 \vec{R}_{ab} \cdot \vec{v}_a \vec{R}_{ab} \cdot \vec{v}_c + 4 \vec{R}_{ab} \cdot \vec{v}_c \vec{R}_{ba} \cdot \vec{v}_b - 2 (\vec{R}_{ab} \cdot \vec{v}_c)^2 - 2 (\vec{R}_{ab} \cdot \vec{v}_a)^2 \right) \\
&\qquad + \frac{1}{R_{ab}} \left(2 \vec{v}_a^2 - 4 \vec{v}_a \cdot \vec{v}_c - 2 \vec{R}_{ab} \cdot \dot{\vec{v}}_a \right) \Bigg) + 5 \text{ other permutations of } \{a, b, c\} \Bigg] \Bigg) \\
&+ G_N^3 M_a M_b M_c \Bigg(\Bigg[\frac{(M_a + M_c) R_{ab}^2}{R_{ac}^2 R_{bc}^3} + \frac{2 M_b R_{ab}}{R_{ac} R_{bc}^3} - \frac{3 M_a}{R_{ab}^3} \\
&\quad - \frac{1}{R_{ab} R_{ac}^2} \left(M_a + \frac{3}{2} M_c \right) + 5 \text{ other permutations of } \{a, b, c\} \Bigg] \\
&\quad + \frac{1}{16 \pi^2} \left(M_a I_{22}[a, a, b, c] + M_b I_{22}[b, b, a, c] + M_c I_{22}[c, c, a, b] \right) - 2 \left(\frac{M_a}{R_{bc}^3} + \frac{M_b}{R_{ac}^3} + \frac{M_c}{R_{ab}^3} \right) \Bigg) \Bigg\}
\end{aligned}
$$

O[(v/c)^6]: 3 PN Feynman diagrams
3 (distinct) bodies: I or II

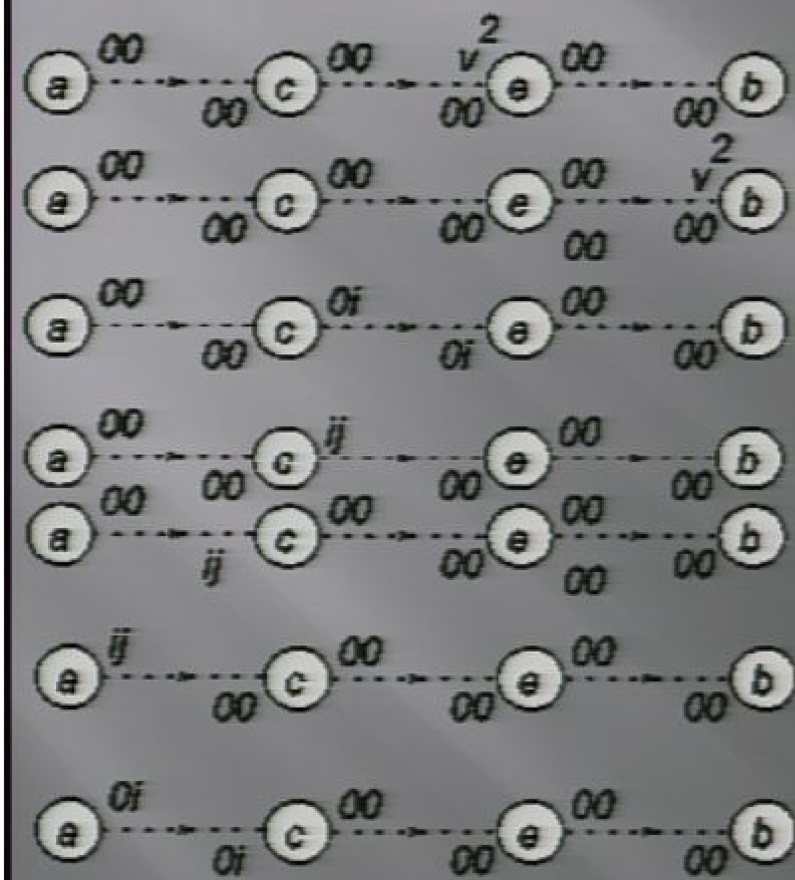
$O[(v/c)^6]$: 3 PN Feynman diagrams 3 (distinct) bodies: II of II

