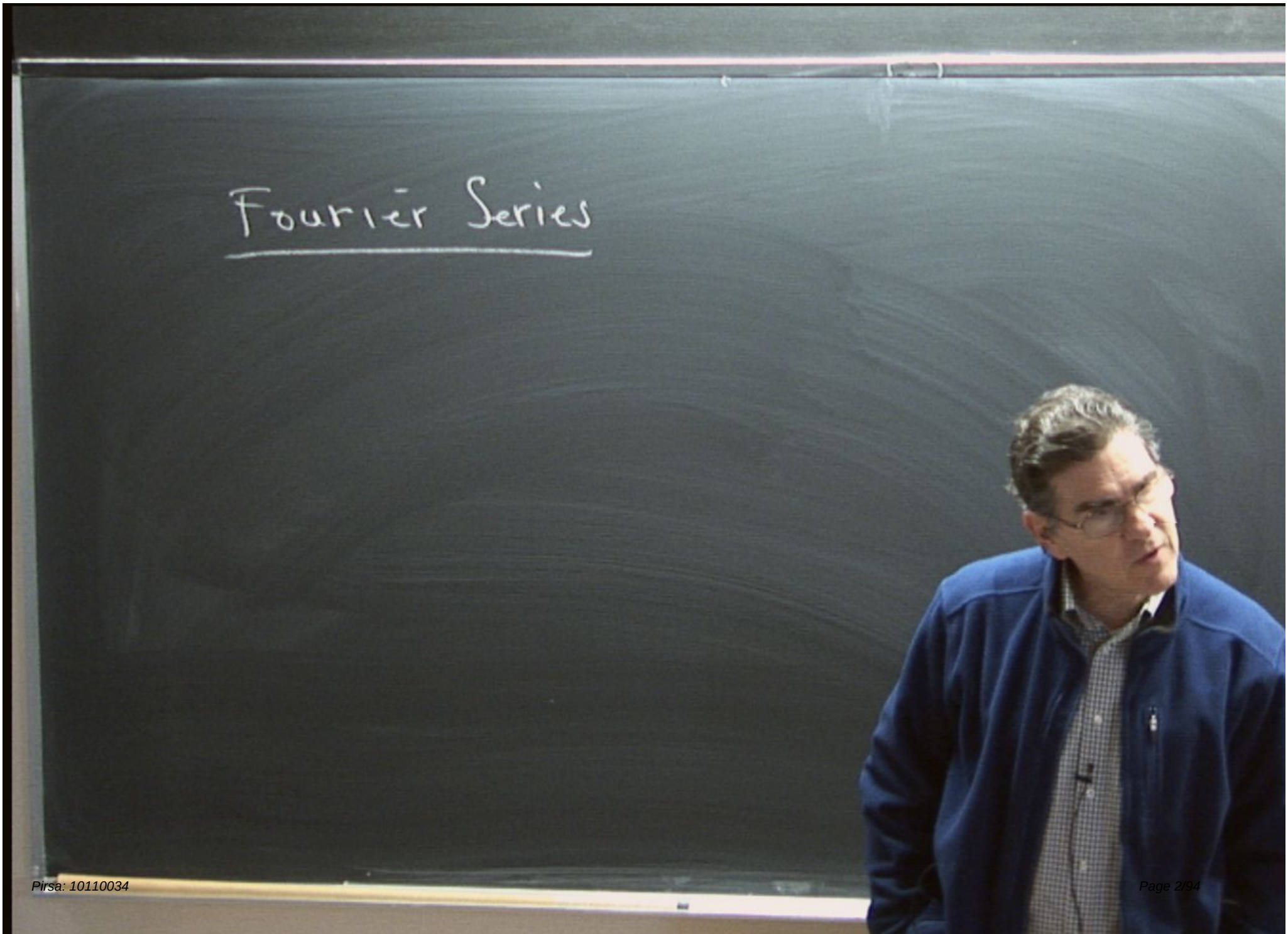
Title: Mathematical Physics (PHYS 624) - Lecture 5

Date: Nov 26, 2010 10:50 AM

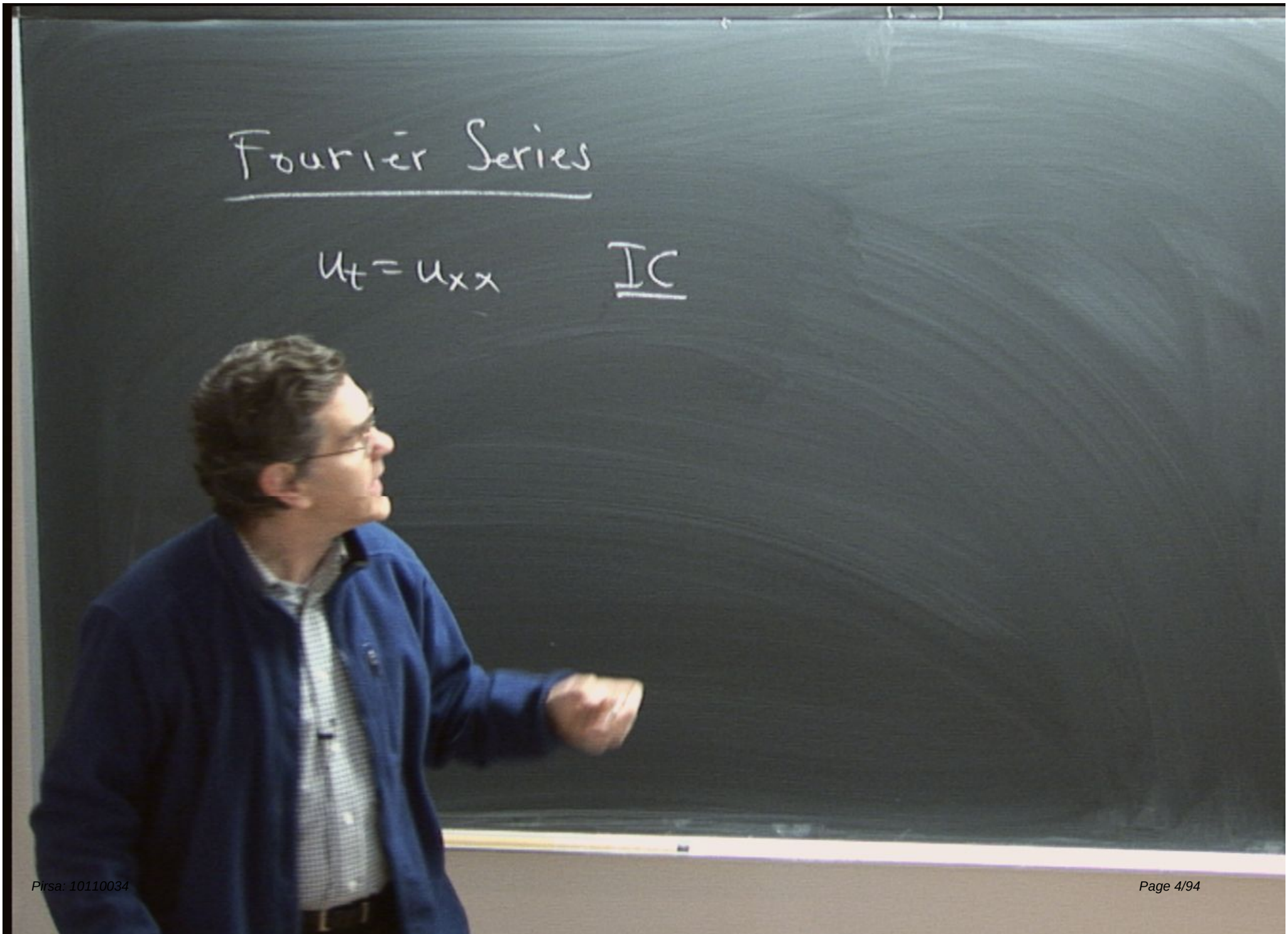
URL: http://pirsa.org/10110034

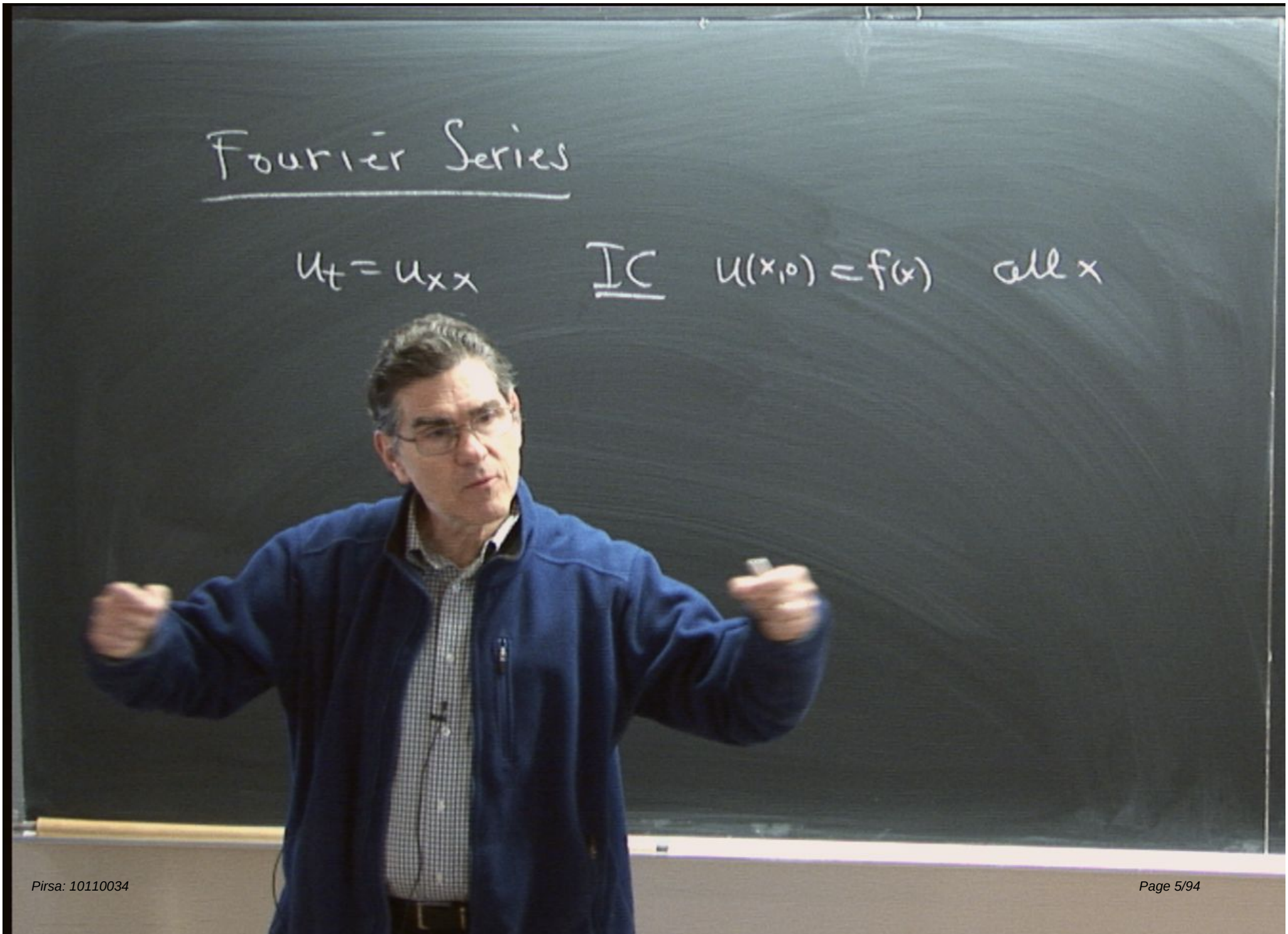
Abstract:


Fourier Series

Fourier Series

$$u_t = u_{xx}$$


Fourier Series
$u_t = u_{xx}$
IC


Fourier Series
$u_t = u_{xx}$
IC $u(x,0) = f(x)$ all x

Fourier Series

$u(x,t)$ = Temp at x, at time t.

$u_t = u_{xx}$

IC $u(x,0) = f(x)$ all x

Cauchy prob.

Fourier Series

$u(x,t)$ = Temp at x, at time t.

$u_t = u_{xx}$

IC $\cancel{u(x,0) = f(x) \quad \text{all } x}$

$\cancel{\text{Cauchy prob.}}$

$u(x,0) = f(x) \qquad 0 \le x \le \pi$

Fourier Series

$u(x,t)$ = Temp at x, at time t.

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$\cancel{\text{Cauchy prob.}}$

IC $u(x,0) = f(x) \qquad 0 \le x \le \pi$

BC

Fourier Series

$u(x,t)$ = Temp at x, at time t.

$u_t = u_{xx}$

IC $\cancel{u(x,0) = f(x) \quad \text{all } x}$

~~Cauchy prob.~~

IC $u(x,0) = f(x) \qquad 0 \le x \le \pi$

$0 \qquad x \qquad \pi$

BC $u(0,t) = g(t)$

Fourier Series

$u(x,t)$ = Temp at x, at time t.

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~~IC $u(x,0) = f(x)$ all x~~

~~Cauchy prob.~~

IC $u(x,0) = f(x)$ $\quad 0 \le x \le \pi$

BC $u(0,t) = g(t)$

$u(\pi,t) = h(t)$

Fourier Series

$u(x,t)$ = Temp at x, at time t.

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~~IC $u(x,0) = f(x)$ all x~~

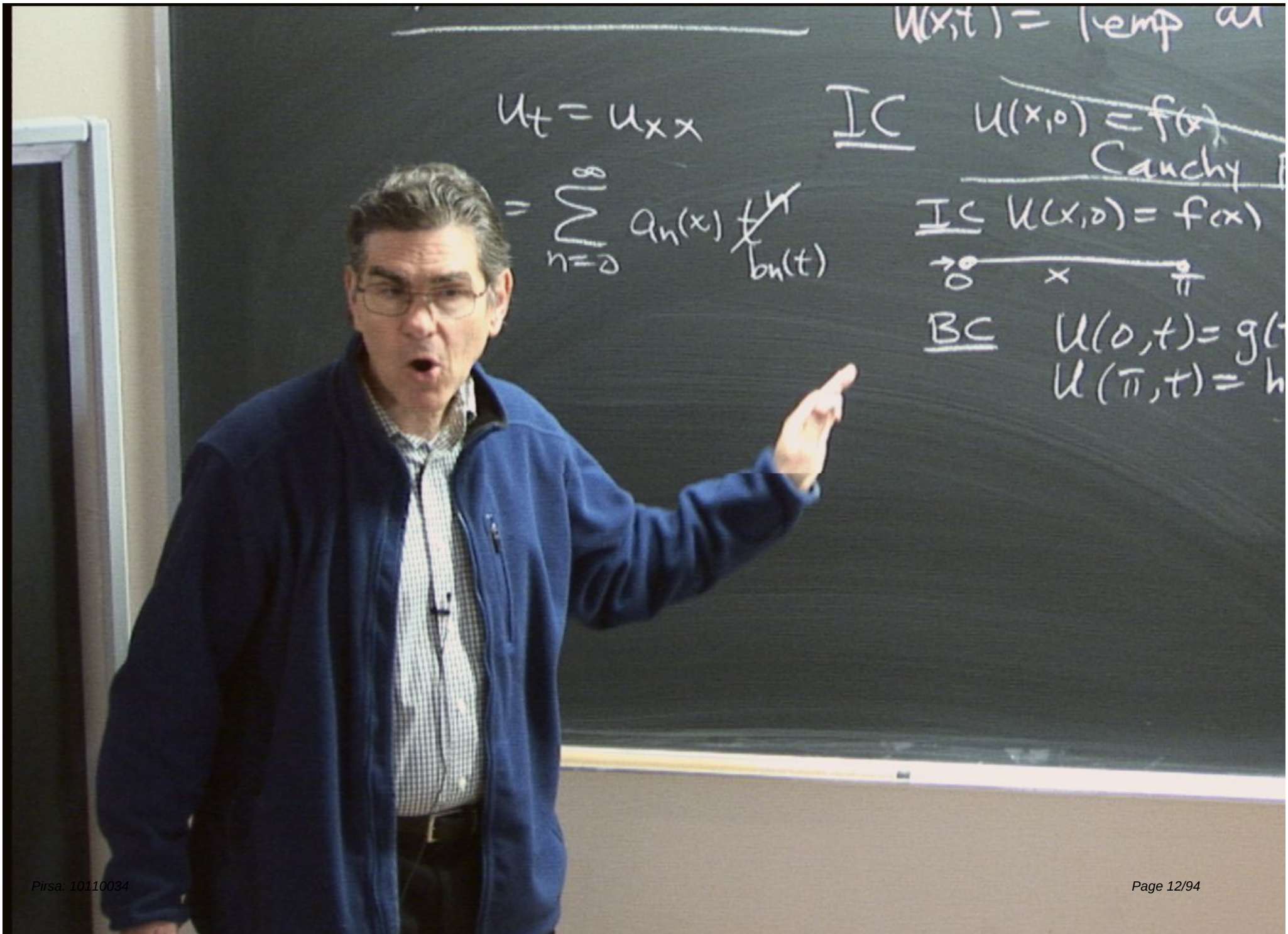
~~Cauchy prob.~~

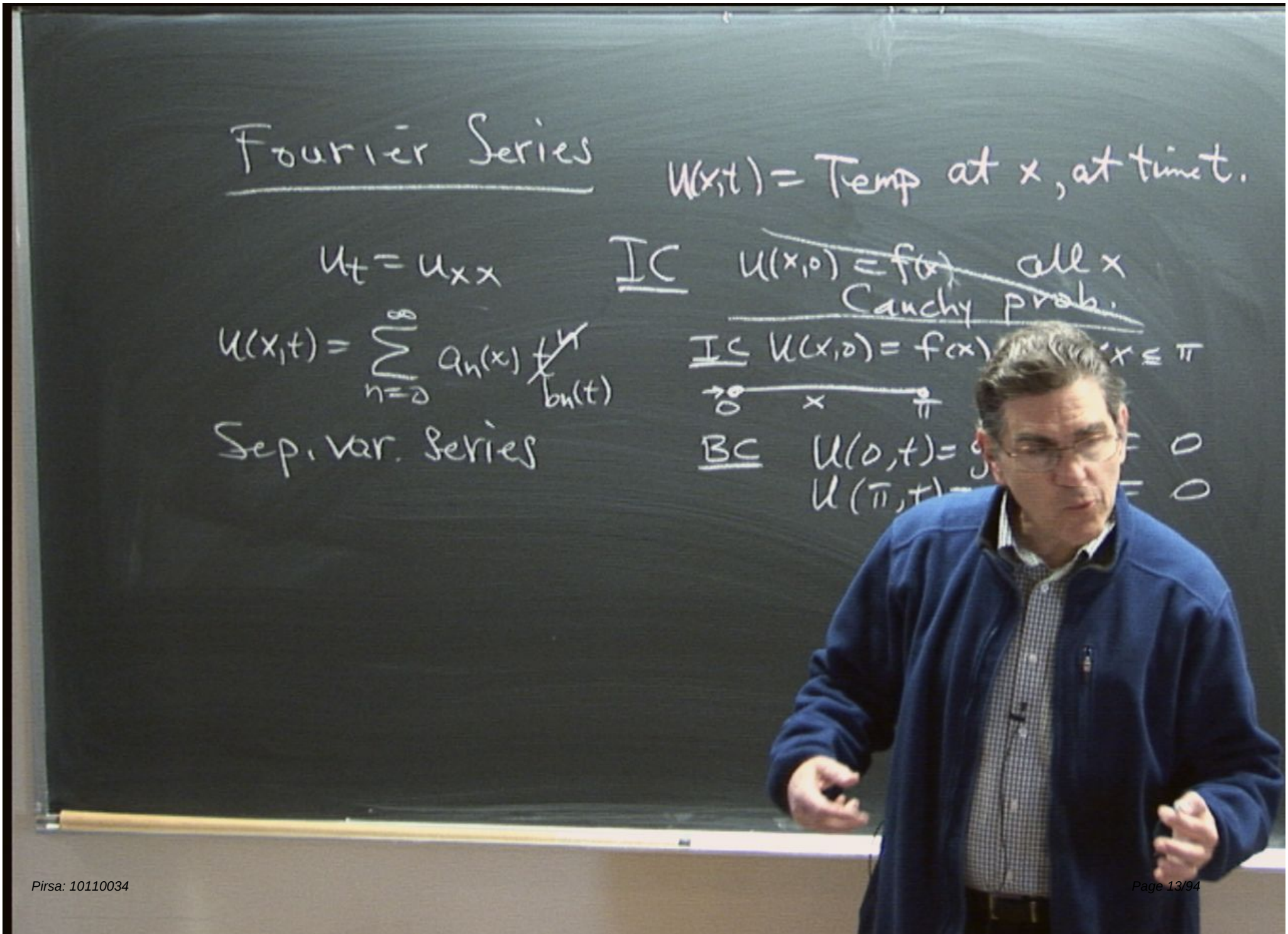
IC $u(x,0) = f(x)$ $\quad 0 \leq x \leq \pi$

