Title: Specker's parable of the overprotective seer: Implications for Contextuality, Nonlocality and Complementarity

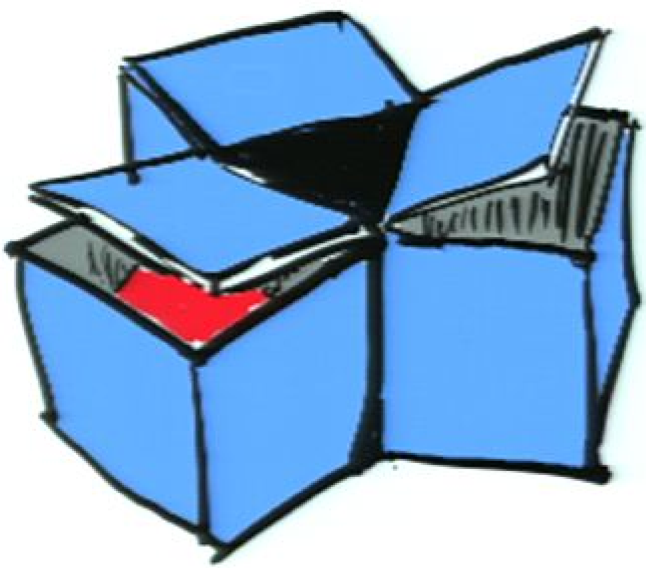
Date: Oct 12, 2010 04:00 PM

URL: http://pirsa.org/10100060

Abstract: I revisit an example of stronger-than-quantum correlations that was discovered by Ernst Specker in 1960. The example was introduced as a parable wherein an over-protective seer sets a simple prediction task to his daughter's suitors. The challenge cannot be met because the seer asks the suitors for a noncontextual assignment of values but measures a system for which the statistics are inconsistent with such an assignment. I will show how by generalizing these sorts of correlations, one is led naturally to some well-known proofs of nonlocality and contextuality, and to some new ones. Specker's parable involves a kind of complementarity that does not arise in quantum theory - three measurements that can be implemented jointly pairwise but not triplewise -- and therefore prompts the question of what sorts of foundational principles might rule out this kind of complementarity. This is joint work with Howard Wiseman and Yeong-Cherng Liang.

Specker's parable of the overprotective seer: Implications for Contextuality, Nonlocality and Complementarity

Robert Spekkens



Joint work with:
Howard Wiseman
(Griffith University, Brisbane)
Yeong-Cherng Liang
(University of Sydney, now Geneva)

Funding by: the PIAF collaboration

Industry Canada Industrie Canada
Canada

MINISTRY OF RESEARCH & INNOVATION

See: arXiv:1010.1273

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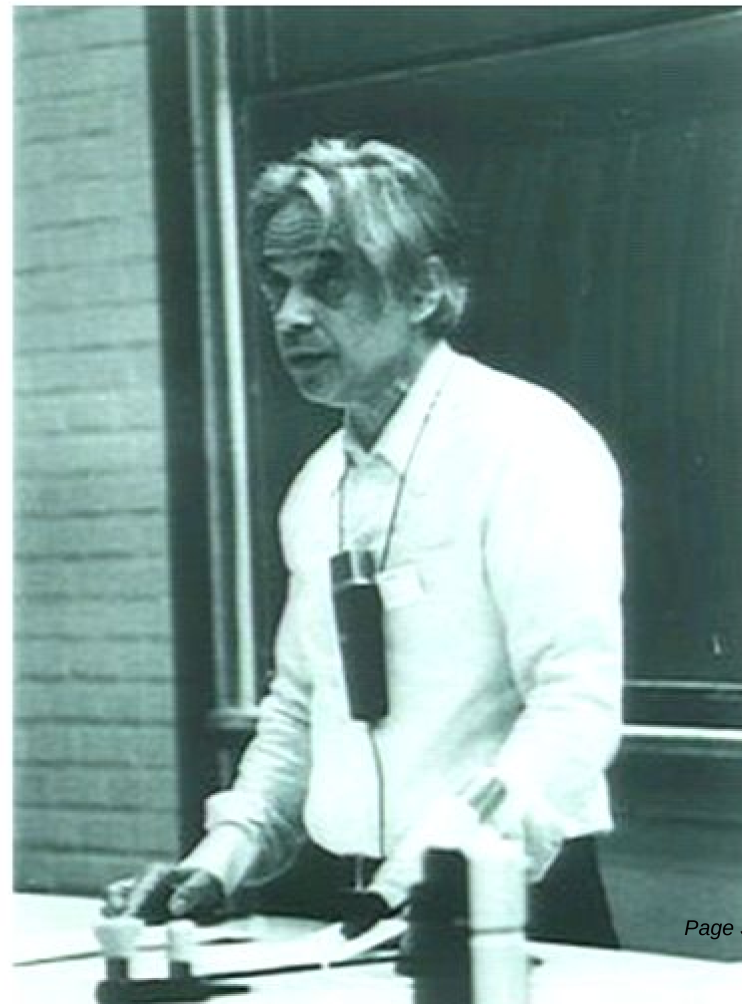
(“The logic of propositions which are not simultaneously decidable”)



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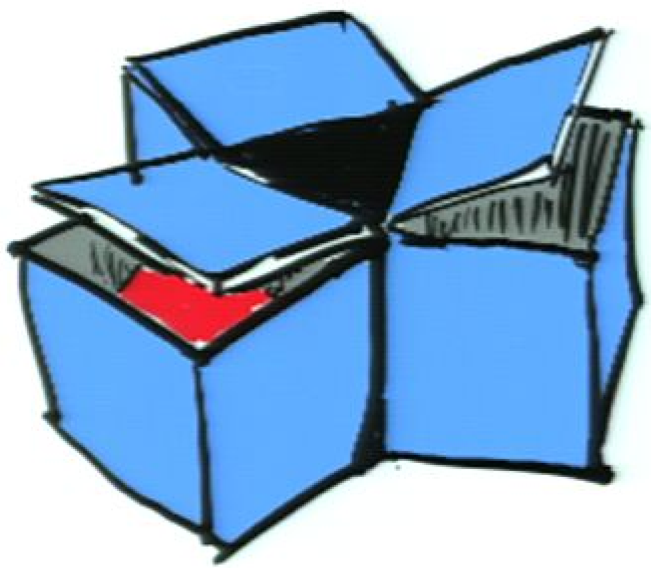

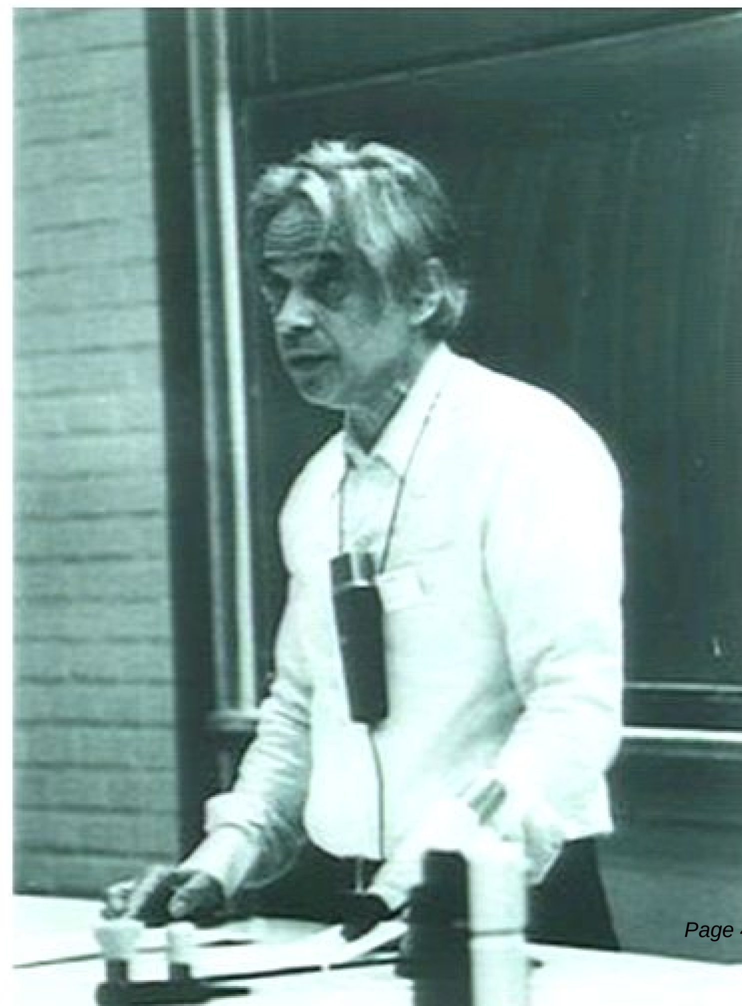

« La logique est d’abord une science naturelle. » -- F. Gonseth
(“Logic is primarily a natural science.”)

Joint measurability of POVMs

Noisy z-spin $\{E^z_+, E^z_-\}$

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$$E^z_\pm = \eta|\pm z\rangle\langle\pm z| + (1-\eta)\frac{I}{2}$$

η = sharpness factor

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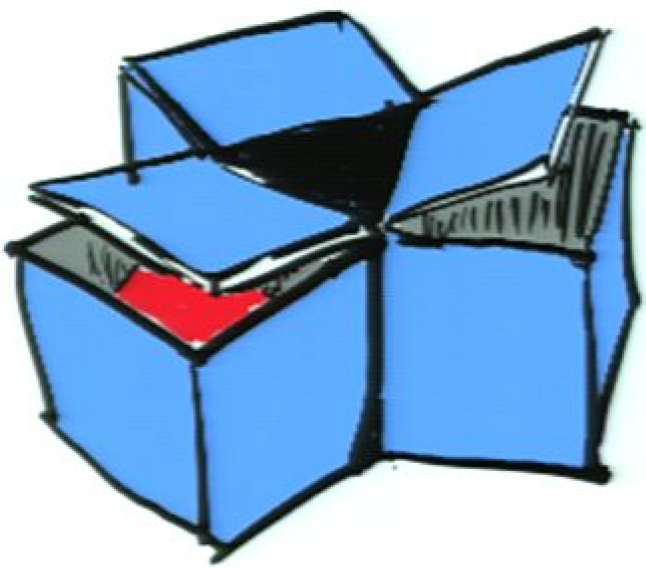

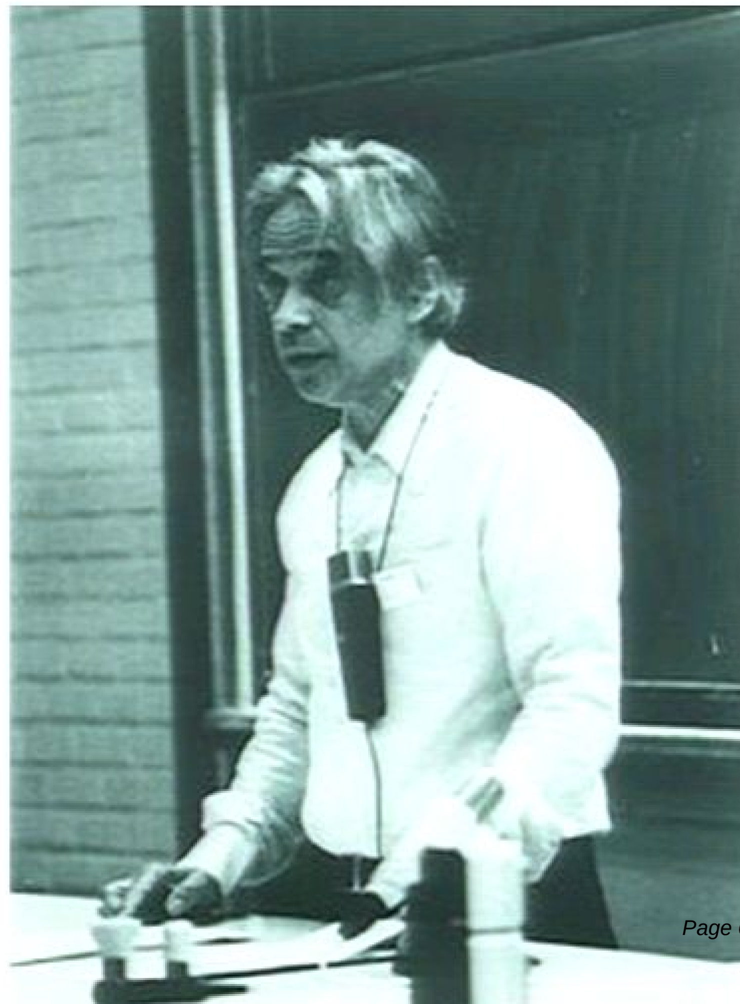

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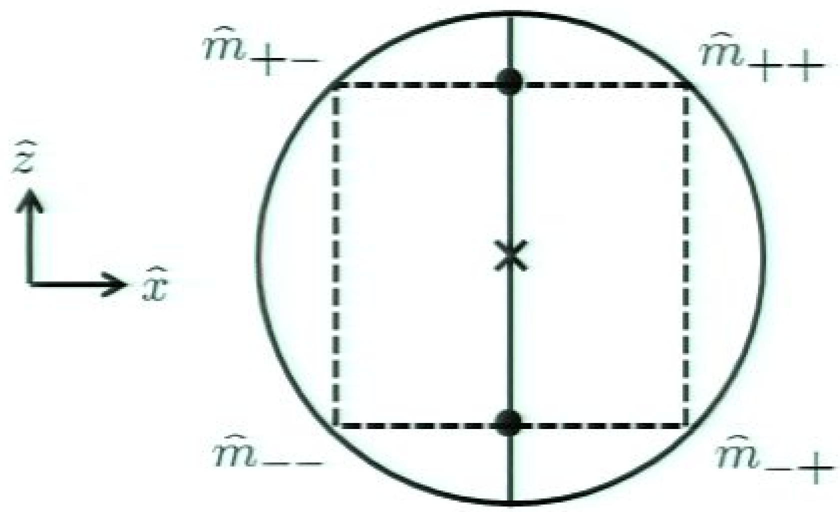

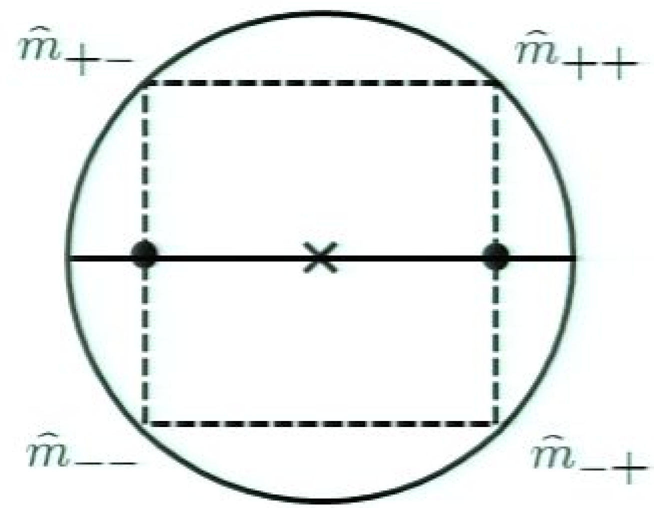

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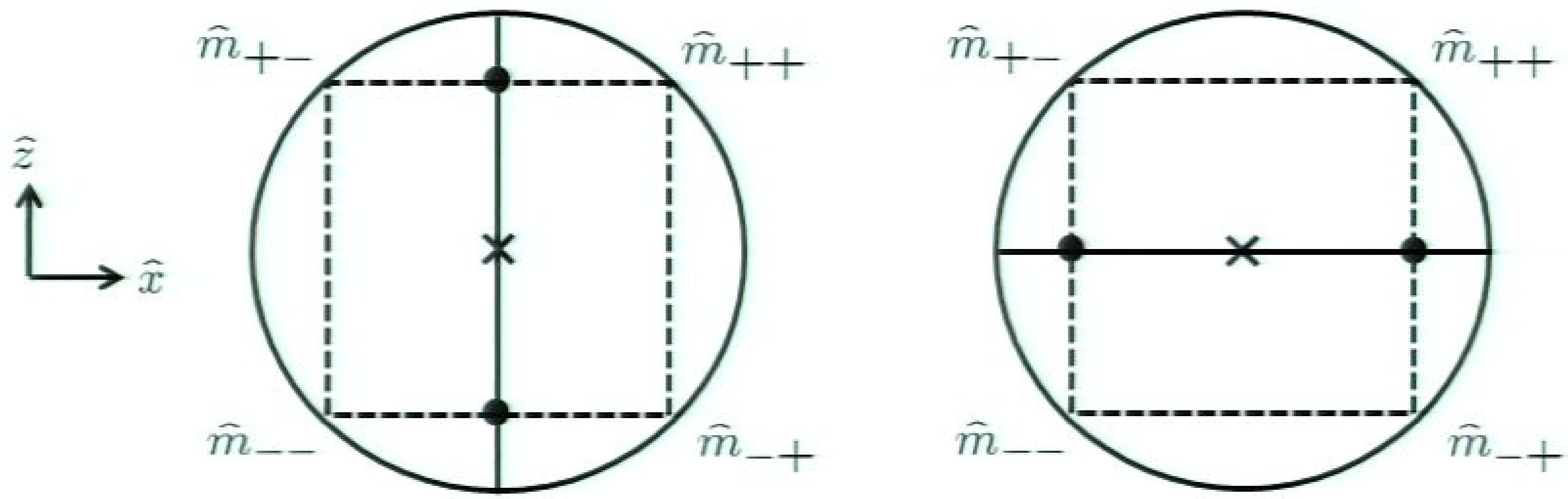

Pairwise JM iff $\quad \eta \leq \frac{1}{\sqrt{2}}$

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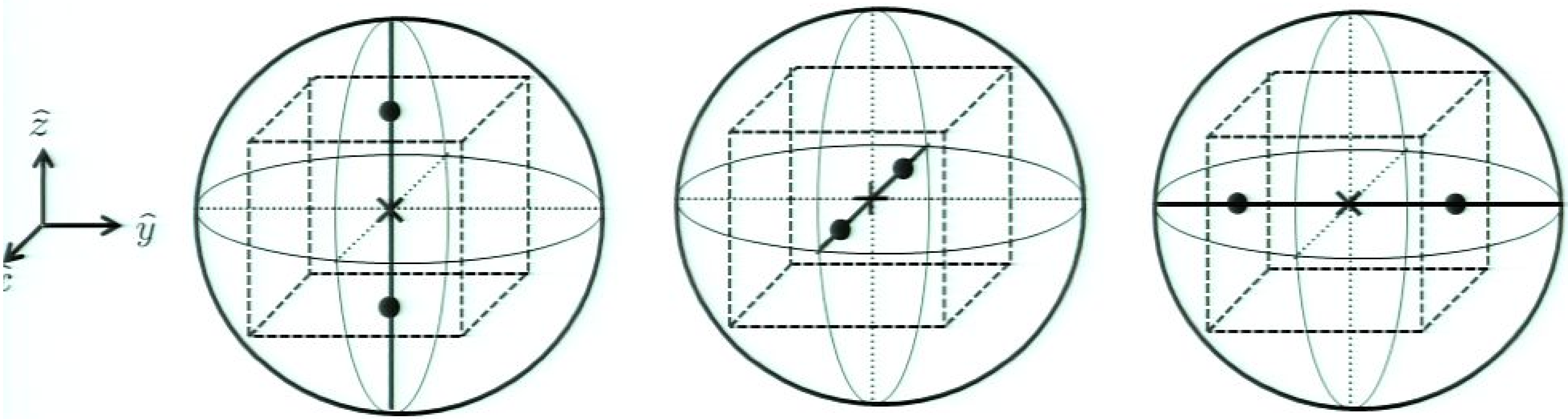

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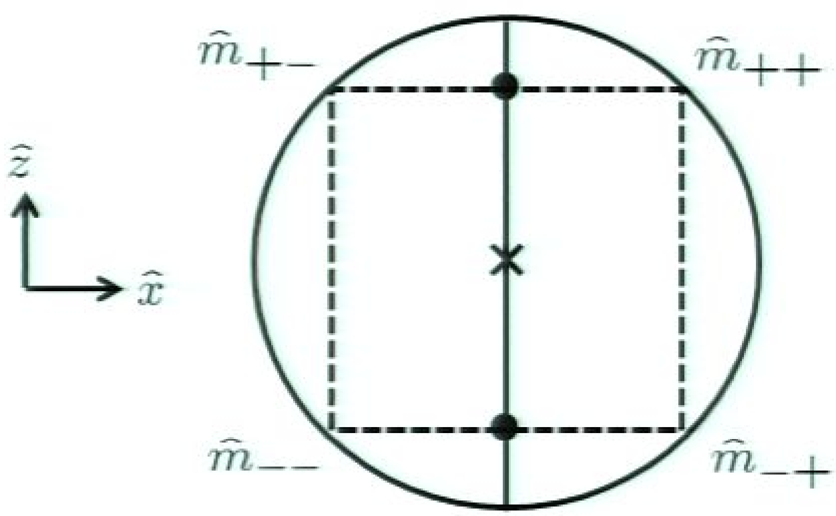