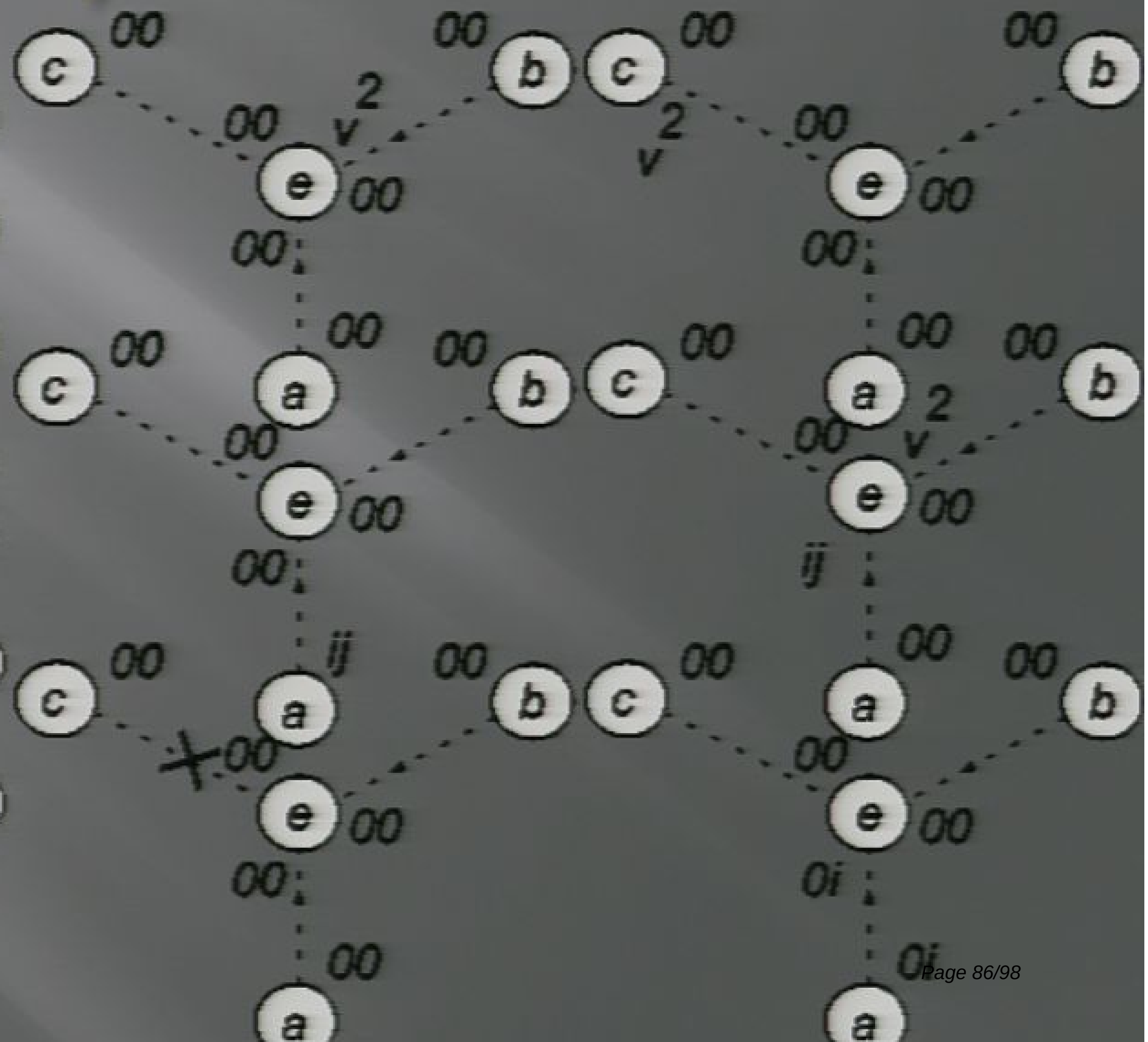


$O[(v/c)^6]$: 3 PN Feynman diagrams
4 (distinct) bodies: I of II



$O[(v/c)^6]$: 3 PN Feynman diagrams 4 (distinct) bodies: II of II

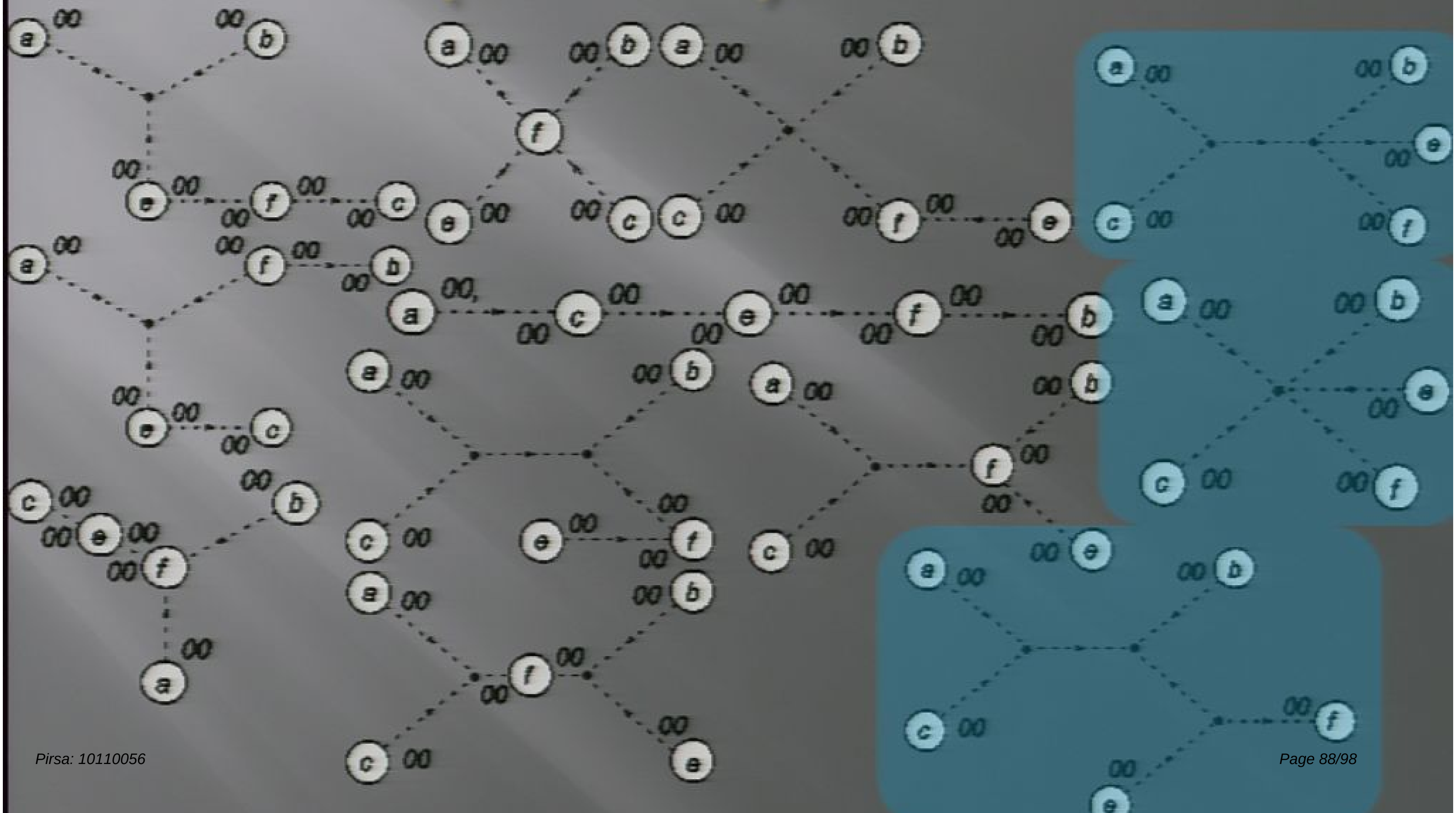




The road ahead

- N-body problem
  - Rotation, multipoles, tails, etc.
  - Gravitational radiation
- Higher PN computation:
  - Different field variables: ADM, Kol-Smolkin-Kaluza-Klein.
  - Different gravitational lagrangian: Bern-Grant.
  - Are there recursion relations for off-shell gravitational amplitudes?
  - Software development.