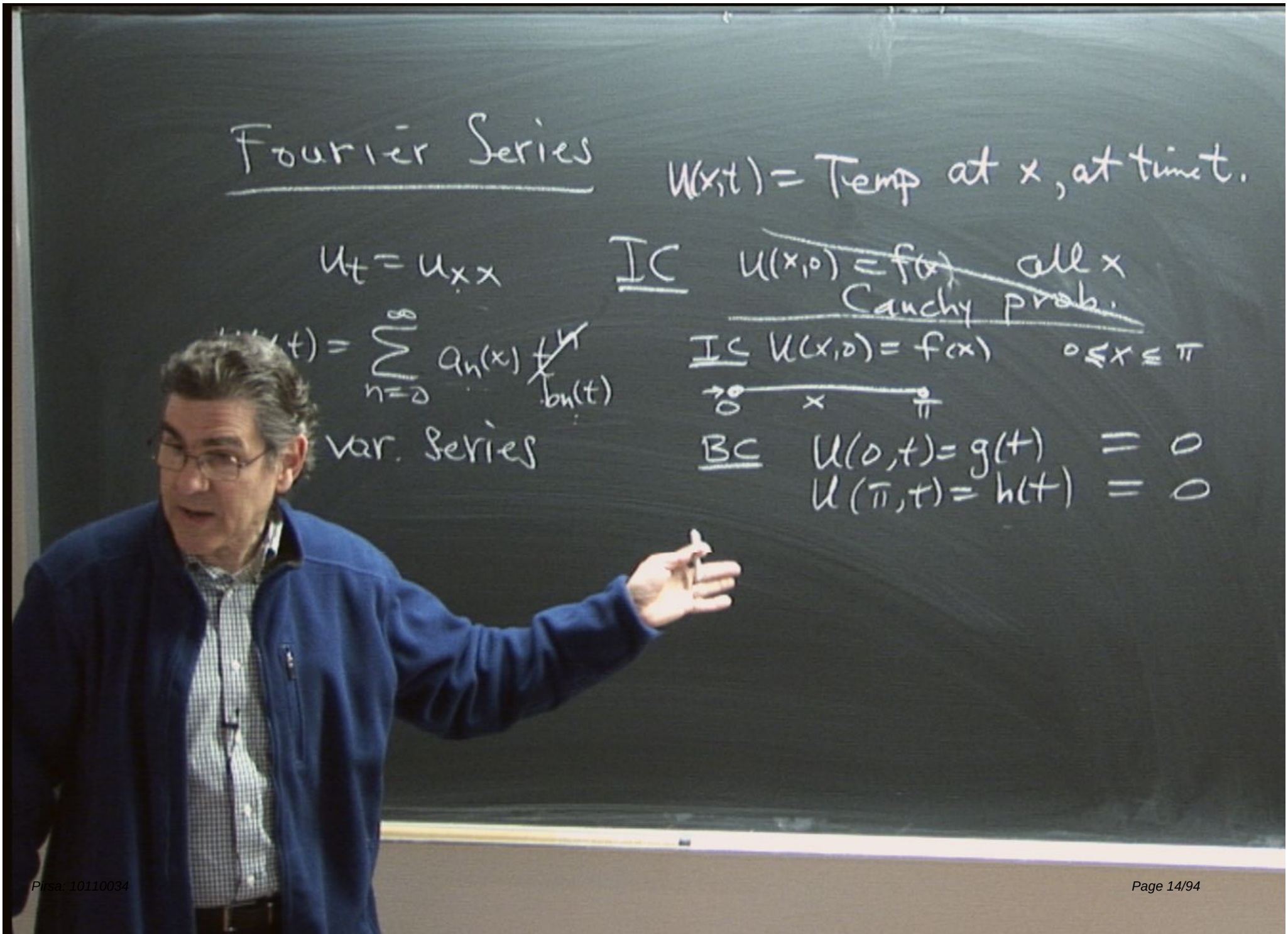
$0 \quad x \quad \pi$

BC $u(0,t) = g(t) = 0$

$u(\pi,t) = h(t) = 0$


u(x,t) = temp
u_t = u_xx
IC u(x,0) = f(x)
Cauchy
= Σ_{n=0}^{∞} a_n(x) b_n(t)
IC u(x,0) = f(x)
0 x π
BC u(0,t) = g(
u(π,t) = h


Fourier Series
u(x,t) = Temp at x, at time t.
u_t = u_xx
IC u(x,0) = f(x) all x
Cauchy prob.
u(x,t) = Σ_{n=0}^{∞} a_n(x) b_n(t)
IC u(x,0) = f(x)
0 x π
Sep. var. Series
BC u(0,t) = 0
u(π,t) = 0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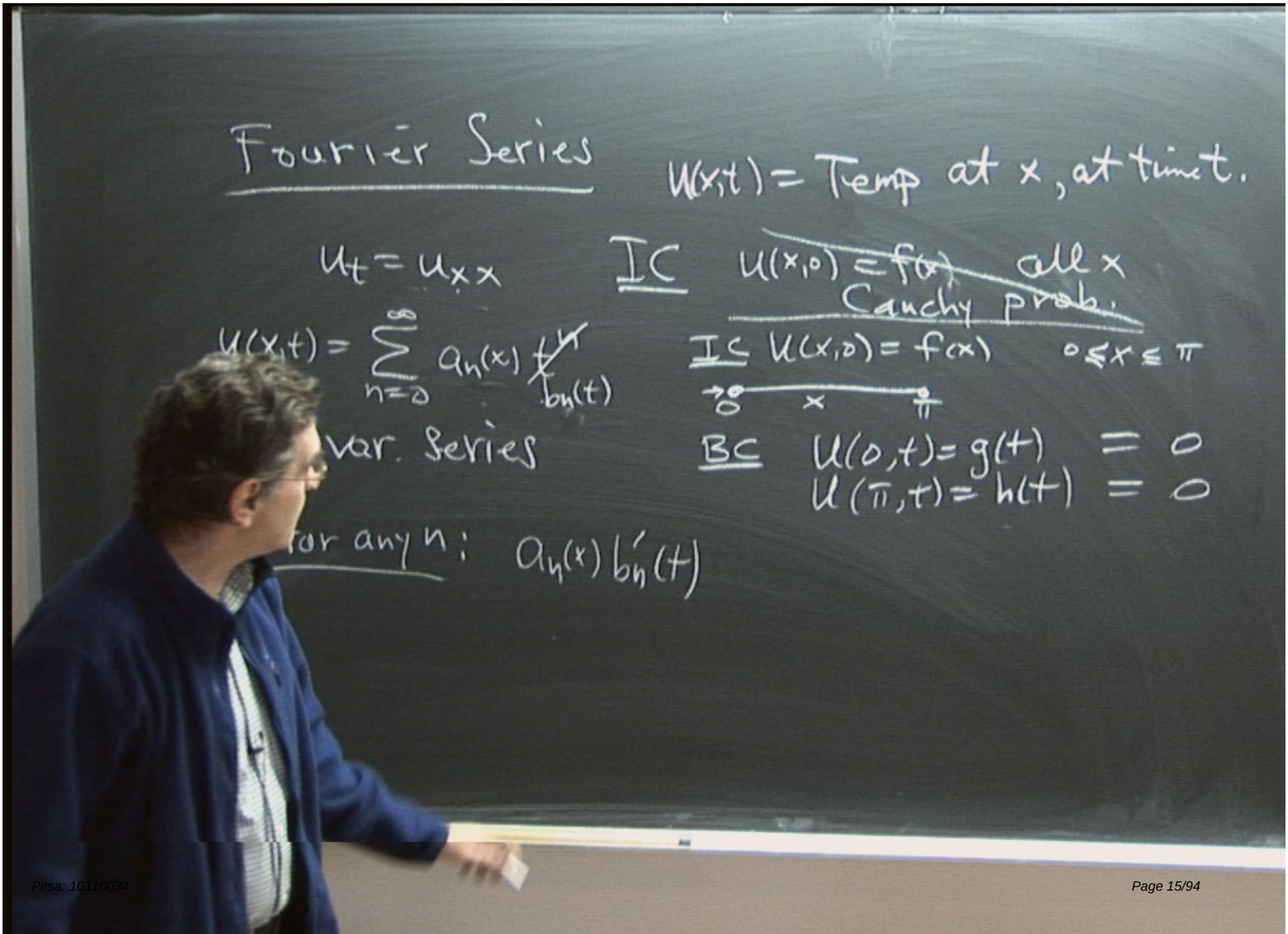
Cauchy prob.
t) = Σ_{n=0}^{∞} a_n(x) b_n(t)
var. Series
IC u(x,0) = f(x) 0 ≤ x ≤ π
0 x π
BC u(0,t) = g(t) = 0
u(π,t) = h(t) = 0


Fourier Series
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IC u(x,0) = f(x) all x
Cauchy prob.
u(x,t) = Σ_{n=0}^{∞} a_n(x) b_n(t)
IC u(x,0) = f(x) 0 ≤ x ≤ π
0 x π
var. Series
BC u(0,t) = g(t) = 0
u(π,t) = h(t) = 0
for any n: a_n(x) b_n'(t)

Fourier Series

$u(x,t)$ = Temp at x, at time t.

$u_t = u_{xx}$

~~IC $u(x,0) = f(x)$ all x~~

~~Cauchy prob.~~

$$u(x,t) = \sum_{n=0}^{\infty} a_n(x)\, \cancel{t^n}\, b_n(t)$$

IC $u(x,0) = f(x)$ $\quad 0 \le x \le \pi$

Sep. var. Series

BC $u(0,t) = g(t) = 0$

$u(\pi,t) = h(t) = 0$

for any n: $a_n(x)\, b_n'(t) = a_n''(x)\, b_n(t)$

Fourier Series

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for any n: $a_n(x)\, b_n'(t) = a_n''(x)\, b_n(t)$

$$\frac{b_n'(t)}{b_n(t)} = \frac{a_n''(x)}{a_n(x)}$$

Fourier Series

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$$\frac{b_n'(t)}{b_n(t)} = \frac{a_n''(x)}{a_n(x)} = C$$

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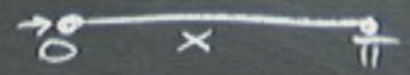


Sep. var. Series

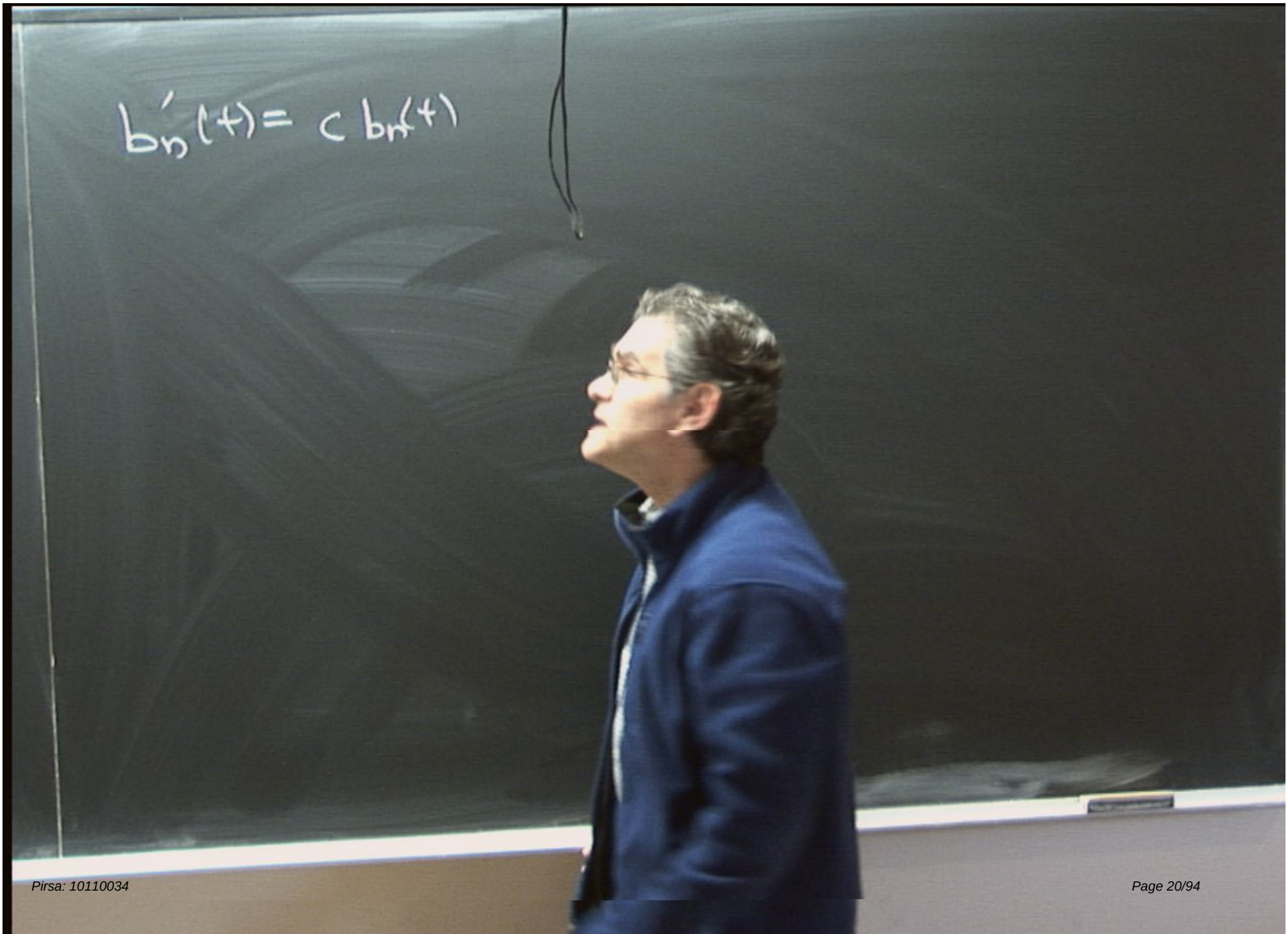
BC $u(0,t) = g(t) = 0$

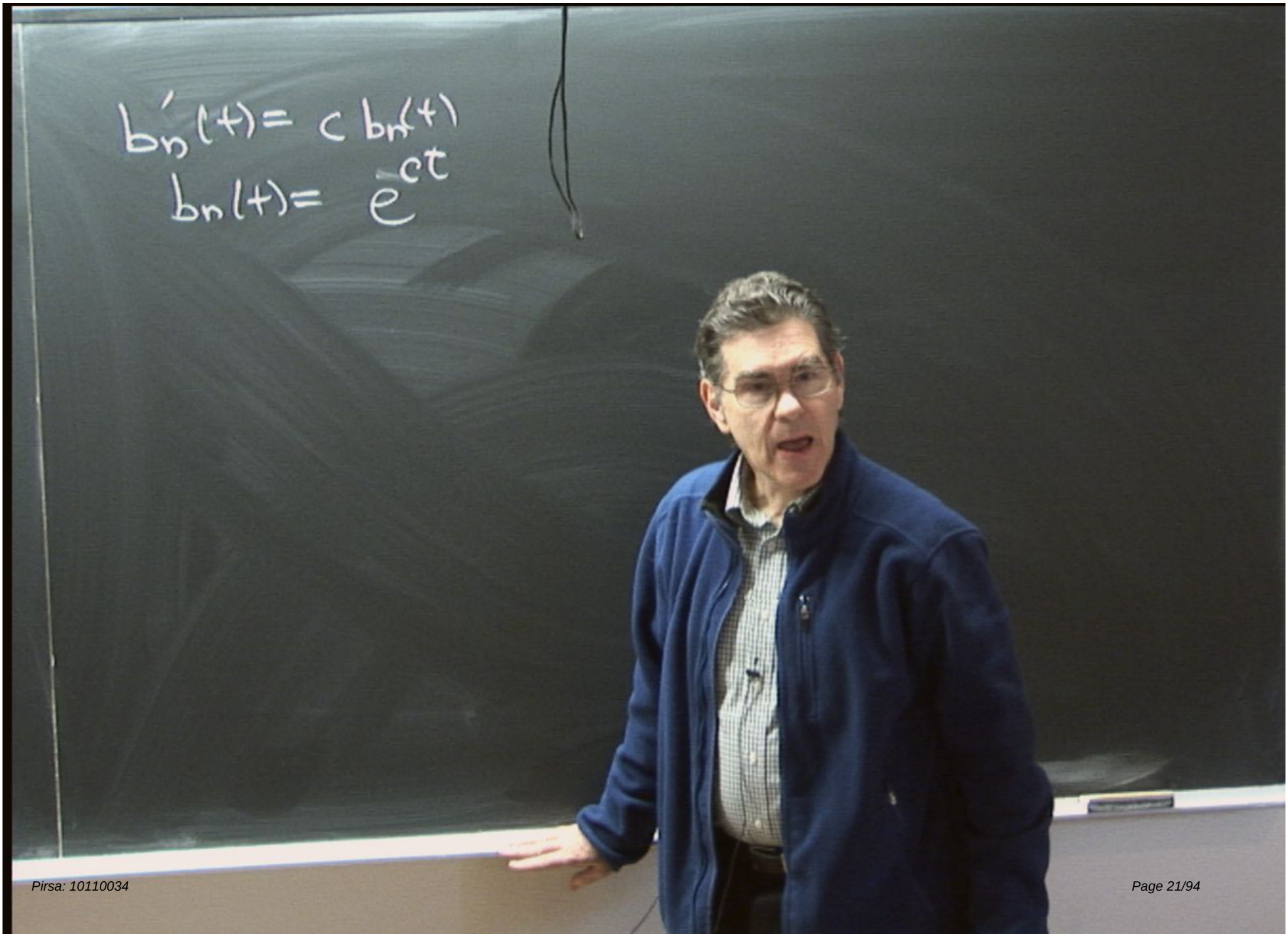
$u(\pi,t) = h(t) = 0$

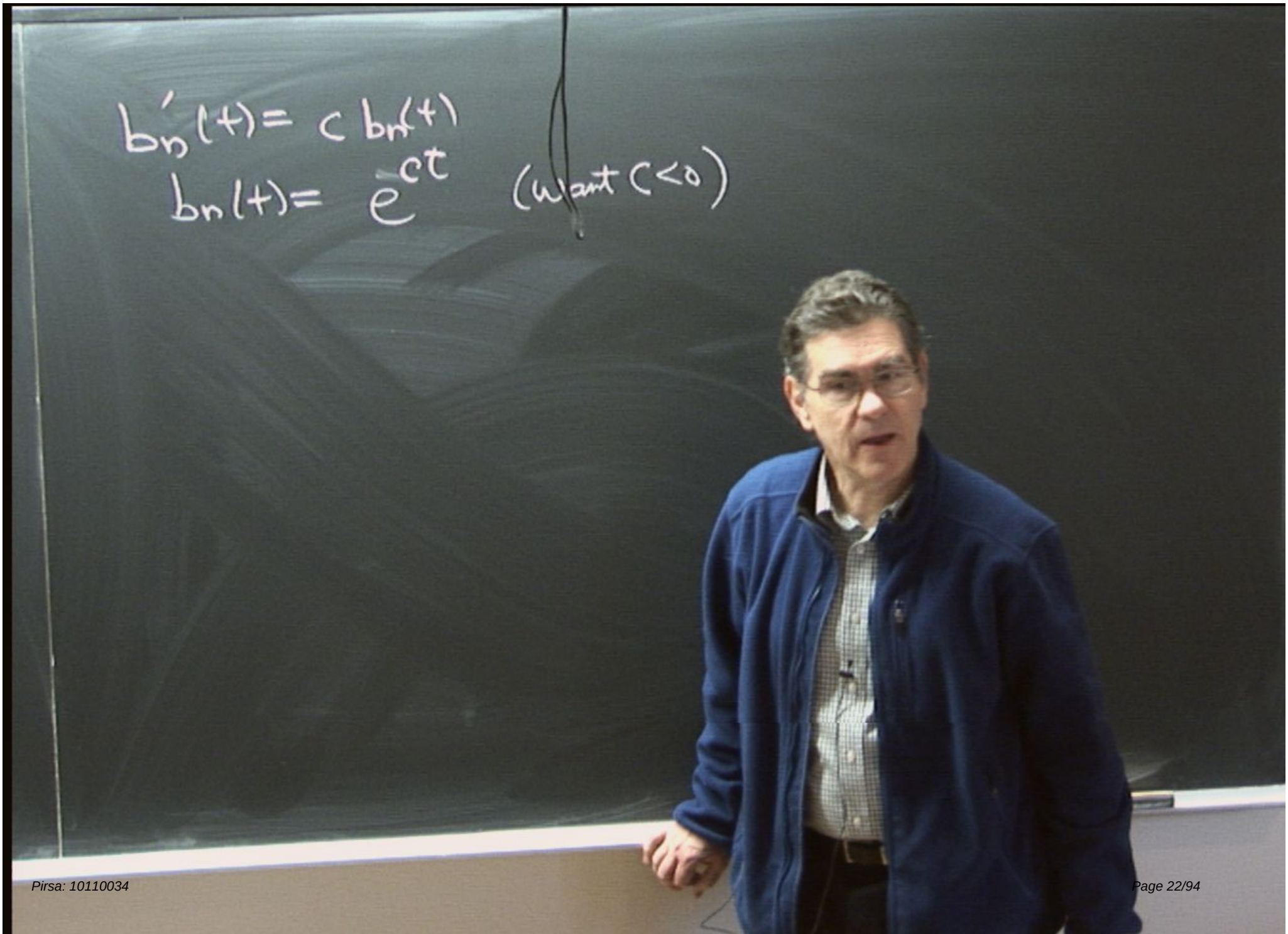
for any n: $a_n(x)\, b_n'(t) = a_n''(x)\, b_n(t)$