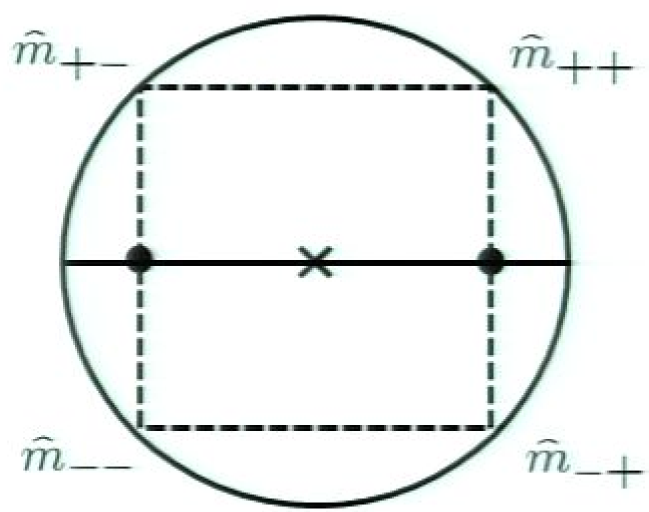

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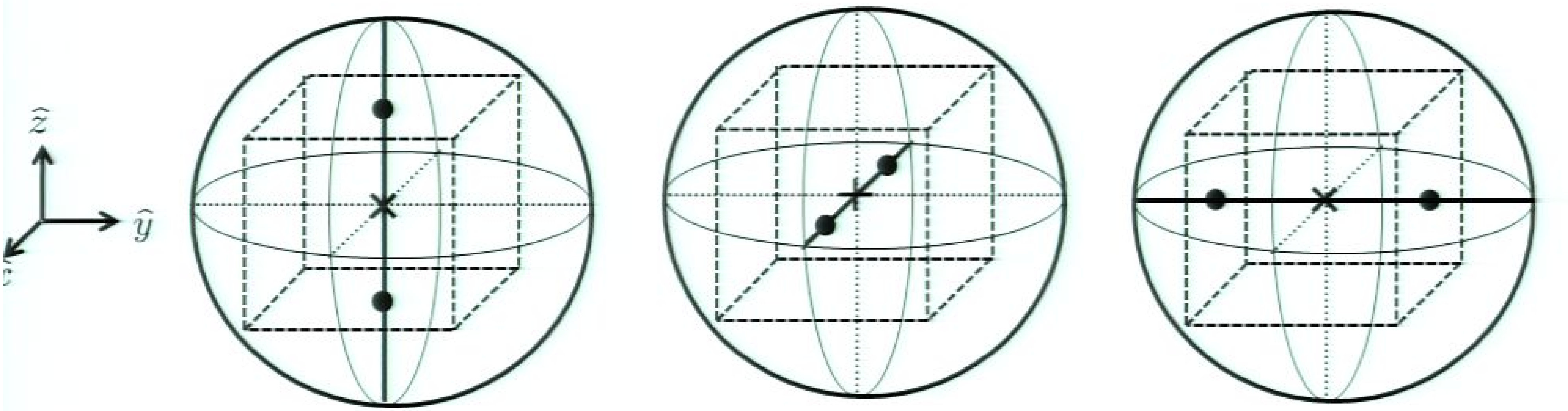

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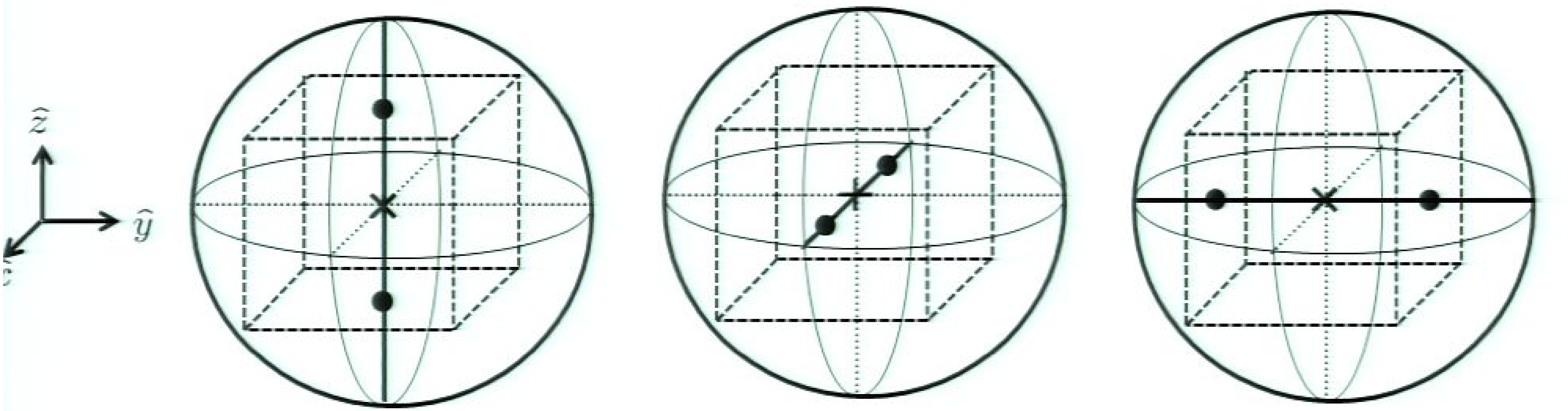

Pairwise JM but not triplewise JM iff $\frac{1}{\sqrt{2}} \leq \eta \leq \frac{1}{\sqrt{3}}$

Using a notion of noncontextuality for POVMs in
RWS, PRA 71, 052108 (2005)
we find that these correlations are
consistent with a noncontextual model

Therefore, we move on to projective measurements

Frustrated Networks

Nodes are binary variables

Edges imply joint measurability

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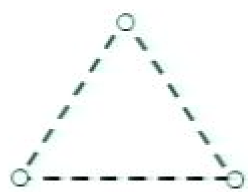

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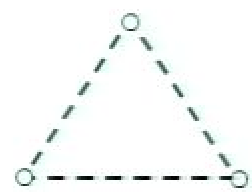

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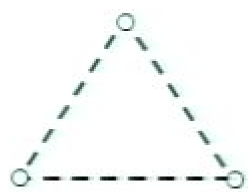

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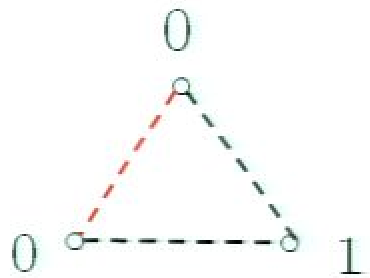
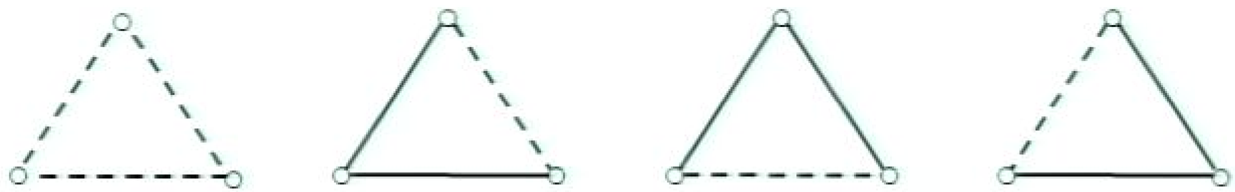
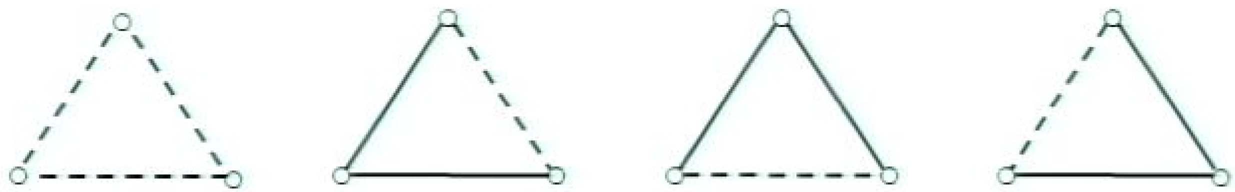


$$R \leq \frac{2}{3}$$

Kochen-Specker inequality



All 3-node frustrated networks
Equivalent under relabelling of the outcomes



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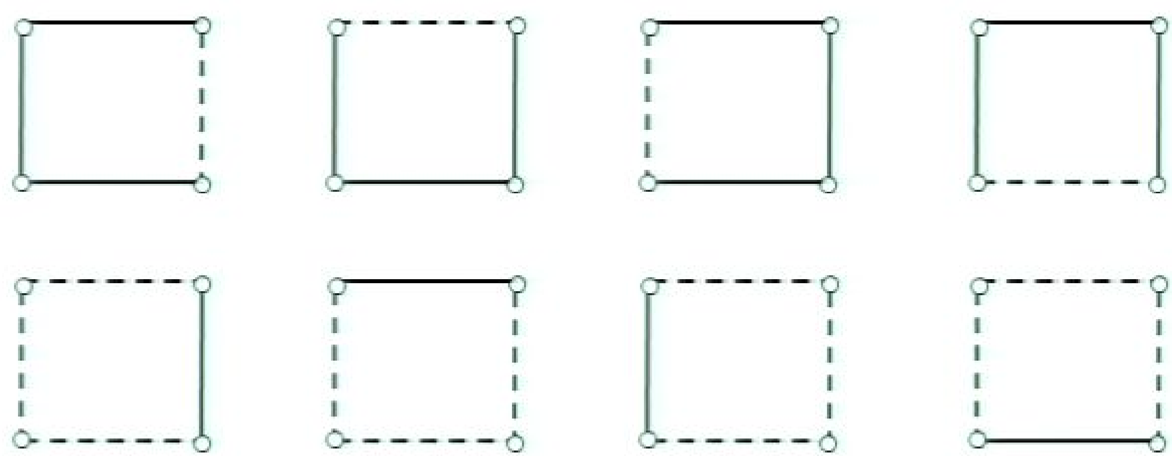
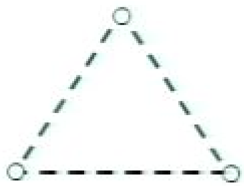
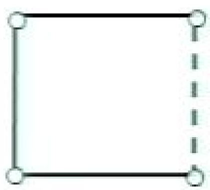
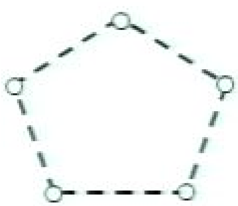
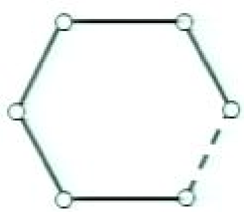
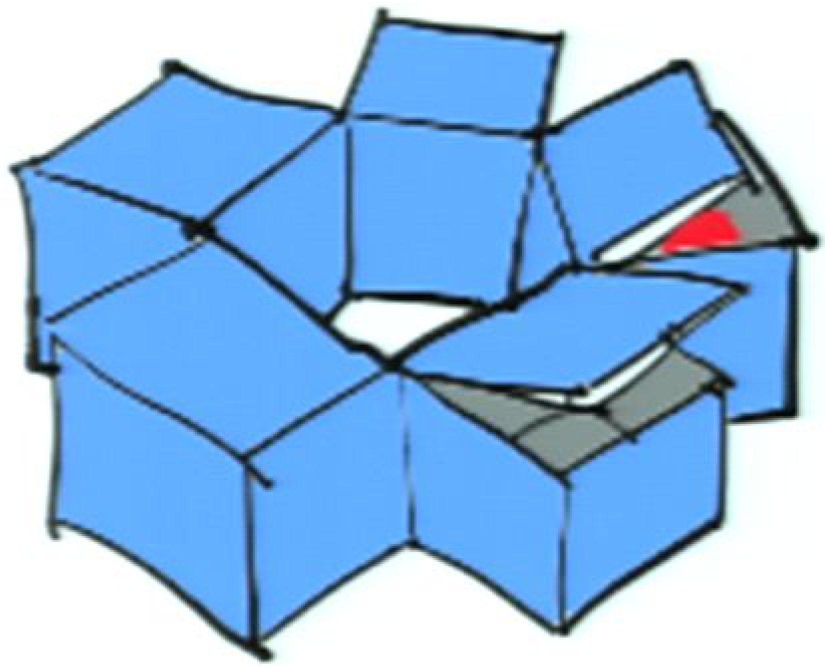


All 4-node frustrated networks
Equivalent under relabelling of the outcomes

	Kochen-Specker bound	Quantum bound	Maximum value	
	$R \leq \frac{2}{3}$	N/A	$R = 1$	Overprotective Seer (OS) correlations
	$R \leq \frac{3}{4}$	$R \leq \frac{1}{2} + \frac{1}{\sqrt{2}}$ $\simeq 0.85$	$R = 1$	CHSH correlations & PR box
	$R \leq \frac{4}{5}$	$R \leq \frac{2}{\sqrt{5}}$ $\simeq 0.89$	$R = 1$	Klyachko correlations
	$R \leq \frac{5}{6}$	$R \leq \frac{1}{2} + \frac{\sqrt{3}}{4}$ $\simeq 0.93$	$R = 1$	Vaidman's necklace correlations
n odd 	$R \leq \frac{n-1}{n}$	$R \leq \frac{2\cos(\frac{\pi}{n})}{1+\cos(\frac{\pi}{n})}$	$R = 1$	Analogue of Chained Bell

Double-query 5-box system allowing only **adjacent** queries



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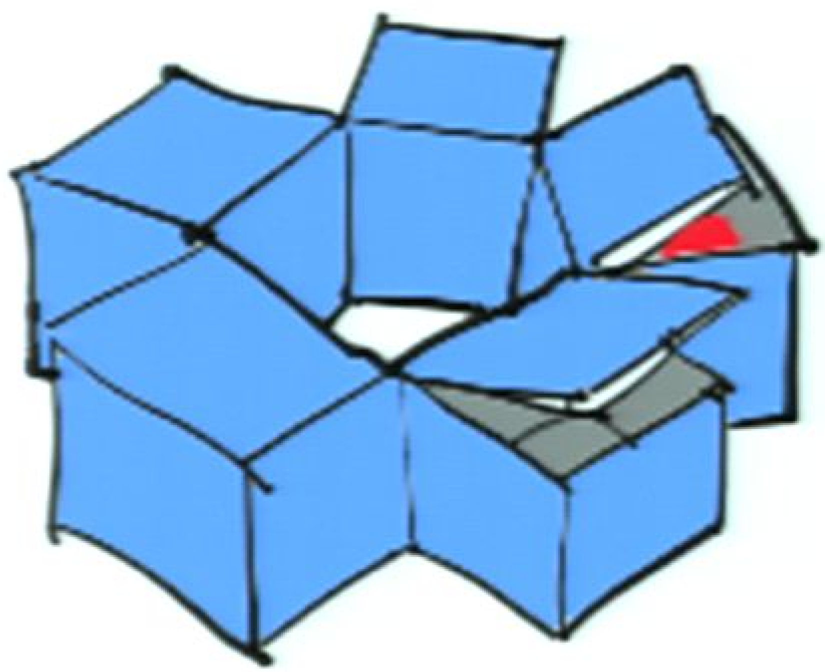

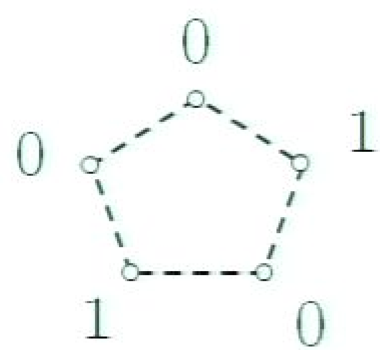

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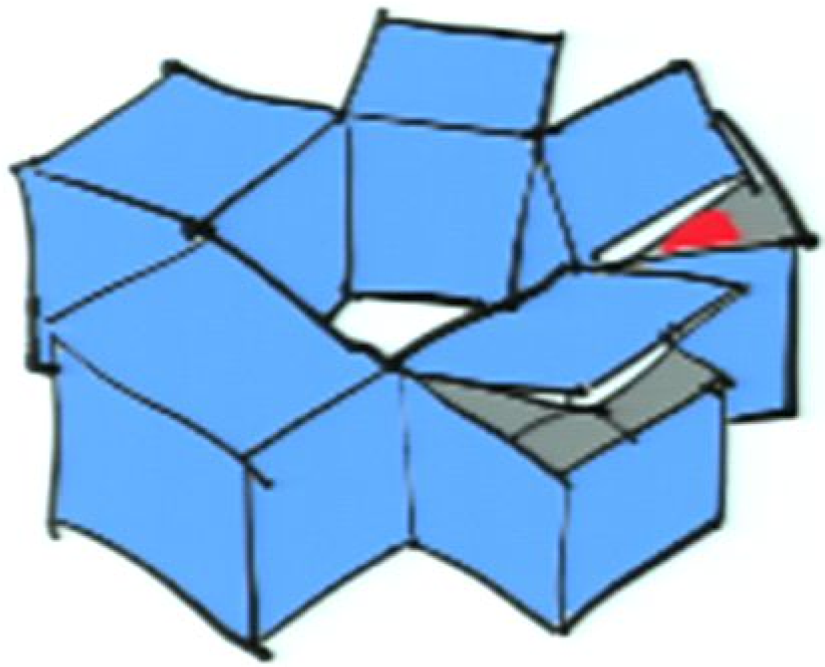

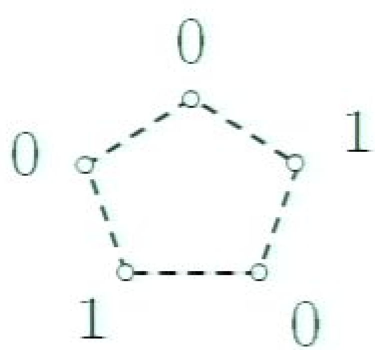

$$R \leq \frac{4}{5}$$

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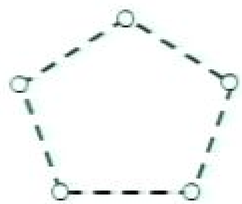

( This one is not)

E.g. 5 projective mmts:

$$\{\Pi_1 \, , \, I - \Pi_1\}$$
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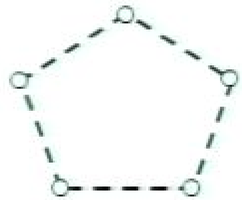
$$\{\Pi_4 \, , \, I - \Pi_4\}$$
$$\{\Pi_5 \, , \, I - \Pi_5\}$$

where $[\Pi_i, \Pi_{i \oplus 1}] = 0 \quad i \in \{1, \ldots, 5\}$

Klyachko's proof of the Kochen-Specker theorem

Klyachko, arXiv:quant-ph/0206012

Klyachko's proof of the Kochen-Specker theorem



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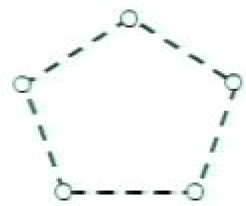

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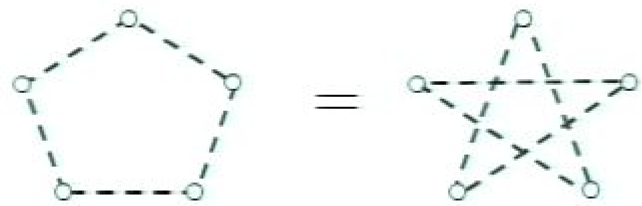

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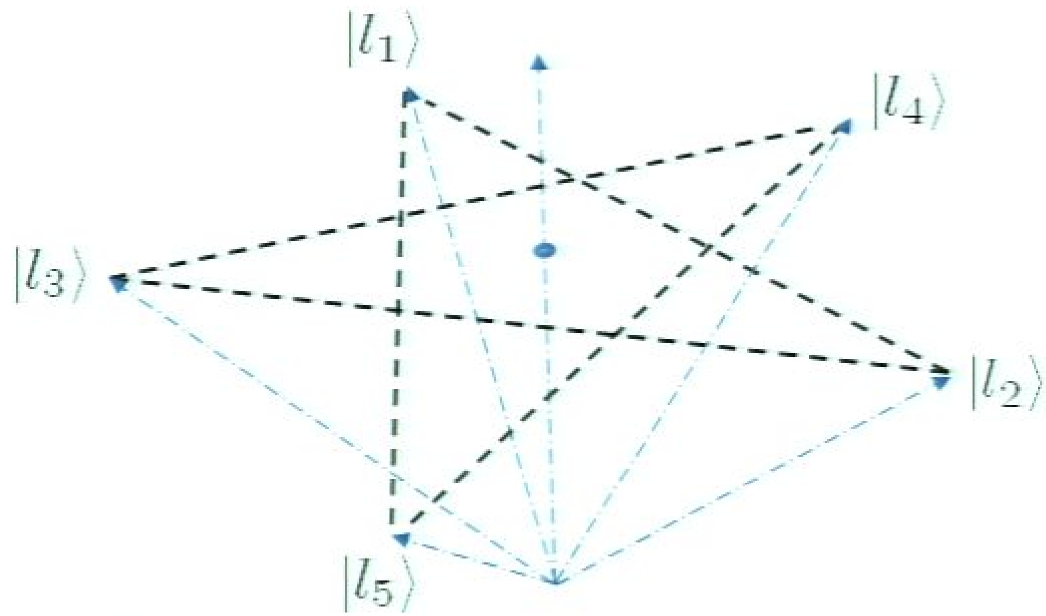

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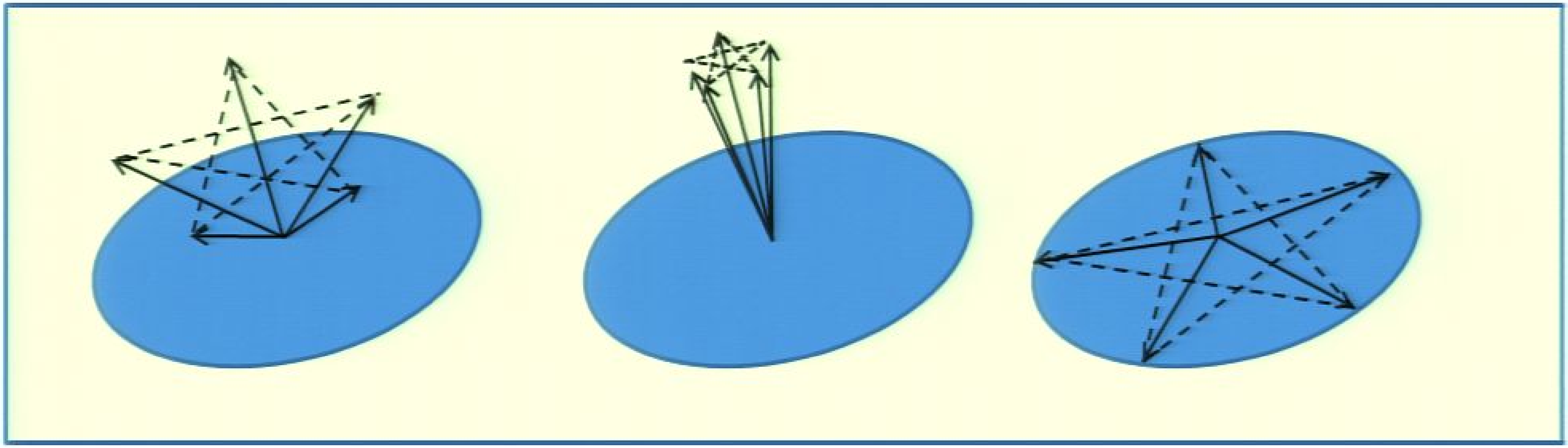

