Title: Mutual information and the structure of entanglement in quantum field theory

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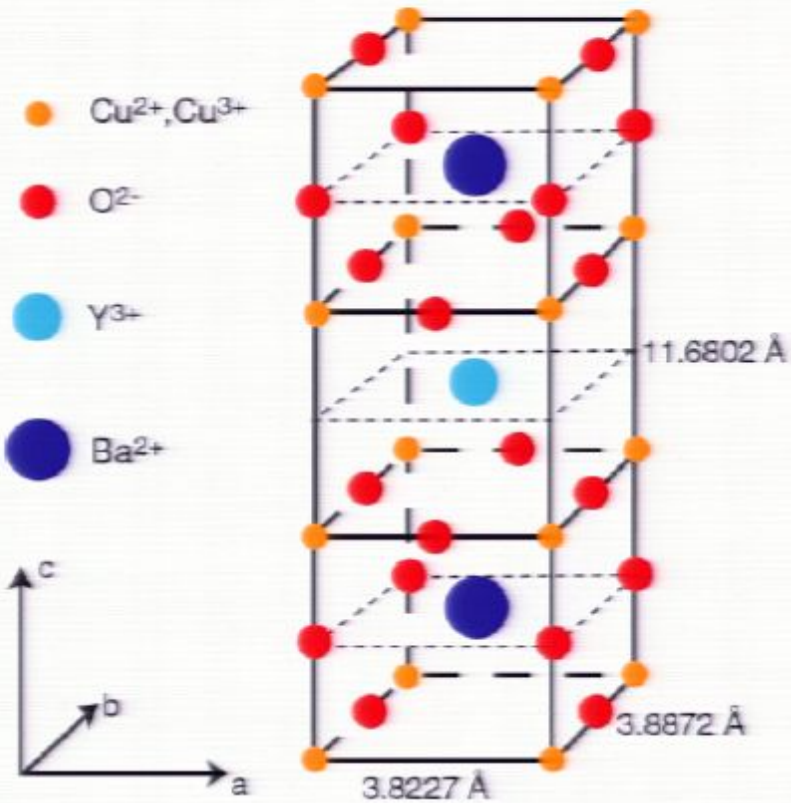
Abstract: In this talk I will describe my recent work on the structure of entanglement in field theory from the point of view of mutual information. I will give some basic scaling intuition for the entanglement entropy and then describe how this intuition is better captured by the mutual information. I will also describe a proposal for twist operators that can be used to calculate the mutual information using the replica method. Finally, I will discuss the relevance of my results for holographic duality and entanglement based simulation methods for many body systems.

# Mutual information and the structure of entanglement in quantum field theory

Brian G. Swingle, MIT

Perimeter Institute, October 26, 2010

# Quantum many body physics




What happens when many local quantum degrees of freedom interact?

$$\mathcal{H}_{tot} = \otimes_{i=1}^{N} \mathcal{H}_i$$

# The many body state

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## Mean field theory

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Mean field theory can fail qualitatively, for example, in the fractional quantum hall effect.

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## Entanglement

For a system of $N$ spins, the many body Hilbert space is enormous: $\dim(\mathcal{H}_{tot}) = 2^N$. Most states are entangled: $|G\rangle \neq \otimes_{i=1}^{N} |g_i\rangle$.

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- The unknown: a new way to do many body quantum physics?

We don't really have a choice!

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## Quantum states

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Does the entanglement structure of quantum field theory tell us anything about what sorts of states to focus on?

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What kind of structure exists on the space of quantum field theories, for example, can we order them in some way?

## Holographic duality

Does the duality provide access to field theories that are in some sense generic?

