Title: Mathematical Physics (PHYS 624) - Lecture 1

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Abstract:

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international

How to sum a series if it CONVERGES —and— How to sum a series if it DIVERGES

Carl Bender
Physics Department
Washington University in St. Louis

Perimeter Institute, 2010

How to sum a series if it CONVERGES —and— How to sum a series if it DIVERGES

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Pertur bation theory

HP

Pertur bation theory

HP

(1) ϵ

Perturbation theory

HP

(1) $\epsilon \longrightarrow HP(\epsilon)$

Pertur bation theory

HP

(1) $\epsilon \longrightarrow HP(\epsilon)$

(2) $ANS(\epsilon) = \sum a_n \epsilon^n$

Pertur bation theory

HP

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(2) $ANS(\epsilon) = \sum a_n \epsilon^n$

calc. a_n

(3) $\epsilon = 1$

Pertur bation theory

HP

(1) $\epsilon \longrightarrow HP(\epsilon)$

2) $ANS(\epsilon) = \sum a_n \epsilon^n$

calc. a_n

3) $\epsilon = 1$, sum series

ANS!

Pertur bation theory

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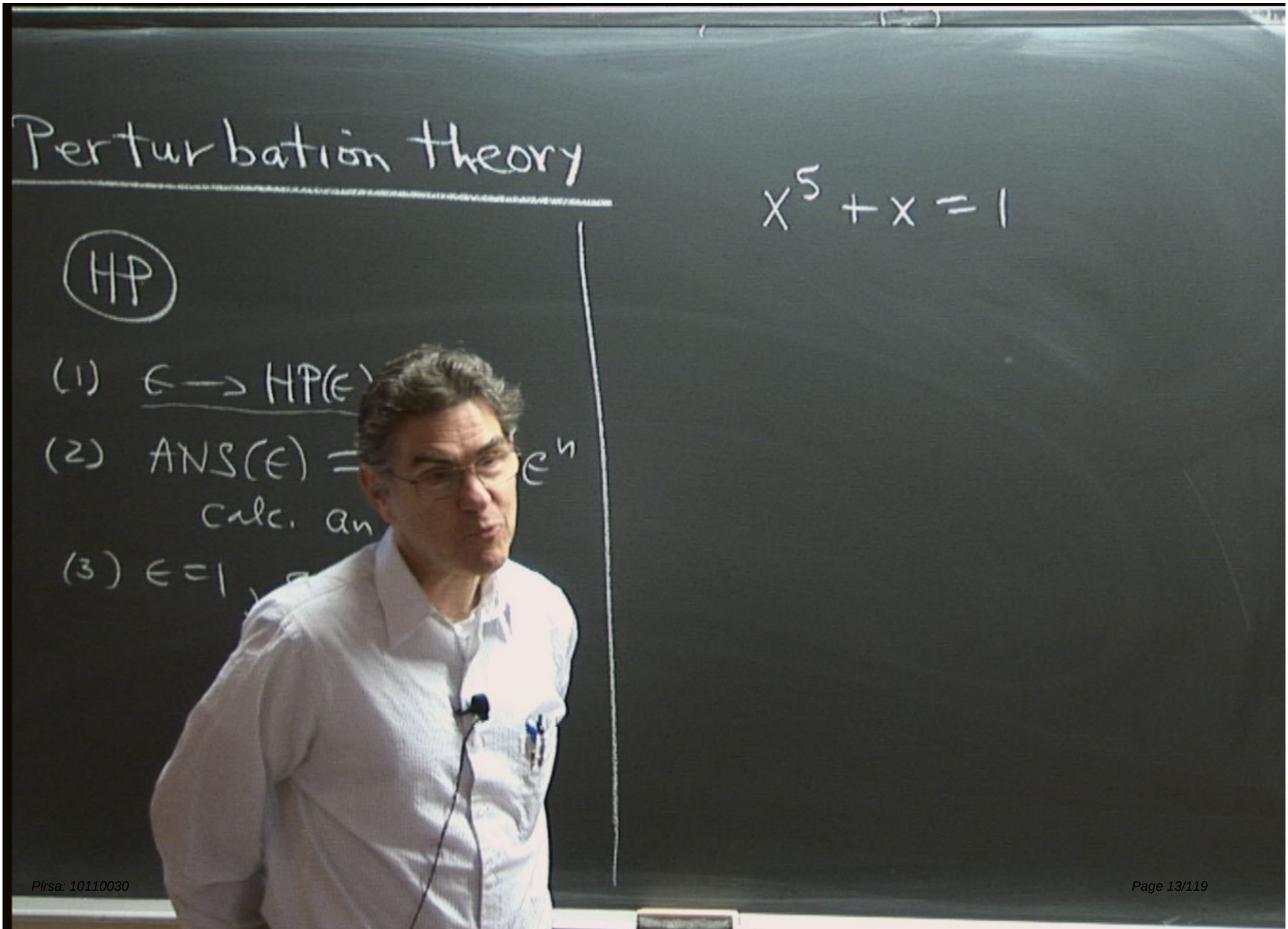
ANS!

$x^5 + x = 1$


Perturbation theory
HP
(1) ε → HP(ε)
(2) ANS(ε) =
ε^n
Calc. an
(3) ε = 1
x^5 + x = 1

Pertur bation theory

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(1) $\epsilon \longrightarrow HP(\epsilon)$

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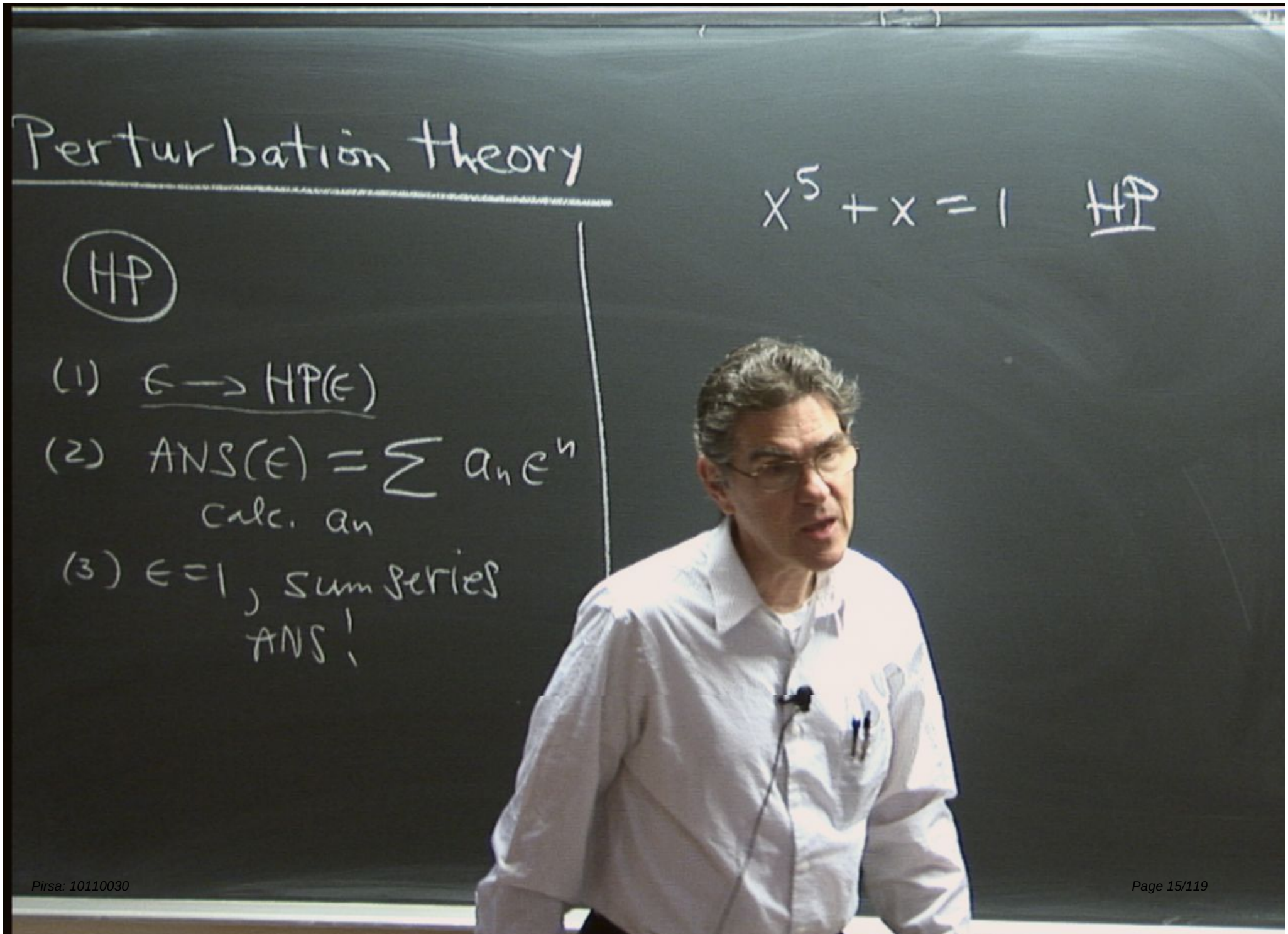
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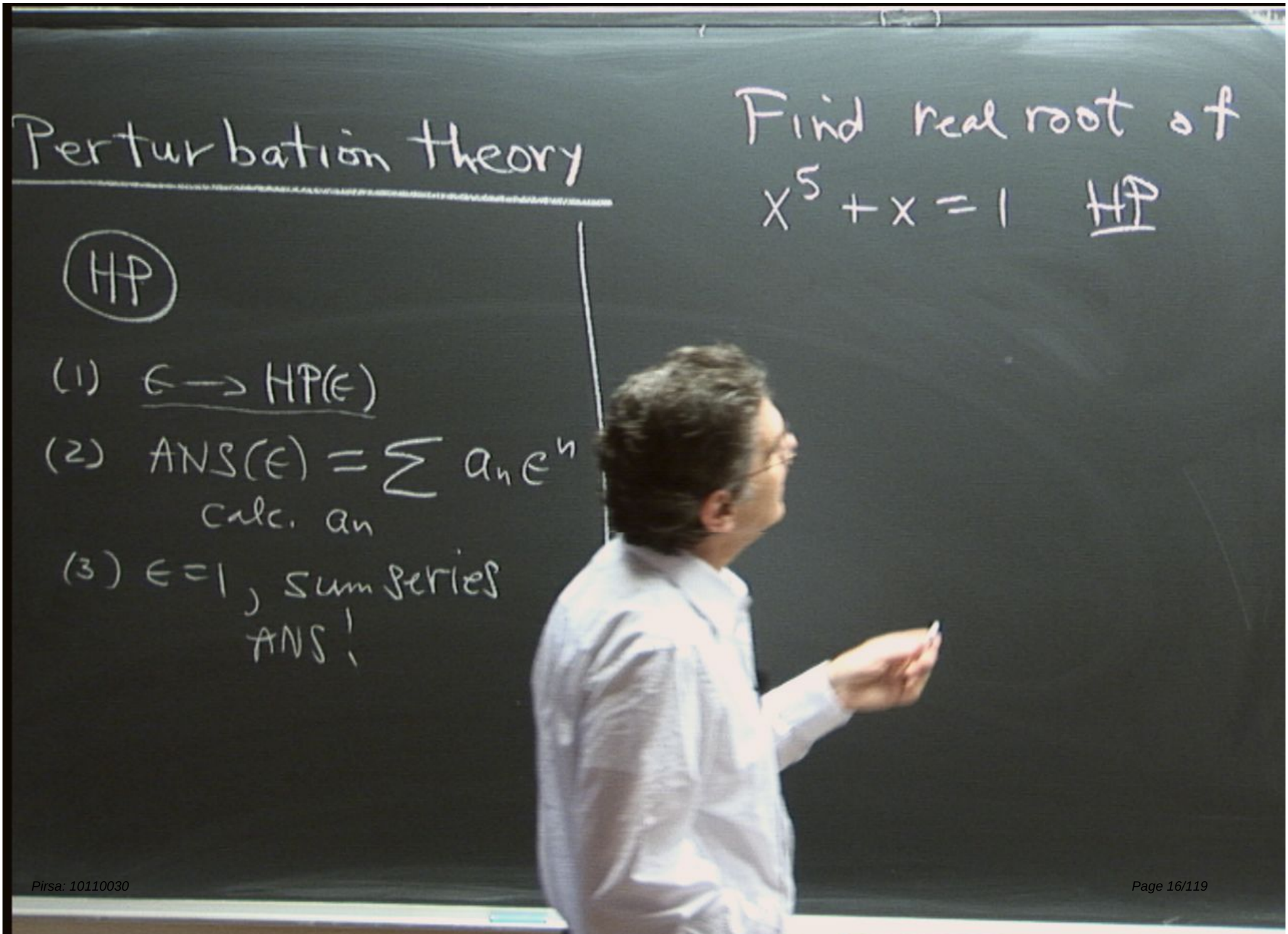
ANS!
x^5 + x = 1 HP


Pertur bation theory
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(1) ε → HP(ε)
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calc. a_n
(3) ε = 1, sum series
ANS!
Find real root of
x^5 + x = 1 HP

ion theory

$P(\epsilon)$

$= \sum a_n \epsilon^n$

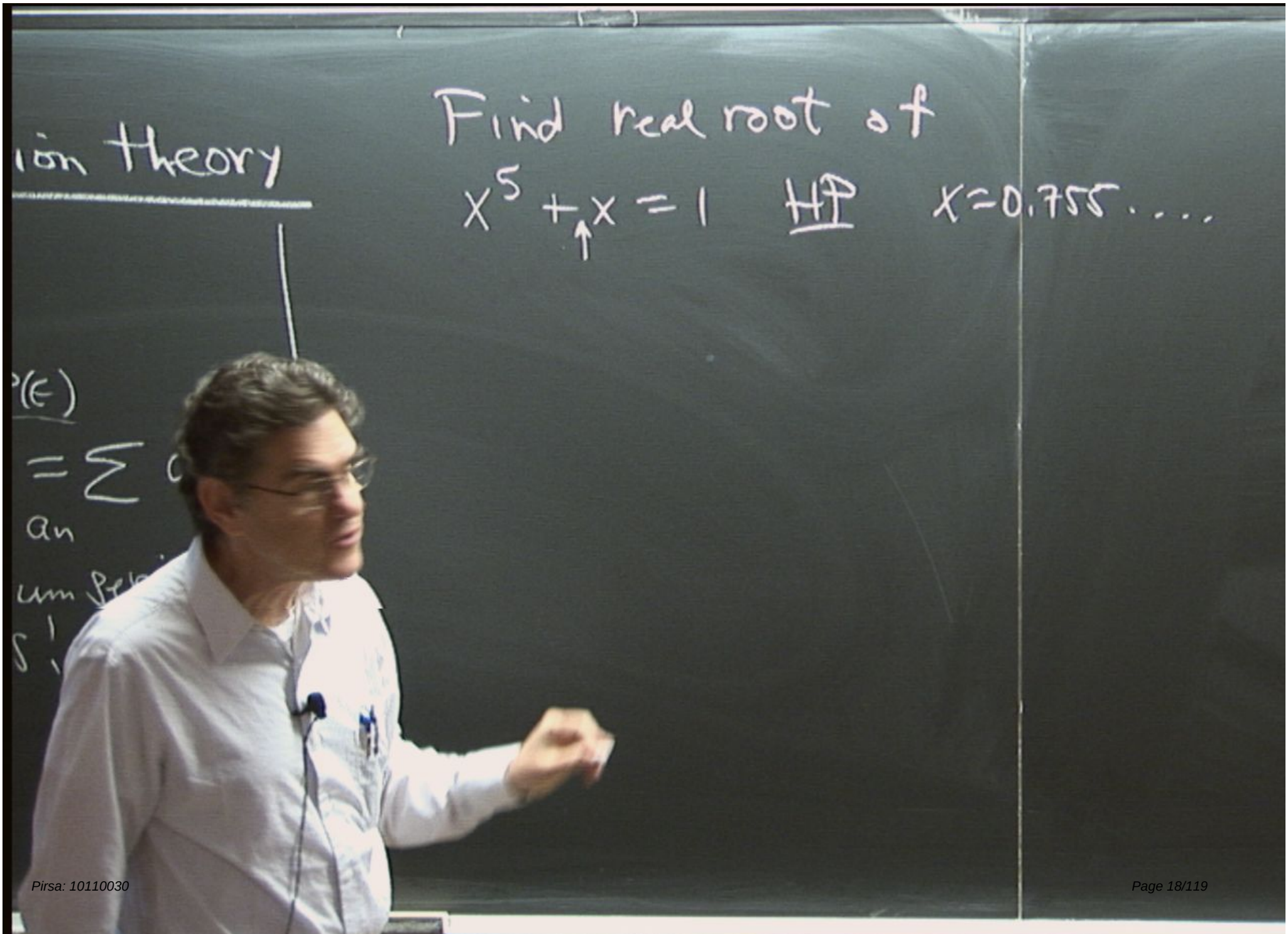
a_n

um series

s!

Find real root of

$x^5 + x = 1$ HP $x = 0.755\ldots$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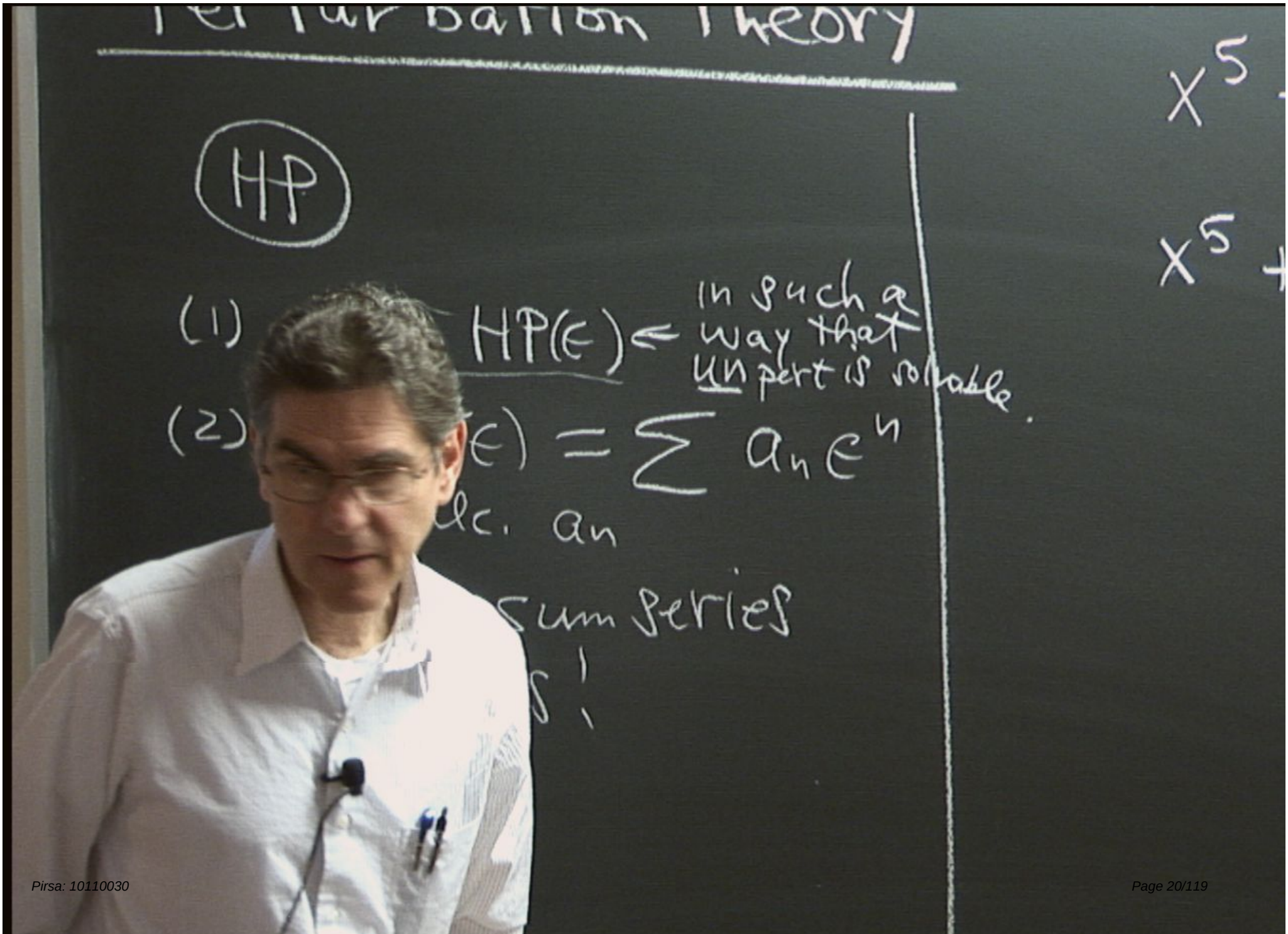
$x^5 + \epsilon x = 1$ ("strong coupl." app


Perturbation Theory
HP
(1) HP(ε) ← in such a way that unpert is solvable.
(2) ε) = Σ a_n ε^n
lc. a_n
sum series
s!
x^5
x^5 +

theory

n such a
way that
n part is solvable.

$a_n \epsilon^n$

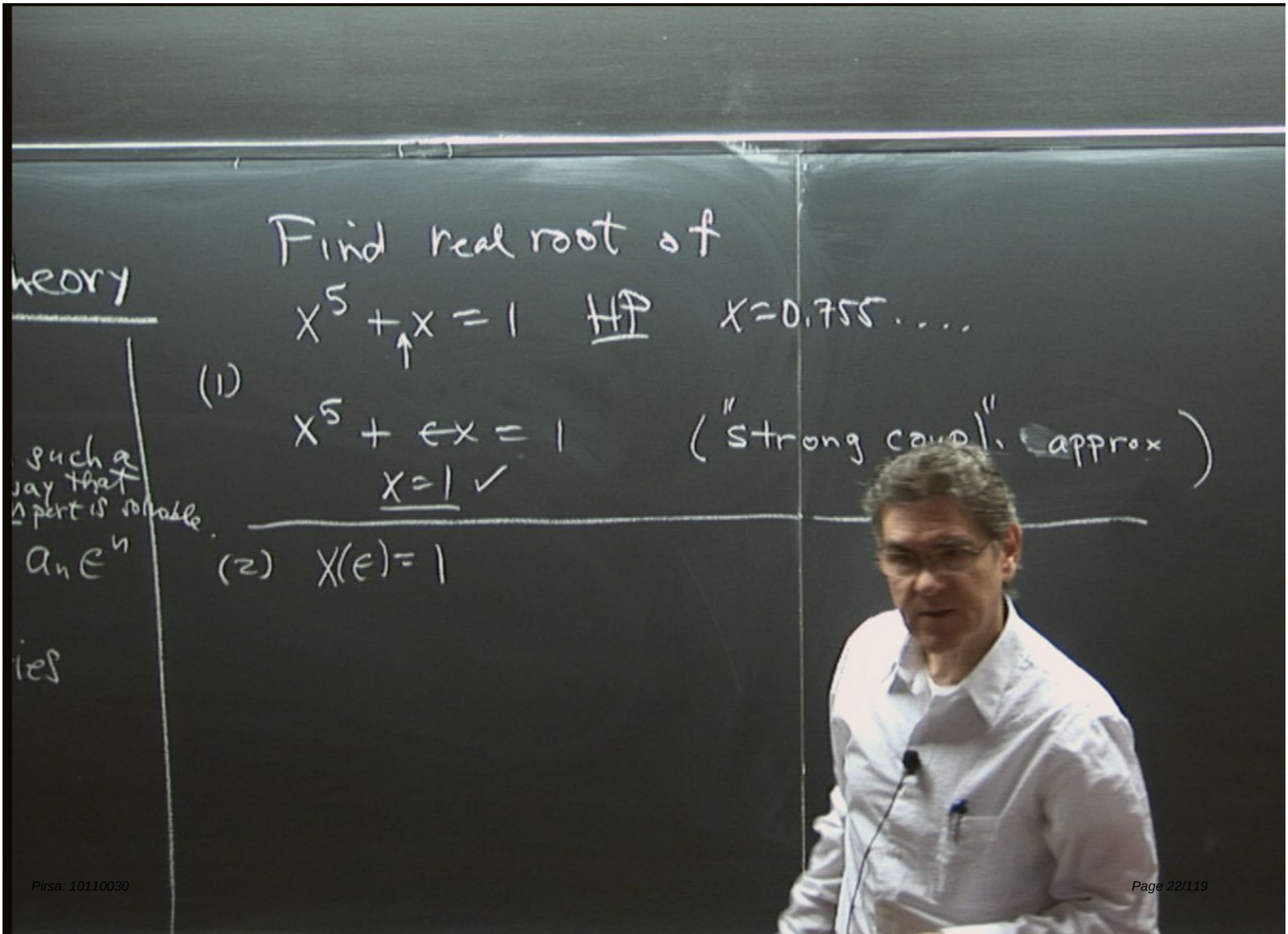
ies

Find real root of

$x^5 + x = 1$ HP $x = 0.755\ldots$

$x^5 + \epsilon x = 1$ ("strong coupl".

