Title: Making sense of non-Hermitian Hamiltonians

Date: Nov 24, 2010 02:00 PM

URL: http://pirsa.org/10110055

Abstract: The average quantum physicist on the street believes that a quantum-mechanical Hamiltonian must be Dirac Hermitian (invariant under combined matrix transposition and complex conjugation) in order to guarantee that the energy eigenvalues are real and that time evolution is unitary. However, the Hamiltonian $H=p^2+ix^3$, which is obviously not Dirac Hermitian, has a real positive discrete spectrum and generates unitary time evolution, and thus it defines a fully consistent and physical quantum theory.

Evidently, the axiom of Dirac Hermiticity is too restrictive. While $H=p^2+ix^3$ is not Dirac Hermitian, it is PT symmetric; that is, invariant under combined space reflection P and time reversal T. The quantum mechanics defined by a PT-symmetric Hamiltonian is a complex generalization of ordinary quantum mechanics. When quantum mechanics is extended into the complex domain, new kinds of theories having strange and remarkable properties emerge. Some of these properties have recently been verified in laboratory experiments. If one generalizes classical mechanics into the complex domain, the resulting theories have equally remarkable properties.

Making Sense of Immanent Rhino Inhalations

Crab Lender
Washing Nervy Tuitions

Permit True Entities

Making Sense of Non-Hermitian Hamiltonians

Carl Bender
Washington University

Perimeter Institute

Quantum mechanics is a strange animal!

Quantum mechanics

- “Anyone who thinks he can contemplate quantum mechanics without getting dizzy hasn’t properly understood it.” – Niels Bohr
- “Anyone who thinks they know quantum mechanics doesn’t.” – Richard Feynman
- “I don’t like it, and I’m sorry I ever had anything to do with it.” – Erwin Schrödinger

Assumptions of quantum mechanics

- causality
- locality
- relativistic invariance
- existence of a ground state
- conservation of probability (unitarity)
- positive real energies
- Hermitian Hamiltonian

The point of this talk:

Dirac Hermiticity is too strong an axiom of quantum mechanics!

$$H = H^{\dagger}$$

$\dagger$ means transpose + complex conjugate

- guarantees real energy and conserved probability
- but … is a **mathematical** axiom and not a **physical** axiom of quantum mechanics

$$H = p^2 + ix^3$$

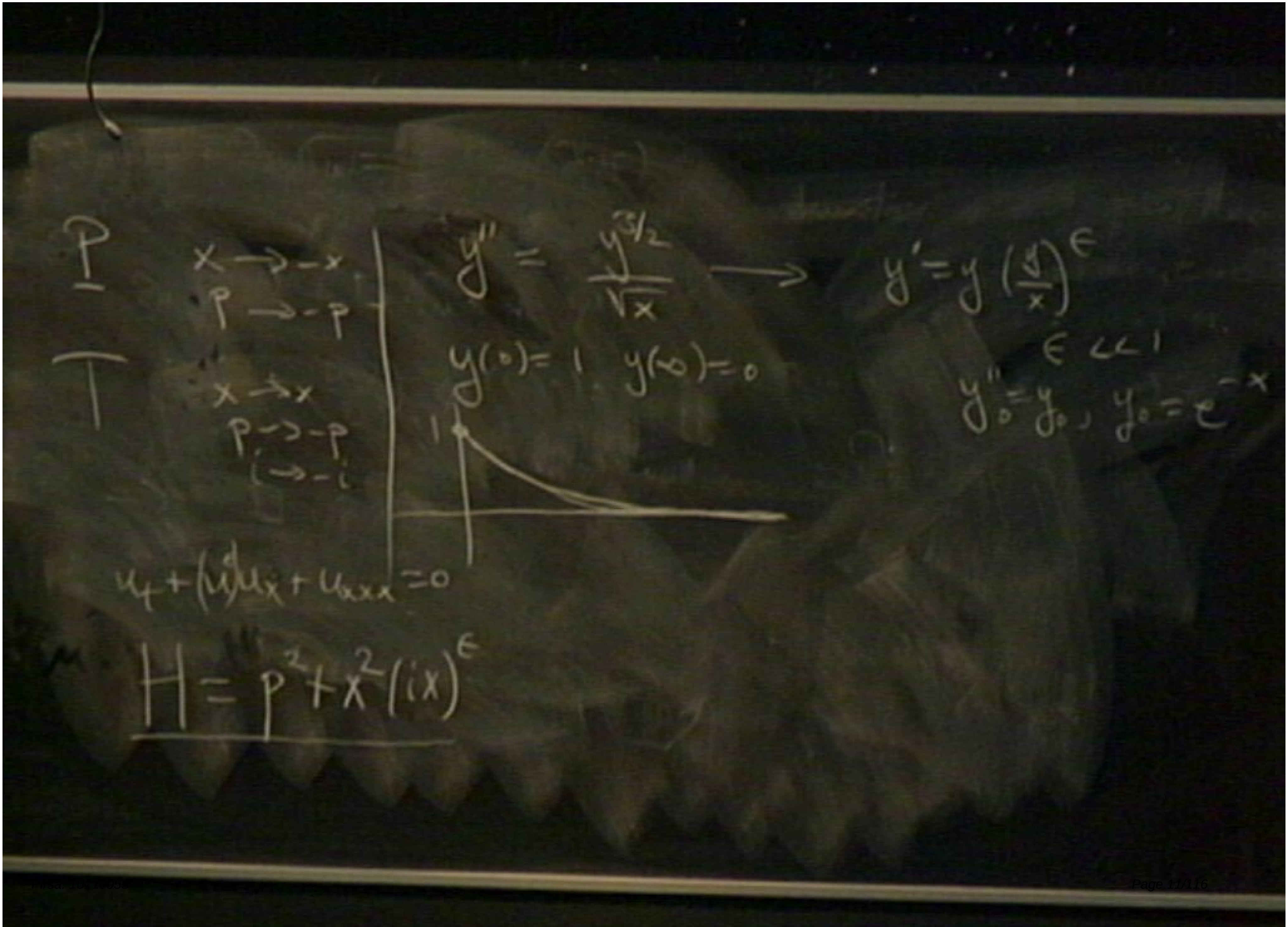
$$H = p^2 + ix^3$$

Wait a minute…
this Hamiltonian has
PT symmetry!

P = parity

T = time reversal

[redacted]


P
x → −x
p → −p
T
x → x
p → −p
i → −i
y'' = y^{3/2}/√x
y(0) = 1 y(∞) = 0
y' = y (y/x)^ε
ε << 1
y''_0 = y_0 , y_0 = e^{−x}
1
u_t + (u)^d u_x + u_xxx = 0
H = p^2 + x^2 (ix)^ε

$$H = p^2 + ix^3$$

Wait a minute…
this Hamiltonian has
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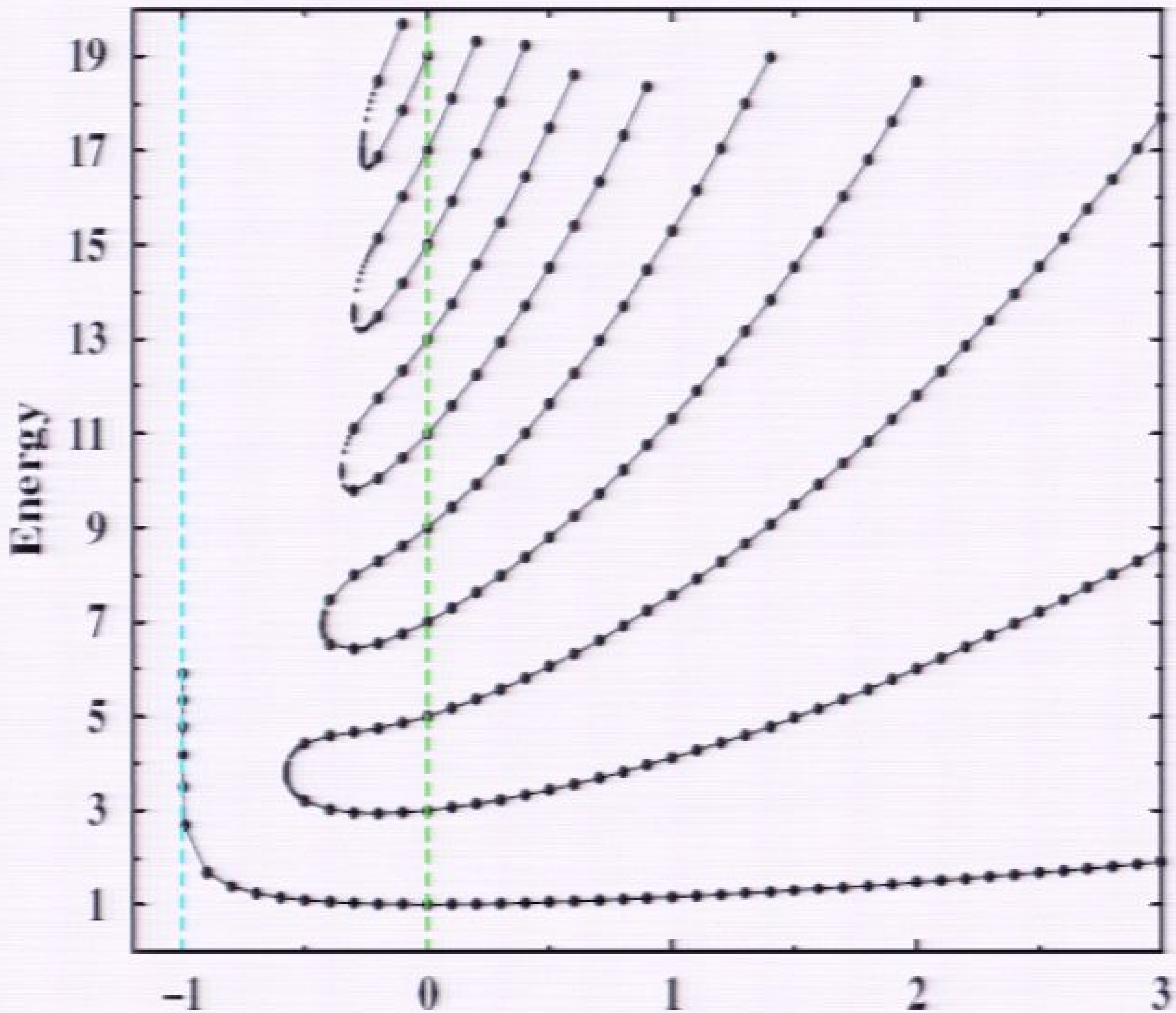
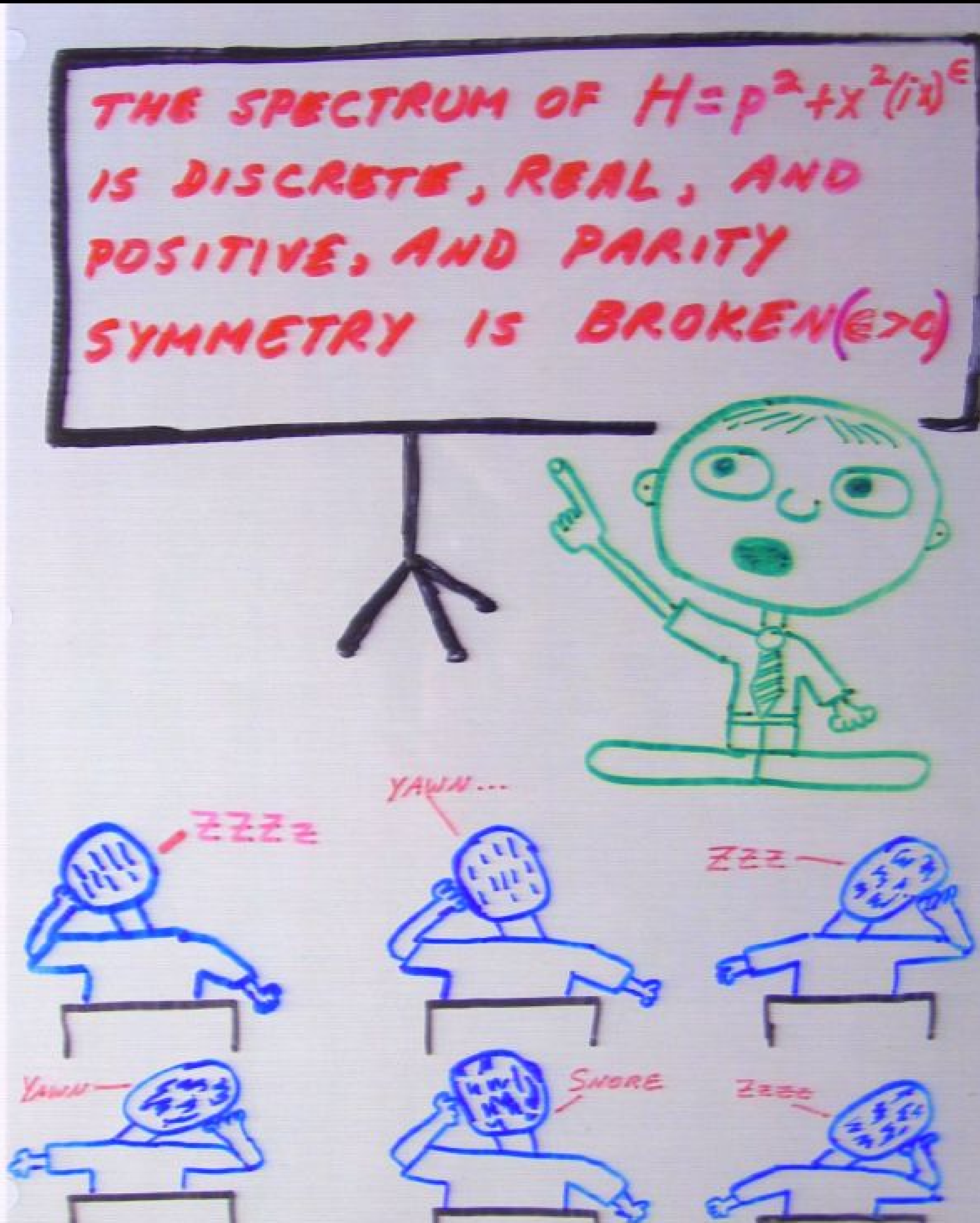
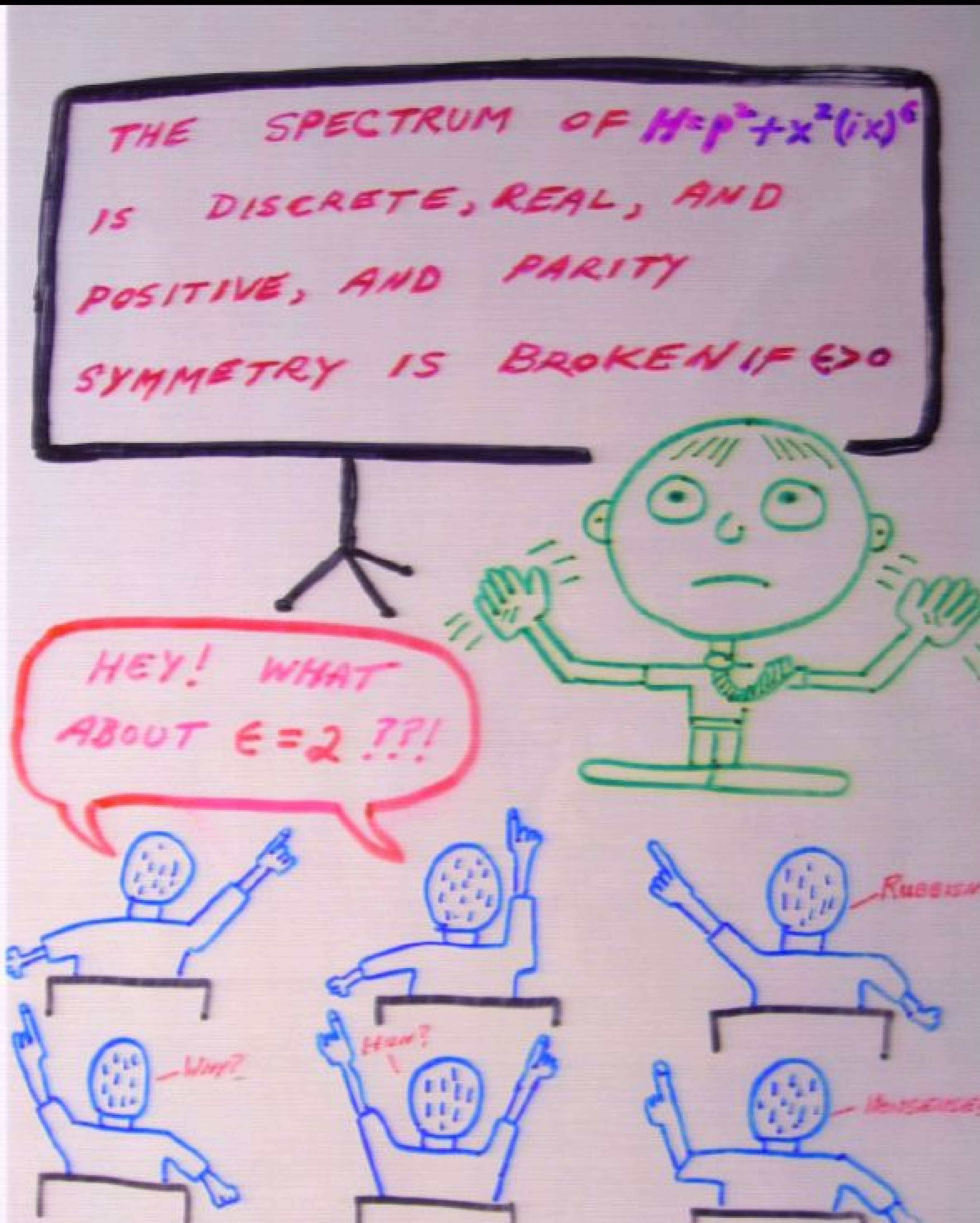
P = parity