# Outline

Entanglement ABCs

Useful tools

Mutual information

Twist fields

Applications

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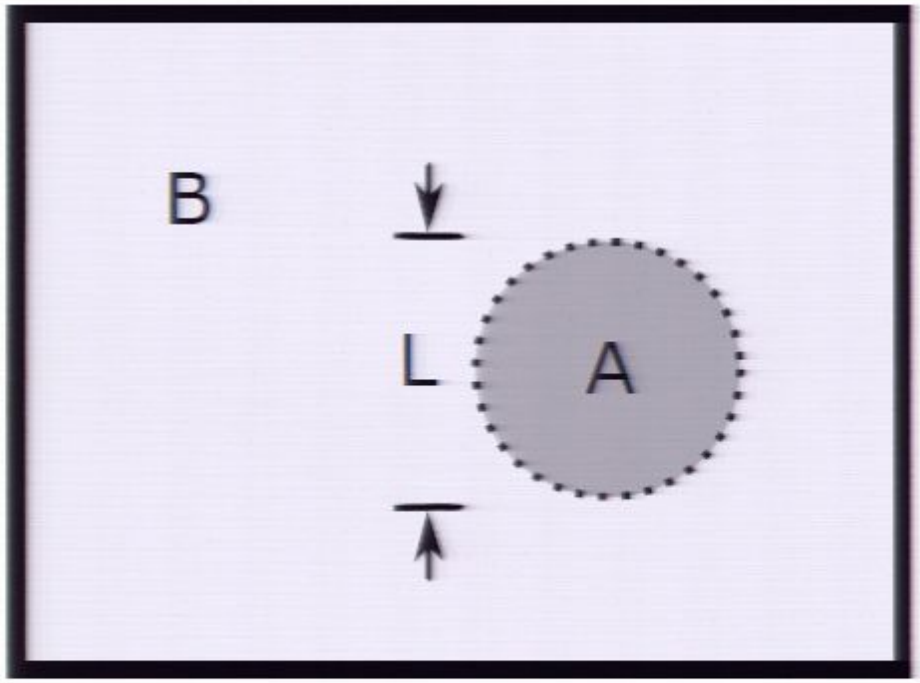
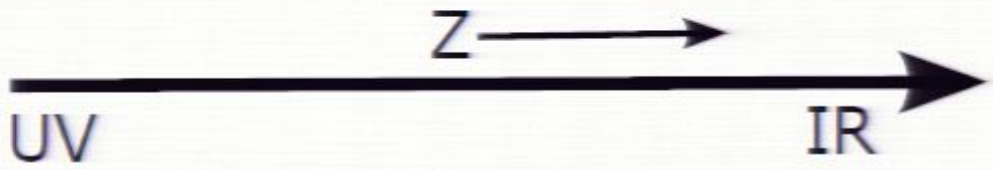
# Entanglement entropy




$$\rho_A = \mathrm{tr}_B\left(|G\rangle\langle G|\right) \Rightarrow S_A = -\mathrm{tr}_A\left(\rho_A \log\left(\rho_A\right)\right)$$

How does $S_A$ behave in a many body system?

# Renormalization group flow



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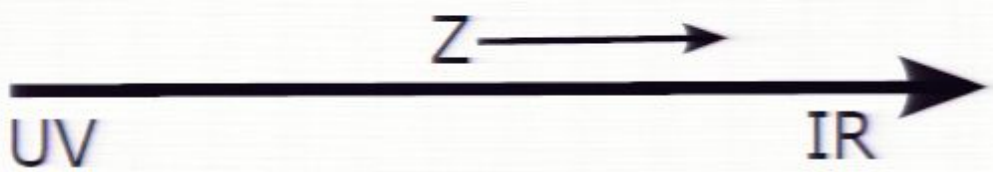


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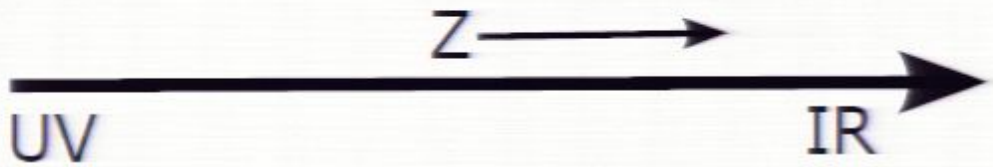


- Renormalization group: $z$ labels the length scale of interest with $z = \epsilon$ in the UV and $z \to \infty$ in the IR
- Renormalization group (RG) transformations typically flow towards zero momentum i.e. momentum shell RG

# Understanding the boundary law

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Change $z \to ze^{d\ell}$ giving $d\ell = \frac{dz}{z}$. The RG integration measure is $\frac{dz}{z}$ (logarithmic intervals).

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## Entanglement "per scale"

Local interactions at scale $z$ entangle $A$ and its environment along the boundary of $A$.

Entanglement scales as the boundary of $A$ measured in units of $z$: $\left(\frac{L}{z}\right)^{d-1}$

# Understanding the boundary law

$$S(L) \sim \int_{\epsilon}^{\text{IR cutoff}} \left(\frac{L}{z}\right)^{d-1} \frac{dz}{z}$$

CFT: $z_{IR} = L$

Gap: $z_{IR} = \xi$

$$S(L) \sim \left(\frac{L}{\epsilon}\right)^{d-1} \quad d > 1$$

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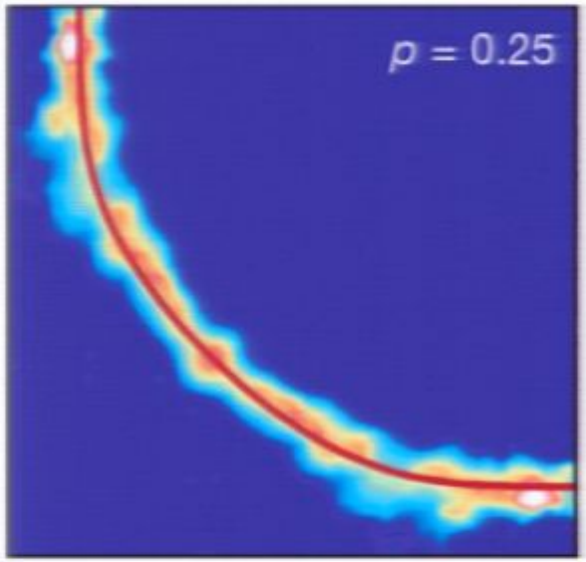
$$S(L) \sim \log\left(\frac{L}{\epsilon}\right) \quad d = 1$$

$$S(L) \sim \log\left(\frac{\xi}{\epsilon}\right) \quad d = 1$$

# Exception: the Fermi surface

## CFT on the Fermi surface

Fermions with a finite codimension $d_c = 1$ Fermi surface violate the boundary law for entanglement entropy: $S(L) \sim k_F L \log L$. (Wolf 2006, Klich-Gioev 2006)