$x = 1$ ✓


Find real root of
$x^5 + x = 1$ HP $x = 0.755\ldots$
(1)
$x^5 + \epsilon x = 1$ ("strong coupl." approx)
$x = 1$ ✓
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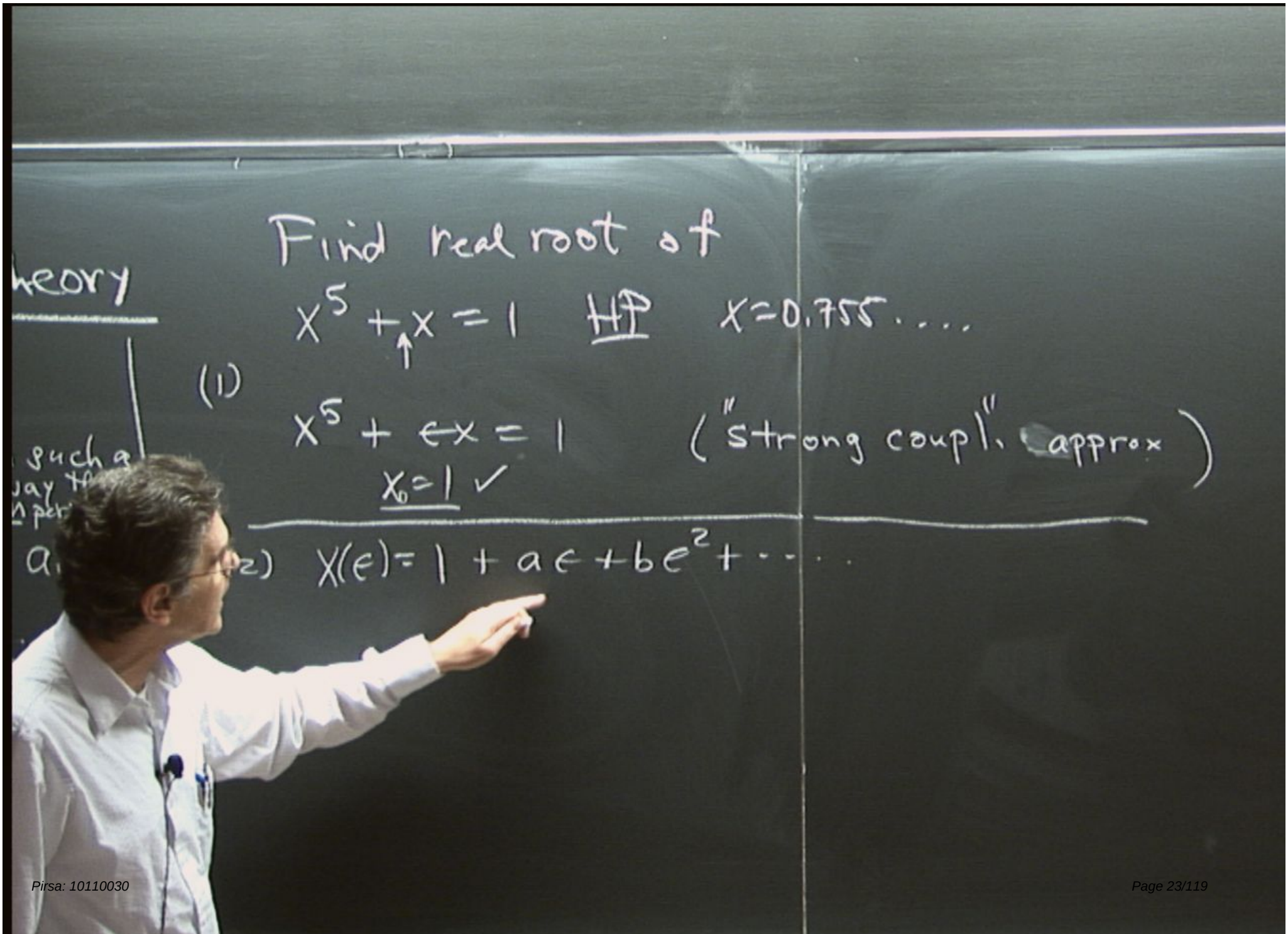
Find real root of
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$x^5 + \epsilon x = 1$ ("strong coupl." approx)
$x_0 = 1$ ✓
2) $x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$

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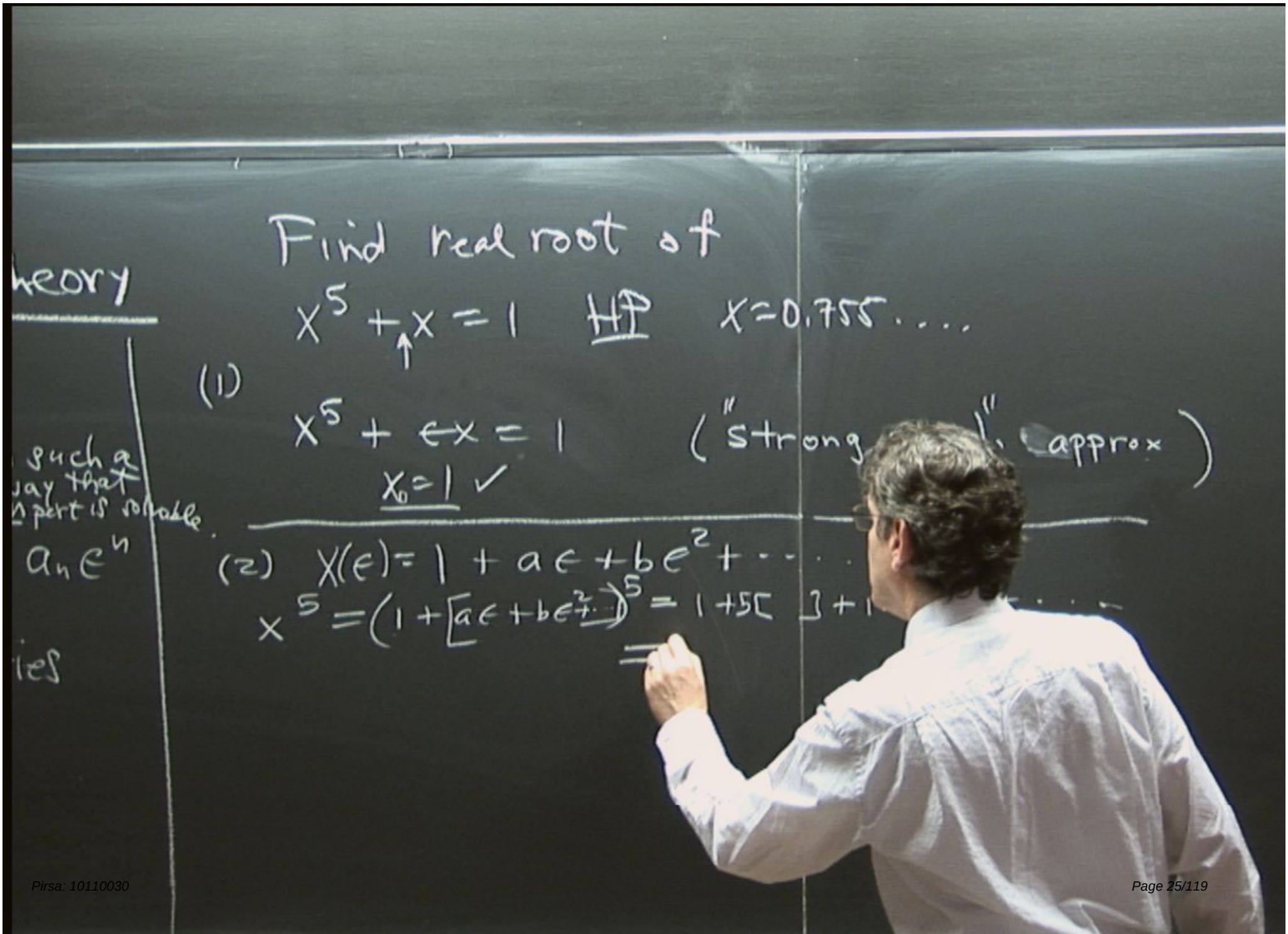
$x^5 = (1 + a\epsilon + b\epsilon^2)^5$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$x^5 = (1 + [a\epsilon + b\epsilon^2 + \cdots])^5 = 1 + 5[\ ] + 1$
heory
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$= 1 + 5a\epsilon + 5b\epsilon^2 + \cdots + 10a^2\epsilon^2$

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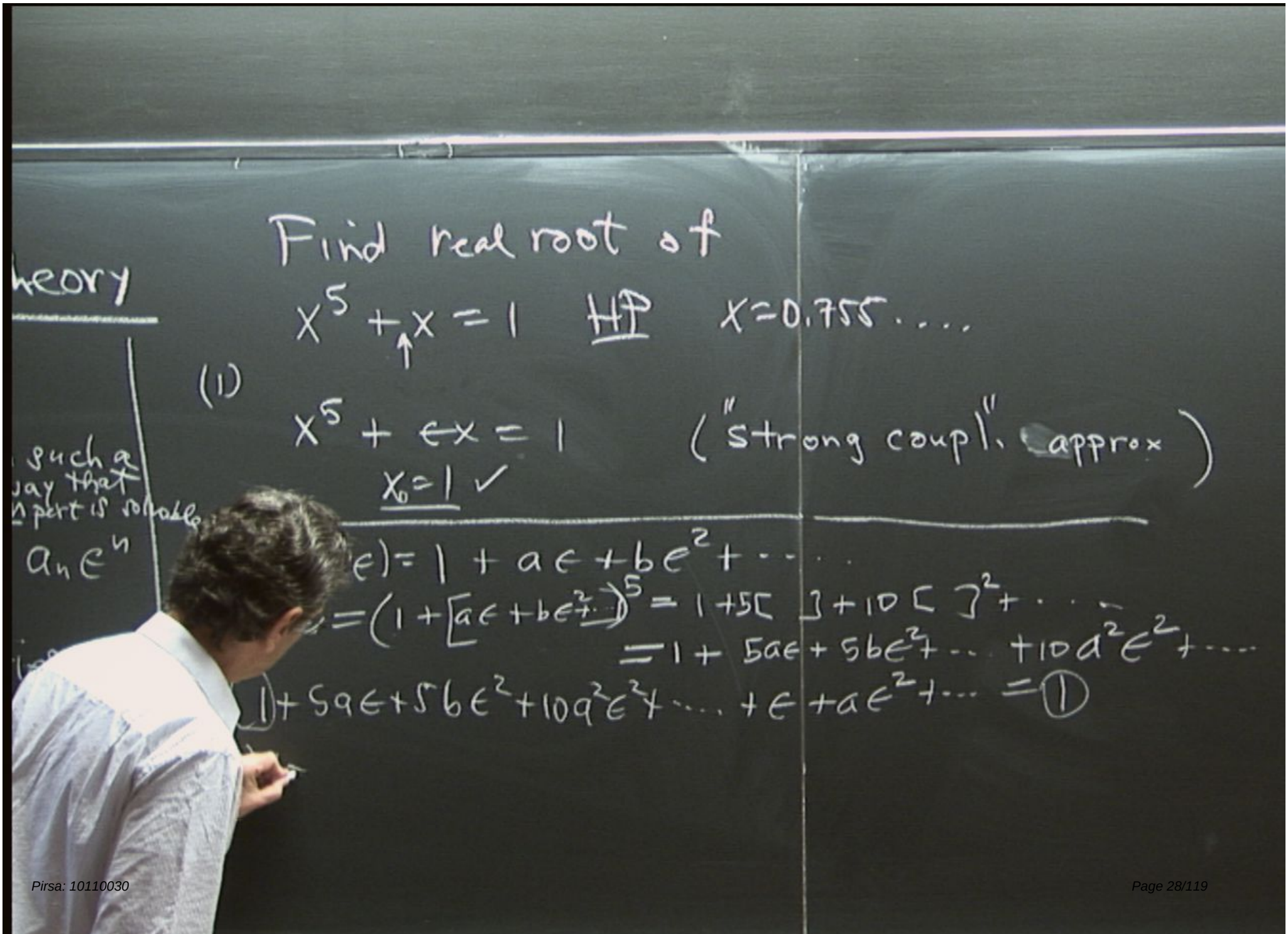
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$x^5 + \epsilon x = 1$ ("strong coupl." approx)
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$\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$
$= (1 + [a\epsilon + b\epsilon^2 + \cdots])^5 = 1 + 5[\ ] + 10[\ ]^2 + \cdots$
$= 1 + 5a\epsilon + 5b\epsilon^2 + \cdots + 10a^2\epsilon^2 + \cdots$
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$\epsilon^0)\ 1 = 1$ ✓

$\epsilon^1)\ 5a + 1 = 0$

$\epsilon^2)\ 5b + 10a^2 + a = 0$

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$\epsilon^0)\ 1 = 1$ ✓

$\epsilon^1)\ 5a + 1 = 0 \longrightarrow a = -\frac{1}{5}$

$\epsilon^2)\ 5b + 10a^2 + a = 0$

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$1 + 5a\epsilon + 5b\epsilon^2 + 10a^2\epsilon^2 + \cdots + \epsilon + a\epsilon^2 + \cdots = 1$

$\epsilon^0)\ 1 = 1$ ✓

$\epsilon^1)\ 5a + 1 = 0 \longrightarrow a = -\frac{1}{5}$

$\epsilon^2)\ 5b + 10a^2 + a = 0 \longrightarrow 5b + \frac{10}{25} - \frac{1}{5} = 0$

real root of

$+x = 1$ HP $x = 0.755\ldots$

$+ \epsilon x = 1$ ("strong coupl." approx)

$x_0 = 1$ ✓

$) = 1 + a\epsilon + b\epsilon^2 + \cdots$

$(1 + [a\epsilon + b\epsilon^2 + \cdots])^5 = 1 + 5[\ ] + 10[\ ]^2 + \cdots$

$= 1 + 5a\epsilon + 5b\epsilon^2 + \cdots + 10a^2\epsilon^2 + \cdots$

$\epsilon + 5b\epsilon^2 + 10a^2\epsilon^2 + \cdots + \epsilon + a\epsilon^2 + \cdots = 1$

$+1 = 0 \longrightarrow a = -\frac{1}{5}$

$+10a^2 + a = 0 \longrightarrow$

$5b + \frac{10}{25} - \frac{1}{5} = 0$ $\quad 5b + \frac{1}{5} = 0$ $\quad b = -\frac{1}{25}$

$= 0.755 \ldots$

strong coupl'n approx)

$5[\quad] + 10[\quad]^2 + \cdots$

$5a\epsilon + 5b\epsilon^2 + \cdots \quad + 10a^2\epsilon^2 + \cdots$

$+\epsilon + a\epsilon^2 + \cdots = 1$

$5b + \frac{10}{25} - \frac{1}{5} = 0 \qquad 5b + \frac{1}{5} = 0 \qquad b = -\frac{1}{25} \qquad c = -\frac{1}{125}$

$= 0.755 \ldots$

strong coupl'n approx)

$\ldots\ldots$

$5[\quad] + 10[\quad]^2 + \ldots$

$5a\epsilon + 5b\epsilon^2 + \ldots + 10a^2\epsilon^2 + \ldots$

$+\epsilon + a\epsilon^2 + \ldots = 1$

$5b + \frac{10}{25} - \frac{1}{5} = 0 \quad 5b + \frac{1}{5} = 0 \quad b = -\frac{1}{25} \quad c = -\frac{1}{125} \quad d =$

$= 0.755 \ldots$

strong coupl'n approx)

$\ldots\ldots$

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$5a\epsilon + 5b\epsilon^2 + \ldots \quad + 10a^2\epsilon^2 + \ldots$

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$5b + \frac{10}{25} - \frac{1}{5} = 0 \qquad 5b + \frac{1}{5} = 0 \qquad b = -\frac{1}{25} \qquad c = -\frac{1}{125} \qquad d \cancel{A} \frac{1}{625}$

$=0.755\ldots$

(3)

$$x(\epsilon) = 1 - \frac{\epsilon}{5} - \frac{\epsilon^2}{25} - \frac{\epsilon^3}{125} \cdots$$

strong coupl'n approx)

$\ldots\ldots$

$5[\quad] + 10[\quad]^2 + \cdots$

$5a\epsilon + 5b\epsilon^2 + \cdots + 10a^2\epsilon^2 + \cdots$

$+\epsilon + a\epsilon^2 + \cdots = 1$

$5b + \frac{10}{25} - \frac{1}{5} = 0 \quad 5b + \frac{1}{5} = 0 \quad b = -\frac{1}{25} \quad c = -\frac{1}{125} \quad d \ldots$

$=0.755\ldots$

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$$x(\epsilon) = 1 - \frac{\epsilon}{5} - \frac{\epsilon^2}{25} - \frac{\epsilon^3}{125} \cdots$$

$\underline{\epsilon = 1}$

strong coupl'n approx)

$$x(1) = 1 - .2$$

$\ldots\ldots$

$5[\ \ ] + 10[\ \ ]^2 + \cdots$

$5a\epsilon + 5b\epsilon^2 + \cdots + 10a^2\epsilon^2 + \cdots$

$+\epsilon + a\epsilon^2 + \cdots = 1$

$5b + \frac{10}{25} - \frac{1}{5} = 0 \quad 5b + \frac{1}{5} = 0 \quad b = -\frac{1}{25} \quad c = -\frac{1}{125}$

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$5a\epsilon + 5b\epsilon^2 + \ldots + 10a^2\epsilon^2 + \ldots$

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$5b + \frac{10}{25} - \frac{1}{5} = 0 \quad 5b + \frac{1}{5} = 0 \quad b = -\frac{1}{25} \quad c = -\frac{1}{125} \quad d \cancel{A} \frac{1}{625}$

(3)

$$x(\epsilon) = 1 - \frac{\epsilon}{5} - \frac{\epsilon^2}{25} - \frac{\epsilon^3}{125} \ldots$$

$\epsilon = 1$

$$x(1) = 1 - .2 - .04 - .008$$
$$= 1 - .248$$
$$= .752$$

(3)

$$X(\epsilon) = 1 - \frac{\epsilon}{5} - \frac{\epsilon^2}{25} - \frac{\epsilon^3}{125} \cdots$$

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$$X(1) = 1 - .2 - .04 - .008$$
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$$= .752$$

$= 0.755 \ldots$

strong coupl'n approx)

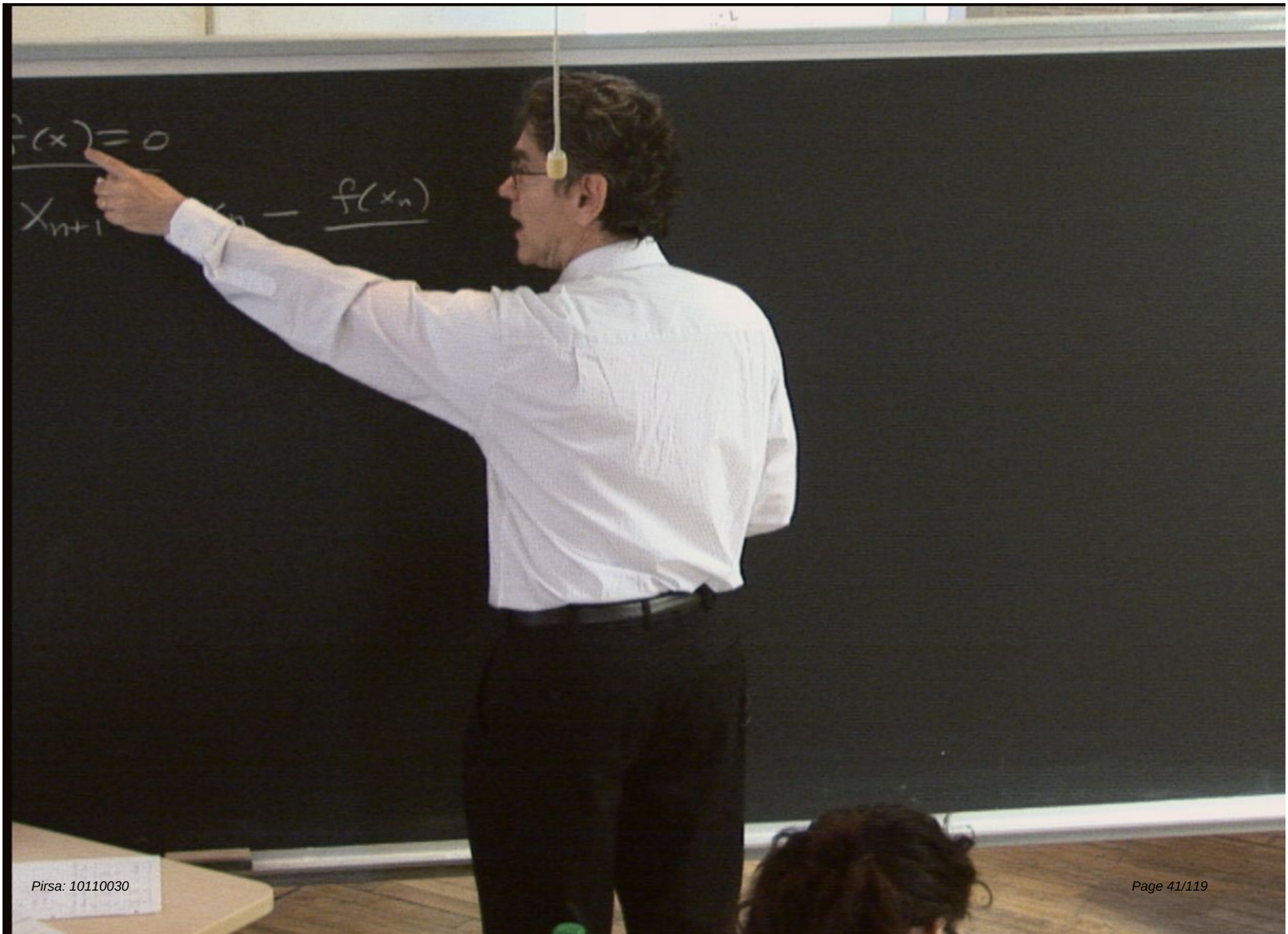
$\ldots$

$5[\ ] + 10[\ ]^2 + \cdots$

$5a\epsilon + 5b\epsilon^2 + \cdots + 10a^2\epsilon^2 + \cdots$

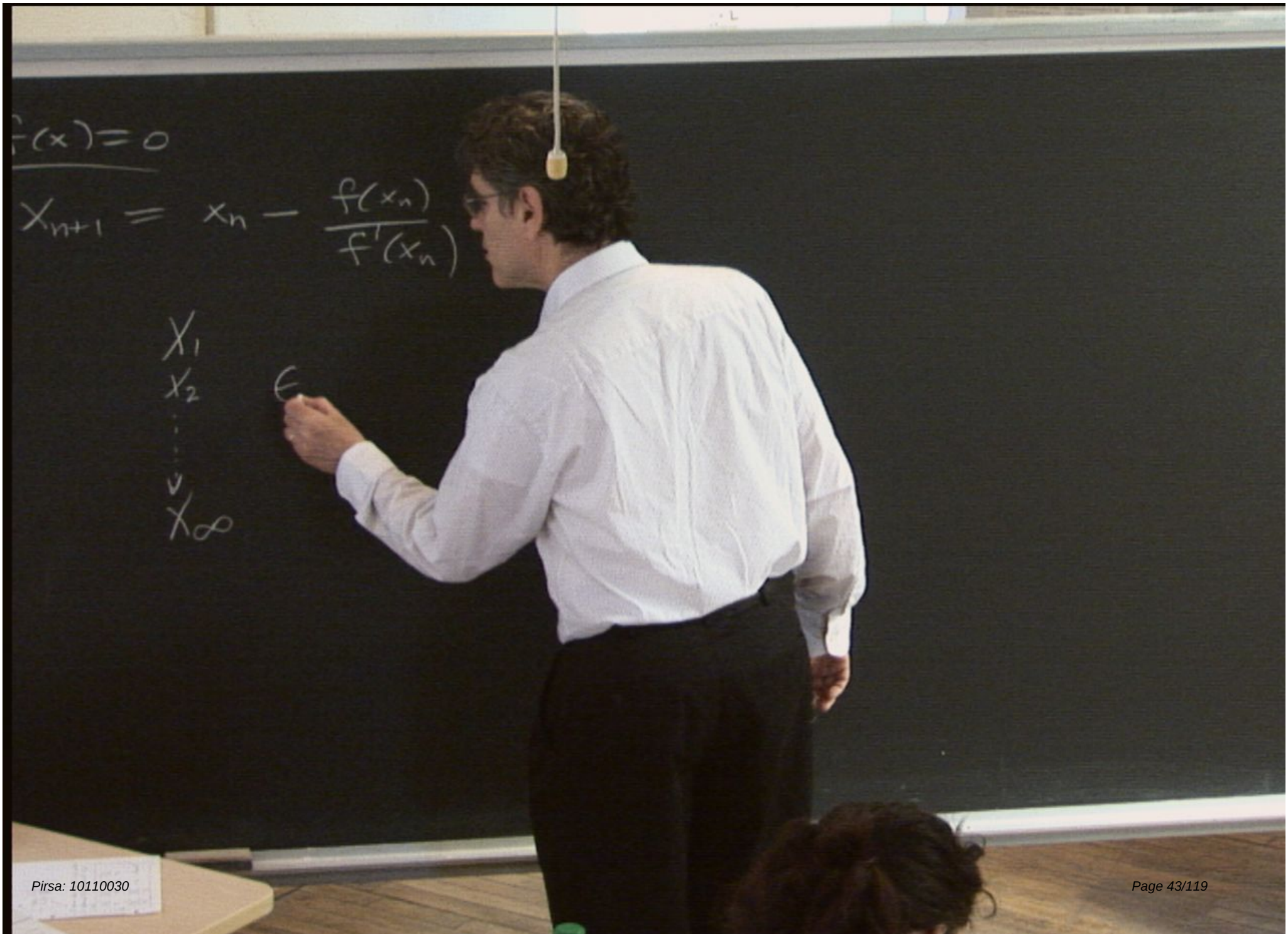
$+\epsilon + a\epsilon^2 + \cdots = 1$

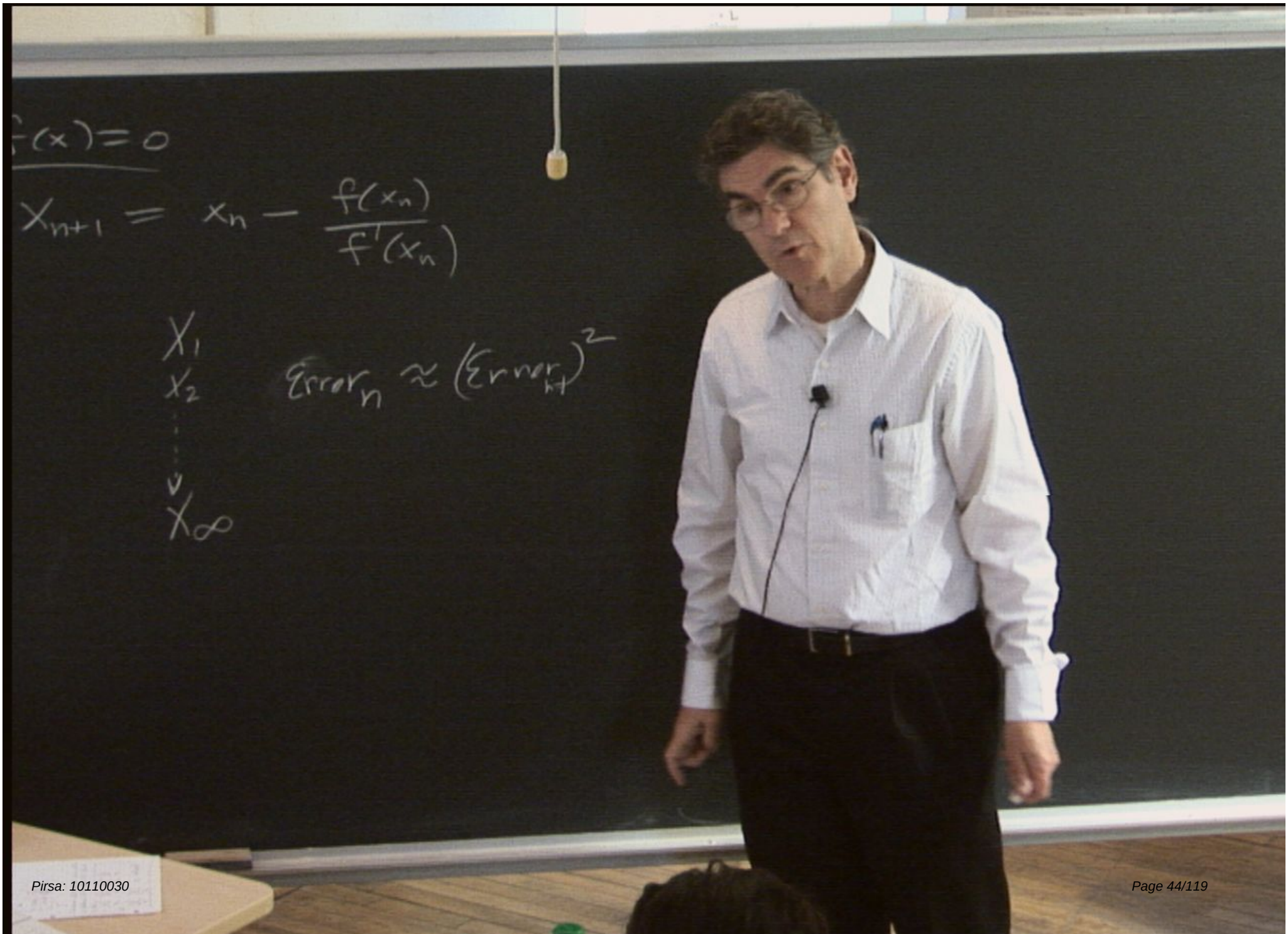
$5b + \frac{10}{25} - \frac{1}{5} = 0 \quad 5b + \frac{1}{5} = 0 \quad b = -\frac{1}{25} \quad c = -\frac{1}{125} \quad d \cancel{A} \frac{1}{625}$


f(x)=0
f(xn)

$f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$


f(x)=0
$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
x_1
x_2
x_∞


f(x) = 0
$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
x_1
x_2
x_∞
$\text{Error}_n \approx (\text{Error}_{n-1})^2$

$f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

x_1

x_2

$\vdots$

x_∞

$$\text{Error}_n \approx (\text{Error}_{n-1})^2$$

$RAC \cong 1.6$

(3)

$$x(\epsilon) = 1 - \frac{\epsilon}{5} - \frac{\epsilon^2}{25} - \frac{\epsilon^3}{125} \cdots$$

$\epsilon = 1$

$$x(1) = 1 - .2 - .04 - .008$$
$$= 1 - .248$$
$$= .752$$

...