T = time reversal

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$




THE SPECTRUM OF $H=p^2+x^2(ix)^\epsilon$
IS DISCRETE, REAL, AND
POSITIVE, AND PARITY
SYMMETRY IS BROKEN ($\epsilon>0$)
YAWN...
ZZZZ
ZZZ
YAWN
SNORE
Zzzz


THE SPECTRUM OF $H=p^2+x^2(ix)^\epsilon$
IS DISCRETE, REAL, AND
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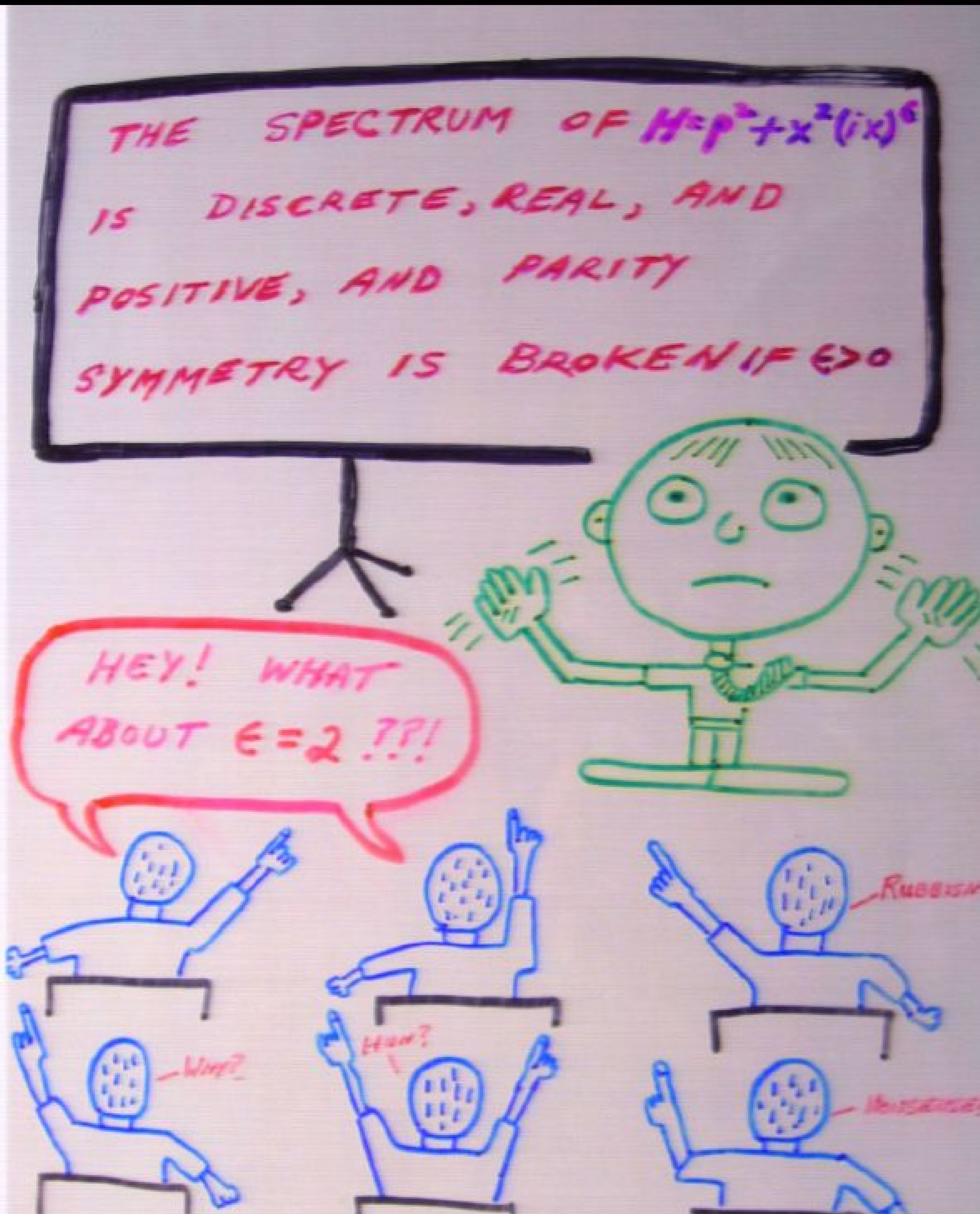
SYMMETRY IS BROKEN IF $\epsilon>0$
HEY! WHAT ABOUT $\epsilon=2$??!
RUBBISH

[redacted]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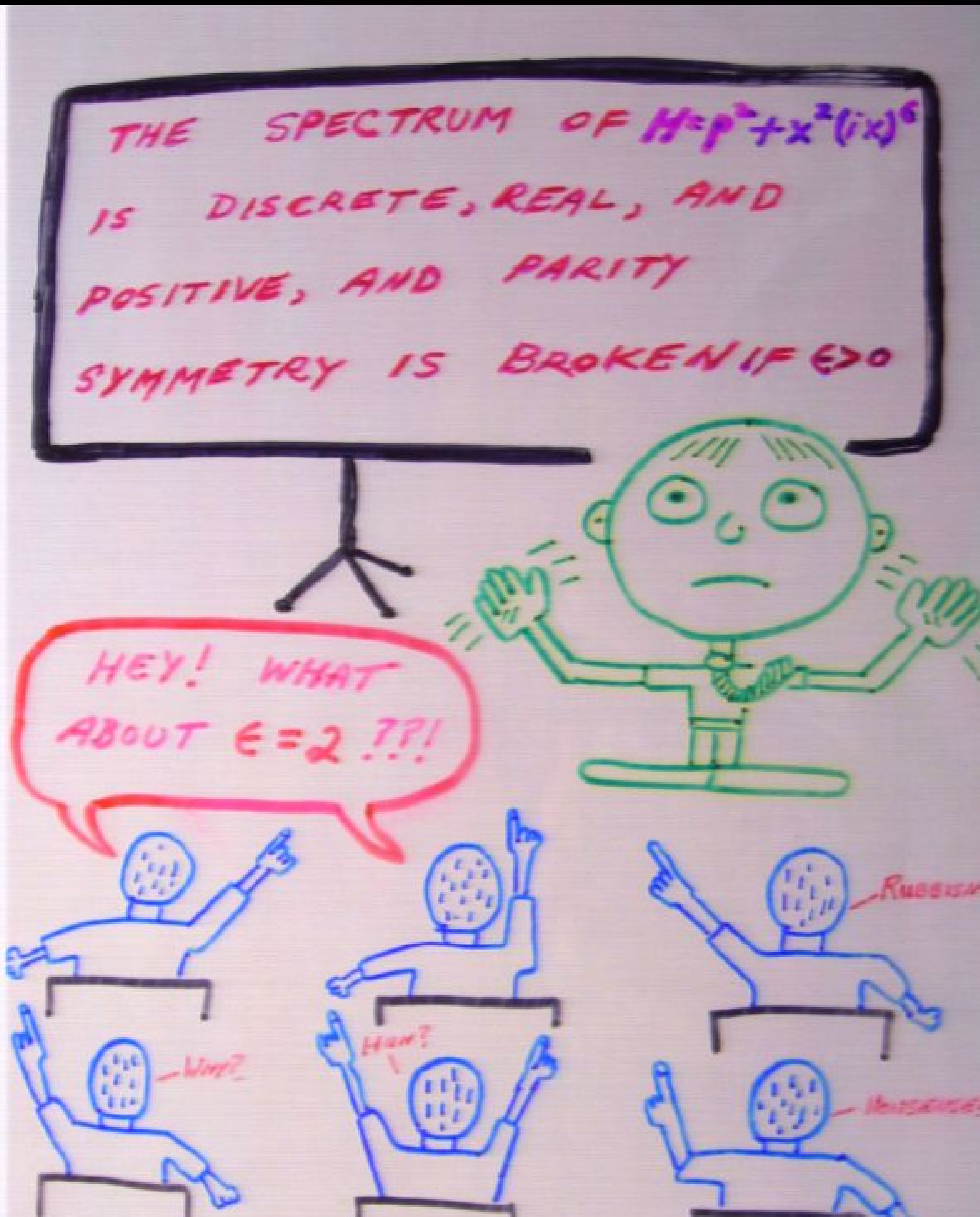
HEY! WHAT ABOUT $\epsilon=2$???!
RUBBISH
WHY?
HUH?

[redacted]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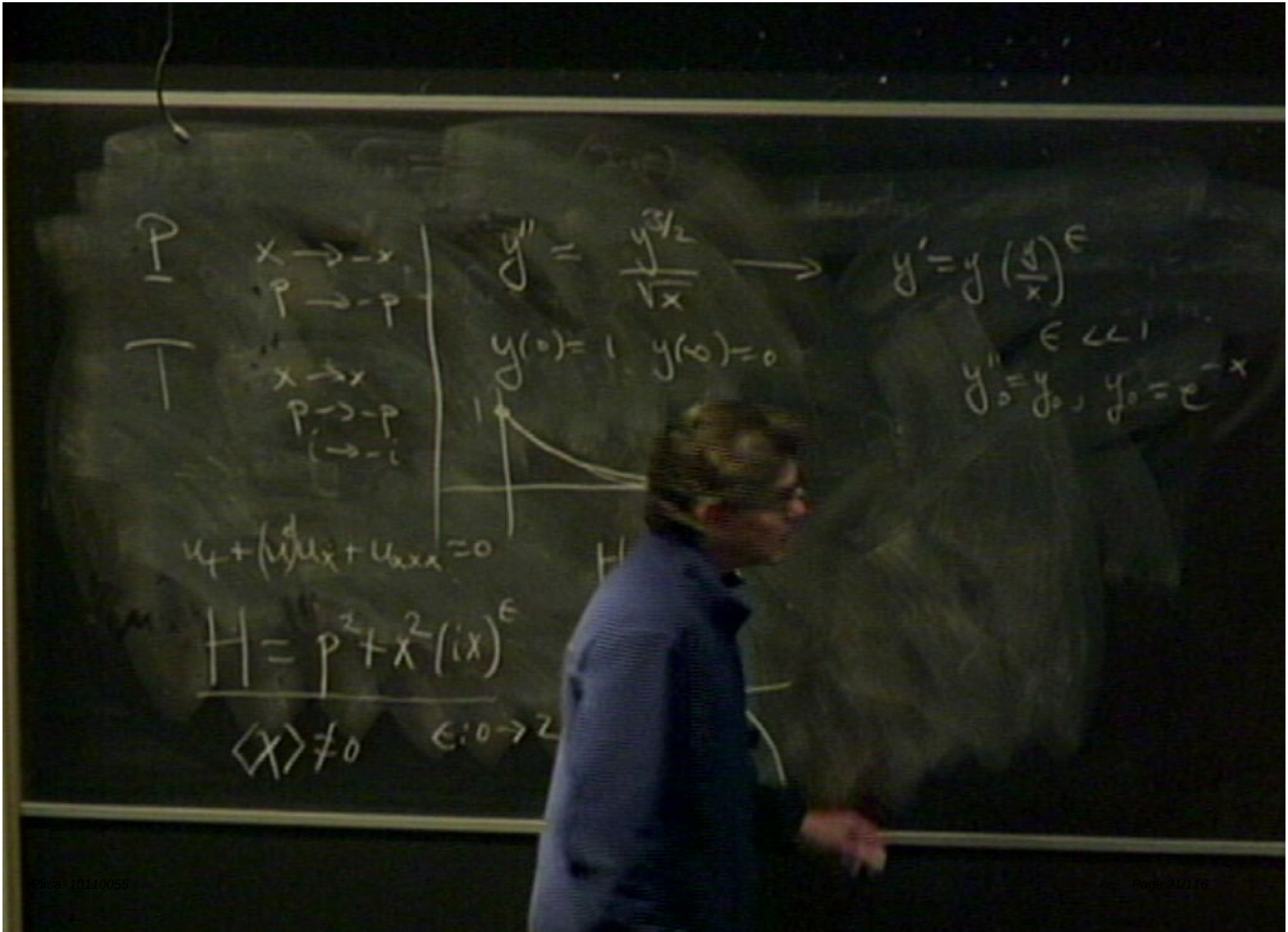
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T
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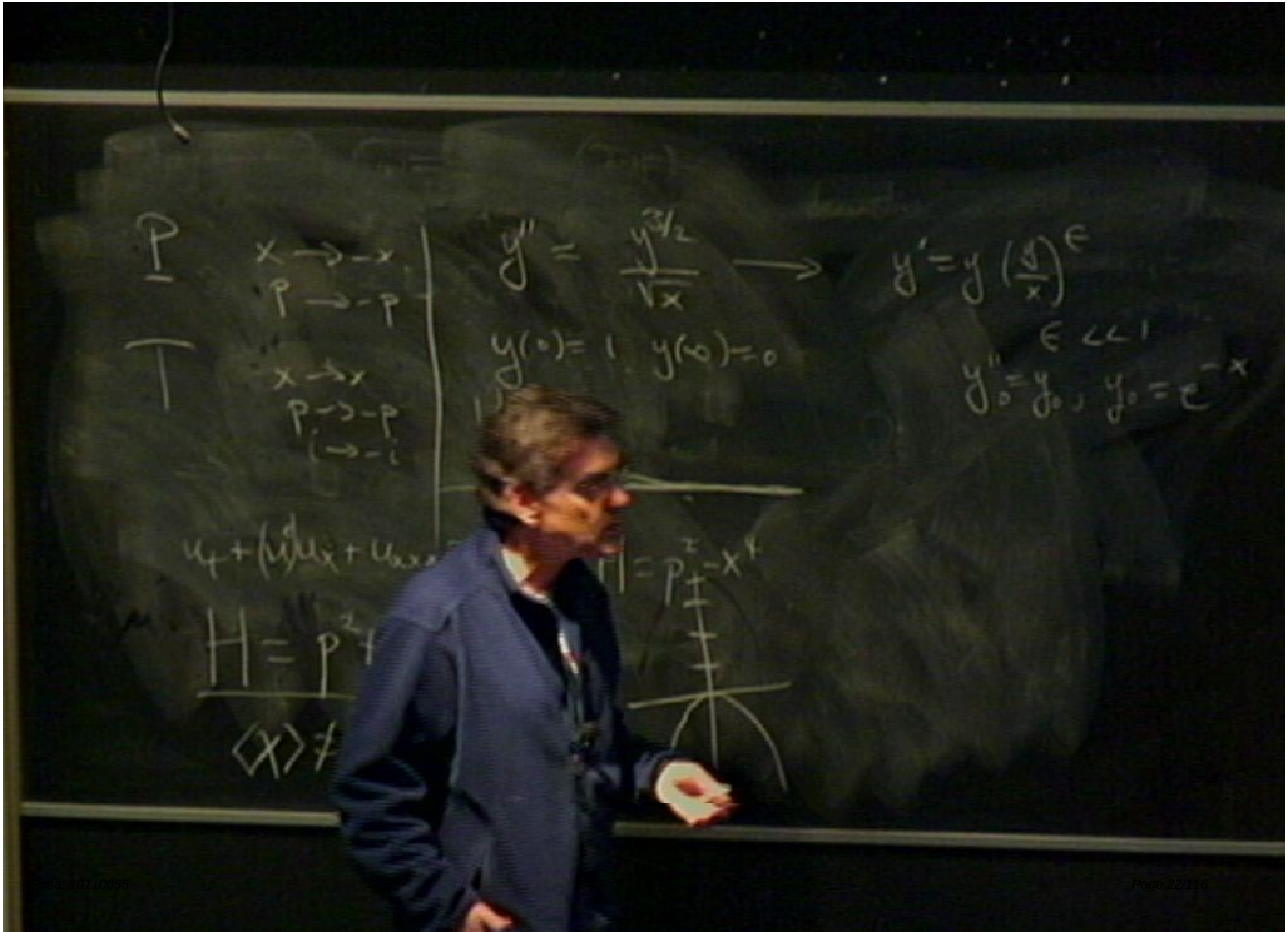
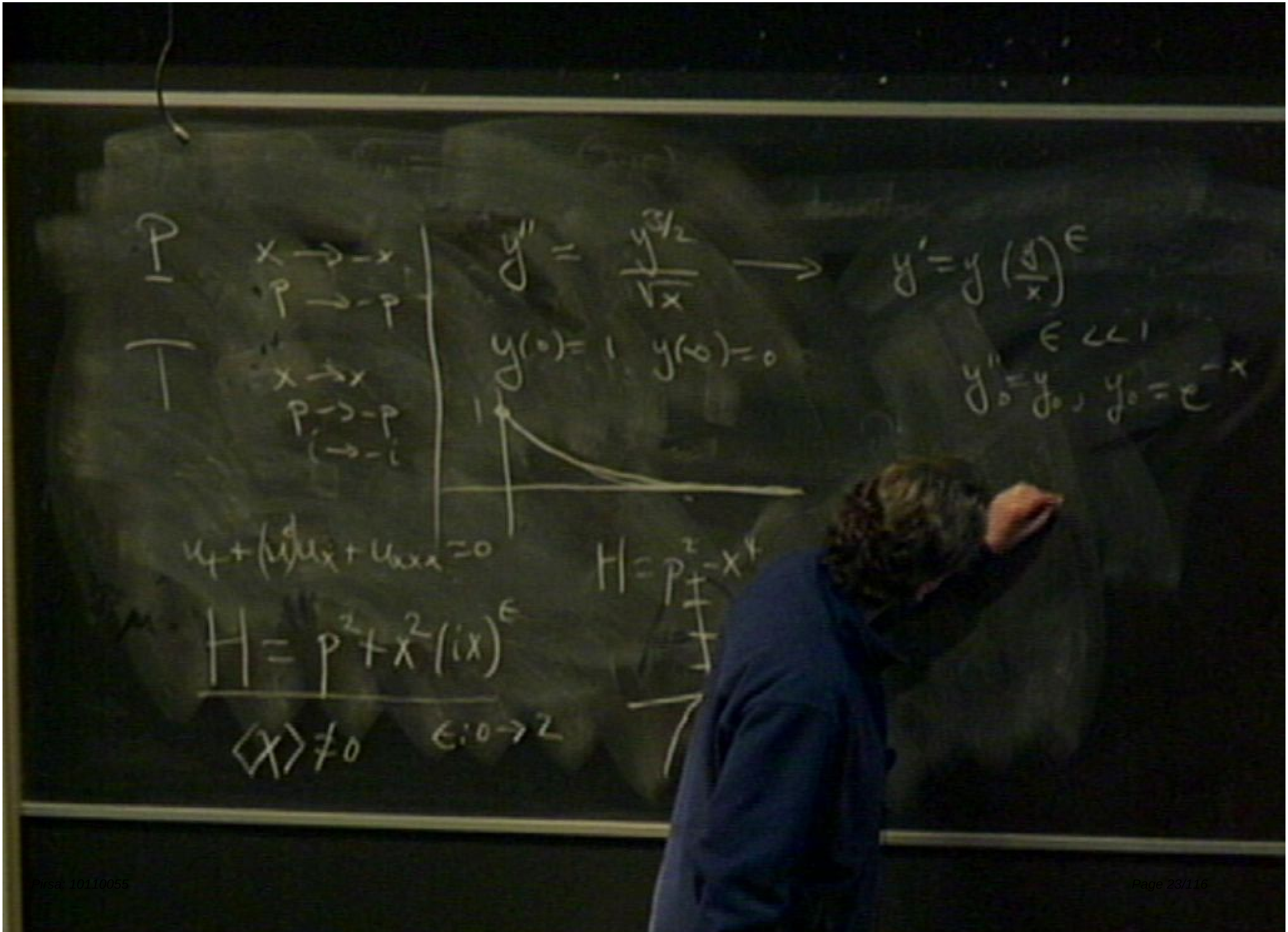
y(0) = 1 y(∞) = 0
y' = y (y/x)^ε
ε << 1
y''₀ = y₀ , y₀ = e^{−x}
u_t + (u)^ε u_x + u_xxx = 0
H = p² + x²(ix)^ε
⟨x⟩ ≠ 0
ε: 0 → 2