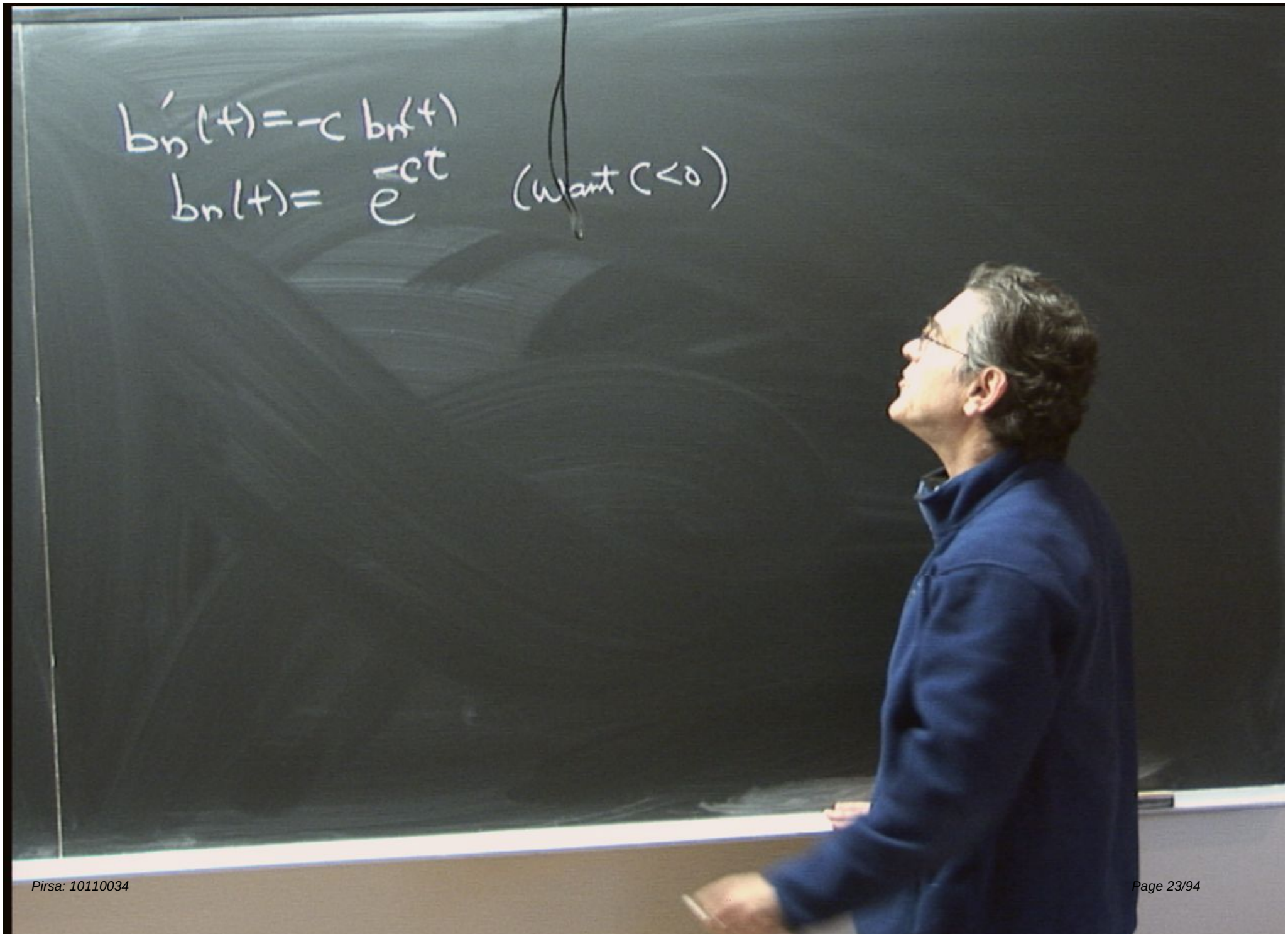
$$\frac{b_n'(t)}{b_n(t)} = \frac{a_n''(x)}{a_n(x)} = c$$

sep'n const


$b_n'(t) = c\, b_n(t)$

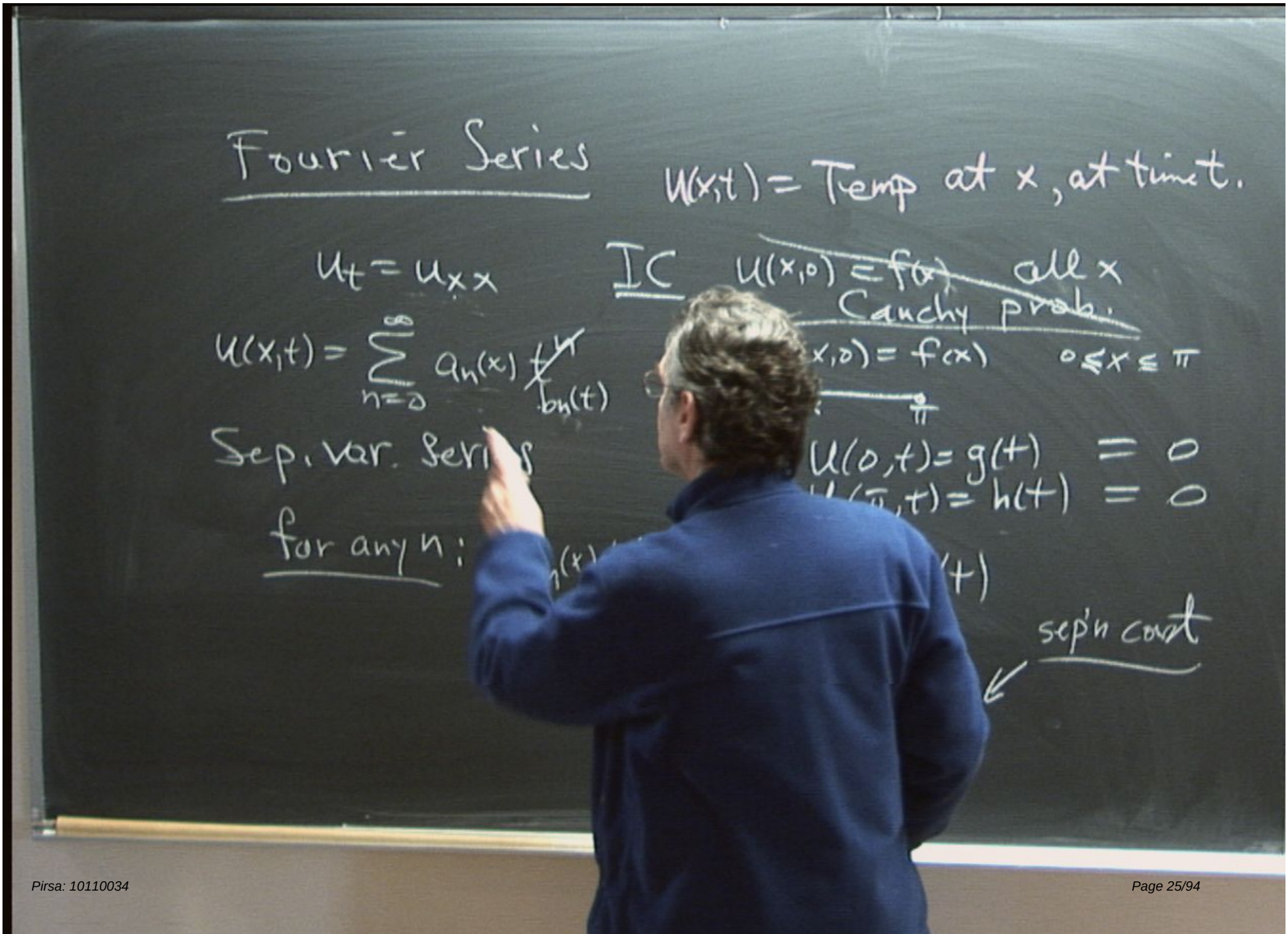

$b_n'(t) = c\, b_n(t)$
$b_n(t) = e^{ct}$


$b_n'(t) = c\, b_n(t)$
$b_n(t) = e^{ct}$ (want $c<0$)


$b_n'(t) = -c\, b_n(t)$
$b_n(t) = e^{-ct}$ (want $c<0$)

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$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u(x,t) = Σ_{n=0}^{∞} a_n(x) b_n(t)
x,0) = f(x) 0 ≤ x ≤ π
Sep. var. series
u(0,t) = g(t) = 0
u(π,t) = h(t) = 0
for any n:
sep'n const

Fourier Series

$u(x,t)$ = Temp at x, at time t.

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IC $\cancel{u(x,0) = f(x) \quad \text{all } x}$

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$$u(x,t) = \sum_{n=0}^{\infty} a_n(x)\, \cancel{t^n}\, b_n(t)$$

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for any n: $a_n(x)\, b_n'(t) = a_n''(x)\, b_n(t)$

$$\frac{b_n'(t)}{b_n(t)} = \frac{a_n''(x)}{a_n(x)} = -c$$

sep'n const

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC ___ term-by-term (!!)

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC ___ term-by-term (!!!)

$$a_n''(x) + c\, a_n(x) = 0$$

$$a_n(x) = \sin(\sqrt{c}\, x) \quad \cos(\sqrt{c}\, x)$$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC ___ term-by-term (!!!)

$$a_n''(x) + c\, a_n(x) = 0$$

$$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \underline{\beta} \cos(\sqrt{c}\, x)$$

$a_n(x) = 0$ at $x = 0$.

$\Longrightarrow \beta = 0$.

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC — term-by-term (!!)

$$a_n''(x) + c\, a_n(x) = 0$$

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$a_n(x) = 0$ at $x = 0$.

$\Rightarrow \beta = 0$.

$a_n(x) = 0$ at $x = \pi$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC ___ term-by-term (!!)

$$a_n''(x) + c\, a_n(x) = 0$$

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$a_n(x) = 0$ at $x = 0$.

$\Longrightarrow \beta = 0$.

$$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} = n$$

$a_n(x) = 0$ at $x = \pi$

$n = 1, 2, 3, \ldots$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC — term-by-term (!!)

$$a_n''(x) + c\, a_n(x) = 0$$

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$\Longrightarrow \beta = 0$.

$$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} = n$$

$a_n(x) = 0$ at $x = \pi$

$n = 1, 2, 3, \ldots$

$$\underline{c = n^2}$$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC ___ term-by-term (!!)

$$a_n''(x) + c\, a_n(x) = 0$$

$$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \cancel{\beta \cos(\sqrt{c}\, x)}$$

$a_n(x) = 0$ at $x = 0$.

$\Longrightarrow \beta = 0$

$$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} = n$$

$a_n(x) = 0$ at $x = \pi$

$n = 1, 2, 3, \ldots$

$$\underline{c = n^2}$$

$$a_n(x) = \alpha \sin(nx)$$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC — term-by-term (!!!)

$$a_n''(x) + c\, a_n(x) = 0$$

$$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \cancel{\beta \cos(\sqrt{c}\, x)}$$

$a_n(x) = 0$ at $x = 0$.

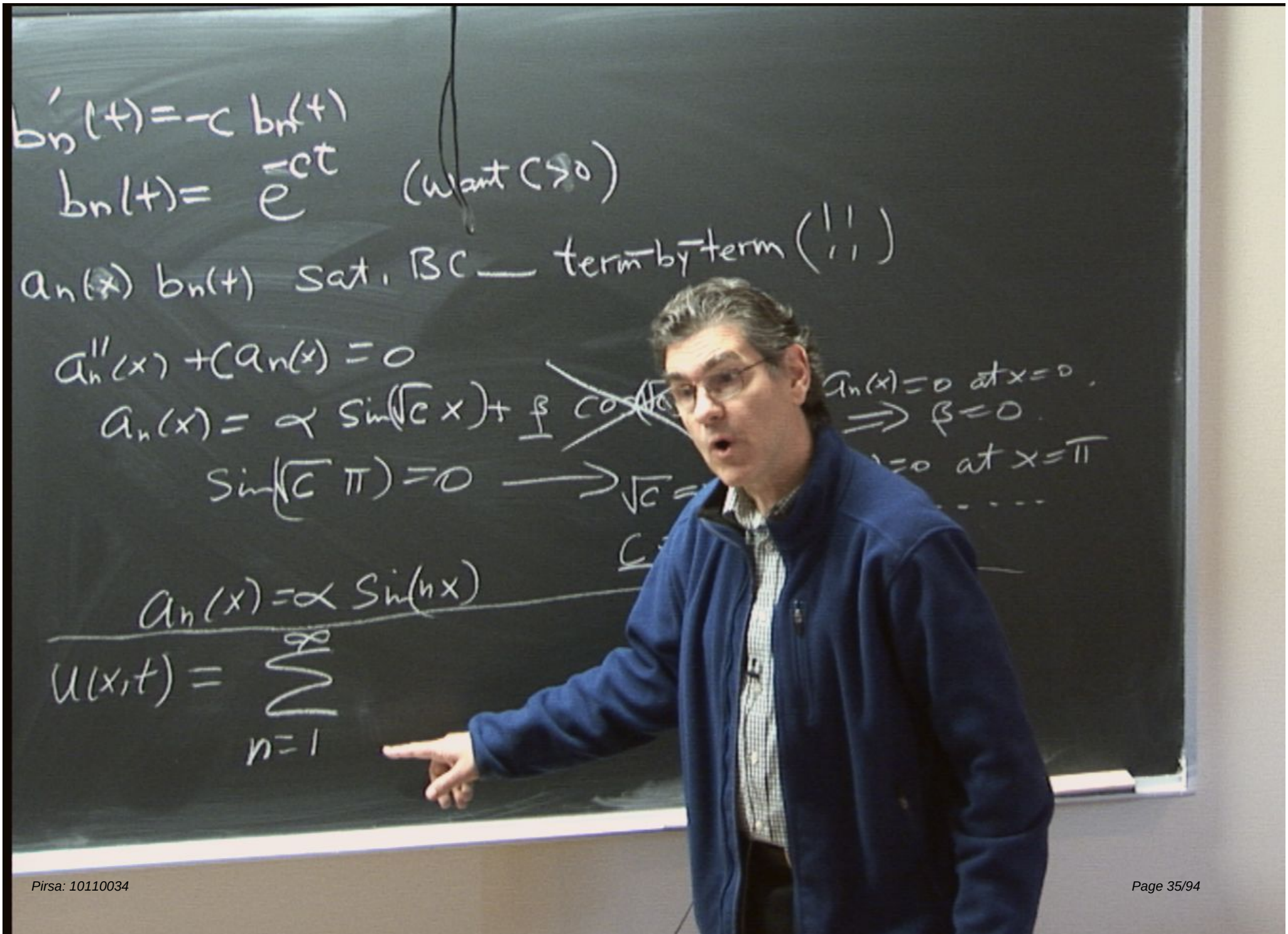
$\Longrightarrow \beta = 0$

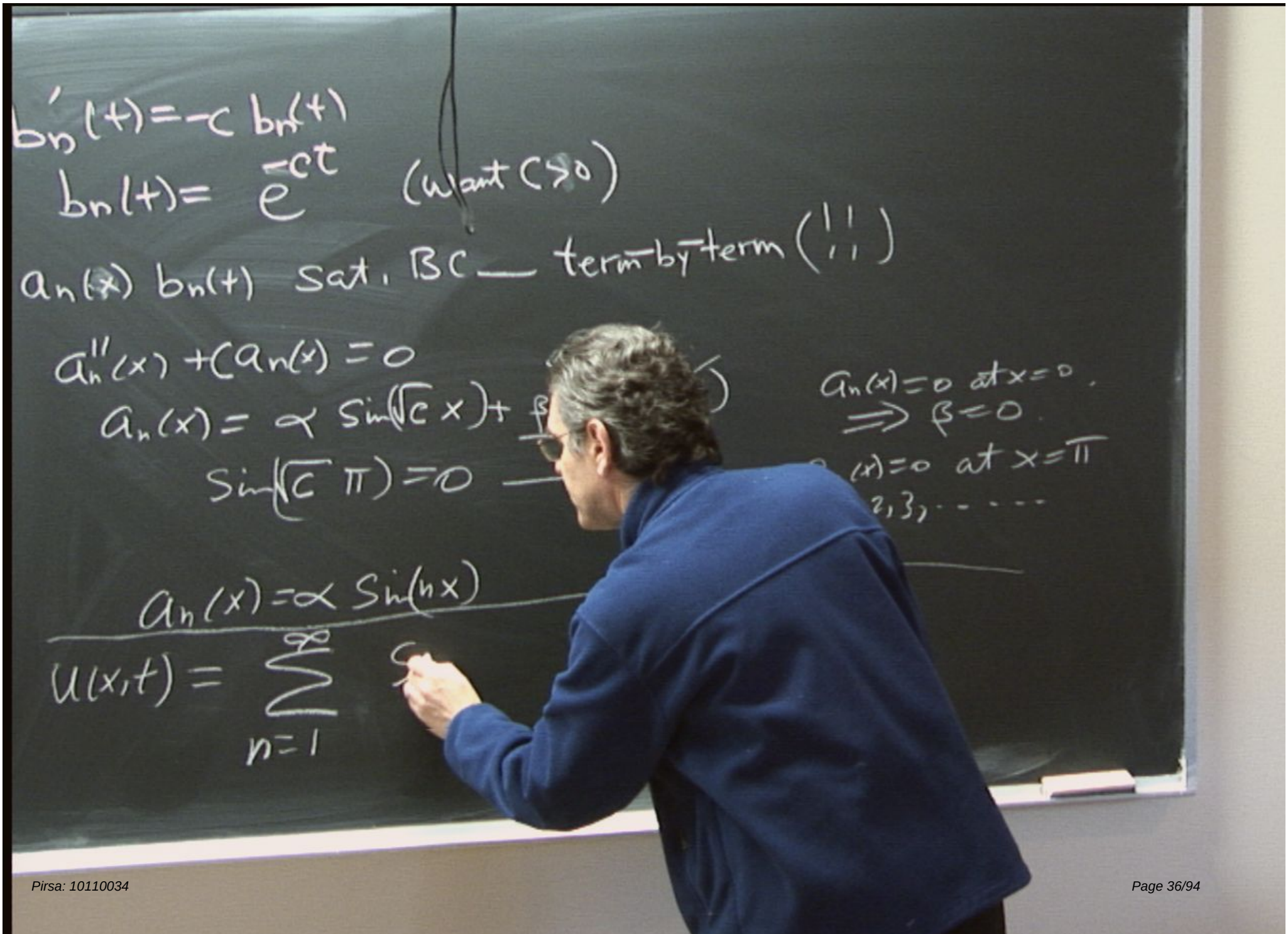
$$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} = n, \quad \underline{c = n^2}$$