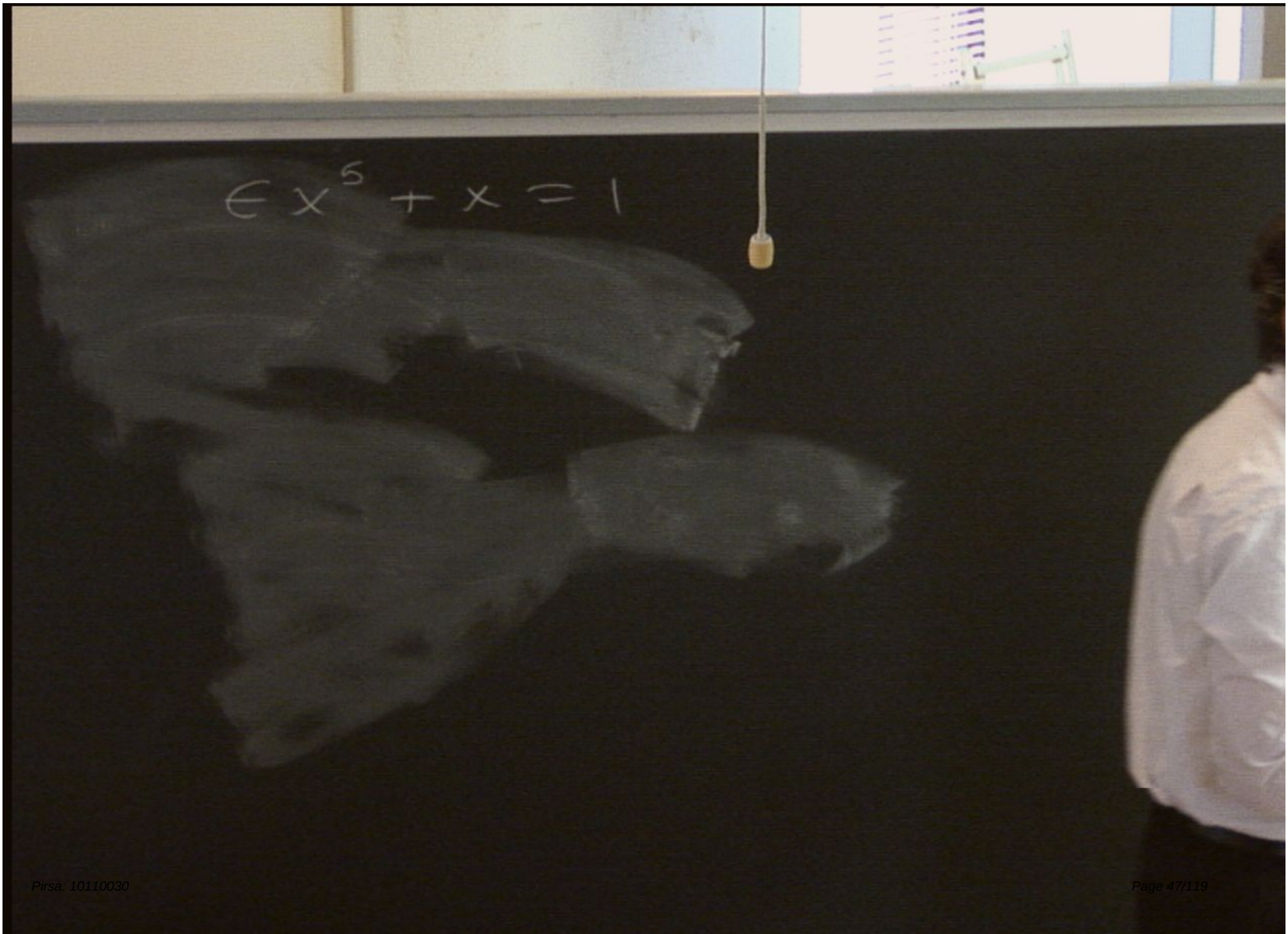
oupl'. approx)

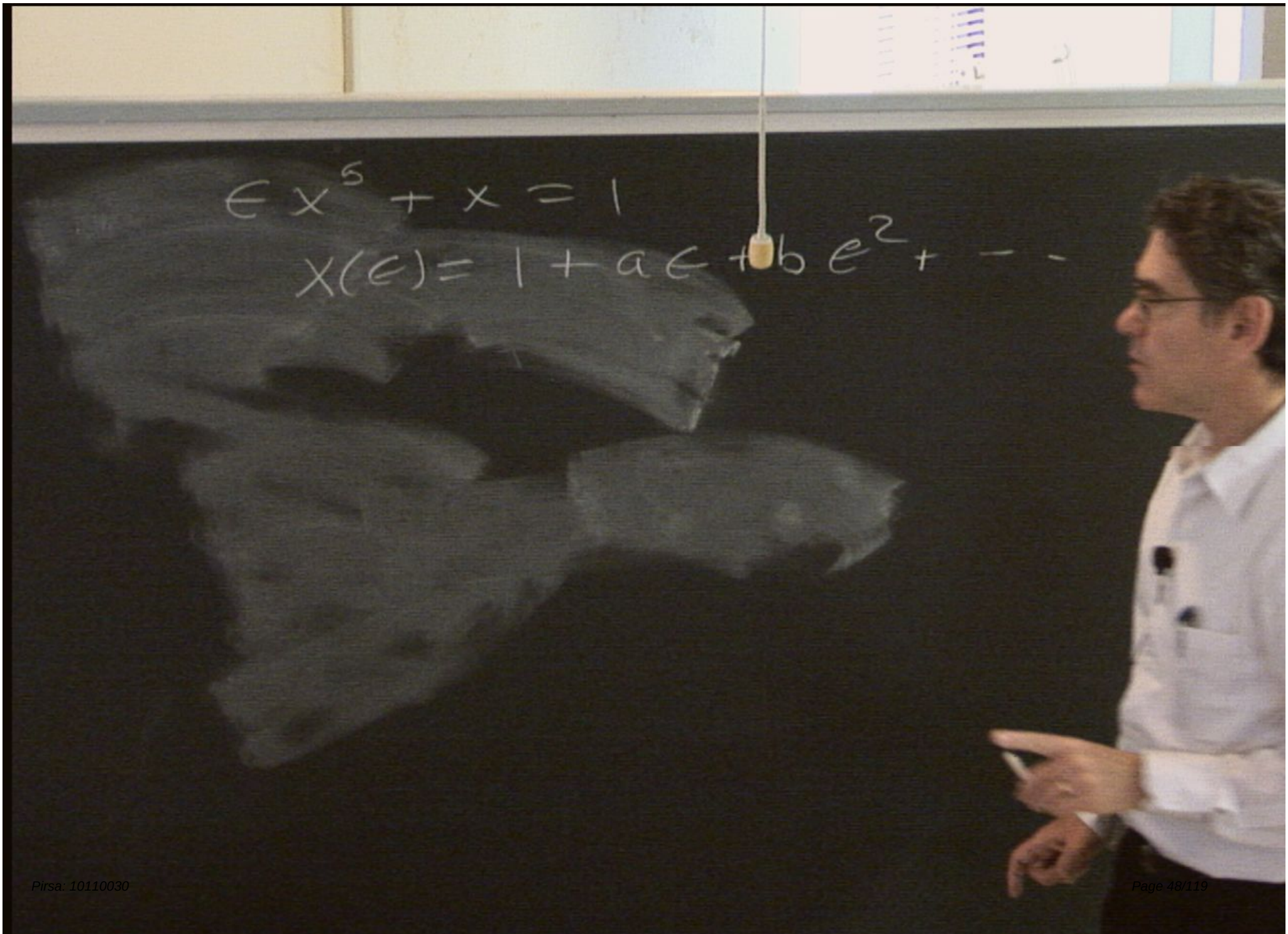
$[\ ]^2 + \cdots$

$^2 + \cdots + 10a^2\epsilon^2 + \cdots$

$^2 + \cdots = 1$

$+ \frac{10}{25} - \frac{1}{5} = 0 \quad 5b + \frac{1}{5} = 0 \quad b = -\frac{1}{25} \quad c = -\frac{1}{125} \quad d \cancel{A} \frac{1}{625}$


$\epsilon x^5 + x = 1$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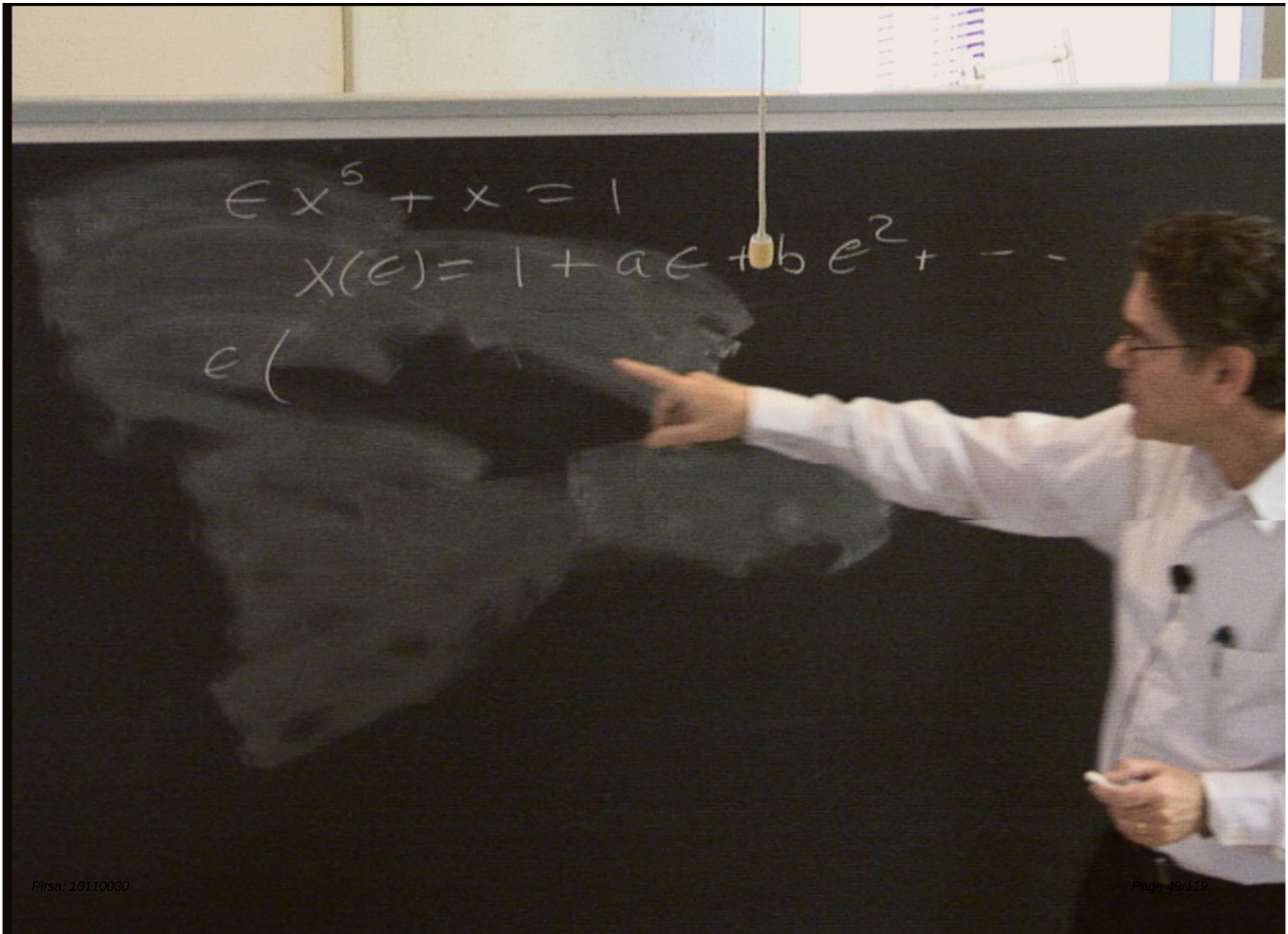
$x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$


$\epsilon x^5 + x = 1$
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$\epsilon($

$$\epsilon x^5 + x = 1$$

$$x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$$

$$\epsilon(1 + 5a\epsilon)$$

$$\epsilon x^5 + x = 1$$

$$x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$$

$$\epsilon(1 + 5a\epsilon) + 1 + a\epsilon + b\epsilon^2 + \cdots$$

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$$\epsilon + 5a\epsilon^2 + 1 + a\epsilon + b\epsilon^2 \cdots$$

$$x^5 + x = 1$$

$$x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$$

$$(1 + 5a\epsilon) + 1 + a\epsilon + b\epsilon^2 + \cdots = 1$$

$$\underline{\epsilon} + 5a\epsilon^2 + \cancel{1} + \underline{a\epsilon} + b\epsilon^2 \cdots = \cancel{1}$$

$$\epsilon^1)\ a + 1 = 0 \qquad \underline{a = -1}$$

$$x^5 + x = 1$$

$$x(\epsilon) = 1 + a\epsilon + b\epsilon^2 + \cdots$$

$$(1 + 5a\epsilon) + 1 + a\epsilon + b\epsilon^2 + \cdots = 1$$

$$\underline{\epsilon} + 5a\epsilon^2 + \cancel{1} + \underline{a\epsilon} + b\epsilon^2 \cdots = \cancel{1}$$

$$\epsilon^1)\ a + 1 = 0 \qquad \underline{a = -1}$$

$$\epsilon^2)\ 5a + b = 0 \rightarrow b = 5$$

$$x(\epsilon) = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 \ldots$$

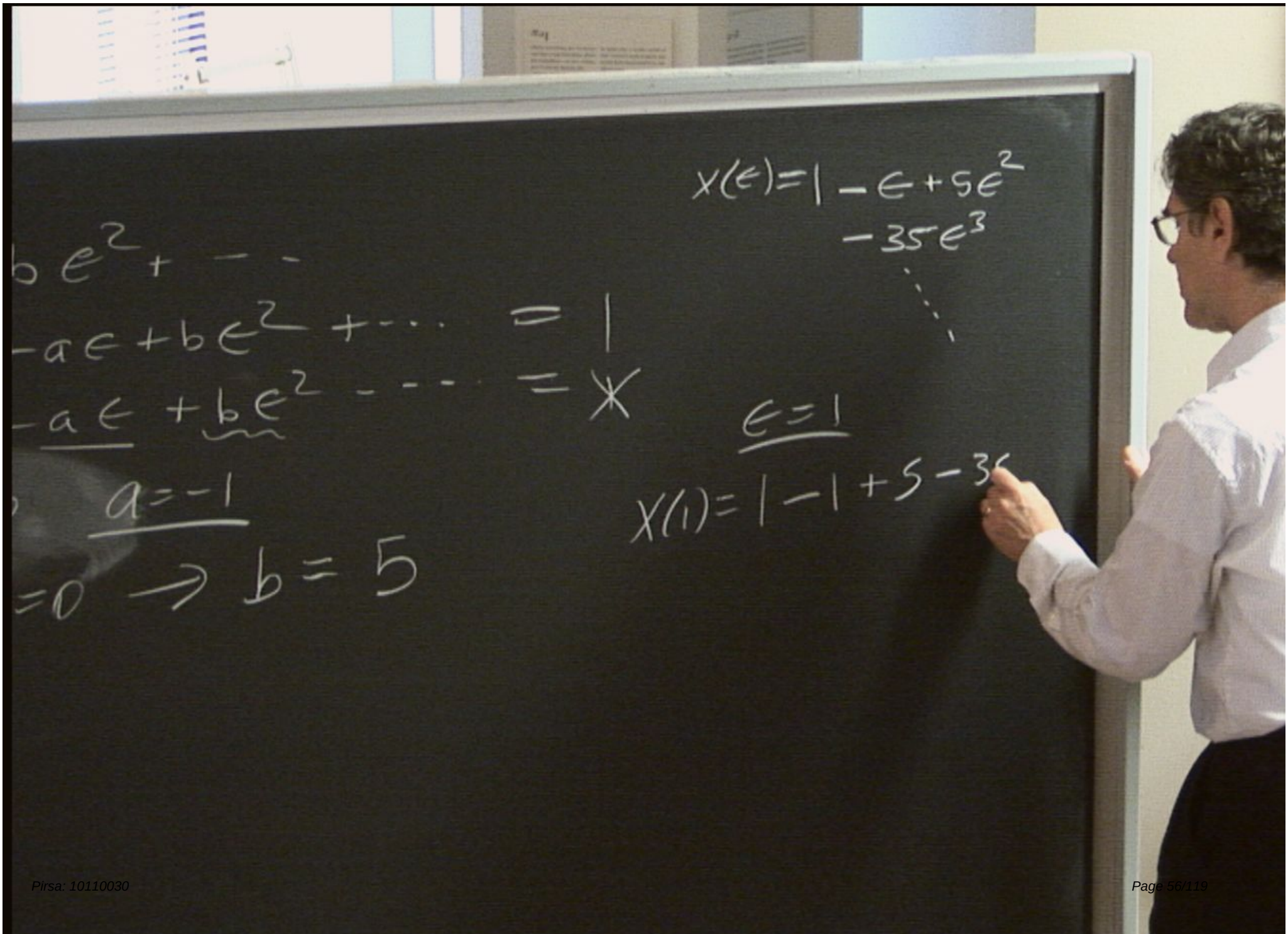
$$\ldots e^2 + \ldots$$

$$-a\epsilon + b\epsilon^2 + \ldots = 1$$

$$\underline{-a\epsilon} + \underline{b}\epsilon^2 \ldots = x$$

$$\underline{a = -1}$$

$$= 0 \rightarrow b = 5$$


$x(\epsilon) = 1 - \epsilon + 5\epsilon^2$
$- 35\epsilon^3$
$b\epsilon^2 + \cdots$
$-a\epsilon + b\epsilon^2 + \cdots = 1$
$-a\epsilon + b\epsilon^2 \cdots = x$
$\epsilon = 1$
$a = -1$
$x(1) = 1 - 1 + 5 - 35$
$= 0 \rightarrow b = 5$

$$x(\epsilon) = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 \ldots$$

$$b\epsilon^2 + \ldots$$

$$-a\epsilon + b\epsilon^2 + \ldots = 1$$

$$-a\epsilon + b\epsilon^2 \ldots = x$$

$$\underline{\epsilon = 1}$$

$$x(1) = 1 - 1 + 5 - 35 \ldots$$

$$\underline{a = -1}$$

$$= 0 \rightarrow b = 5$$

$b\epsilon^2 + \cdots$

$-a\epsilon + b\epsilon^2 + \cdots = 1$

$\underline{-a\epsilon} + \underline{b\epsilon^2} \cdots = X$

$\underline{a = -1}$

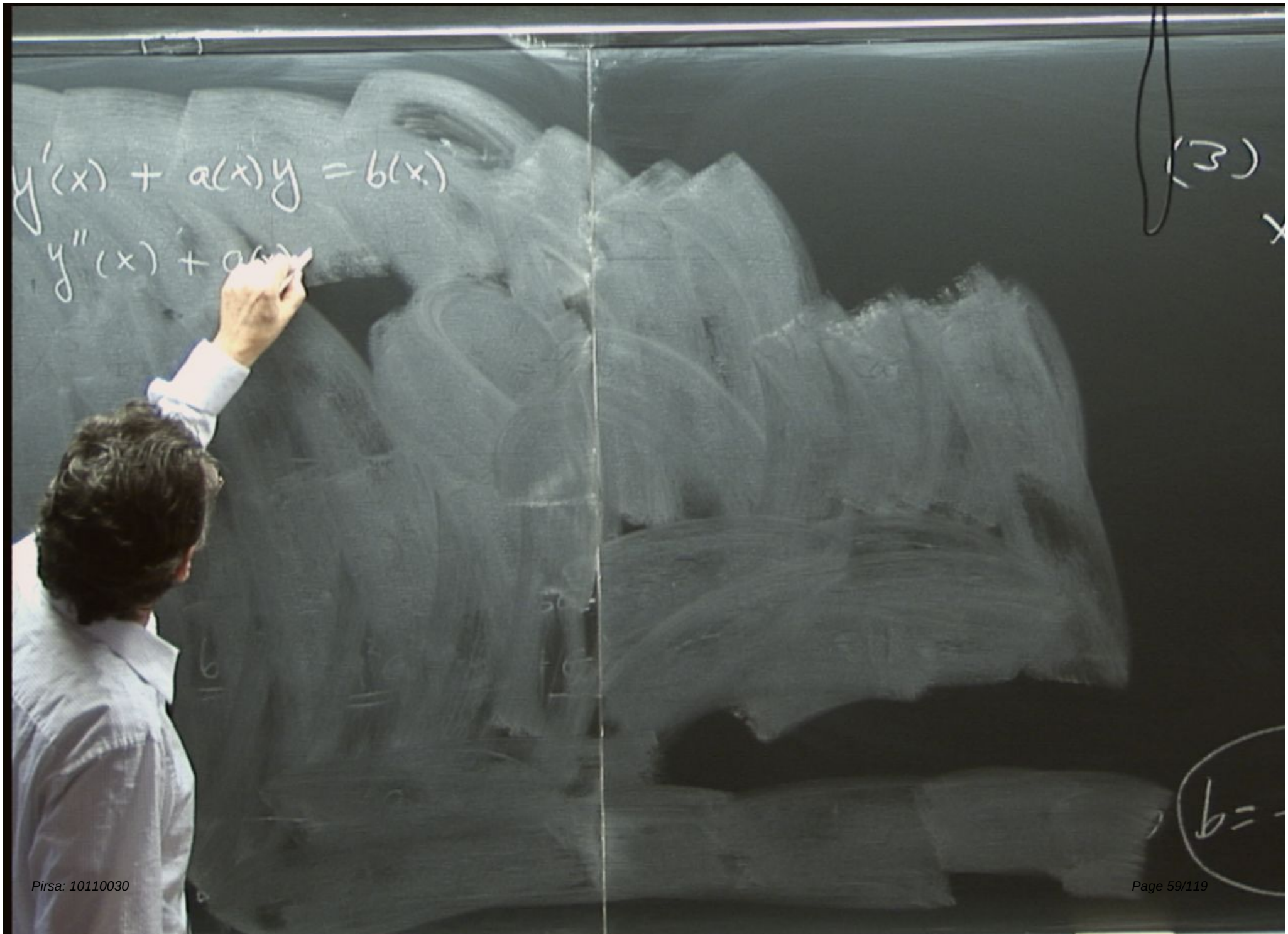
$= 0 \rightarrow b = 5$

R of C

$X(\epsilon) = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 \cdots$

$\underline{\epsilon = 1}$

$X(1) = 1 - 1 + 5 - 35 \cdots$


y'(x) + a(x)y = b(x)
y''(x) +
(3)
b=

$$y'(x) + a(x)y = b(x)$$

$$y''(x) + a(x)y' + b(x)y = 0$$

(3)