P
x → -x
p → -p
T
x → x
p → -p
i → -i
y'' = y^3/2 / √x
y(0) = 1   y(∞) = 0
y'' = y (y/x)^ε
ε << 1
y0'' = y0 , y0 = e^-x
H = p² - x⁴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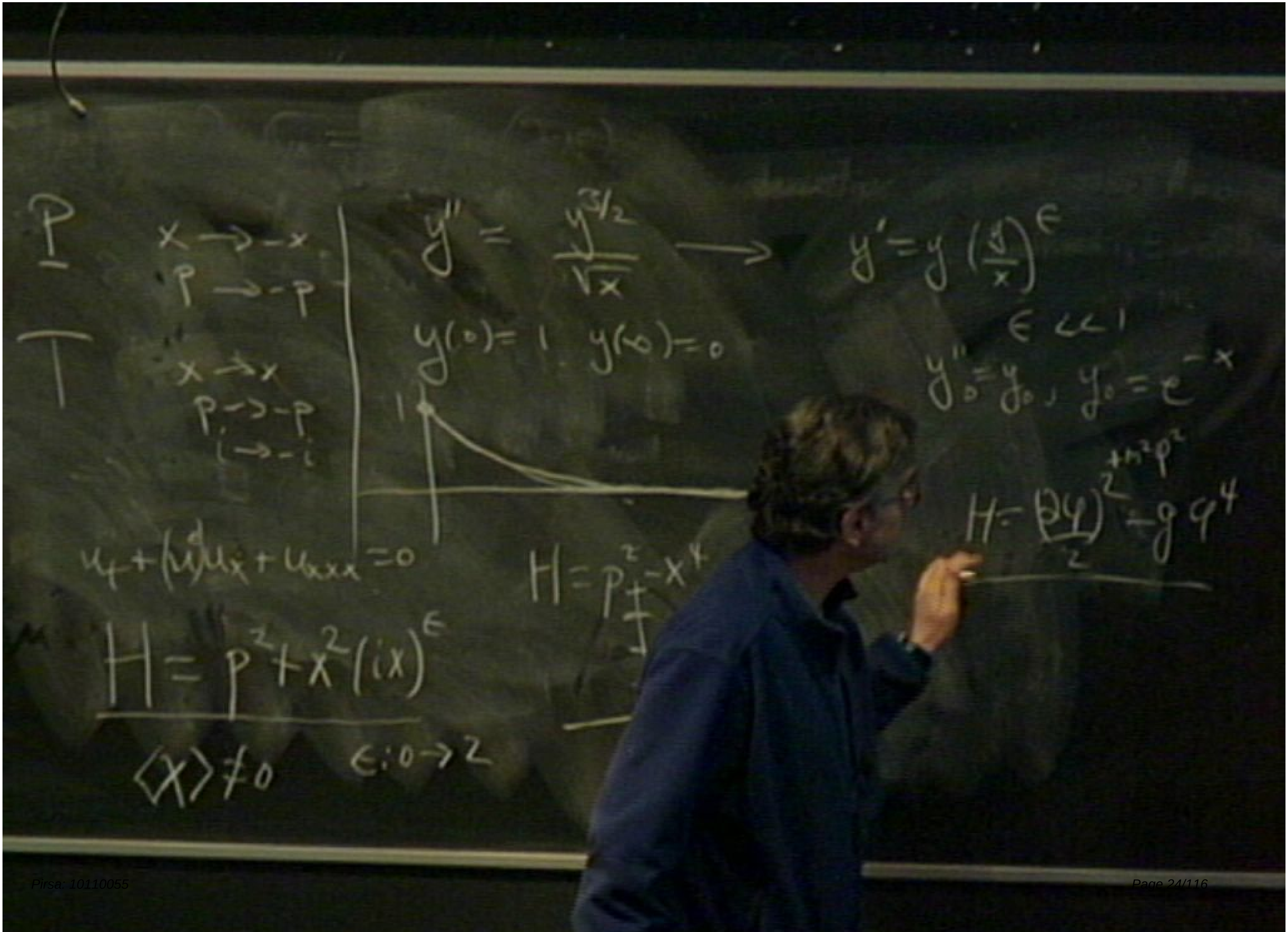
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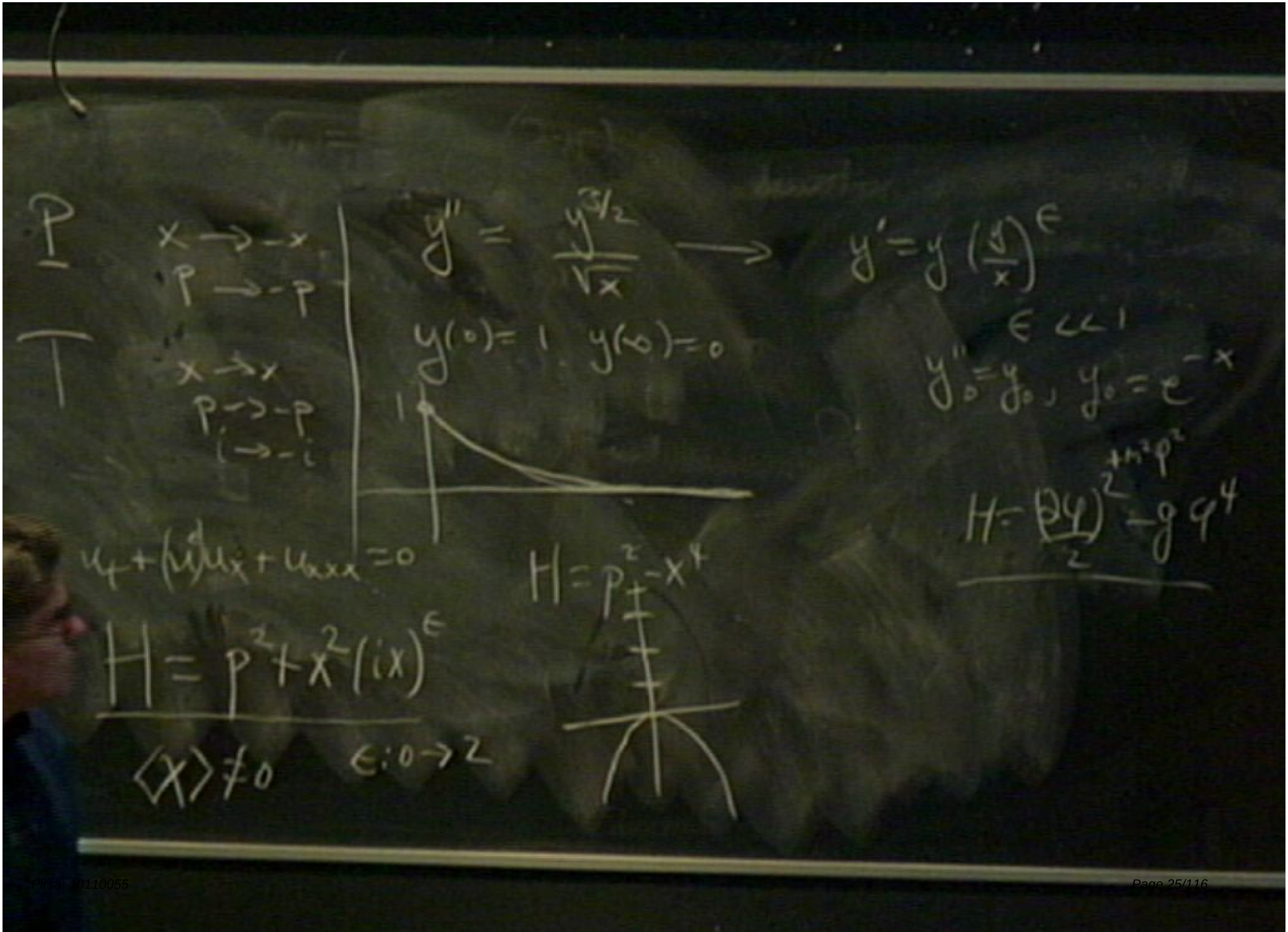
<x> ≠ 0
ε : 0 → 2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x → x
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y'' = y^{3/2} / √x
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y'' = y (y/x)^ε
ε << 1
y''₀ = y₀ , y₀ = e^{-x}
H = (∂φ)²/2 + m²φ² - g φ⁴
u_t + (iu)^ε u_x + u_xxx = 0
H = p² - x⁴
H = p² + x²(ix)^ε
⟨x⟩ ≠ 0
ε: 0 → 2

Some references …

- CMB and S. Boettcher, *Physical Review Letters* **80**, 5243 (1998)
- CMB, D. Brody, H. Jones, *Physical Review Letters* **89**, 270401 (2002)
- CMB, D. Brody, and H. Jones, *Physical Review Letters* **93**, 251601 (2004)
- CMB, D. Brody, H. Jones, B. Meister, *Physical Review Letters* **98**, 040403 (2007)
- CMB and P. Mannheim, *Physical Review Letters* **100**, 110402 (2008)
- CMB, D. Hook, P. Meisinger, Q. Wang, *Physical Review Letters* **104**, 061601 (2010)
- CMB and S.Klevansky, *Physical Review Letters* **105**, 031602 (2010)

- CMB, *Reports on Progress in Physics* **70**, 947 (2007)
- P. Dorey, C. Dunning, and R. Tateo, *Journal of Physics A* **34**, 5679 (2001)
- P. Dorey, C. Dunning, and R. Tateo, *Journal of Physics A* **40**, R205 (2007)

Some references ...

- CMB and S. Boettcher, *Physical Review Letters* **80**, 5243 (1998)
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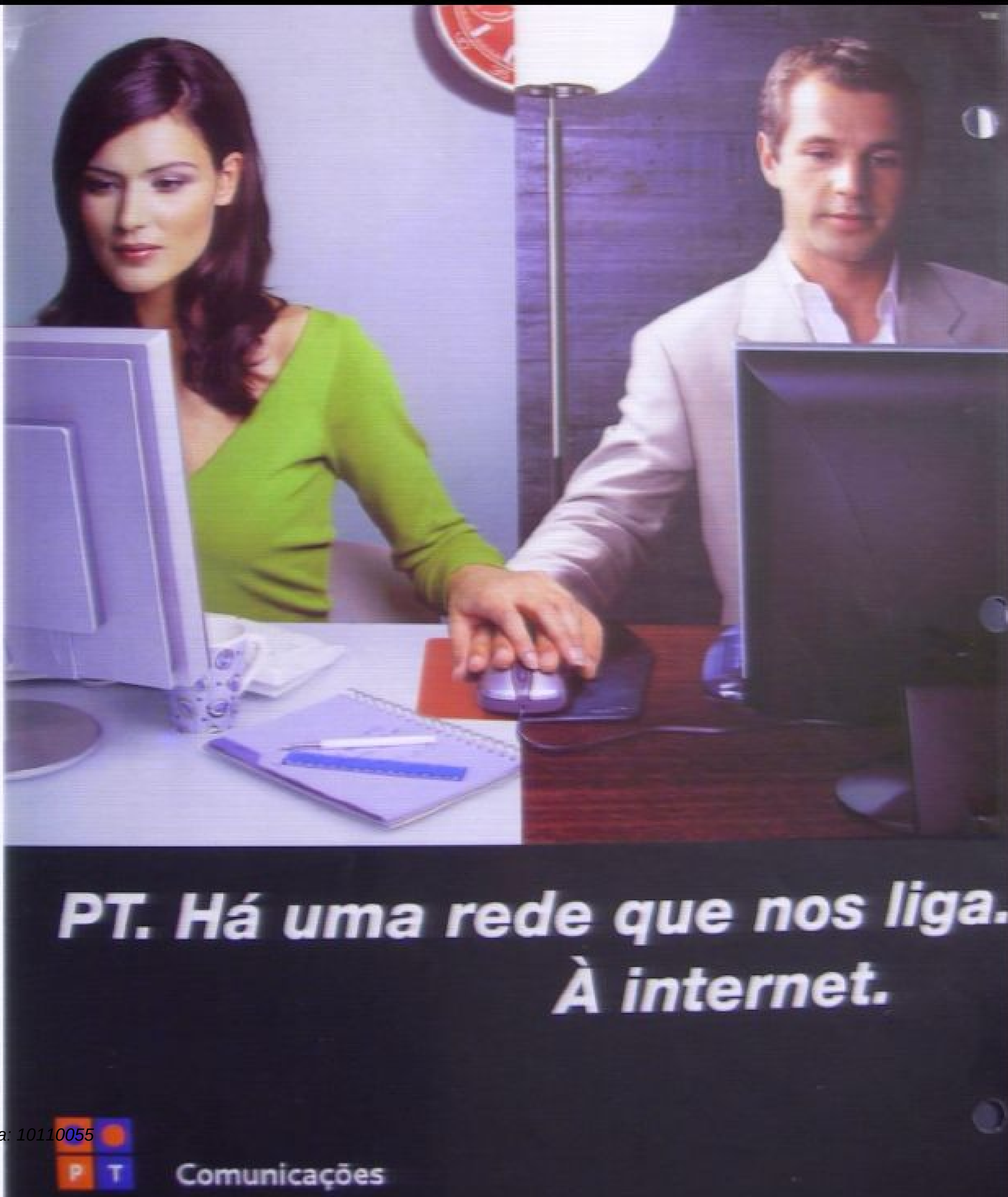
- P. Dorey, C. Dunning, and R. Tateo, *Journal of Physics A* **40**, R205 (2007)

Other recent PT papers …

- K. Makris, R. El-Ganainy, D. Christodoulides, and Z. Musslimani, *Phyical Review Letters* **100**, 103904 (2008)
- Z. Musslimani, K. Makris, R. El-Ganainy, and D. Christodoulides, *Physical Review Letters* **100**, 030402 (2008)
- U. Günther and B. Samsonov, *Physical Review Letters* **101**, 230404 (2008)
- E. Graefe, H. Korsch, and A. Niederle, *Physical Review Letters* **101**, 150408 (2008)
- S. Klaiman, U. Günther, and N. Moiseyev, *Physical Review Letters* **101**, 080402 (2008)

- U. Jentschura, A. Surzhykov, and J. Zinn-Justin, *Physical Review Letters* **102**, 011601 (2009)
- A. Mostafazadeh, *Physical Review Letters* **102**, 220402 (2009)
- O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, *Physical Review Letters* **103**, 030402 (2009)
- S. Longhi, *Physical Review Letters* **103**, 123601 (2009)
- A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, *Physical Review Letters* **103**, 093902 (2009)

- H. Schomerus, *Physical Review Letters* **104**, 233601 (2010)
- S. Longhi, *Physical Review Letters* **105**, 013903 (2010)
- C. West, T. Kottos, T. Prosen, *Physical Review Letters* **104**, 054102 (2010)
- S. Longhi, *Physical Review Letters* **105**, 013903 (2010)
- T. Kottos, *Nature Physics* **6**, 166 (2010)
- C. Ruter, K. Makris, R. El-Ganainy, D. Christodoulides, M. Segev, and D. Kip, *Nature Physics* **6**, 192 (2010)



Translation:

PT. There is a network that ties us together.