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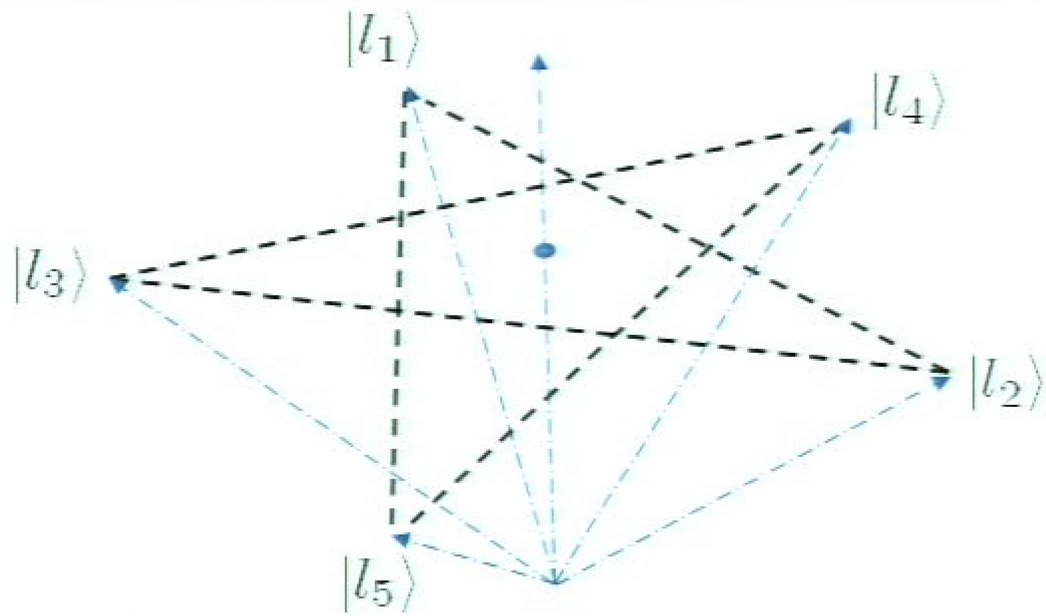

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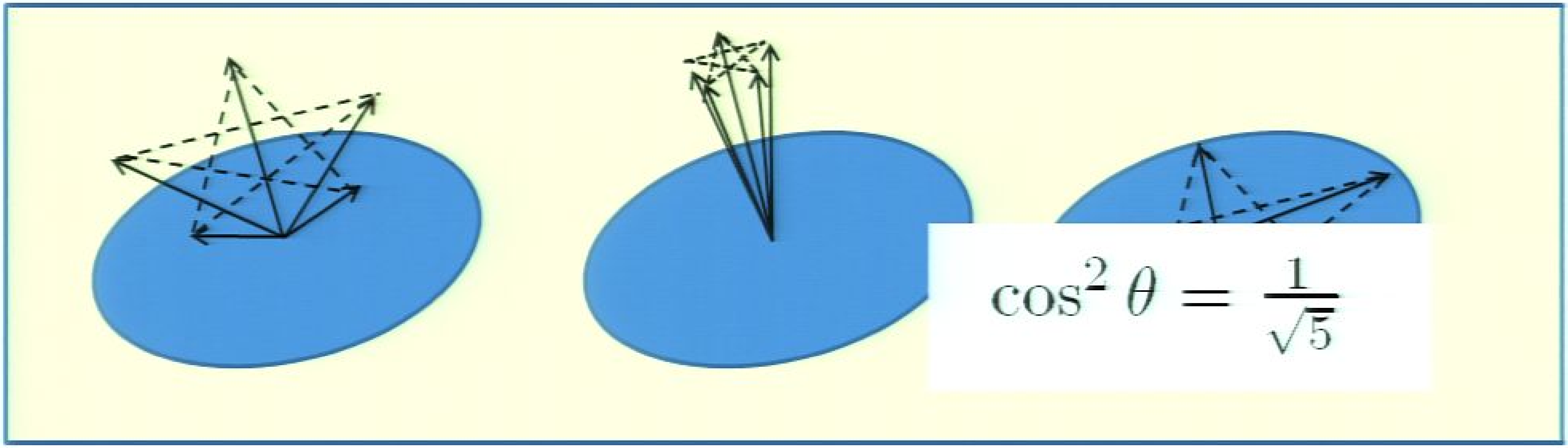

$$\cos^2\theta = \frac{1}{\sqrt{5}}$$

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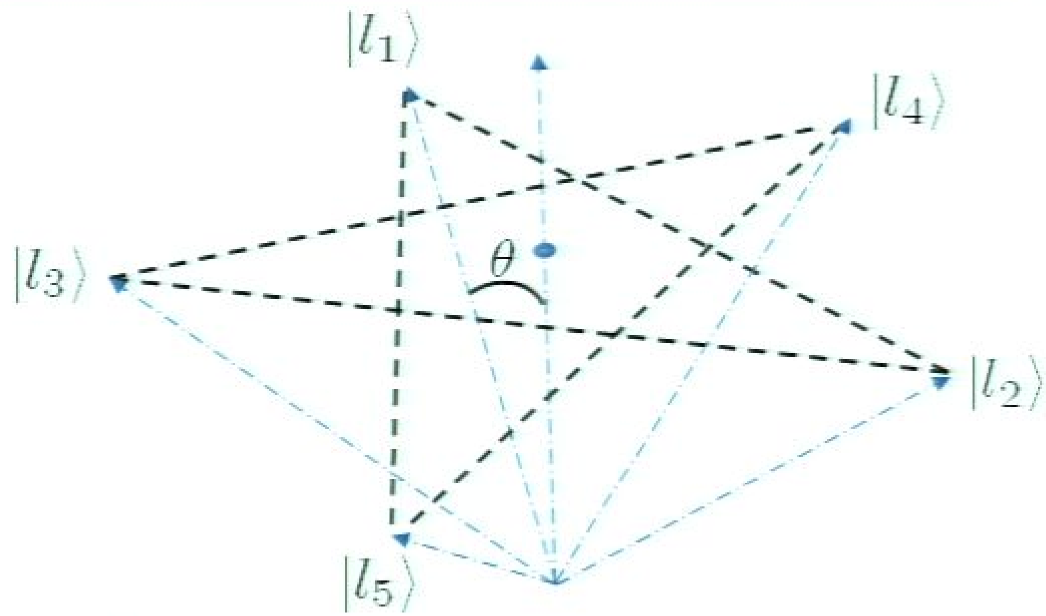

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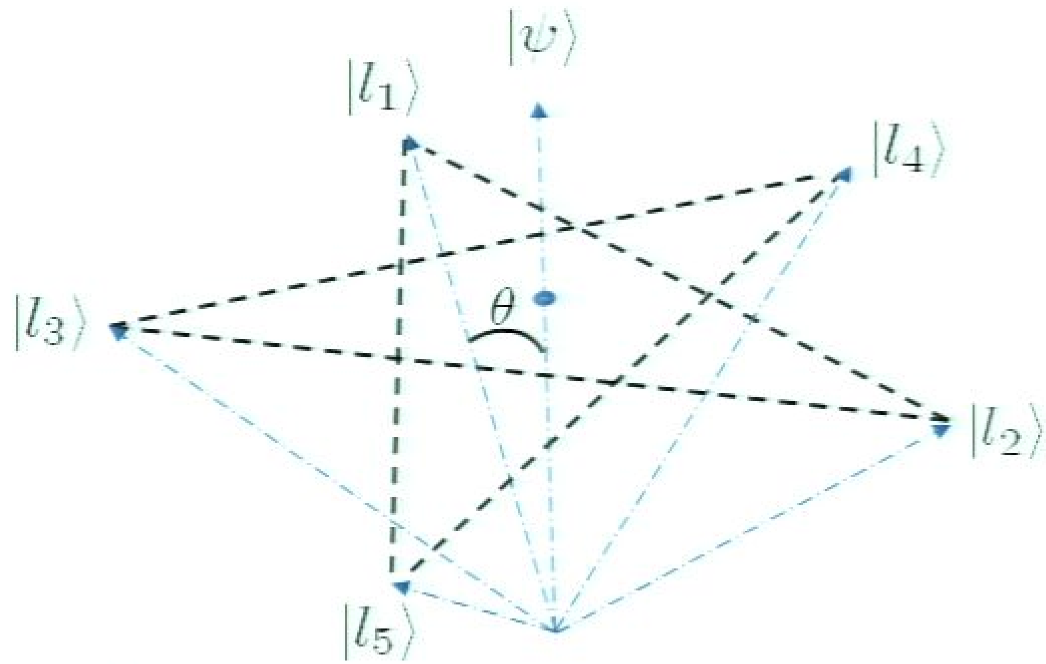

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State-specific : prepare ψ on symmetry axis

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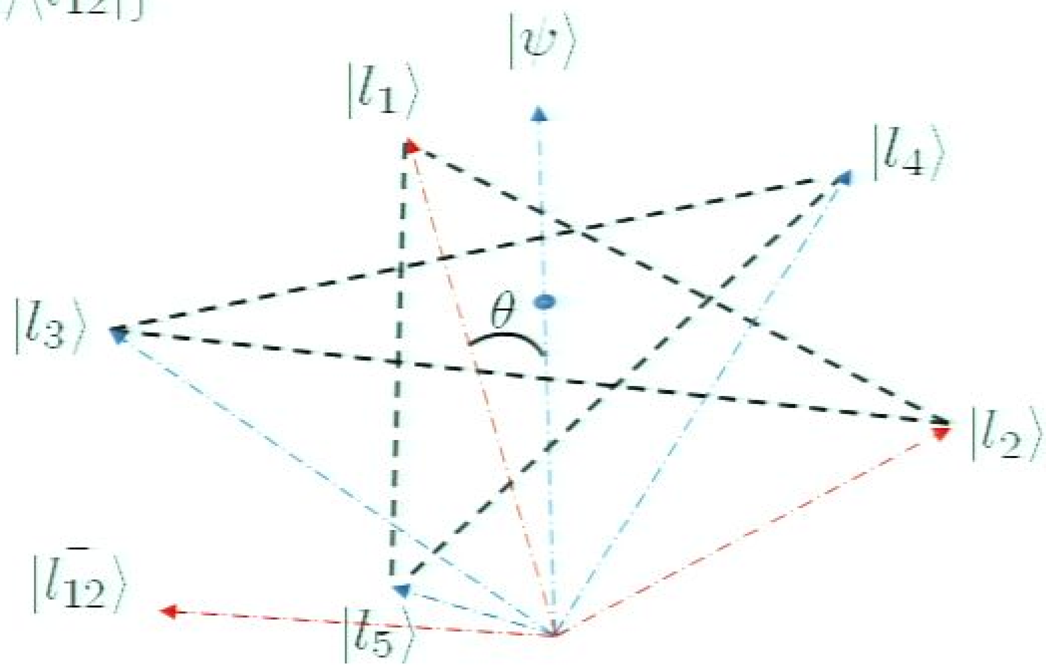

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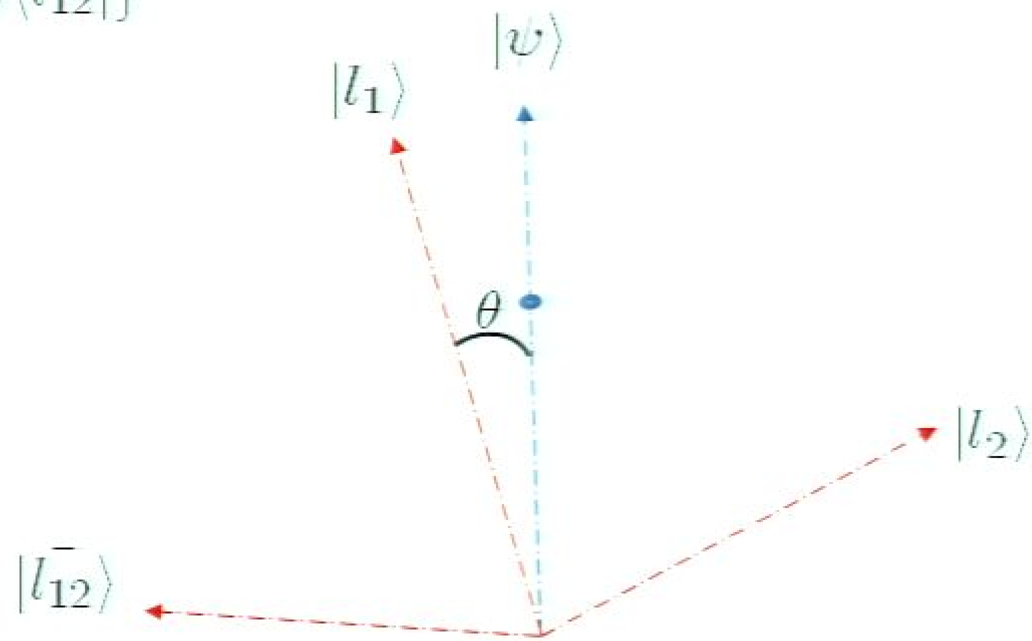
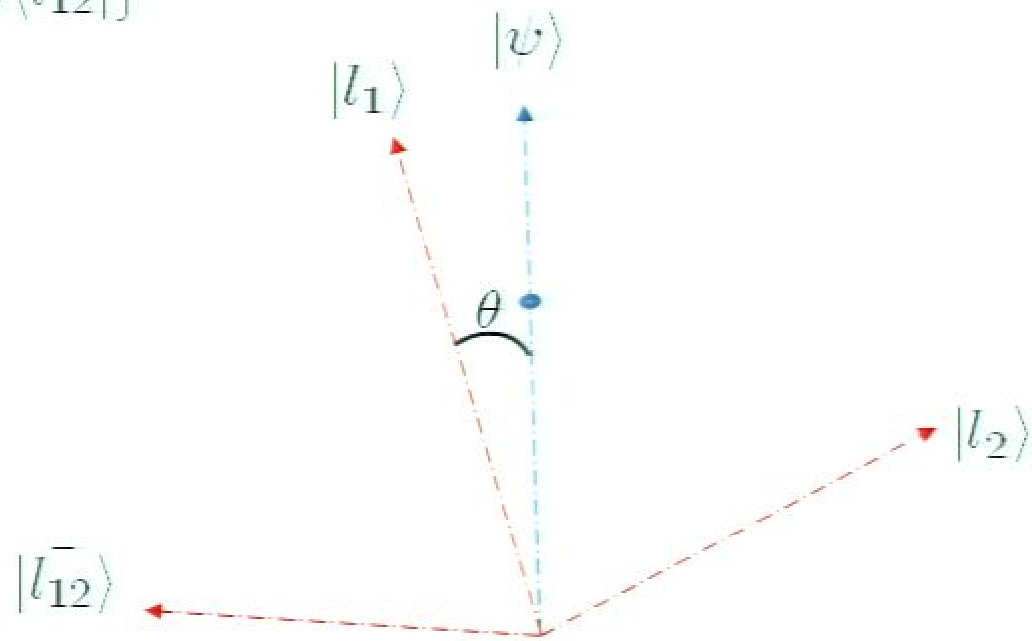


Klyachko's proof of the Kochen-Specker theorem

$$\cos^2\theta = \frac{1}{\sqrt{5}}$$

Consider: $\{|l_1\rangle\langle l_1| \;,\; |l_2\rangle\langle l_2| \;,\; |\bar{l_{12}}\rangle\langle \bar{l_{12}}|\}$



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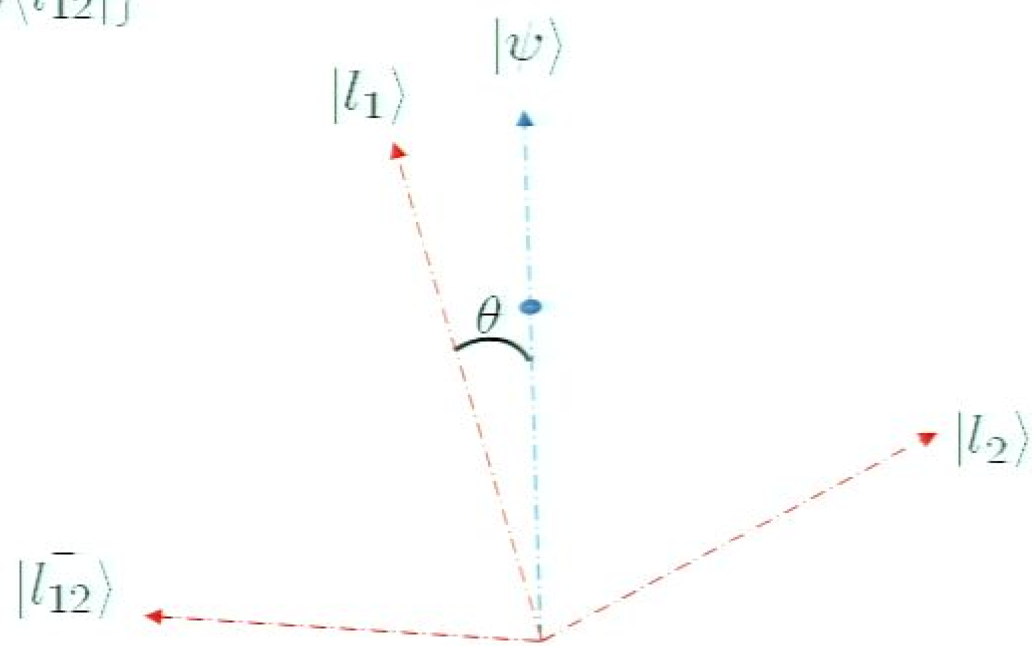

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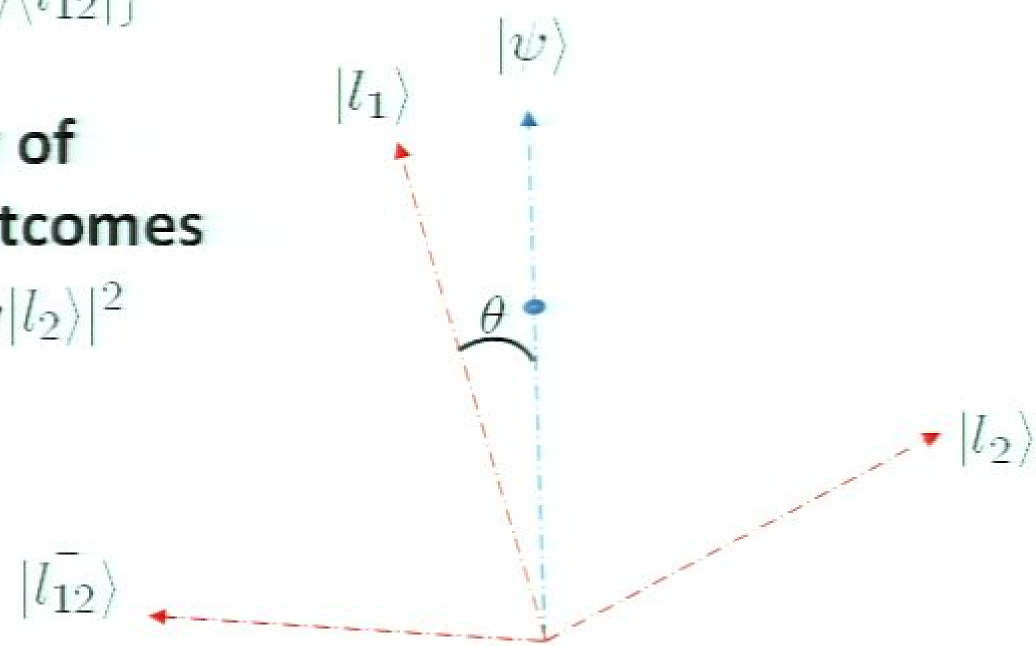

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Probability of anticorreled outcomes

$$|\langle\psi|l_1\rangle|^2 + |\langle\psi|l_2\rangle|^2 = \frac{2}{\sqrt{5}}$$

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Similarly for any pair of measurements...