$b =$

$$y'(x) + a(x)y = b(x)$$

$$y''(x) + a(x)y' + b(x)y = 0$$

$$y(x) = e^{rx}$$

(3)

x

$b =$

$$y'(x) + a(x)y = b(x)$$

$$y''(x) + a(x)y' + b(x)y = 0$$

$$y(x) = e^{rx}$$

$$\cancel{e^{rx}} r^2 + a(x) r \cancel{e^{rx}} + b(x) \cancel{e^{rx}} = 0$$

(3)

x

$b =$

$$y'(x) + a(x)y = b(x)$$

$$y''(x) + \cancel{a(x)y'} + b(x)y = 0$$

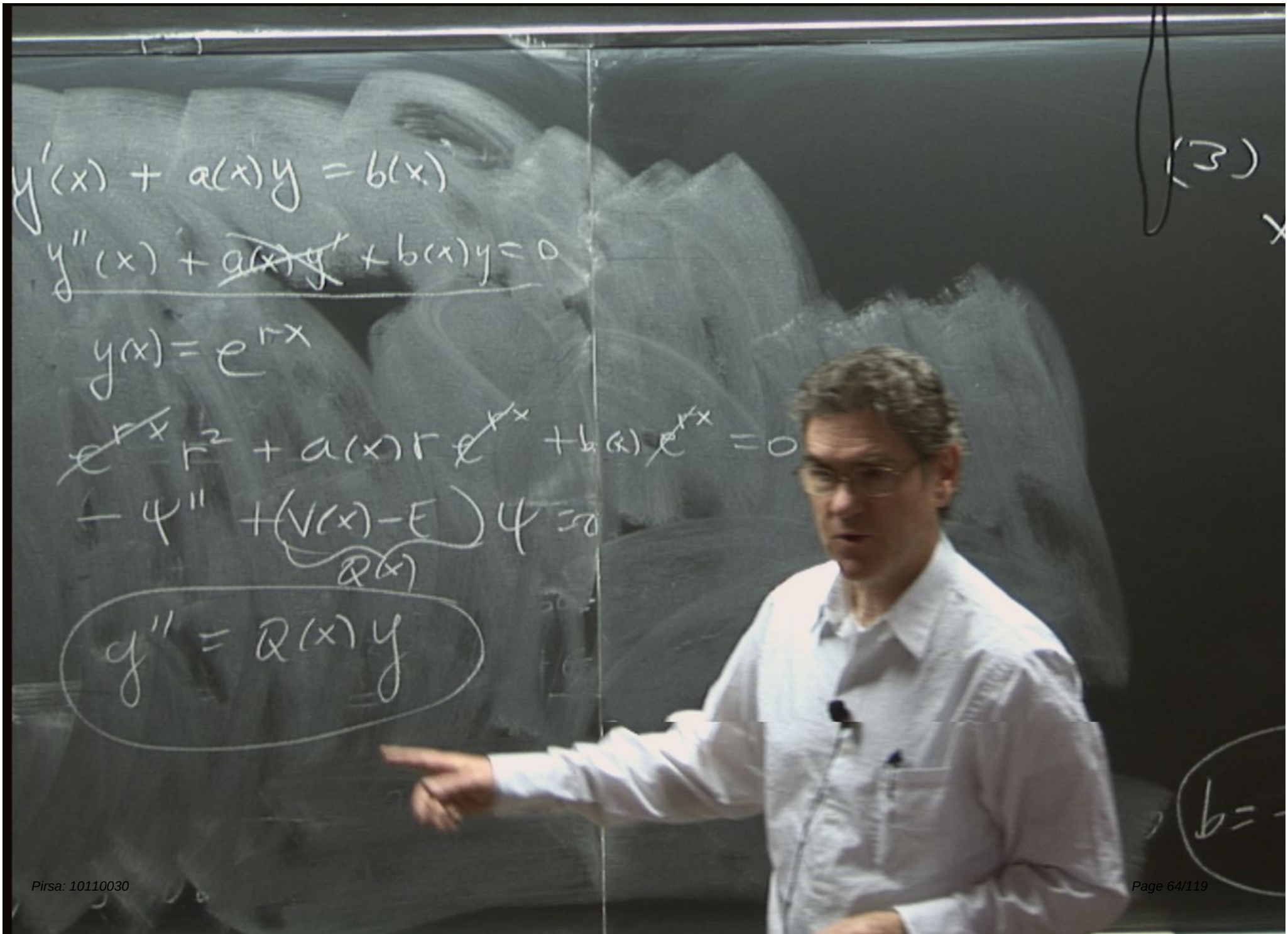
$$y(x) = e^{rx}$$

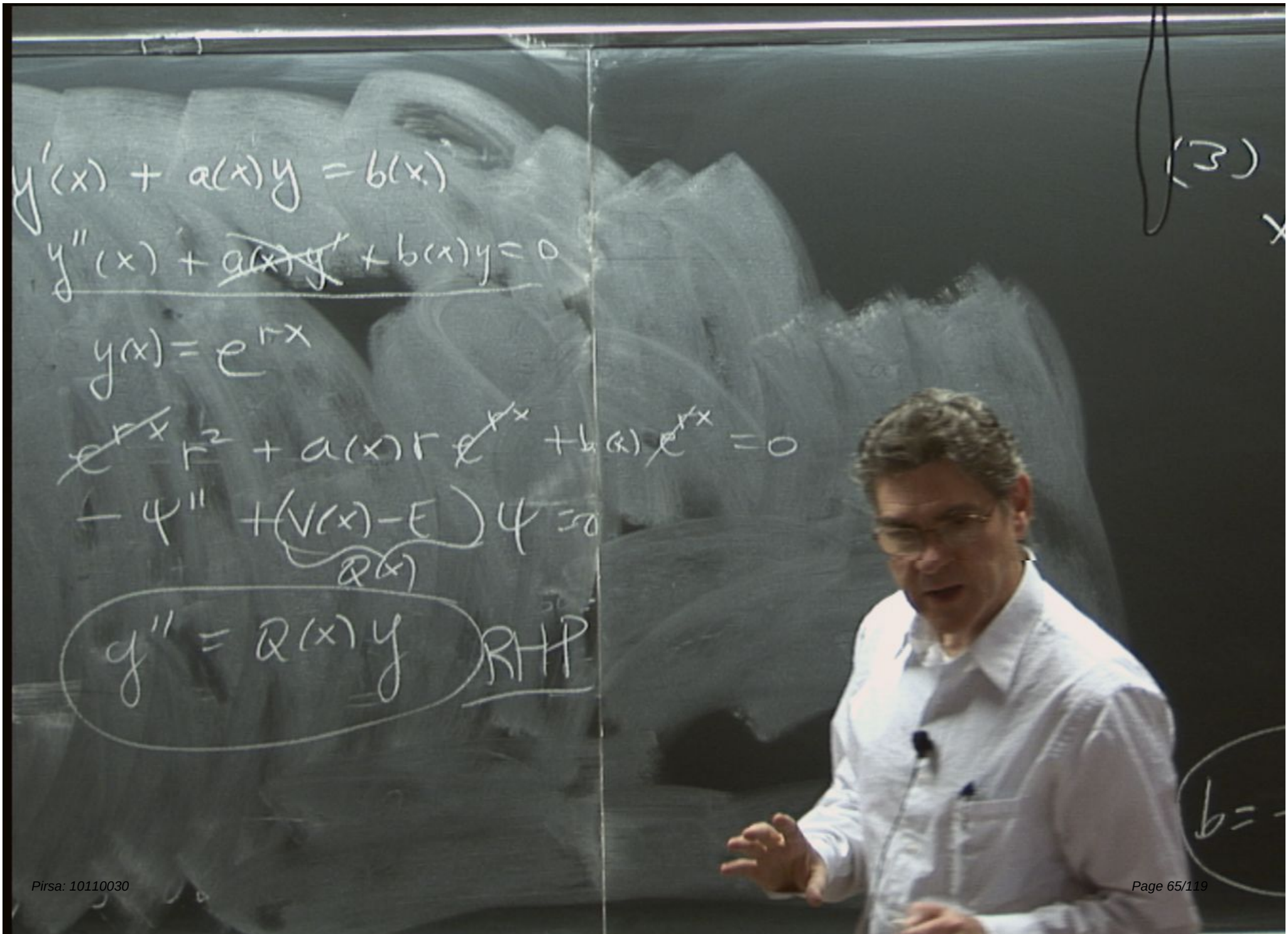
$$\ldots + a(x)r\cancel{e^{rx}} + b(x)\cancel{e^{rx}} = 0$$

$$\ldots + (V(x) - E)\psi = 0$$

(3)

$b = \ldots$


y'(x) + a(x)y = b(x)
y''(x) + a(x)y' + b(x)y = 0
y(x) = e^{rx}
e^{rx} r^2 + a(x) r e^{rx} + b(x) e^{rx} = 0
-ψ'' + (V(x) - E) ψ = 0
Q(x)
y'' = Q(x) y
(3)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y''(x) + a(x)y' + b(x)y = 0
y(x) = e^{rx}
e^{rx} r^2 + a(x) r e^{rx} + b(x) e^{rx} = 0
-ψ'' + (V(x) - E) ψ = 0
Q(x)
y'' = Q(x) y
RHP
(3)

in such a way that unpert is solvable

$= \sum a_n \epsilon^n$

series

$y(x) = e^{rx}$

$\cancel{e^{rx}} r^2 + a(x) r \cancel{e^{rx}} + b(x) \cancel{e^{rx}} = 0$

$-\psi'' + \underbrace{(V(x) - E)}_{Q(x)} \psi = 0$

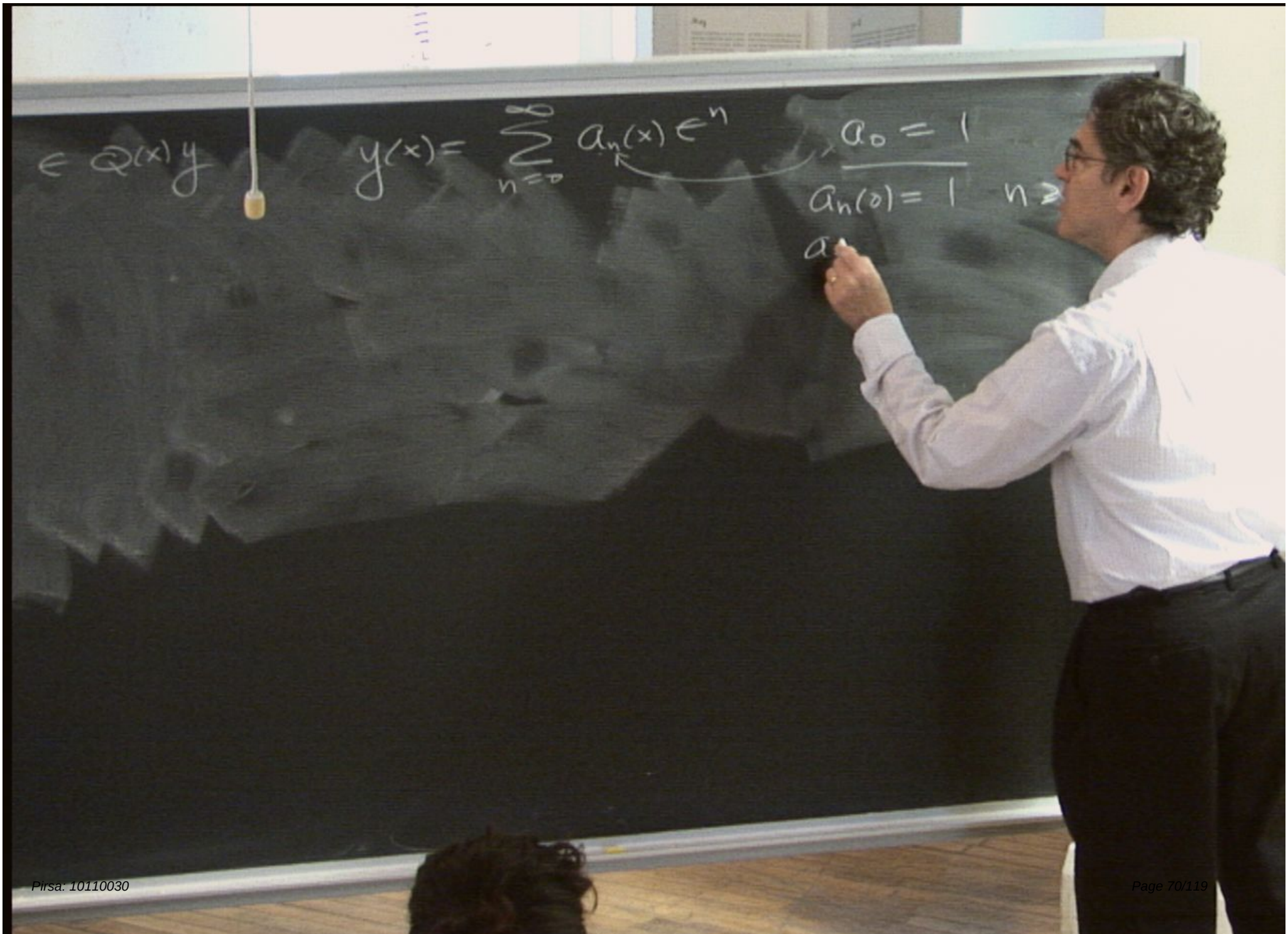
$y'' = Q(x) y$ RHP

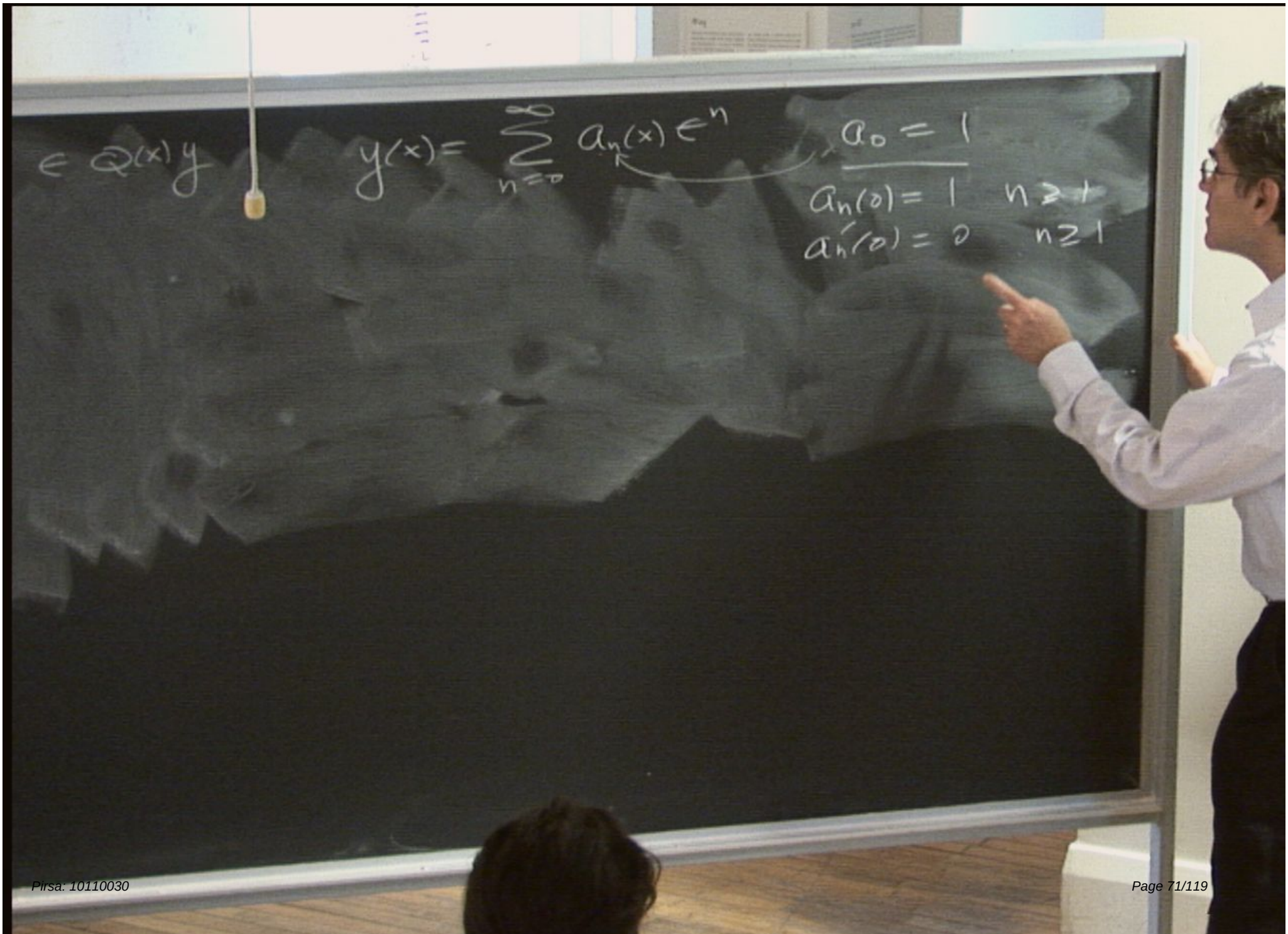
$y(0) = 1 \quad y'(0) = 0$

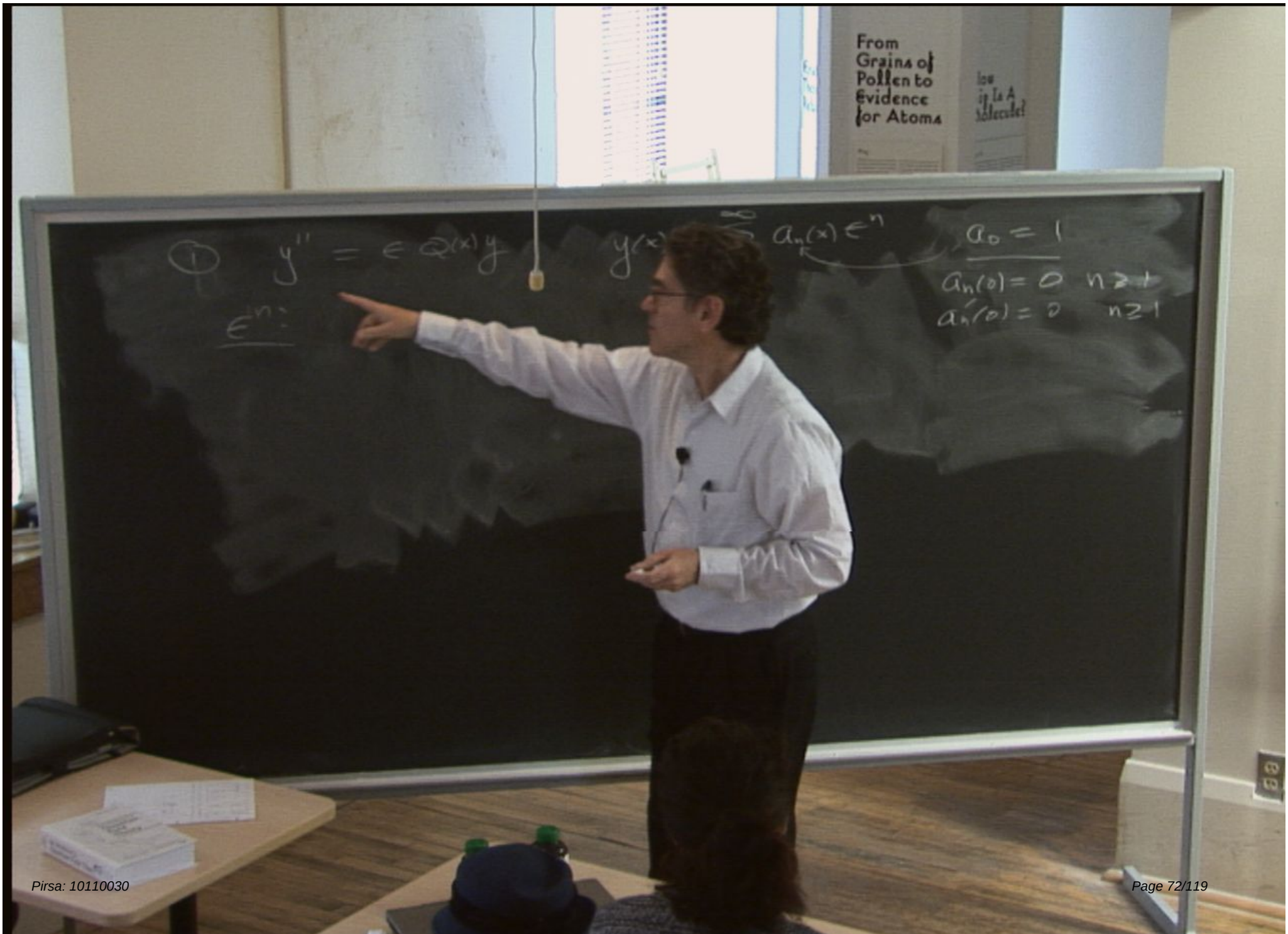
(1) $y'' = \epsilon Q(x) y$ $\qquad y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$

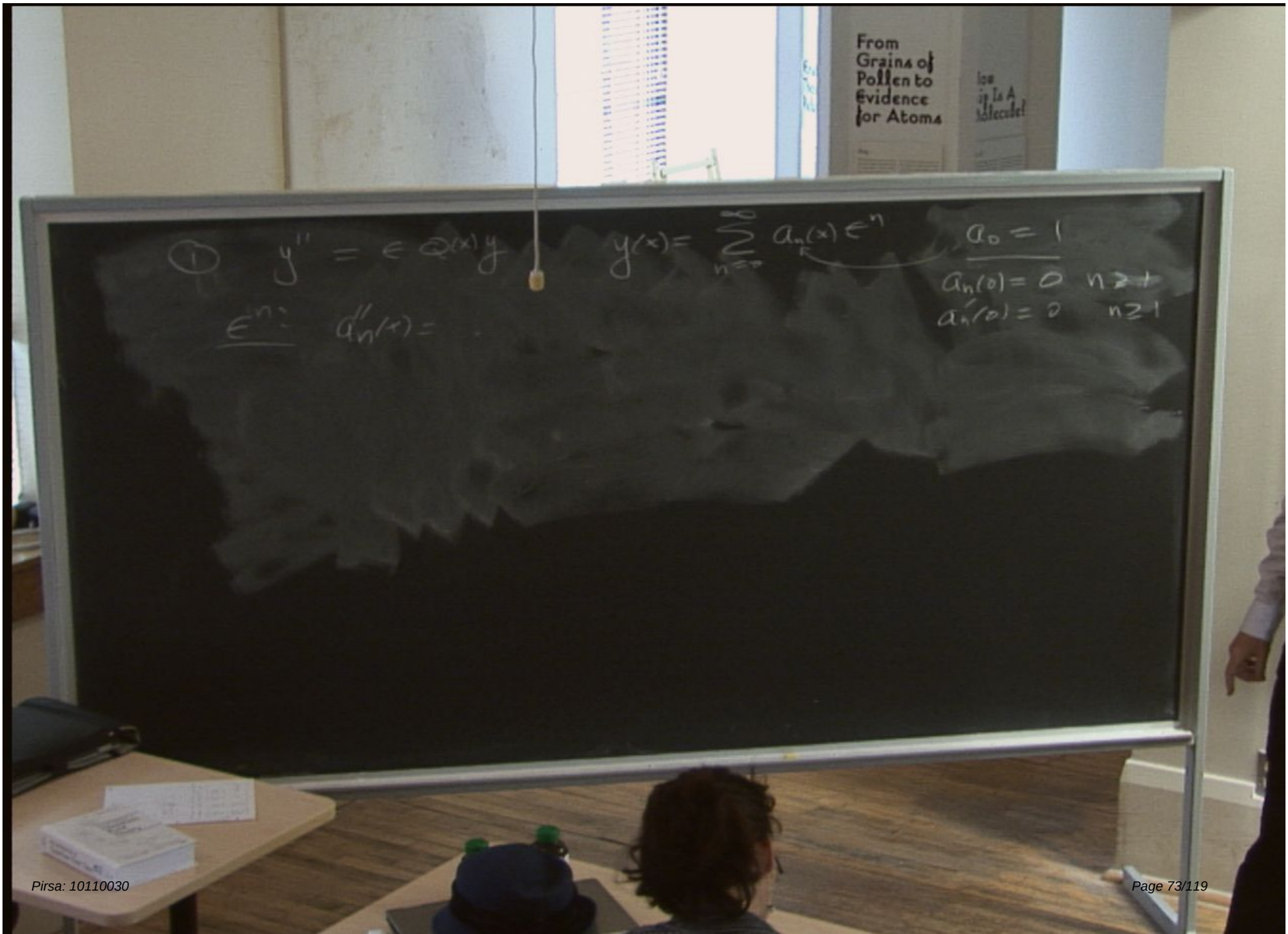
$\epsilon\, Q(x)\, y$ $\quad y(x) = \sum_{n=0}^{\infty} a_n(x)\, \epsilon^n$ $\quad , \underline{a_0 = 1}$

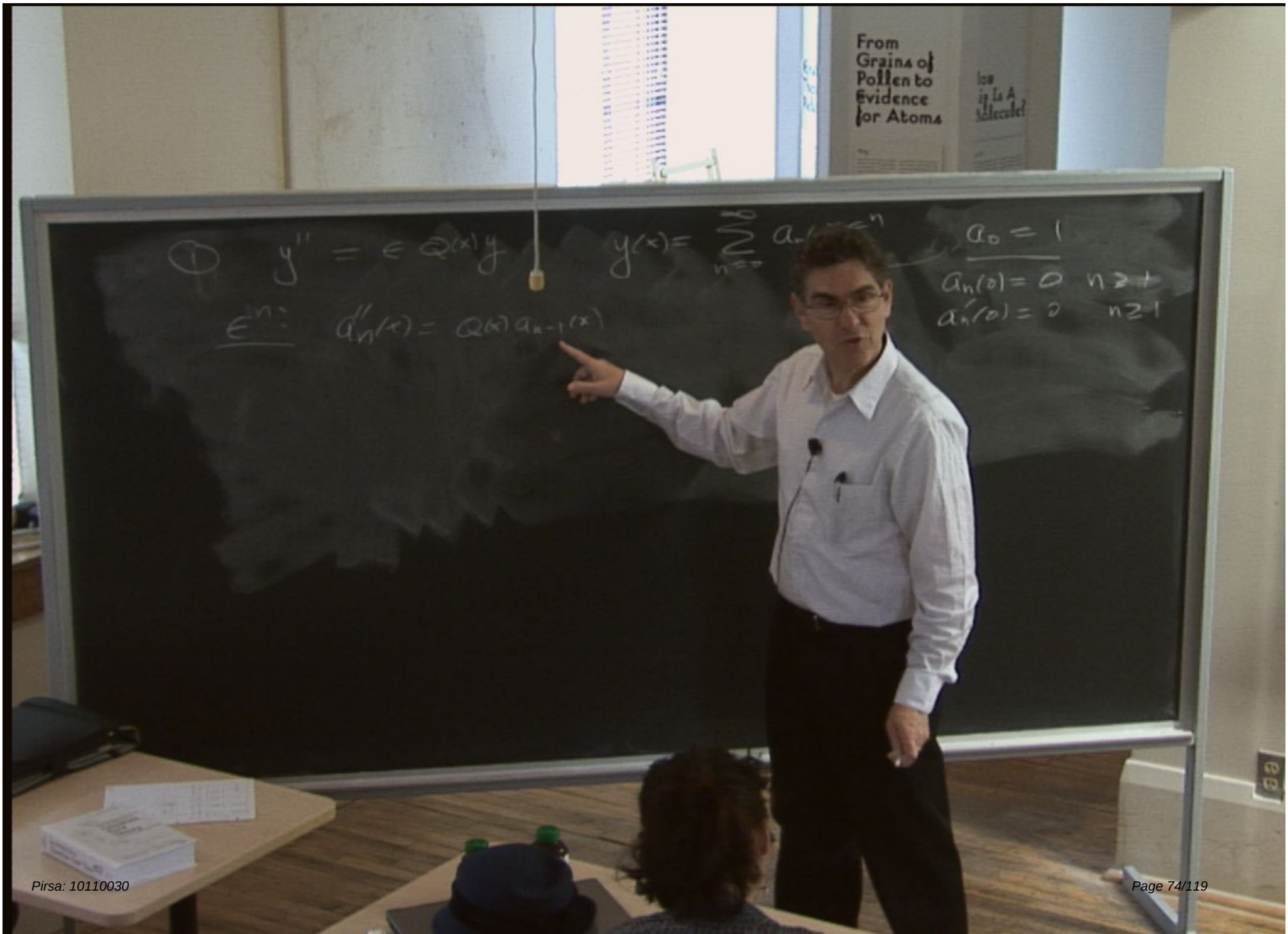
$$\epsilon\, Q(x)\, y \qquad y(x) = \sum_{n=0}^{\infty} a_n(x)\, \epsilon^n \qquad \underline{a_0 = 1}$$