The original discoverers of **PT** symmetry:

"It's only performance art, you know. Rhetoric. They used to teach it in ancient times, like PT."
--- *Arcadia*, Tom Stoppard


JONES
BERRY
MEISINGER
MILTON
BOETTCHER
The PT BUNCH
DUNNE
SAVAGE
WANG


PT
Posteitaliane
POSTAMAT
SE VOCÊ
PT
QUERO FALAR
COM VOCÊ.


PALAZZO
PITTI
2
INGRESSO PRINCIPALE
MAIN ENTRANCE
1
1 BIGLIETTERIA
TICKETS
2 INGRESSO MUSEI
MUSEUMS ENTRANCE
PT


CRUISER
PT
SE VOCÊ

PT in China

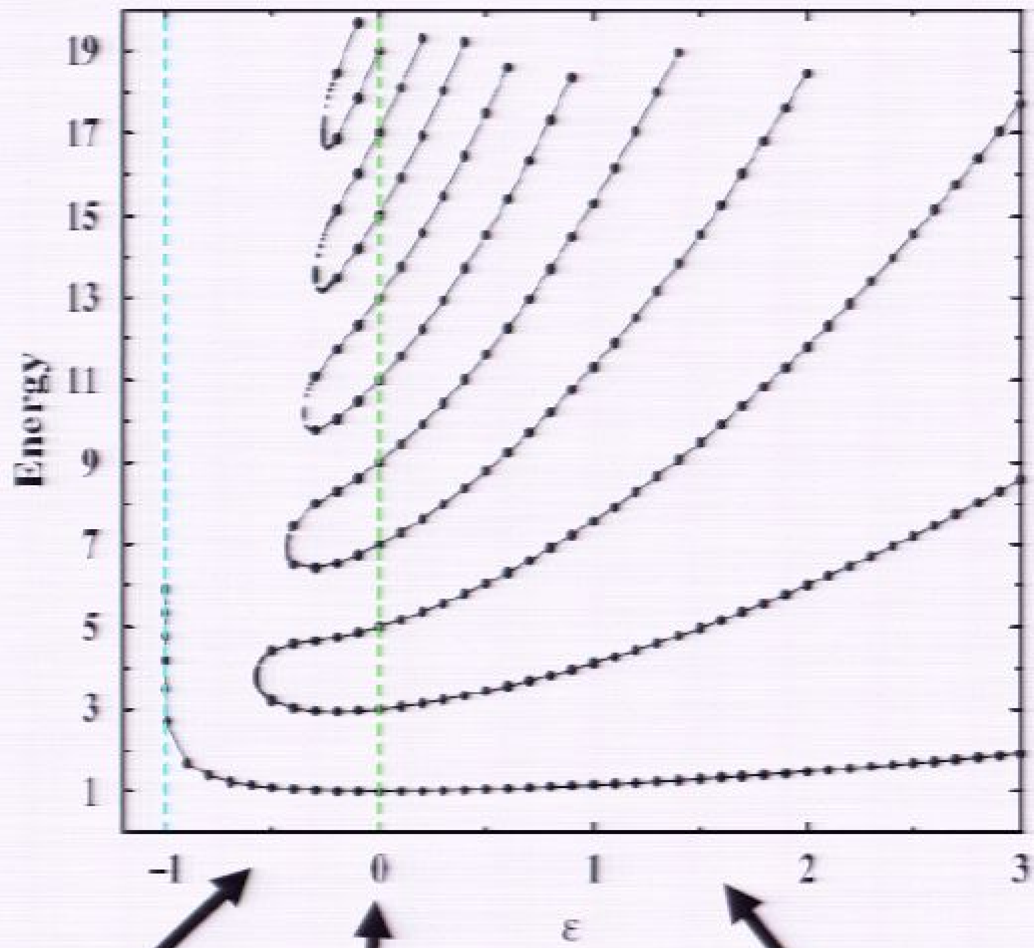

60m
上车点
Get On Here 차를 타는 곳
上り停留所 สถานที่ขึ้นรถ
厕所
Toilets 화장실
トイレ ห้องน้ำ

How to prove that the eigenvalues are real

The proof is difficult! It uses techniques from conformal field theory and statistical mechanics:

(1) Bethe ansatz
(2) Monodromy group
(3) Baxter T-Q relation
(4) Functional Determinants

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$



Region of ***broken***
PT symmetry

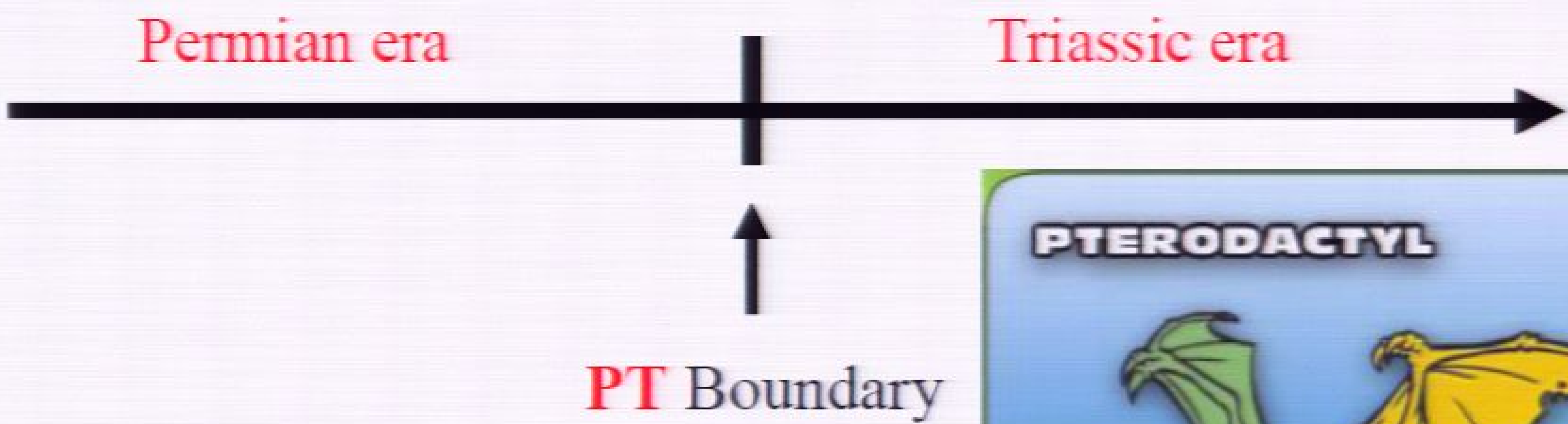
PT Boundary

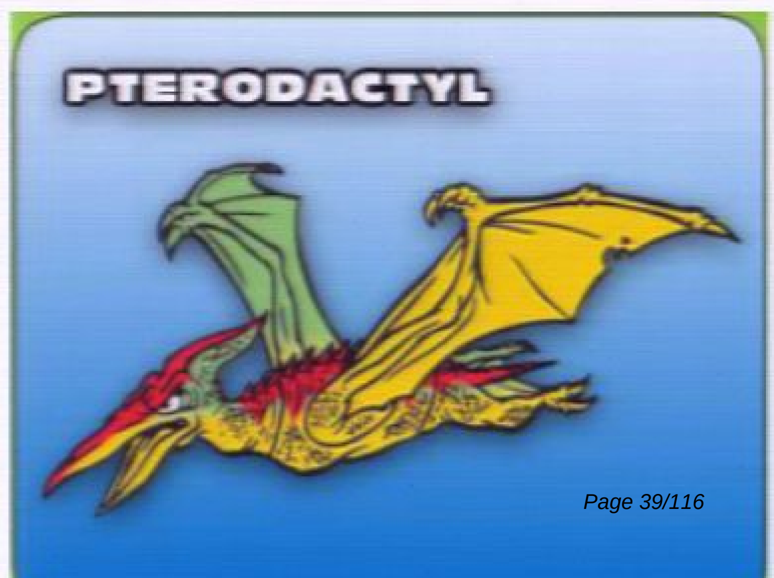
Region of ***unbroken***
PT symmetry

Broken PT symmetry in Paris

PT Boundary

Greatest murder mystery of all time…
Extinction of over 90% of species!





The PT Boundary is a phase transition – at the *classical* level

(To be explained at the end of this talk if there's time!)

OK, so the eigenvalues are real …
But is this quantum mechanics??

- Probabilistic interpretation??
- Hilbert space with a positive metric??
- Unitarity??

P. A. M. Dirac: Bakerian Lecture,

Proceedings of the Royal Society A (1941)

Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative. Thus negative energies and probabilities should be considered simply as things which do not appear in experimental results. The physical interpretation of relativistic quantum mechanics that one gets by a natural development of the non-relativistic theory involves these things and is thus in contradiction with experiment. We therefore have to consider ways of modifying or supplementing this interpretation.

The Hamiltonian determines its own adjoint

$$[\mathcal{C}, \mathcal{PT}] = 0,$$

$$[\mathcal{C}^2 = 1],$$

$$[\mathcal{C}, H] = 0$$

Replace $\dagger$ by $\mathcal{CPT}$

Unitarity

With respect to the **CPT** adjoint the theory has UNITARY time evolution.

Norms are strictly positive!
Probability is conserved!


cpt
LAM
Semplice
Comoda
Rapida
Frequente
80
100
cpt 3202
AG 119 ZV

OK, we have unitarity…
But is PT quantum mechanics useful??

- Revives quantum theories that were thought to be dead
- Beginning to be observed experimentally

Lee Model

$$V \to N + \theta, \qquad N + \theta \to V.$$

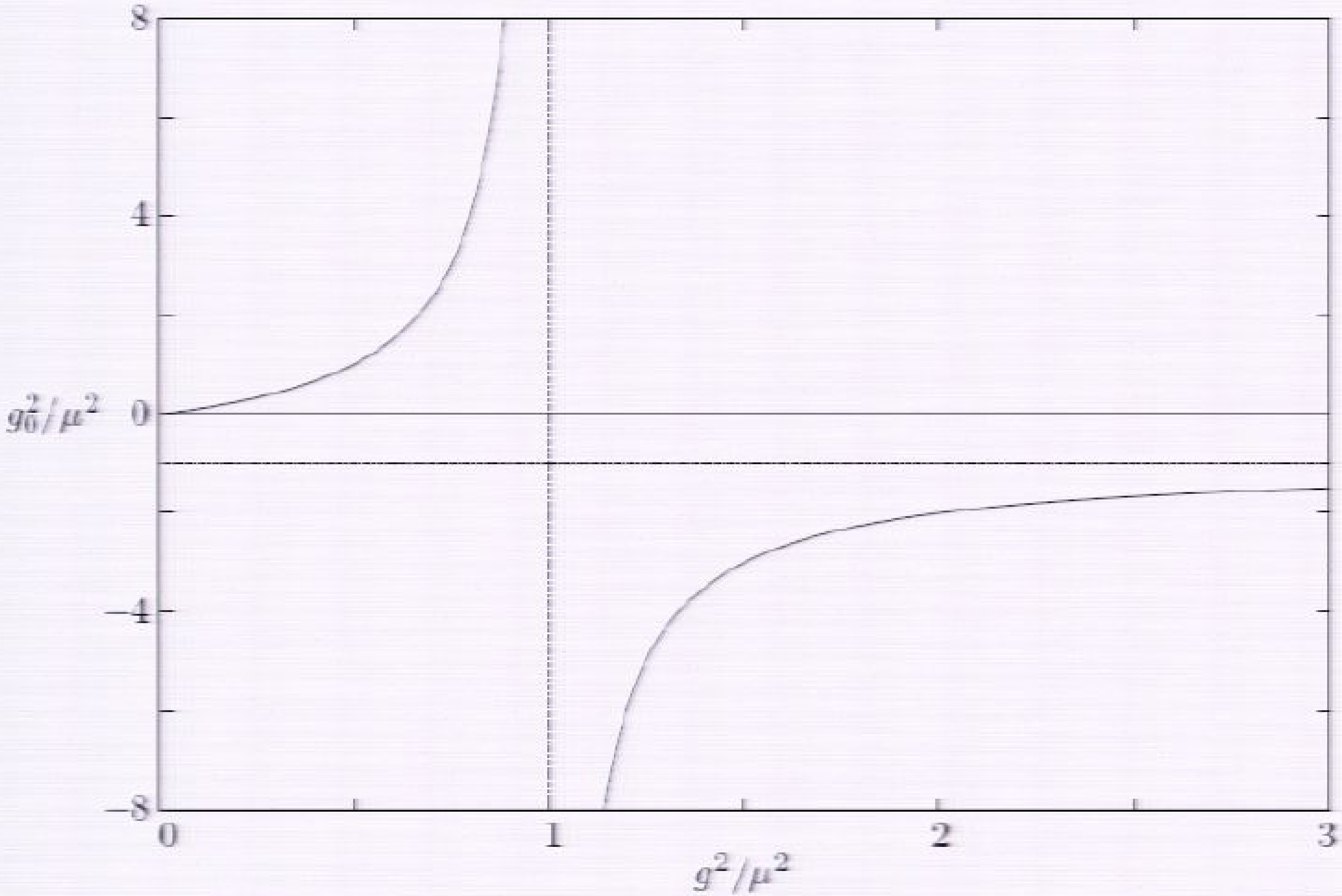
$$H = H_0 + g_0 H_1,$$

$$H_0 = m_{V_0} V^\dagger V + m_N N^\dagger N + m_\theta a^\dagger a,$$
$$H_1 = V^\dagger N a + a^\dagger N^\dagger V.$$