Probability of anticorreled outcomes

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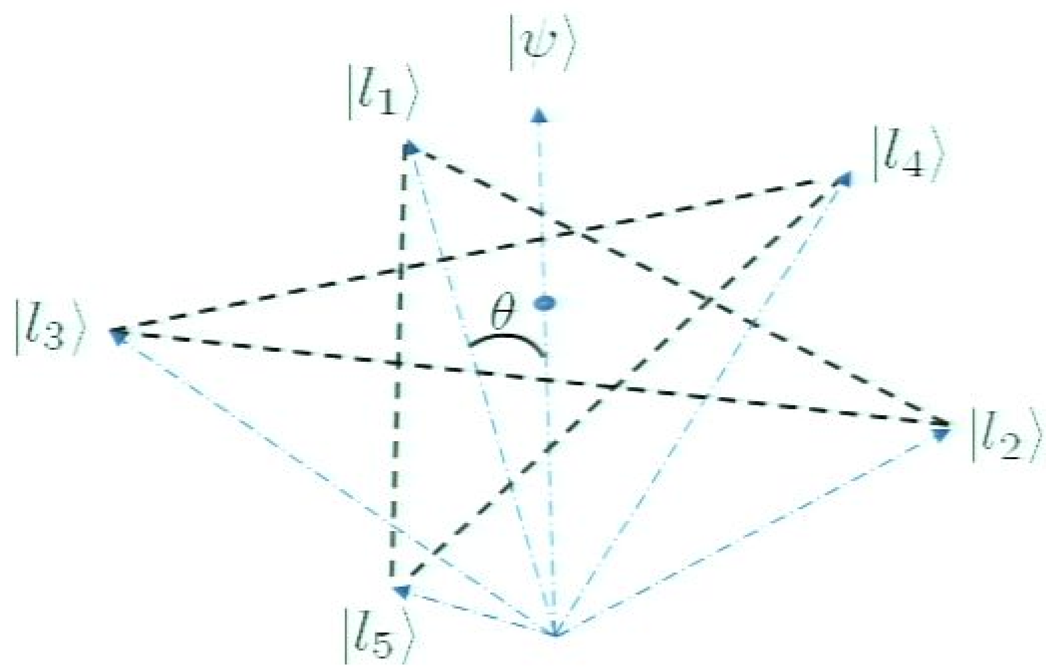

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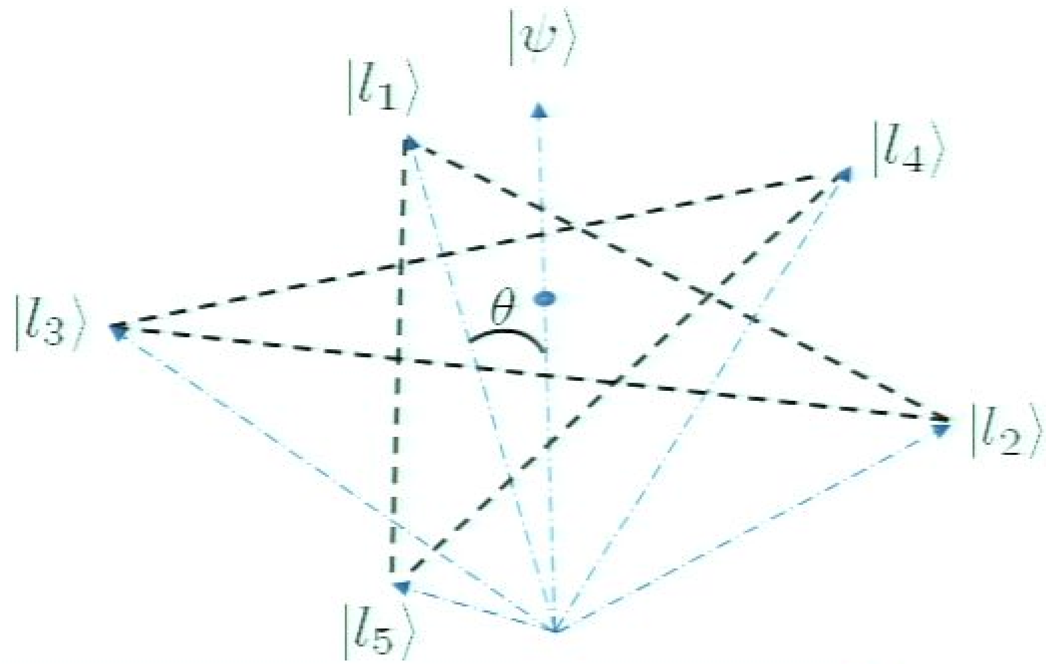

Quantum violation of Kochen-Specker inequality

$$R = \frac{2}{\sqrt{5}} \simeq 0.89 > \frac{4}{5}$$

Double-query **n**-box system allowing only adjacent queries

n odd

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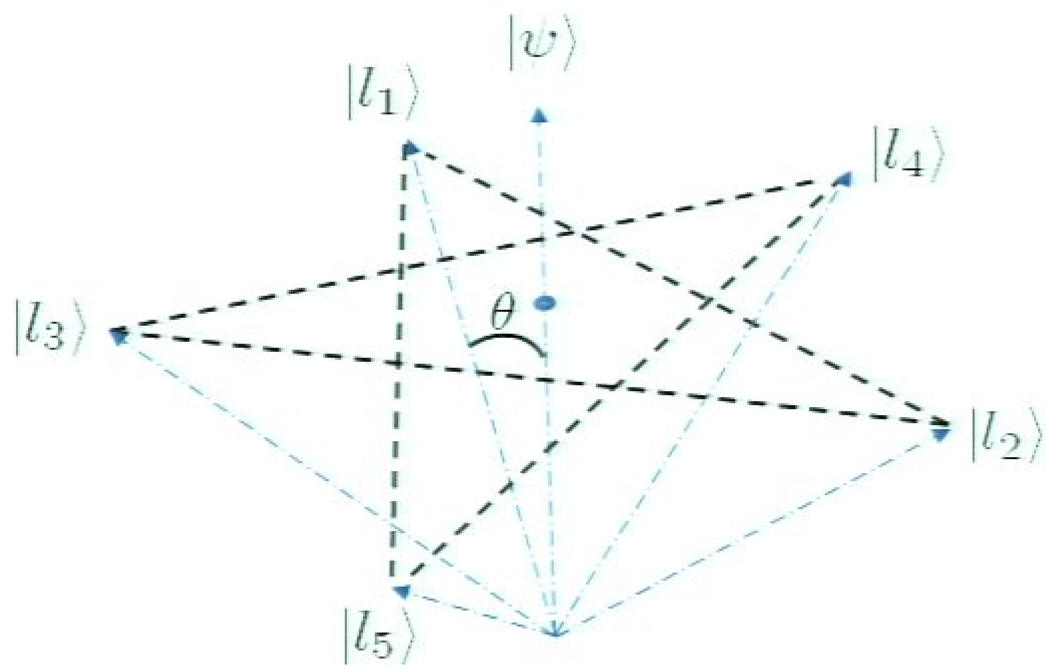

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Generalization of Klyachko's proof

n projective mmts:

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where $\langle l_i | l_{i\oplus 1}\rangle = 0 \quad i \in \{1, \ldots, n\}$

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$$R = \frac{2\cos(\frac{\pi}{n})}{1+\cos(\frac{\pi}{n})} > 1 - \frac{1}{n}$$

How the seer can achieve his ends in a quantum world

n odd

	Kochen-Specker bound	Quantum bound
	$R \leq \frac{n-1}{n} = 1 - \frac{1}{n}$	$R \leq \frac{2\cos(\frac{\pi}{n})}{1+\cos(\frac{\pi}{n})} \simeq 1 - \frac{\pi^2}{4n^2}$

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Suppose $n \ll \text{no. of suitors} \ll n^2$

Suitors believe it is very likely that one of them will win

In fact it is very likely that none of them will win

The failure of transitivity of implication

Note: noncontextuality implies transitivity of implication

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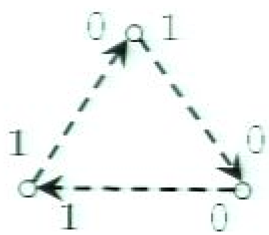

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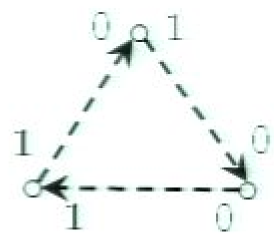
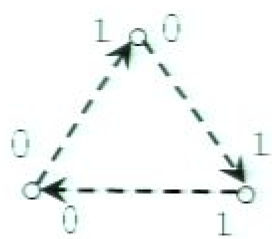
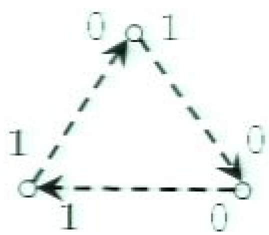


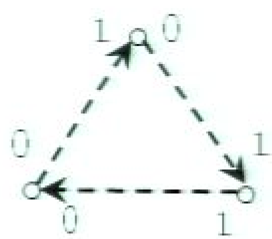


CONTRADICTION!

The failure of transitivity of implication

Note: noncontextuality implies transitivity of implication

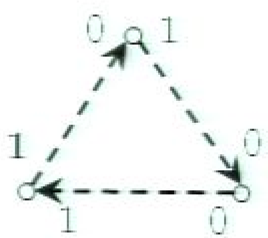




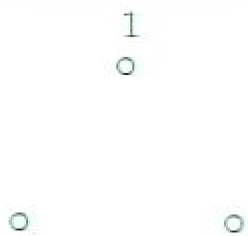
CONTRADICTION!

Alternatively:

Always



Sometimes



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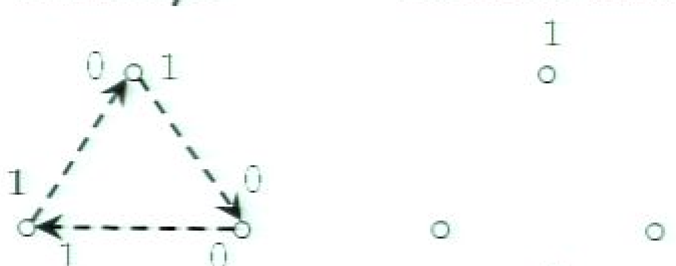

CONTRADICTION!

OS correlations give unbiased
outcomes for each mmt

A novel proof of the Kochen-Specker theorem based on the failure of transitivity of implication

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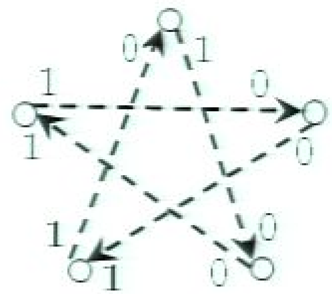
Always

Form of the proof:



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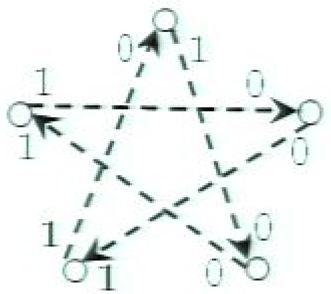

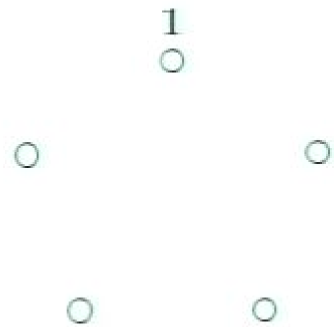

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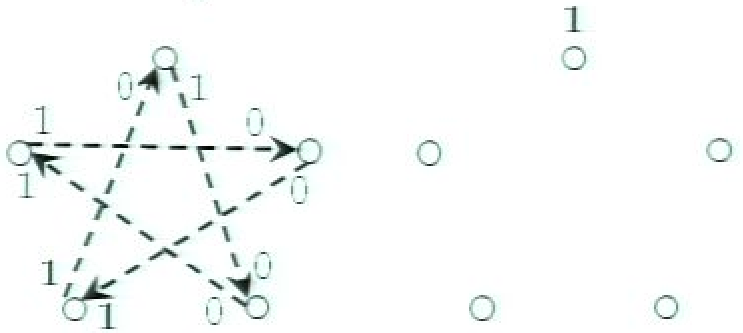

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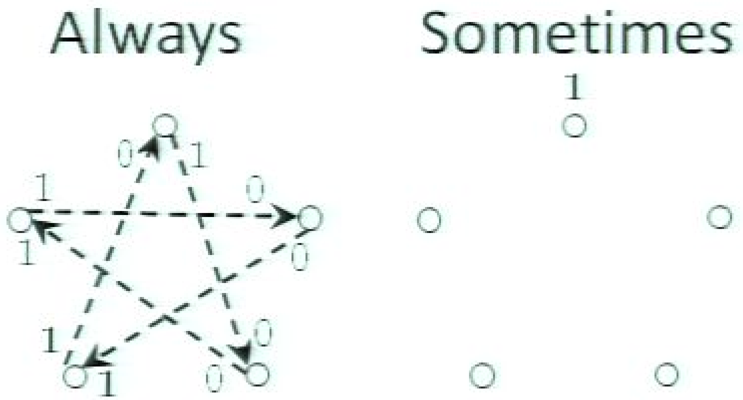
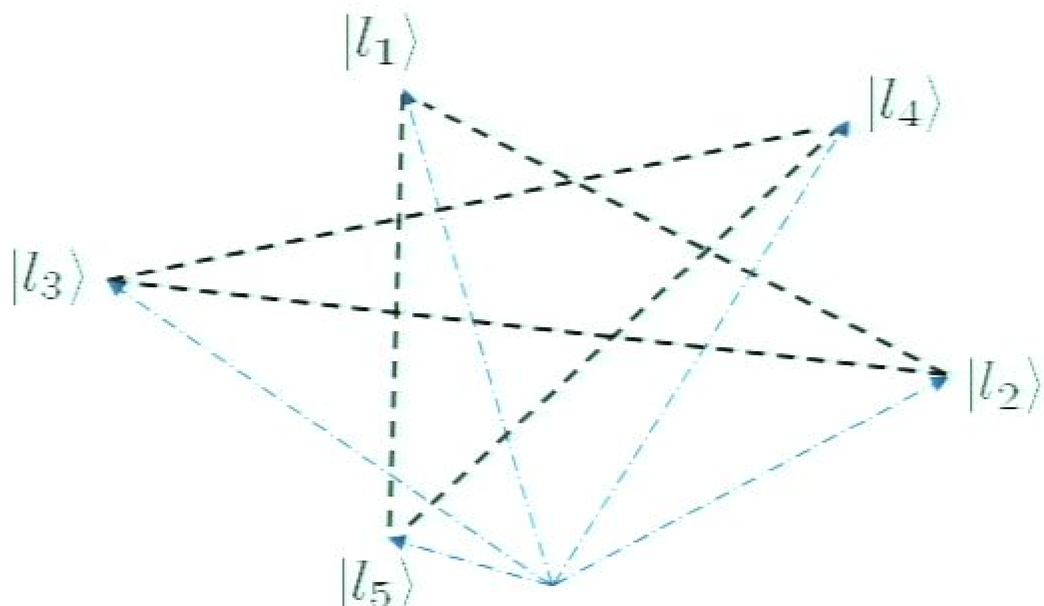


CONTRADICTION!

We use:



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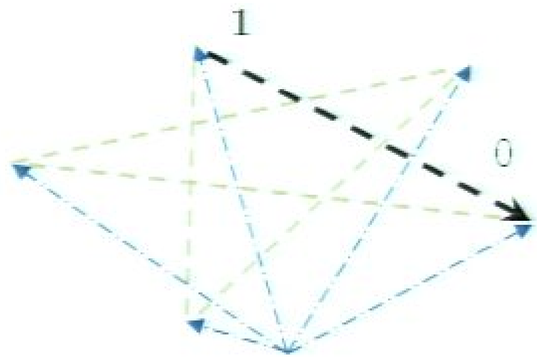
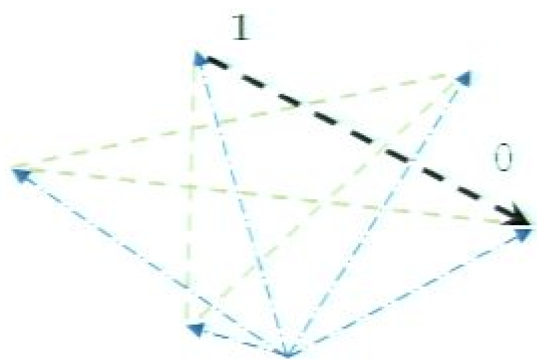


For all states

If $|l_1\rangle \perp |l_2\rangle$
Then $v(|l_1\rangle\langle l_1|) = 1 \implies v(|l_2\rangle\langle l_2|) = 0$

A novel proof of the Kochen-Specker theorem based on the failure of transitivity of implication



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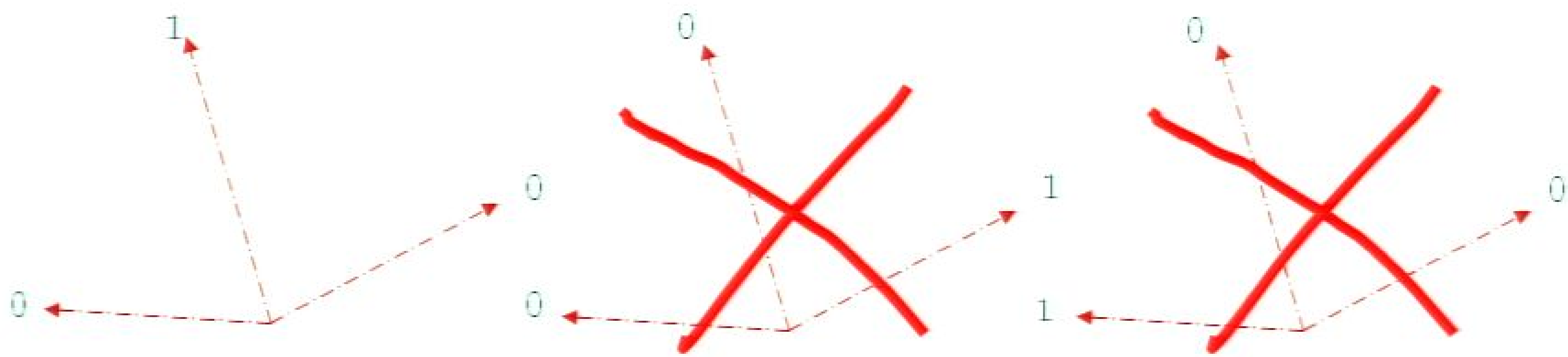

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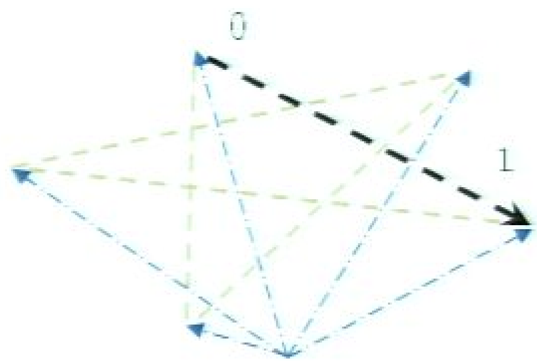

BUT NOT THE CASE THAT
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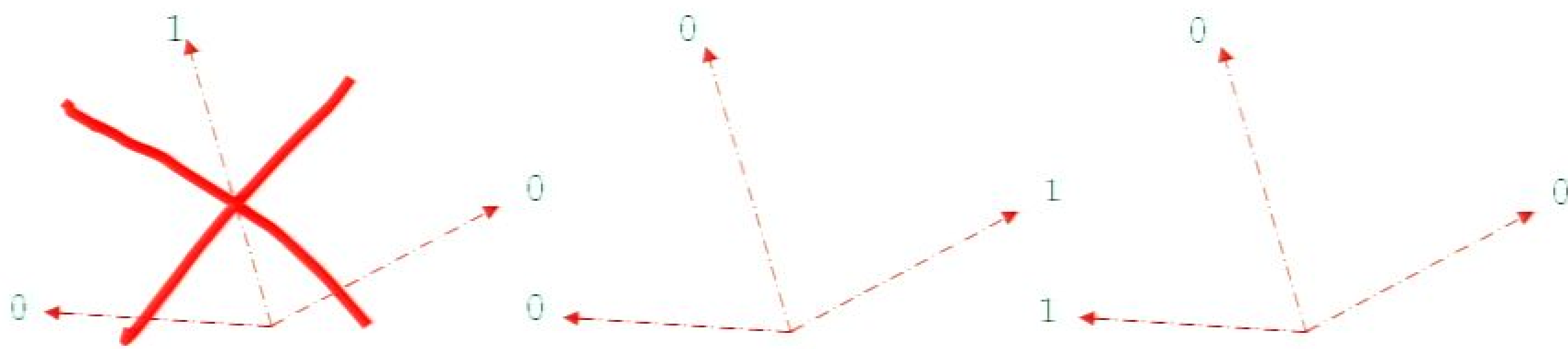

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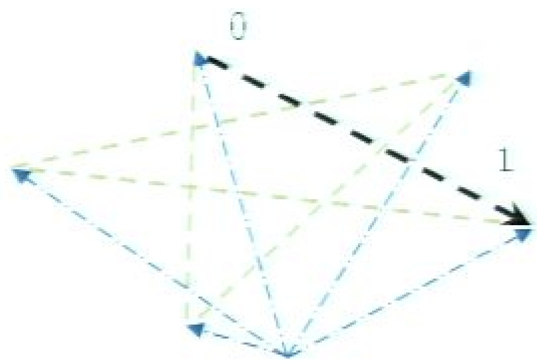

NONETHELESS

for all states $|\psi\rangle \in \text{span}(|l_1\rangle, |l_2\rangle)$

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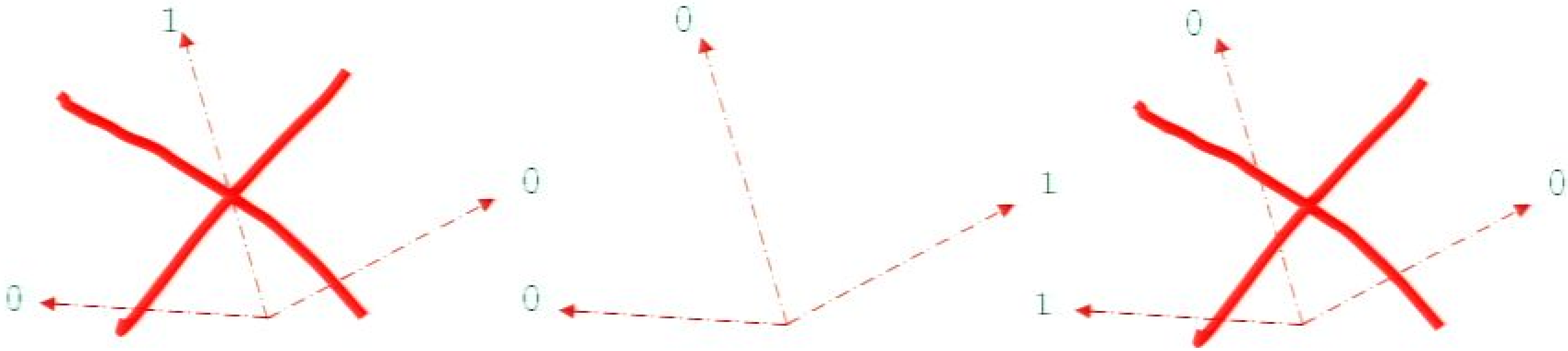