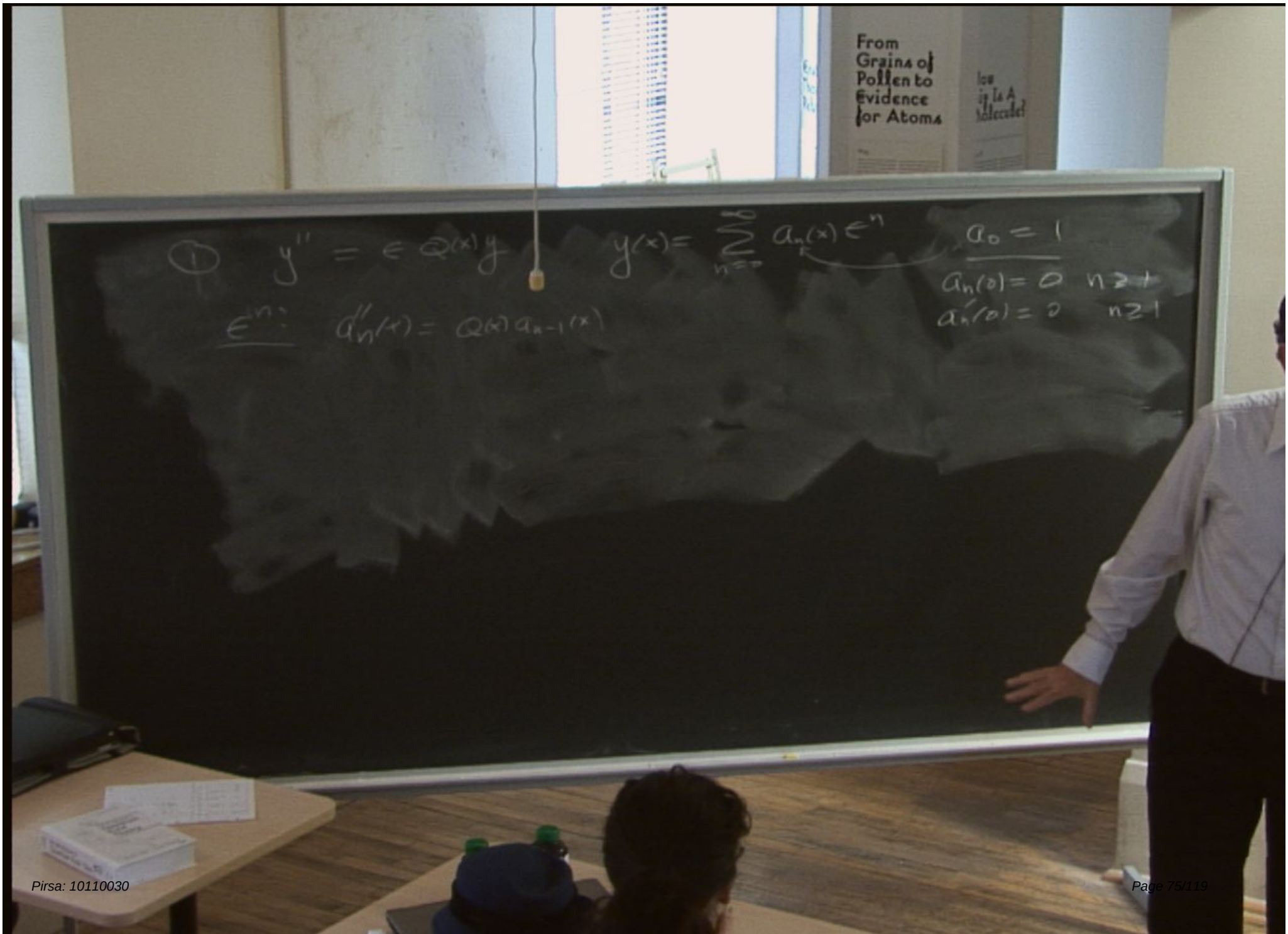

$\epsilon Q(x) y$
$y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$
$a_0 = 1$
$a_n(0) = 1$ $n \geq$


$\epsilon Q(x) y$
$y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$
$a_0 = 1$
$a_n(0) = 1 \quad n \geq 1$
$a_n'(0) = 0 \quad n \geq 1$


From
Grains of
Pollen to
Evidence
for Atoms
$y'' = \epsilon Q(x) y$
$a_0 = 1$
$a_n(0) = 0 \quad n \geq 1$
$a_n'(0) = 0 \quad n \geq 1$


From Grains of Pollen to Evidence for Atoms
Ice is A Molecule!
① $y'' = \epsilon Q(x) y$
$y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$
$a_0 = 1$
$a_n(0) = 0 \quad n \geq 1$
$a_n'(0) = 0 \quad n \geq 1$
ϵ^n: $a_n''(x) =$


From
Grains of
Pollen to
Evidence
for Atoms
$y'' = \epsilon Q(x) y$
$y(x) = \sum_{n=0}^{\infty} a_n$
$a_0 = 1$
$a_n(0) = 0 \quad n \geq 1$
$a_n'(0) = 0 \quad n \geq 1$
ϵ^n: $a_n''(x) = Q(x) a_{n-1}(x)$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$y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$
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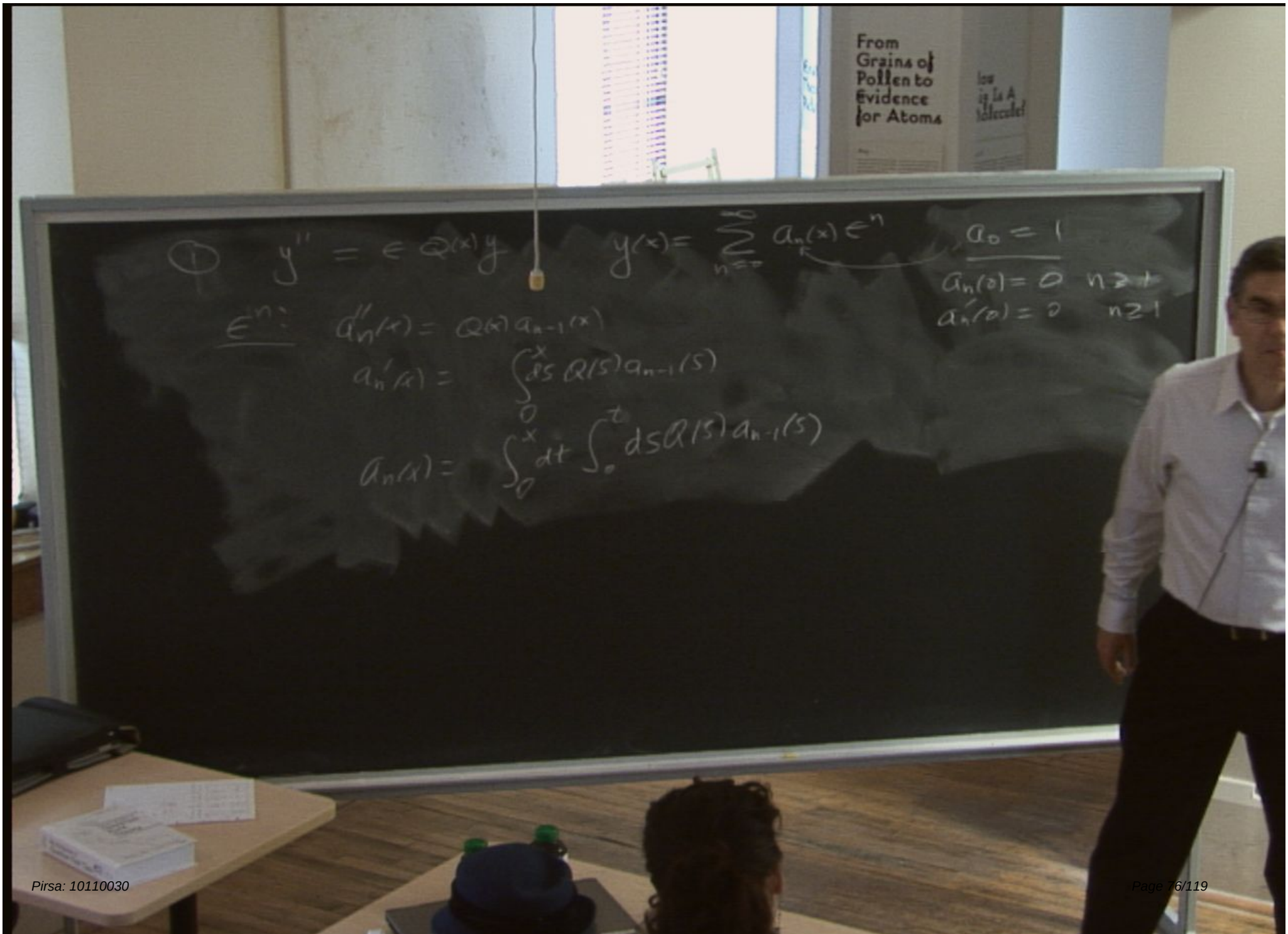
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$a_n(x) = \int_0^x dt \int_0^t ds\, Q(s) a_{n-1}(s)$

(1) $y'' = \epsilon Q(x) y$ $\quad y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$ $\quad a_0 = 1$

$a_n(0) = 0 \quad n \geq$

$a_n'(0) = 0 \quad n$

$\underline{\epsilon^n:}$ $\quad a_n''(x) = Q(x) a_{n-1}(x)$

$$a_n'(x) = \int_0^x ds\, Q(s) a_{n-1}(s)$$

$$a_n(x) = \int_0^x dt \int_0^t ds\, Q(s) a_{n-1}(s)$$

$$= \epsilon Q(x) y \qquad y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$$

$$a_0 = 1$$

$$a_n(0) = 0 \quad n \geq 1$$

$$a_n'(0) = 0 \quad n \geq 1$$

$$a_n''(x) = Q(x)\, a_{n-1}(x)$$

$$a_n'(x) = \int_0^x ds\, Q(s)\, a_{n-1}(s)$$

$$a_n(x) = \int_0^x dt \int_0^t ds\, Q(s)\, a_{n-1}(s)$$

$$a_1(x) = \int_0^x dt \int_0^t ds\, Q(s)$$

$$a_2(x) = \iint Q \iint Q$$

$$a_3(x) = \iint Q \iint Q$$

$$= \epsilon Q(x) y \qquad y(x) = \sum_{n=0}^{\infty} a_n(x) \epsilon^n$$

$$a_0 = 1$$

$$a_n(0) = 0 \quad n \geq 1$$

$$a_n'(0) = 0 \quad n \geq 1$$

$$a_n''(x) = Q(x)\, a_{n-1}(x)$$

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$$a_1(x) = \int_0^x dt \int_0^t ds\, Q(s)$$

$$a_2(x) = \iint Q \iint Q$$

$$a_3(x) = \iint Q \iint Q \iint Q$$

rturbation theory

HP

) $\epsilon \longrightarrow HP(\epsilon)$ ← in such a way that un part is simple

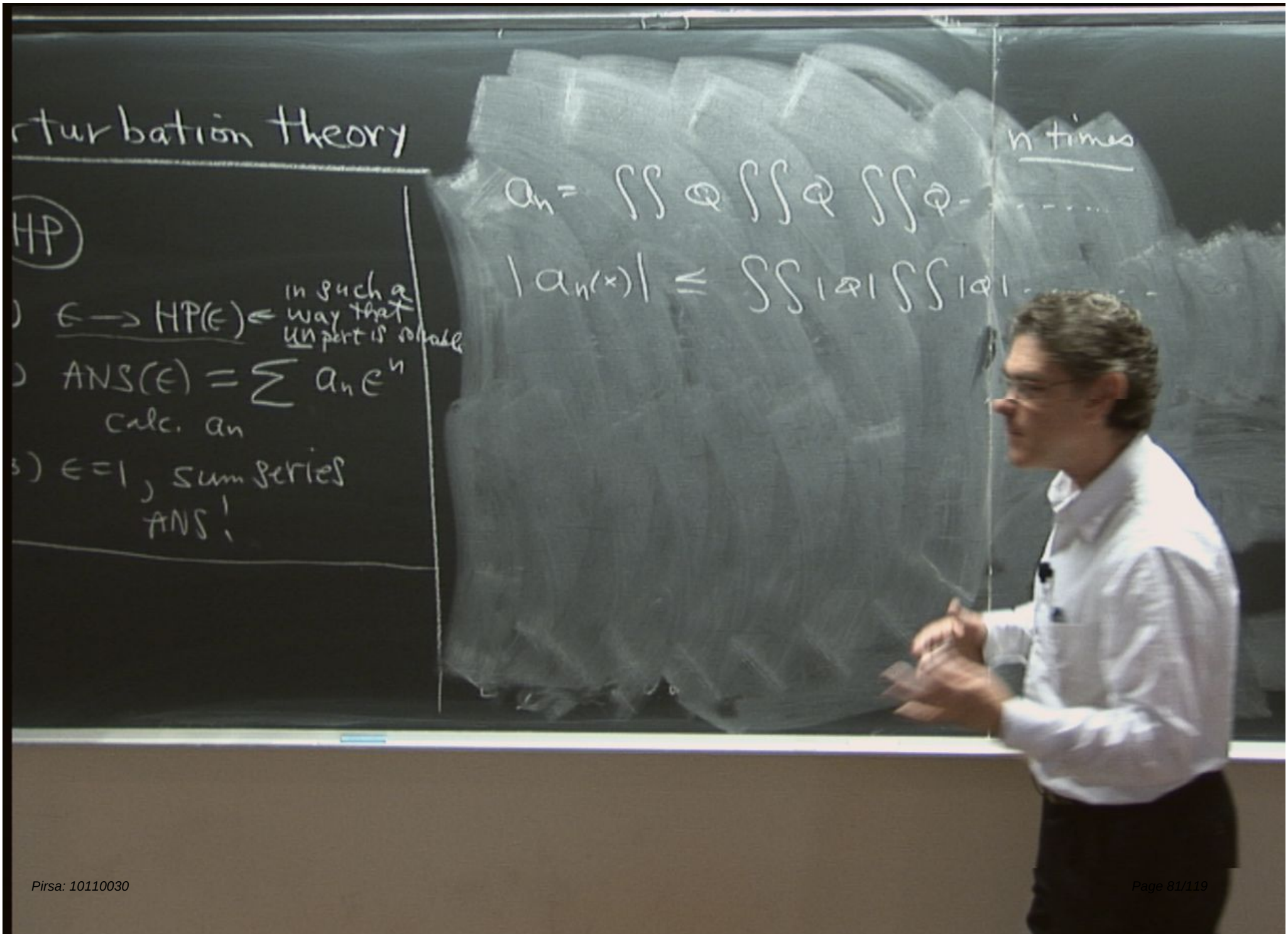
) $ANS(\epsilon) = \sum a_n \epsilon^n$

calc. a_n

s) $\epsilon = 1$, sum series

ANS!

$a_n = \int\int Q \int\int Q \int\int Q \cdots$ n times


rturbation theory
HP
) ε → HP(ε) ← in such a way that un part is small
) ANS(ε) = Σ a_n ε^n
calc. a_n
) ε=1, sum series
ANS!
n times
a_n = ∫∫ Q ∫∫ Q ∫∫ Q
|a_n(x)| ≤ ∫∫ |Q| ∫∫ |Q|

rturbation theory

HP

) $\epsilon \to HP(\epsilon)$ ← in such a way that unpert is simple

) $ANS(\epsilon) = \sum a_n \epsilon^n$

calc. a_n

s) $\epsilon = 1$, sum series

ANS!

n times

$a_n = \int\int Q \int\int Q \int\int Q \ldots\ldots$

$|a_n(x)| \leq \int\int |Q| \int\int |Q| \ldots\ldots$

$|Q(x)| \leq M$

rturbation theory

HP

) $\epsilon \longrightarrow HP(\epsilon)$ ← in such a way that un part is small

) $ANS(\epsilon) = \sum a_n \epsilon^n$

calc. a_n

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ANS!

n times

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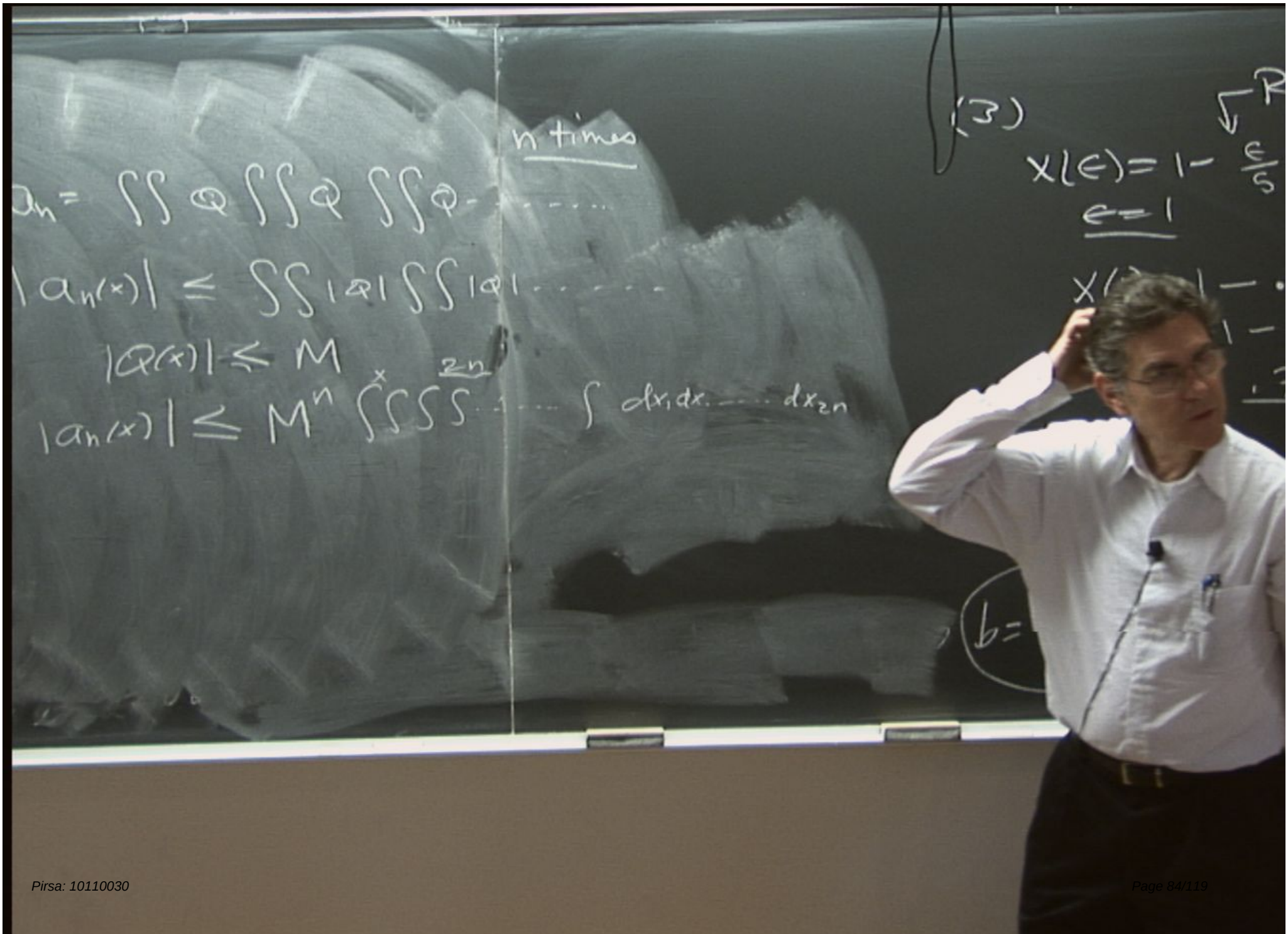
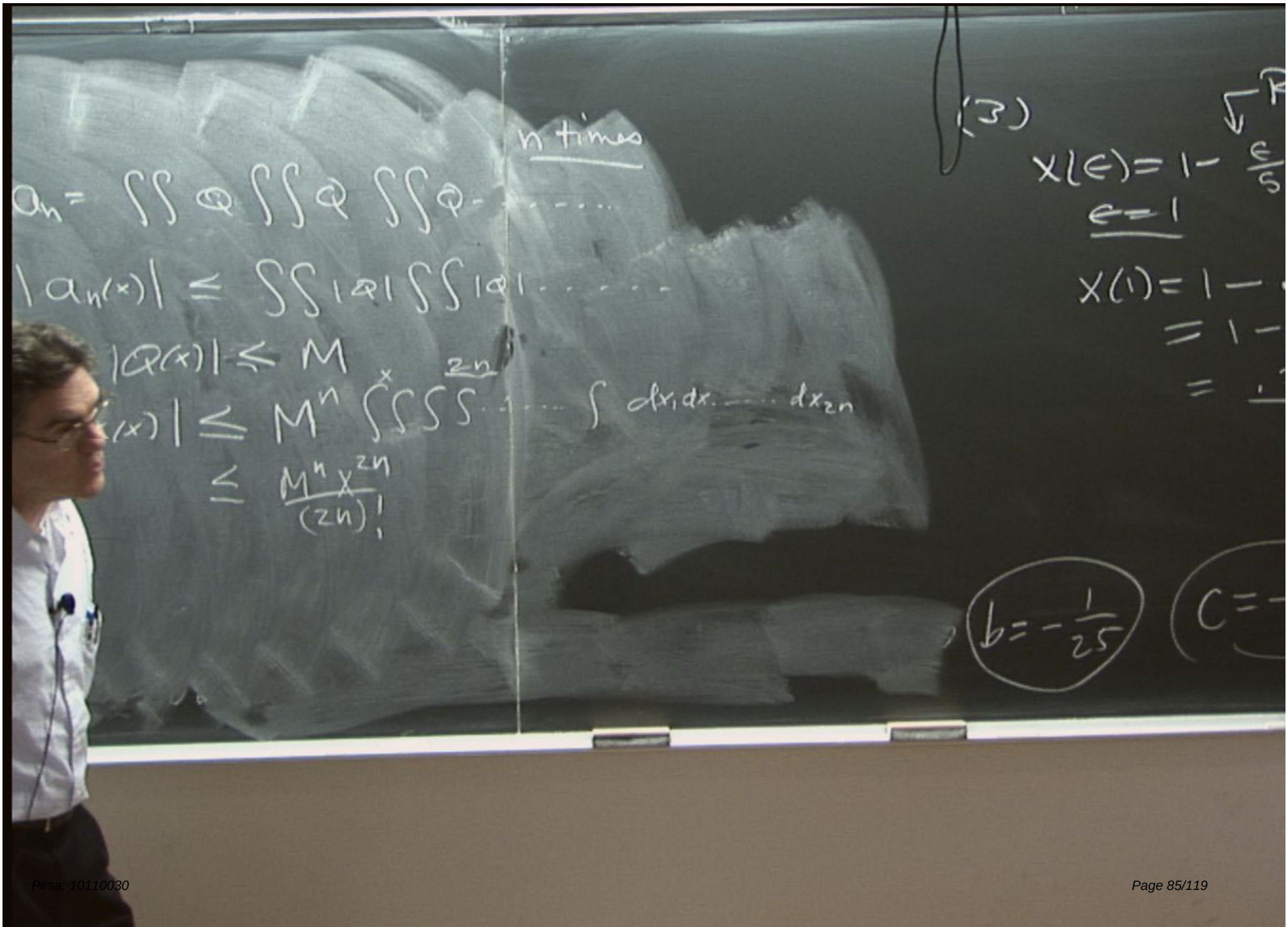
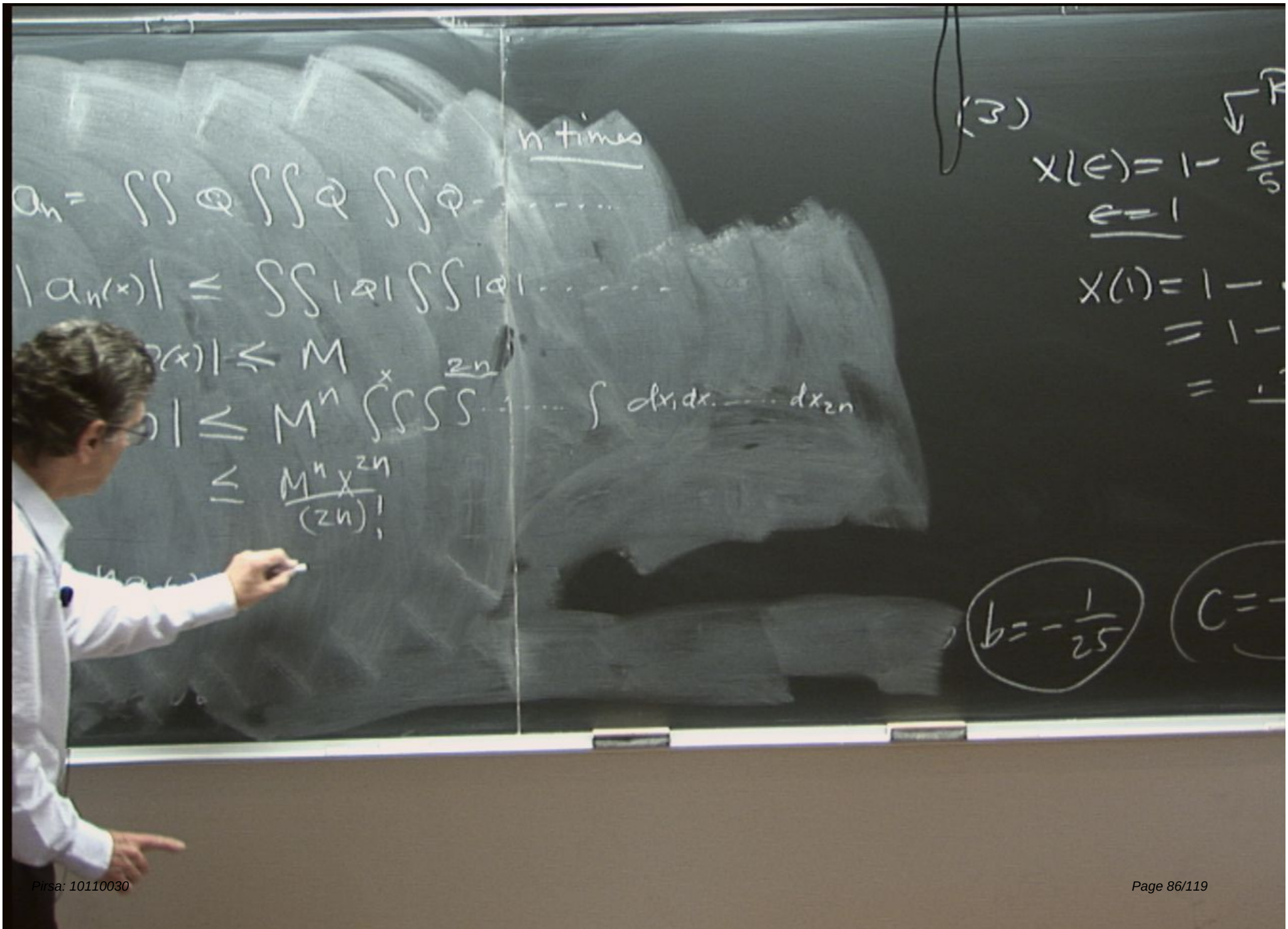
$|a_n(x)| \leq \int\int |Q| \int\int |Q| \cdots\cdots$