T. D. Lee, Phys. Rev. **95**, 1329 (1954)

G. Källén and W. Pauli, Dan. Mat. Fys. Medd. **30**, No. 7 (1955)

The problem with the Lee Model:



$$g_0^2 = g^2 / \left(1 - g^2/\mu^2\right)$$

Lee Model

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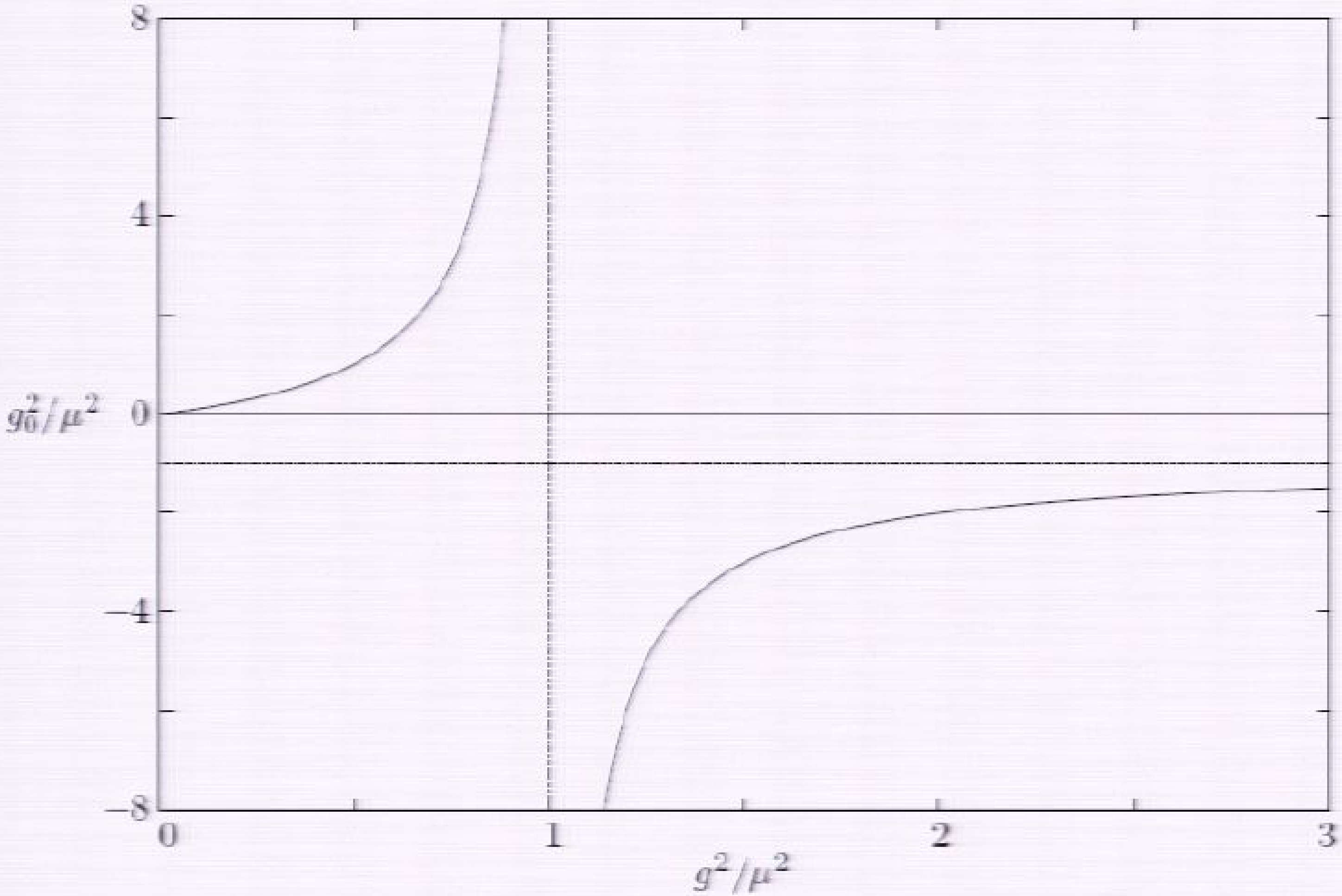


$$g_0^2 = g^2 / \left(1 - g^2/\mu^2\right)$$

MR0076639 (17,927d) 81.0X

Källén, G.; Pauli, W.

On the mathematical structure of T. D. Lee's model of a renormalizable field theory.

Danske Vid. Selsk. Mat.-Fys. Medd. **30** (1955), *no. 7, 23 pp.*

Lee [Phys. Rev. (2) **95** (1954), 1329–1334; MR0064658 (16,317b)] has recently suggested perhaps the first non-trivial model of a field-theory which can be explicitly solved. Three particles (V, N and θ) are coupled, the explicit solution being secured by allowing reactions $V \rightleftarrows N + \theta$ but forbidding $N \rightleftarrows V + \theta$. The theory needs conventional mass and charge renormalizations which likewise can be explicitly calculated. The renormalized coupling constant g is connected to the unrenormalized constant g_0 by the relation $g^2/g_0{}^2 = 1 - Ag^2$, where A is a divergent integral. This can be made finite by a introducing a cut-off.

The importance of Lee's result lies in the fact that Schwinger (unpublished) had already proved on very general principles, that the ratio $g^2/g_0{}^2$ should lie between zero and one. [For published proofs of Schwinger's result, see Umezawa and Kamefuchi, Progr. Theoret. Phys. **6** (1951), 543–558; MR0046306 (13,713d); Källén, Helv. Phys. Acta **25** (1952), 417–434; MR0051156 (14,435l); Lehmann, Nuovo Cimento (9) **11** (1954), 342–357; MR0072756 (17,332e); Gell-Mann and Low, Phys. Rev. (2) **95** (1954), 1300–1312; MR0064652 (16,315e)]. The results of Lee and Schwinger can be reconciled only if (i) there is a cut-off in Lee's theory and (ii) if g lies below a critical value g_{crit}. The present paper is devoted to investigation of physical consequences if these two conditions are not satisfied.

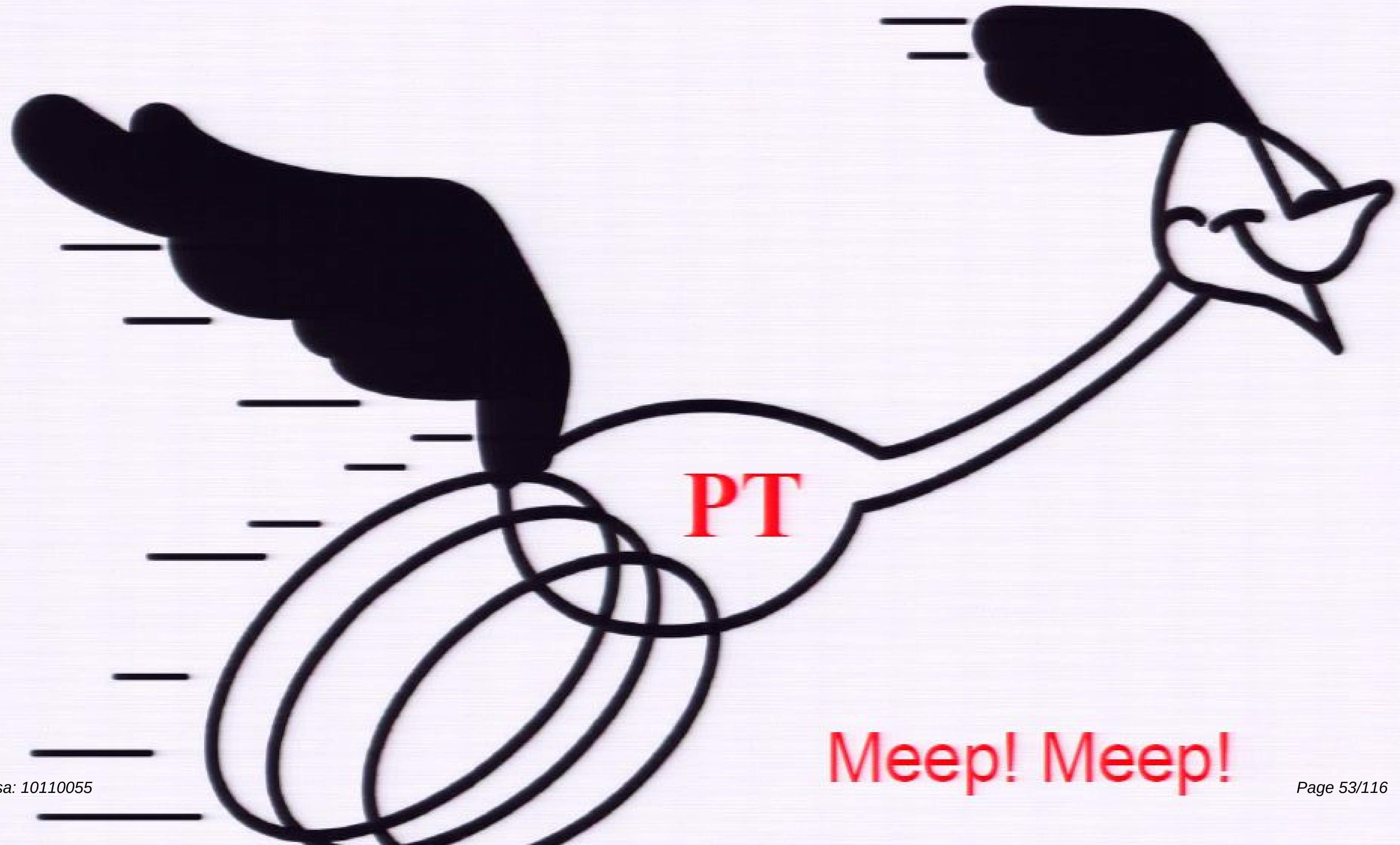
The authors discover the remarkable result that if $g > g_{\text{crit}}$ there is exactly one new eigenstate for the physical V-particle having an energy that is below the mass of the normal V-particle. It is further shown that the S-matrix for Lee's theory is not unitary when $g > g_{\text{crit}}$ and that the probability for an incoming V-particle in the normal state and a θ-meson, to make a transition to an outgoing V-particle in the new ("ghost") state, must be negative if the sum of all transition probabilities for the in-coming state shall add up to one. The possible implication of Källén and Pauli's results for quantum-electrodynamics, where in perturbation theory $(e/e_0)^2$ has a behaviour similar to $(g/g_0)^2$ in Lee's theory, need not be stressed.

Reviewed by *A. Salam*

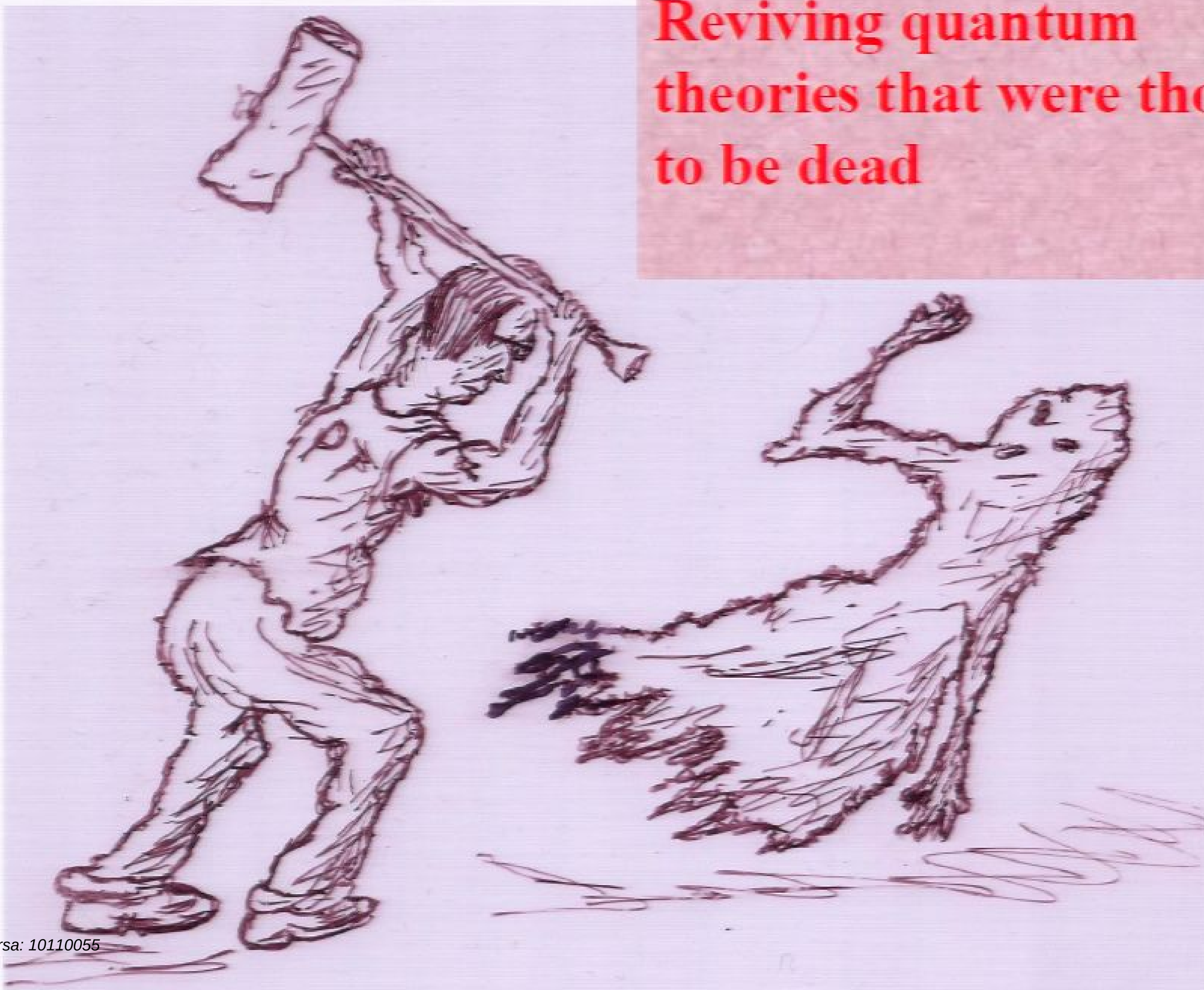
"A non-Hermitian Hamiltonian is unacceptable partly because it may lead to complex energy eigenvalues, but chiefly because it implies a non-unitary S matrix, which fails to conserve probability and makes a hash of the physical interpretation."

G. Barton, *Introduction to Advanced Field Theory* (John Wiley & Sons, New York, 1963)

PT quantum mechanics to the rescue…



GHOSTBUSTING: Reviving quantum theories that were thought to be dead



Pais-Uhlenbeck action

$$I = \frac{\gamma}{2} \int dt \left[\ddot{z}^2 - \left(\omega_1^2 + \omega_2^2\right) \dot{z}^2 + \omega_1^2 \omega_2^2 z^2 \right]$$

Gives a fourth-order field equation:

$$z''''(t) + (\omega_1^2 + \omega_2^2) z''(t) + \omega_1^2 \omega_2^2 z(t) = 0$$

The problem: A fourth-order field equation gives a propagator like

$$G(E) = \frac{1}{(E^2 + m_1^2)(E^2 + m_2^2)}$$

$$G(E) = \frac{1}{m_2^2 - m_1^2}\left(\frac{1}{E^2 + m_1^2} - \frac{1}{E^2 + m_2^2}\right)$$

↑

GHOST!

Two possible realizations...

(I) If a_1 and a_2 annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2|\Omega\rangle = 0,$$

then the energy spectrum is real and bounded below. The state $|\Omega\rangle$ is the ground state of the theory and it has zero-point energy $\frac{1}{2}(\omega_1 + \omega_2)$. The problem with this realization is that the excited state $a_2^\dagger|\Omega\rangle$, whose energy is ω_2 above ground state, has a *negative Dirac norm* given by $\langle\Omega|a_2 a_2^\dagger|\Omega\rangle$.

(II) If a_1 and $a_2^\dagger$ annihilate the 0-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle = 0, \qquad a_2^\dagger|\Omega\rangle = 0,$$

then the theory is free of negative-norm states. However, this realization has a different and equally serious problem; namely, that the energy spectrum is unbounded below.

There can be other realizations as well!

Calculate the equivalent Dirac Hermitian Hamiltonian:

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2$$

No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck model, CMB and P. Mannheim, *Physical Review Letters* **100**, 110402 (2008)

CMB and P. Mannheim, *Physical Review D* **78**, 025002 (2008)

Totalitarian principle

"Everything which is not forbidden is compulsory."

---M. Gell-Mann

Laboratory verification using table-top optics experiments!