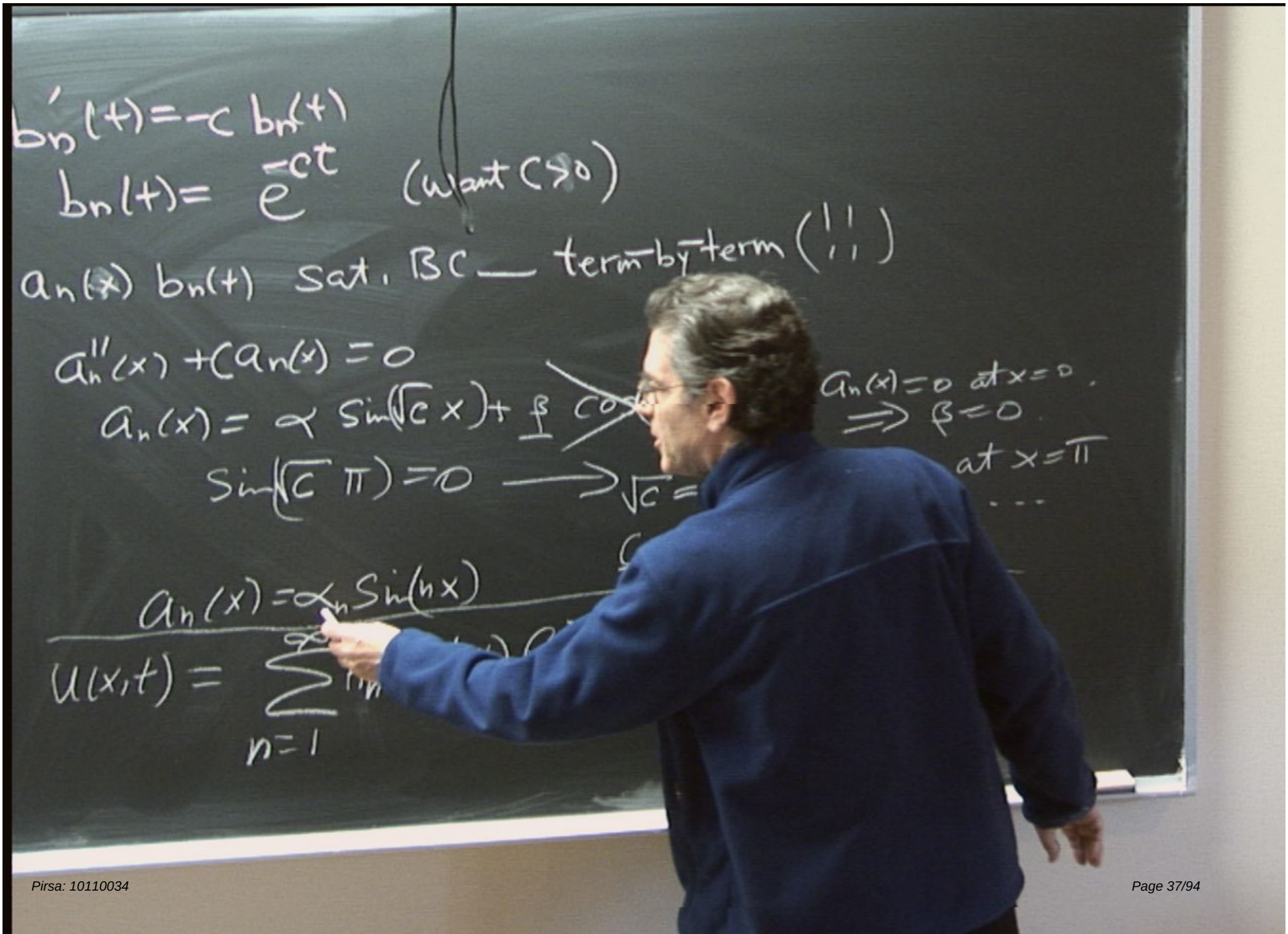
$a_n(x) = 0$ at $x = \pi$

$n = 1, 2, 3, \ldots$

$$a_n(x) = \alpha \sin(nx)$$


b_n'(t) = -c b_n(t)
b_n(t) = e^{-ct} (want c > 0)
a_n(x) b_n(t) sat. BC — term-by-term
a_n''(x) + c a_n(x) = 0
a_n(x) = α sin(√c x) + β cos(√c
a_n(x) = 0 at x = 0.
⟹ β = 0
sin(√c π) = 0 ⟶ √c =
=0 at x = π
a_n(x) = α sin(nx)
u(x,t) = Σ_{n=1}^{∞}


$b_n'(t) = -c\, b_n(t)$
$b_n(t) = e^{-ct}$ (want $c > 0$)
$a_n(x)\, b_n(t)$ sat. BC — term-by-term
$a_n''(x) + c\, a_n(x) = 0$
$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \beta$
$\sin(\sqrt{c}\, \pi) = 0$
$a_n(x) = 0$ at $x = 0$.
$\Rightarrow \beta = 0$
$(x) = 0$ at $x = \pi$
$2, 3, \ldots$
$a_n(x) = \alpha \sin(nx)$
$u(x,t) = \sum_{n=1}^{\infty}$


$b_n'(t) = -c\, b_n(t)$
$b_n(t) = e^{-ct}$ (want $c > 0$)
$a_n(x)\, b_n(t)$ sat. BC — term-by-term
$a_n''(x) + c\, a_n(x) = 0$
$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \beta \cos$
$a_n(x) = 0$ at $x = 0$.
$\Rightarrow \beta = 0$
$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} =$
at $x = \pi$
$a_n(x) = \alpha_n \sin(nx)$
$u(x,t) = \sum_{n=1}^{\infty}$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC — term-by-term (! !)

$$a_n''(x) + c\, a_n(x) = 0$$

$$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \cancel{\beta \cos(\sqrt{c}\, x)}$$

$a_n(x) = 0$ at $x = 0$.

$\Longrightarrow \beta = 0$.

$$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} = n, \quad \underline{c = n^2}$$

$a_n(x) = 0$ at $x = \pi$

$n = 1, 2, 3, \ldots$

$$a_n(x) = \alpha_n \sin(nx)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\, e^{-n^2 t}$$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC — term-by-term (!!)

$$a_n''(x) + c\, a_n(x) = 0$$

$$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \cancel{\beta \cos(\sqrt{c}\, x)}$$

$a_n(x) = 0$ at

$\Longrightarrow \beta = 0$

$$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} = n, \quad \underline{c = n^2}$$

$a_n(x) = 0$ at x

$n = 1, 2, 3, \ldots$

$$a_n(x) = \alpha_n \sin(nx)$$

PDE ✓

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(nx)\, e^{-n^2 t}$$

$$b_n'(t) = -c\, b_n(t)$$

$$b_n(t) = e^{-ct} \quad (\text{want } c > 0)$$

$a_n(x)\, b_n(t)$ sat. BC — term-by-term $\left(\begin{matrix} | & | \\ | & | \end{matrix}\right)$

$$a_n''(x) + c\, a_n(x) = 0$$

$$a_n(x) = \alpha \sin(\sqrt{c}\, x) + \cancel{\beta \cos(\sqrt{c}\, x)}$$

$a_n(x) = 0$ at $x = 0$.

$\Longrightarrow \beta = 0$.

$$\sin(\sqrt{c}\, \pi) = 0 \longrightarrow \sqrt{c} = n, \quad \underline{c = n^2}$$

$a_n(x) = 0$ at $x = \pi$

$n = 1, 2, 3, \ldots$

$$a_n(x) = \alpha_n \sin(nx)$$

$$u(x,t) = \sum_{n=1}^{\infty} \underline{A_n} \sin(nx)\, e^{-n^2 t}$$

PDE ✓

BC ✓

Fourier Series

$u(x,t)$ = Temp at x, at time t.

$u_t = u_{xx}$

IC $\cancel{u(x,0) = f(x) \quad \text{all } x}$

~~Cauchy prob.~~

$$u(x,t) = \sum_{n=0}^{\infty} a_n(x)\, \cancel{t^n}\, b_n(t)$$

IC $u(x,0) = f(x) \qquad 0 \le x \le \pi$

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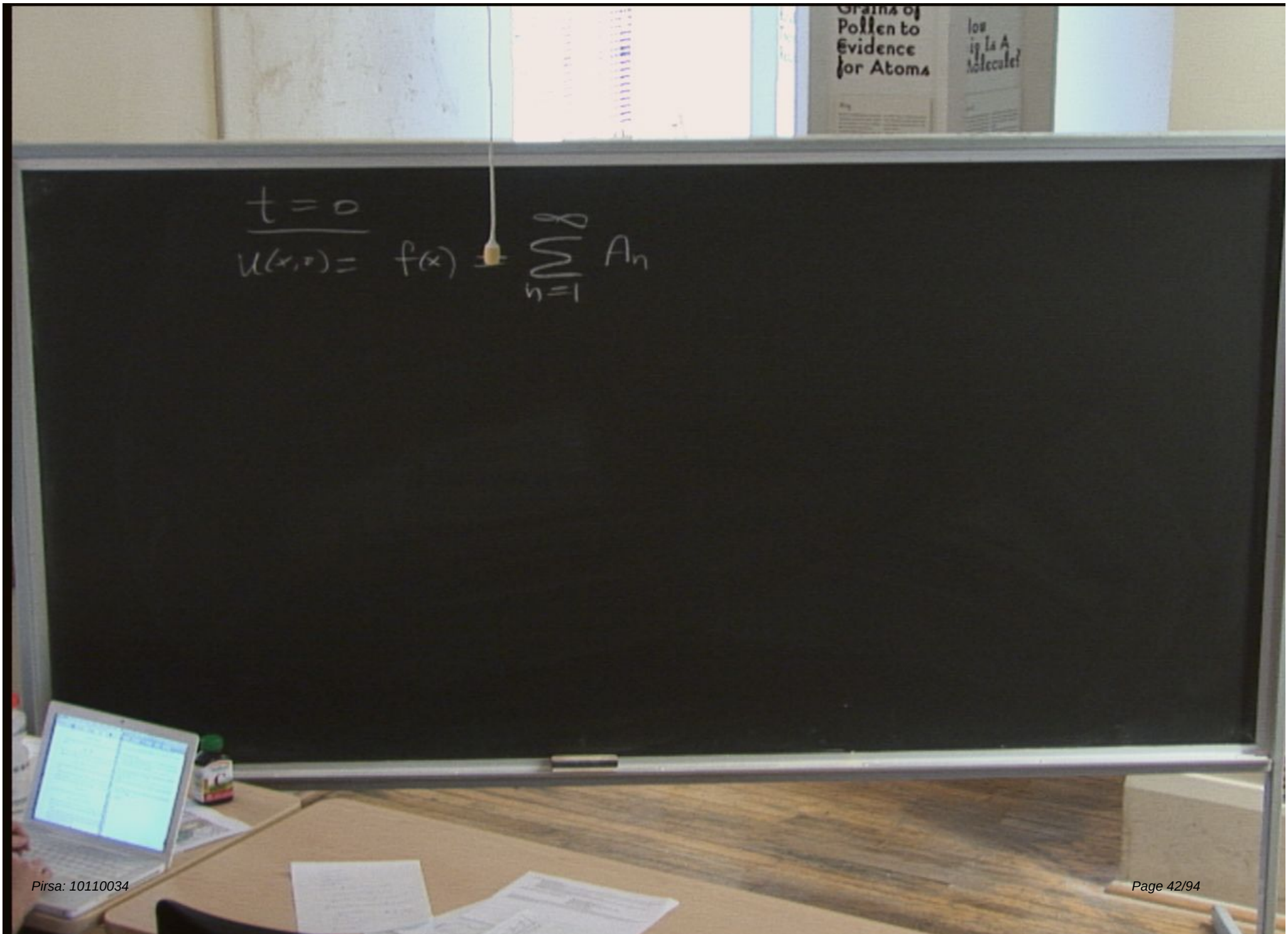
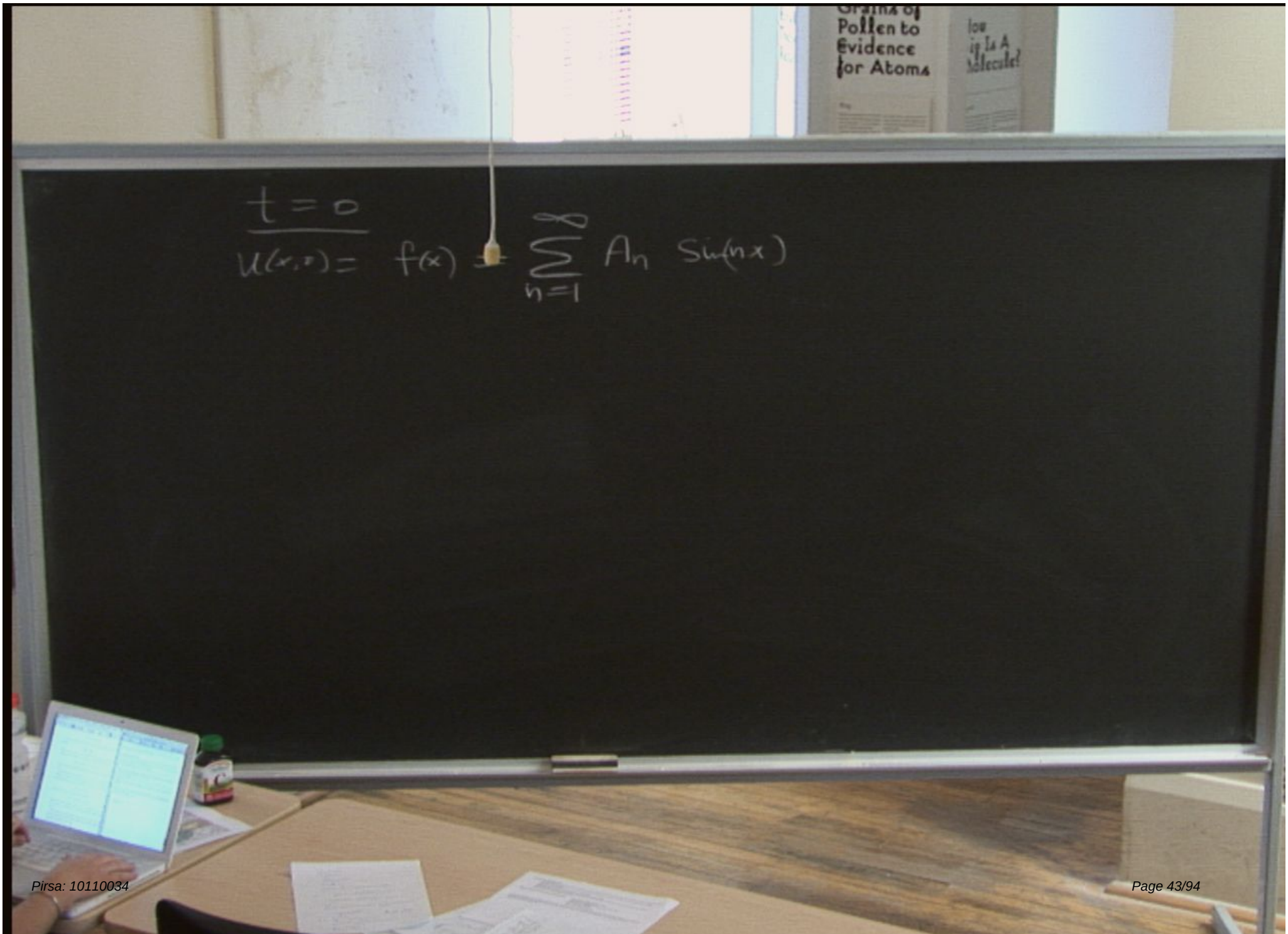
$u(\pi,t) = h(t) = 0$

for any n: $a_n(x)\, b_n'(t) = a_n''(x)\, b_n(t)$

$$\frac{b_n'(t)}{b_n(t)} = \frac{a_n''(x)}{a_n(x)} = -c$$

sep'n const


Grains of
Pollen to
Evidence
for Atoms
How
Is A
Molecule?
t = 0
u(x,0) = f(x) = Σ_{n=1}^{∞} A_n


Pollen to
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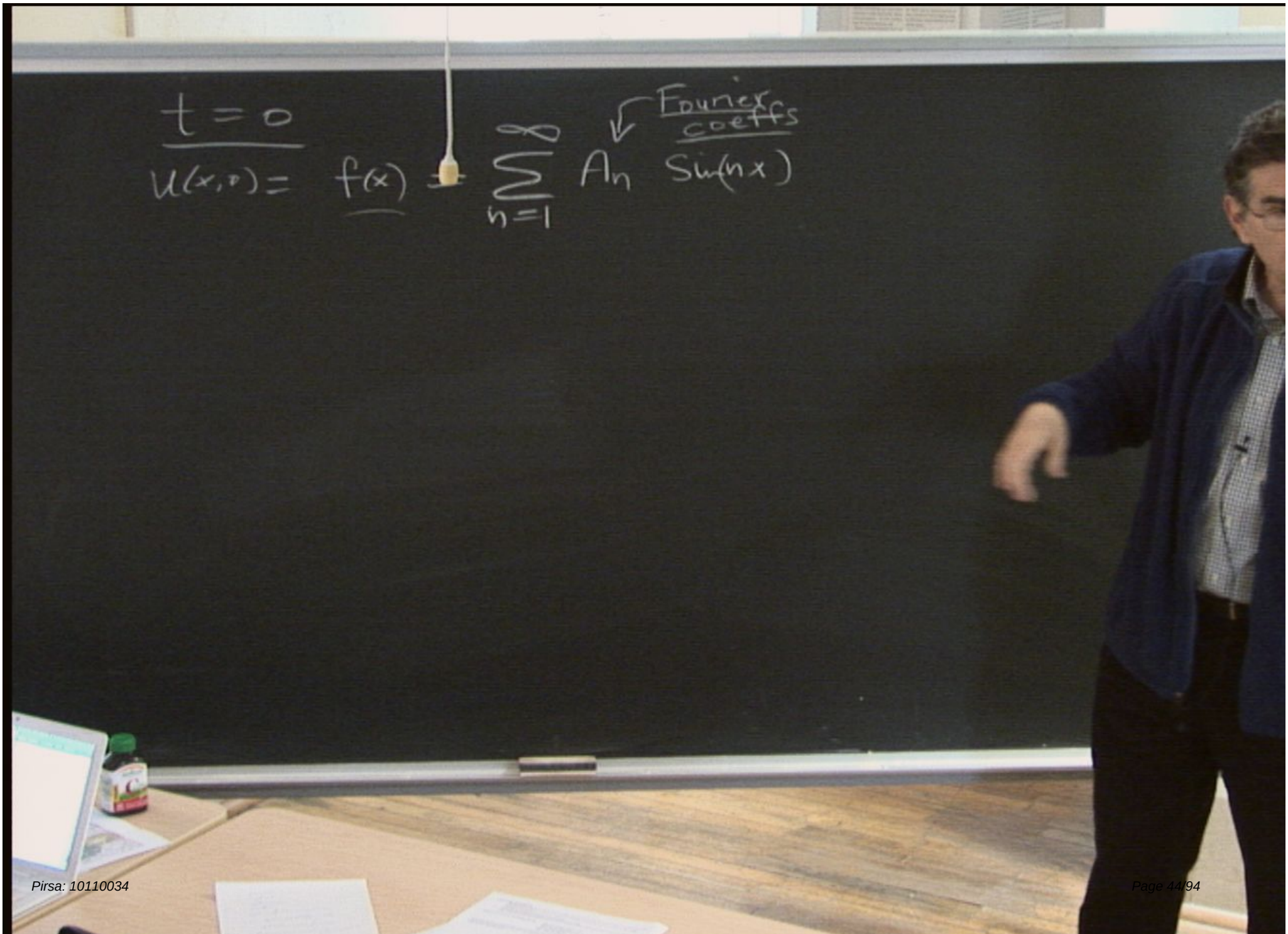
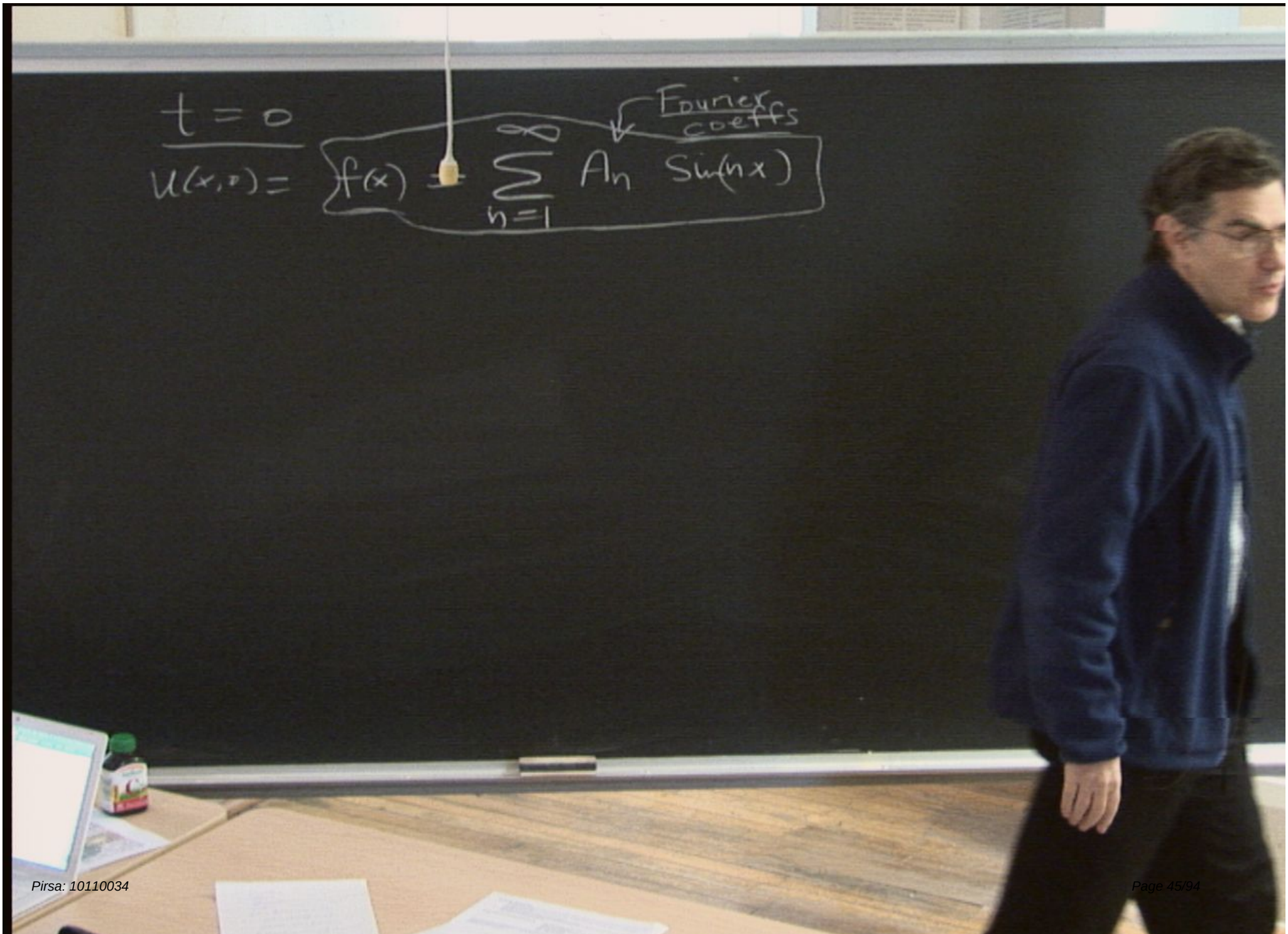
t = 0
u(x,0) = f(x) = Σ_{n=1}^{∞} A_n sin(nx)


t = 0
u(x,0) = f(x) = Σ_{n=1}^{∞} A_n sin(nx)
Fourier coeffs


t = 0
u(x,0) = f(x) = Σ_{n=1}^{∞} A_n sin(nx)
Fourier coeffs

$$\underline{t=0}$$

$$u(x,0) = \boxed{f(x) = \sum_{n=1}^{\infty} A_n \, \underline{\sin(nx)}}$$

Fourier coeffs

$t = 0$

$(x,0) = \boxed{f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)}$

Fourier coeffs

$$\int_0^{\pi} dx \, \sin(nx) \sin(mx) = \begin{cases} 0 & n \neq m. \end{cases}$$