The Fermi surface is a "bundle" of gapless $1+1$ dimensional modes! (BGS 2009,2010)

# Outline

Entanglement ABCs

Useful tools

Mutual information

Twist fields

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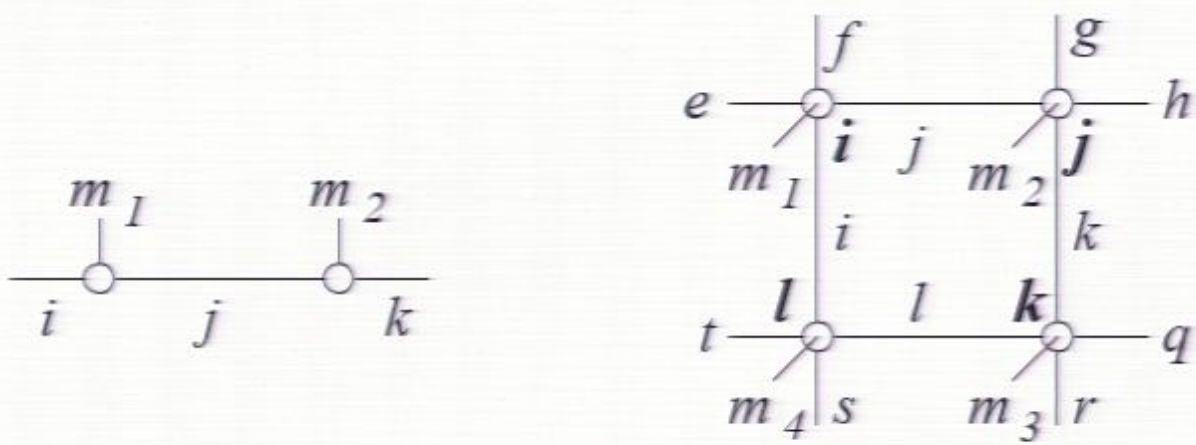
# Tensor network states

## Using the area law

Tensor network states are a class of variational states that automatically incorporate the boundary law for entanglement entropy and appear to be in the right "corner" of Hilbert space to approximate ground states of local Hamiltonians.




(a) (b)

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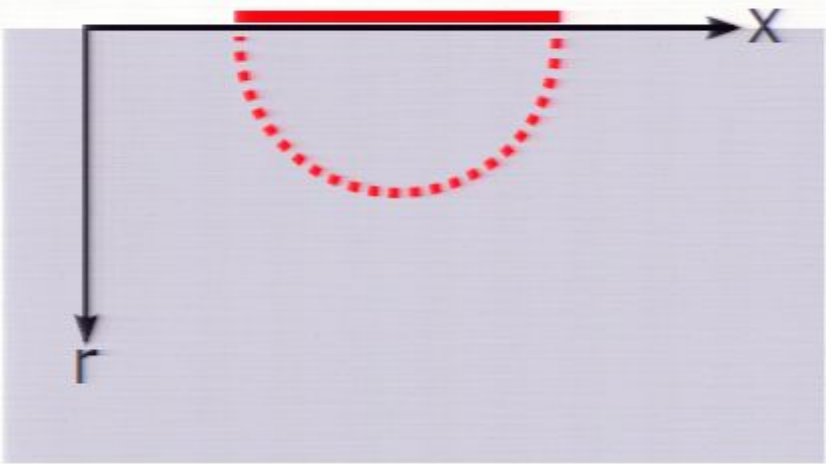
## Qualitative improvement

Tensor network states can describe topological phases beyond mean field theory. (Gu-Levin-Wen-BGS 2009, Konig-Reichardt-Vidal 2009) One can in principle extract topological data from tensor network states. (BGS-Wen 2010, Schuch-Cirac-Perez Garcia 2010)

# Holographic duality: $CFT_{1+1}/AdS_{2+1}$




The emergent radial coordinate plays the role of energy scale in the field theory. Space and the emergent radial coordinate are shown.
Entanglement entropy can be calculated in terms of minimal hypersurfaces in the bulk. (Ryu-Takayanagi 2006)

# Holographic duality

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## Strongly coupled entanglement

There are strongly coupled systems that violate the boundary law for entanglement entropy. The same effectively one dimensional physics of the Fermi liquid is at work. (D'Hoker-Kraus 2010, BGS 2010)

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## Low energy entanglement?

Many body systems are described by effective field theories at low energy. For studying universal properties, it would be desirable to isolate as much possible the low energy structure of entanglement.