Observing **PT** symmetry using optical wave guides:

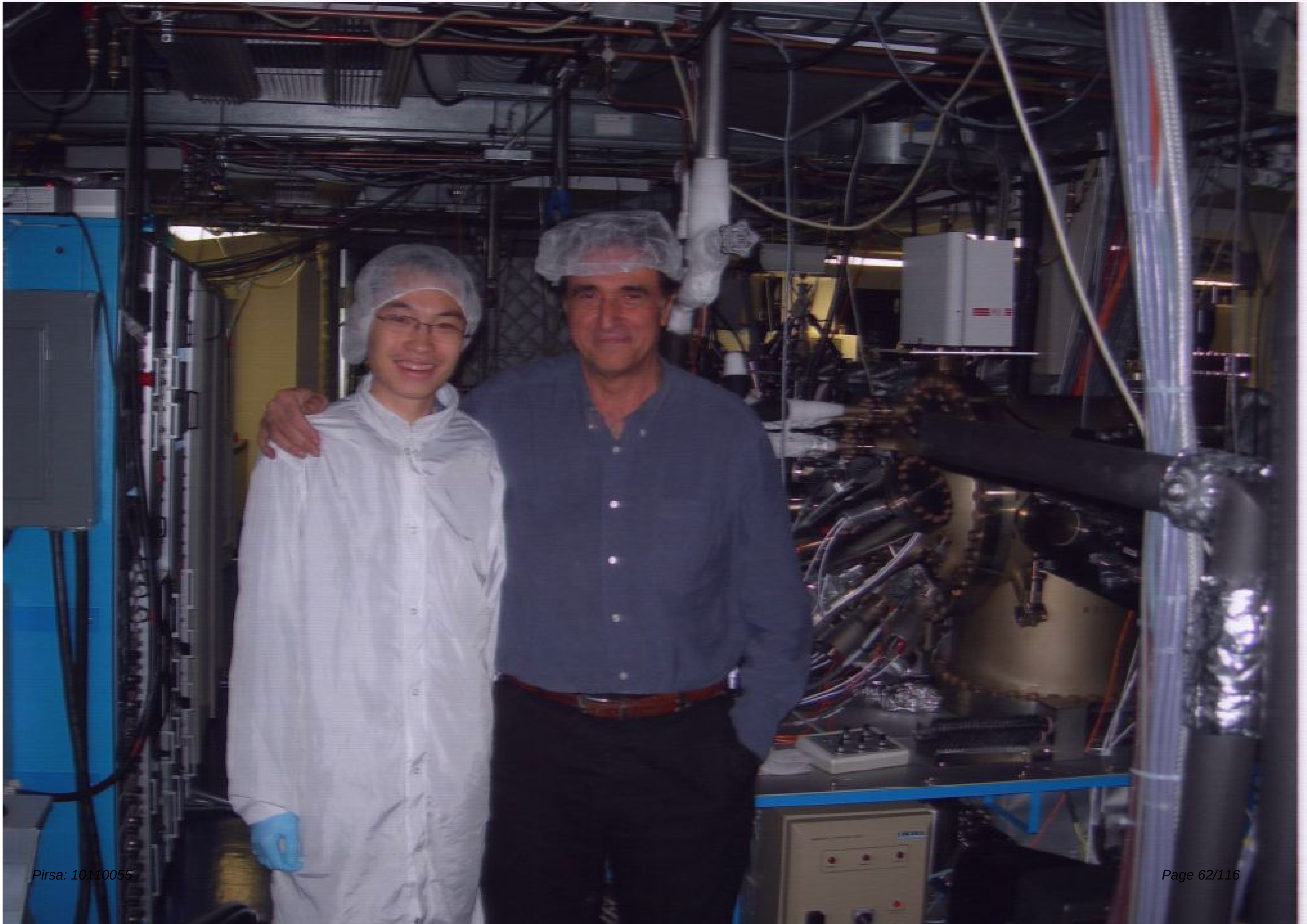
- Z. Musslimani, K. Makris, R. El-Ganainy, and D. Christodoulides, *Physical Review Letters* **100**, 030402 (2008)
- K. Makris, R. El-Ganainy, D. Christodoulides, and Z. Musslimani, *Physical Review Letters* **100**, 103904 (2008)
- A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, *Physical Review Letters* **103**, 093902 (2009)
- C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, *Nature Physics* **6**, 192 (2010)

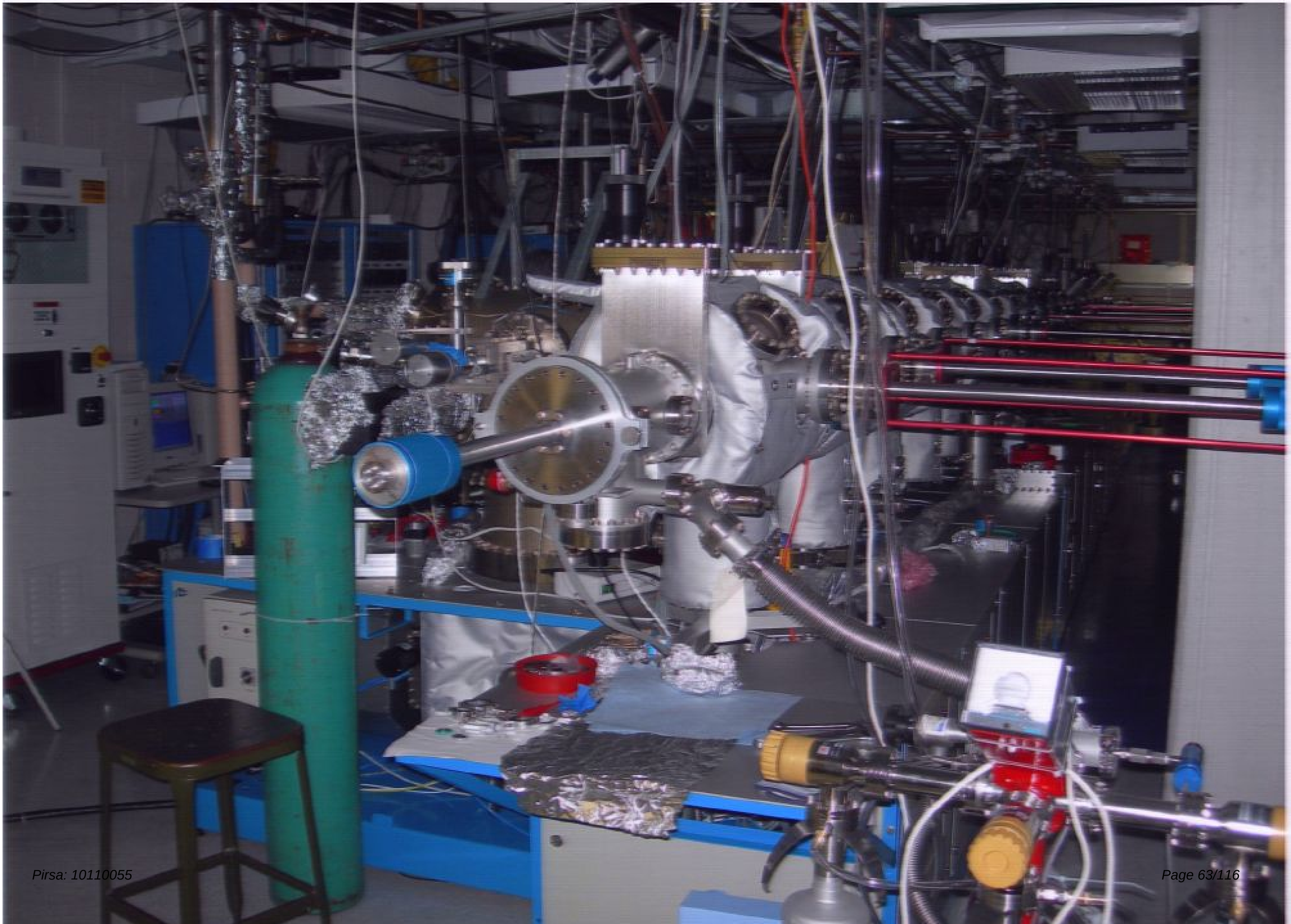
Date: Thu, 13 Mar 2008 23:04:45 -0400
From: Demetrios Christodoulides <demetri@creol.ucf.edu>
To: Carl M. Bender <cmb@wuphys.wustl.edu>
Subject: Re: Benasque workshop on non-Hermitian Hamiltonians

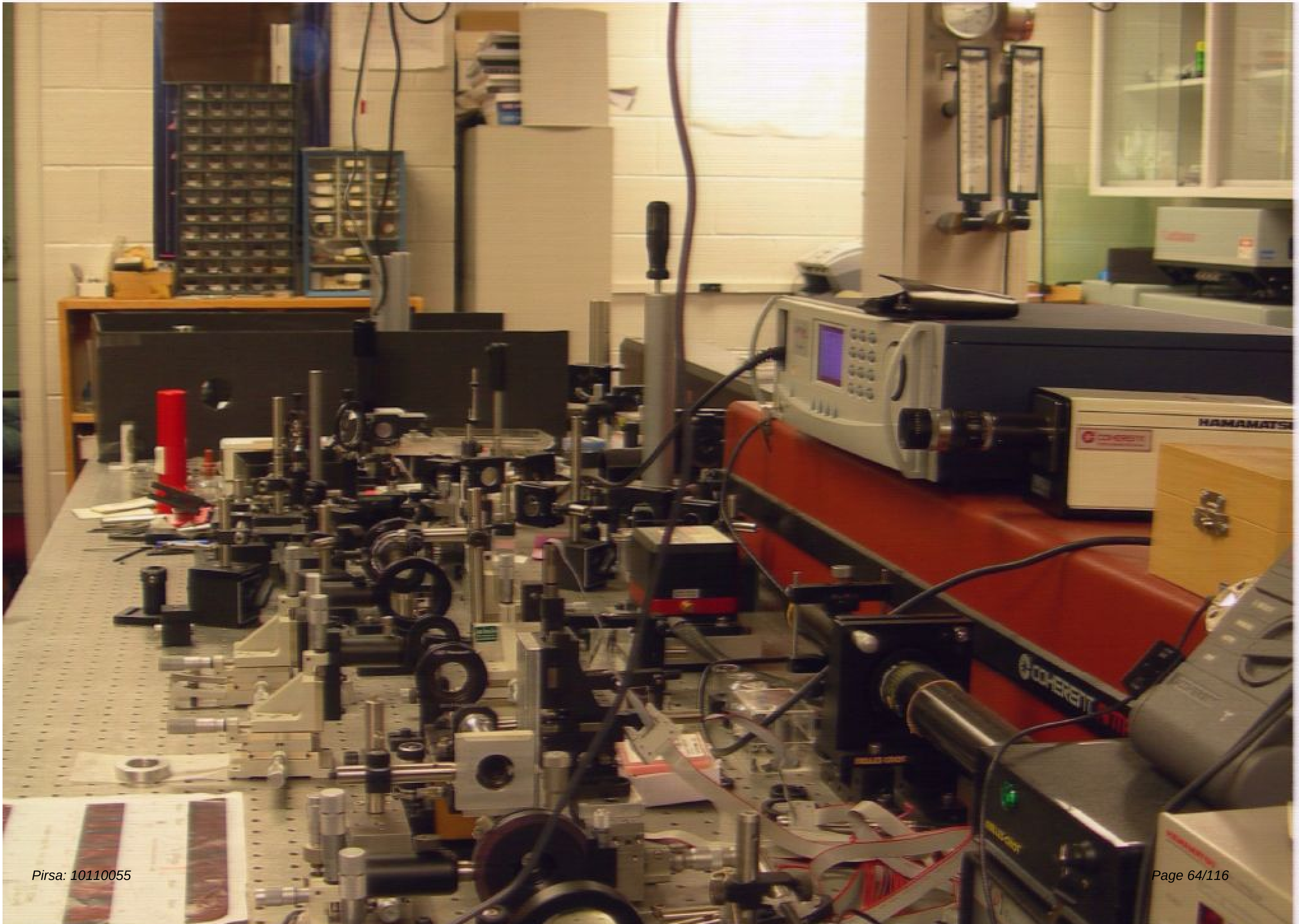
Dear Carl,
I have some good news from Greg Salamo (U. of Arkansas). His students (who are now visiting us here in Florida) have just observed a PT phase transition in a passive AlGaAs waveguide system. We will be submitting soon these results as a post-deadline paper to CLEO/QELS and subsequently to a regular journal. We are still fighting against the Kramers-Kronig relations, but the phase transition effect is definitely there. We expect even better results under TE polarization conditions. I will bring them over to Israel.

In close collaboration with us, more teams (also best friends!) are moving ahead in this direction. Moti Segev (from Technion) is planning an experiment in an active-passive dual core optical fiber -- fabricated in Southampton, England. More experiments will be carried later in Germany by Detlef Kip. Christian (his post doc) just left from here with a possible design. If everything goes well, with a bit of luck we may have an experimental explosion in the PT area. I wish the funding situation was a bit better. So far everything is done on a shoe-string budget (it is subsidized by other projects). Let us see...

All the best
Demetri



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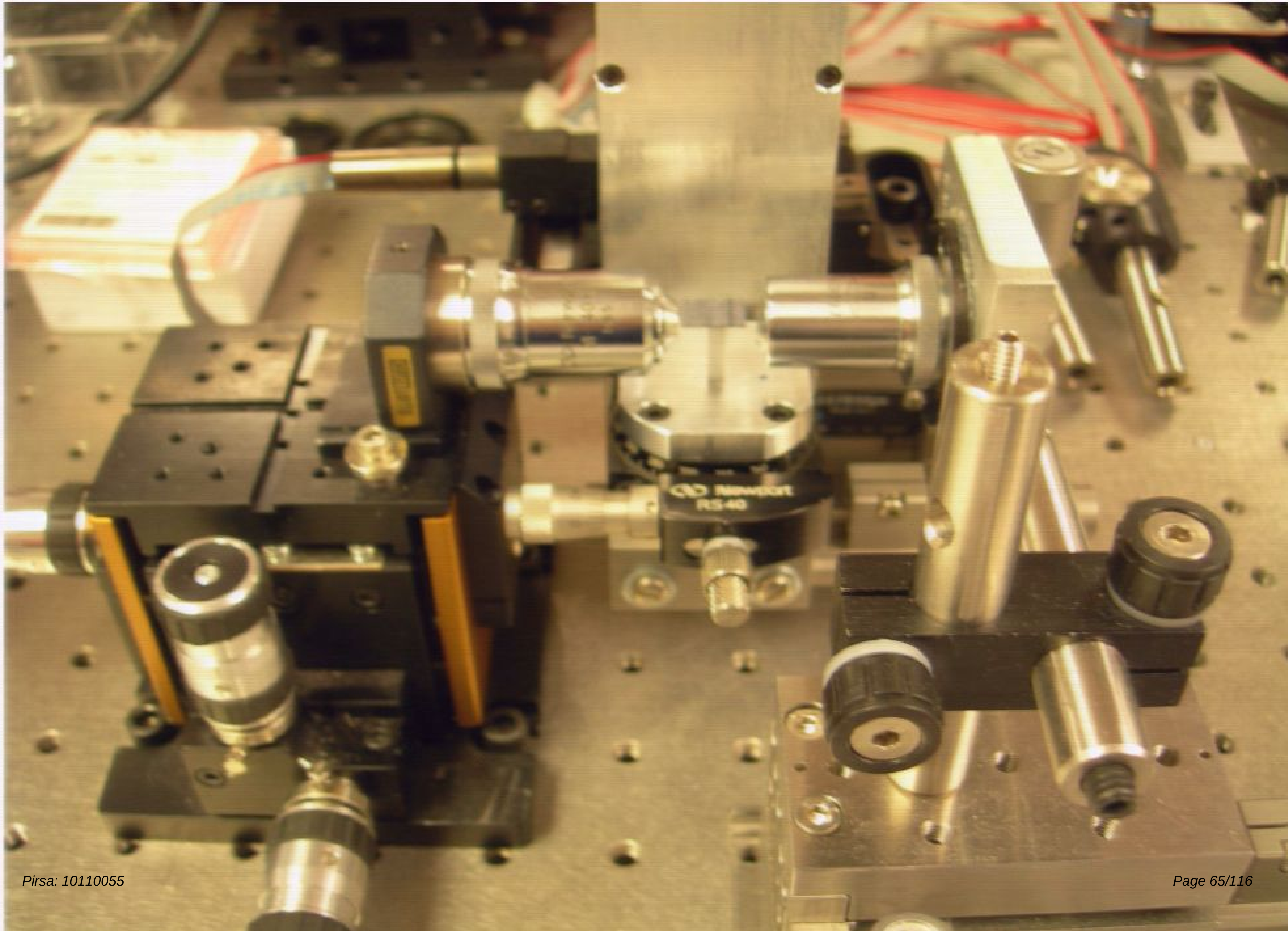
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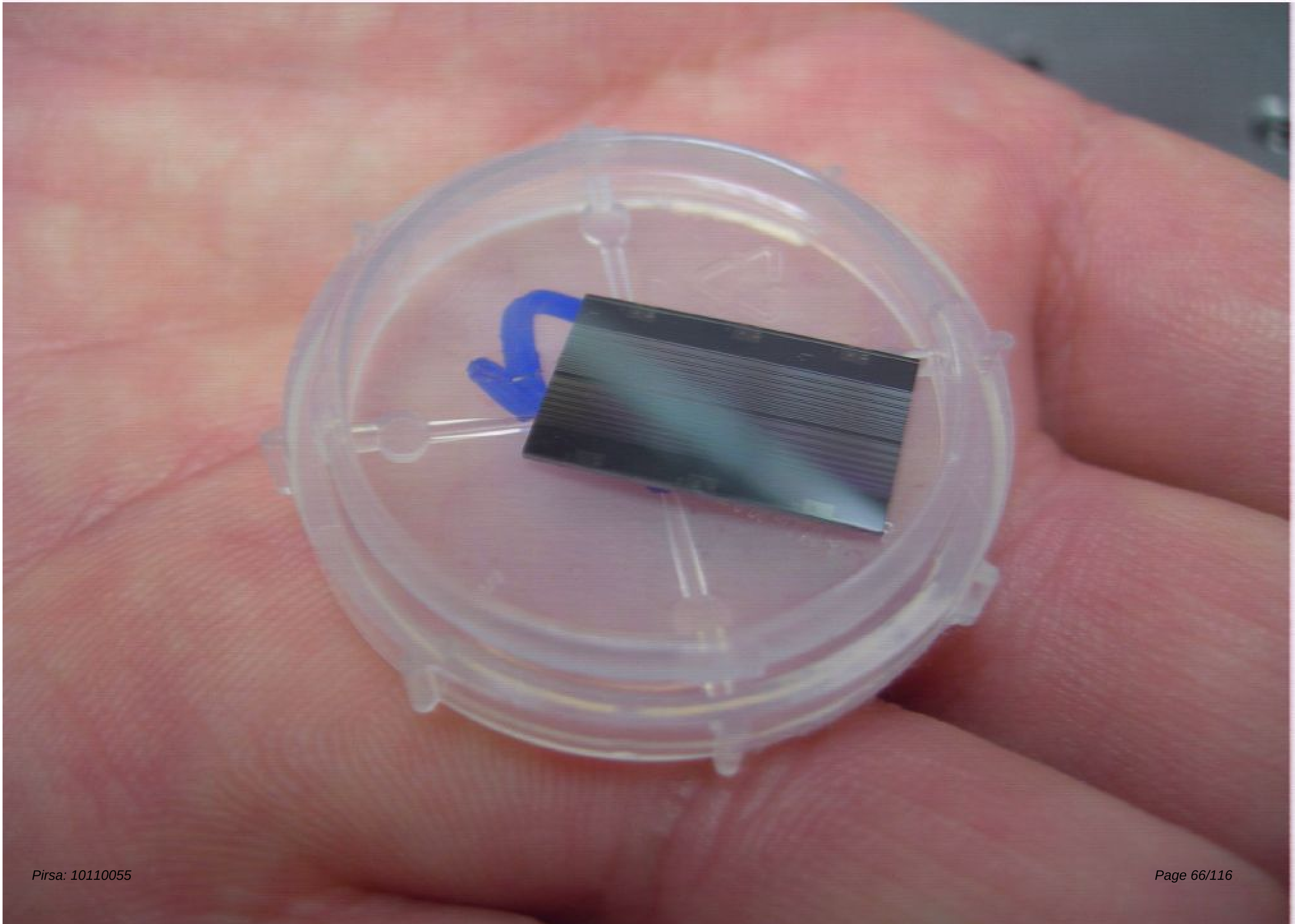
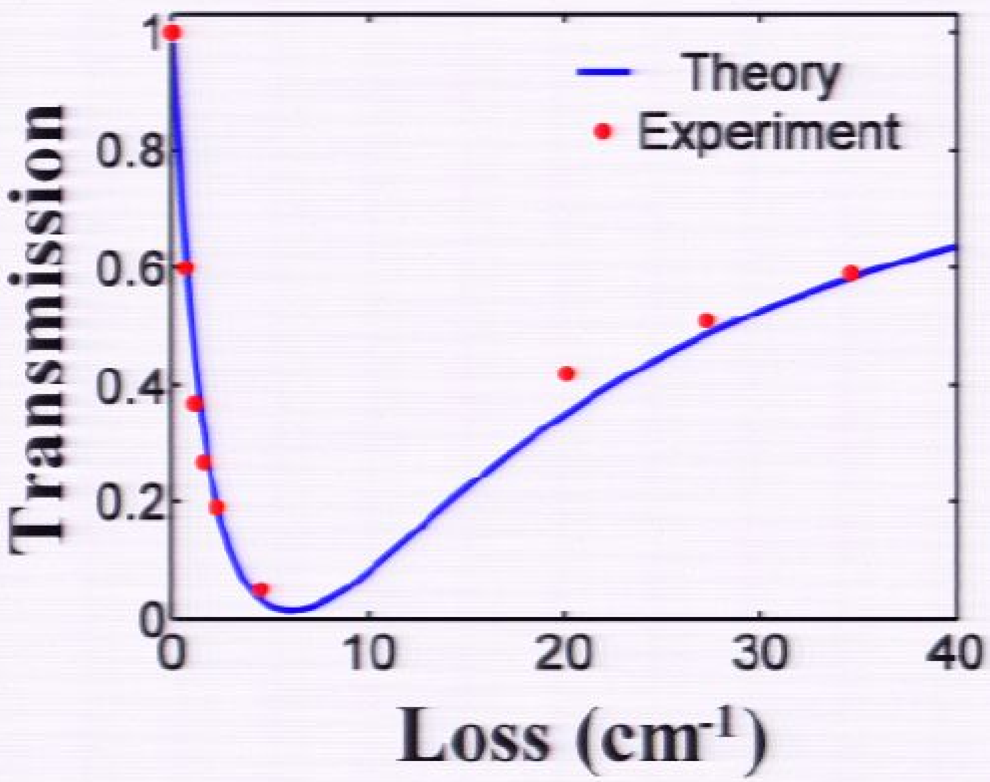