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Fourier coeffs

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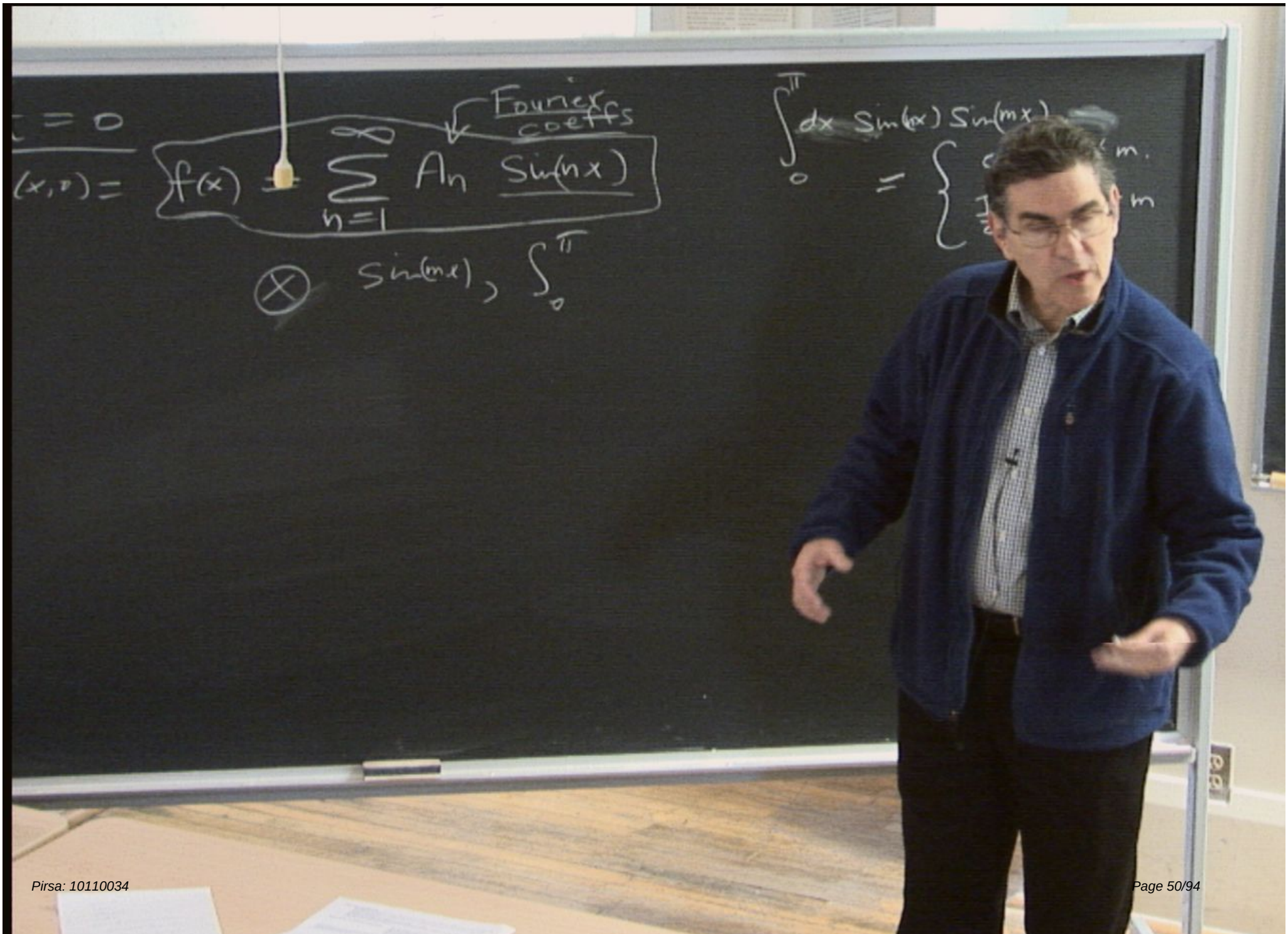
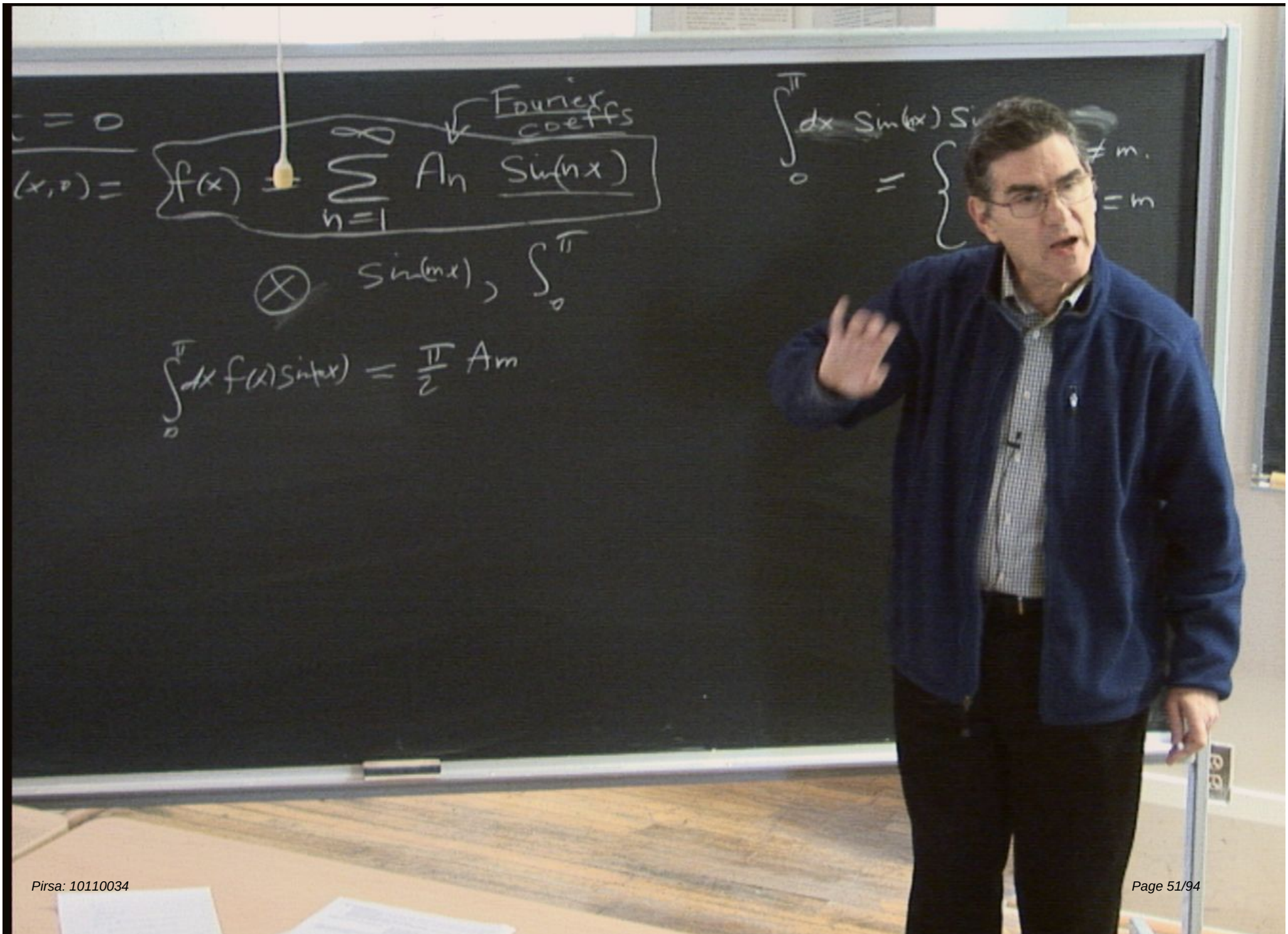
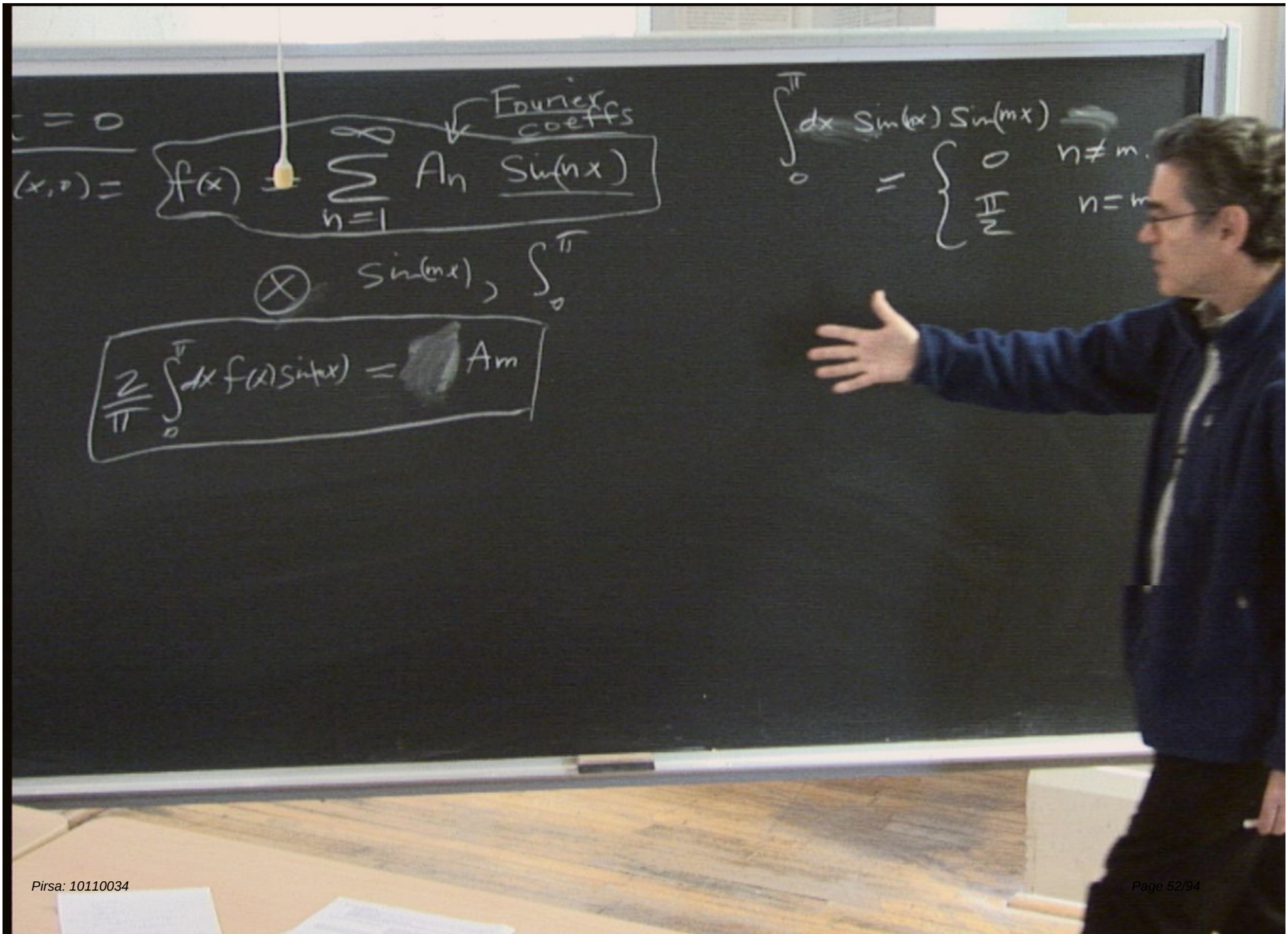

Fourier coeffs
f(x) = Σ An Sin(nx)
n=1
sin(mx), ∫0π
∫0π dx Sin(nx) Sin(mx)


Fourier coeffs
$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$
$\sin(mx)$, $\int_0^{\pi}$
$\int_0^{\pi} dx\, f(x) \sin(mx) = \frac{\pi}{2} A_m$


Fourier coeffs
$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$
$\int_0^{\pi} dx \sin(nx) \sin(mx) = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$
$\sin(mx)$, $\int_0^{\pi}$
$\frac{2}{\pi} \int_0^{\pi} dx\, f(x) \sin(mx) = A_m$

$t = 0$

$(x,0) = \boxed{f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)}$

Fourier coeffs

$\otimes \sin(mx), \int_0^{\pi}$

$$\boxed{\frac{2}{\pi}\int_0^{\pi} dx\, f(x)\sin(mx) = A_m}$$

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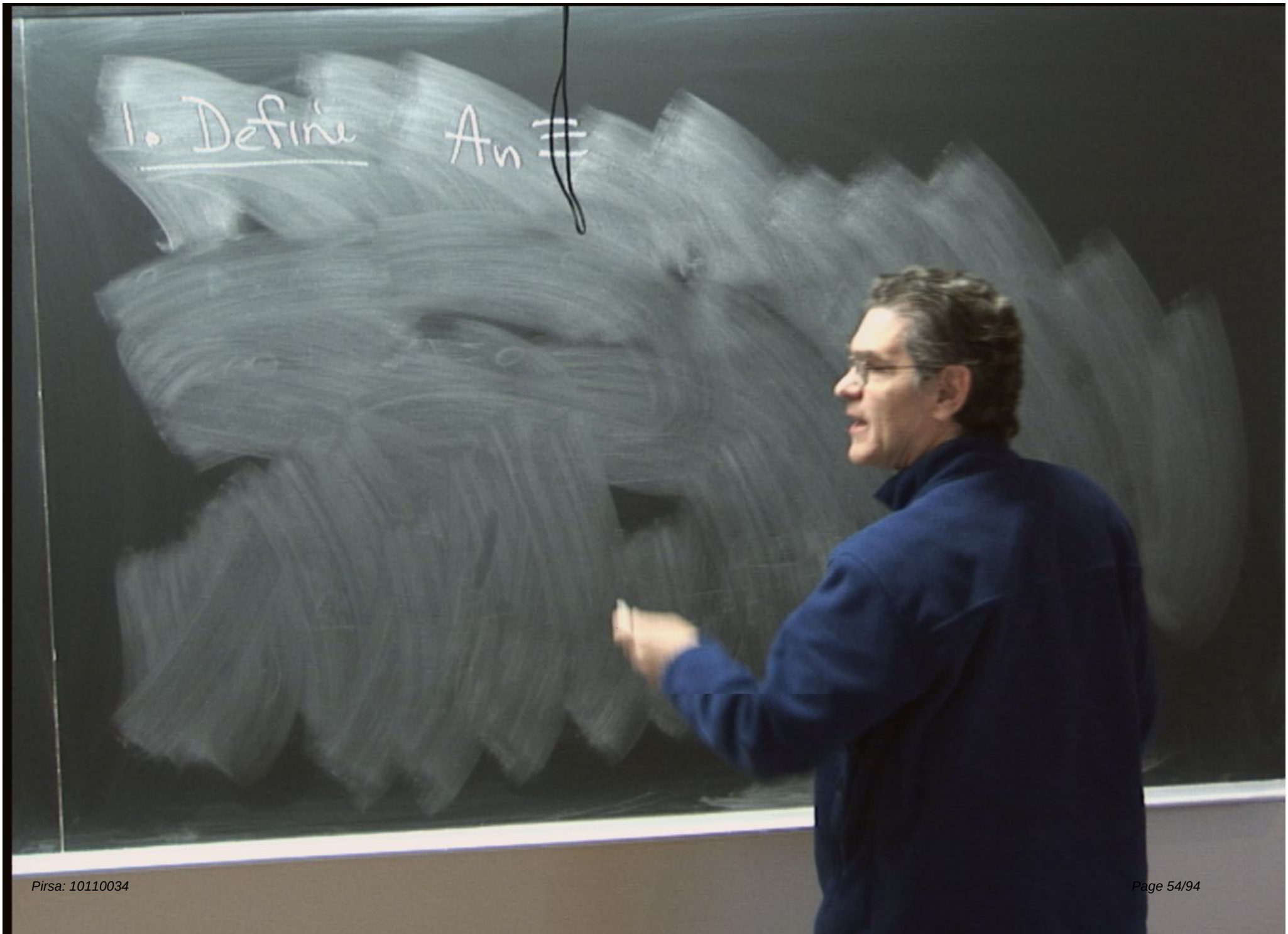
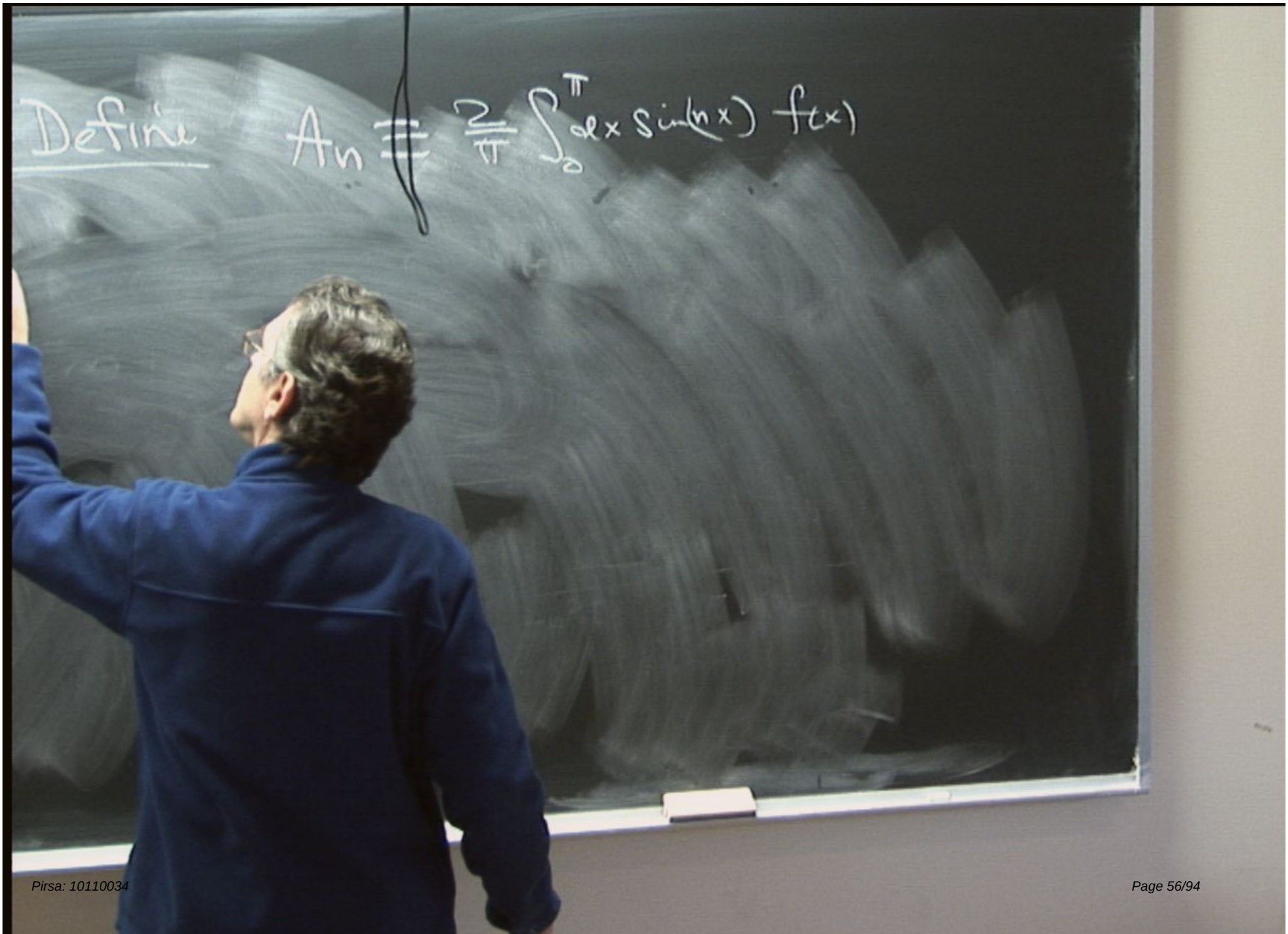