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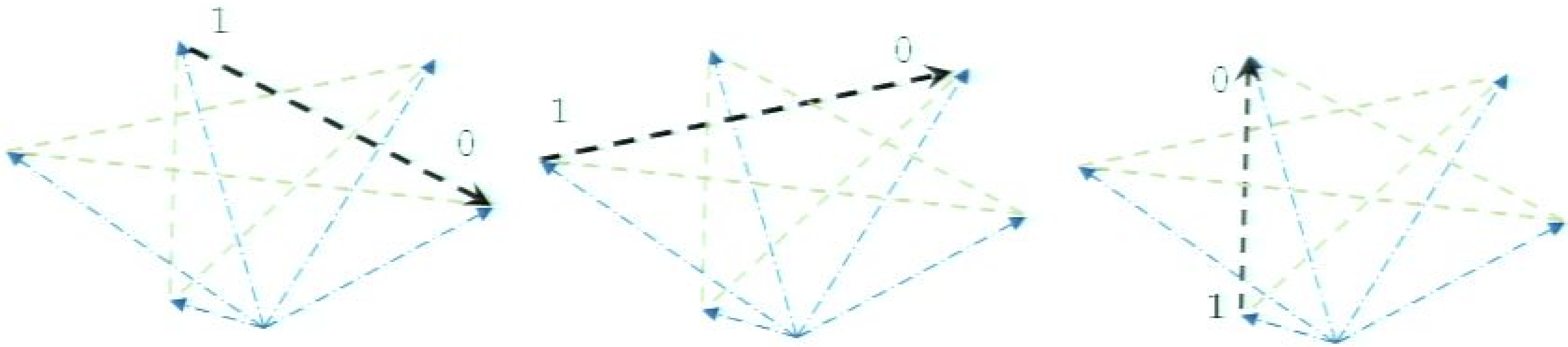
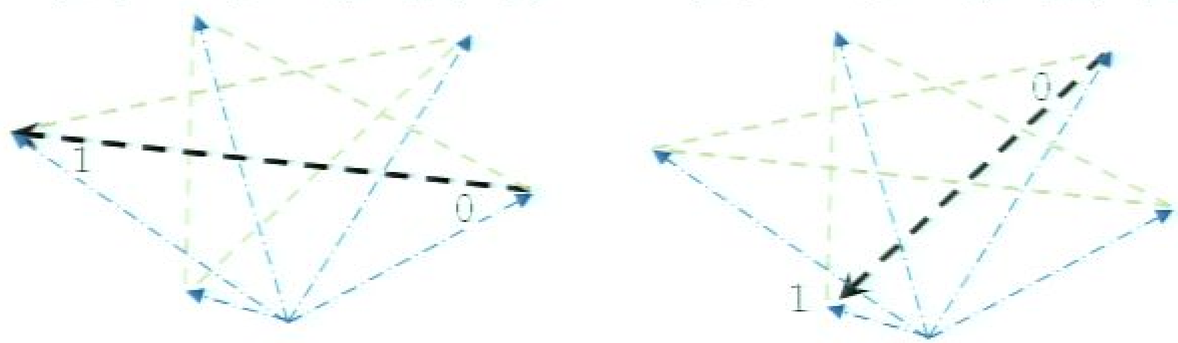


$\forall |\psi\rangle \in \mathrm{span}(|l_2\rangle, |l_3\rangle)$ $\quad$ $\forall |\psi\rangle \in \mathrm{span}(|l_4\rangle, |l_5\rangle)$



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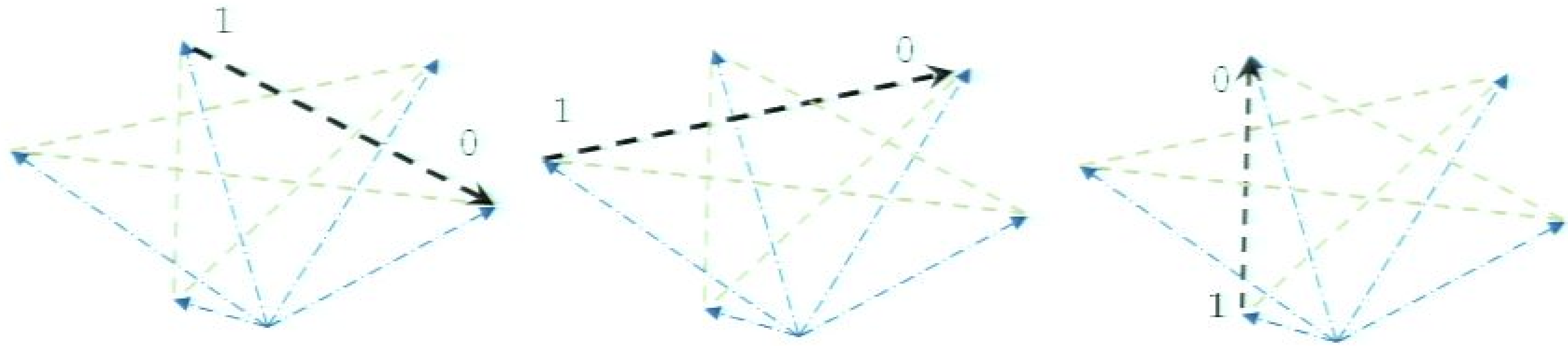

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Therefore choose $|\psi\rangle \in \mathrm{span}(|l_2\rangle,|l_3\rangle) \cap \mathrm{span}(|l_4\rangle,|l_5\rangle)$

A novel proof of the Kochen-Specker theorem based on the failure of transitivity of implication

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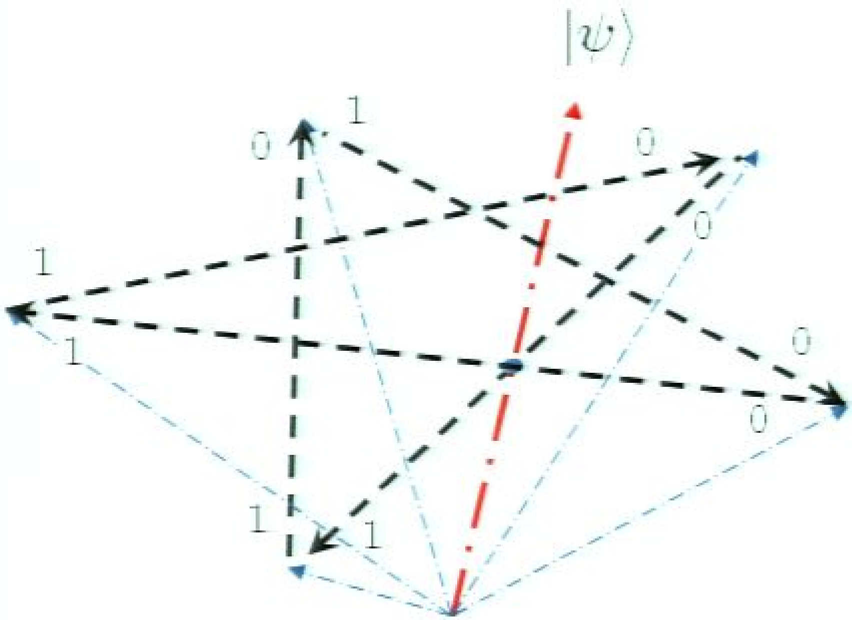

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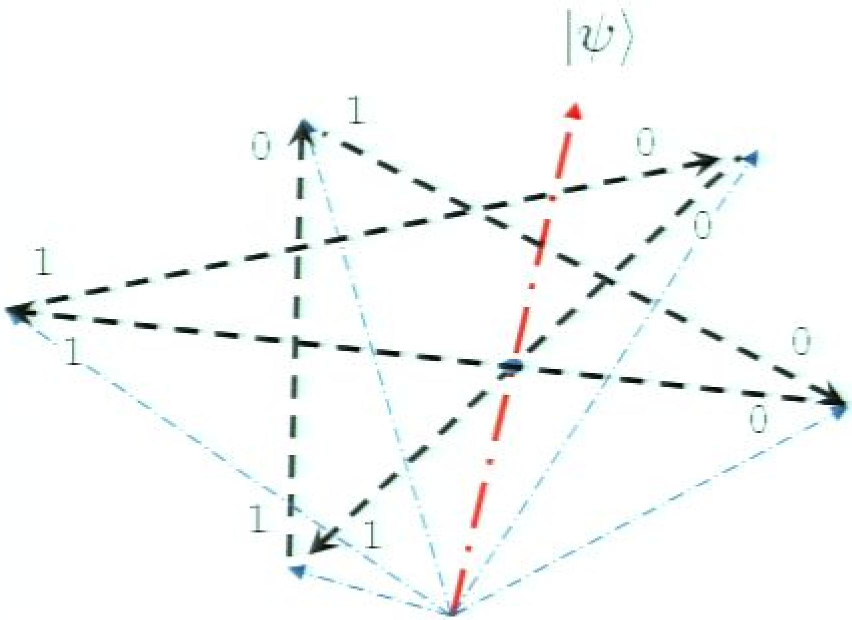

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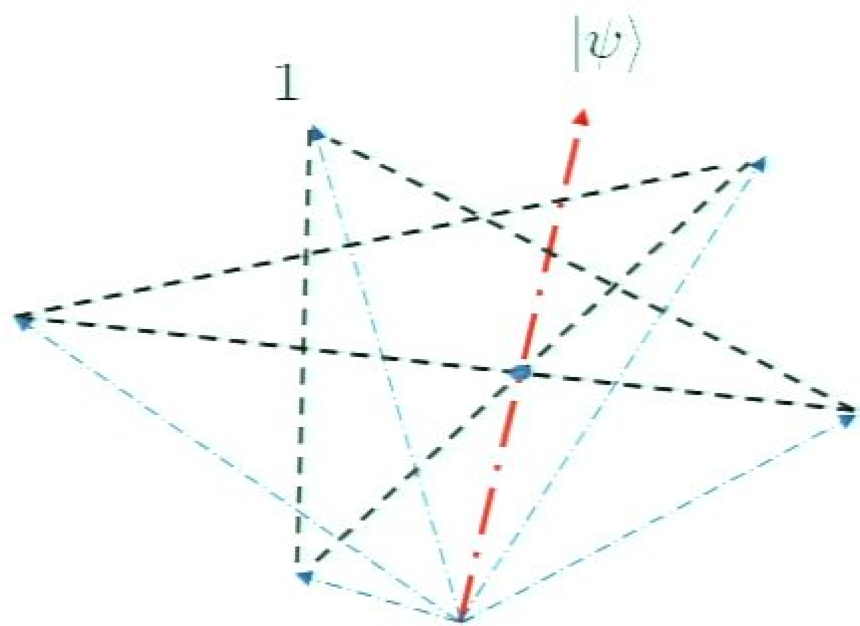

i.e. with probability

$$|\langle l_1|\psi\rangle|^2 = 1 - \frac{2}{\sqrt{5}} \simeq 0.1056$$

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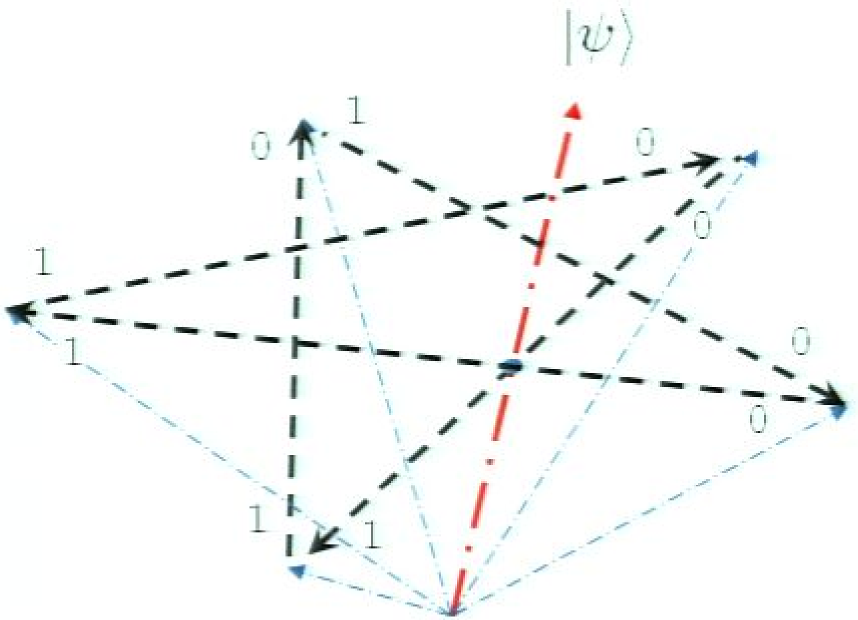

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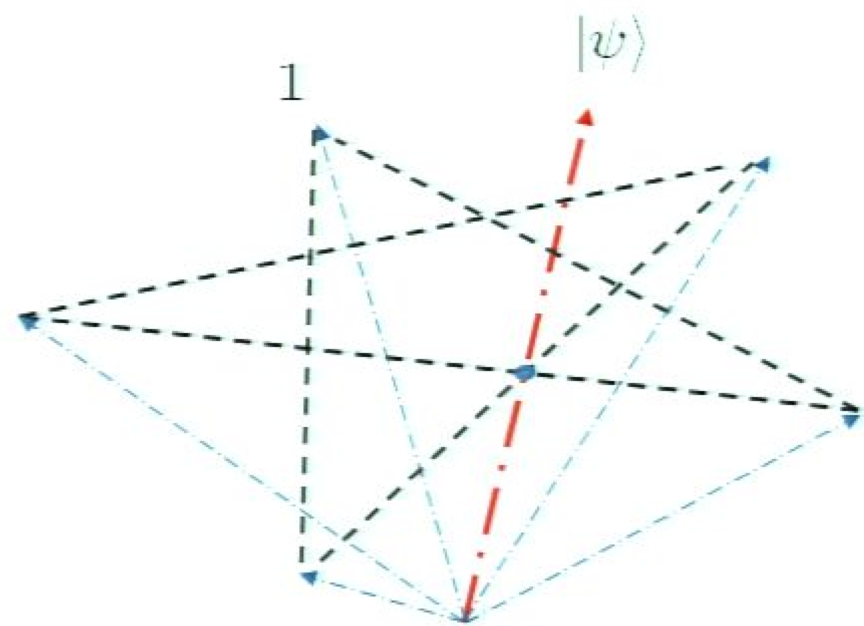

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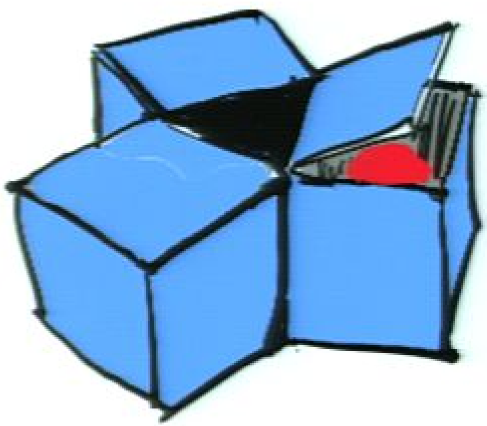
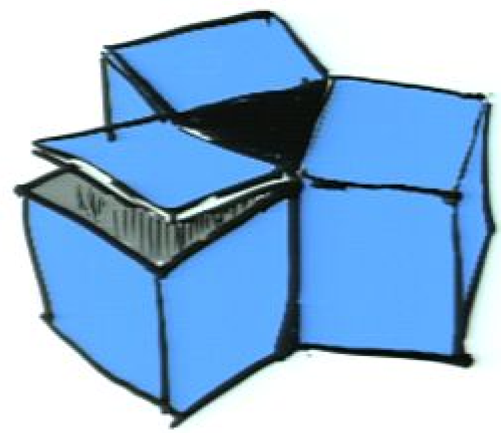
A separated pair of single-query 3-box systems

Abydos



Babylon



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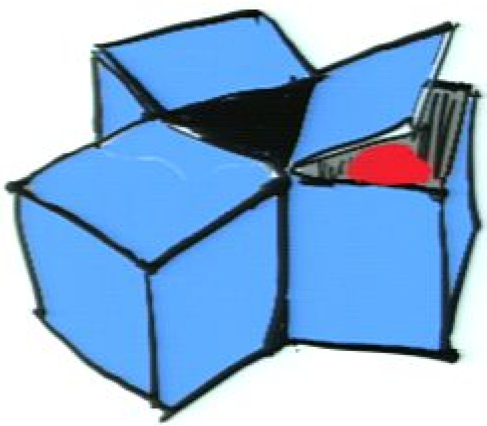
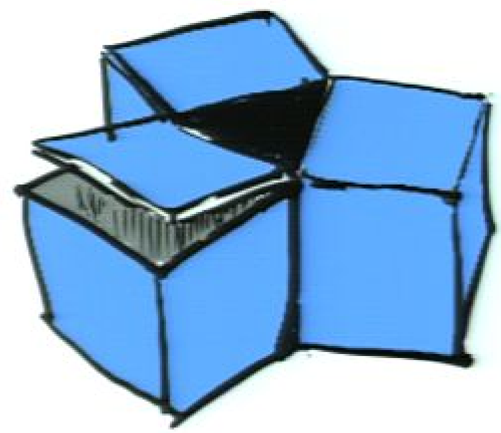
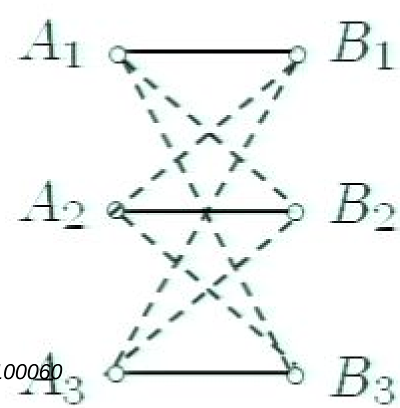


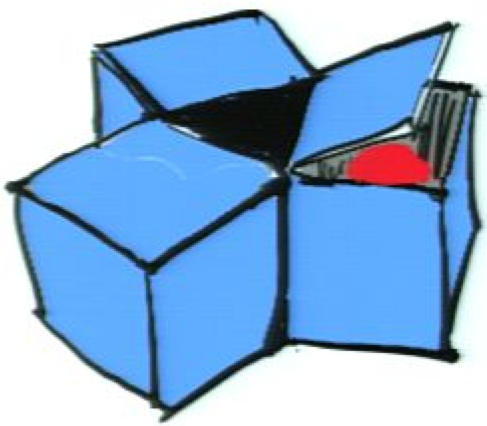
Babylon





A separated pair of single-query 3-box systems

Abydos



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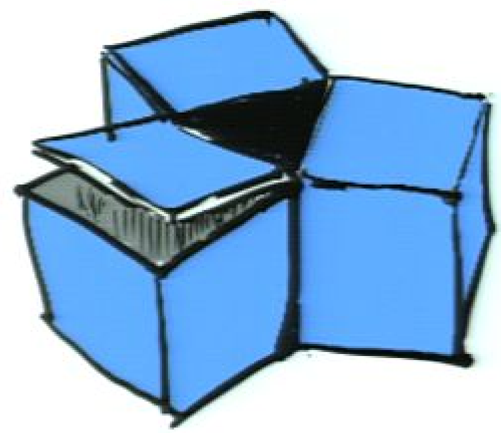

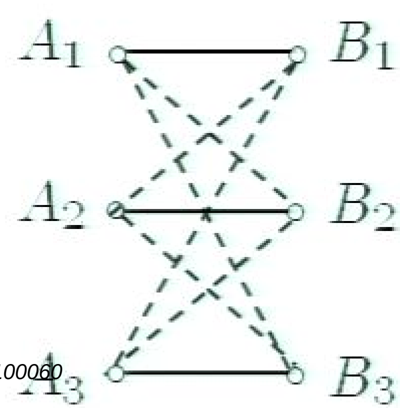

frustrated

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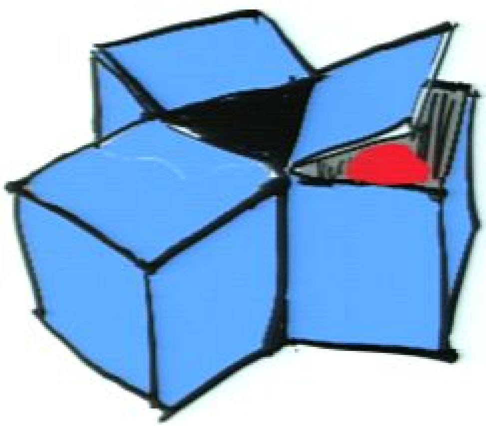
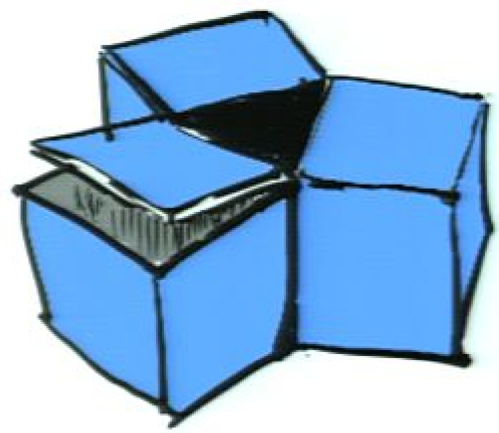
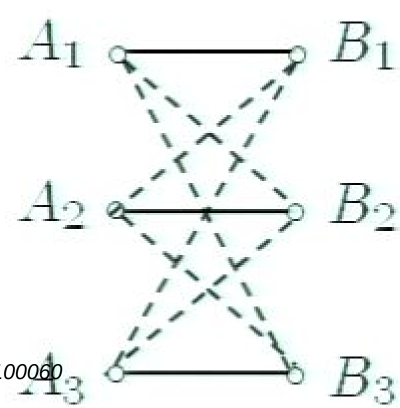
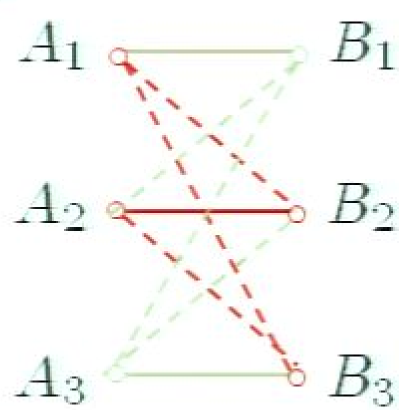