The observed **PT** phase transition

Figure 4: Experimental observation of spontaneous passive $\mathcal{PT}$-symmetry breaking. Output transmission of a passive $\mathcal{PT}$ complex system as the loss in the lossy waveguide arm is increased. The transmission attains a minimum at 6 cm^{-1}.



nature physics

LETTERS

PUBLISHED ONLINE: 24 JANUARY 2010 | DOI: 10.1038/NPHYS1515

Observation of parity–time symmetry in optics

Christian E. Rüter[1], Konstantinos G. Makris[2], Ramy El-Ganainy[2], Demetrios N. Christodoulides[2], Mordechai Segev[3] and Detlef Kip[1]*

One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables[1]. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity–time (*PT*) symmetry[2–7]. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories[8], non-Hermitian Anderson models[9] and open quantum systems[10,11], to mention a few. Although the impact of *PT* symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where *PT*-related notions can be implemented and experimentally investigated[12–15]. In this letter we report the first observation of the behaviour of a *PT* optical coupled system that judiciously involves a complex index potential. We observe both spontaneous *PT* symmetry breaking and power oscillations violating left-right symmetry. Our results may pave the way towards a new class of *PT*-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and

($\varepsilon > \varepsilon_{th}$), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous *PT* symmetry-breaking, that is, a 'phase transition' from the exact to broken-*PT* phase[7,20].

In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in *PT*-symmetric complex potentials. In fact, such *PT* 'optical potentials' can be realized through a judicious inclusion of index guiding and gain/loss regions[7,12–14]. Given that the complex refractive-index distribution $n(x) = n_R(x) + i n_I(x)$ plays the role of an optical potential, we can then design a *PT*-symmetric system by satisfying the conditions $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$.

In other words, the refractive-index profile must be an even function of position x whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope E of the optical beam is governed by the paraxial equation of diffraction[13]:

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0[n_R(x) + i n_I(x)]E = 0$$

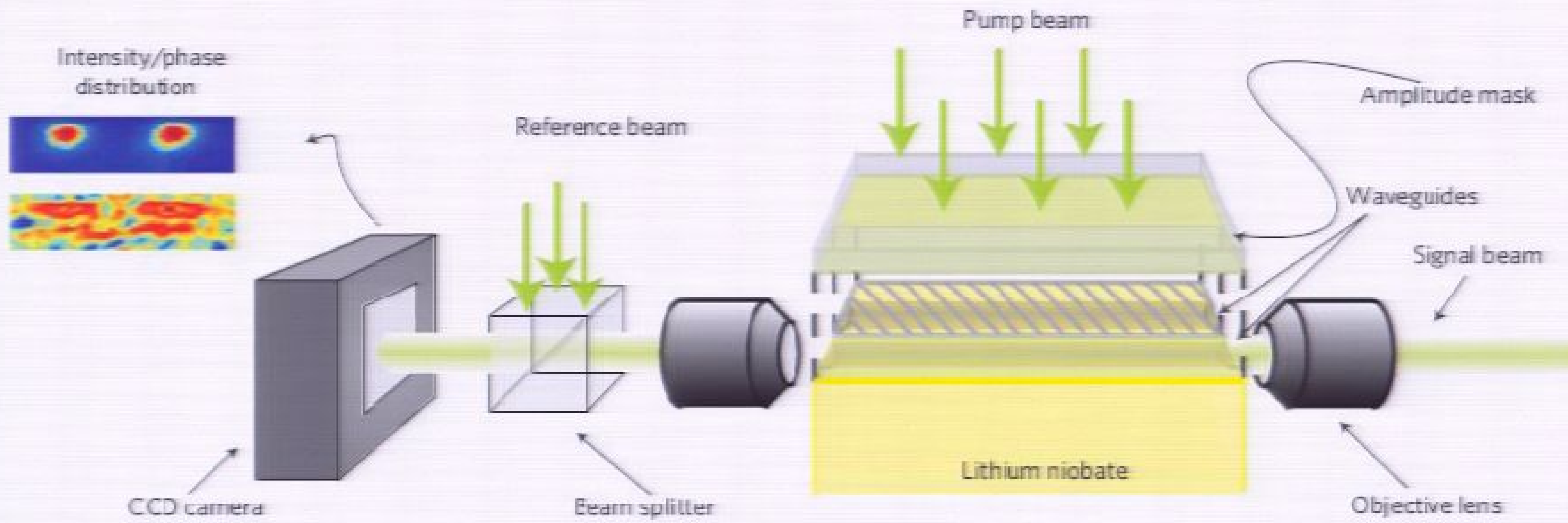


Figure 2 | Experimental set-up. An Ar^+ laser beam (wavelength 514.5 nm) is coupled into the arms of the structure fabricated on a photorefractive $LiNbO_3$ substrate. An amplitude mask blocks the pump beam from entering channel 2, thus enabling two-wave mixing gain only in channel 1. A CCD camera is used to monitor both the intensity and phases at the output.

NATURE PHYSICS DOI: 10.1038/NPHYS1515

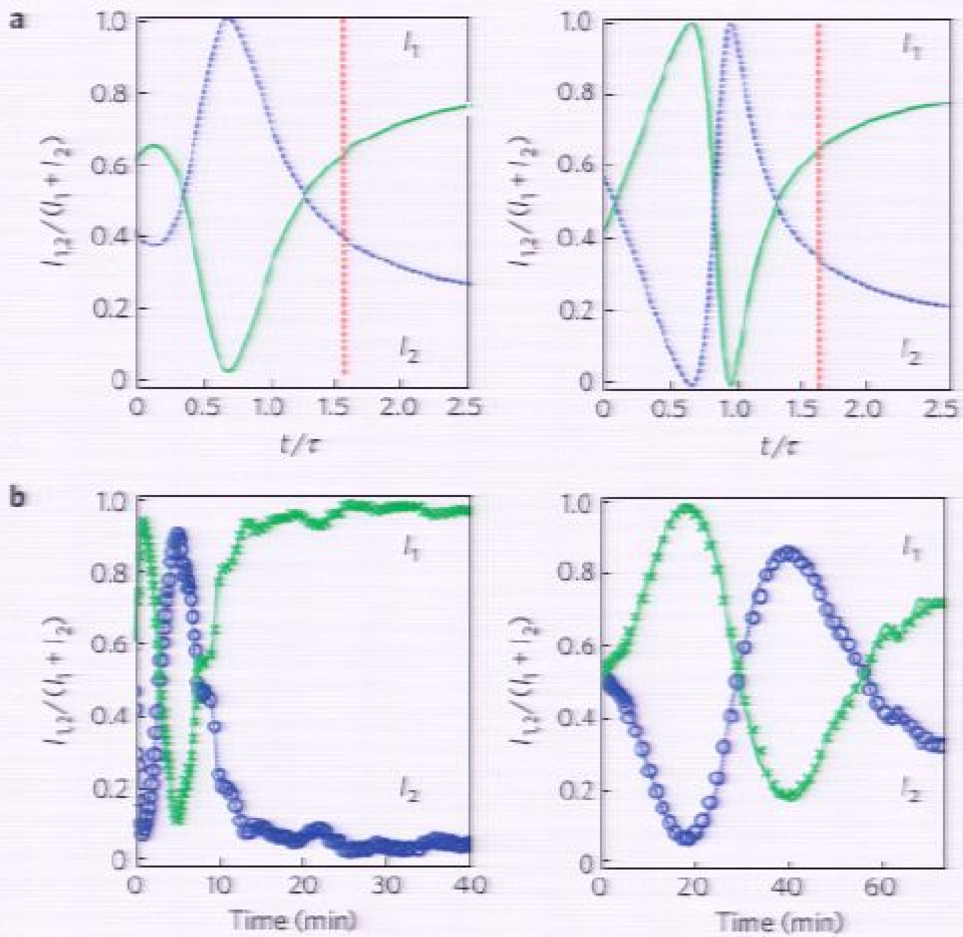


Figure 3 | Computed and experimentally measured response of a *PT*-symmetric coupled system. a, Numerical solution of the coupled equations (1) describing the *PT*-symmetric system. The left (right) panel shows the situation when light is coupled into channel 1 (2). Red dashed lines mark the symmetry-breaking threshold. Above threshold, light is predominantly guided in channel 1 experiencing gain, and the intensity of channels 1 and 2 depends solely on the magnitude of the gain.
b, Experimentally measured (normalized) intensities at the output facet

Another experiment...

"Enhanced magnetic resonance signal of spin-polarized Rb Atoms near surfaces of coated cells"
K. F. Zhao, M. Schaden, and Z. Wu
Physical Review A **81**, 042903 (2010)

And another experiment...

pontaneous PT-symmetry breakdown in superconducting weak inks

. M. Chtchelkatchev, A. A. Golubov, T. I. Baturina, V. M. Vinokur

rXiv:1008.3590v2 [cond-mat.supr-con], submitted on 21 Aug 2010 (v1), last revised 1 Sep 2010(v2))

bstract: We formulate a description of transport in a superconducting weak link in erms of the non-Hermitian quantum mechanics. We find that the applied electric field xceeding a certain critical value change the topological structure of the effective non-Iermitian Hamiltonian of the weak link in the Hilbert space causing the parity eflection – time reversal symmetry (PT-symmetry) breakdown. We derive the xpression of the critical electric field and show that the PT-symmetry breakdown gives ise to the switching instability in the current-voltage characteristic of the weak link. 'aking into account superconducting fluctuations we quantitatively describe the xperimentally observed differential resistance of the weak link in the vicinity of the ritical temperature.

And yet another...

Spontaneous Parity--Time Symmetry Breaking and Stability of Solitons in Bose-Einstein Condensates

Zhenya Yan, Bo Xiong, Wu-Ming Liu

(arXiv:1009.4023v1 [cond-mat.quant-gas], submitted on 21 Sep 2010)

Abstract: We report explicitly a novel family of exact PT-symmetric solitons and further study their spontaneous PT symmetry breaking, stabilities and collisions in Bose-Einstein condensates trapped in a PT-symmetric harmonic trap and a Hermite-Gaussian gain/loss potential. We observe the significant effects of mean-field interaction by modifying the threshold point of spontaneous PT symmetry breaking in Bose-Einstein condensates. Our scenario provides a promising approach to study PT-related universal behaviors in non-Hermitian quantum system based on the manipulation of gain/loss potential in Bose-Einstein condensates.

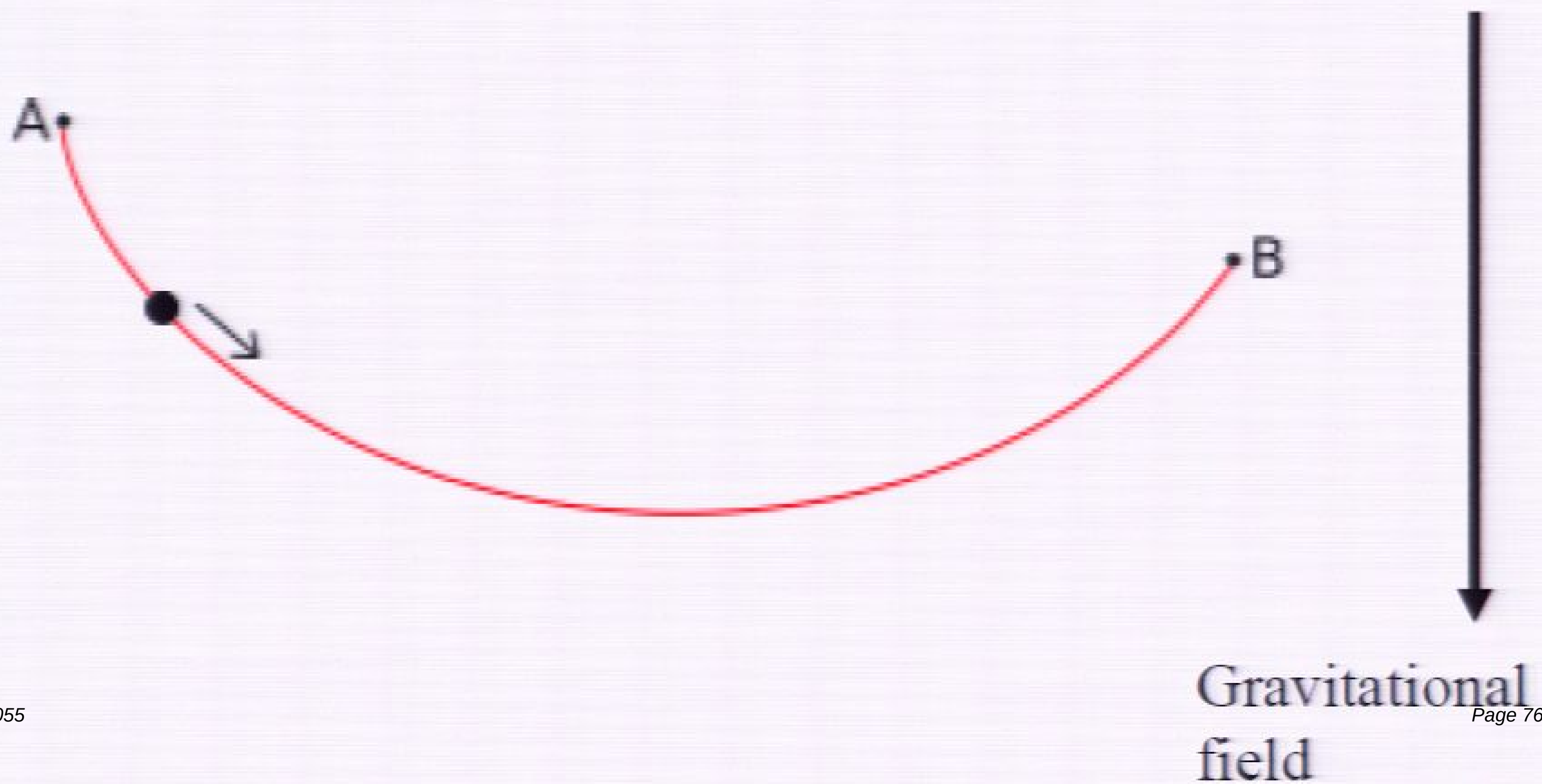
OK, but how do we interpret a non-Hermitian Hamiltonian??