1. Define
An ≡

1. Define $A_n \equiv \frac{2}{\pi} \int_0^{\pi} dx \, \sin(nx) \, f(x)$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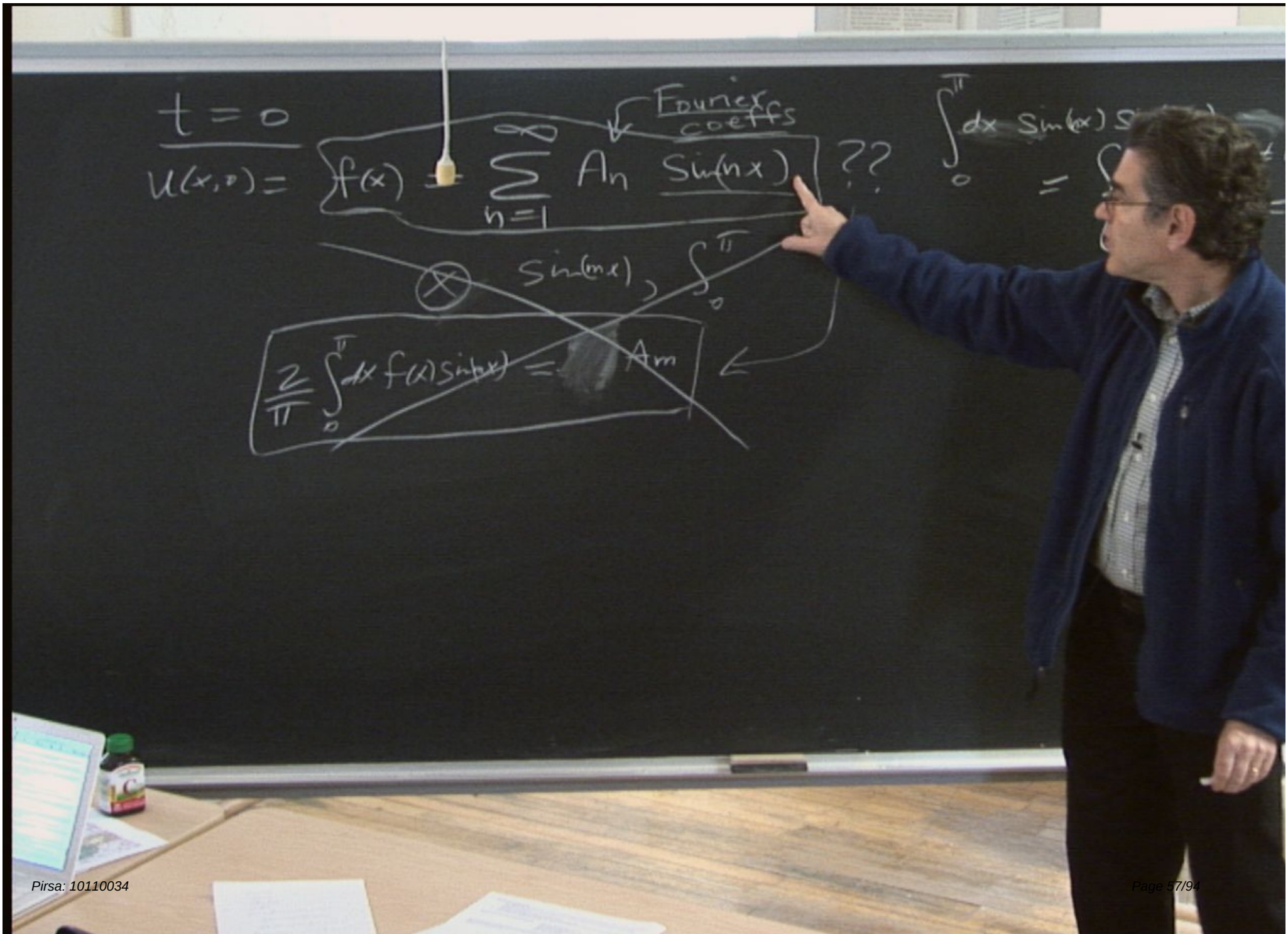
t = 0
u(x,0) = f(x) = Σ_{n=1}^{∞} An sin(nx) ??
Fourier coeffs
sin(mx), ∫_0^π
2/π ∫_0^π dx f(x) sin(mx) = Am

1. Define $A_n \equiv \frac{2}{\pi} \int_0^{\pi} dx \, \sin(nx) \, f(x)$

2. How does A_n behave as $n \to \infty$.

$$\underline{t=0}$$

$$u(x,0) = \boxed{f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)} \quad ??$$

Fourier coeffs

$$\int_0^{\pi} dx \, \sin(nx)\sin(mx) = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$$\cancel{\otimes \quad \sin(mx), \int_0^{\pi}}$$

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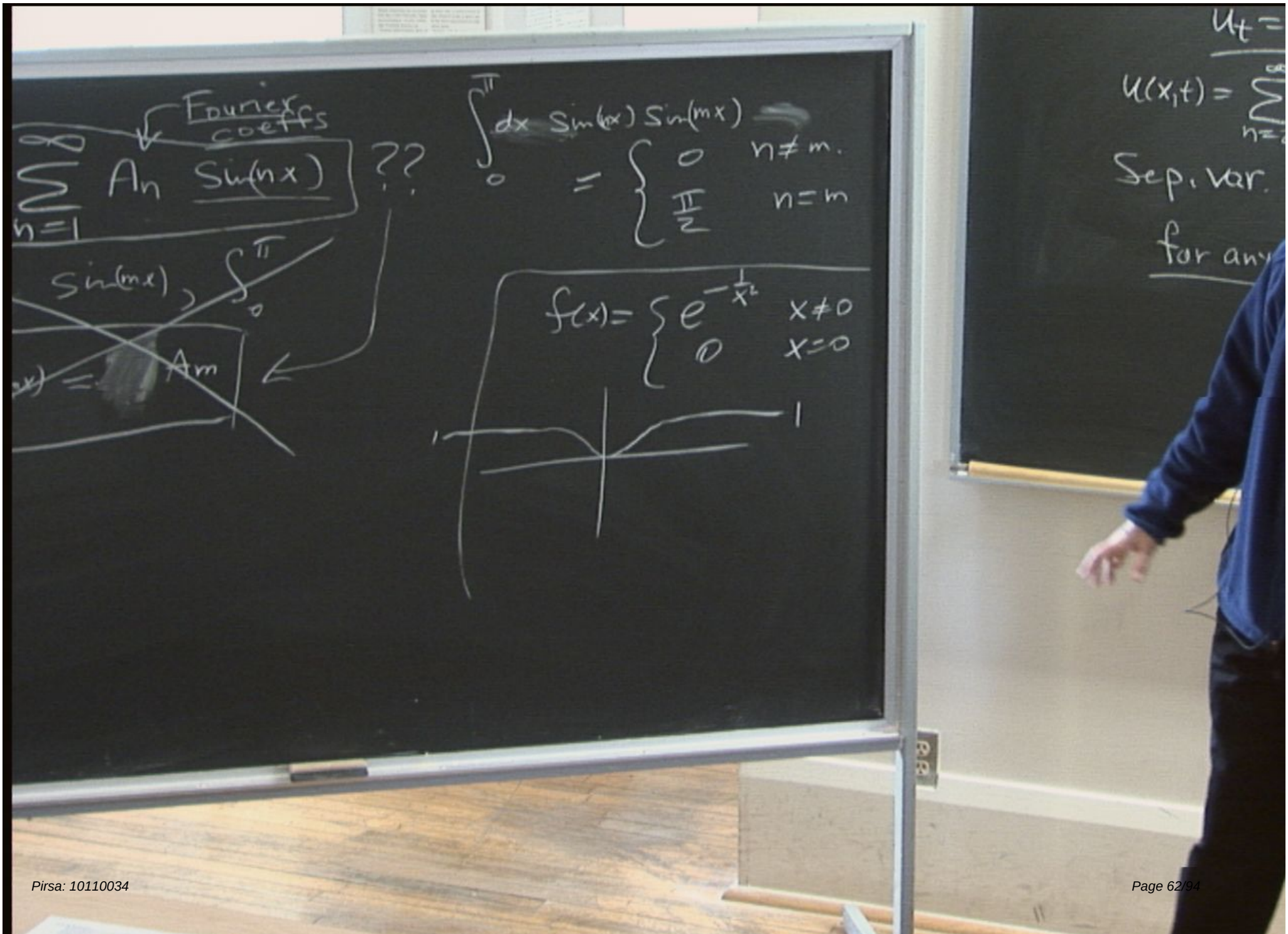
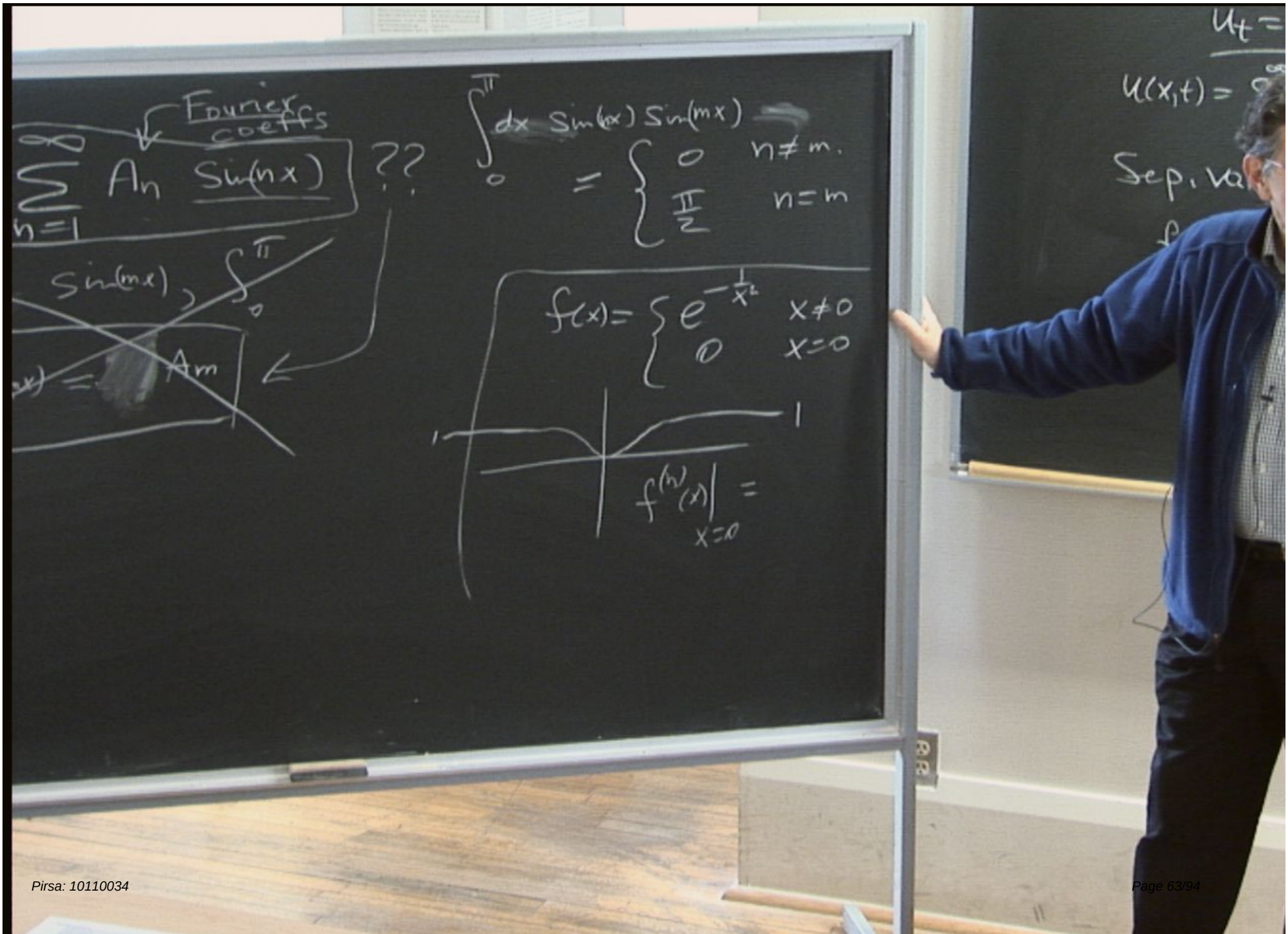