Babylon





frustrated



PR box

Bell inequality

Recall: Bell locality + perfect correlation

$\Rightarrow$ determinstic noncontextual values

Bell inequality

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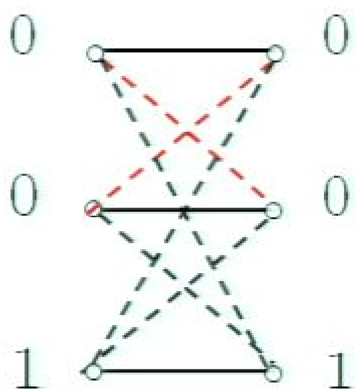
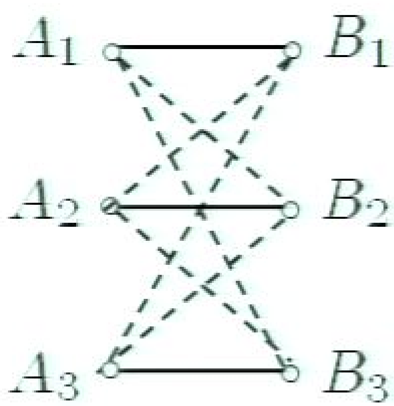


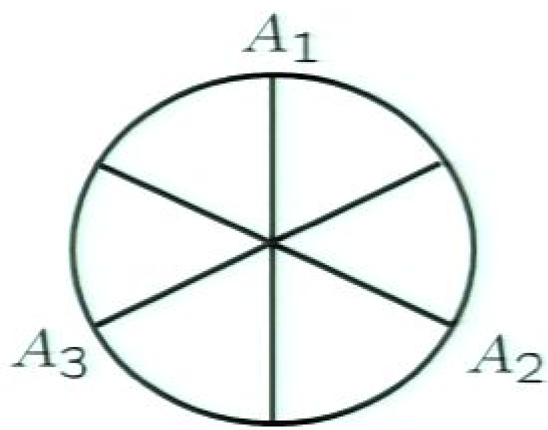
$$R \leq \frac{7}{9}$$

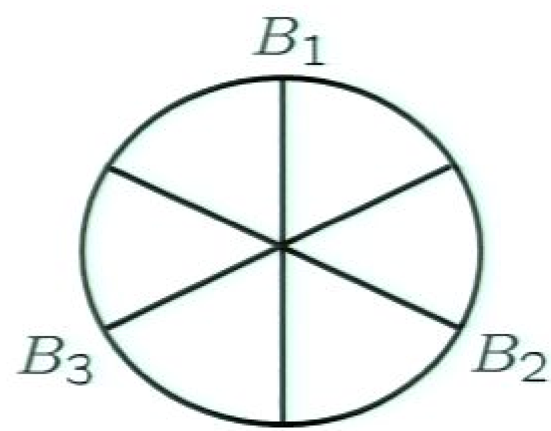
Local bound

Mermin's proof of Bell's theorem

Mermin, Phys. Today 38 (4), 38 (1985)







$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

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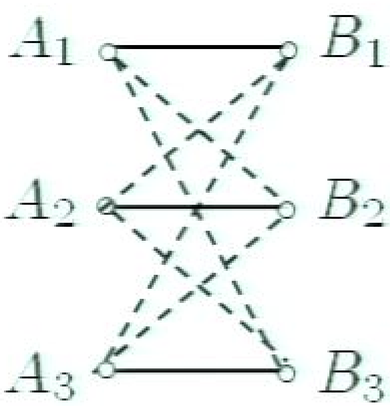

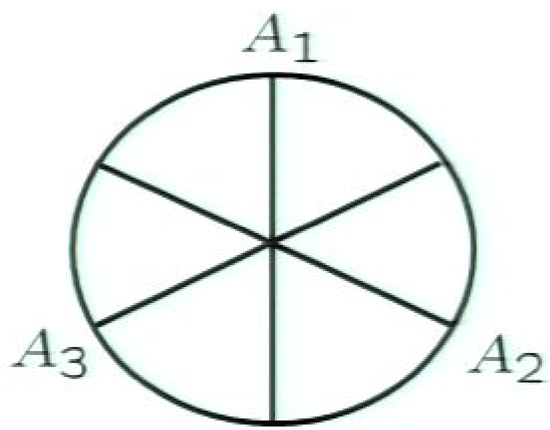

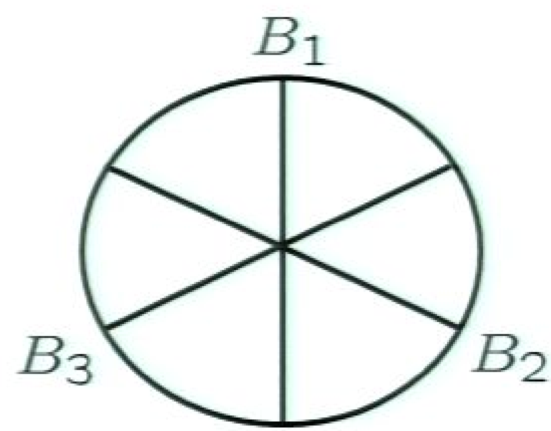

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$$p(A_k = B_k) = 1$$

$$p(A_k \neq B_{k\oplus 1}) = \cos^2(\tfrac{\pi}{6}) = \tfrac{3}{4}$$

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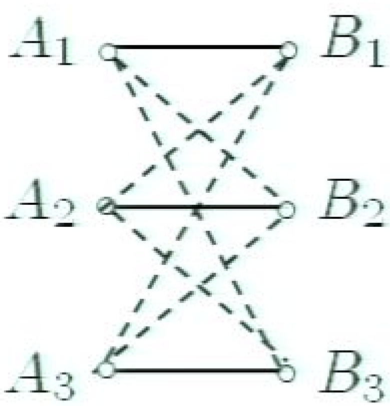

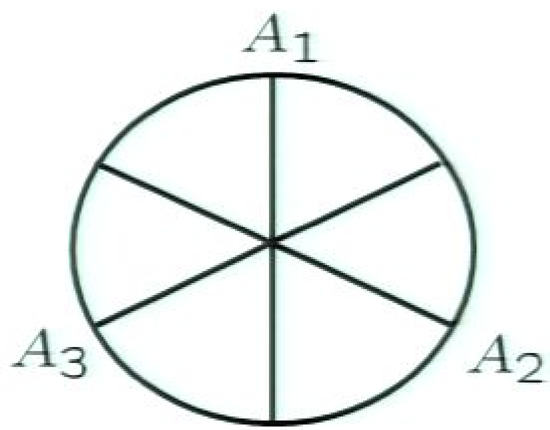

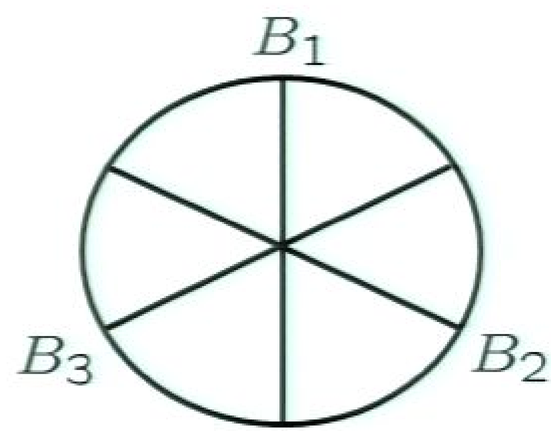

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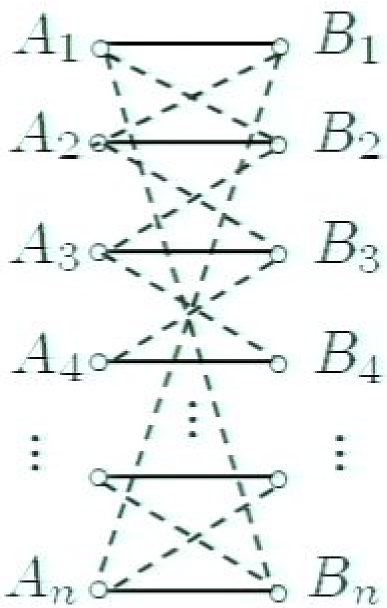
Quantum violation of Bell inequality

$$R = \tfrac{5}{6} > \tfrac{7}{9}$$

A separated pair of single-query **n**-box systems



n odd

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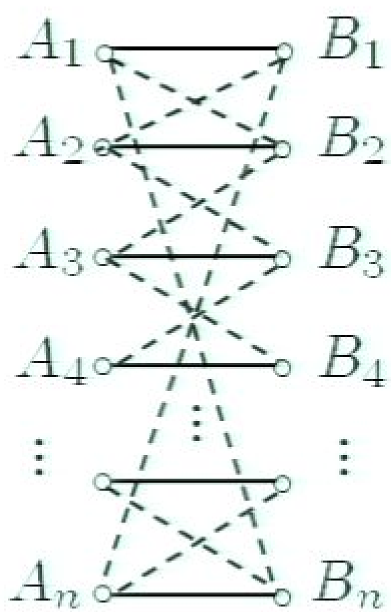

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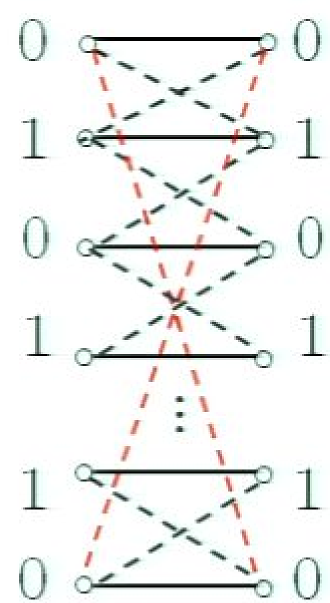
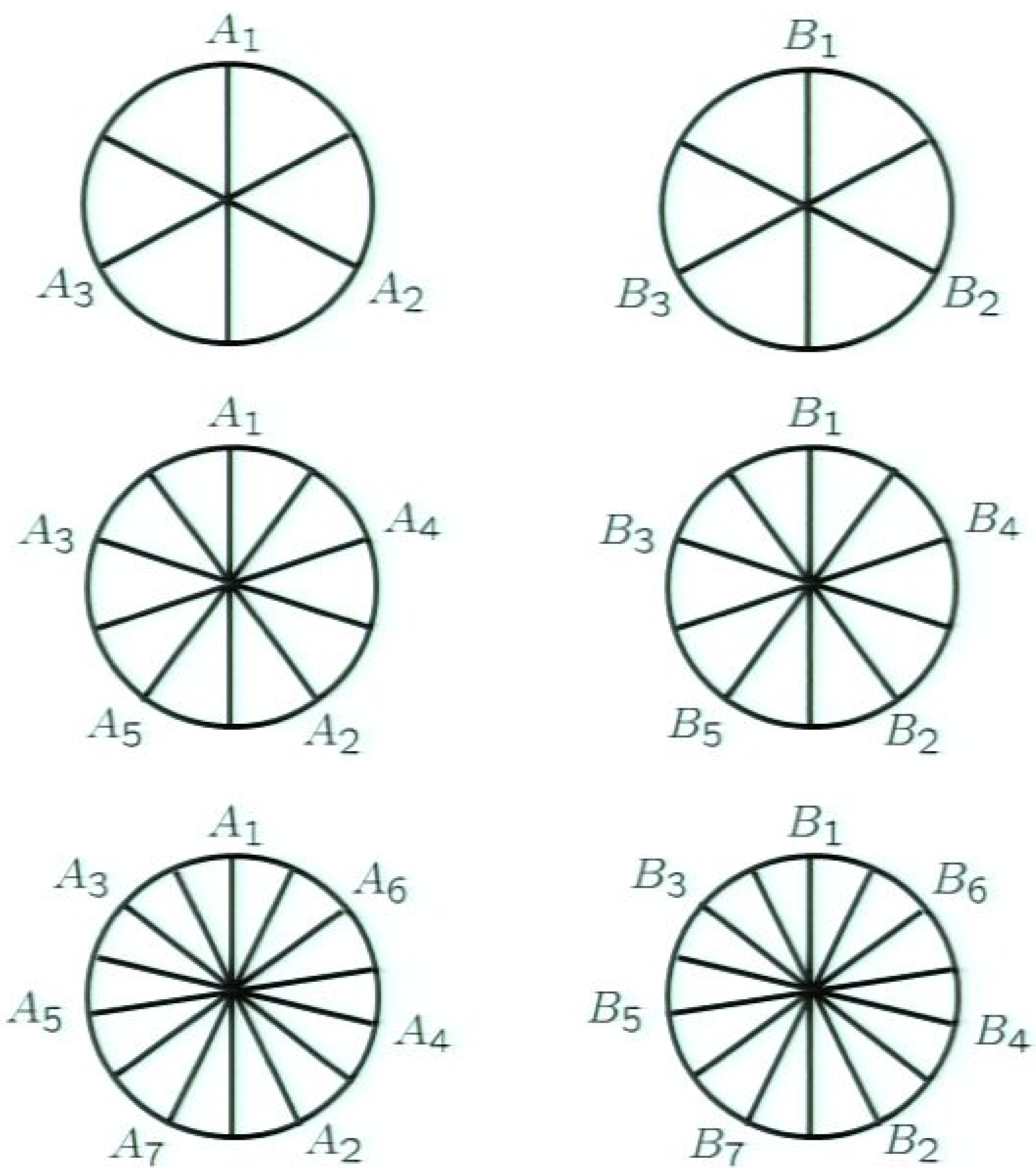


Generalization of Mermin's proof



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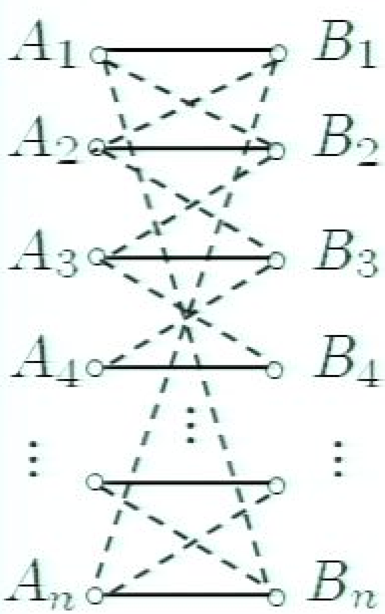

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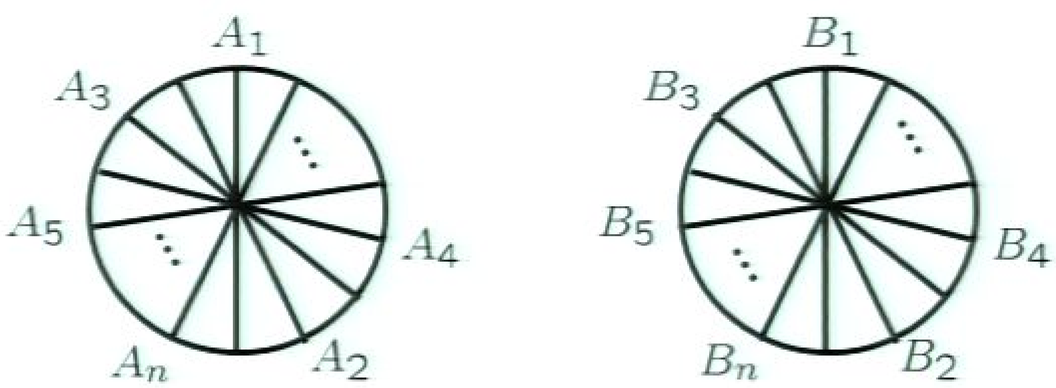

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$$\simeq 1 - \tfrac{\pi^2}{6n^2} \quad \text{as } n \to \infty$$

Quantum violation of Bell inequality

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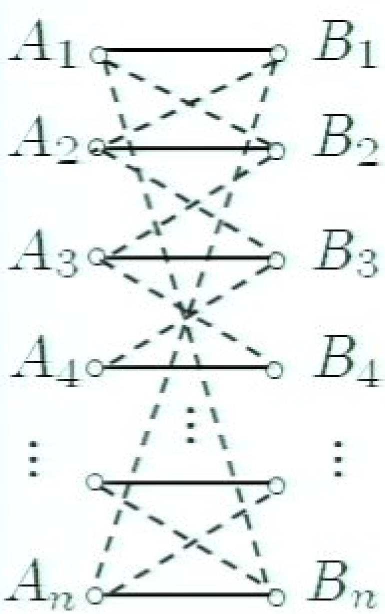

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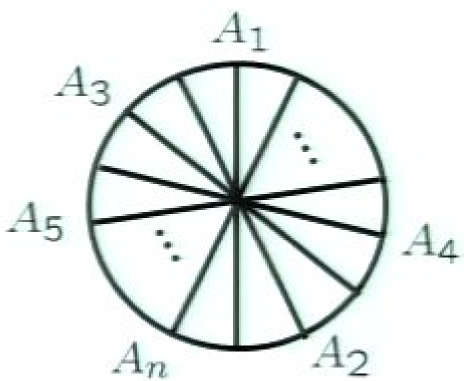

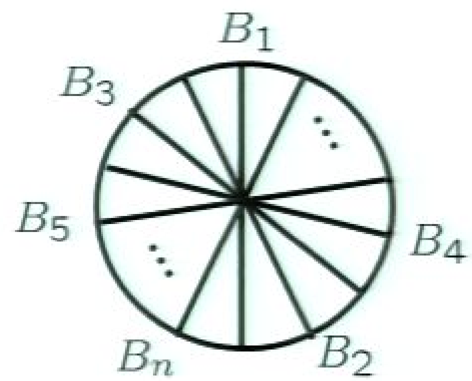

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Compare to:
Chained Bell
inequalities

Quantum violation of Bell inequality

$$R = \frac{1}{3} + \frac{2}{3}\cos^2(\frac{\pi}{3}) > 1 - \frac{2}{3}$$

Hardy's proof of Bell's theorem

Hardy, PRL 71, 1665 (1993)

2 settings at each wing

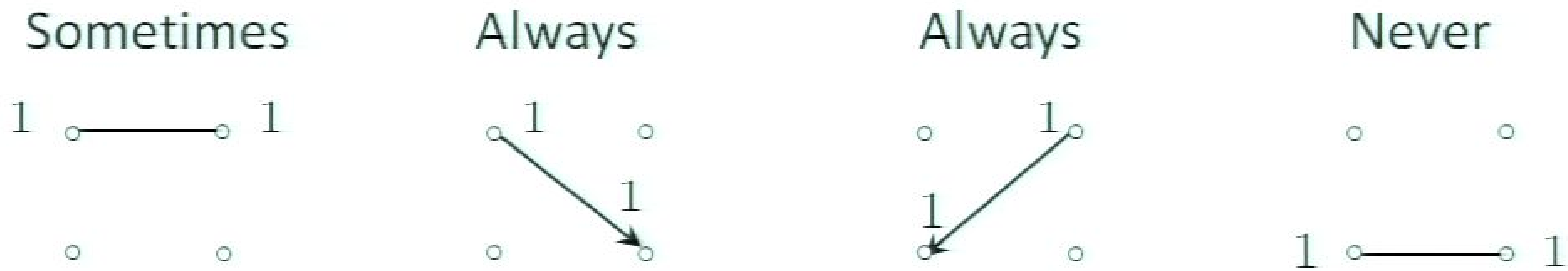
2 outcomes for each measurement

$A_1 \circ \qquad \circ B_1$

$A_2 \circ \qquad \circ B_2$

Hardy's proof of Bell's theorem

It is possible to find A_1, A_2, B_1 and B_2 and a state such that



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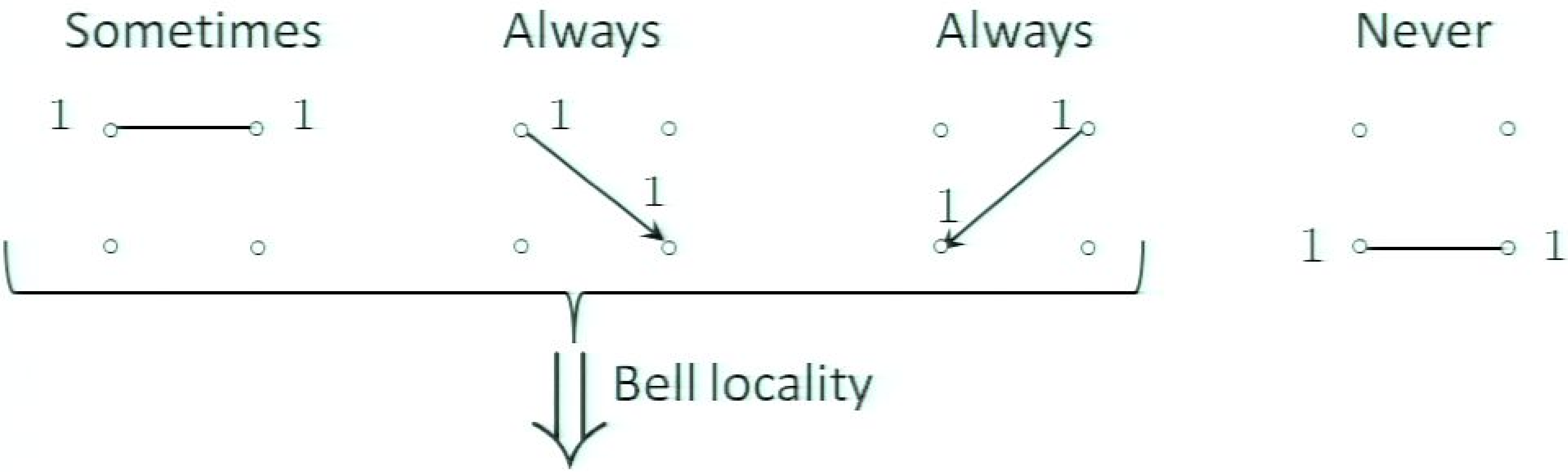
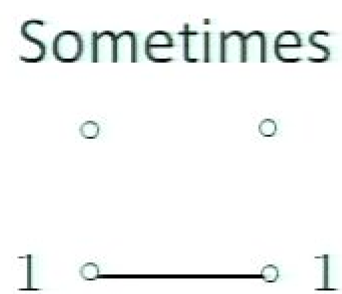
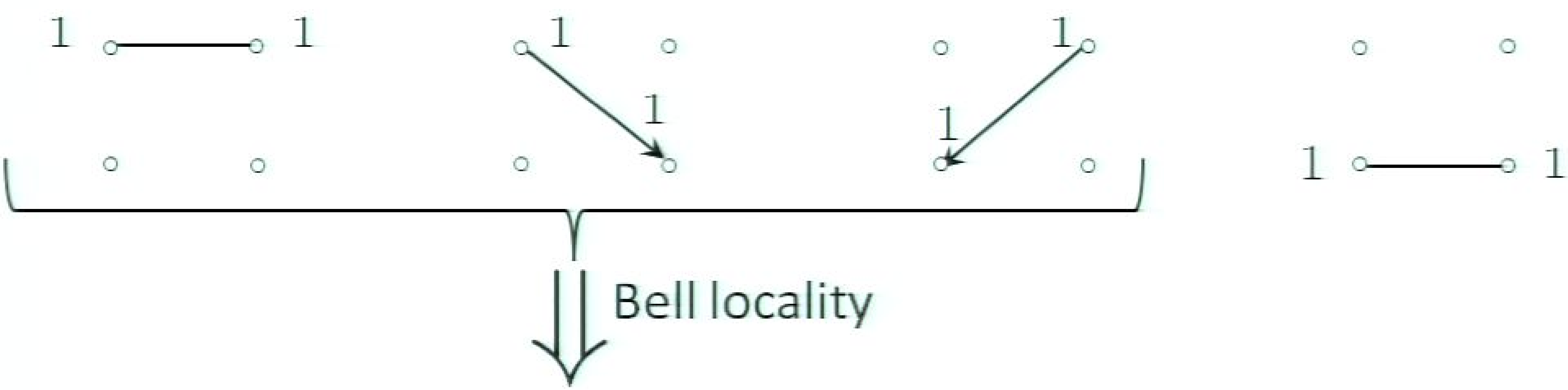




Hardy's proof of Bell's theorem

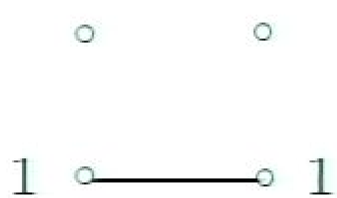
It is possible to find A_1, A_2, B_1 and B_2 and a state such that

Sometimes | Always | Always | Never



CONTRADICTION!