O[(v/c)^6]: 3 PN Feynman diagrams 5 (distinct) bodies



The road ahead

- N-body problem
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$O[(v/c)^6]$: 3 PN Feynman diagrams
3 (distinct) bodies: I or II

O[(v/c)^4]: 2 PN
n = 4 Body Problem

$$
\begin{aligned}
L_4^{4\,\text{Body}} \equiv \frac{1}{4!} \sum_{\substack{1\le a,b,c,e\le n \\ a,b,c,e \text{ distinct}}} & G_N^3 M_a M_b M_c M_e \\
& \times \left\{ \frac{I_{22}[a,b,c,e]}{8\pi^2} - 3\left(\frac{1}{R_{ab}R_{ac}R_{ae}} + \frac{1}{R_{ba}R_{bc}R_{be}} + \frac{1}{R_{ca}R_{cb}R_{ce}} + \frac{1}{R_{ea}R_{eb}R_{ec}}\right) \right. \\
& \quad + \left[\frac{1}{R_{ab}+R_{ac}+R_{bc}}\left(\frac{R_{bc}}{R_{ab}R_{ac}R_{ae}} + \frac{2R_{ae}^2}{R_{ab}R_{be}^3} - \frac{2R_{ab}}{R_{ae}^3}\right)\right. \\
& \qquad \left.\left. +23 \text{ other permutations of } \{a,b,c,e\} \right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
I_{22} \sim \int d^{3-2\varepsilon} y \int d^{3-2\varepsilon} z\, \delta^{ij} \delta^{mn} \partial_i \,|\bar{x}_a - \bar{y}|^{-1+2\varepsilon}\, \partial_m \,|\bar{x}_b - \bar{y}|^{-1+2\varepsilon} \\
\times |\bar{y} - \bar{z}|^{-1+2\varepsilon}\, \partial_j \,|\bar{x}_c - \bar{z}|^{-1+2\varepsilon}\, \partial_n \,|\bar{x}_e - \bar{z}|^{-1+2\varepsilon}
\end{aligned}
$$

O[(v/c)⁴]; 2 PN
n = 4 Body Problem

$L_4^{\text{3 Body}}$

$$\equiv \frac{1}{3!}\sum_{\substack{1\le a,b,c\le n\\ a,b,c\text{ distinct}}} \Big\{ G_N^2 M_a M_b M_c$$

$$\times\Bigg(\frac{1}{R_{ab}R_{ac}}\left(\frac{9}{2}\vec{v}_a^2 + 8\vec{v}_b\cdot\vec{v}_c\right) + \frac{1}{R_{ab}R_{bc}}\left(\frac{9}{2}\vec{v}_b^2+8\vec{v}_a\cdot\vec{v}_c\right) + \frac{1}{R_{ac}R_{bc}}\left(\frac{9}{2}\vec{v}_c^2 + 8\vec{v}_a\cdot\vec{v}_b\right)$$

$$-\frac{8}{(R_{ab}+R_{ac}+R_{bc})^2}\left(\frac{\vec{R}_{ba}\cdot\vec{v}_b \vec{R}_{ca}\cdot\vec{v}_c}{R_{ab}R_{ac}} + \frac{\vec{R}_{ab}\cdot\vec{v}_a\vec{R}_{cb}\cdot\vec{v}_c}{R_{ab}R_{bc}} + \frac{\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{bc}\cdot\vec{v}_b}{R_{ac}R_{bc}}\right)$$

$$+\frac{4}{R_{ab}+R_{ac}+R_{bc}}\left(\frac{\vec{v}_a^2}{R_{bc}}+\frac{\vec{v}_b^2}{R_{ac}}+\frac{\vec{v}_c^2}{R_{ab}}\right)\Bigg)$$

$$+\Bigg[\frac{1}{2R_{ab}R_{ac}^3}\left(\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{ba}\cdot\vec{v}_b + \vec{R}_{ac}\cdot\vec{v}_a \vec{R}_{ca}\cdot\vec{v}_c + \vec{R}_{ba}\cdot\vec{v}_b\vec{R}_{ca}\cdot\vec{v}_c + 2(\vec{R}_{ca}\cdot\vec{v}_c)^2\right)$$