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- $||\mathcal{O}_A||^2||\mathcal{O}_B||^2\mathcal{I}(A,B) \geq \langle \mathcal{O}_A\mathcal{O}_B\rangle_c^2$

(Hastings-Wolf-Verstraete-Cirac 2008)

# Universal singularities: $\mathrm{CFT}_{1+1}$

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## Dirac fermions

The mutual information for Dirac fermions can be calculated using methods from conformal field theory. For two intervals of length $L$ separated by a distance $x$, $\mathcal{I}(L,x) = \frac{1}{3}\ln\left(\frac{(L+x)^2}{x(2L+x)}\right)$. (Casini-Huerta-Fosco 2005)

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## Entanglement "per scale"

Notice that that the mutual information diverges as $x \to 0$. This logarithmic divergence is a general feature of conformal field theories. The divergence is controlled by the central charge of the conformal field theory. (Cardy-Calabrese 2009)

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$$I \sim \frac{c}{3}\ln\left(\frac{L}{x}\right)$$

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## Collision geometry

Consider two flat parallel $d-1$ dimensional surfaces of size $V_{d-1}$ (part of $\partial A$ and $\partial B$) separated by a distance $x$.

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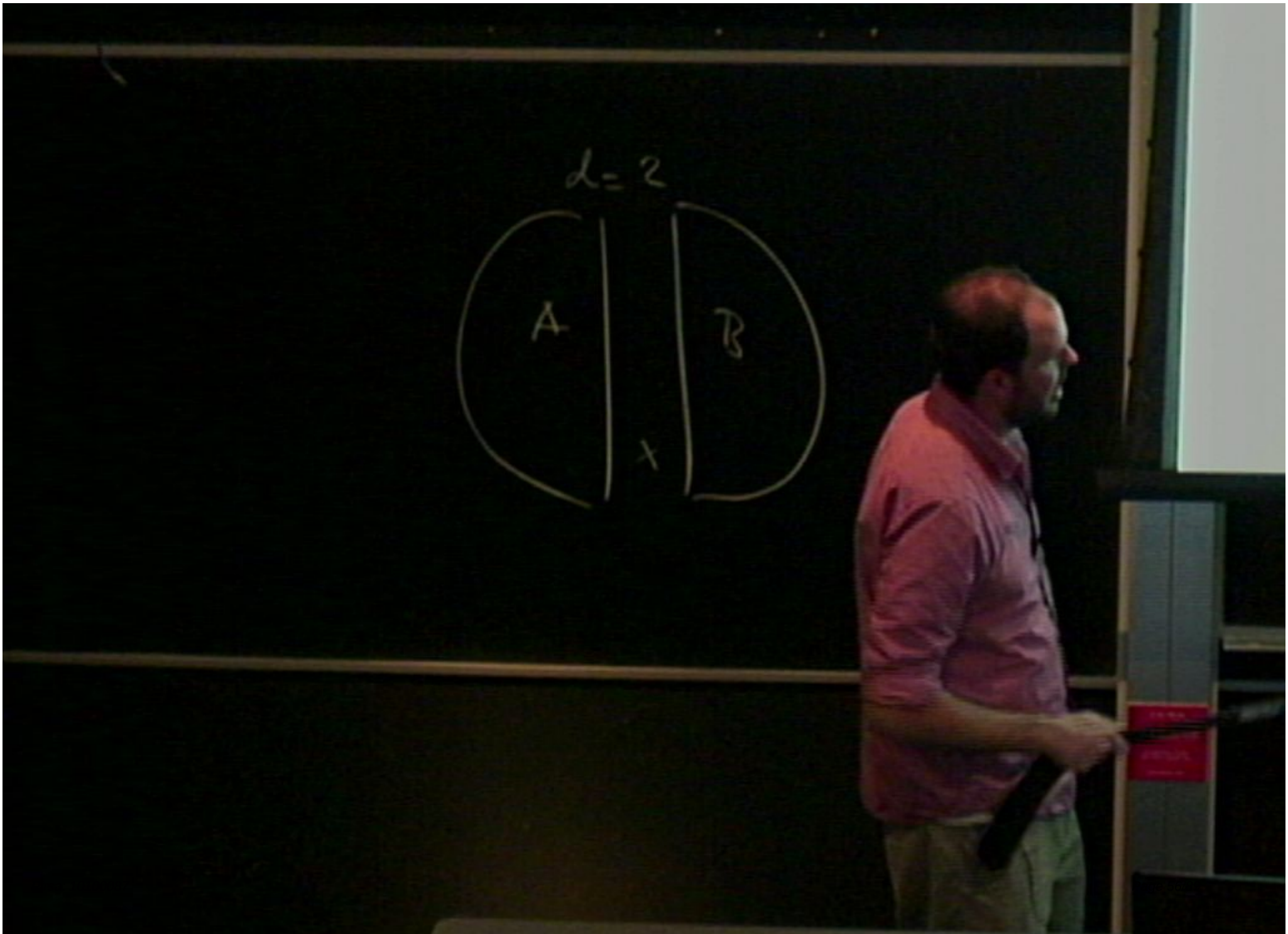
The mutual information $\mathcal{I}(A,B)$ will diverge as $x \to 0$. From the boundary law we have