Sometimes



Recasting Hardy's proof as failure of transitivity

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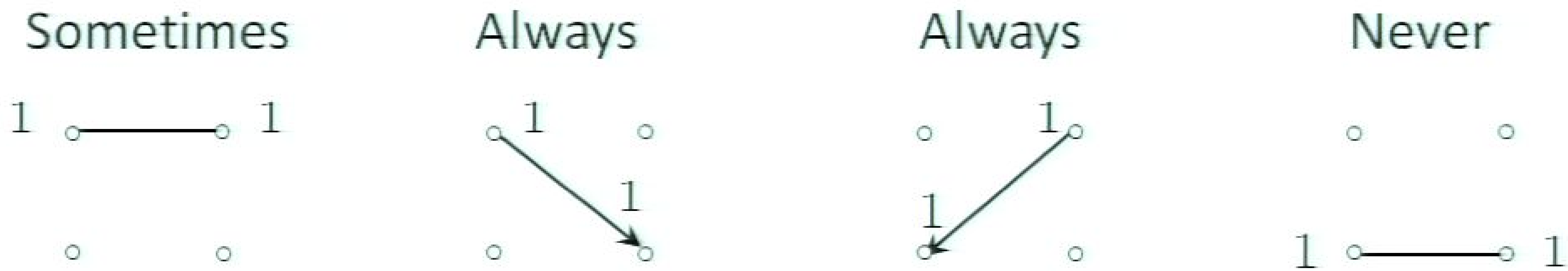

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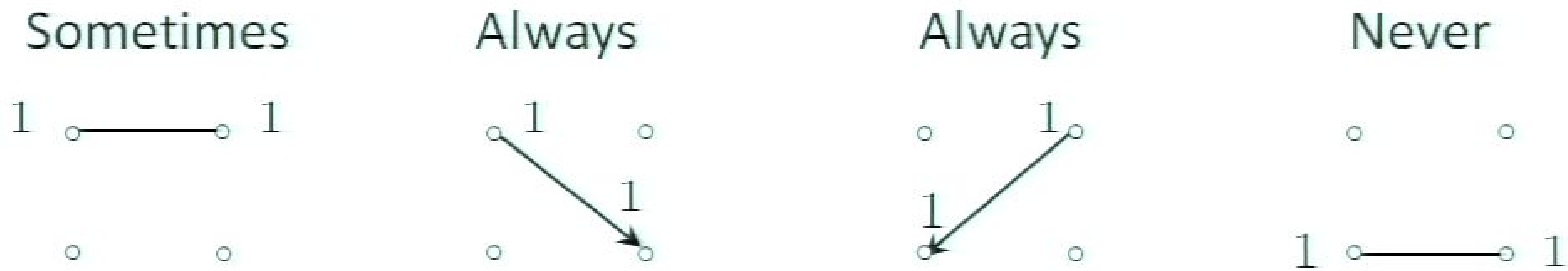

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Sometimes Always Always Always

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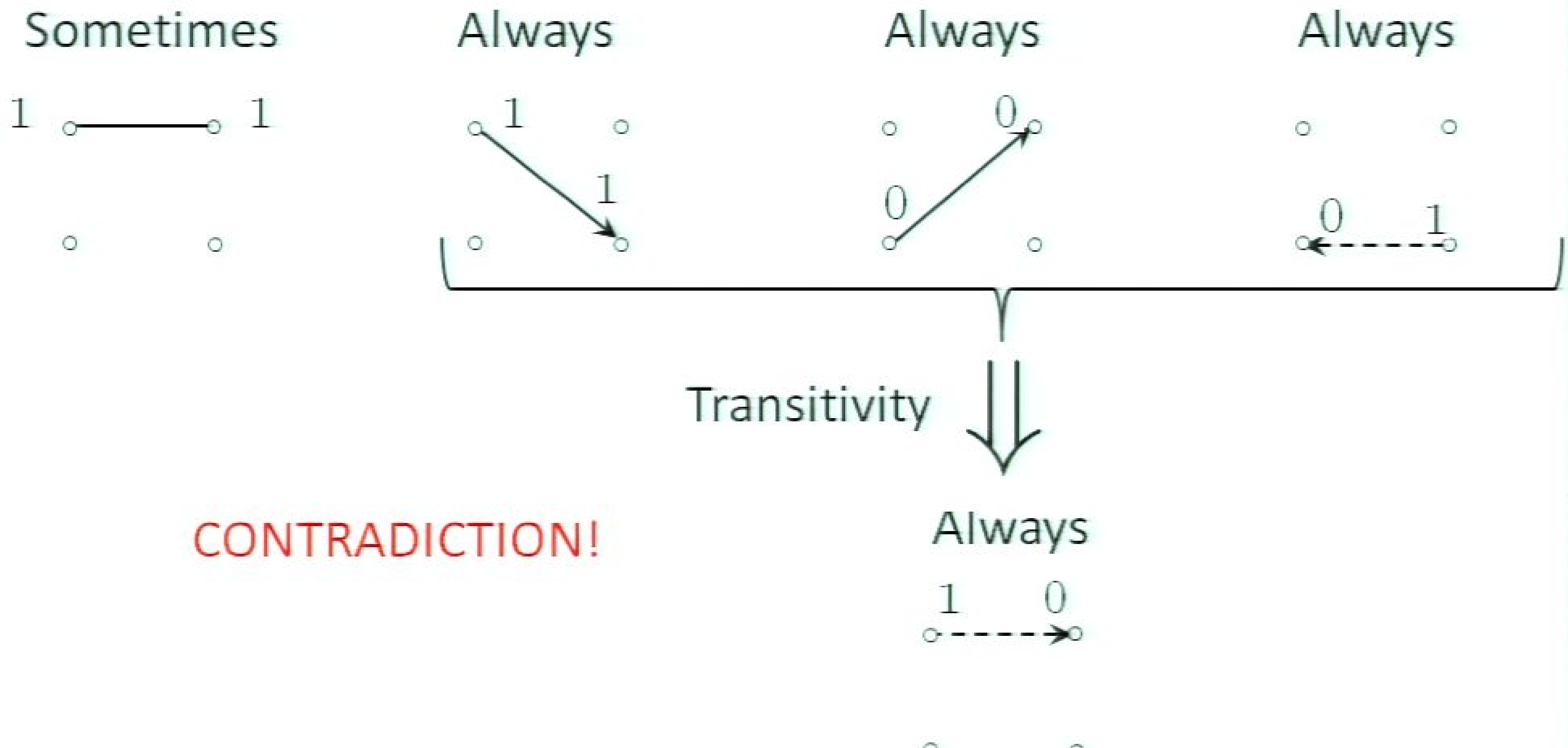
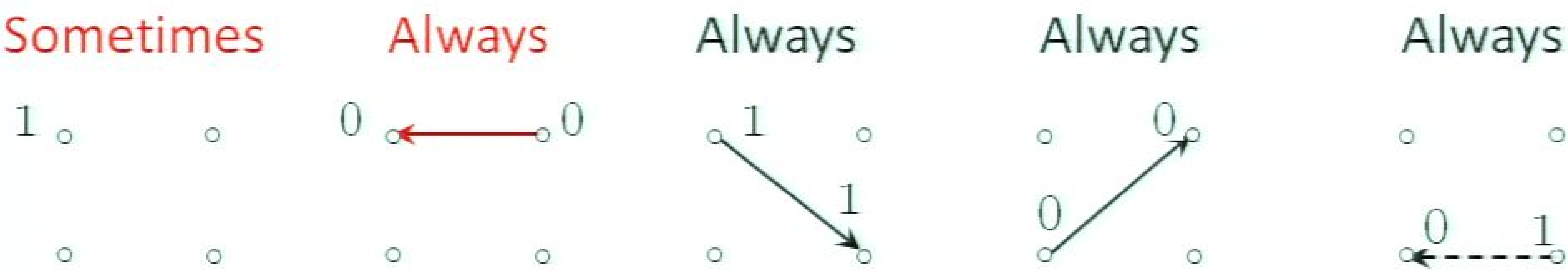


How the implications could be even more striking

Question: Can we find A_1, A_2, B_1 and B_2 and a state such that



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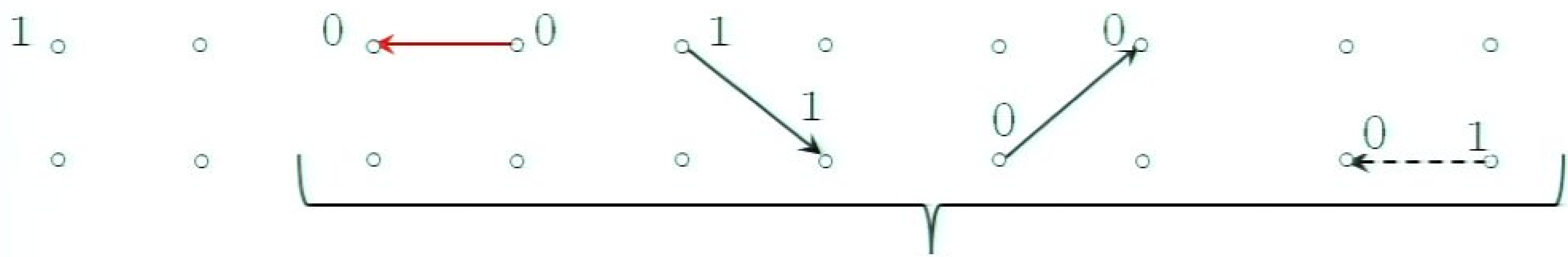
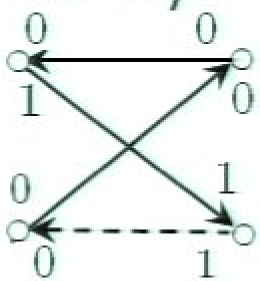


Transitivity $\Downarrow$

Always



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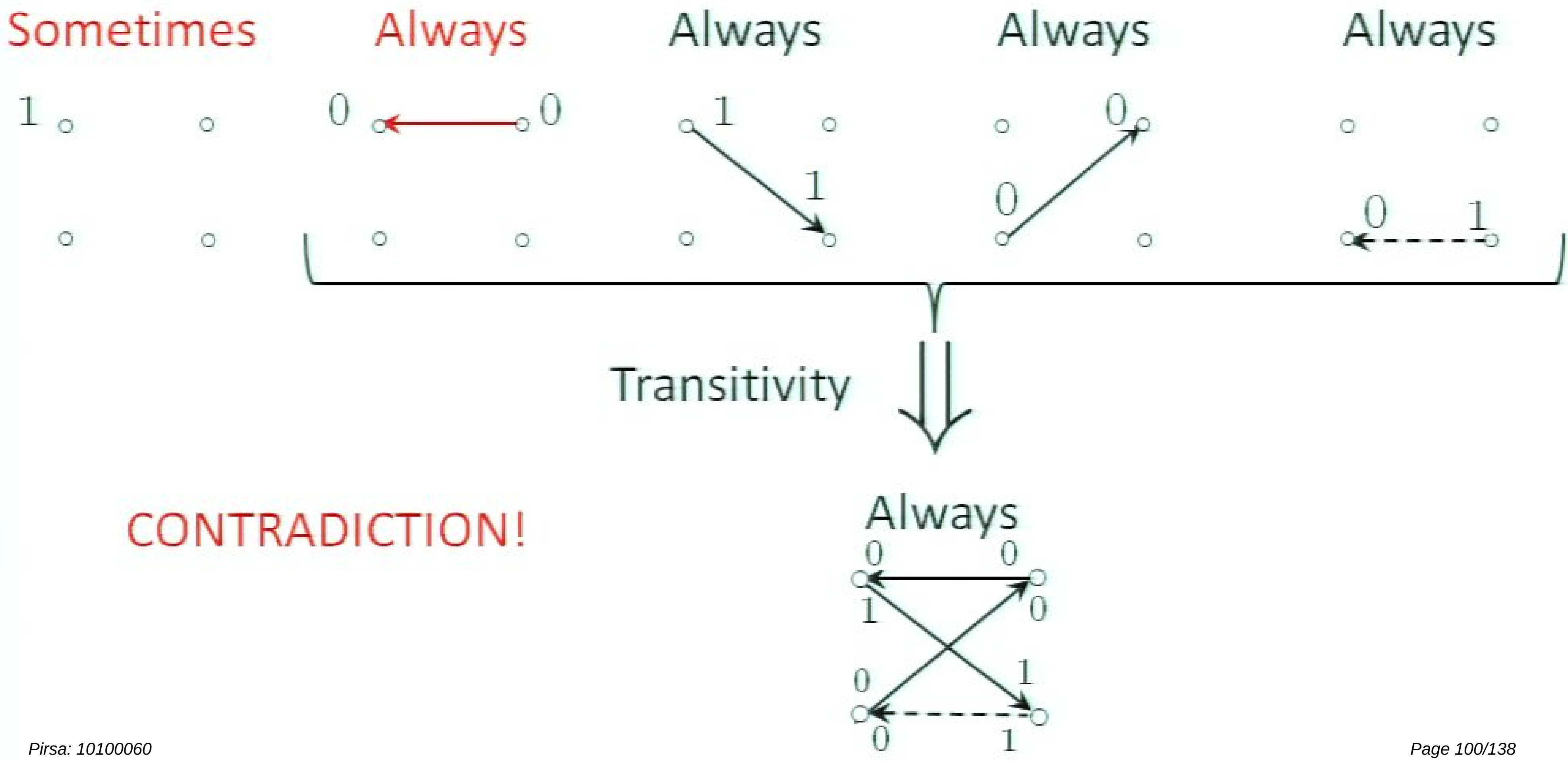

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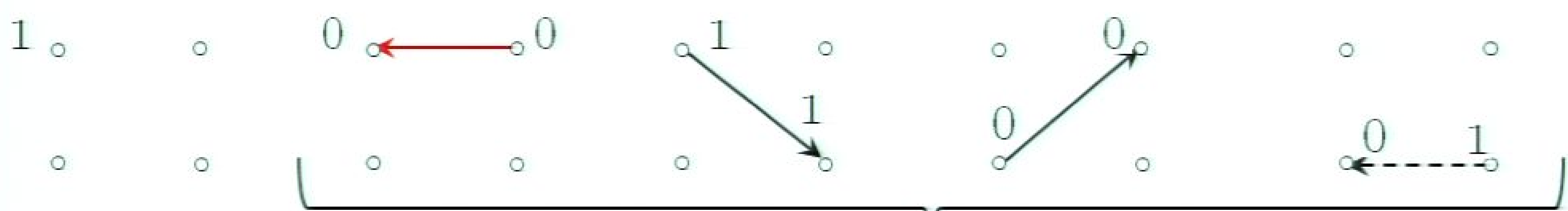
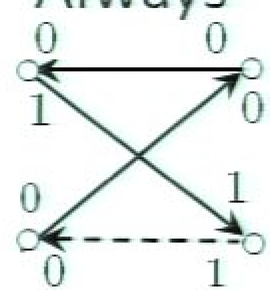


Transitivity ⇓

Always



CONTRADICTION!

Answer: NO!

How the implications could be even more striking

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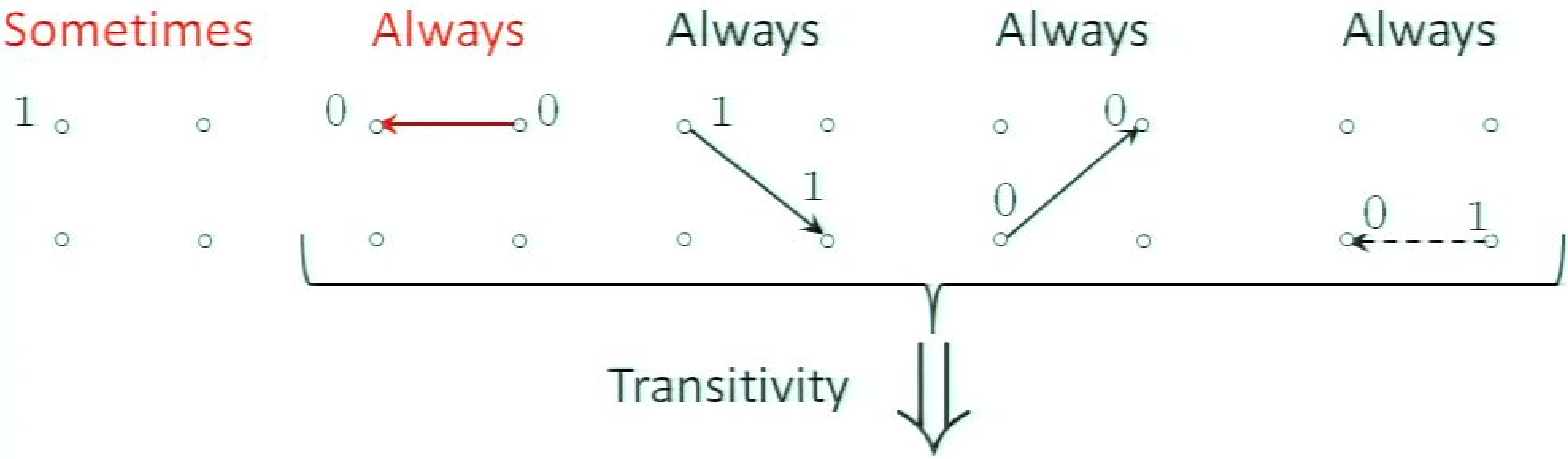

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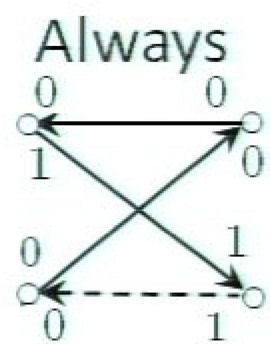

Answer: NO!

Theorem: No bipartite proof of nonlocality can have the form of a chain of implications where the final consequent denies the initial antecedent and the initial antecedent has nonzero probability.

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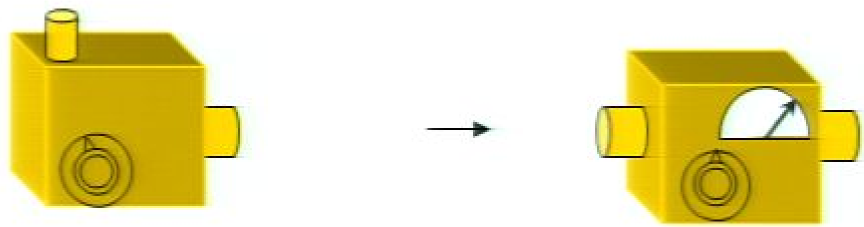
Why do there exist "striking" proofs (via the failure of transitivity) for the Kochen-Specker theorem, but not Bell's theorem?