$$-\frac{1}{R_{ab}R_{ac}}\left(\vec{R}_{ba}\cdot\dot{\vec{v}}_b + \frac{7}{2}\vec{v}_a\cdot\vec{v}_b+\frac{5}{2}\vec{v}_b^2\right)$$

$$+\frac{1}{(R_{ab}+R_{ac}+R_{bc})^2}\Bigg(\frac{1}{R_{ab}^2}\left(4\vec{R}_{ab}\cdot\vec{v}_c\vec{R}_{ba}\cdot\vec{v}_b + 8\vec{R}_{ab}\cdot\vec{v}_a\vec{R}_{ab}\cdot\vec{v}_c - 2(\vec{R}_{ab}\cdot\vec{v}_a)^2 - 2(\vec{R}_{ab}\cdot\vec{v}_c)^2\right)$$

$$+\frac{1}{R_{ab}R_{ac}}\left(4\vec{R}_{ac}\cdot\vec{v}_b\vec{R}_{ba}\cdot\vec{v}_b - 8\vec{R}_{ac}\cdot\vec{v}_b\vec{R}_{ba}\cdot\vec{v}_c + 12\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{ba}\cdot\vec{v}_c - 12\vec{R}_{ac}\cdot\vec{v}_a\vec{R}_{ba}\cdot\vec{v}_b\right)\Bigg)$$

$$+\frac{1}{R_{ab}+R_{ac}+R_{bc}}\Bigg(\frac{1}{R_{ab}^3}\left(8\vec{R}_{ab}\cdot\vec{v}_a\vec{R}_{ab}\cdot\vec{v}_c + 4\vec{R}_{ab}\cdot\vec{v}_c\vec{R}_{ba}\cdot\vec{v}_b - 2(\vec{R}_{ab}\cdot\vec{v}_c)^2 - 2(\vec{R}_{ab}\cdot\vec{v}_a)^2\right)$$

$$+\frac{1}{R_{ab}}\left(2\vec{v}_a^2 - 4\vec{v}_a\cdot\vec{v}_c - 2\vec{R}_{ab}\cdot\dot{\vec{v}}_a\right)\Bigg) + 5\text{ other permutations of }\{a,b,c\}\Bigg]\Bigg)$$

$$+G_N^3 M_aM_bM_c\Bigg(\Bigg[\frac{(M_a+M_c)R_{ab}^2}{R_{ac}^2R_{bc}^3} + \frac{2M_bR_{ab}}{R_{ac}R_{bc}^3} - \frac{3M_a}{R_{ab}^3}$$

$$-\frac{1}{R_{ab}R_{ac}^2}\left(M_a+\frac{3}{2}M_c\right) + 5\text{ other permutations of }\{a,b,c\}\Bigg]$$

$$+\frac{1}{16\pi^2}\left(M_aI_{22}[a,a,b,c] + M_bI_{22}[b,b,a,c]+M_cI_{22}[c,c,a,b]\right) - 2\left(\frac{M_a}{R^3}+\frac{M_b}{R^3}+\frac{M_c}{R^3}\right)\Bigg)\Big\}$$

$O[(v/c)^4]$: 2 PN
n = 4 Body Problem

$$L_4^{4\text{ Body}} \equiv \frac{1}{4!} \sum_{\substack{1 \le a,b,c,e \le n \\ a,b,c,e \text{ distinct}}} G_N^3 M_a M_b M_c M_e$$

$$\times \Bigg\{ \frac{I_{22}[a,b,c,e]}{8\pi^2} - 3\left(\frac{1}{R_{ab}R_{ac}R_{ae}} + \frac{1}{R_{ba}R_{bc}R_{be}} + \frac{1}{R_{ca}R_{cb}R_{ce}} + \frac{1}{R_{ea}R_{eb}R_{ec}} \right)$$

$$+ \Bigg[\frac{1}{R_{ab}+R_{ac}+R_{bc}} \left(\frac{R_{bc}}{R_{ab}R_{ac}R_{ae}} + \frac{2R_{ae}^2}{R_{ab}R_{be}^3} - \frac{2R_{ab}}{R_{ae}^3} \right)$$