$|Q(x)| \leq M$

$|a_n(x)| \leq M^n$


n times
$a_n = \iint Q \iint Q \iint Q \cdots$
$|a_n(x)| \leq \iint |Q| \iint |Q| \cdots$
$|Q(x)| \leq M$
$|a_n(x)| \leq M^n \iint \cdots \int dx_1 dx_2 \cdots dx_{2n}$
(3)
$x(\epsilon) = 1 - \frac{\epsilon}{5}$
$\epsilon = 1$


n times
$a_n = \iint Q \iint Q \iint Q \cdots$
$|a_n(x)| \le \iint |Q| \iint |Q| \cdots$
$|Q(x)| \le M$
$|a_n(x)| \le M^n \overbrace{\iiiint \cdots \int}^{2n} dx_1 dx_2 \cdots dx_{2n}$
$\le \frac{M^n x^{2n}}{(2n)!}$
(3)
$x(\epsilon) = 1 - \frac{\epsilon}{5}$
$\epsilon = 1$
$x(1) = 1 -$
$= 1 -$
$= .$
$b = -\frac{1}{25}$
$c =$


n times
$a_n = \iint Q \iint Q \iint Q \cdots$
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(3)
$x(\epsilon) = 1 - \frac{\epsilon}{5}$
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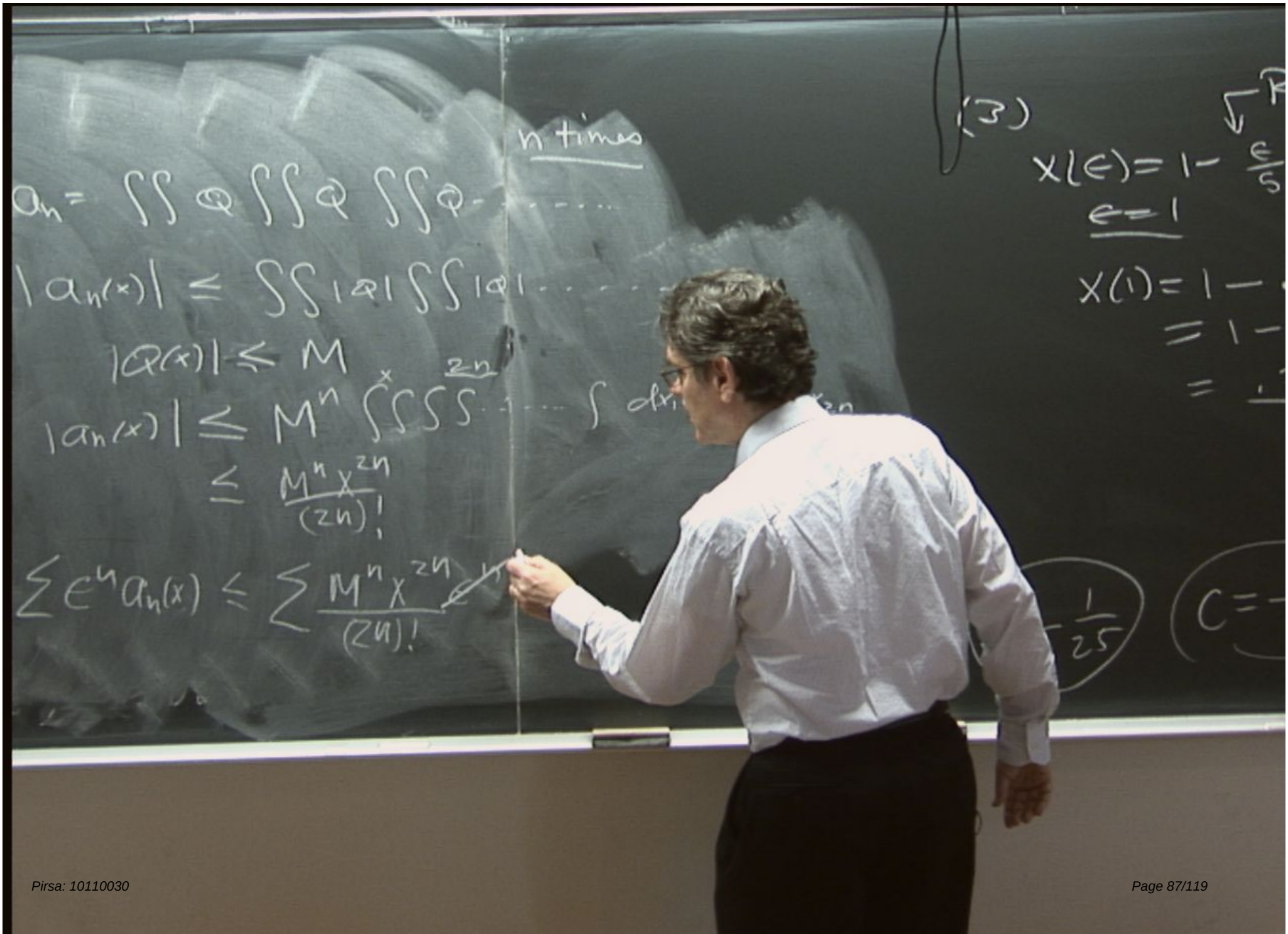
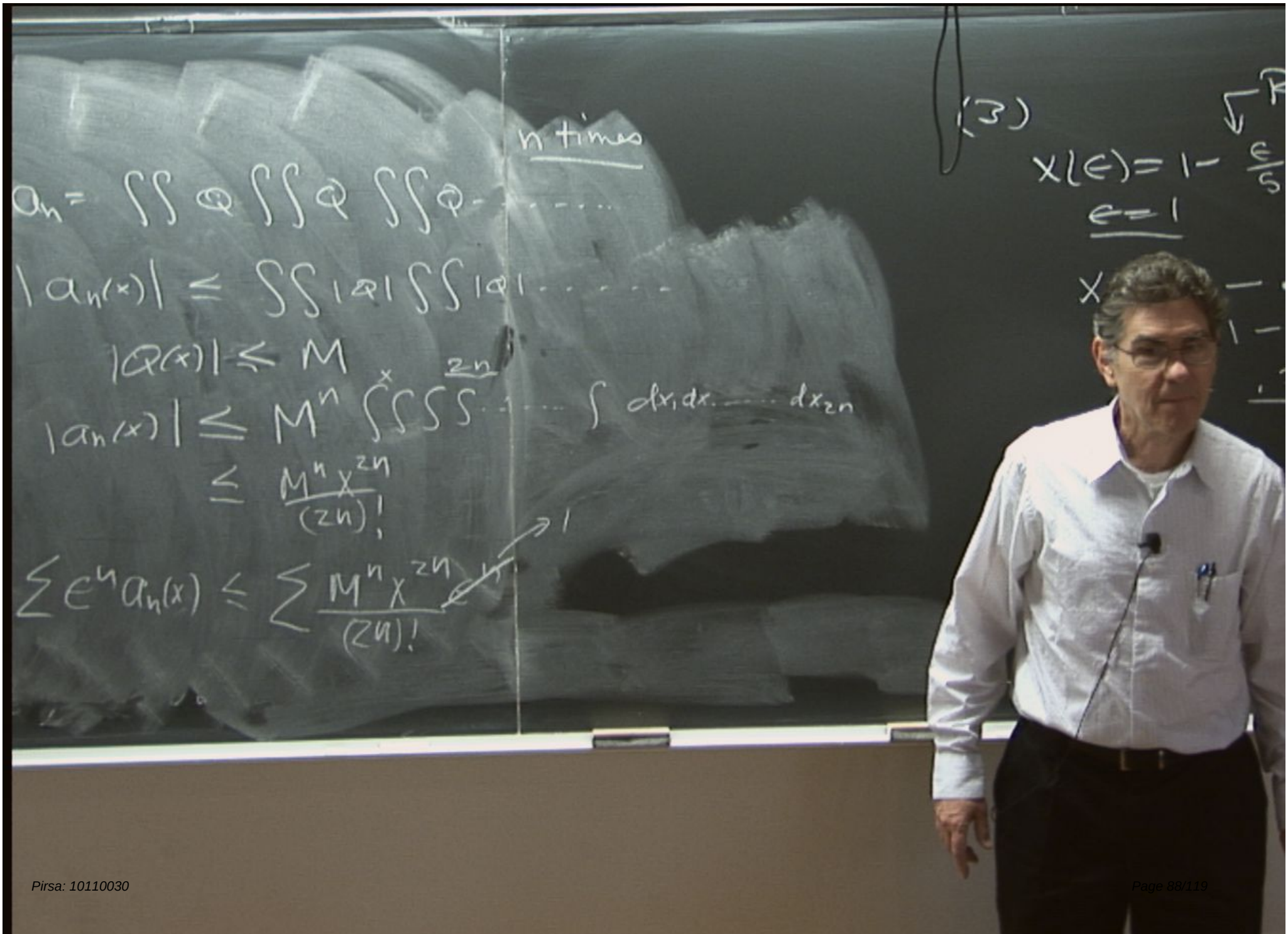

n times
$a_n = \iint Q \iint Q \iint Q \cdots$
$|a_n(x)| \le \iint |Q| \iint |Q| \cdots$
$|Q(x)| \le M$
$|a_n(x)| \le M^n \iiiint \cdots \int dx_1 \cdots$
$\le \frac{M^n x^{2n}}{(2n)!}$
$\sum \epsilon^n a_n(x) \le \sum \frac{M^n x^{2n}}{(2n)!}$
(3)
$x(\epsilon) = 1 - \frac{\epsilon}{5}$
$\epsilon = 1$
$x(1) = 1 -$
$= 1 -$
$= .$
$-\frac{1}{25}$
$C =$


n times
$a_n = \iint Q \iint Q \iint Q \cdots$
$|a_n(x)| \le \iint |Q| \iint |Q| \cdots$
$|Q(x)| \le M$
$|a_n(x)| \le M^n \overbrace{\iiiint \cdots \int}^{2n} dx_1 dx \cdots dx_{2n}$
$\le \frac{M^n x^{2n}}{(2n)!}$
$\sum \epsilon^n a_n(x) \le \sum \frac{M^n x^{2n}}{(2n)!}$
(3)
$x(\epsilon) = 1 - \frac{\epsilon}{5}$
$\epsilon = 1$

$$a_n = \underbrace{\iint Q \iint Q \iint Q \cdots}_{n \text{ times}}$$

$$|a_n(x)| \le \iint |Q| \iint |Q| \cdots$$

$$|Q(x)| \le M$$

$$|a_n(x)| \le M^n \underbrace{\iiiint \cdots \int}_{2n} dx_1 dx_2 \cdots dx_{2n}$$

$$\le \frac{M^n x^{2n}}{(2n)!}$$

$$\sum \epsilon^n a_n(x) \le \sum \frac{M^n x^{2n}}{(2n)!} \epsilon^n \to 1$$

(3)

$$x(\epsilon) = 1 - \frac{\epsilon}{5}$$

$$\epsilon = 1$$

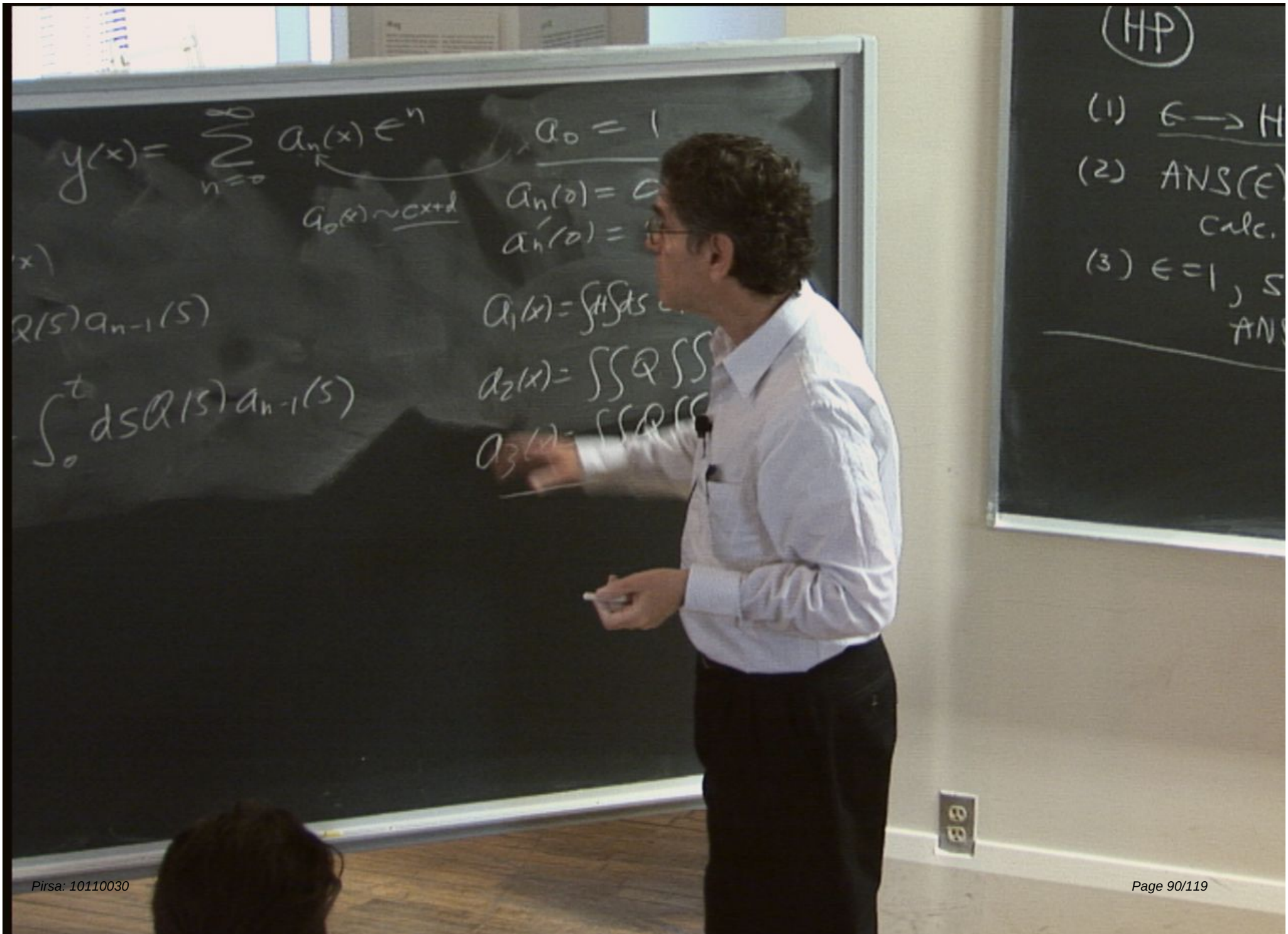
$$x(1) = 1 - \cdots$$

$$= 1 - \cdots$$

$$= \cdots$$

$$b = -\frac{1}{25}$$

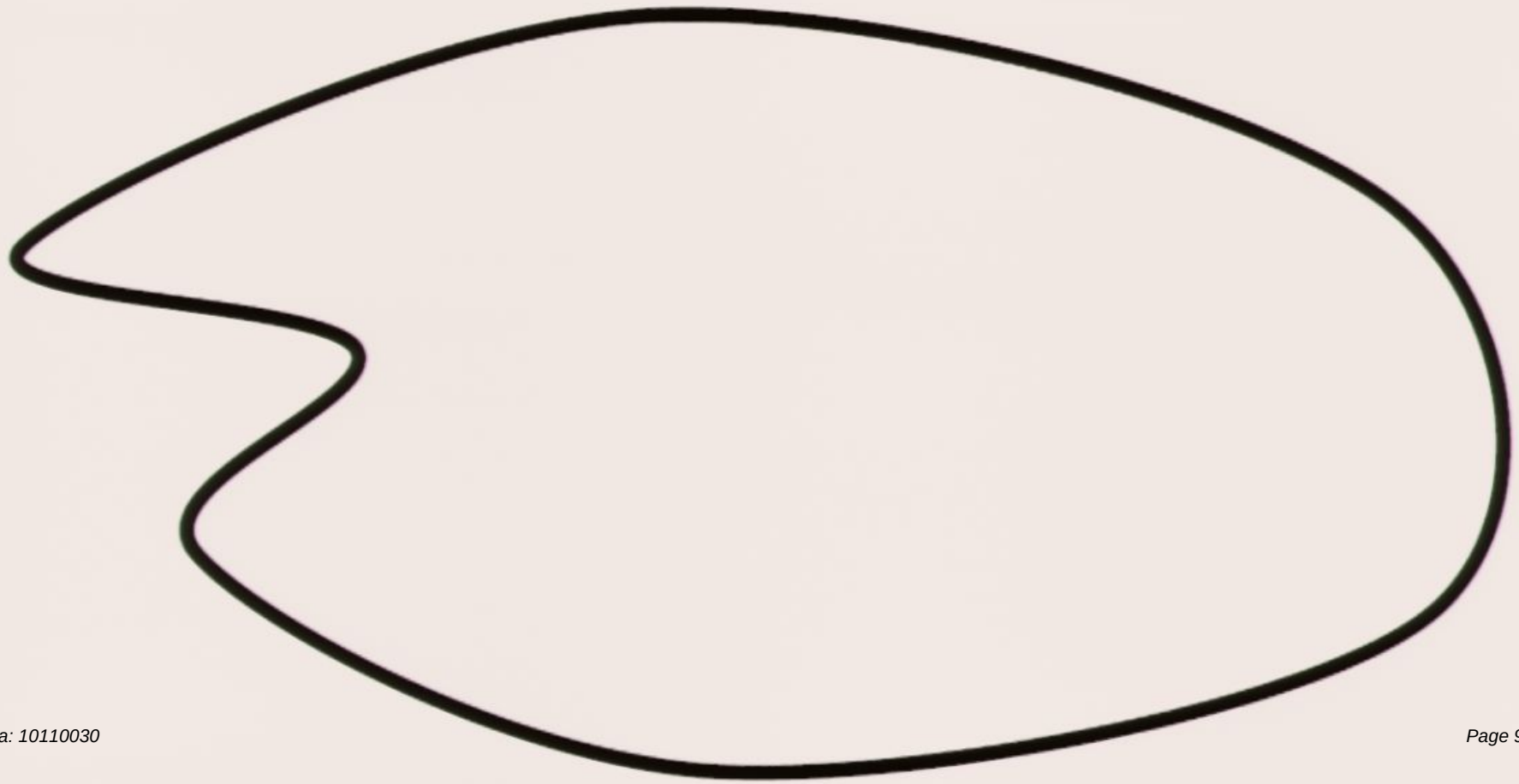
$$c = \cdots$$


HP
(1) ε → H
(2) ANS(ε)
calc.
(3) ε=1, S
ANS
y(x) = Σ_{n=0}^{∞} a_n(x) ε^n
a_0 = 1
a_0(x) ~ cx+d
a_n(0) = c
a_n'(0) =
Q(s) a_{n-1}(s)
∫_0^t ds Q(s) a_{n-1}(s)
a_1(x) = ∫∫ds
a_2(x) = ∫∫Q∫∫
a_3(x) = ∫∫Q∫∫

How to sum a series if it CONVERGES —and— How to sum a series if it DIVERGES

Carl Bender
Physics Department
Washington University in St. Louis

Perimeter Institute 2010



Series arise when you use perturbation theory

Perturbation theory for a HARD PROBLEM:

Step 1. Insert a small parameter ε:

HARD PROBLEM(ε)

Step 2. Expand answer as a perturbation series in powers of ε:

$$\text{ANSWER}(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n$$

Step 3. Set ε=1 and sum the series – **this is not so easy!**

Simple example

HARD PROBLEM: Find the positive root of

$$x^5 + x = 1$$

ANSWER: $x = 0.75487767\cdots$

Step 1. Insert ε: $x^5 + \epsilon x = 1$ (Strong coupling)

Step 2. $x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$

$$a_0 = 1$$

Match powers of ε

$$
\begin{aligned}
5a_1 + 1 &= 0, \\
5a_2 + 10a_1^2 + a_1 &= 0, \\
5a_3 + 20a_1a_2 + a_2 + 10a_1^3 &= 0, \\
5a_4 + 20a_1a_3 + a_3 + 10a_2^2 + 30a_1^2a_2 + 5a_1^4 &= 0.
\end{aligned}
$$

$$a_1 = -\frac{1}{5}, \quad a_2 = -\frac{1}{25}, \quad a_3 = -\frac{1}{125},$$

$$a_4 = 0, \quad a_5 = \frac{21}{15625}, \quad a_6 = \frac{78}{78125}$$

The perturbation series…

$$x(\epsilon) = 1 - \frac{1}{5}\epsilon - \frac{1}{25}\epsilon^2 - \frac{1}{125}\epsilon^3 + \frac{21}{15625}\epsilon^5 + \frac{78}{78125}\epsilon^6 + \cdots$$

Step 3. Sum the series at $x = 1$

Radius of convergence of this series: 1.64938…

Sixth-order result $x(1) = 0.75434$

Exact answer $x = 0.75488$

Another way to insert ε

Step 1. Insert ε: $\epsilon x^5 + x = 1$ (weak coupling)

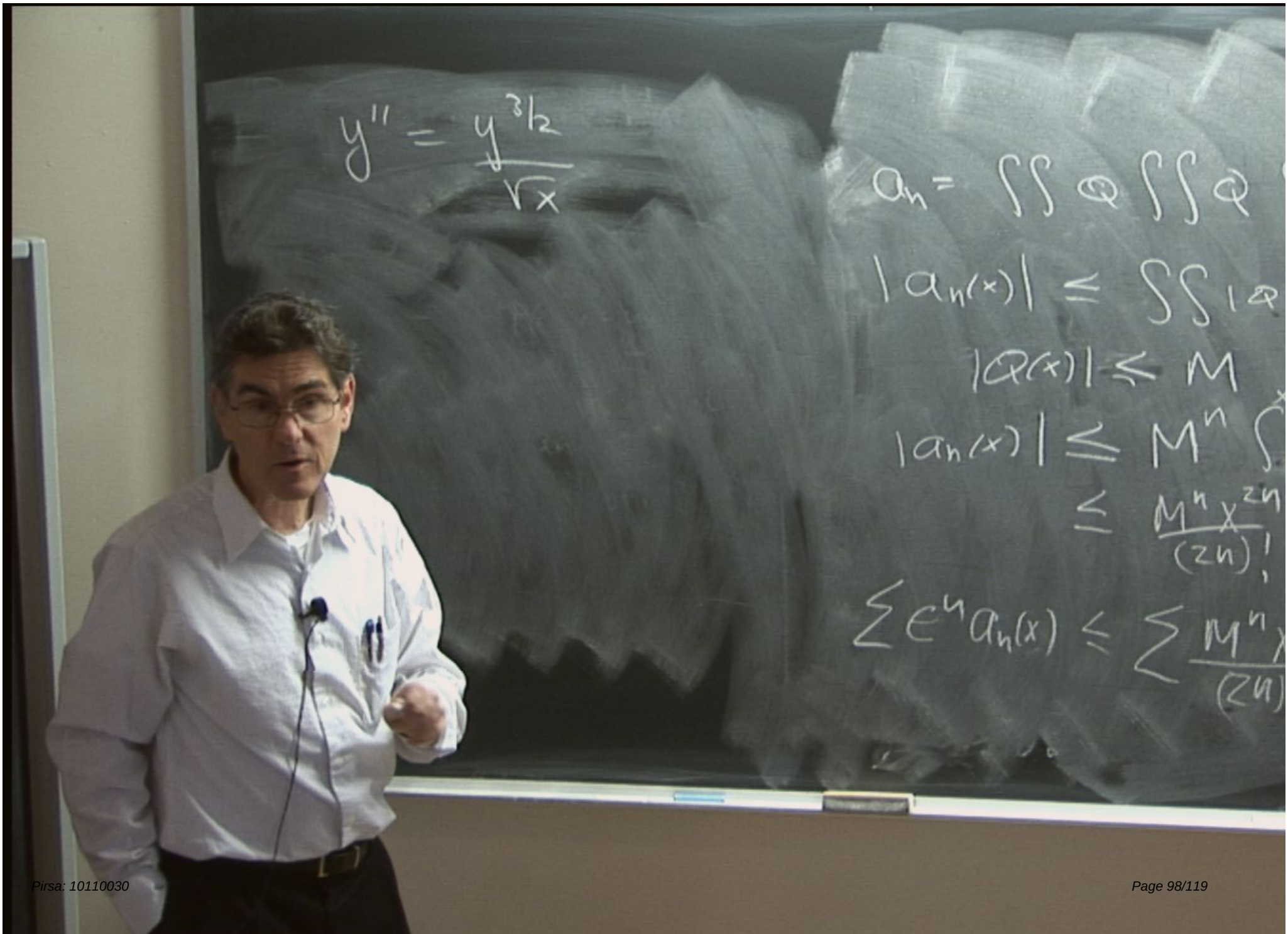
Step 2. Perturbation series

$$x(\epsilon) = 1 - \epsilon + 5\epsilon^2 - 35\epsilon^3 + 285\epsilon^4 - 2530\epsilon^5 + 23751\epsilon^6 - \cdots$$