$$\mathcal{I}(x) = k\frac{V_{d-1}}{x^{d-1}}.$$


d = 2
A
B
x

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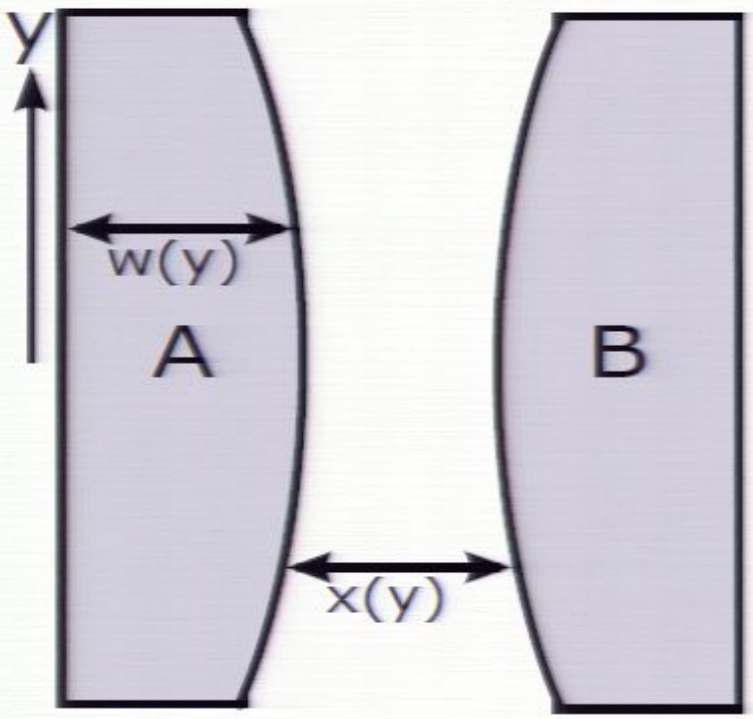
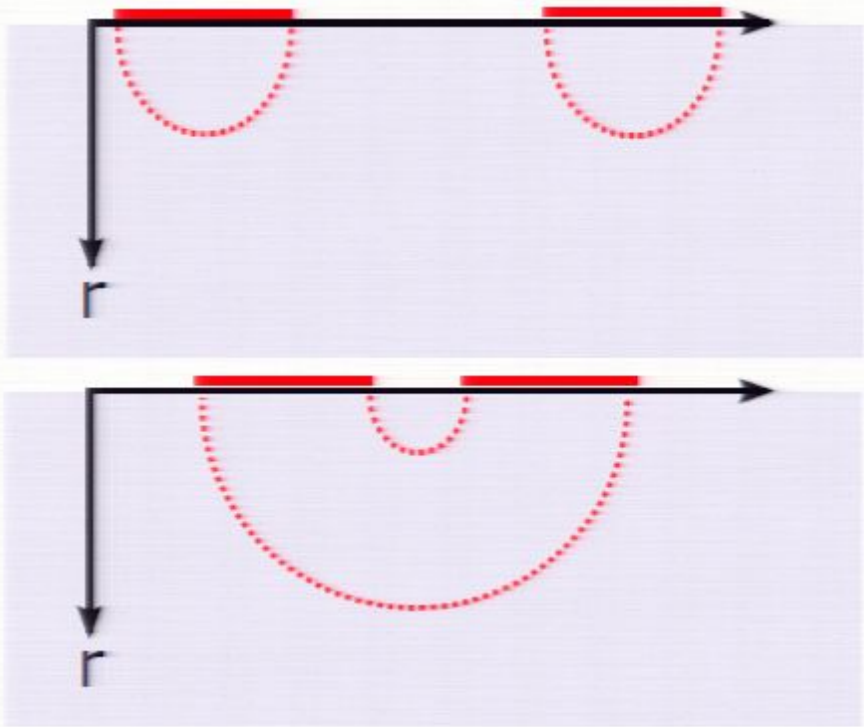
# Other geometries




Generically, colliding regions will touch only at a single point giving $\mathcal{I}(x) \sim \left(\frac{L_c}{x}\right)^{(d-1)/2}$ with $L_c$ the radius of curvature. Corners give a logarithmic divergence in all dimensions. (BGS 2010)

# Holographic computation




All these scaling laws can be reproduced using holographic duality. This check provides confidence and suggests that holographic duality captures well the qualitative features of entanglement. (BGS 2010)