Solve the quantum brachistochrone problem…

Classical brachistochrone

- Newton
- Bernoulli
- Leibniz
- L'Hôpital

Classical brachistochrone is a cycloid



Quantum brachistochrone

$$|\psi_I\rangle \to |\psi_F\rangle = e^{-iHt/\hbar}|\psi_I\rangle$$

$$\text{Constraint: } \omega = E_{\max} - E_{\min}$$

$$|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_F\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Hermitian case

$$H = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix} \qquad (r,\ s,\ u,\ \theta \text{ real})$$

$$H = \tfrac{1}{2}(s+u)\mathbf{1} + \tfrac{1}{2}\omega\sigma\cdot\mathbf{n}$$

$$\mathbf{n} = \frac{1}{\omega}(2r\cos\theta, 2r\sin\theta, s-u) \qquad \omega^2 = (s-u)^2 + 4r^2$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\exp(i\phi\,\sigma\cdot\mathbf{n}) = \cos\phi\,\mathbf{1} + i\sin\phi\,\sigma\cdot\mathbf{n}$$

$$|\psi_F\rangle = e^{-iH\tau/\hbar}|\psi_I\rangle$$

becomes

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{-\frac{1}{2}i(s+u)t/\hbar} \begin{pmatrix} \cos\frac{\omega t}{2\hbar} - i\frac{s-u}{\omega}\sin\frac{\omega t}{2\hbar} \\ -i\frac{2r}{\omega}e^{i\theta}\sin\frac{\omega t}{2\hbar} \end{pmatrix}$$

$$t = \frac{2\hbar}{\omega}\arcsin\frac{\omega|b|}{2r}$$

Minimize t over all positive *r* while maintaining constraint

$$\omega^2 = (s-u)^2 + 4r^2.$$

Minimum evolution time:

$$\omega\tau = 2\hbar \arcsin|b|$$

Looks like uncertainty principle but is merely rate times time = distance

If $a = 0$ and $b = 1$, the smallest time required to transform $\binom{1}{0}$ to the orthogonal state $\binom{0}{1}$ is

$$\tau = \pi\hbar/\omega$$

Non-Hermitian PT-symmetric Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix} \qquad (r,\ s,\ \theta \text{ real})$$

$\mathcal{T}$ is complex conjugation and $\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$E_{\pm} = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta} \quad \text{real if } s^2 > r^2\sin^2\theta$$

$$\mathcal{C} = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}$$

where $\sin\alpha = (r/s)\sin\theta.$

Exponentiate H

$$H = (r\cos\theta)\mathbf{1} + \tfrac{1}{2}\omega\sigma\cdot\mathbf{n},$$

where

$$\mathbf{n} = \frac{2}{\omega}(s, 0, ir\sin\theta)$$

$$\omega^2 = 4s^2 - 4r^2\sin^2\theta.$$

$$e^{-iHt/\hbar}\begin{pmatrix}1\\0\end{pmatrix} = \frac{e^{-itr\cos\theta/\hbar}}{\cos\alpha}\begin{pmatrix}\cos(\frac{\omega t}{2\hbar} - \alpha)\\ -i\sin\left(\frac{\omega t}{2\hbar}\right)\end{pmatrix}$$

The pair of vectors used in the Hermitian case, $|\psi_I\rangle = \binom{1}{0}$ and $|\psi_F\rangle = \binom{0}{1}$, are not orthogonal with respect to the $\mathcal{CPT}$ inner product. The time needed for $|\psi_I\rangle$ to evolve into $|\psi_F\rangle$ is $t = (2\alpha - \pi)\hbar/\omega$. Optimizing this result over α indicates that the optimal time τ is ZERO!

The bottom line…

What does **PT** *symmetry really mean?*

Interpretation…

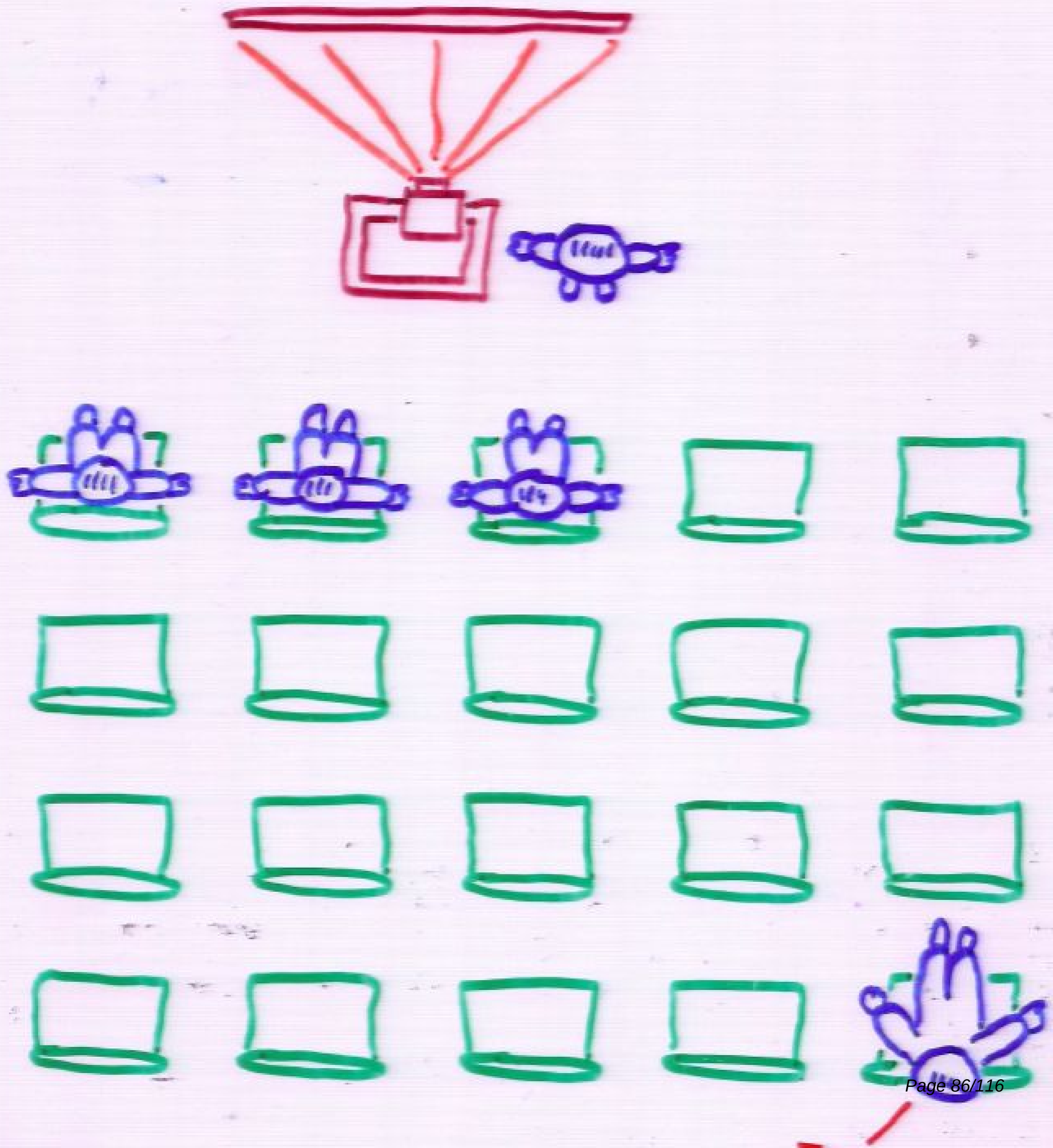
Finding the optimal **PT**-*symmetric Hamiltonian amounts to constructing a wormhole in Hilbert space!*

"The shortest path between two truths in the real domain passes through the complex domain."

-- Jacques Hadamard

The Mathematical Intelligencer **13** (1991)

Overview of talk:

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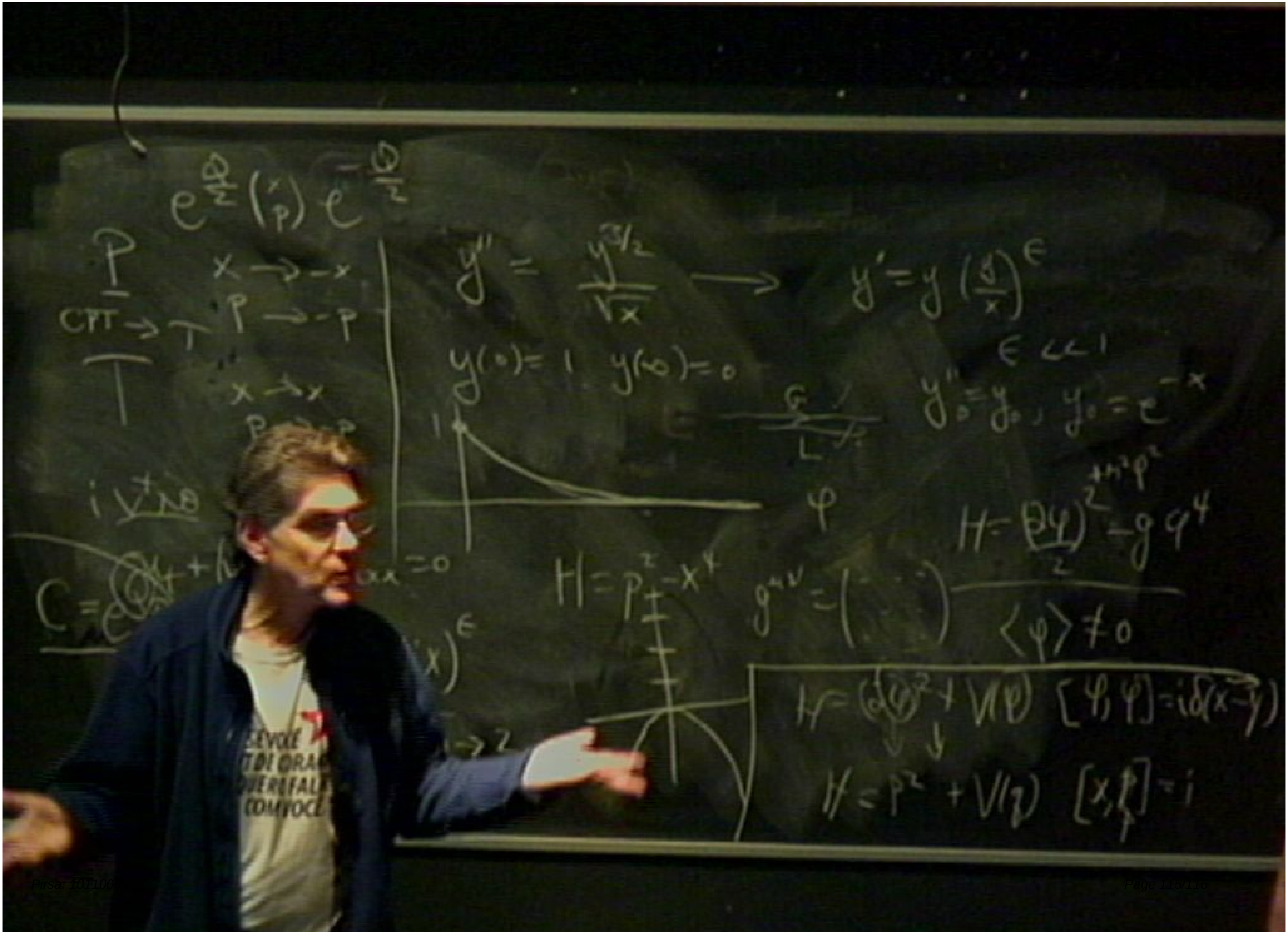
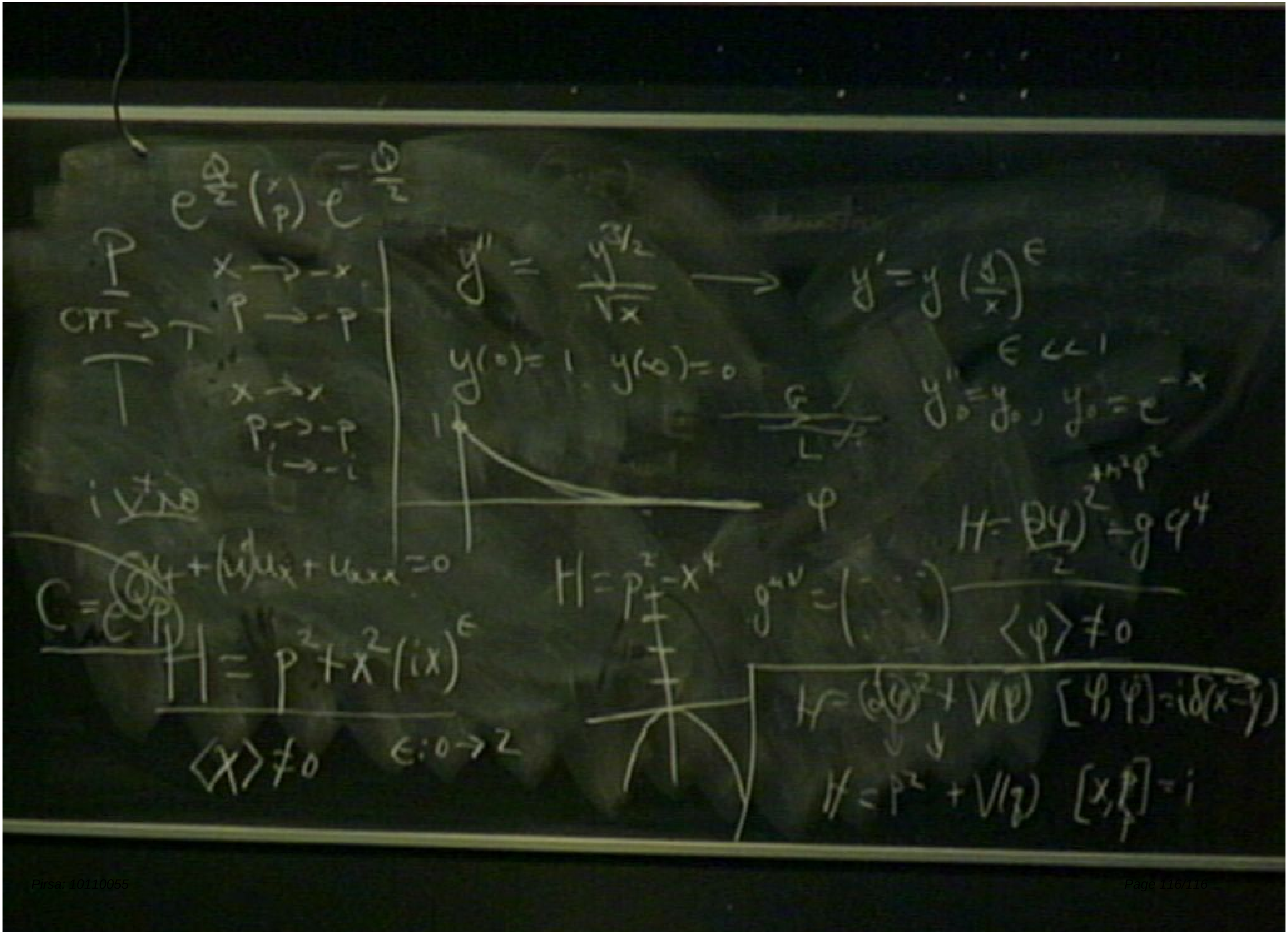

ε << 1
⟨φ⟩ ≠ 0
H = p² − x⁴
H = p² + V(q)   [x, p] = i


CPT → T
y(0) = 1 y(∞) = 0
ε << 1
H = p² − x⁴
H = p² + x²(ix)^ε
⟨x⟩ ≠ 0
ε: 0 → 2
⟨φ⟩ ≠ 0
H = p² + V(q) [x, p] = i