Step 3. Sum the series at $x = 1$

Radius of convergence of this series 0.08192

Result: $x(1) = 21476$


$y'' = \frac{y^{3/2}}{\sqrt{x}}$
$a_n = \iint Q \iint Q$
$|a_n(x)| \leq \iint |Q$
$|Q(x)| \leq M$
$|a_n(x)| \leq M^n$
$\leq \frac{M^n x^{2n}}{(2n)!}$
$\sum \epsilon^n a_n(x) \leq \sum \frac{M^n}{(2n)}$

$$y'' = \frac{y^{3/2}}{\sqrt{x}}$$

$$y(0) = 1 \qquad y(\infty) = 0$$

$$a_n = \iint Q \iint Q$$

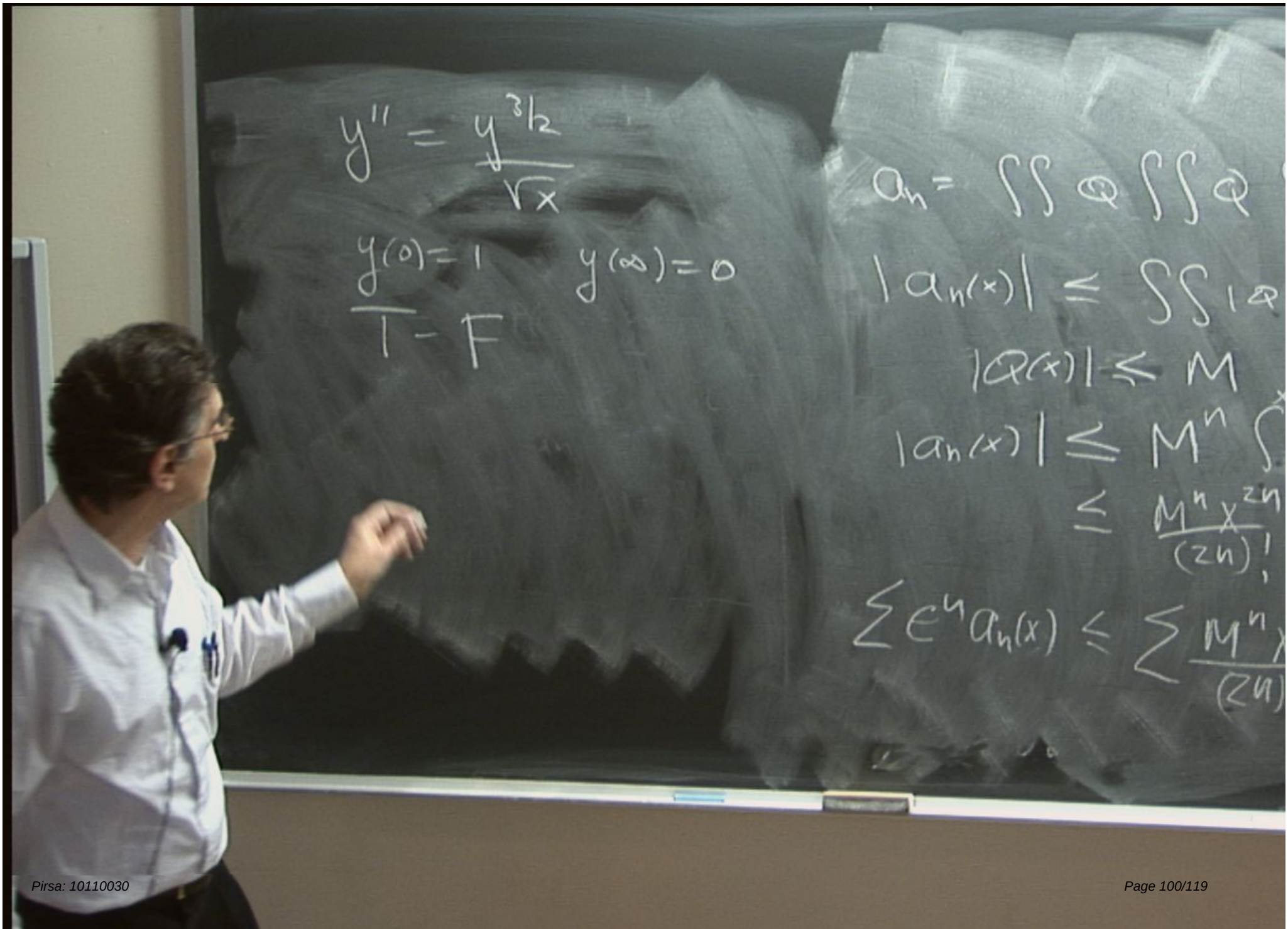
$$|a_n(x)| \leq \iint |Q$$

$$|Q(x)| \leq M$$

$$|a_n(x)| \leq M^n \int$$

$$\leq \frac{M^n x^{2n}}{(2n)!}$$

$$\sum \epsilon^n a_n(x) \leq \sum \frac{M^n x}{(2n)}$$


$y'' = \frac{y^{3/2}}{\sqrt{x}}$
$y(0) = 1 \quad y(\infty) = 0$
T - F
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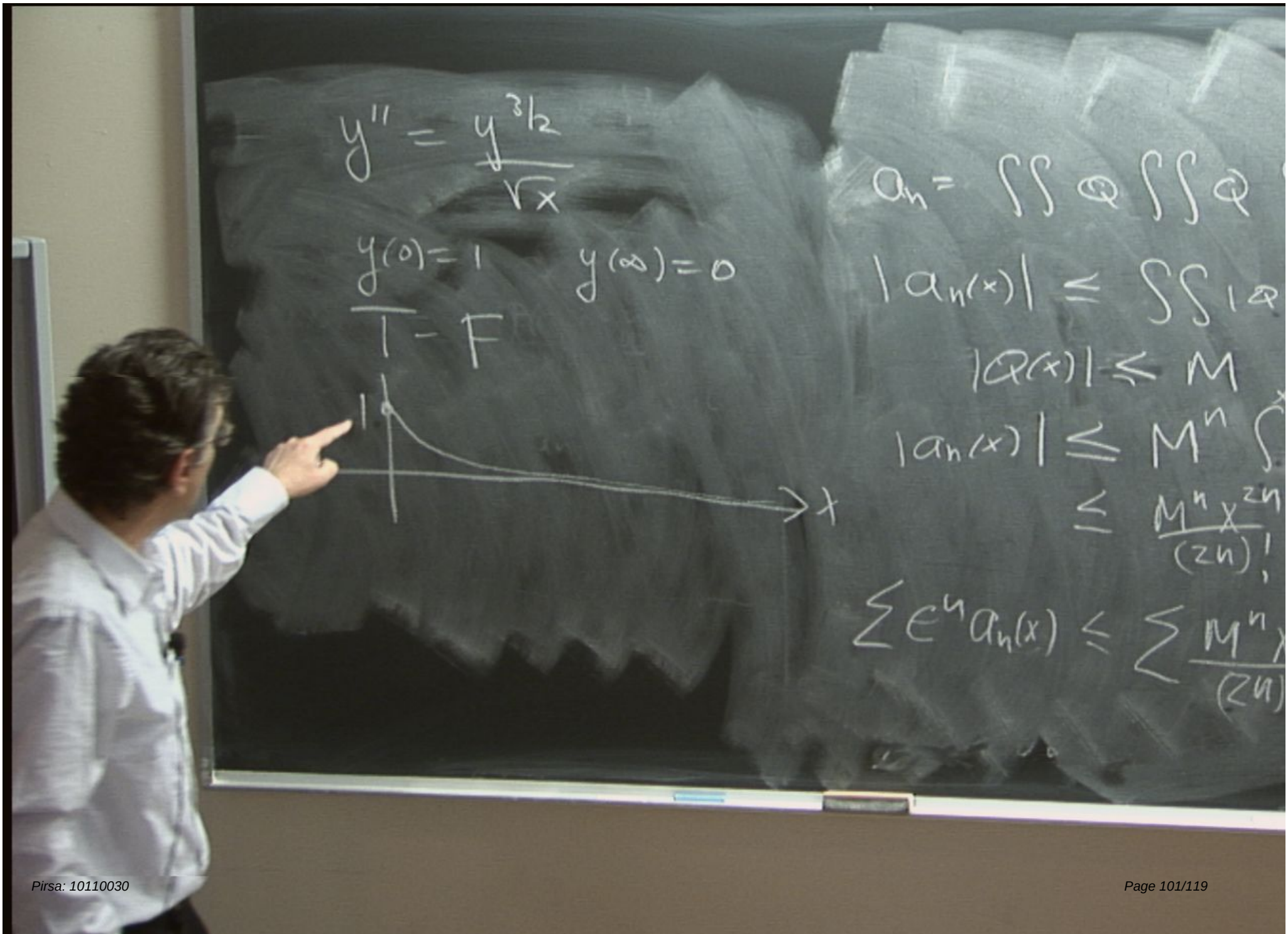
$y'' = \frac{y^{3/2}}{\sqrt{x}}$
$y(0)=1 \quad y(\infty)=0$
T - F
1
x
$a_n = \iint Q \iint Q$
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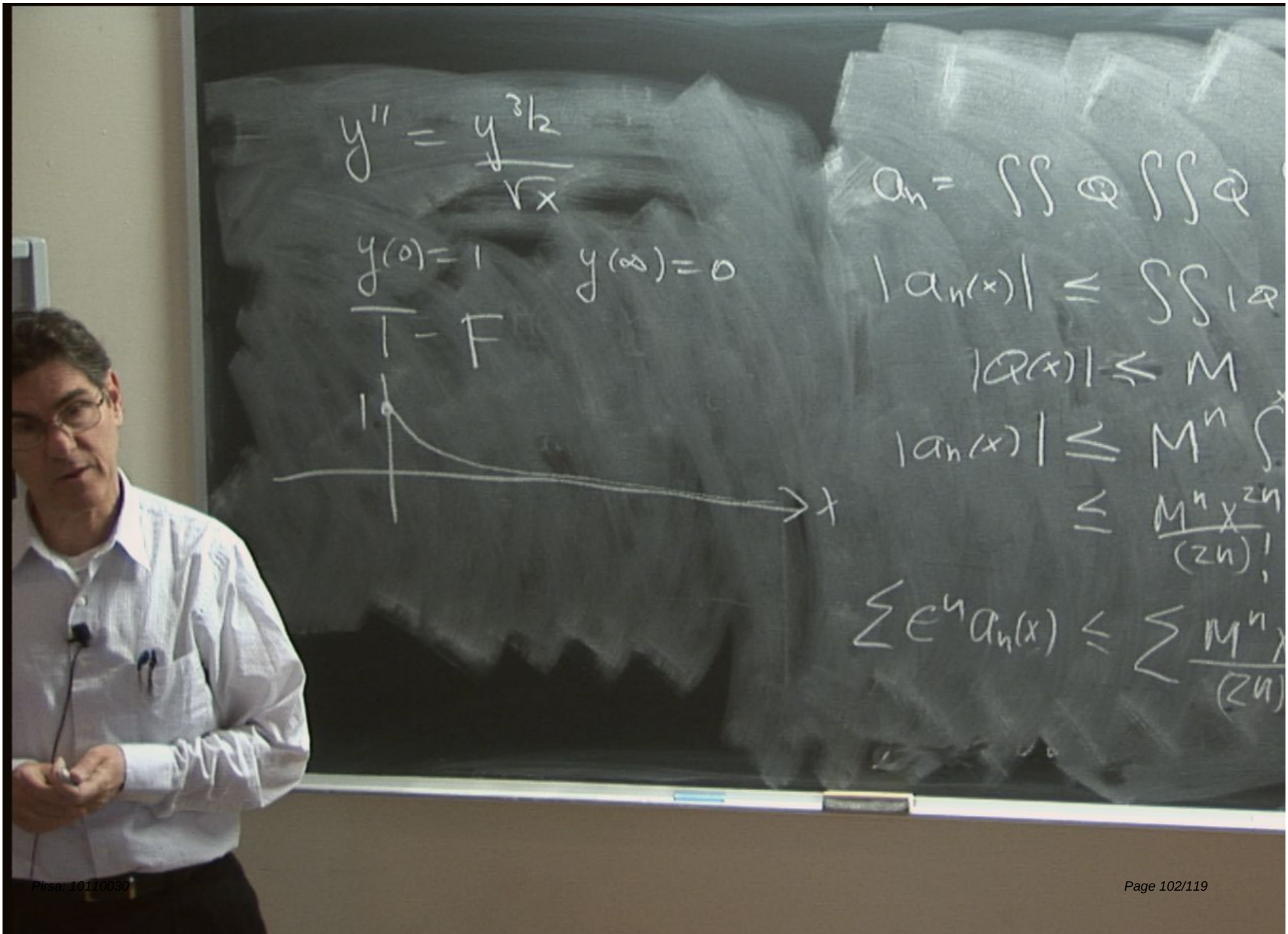
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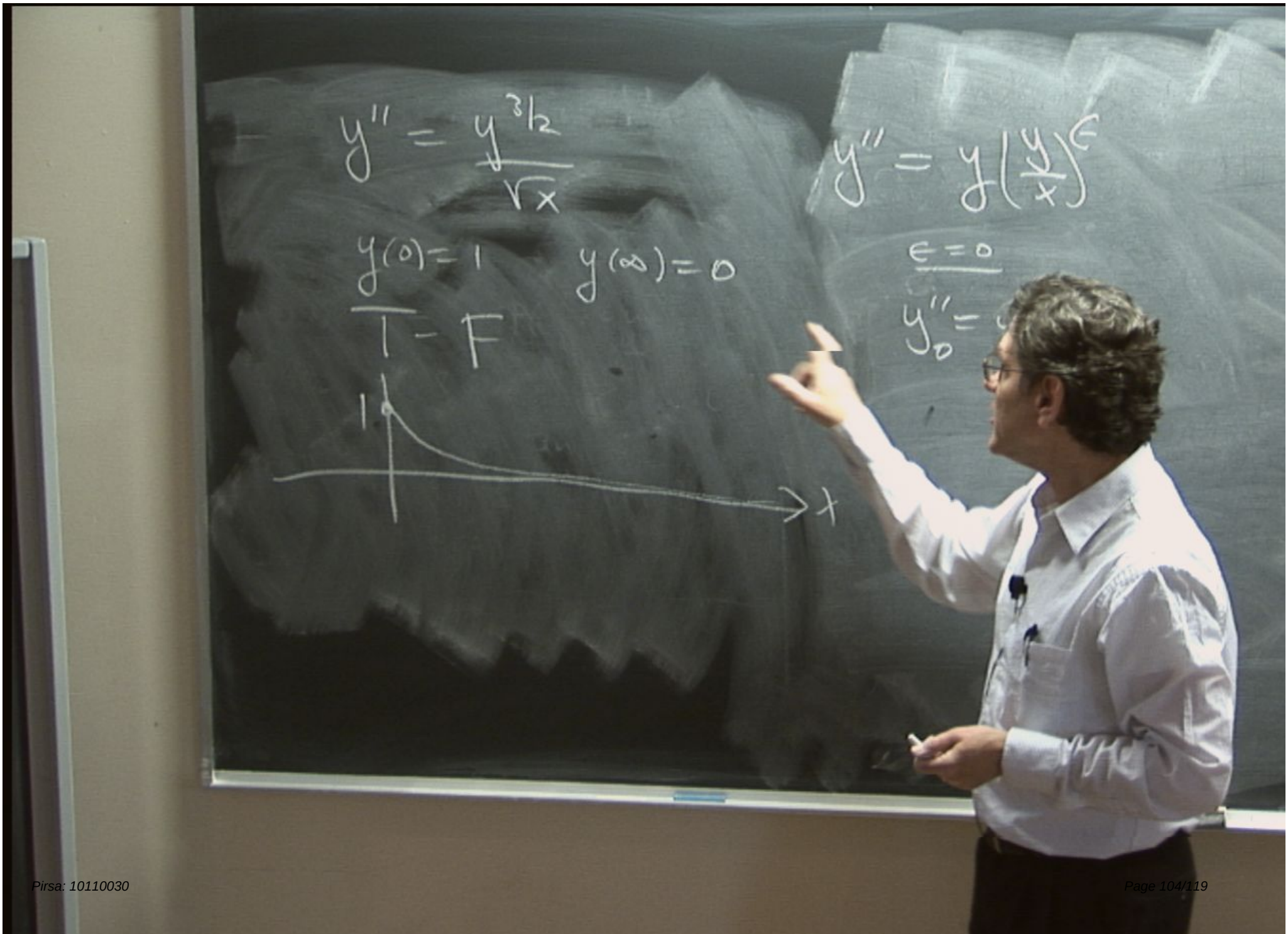
$\sum \epsilon^n a_n(x) \le \sum \frac{M^n x}{(2n)}$

$$y'' = \frac{y^{3/2}}{\sqrt{x}}$$

$$y'' = y$$

$y(0) = 1 \qquad y(\infty) = 0$

T - F


$y'' = \frac{y^{3/2}}{\sqrt{x}}$
$y(0) = 1$
$y(\infty) = 0$
T - F
1
x
$y'' = y\left(\frac{y}{x}\right)^{\epsilon}$
$\epsilon = 0$
$y''_0 =$

$$y'' = \frac{y^{3/2}}{\sqrt{x}}$$

$$y'' = y\left(\frac{y}{x}\right)^{\epsilon}$$

$y(0) = 1$ $\quad y(\infty) = 0$ ✓

$\epsilon = 0$

$y''_0 = y_0$ ✓

T - F

$$y'' = \frac{y^{3/2}}{\sqrt{x}}$$

$y(0) = 1$ $\quad$ $y(\infty) = 0$ ✓

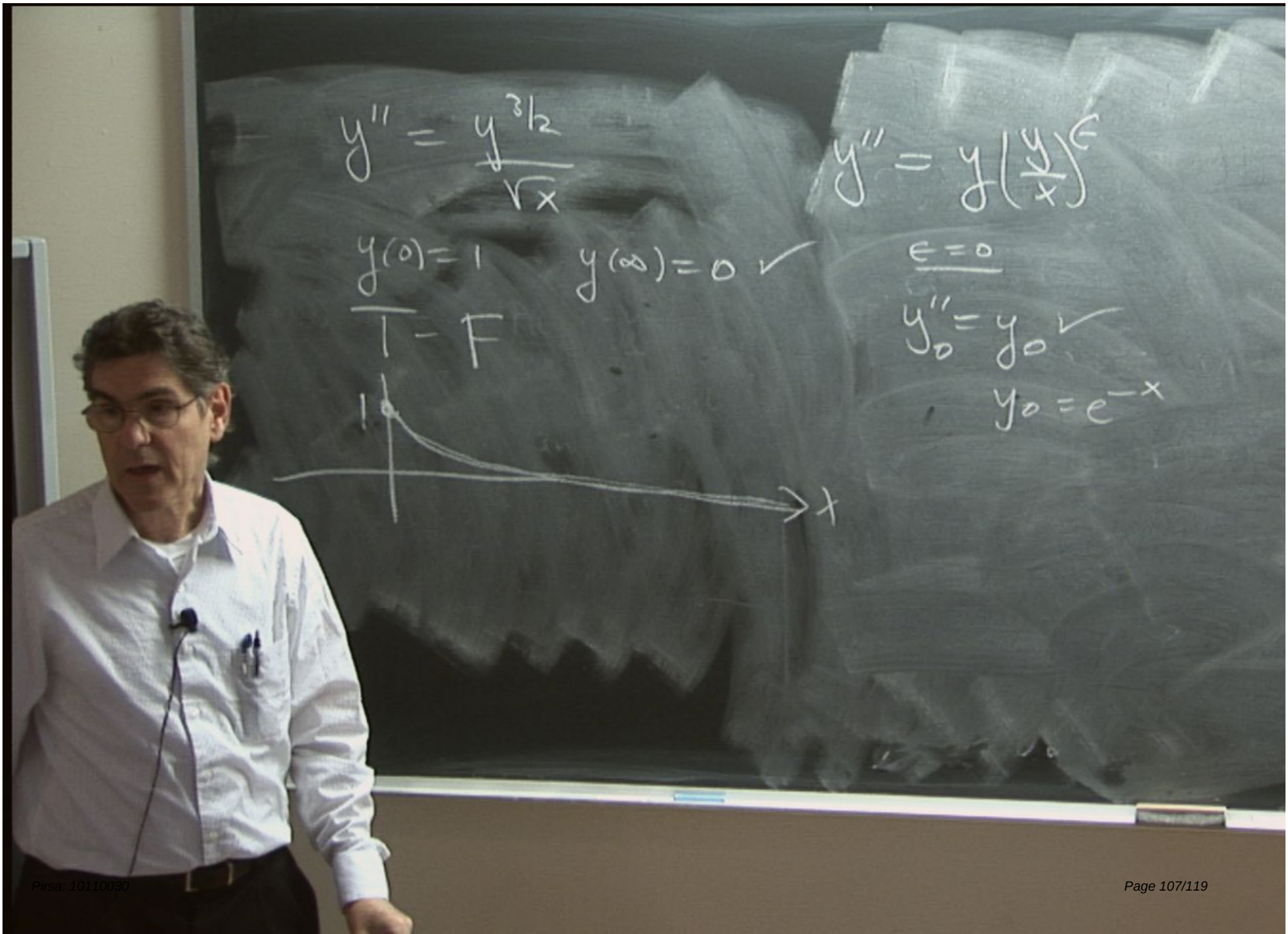
T - F

$$y'' = y\left(\frac{y}{x}\right)^{\epsilon}$$

$\epsilon = 0$

$y_0'' = y_0$ ✓

$y_0 = e^{-x}$


$y'' = \frac{y^{3/2}}{\sqrt{x}}$
$y(0) = 1$
$y(\infty) = 0$ ✓
T - F
1
x
$y'' = y\left(\frac{y}{x}\right)^{\epsilon}$
$\epsilon = 0$
$y_0'' = y_0$ ✓
$y_0 = e^{-x}$

$$y'' = \frac{y^{3/2}}{\sqrt{x}}$$

$y(0) = 1 \qquad y(\infty) = 0$ ✓

T - F

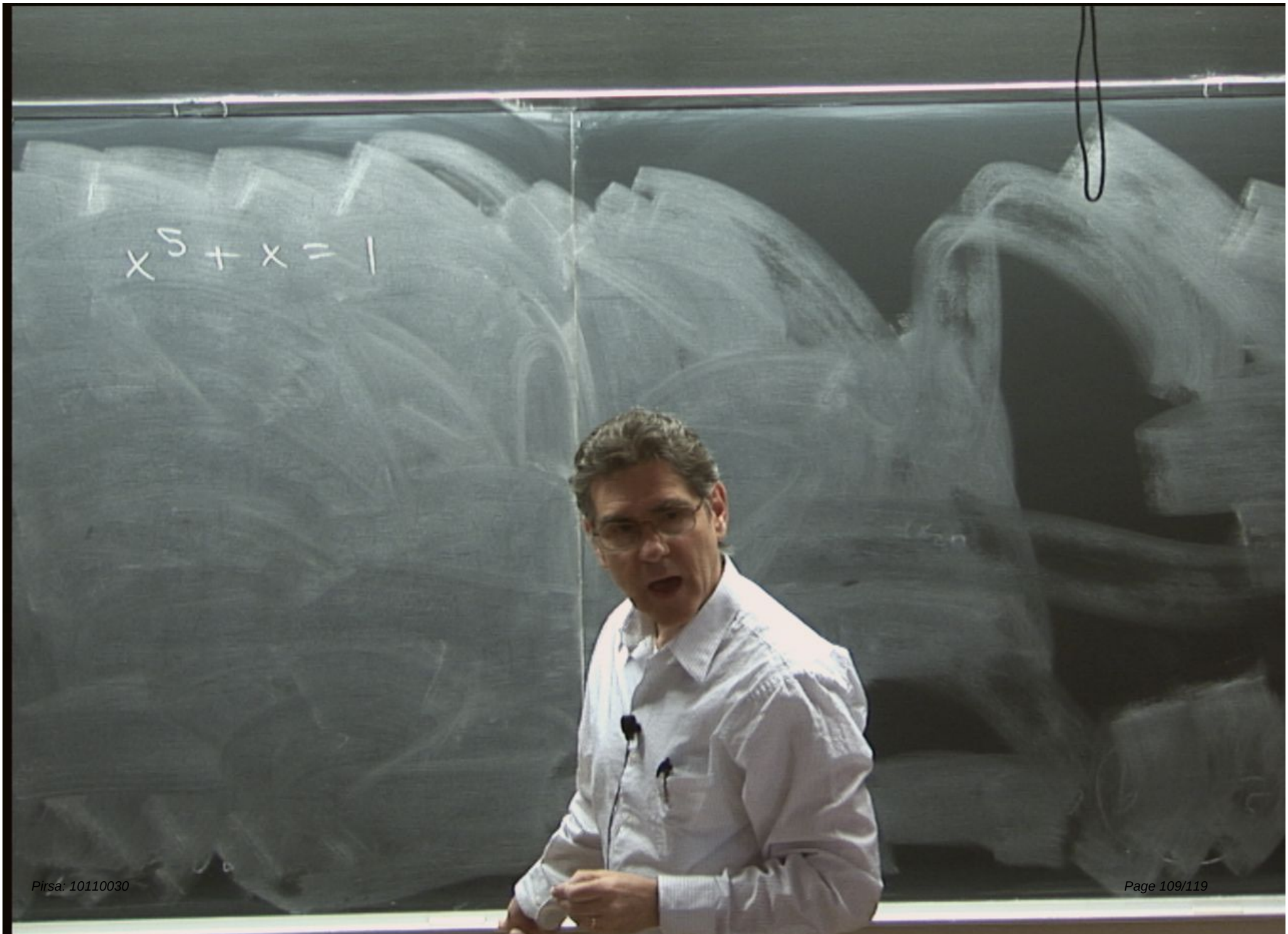
$$y'' = y\left(\frac{y}{x}\right)^{\epsilon}$$