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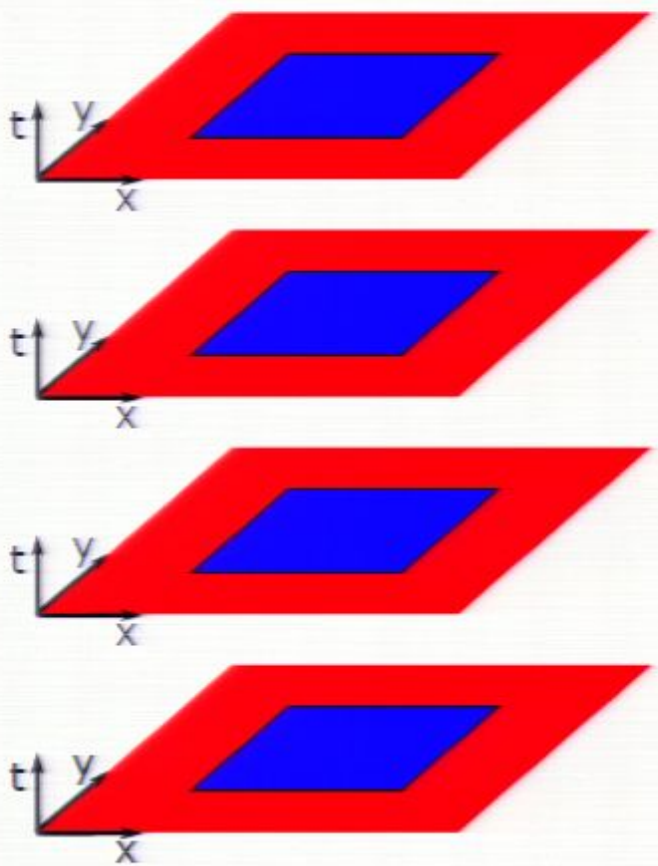
# Field theory computation




## Replica trick

The entanglement entropy for a region $R$ can be obtained from a knowledge of $Z_R(n) = \mathrm{tr}(\rho_R^n)$ via the replica trick: $S_R = -\partial_n Z_R(n)|_{n=1}$.

# Twist fields in $1+1$ dimensions

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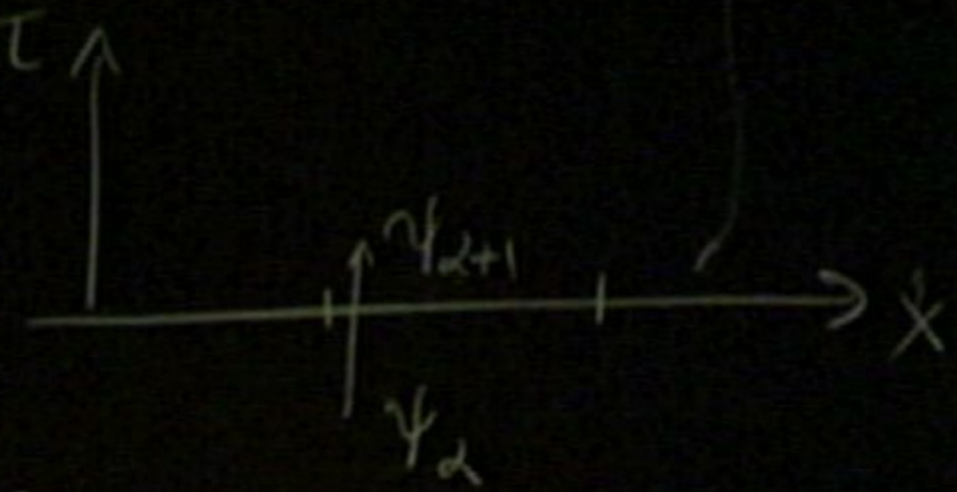
## Single interval

Consider a single interval of length $L$ and $\alpha = 1, ..., n$ copies of each field $\psi_\alpha(x, \tau)$ ($\tau$ is imaginary time). The boundary conditions at $\tau = 0$ are $\psi_\alpha(x \in L, \tau = 0^-) = \psi_{\alpha+1}(x \in L, \tau = 0^+)$.




$\rangle |\uparrow\rangle |\downarrow\rangle \cdots$

$\left(\frac{L}{x}\right)$

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## Twist fields

Because the replica spacetime is locally unexceptional except for the ends of the interval, the effects of the boundary conditions can be subsumed into insertions of local operators called twist fields $Z_R(n) = \langle K_n(0,0) K_n(L,0) \rangle_n$.

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Where the twist fields were point like in $1+1$ dimensions, they become line operators in $2+1$ dimensions and surface operators in $3+1$. The twist fields are defined by $\mathrm{tr}(\rho_R^n) = \langle K_n[R] \rangle_n$.

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## Twist field ansätz

I introduce a vector field $\vec{\phi}$ in terms of which the higher dimensional twist field is

$$K_n[R] = \exp\left( i\lambda_n \int_{\partial R} \hat{n} \cdot \vec{\phi} \right)$$

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## $\phi\phi$ correlator

The two point function for $\phi$ is taken to be

$$\langle \phi^i(x)\phi^j(0)\rangle \sim \frac{\delta^{ij}}{|x|^{2(d-1)}}$$

# Entanglement entropy

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## Setup

Note that $\lambda_n$ must vanish as $n \to 1$, but we assume that $\lambda_n^2$ has a finite first derivative there. The assumption of Gaussian correlations for $\vec{\phi}$ permits calculation of the expectation value of $K_n[R]$.

$$-\partial_n \langle K_n[R] \rangle|_{n=1} = \frac{\partial_n \lambda_n^2|_{n=1}}{2} \int_{\partial R} \int_{\partial R} \hat{n}^i \hat{n}^j \langle \phi_i \phi_j \rangle$$

# Entanglement entropy

## Circle $d = 2$

The formula given above for the entanglement entropy is divergent. Regulating the divergence gives $S = c_1 \frac{L}{\epsilon} + c_2 + O(\epsilon)$.