Fourier coeffs
$\sum_{n=1}^{\infty} A_n \sin(nx)$??
$\int_0^{\pi} dx \sin(nx)\sin(mx) = \begin{cases} 0 & n \neq m. \\ \frac{\pi}{2} & n = m \end{cases}$
$\sin(mx)$, $\int_0^{\pi}$
A_m
$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$
1
1
$u_t =$
$u(x,t) = \sum$
Sep. var.
for any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$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$
$f^{(n)}(x)\Big|_{x=0} =$
$u_t =$
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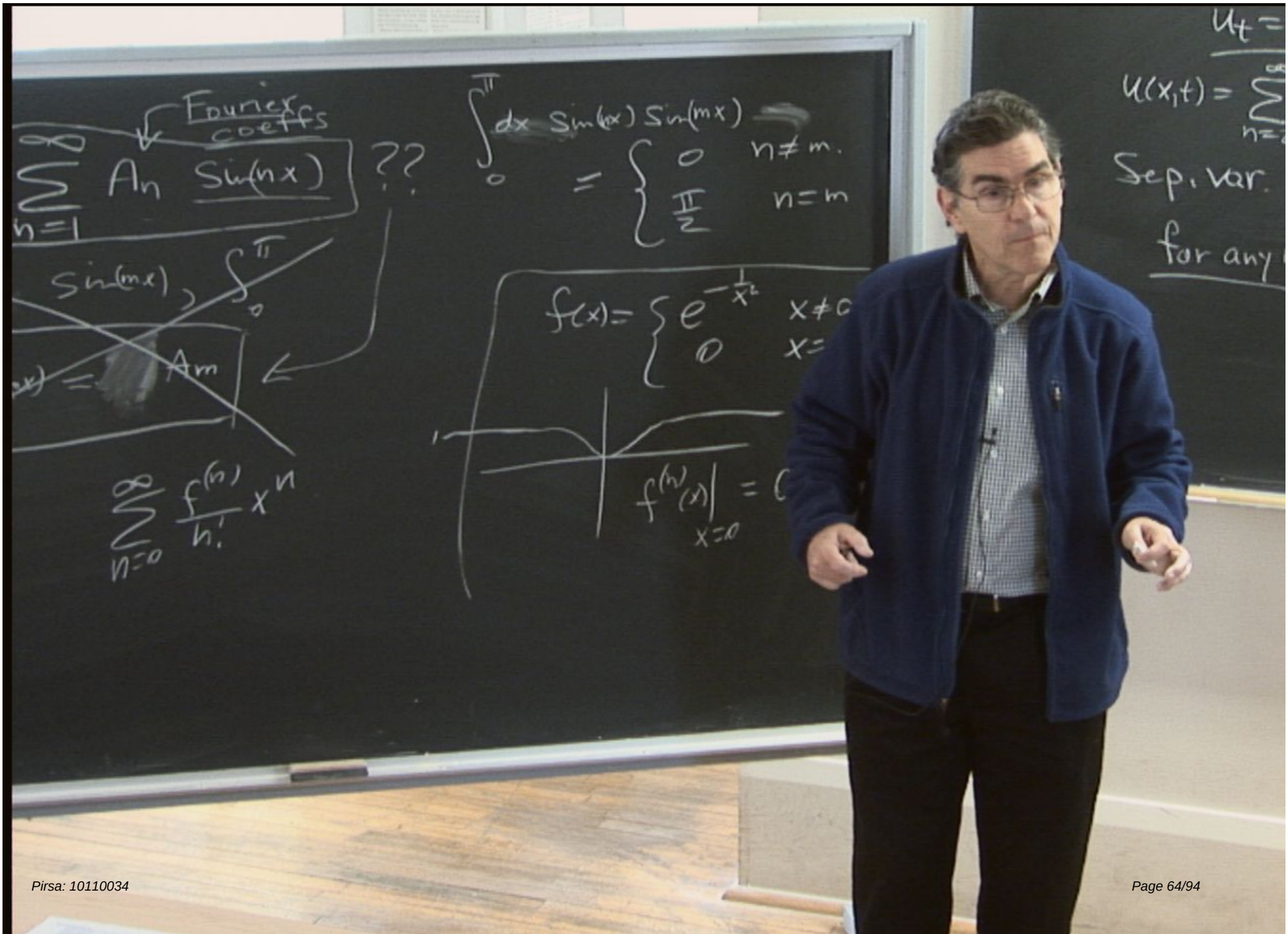
Sep. va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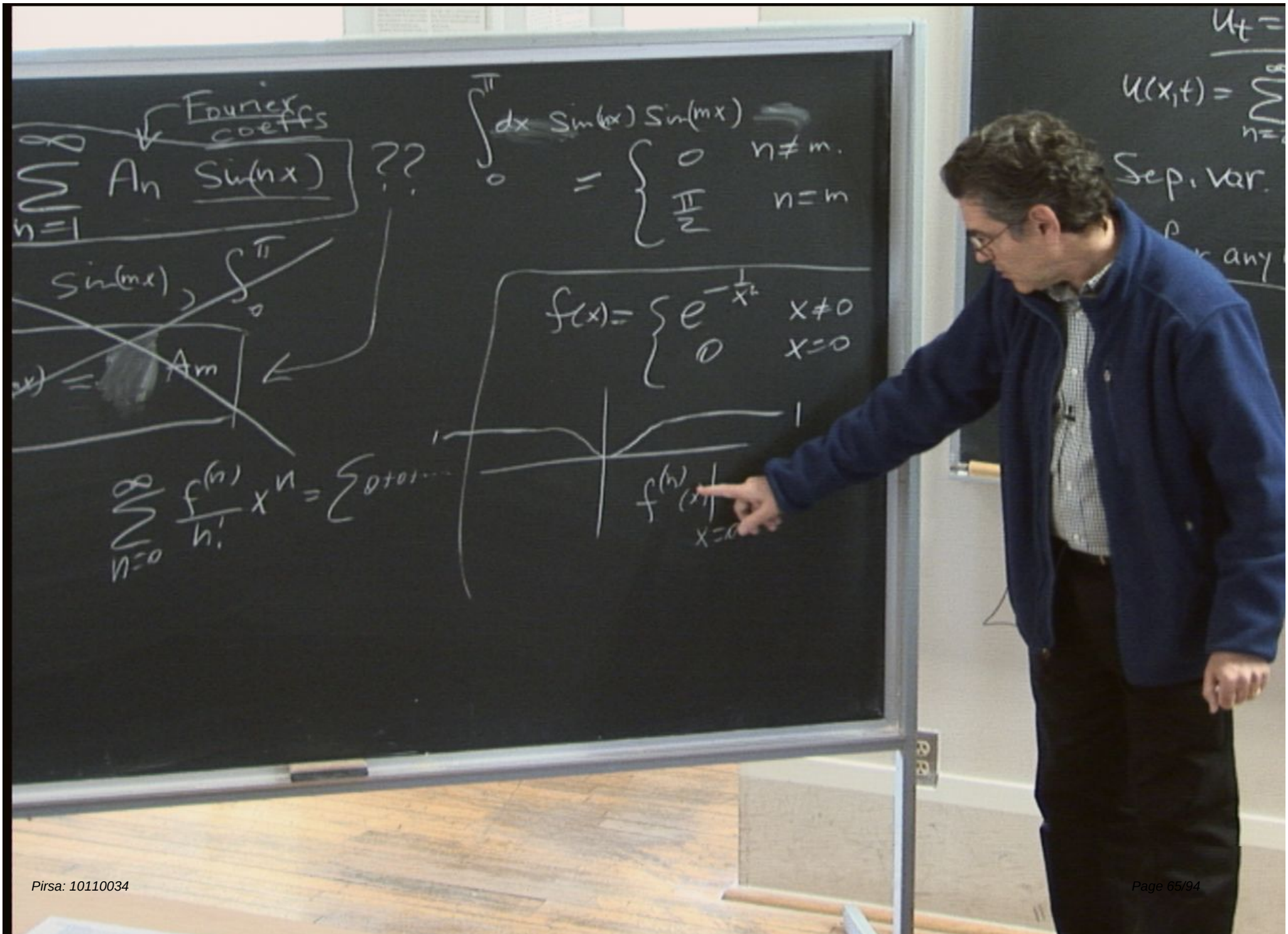
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$f^{(n)}(x)|_{x=0}$
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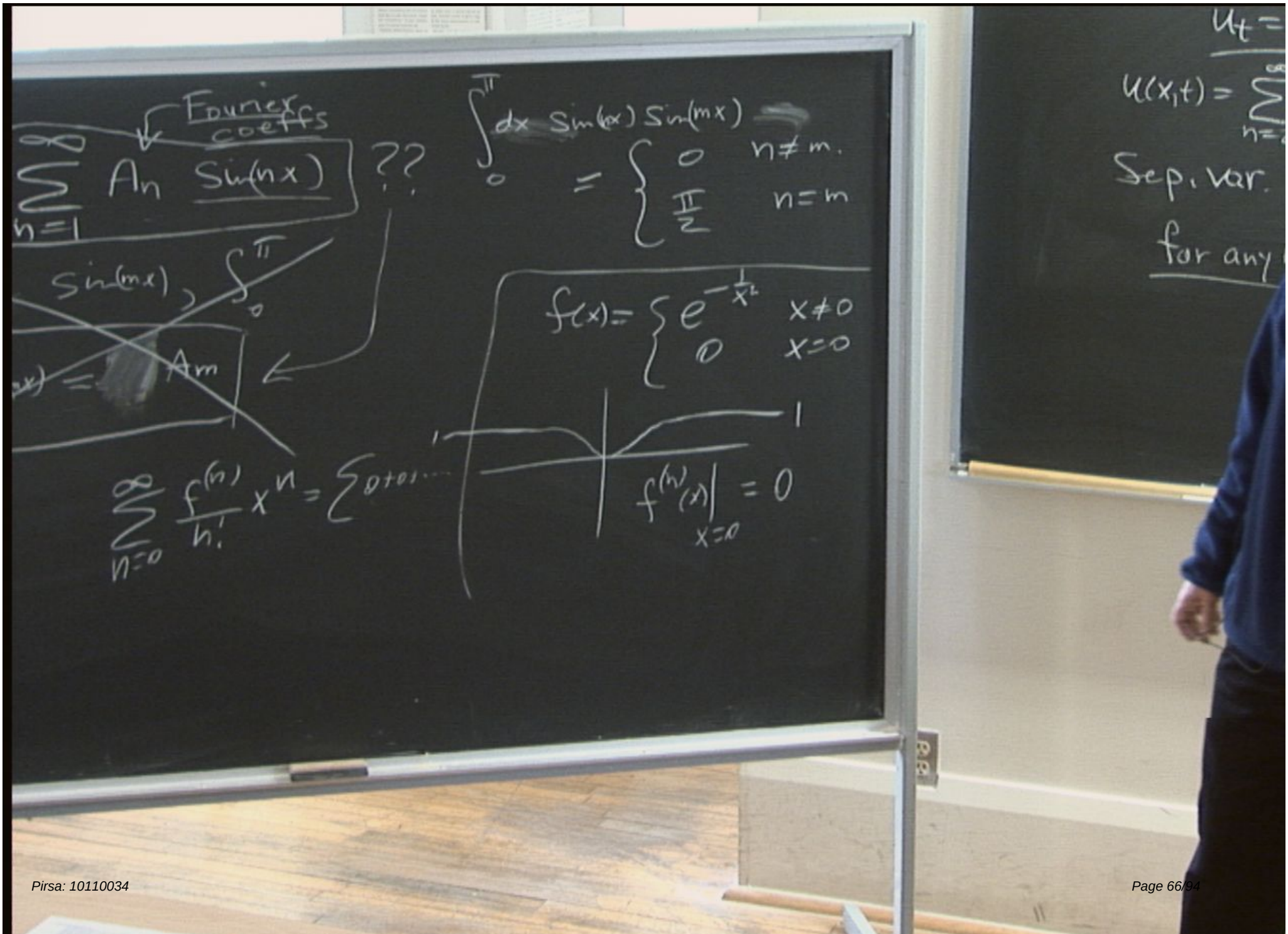
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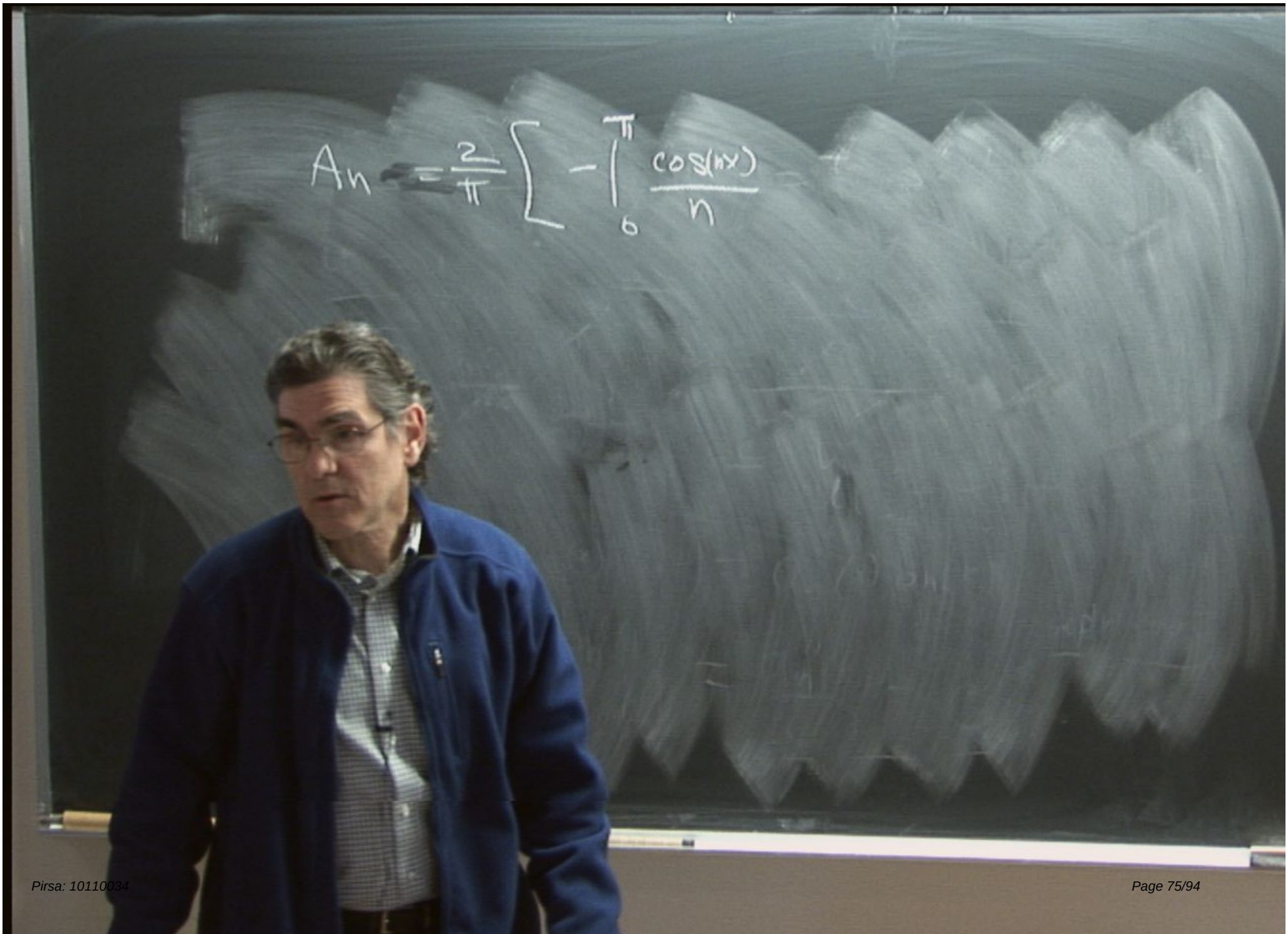
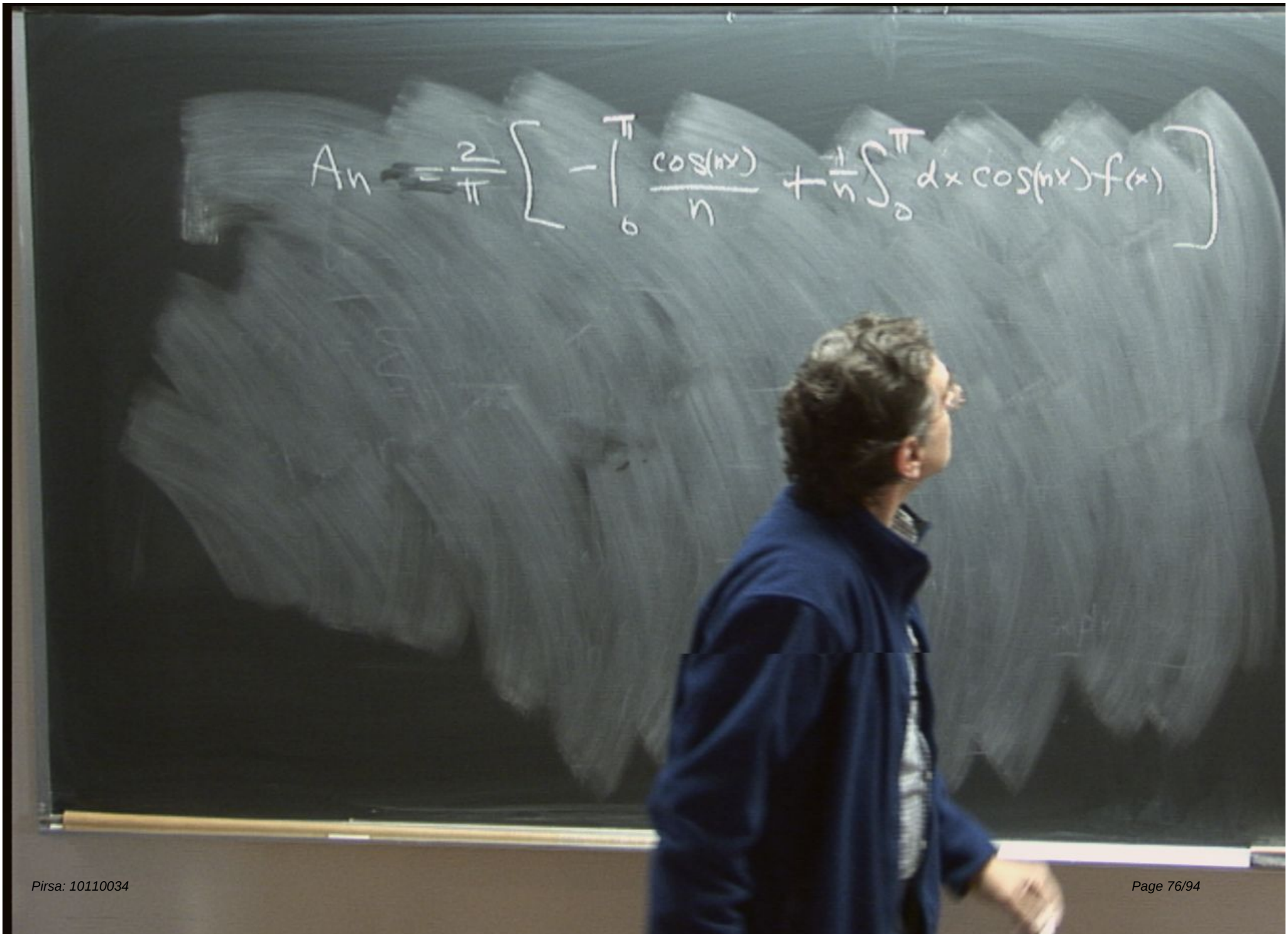
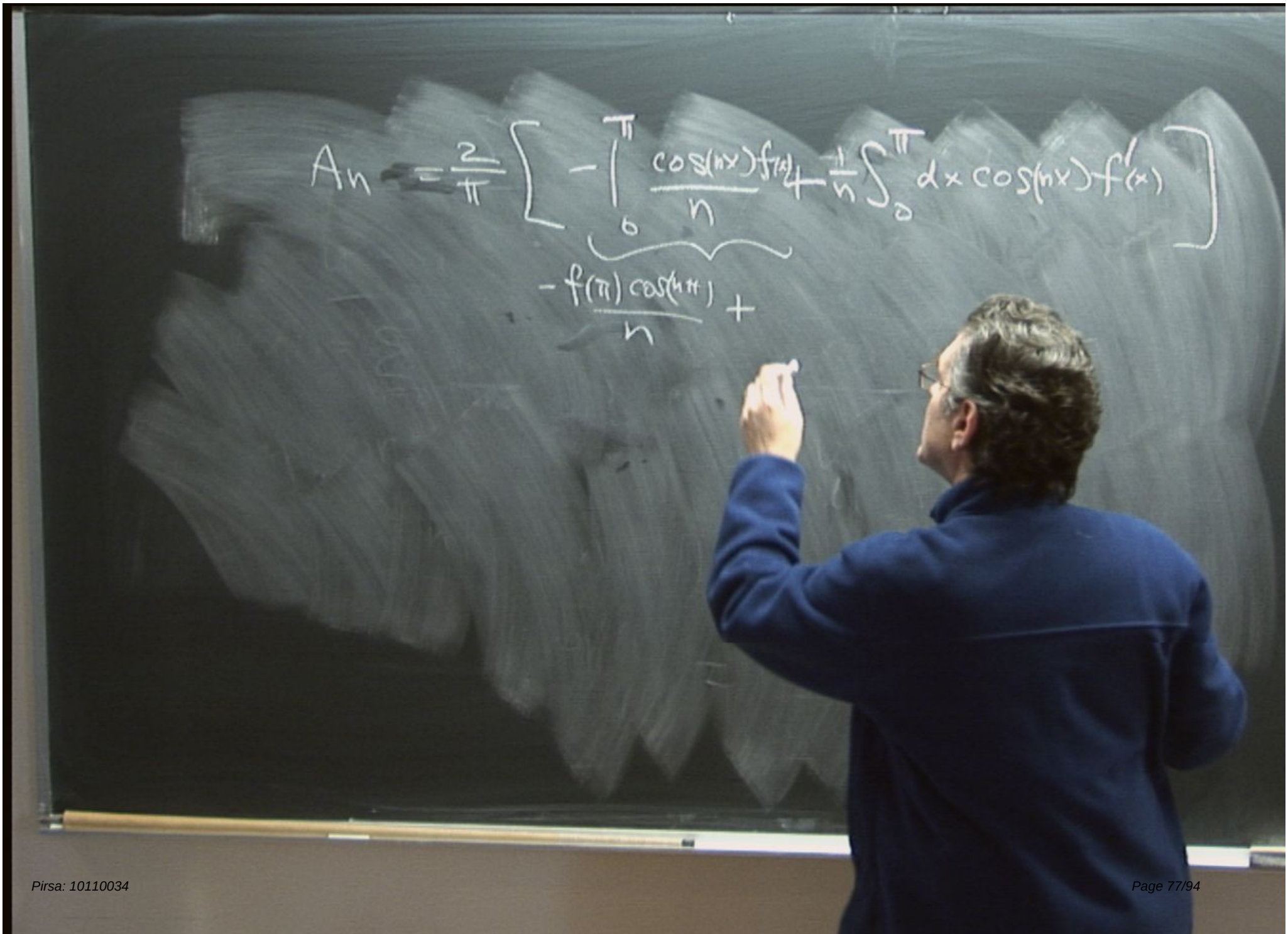
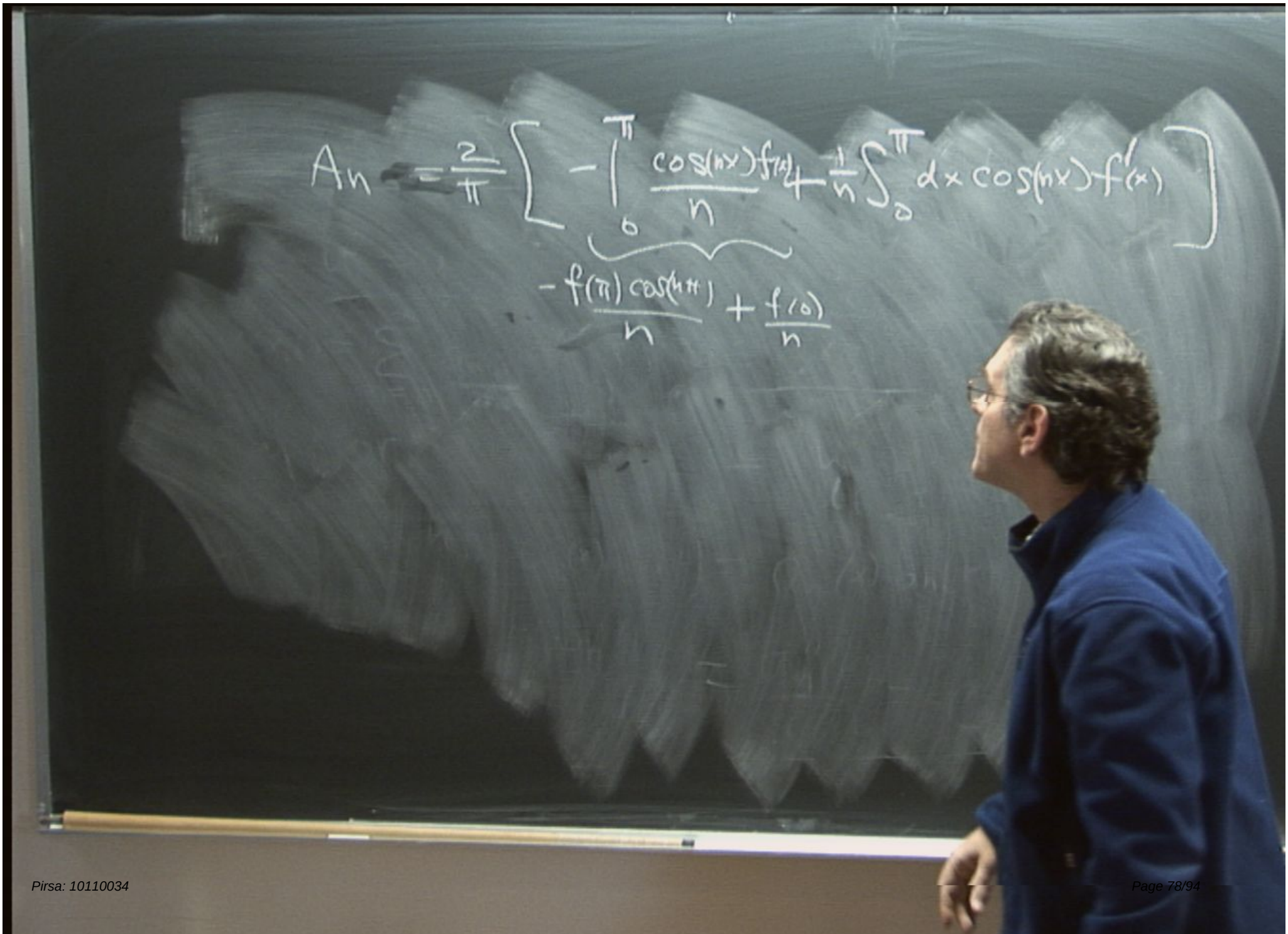

$A_n = \frac{2}{\pi}\left[-\Big|_0^{\pi} \frac{\cos(nx)}{n} \right.$


$A_n = \frac{2}{\pi}\left[-\Big|_0^{\pi} \frac{\cos(nx)}{n} + \frac{1}{n}\int_0^{\pi} dx \cos(nx) f(x) \right]$


$A_n = \frac{2}{\pi}\left[\underbrace{-\left.\frac{\cos(nx) f(x)}{n}\right|_0^{\pi}}_{-\frac{f(\pi)\cos(n\pi)}{n} +} + \frac{1}{n}\int_0^{\pi} dx \cos(nx) f'(x) \right]$


$A_n = \frac{2}{\pi}\left[\underbrace{-\left.\frac{\cos(nx) f(x)}{n}\right|_0^{\pi}}_{-\frac{f(\pi)\cos(n\pi)}{n} + \frac{f(0)}{n}} + \frac{1}{n}\int_0^{\pi} dx \cos(nx) f'(x) \right]$

$$A_n = \frac{2}{\pi}\left[\underbrace{-\left.\frac{\cos(nx) f(x)}{n}\right|_0^{\pi}}_{-\frac{f(\pi)\cdot(-1)^n}{n} + \frac{f(0)}{n}} + \frac{1}{n}\underbrace{\int_0^{\pi} dx \cos(nx) f'(x)}_{RL \to 0.} \right]$$

$$A_n \to 0 \text{ like } \frac{1}{n}$$

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$A_n \to 0$ like $\frac{1}{n}$ unless $f(0)=0$, $f(\pi)=0$.