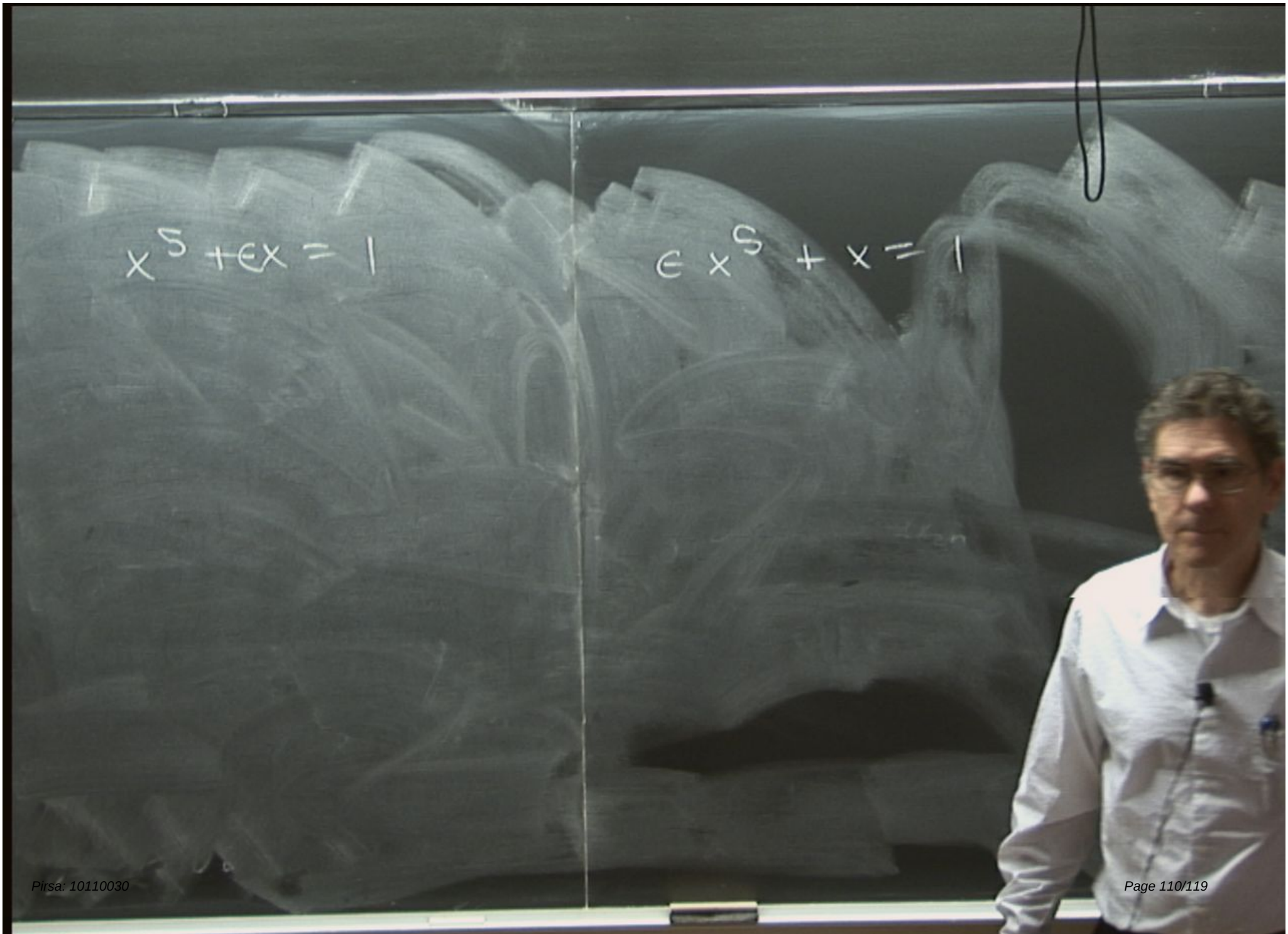
$\underline{\epsilon = 0}$

$y_0'' = y_0$ ✓

$y_0 = e^{-x}$

$y = y_0 + \epsilon y_1 \cdots$

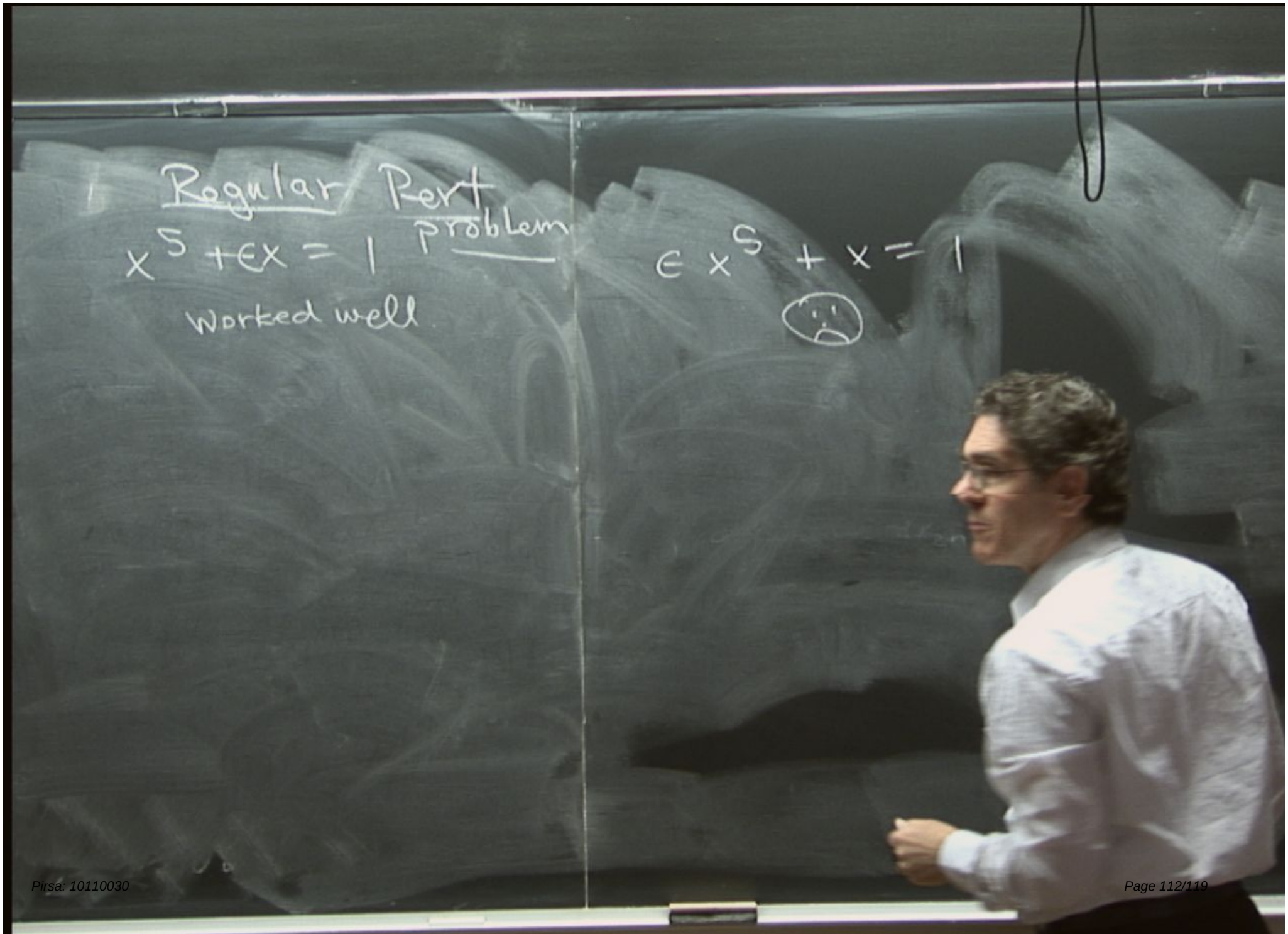

$x^5 + x = 1$

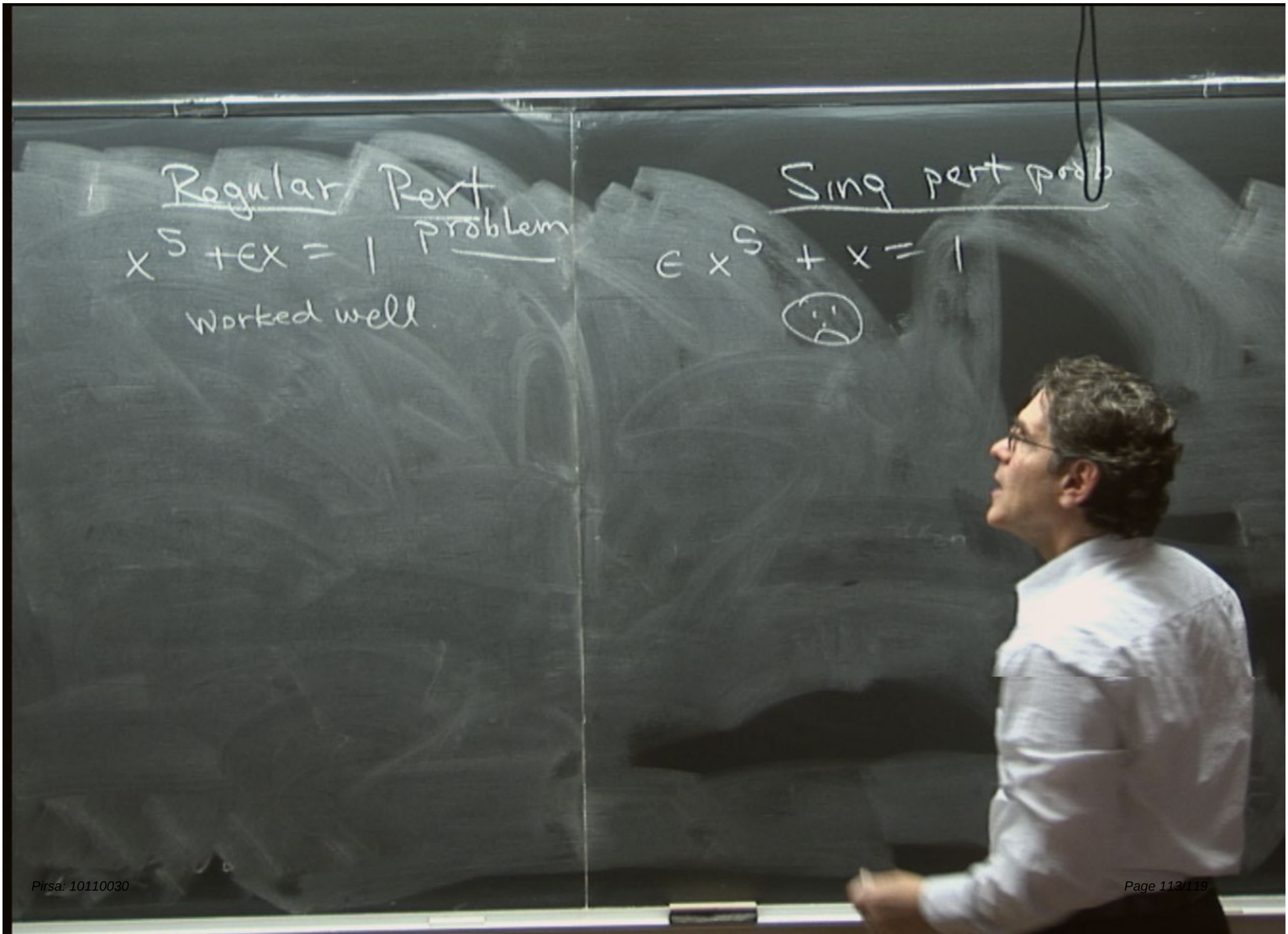

$x^5 + \epsilon x = 1$
$\epsilon x^5 + x = 1$

$x^5 + \epsilon x = 1$

worked well

$\epsilon x^5 + x = 1$


Regular Pert problem
$x^5 + \epsilon x = 1$
worked well
$\epsilon x^5 + x = 1$


Regular Pert problem
$x^5 + \epsilon x = 1$
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Sing pert prob
$\epsilon x^5 + x = 1$

Regular Pert problem

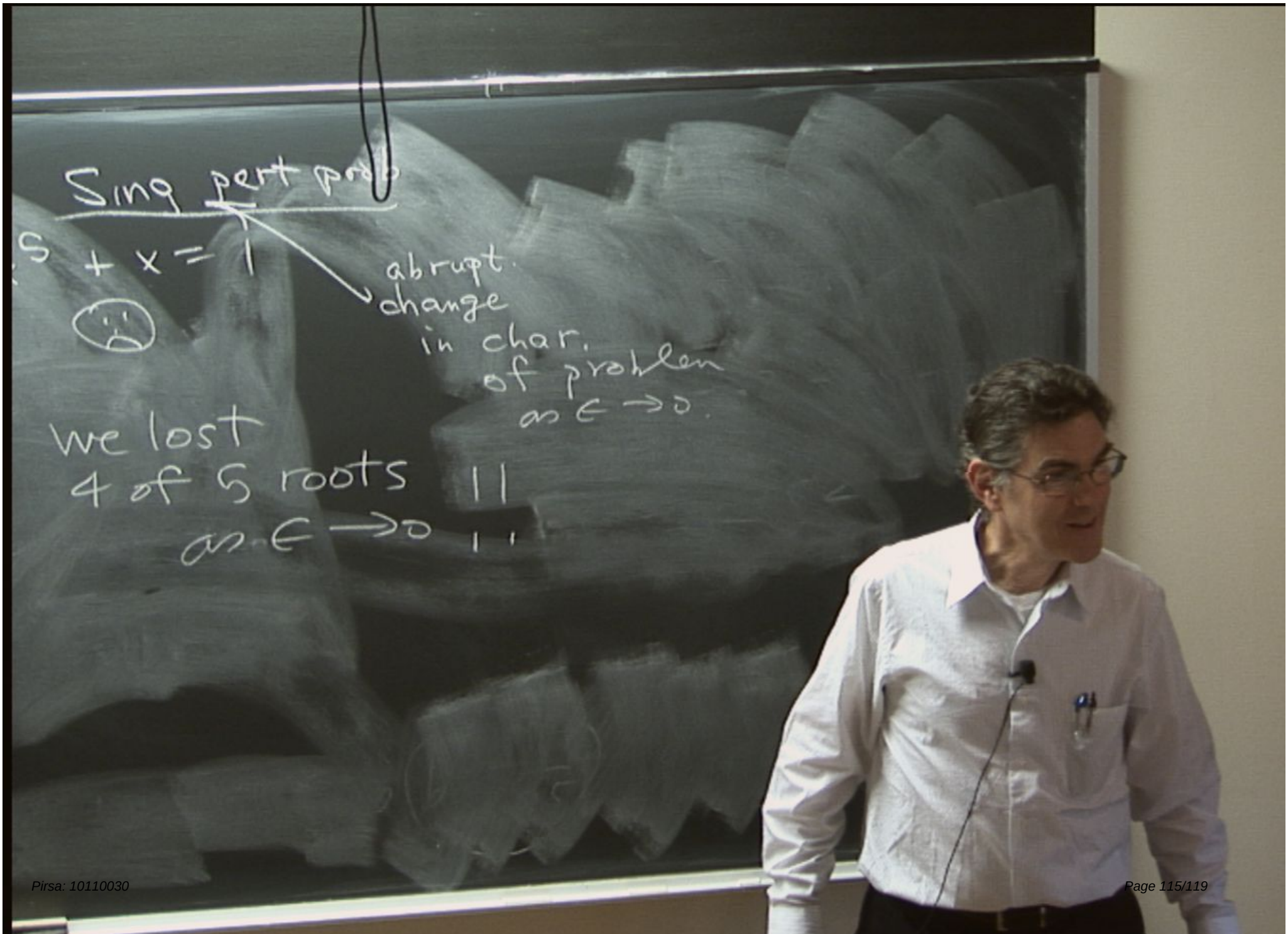
$x^5 + \epsilon x = 1$

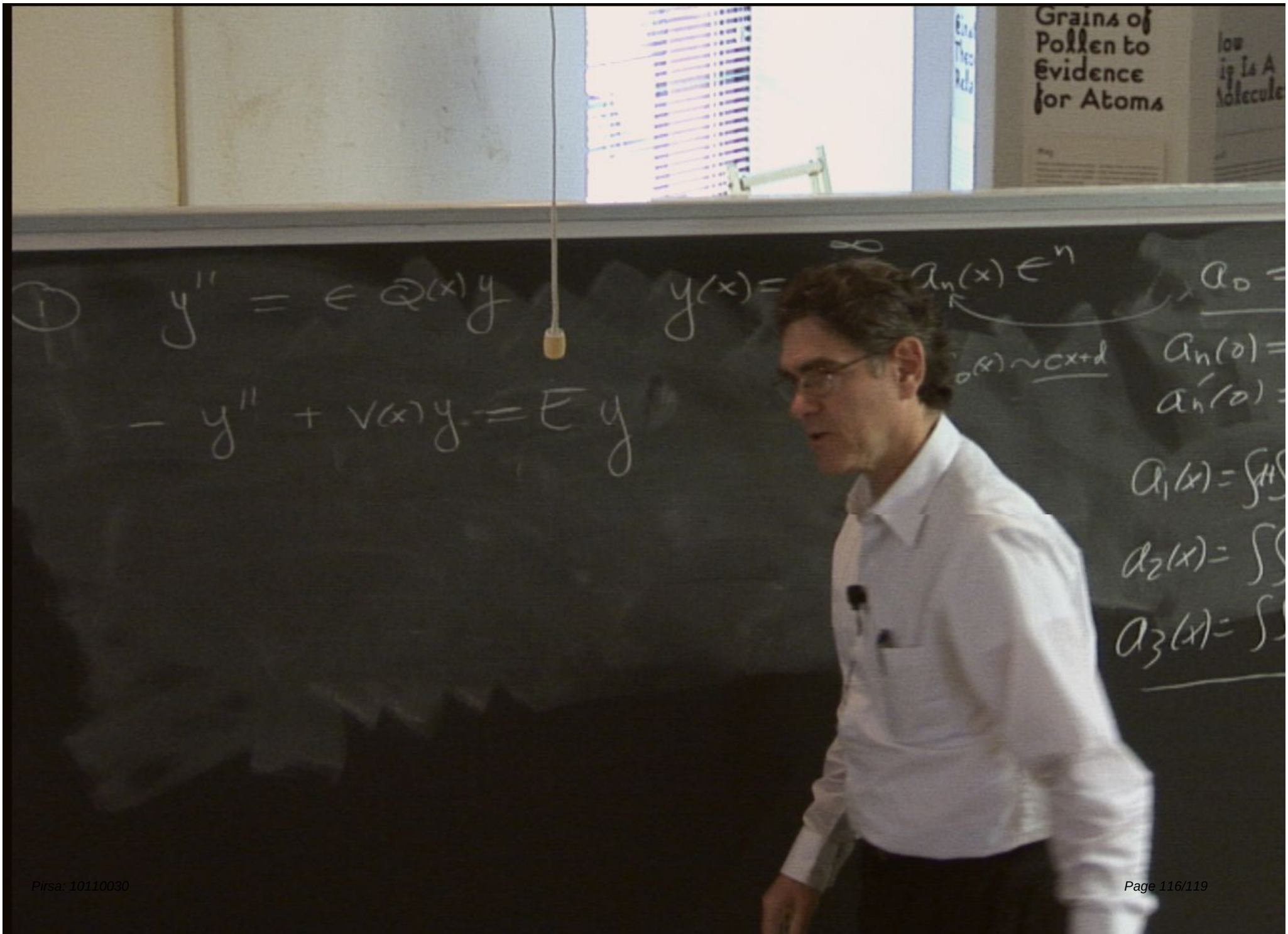
worked well.

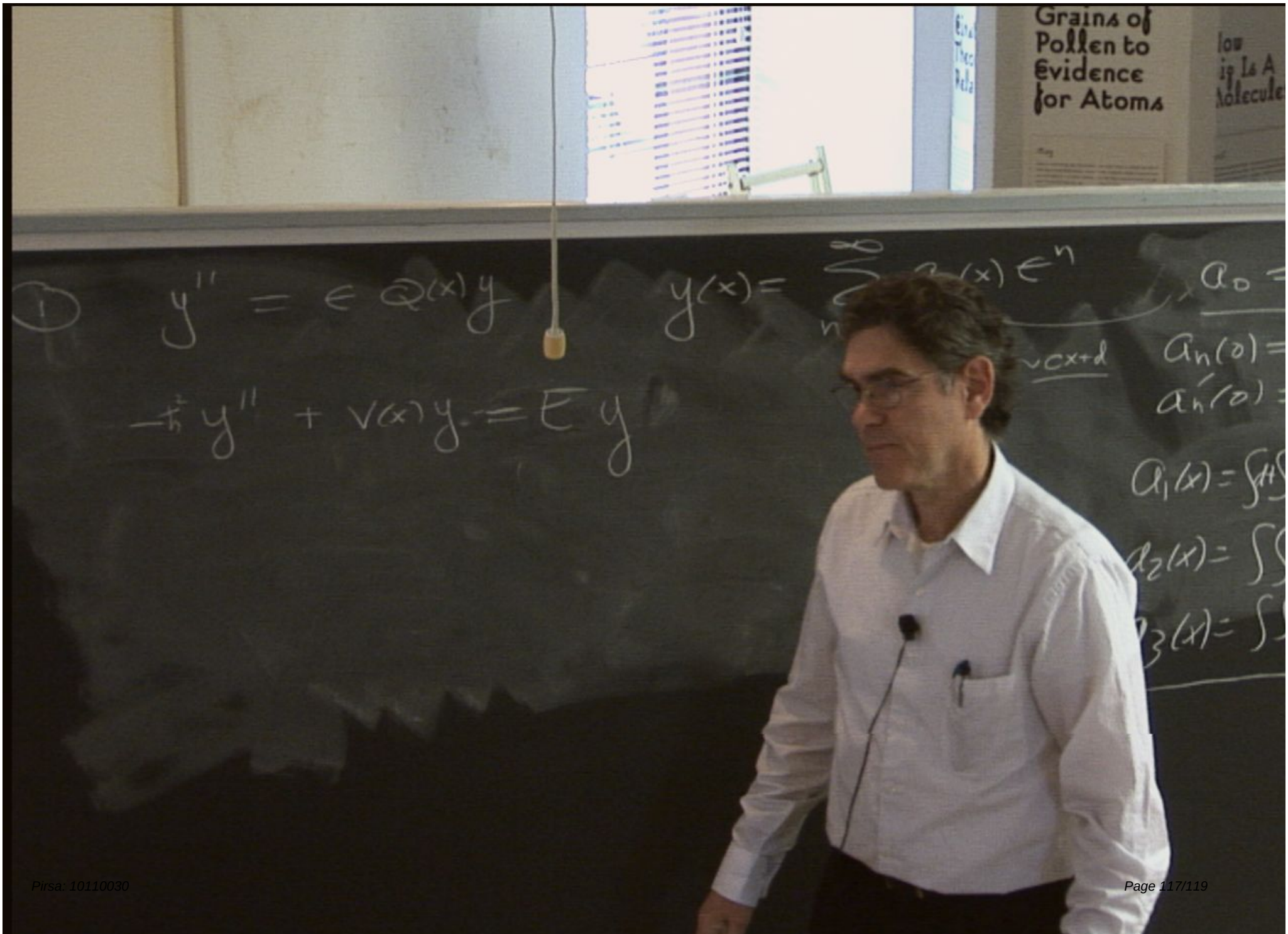
Sing pert prob

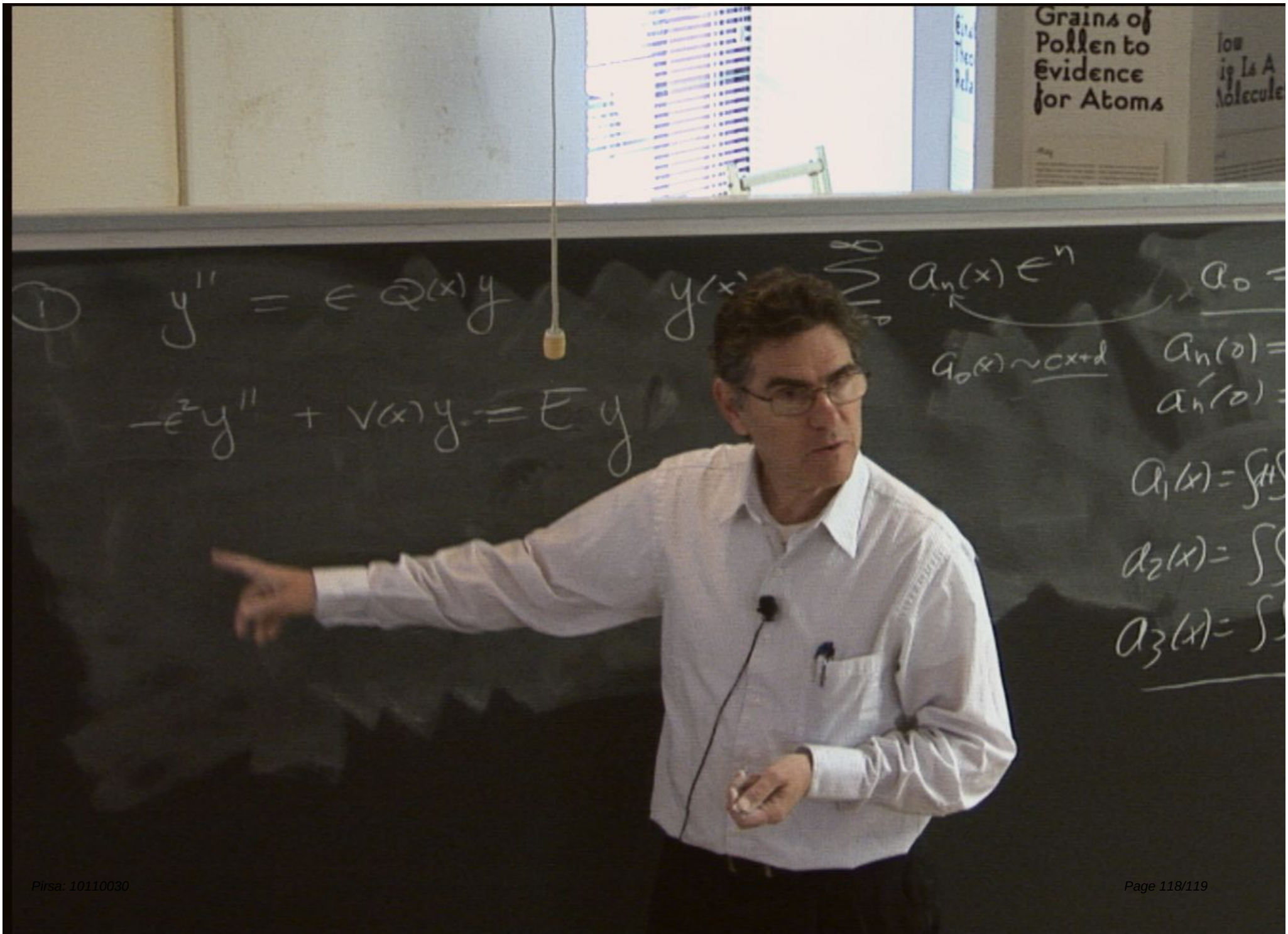
$\epsilon x^5 + x = 1$

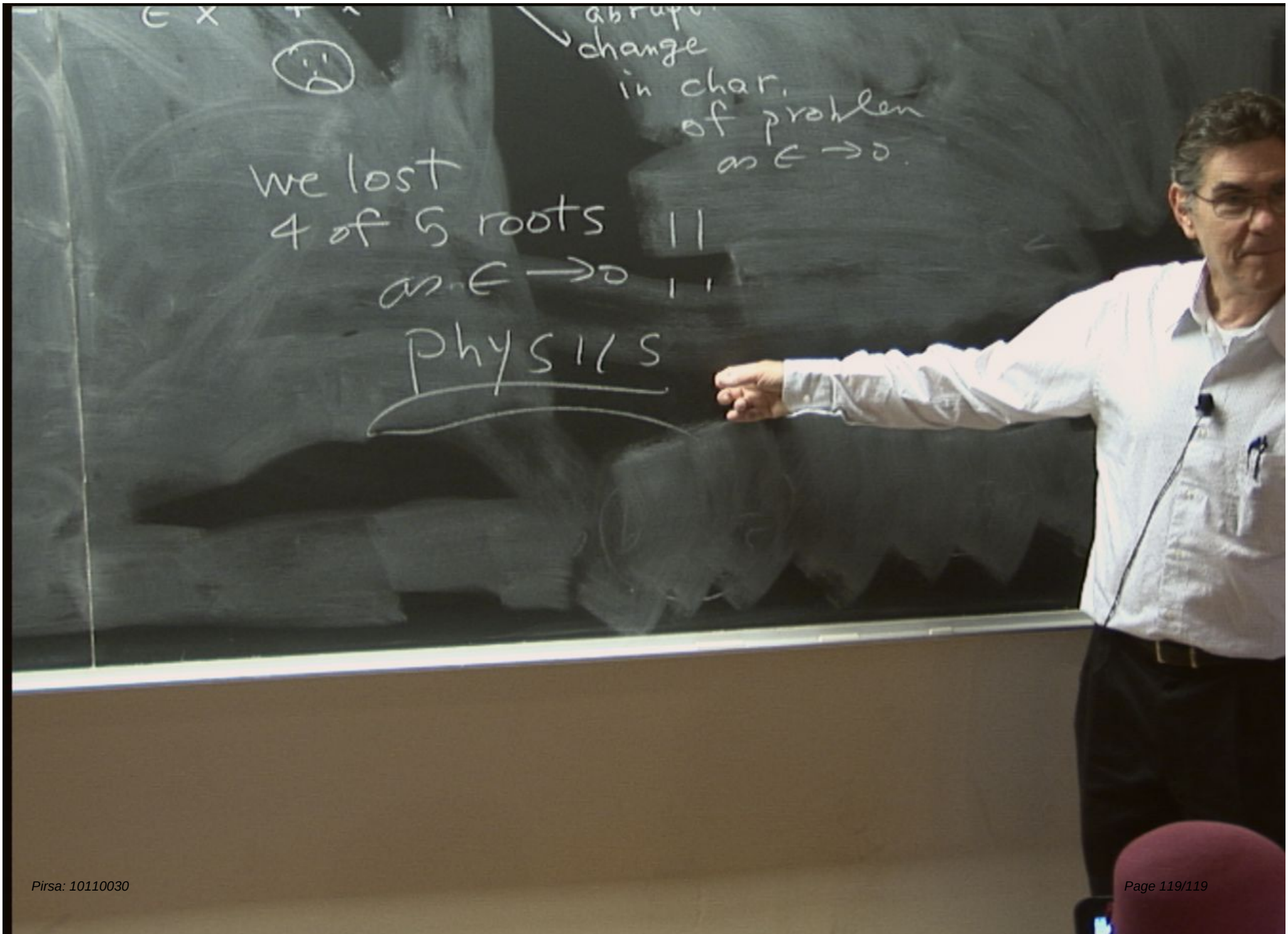
abrupt change in char of p as


Sing pert prob
+ x = 1
abrupt change in char. of problem as ε → 0.
we lost 4 of 5 roots as ε → 0


Grains of Pollen to Evidence for Atoms
$y'' = \epsilon Q(x) y$
$y(x) = \infty \; a_n(x) \epsilon^n$
$a_0 =$
$a_n(0) =$
$a_n'(0) =$
$-y'' + V(x) y = E y$
$a_1(x) =$
$a_2(x) =$
$a_3(x) =$


Grains of Pollen to Evidence for Atoms
$y'' = \epsilon Q(x) y$
$y(x) = \sum_n a_n(x) \epsilon^n$
$-\hbar^2 y'' + V(x) y = E y$
$a_0 =$
$a_n(0) =$
$a_1(x) =$
$a_2(x) =$
$a_3(x) =$


Grains of
Pollen to
Evidence
for Atoms
$y'' = \epsilon Q(x) y$
$-\epsilon^2 y'' + V(x) y = E y$
$a_n(x) \epsilon^n$
$a_0(x) \sim cx+d$
$a_n(0) =$
$a_1(x) =$
$a_2(x) =$
$a_3(x) =$


change
in char.
of problem
as ∈ → 0
we lost
4 of 5 roots
as ∈ → 0
physics