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Here we find $S = c_1 \left(\frac{L}{\epsilon}\right)^2 + c_2 \log\left(\frac{L}{\epsilon}\right) + O(\epsilon)$. Note the logarithmic term, related to anomalies in $3 + 1$ dimensions.

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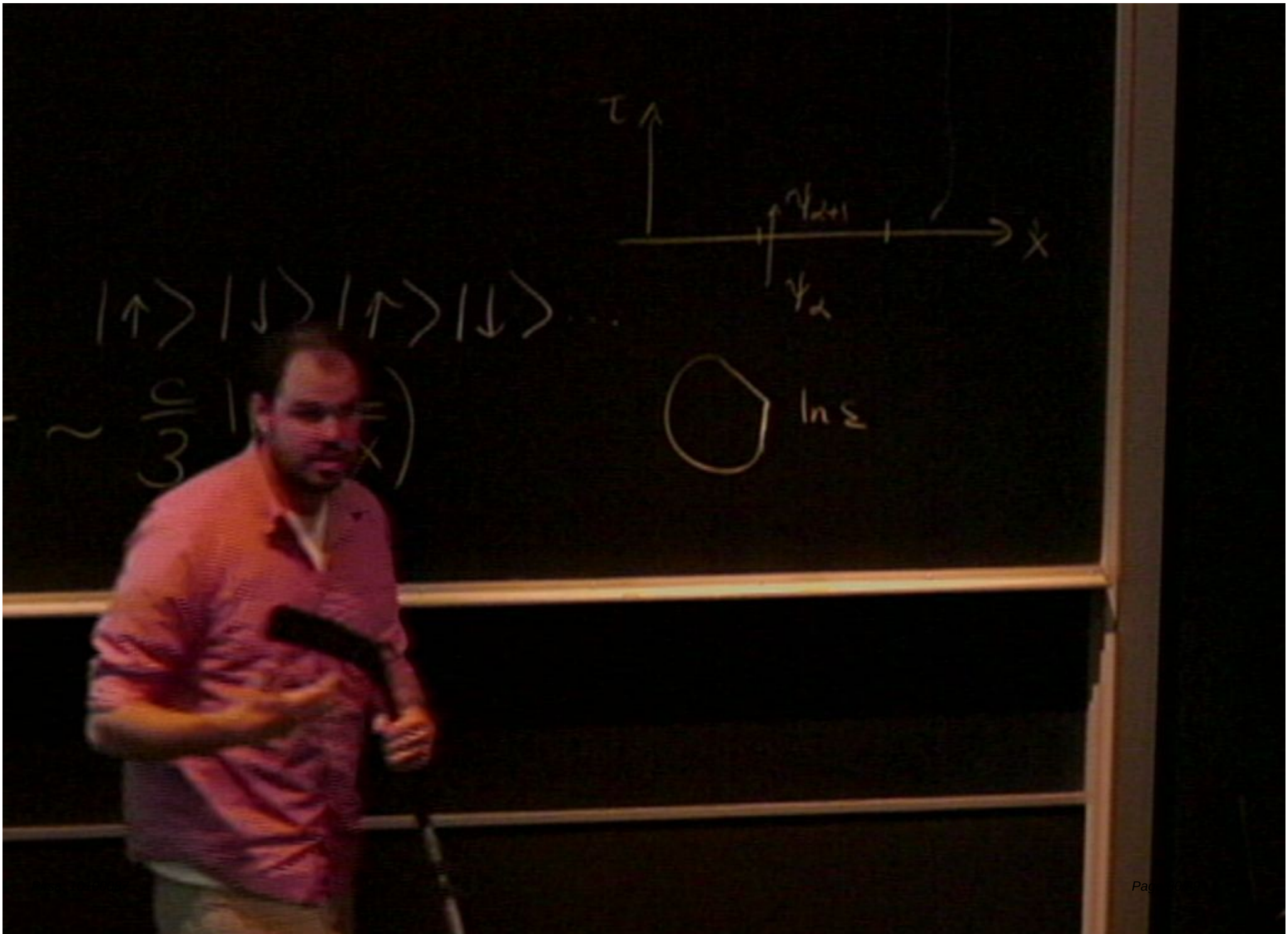
## Corners

The formula also gives a logarithmic correction for corners in any dimension.