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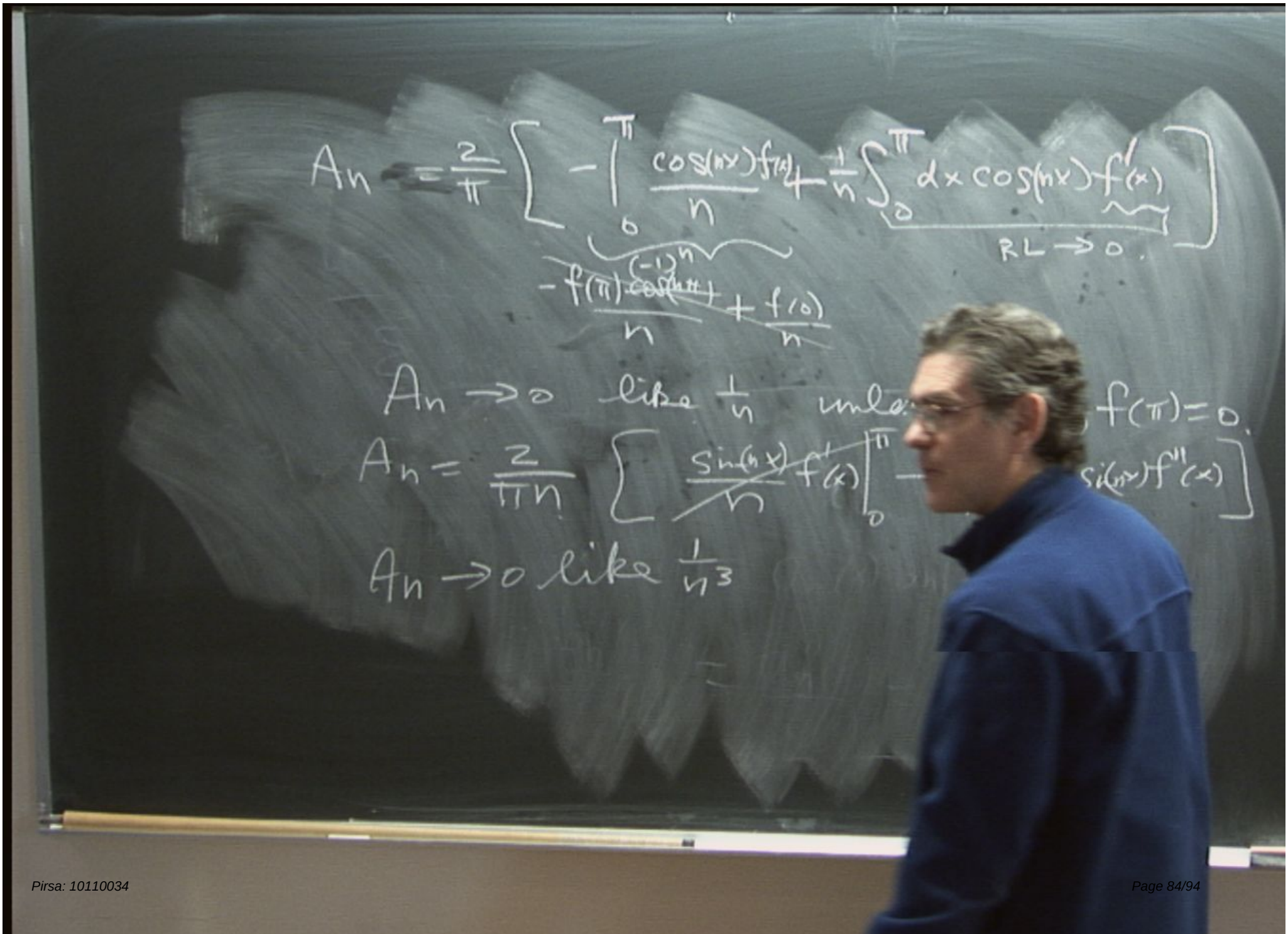
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$(-1)^n$
$-\frac{f(\pi)\cos(n\pi)}{n} + \frac{f(0)}{n}$
RL → 0.
$A_n \to 0$ like $\frac{1}{n}$ unless $f(\pi) = 0$.
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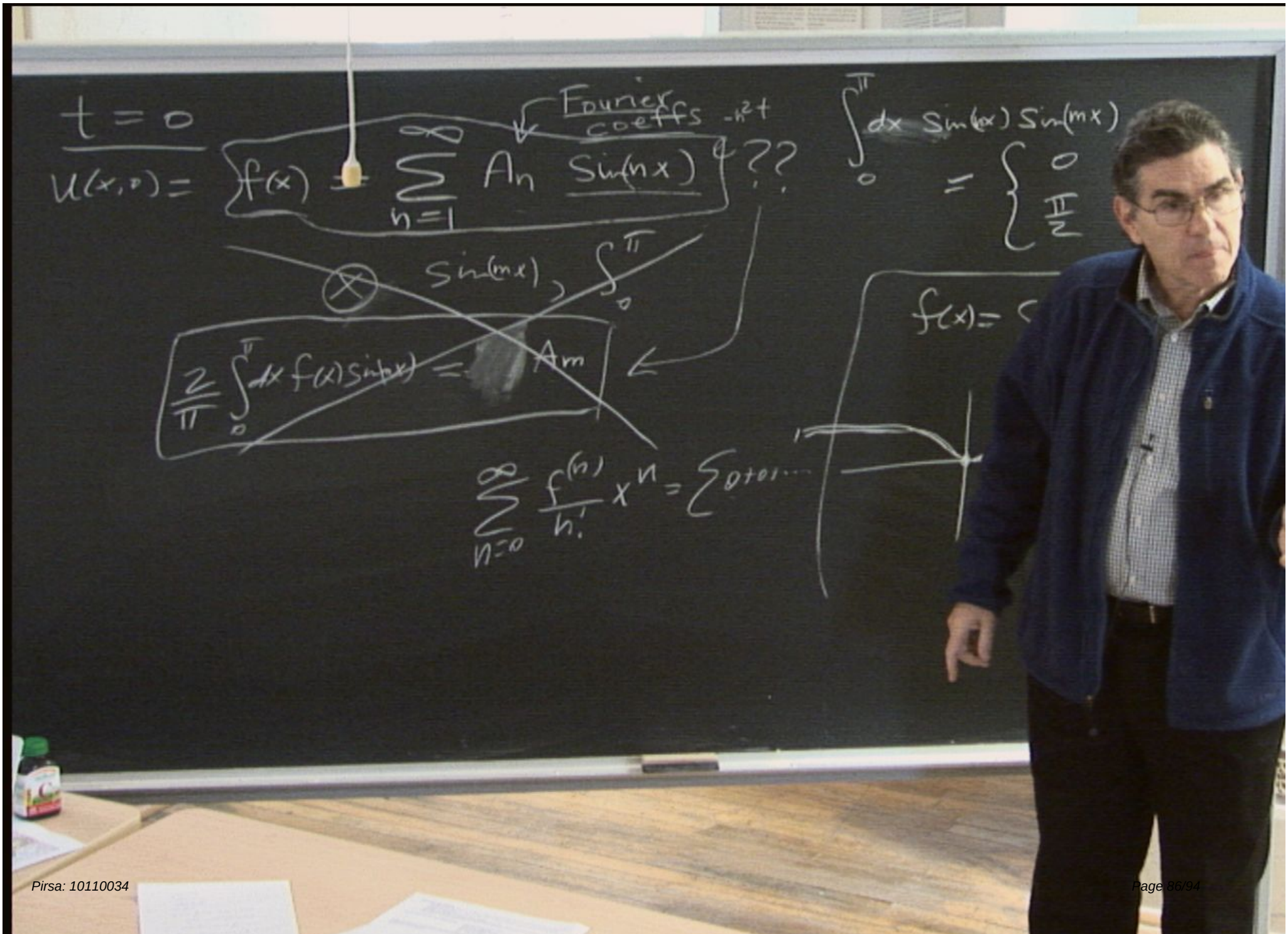
$A_n \to 0$ like $\frac{1}{n^3}$ unless $f''(0)=0$, $f''(\pi)=0$

$A_n \to 0$ like $\frac{1}{n^5}$


t = 0
Fourier coeffs
u(x,0) = f(x) = Σ_{n=1}^{∞} A_n sin(nx) ??
sin(mx), ∫_0^π
2/π ∫_0^π dx f(x) sin(mx) = A_m
∫_0^π dx sin(nx) sin(mx) = { 0, π/2
f(x) = c
Σ_{n=0}^{∞} f^(n)/n! x^n = Σ 0+0+0 ...

$$\underline{t=0}$$

$$u(x,0) = \boxed{f(x) = \sum_{n=1}^{\infty} A_n \, \underline{\sin(nx)}} \; ??$$

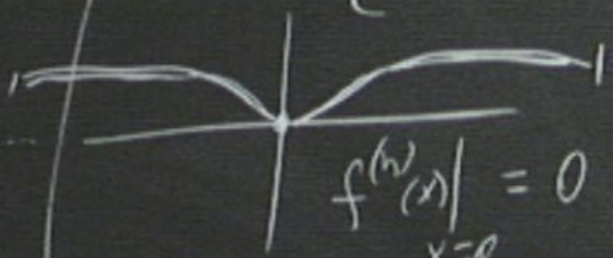
Fourier coeffs $\quad e^{-n^2 t}$

$$\int_0^{\pi} dx \, \sin(nx)\sin(mx) = \begin{cases} 0 & n \neq m. \\ \frac{\pi}{2} & n = m \end{cases}$$

$$\cancel{\otimes \; \sin(mx), \int_0^{\pi}}$$

$$\cancel{\boxed{\frac{2}{\pi}\int_0^{\pi} dx \, f(x)\sin(mx) = A_m}}$$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$$\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = \{0 + 0 + \cdots$$

$$f^{(n)}(x)\Big|_{x=0} = 0$$

Fourier sine series

Typically slowly convergent series!

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

$$0 \leq x \leq \pi$$

Calculate Fourier Coefficients:

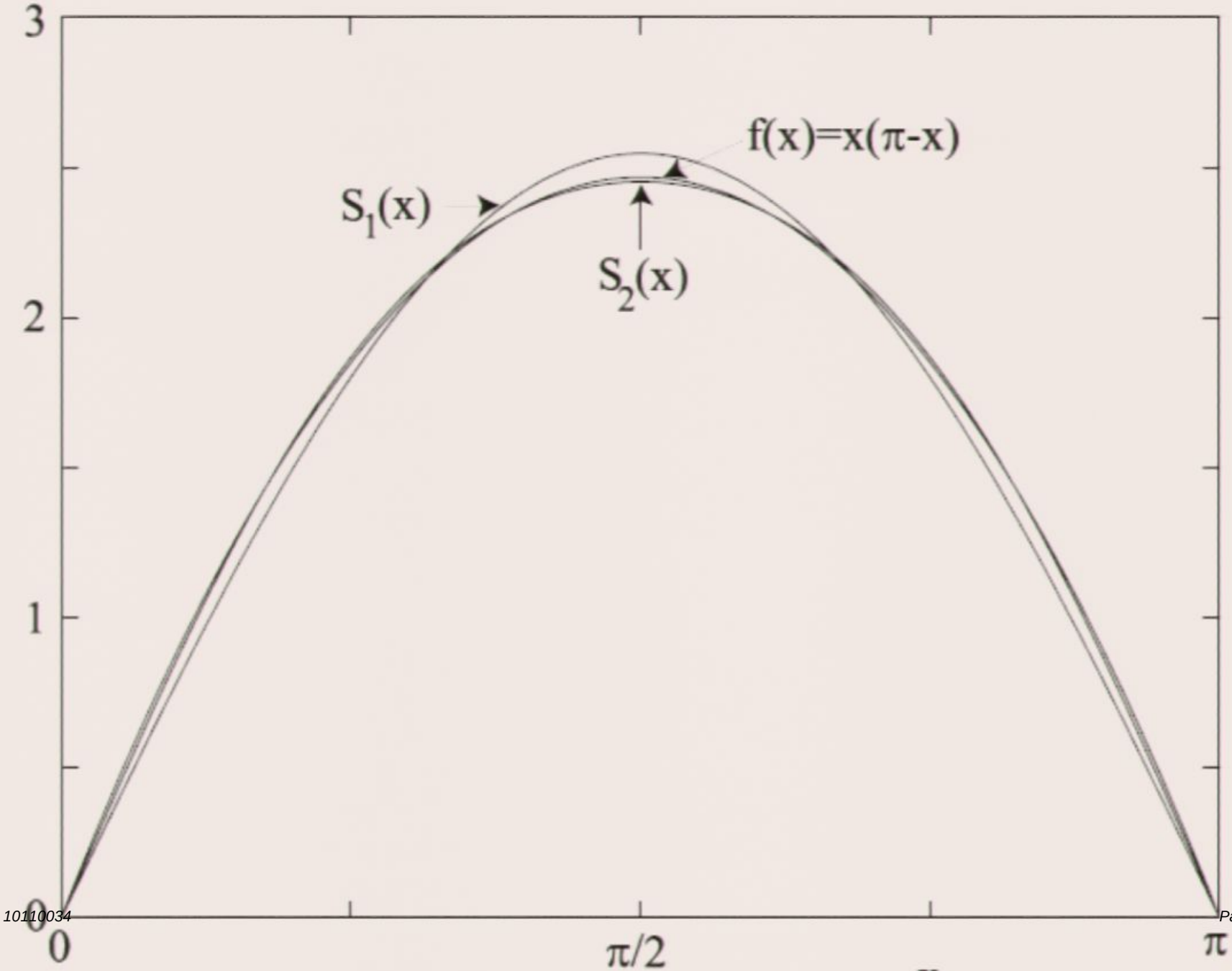
$$\int_0^\pi dx\, \sin(nx)\sin(mx) = \frac{\pi}{2}\,\delta_{nm}$$

$$\boxed{A_m = \frac{2}{\pi}\int_0^\pi dx\, f(x)\sin(mx) \quad (m = 1, 2, \ldots)}$$

Example…

$$f(x) = x(\pi - x)$$

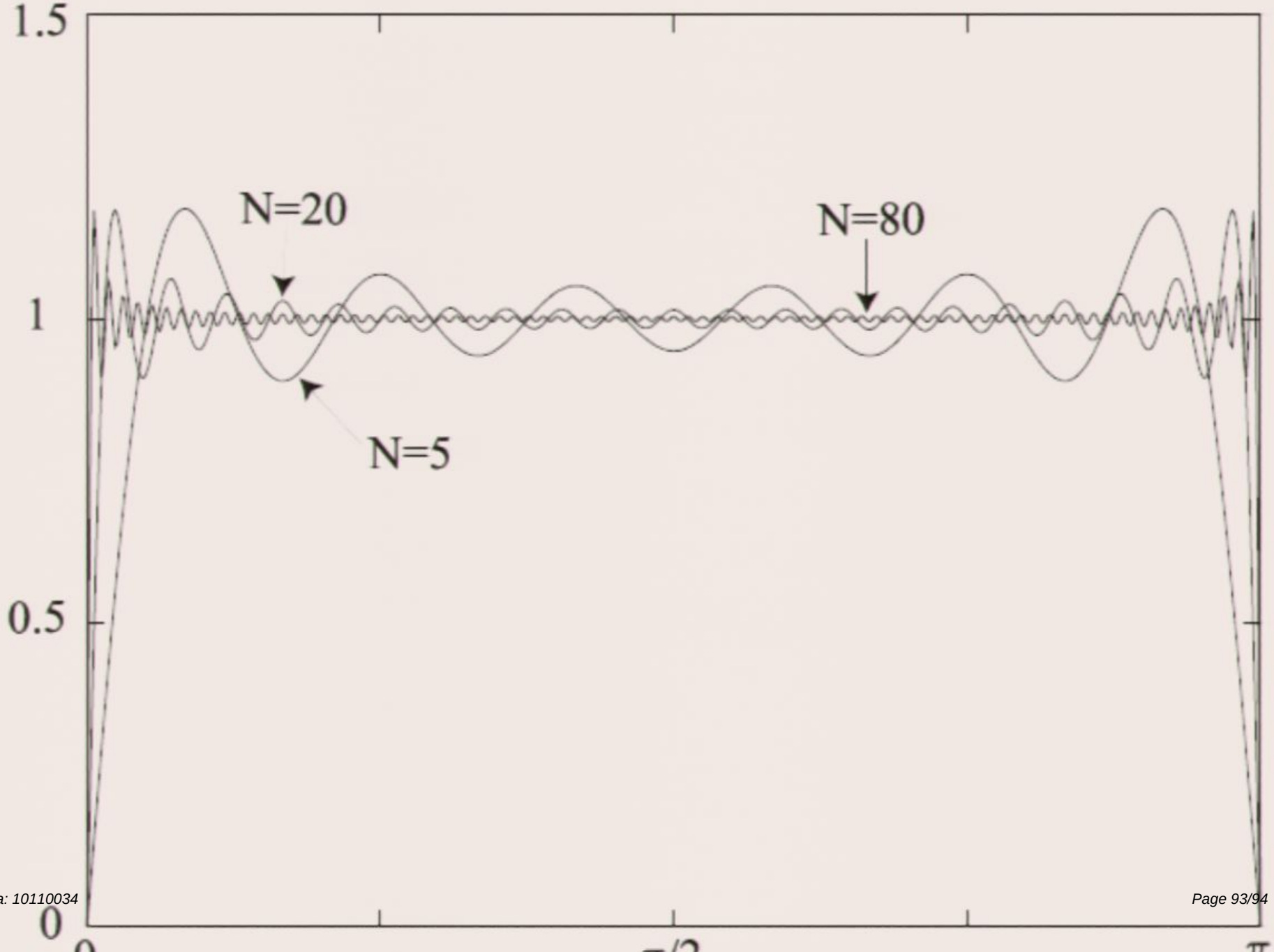
$$A_n = \begin{cases} 8/(\pi n^3) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$


3
2
1
0
0
π/2
π
f(x)=x(π-x)
S1(x)
S2(x)

Another example…

$$f(x) = 1$$

$$A_n = \begin{cases} 4/(\pi n) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

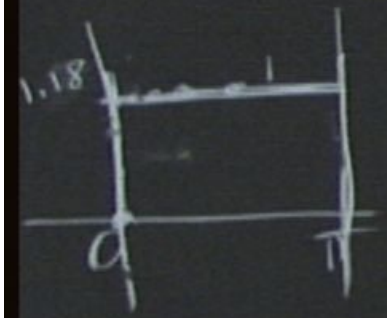

1.5
1
0.5
0
N=20
N=80
N=5

$$\underline{t=0}$$

$$u(x,0) = \boxed{f(x) = \sum_{n=1}^{\infty} A_n \sin(nx)} \; ??$$

Fourier coeffs $e^{-n^2 t}$

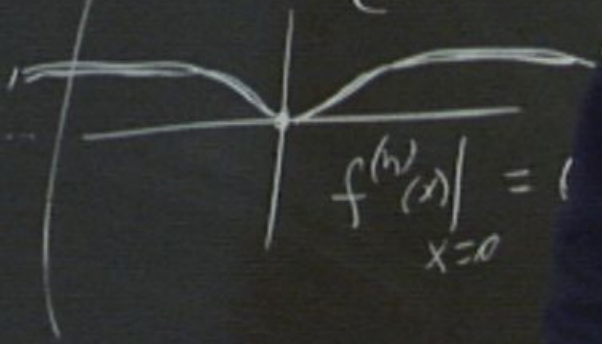
$$\int_0^{\pi} dx \, \sin(nx)\sin(mx) = \begin{cases} 0 & n \neq \\ \frac{\pi}{2} & n = \end{cases}$$



$$\cancel{\otimes \; \sin(mx), \int_0^{\pi}}$$

$$\cancel{\boxed{\frac{2}{\pi}\int_0^{\pi} dx \, f(x)\sin(mx) = A_m}}$$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \\ 0 & x = \end{cases}$$



$$f^{(n)}(x)\Big|_{x=0} = ($$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = \sum 0 + 0 + \cdots$$