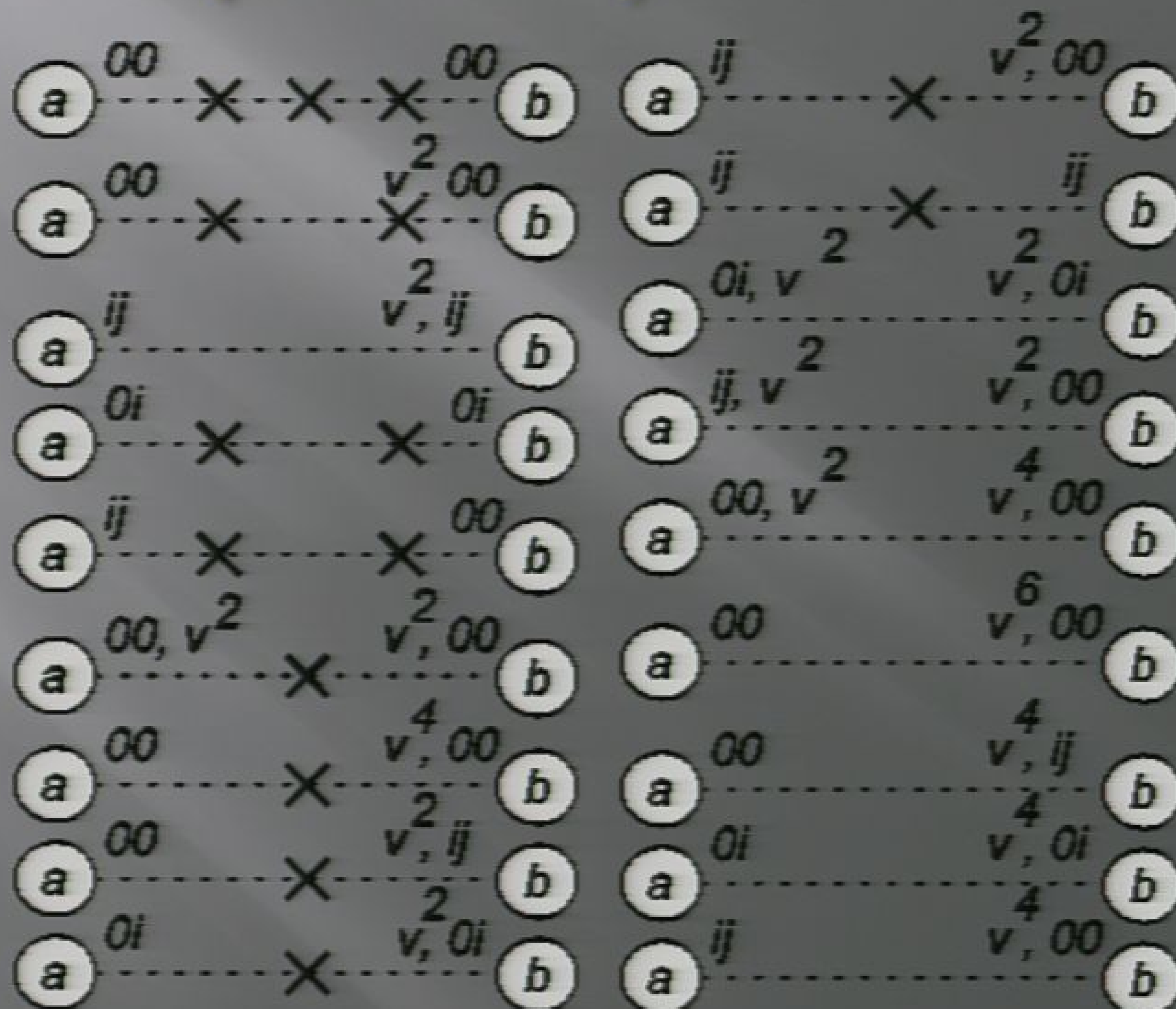
$$+23 \text{ other permutations of } \{a,b,c,e\} \Bigg] \Bigg\}$$

$$I_{22} \sim \int d^{3-2\varepsilon} y \int d^{3-2\varepsilon} z \, \delta^{ij} \delta^{mn} \partial_i |\bar{x}_a - \bar{y}|^{-1+2\varepsilon} \partial_m |\bar{x}_b - \bar{y}|^{-1+2\varepsilon}$$