τ
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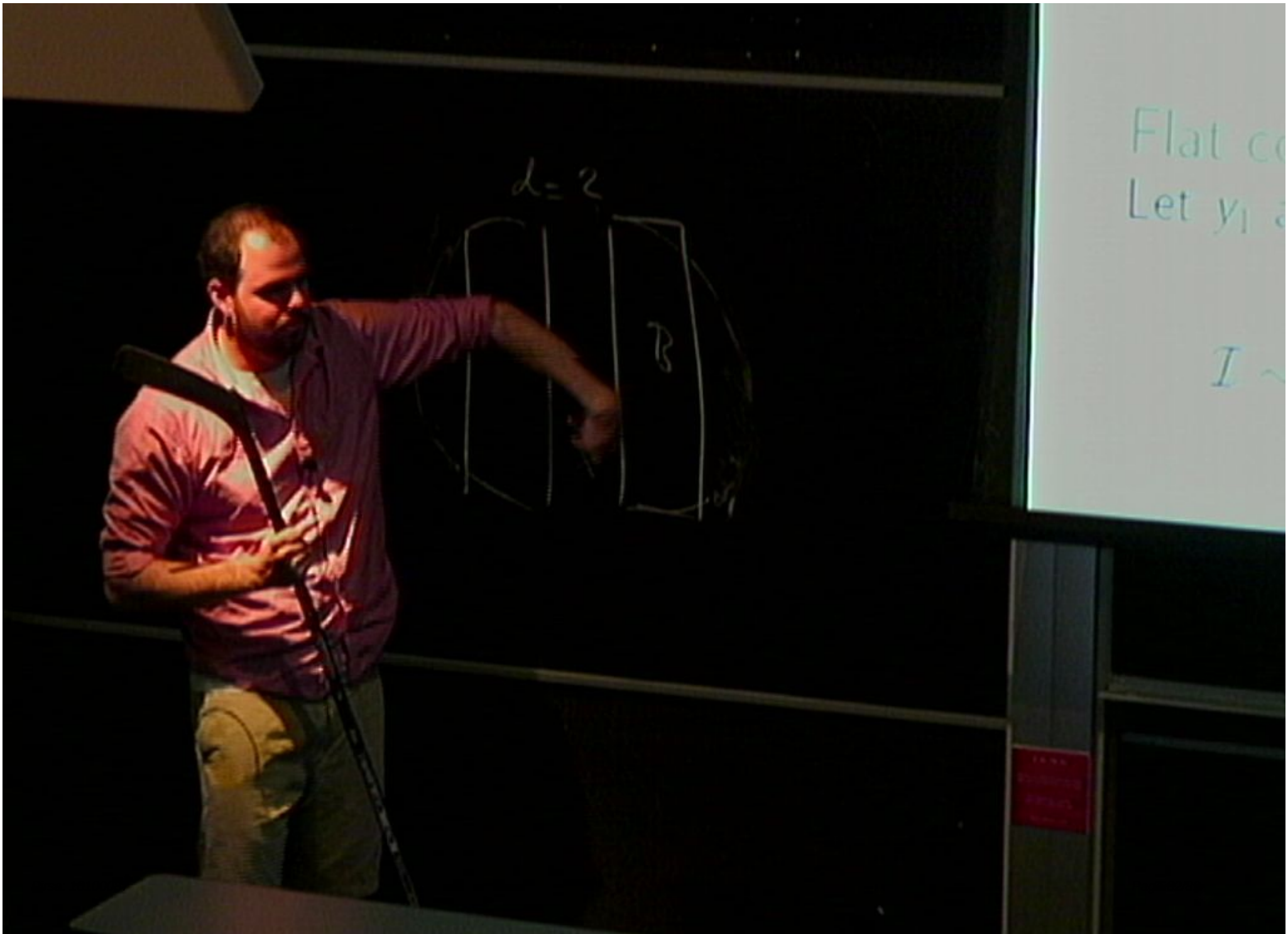
d = 2
B
Flat c
Let y1 a
I ~

# A few applications

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Can one identify quantities that are monotonic under RG flow? There is some evidence that entanglement "per scale" as measured by the mutual information provides such a monotone.

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Can one identify quantities that are monotonic under RG flow? There is some evidence that entanglement "per scale" as measured by the mutual information provides such a monotone.

(Myers-Sinha 2010, BGS 2010)

## Verifying holography

We must know to what extent predictions of the duality can be trusted in more ordinary situations. The basic structure of quantum information and entanglement appears to be captured qualitatively correctly.

# A few applications

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## Tensor networks

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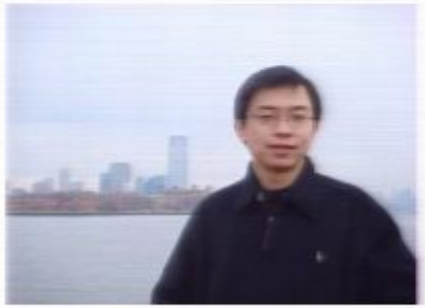
## Finite temperature

A great deal of interesting physics is encoded in the mutual information as a function of temperature. One can obtain critical exponents, operator dimensions, and much more.

# Thank you!