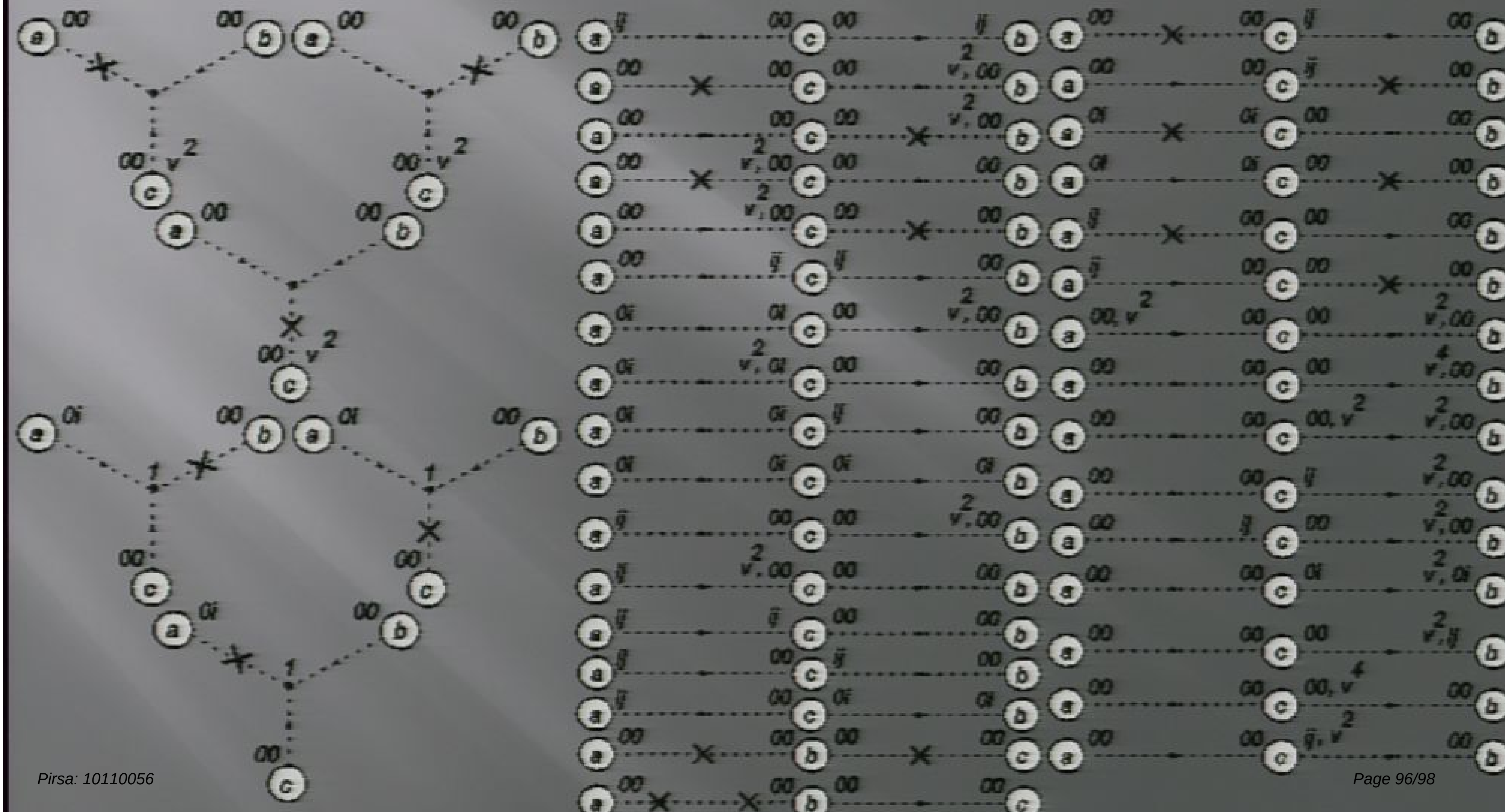
$$\times |\bar{y} - \bar{z}|^{-1+2\varepsilon} \partial_j |\bar{x}_c - \bar{z}|^{-1+2\varepsilon} \partial_n |\bar{x}_e - \bar{z}|^{-1+2\varepsilon}$$

Third Order – $O[(v/c)^6]$ Beyond Newton

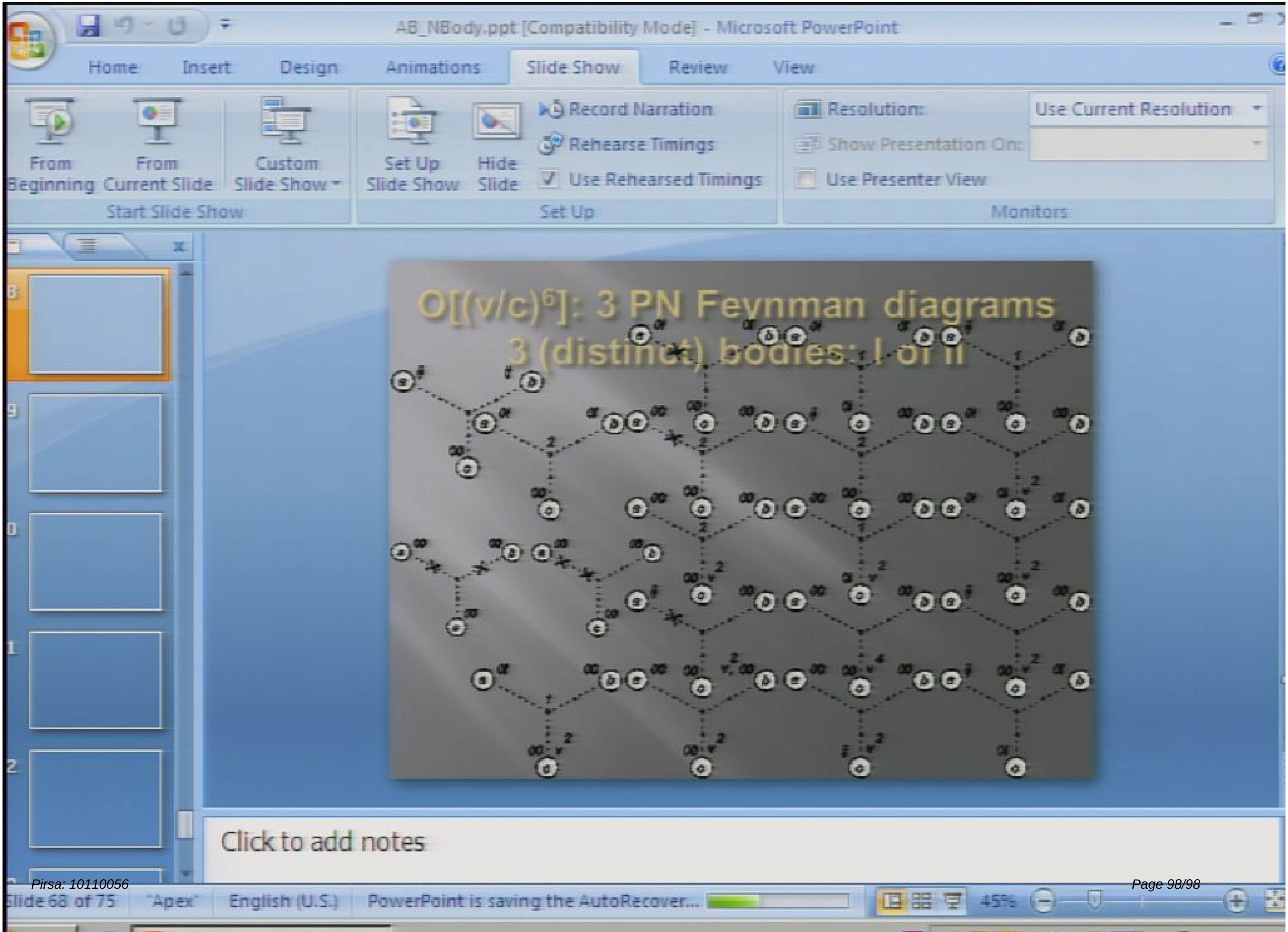
$O[(v/c)^6]$: 3 PN Feynman diagrams 2 (distinct) bodies



$O[(v/c)^6]$: 3 PN Feynman diagrams 3 (distinct) bodies: II of II



$O[(v/c)^6]$: 3 PN Feynman diagrams
3 (distinct) bodies: I or II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