Operational probabilistic theories



Preparation

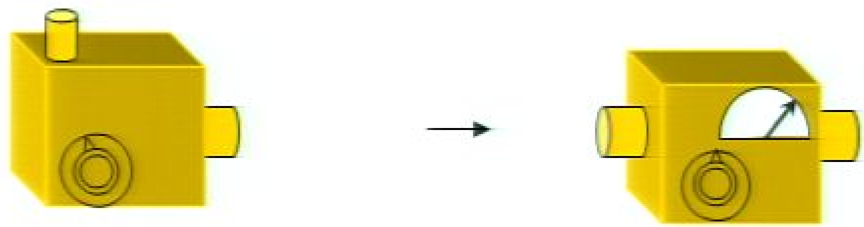
P

Measurement

M

These are defined as lists of instructions

Operational probabilistic theories



Preparation
P

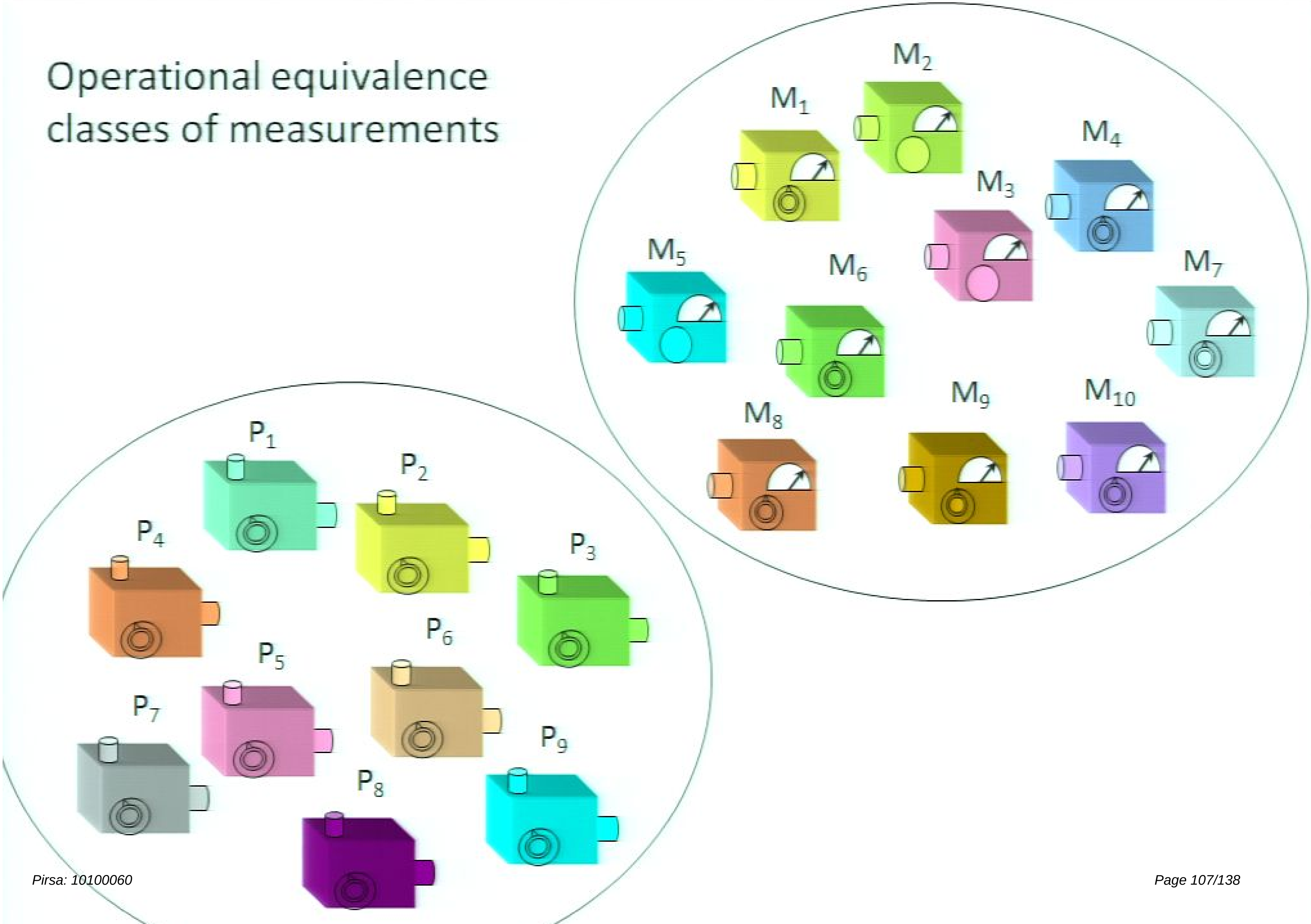
Measurement
M

These are defined as lists of instructions

An operational theory specifies

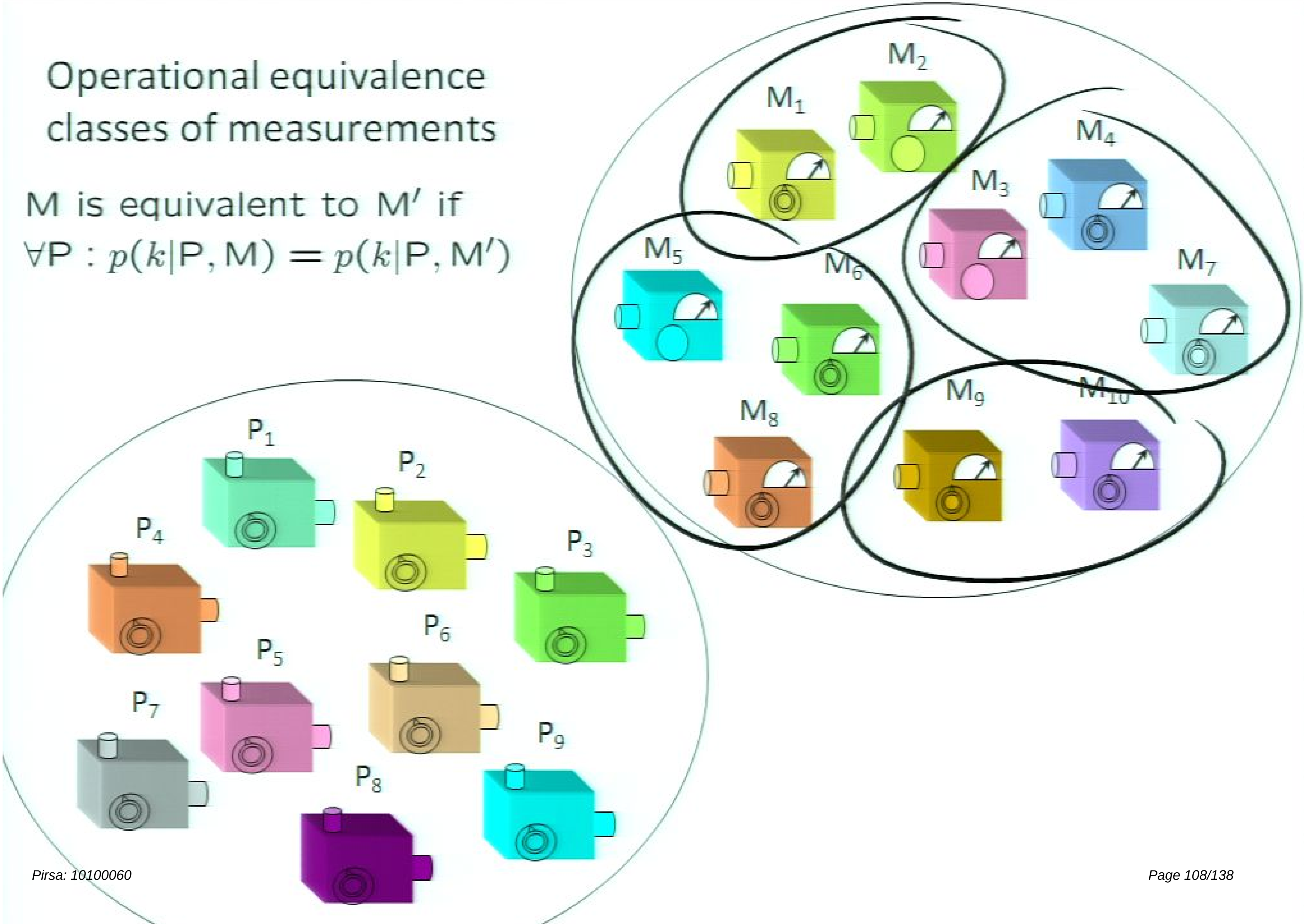
$p(k|\mathsf{P},\mathsf{M}) \equiv$ The probability of outcome k of M given P

Operational equivalence classes of measurements



Operational equivalence classes of measurements

M is equivalent to M′ if
$\forall \mathsf{P} : p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P},\mathsf{M}')$



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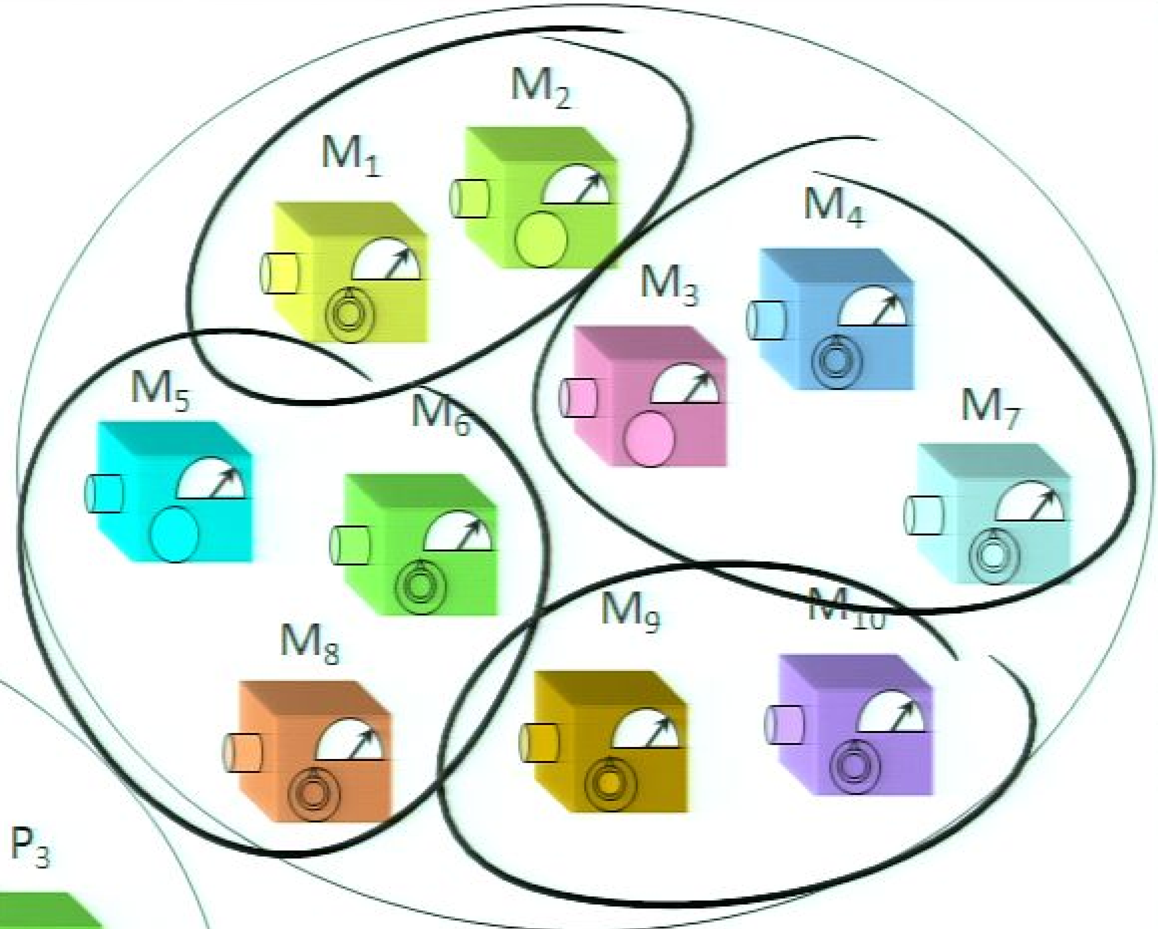
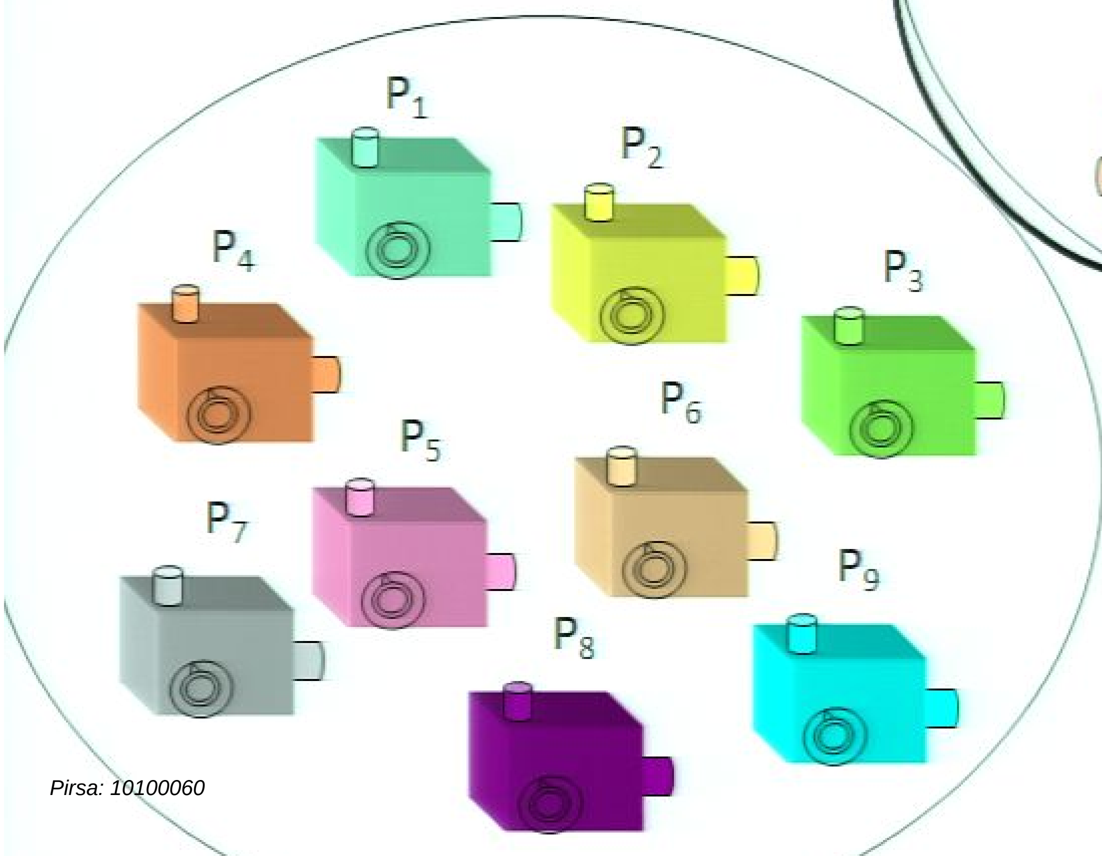
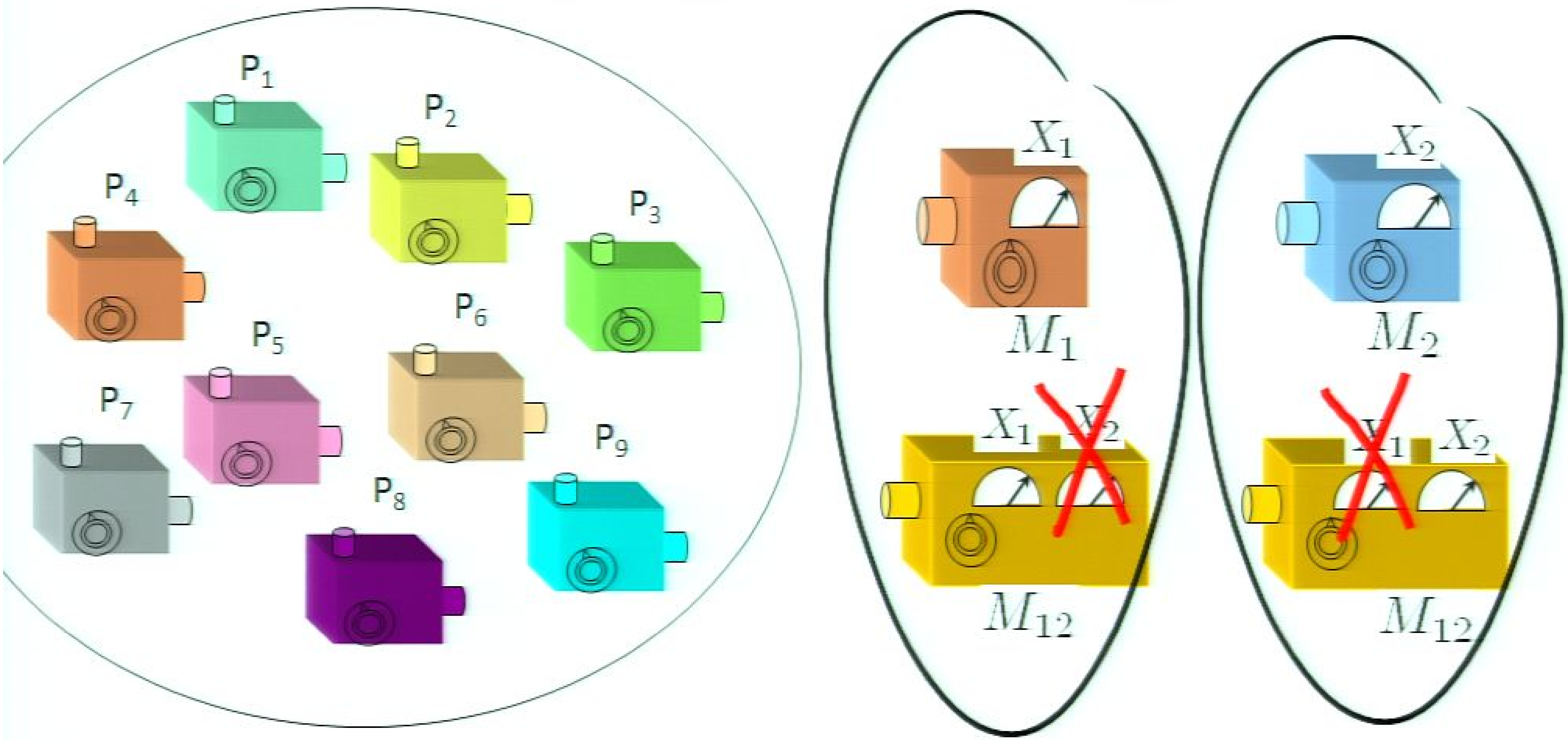




Denote the class of M by $\mathcal{M}$

Joint Measurability #1 (joint simulatibility)



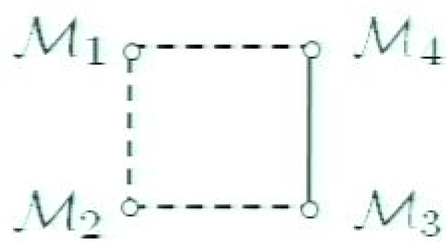
$\mathcal{M}_1$ and $\mathcal{M}_2$ are jointly measurable if $\exists \mathcal{M}_{12}$ such that

$$\forall \mathcal{P} : p(X_1|\mathcal{M}_1, \mathcal{P}) = \sum_{X_2} p(X_1, X_2|\mathcal{M}_{12}, \mathcal{P})$$

$$\forall \mathcal{P} : p(X_2|\mathcal{M}_2, \mathcal{P}) = \sum p(X_1, X_2|\mathcal{M}_{12}, \mathcal{P})$$

Ruling out extremal frustrated network correlations by ruling out OS correlations

Suppose a theory has



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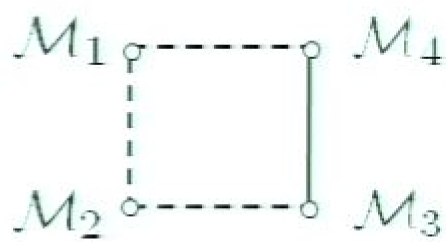
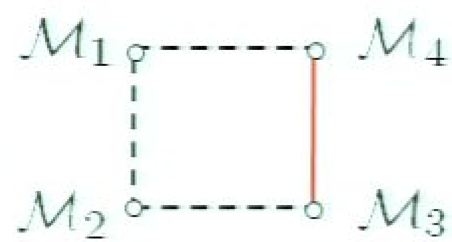


Consider the set of preparations for which



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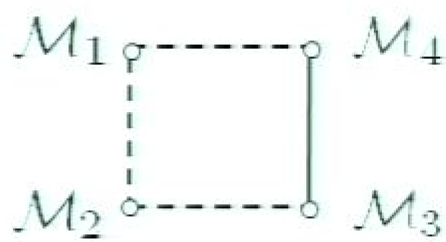

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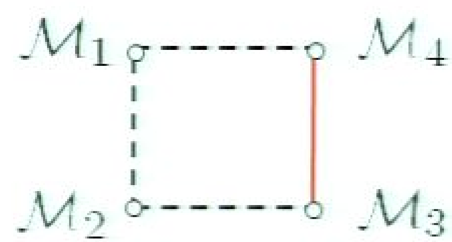

On this set, $\mathcal{M}_3$ and $\mathcal{M}_4$ are operationally equivalent

Ruling out extremal frustrated network correlations by ruling out OS correlations

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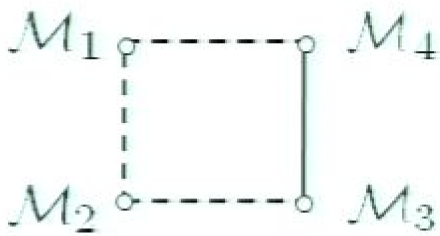

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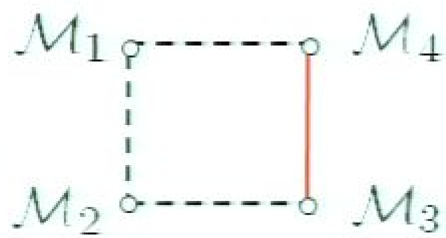
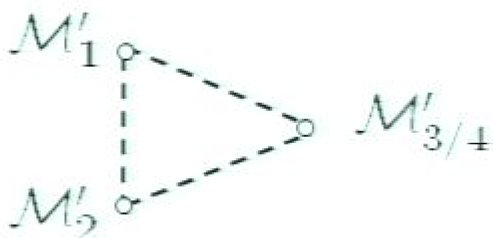
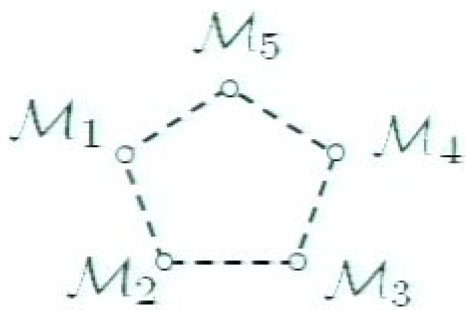


On this set, $\mathcal{M}_3$ and $\mathcal{M}_4$ are operationally equivalent



Ruling out extremal frustrated network correlations by ruling out OS correlations

Suppose a theory has 

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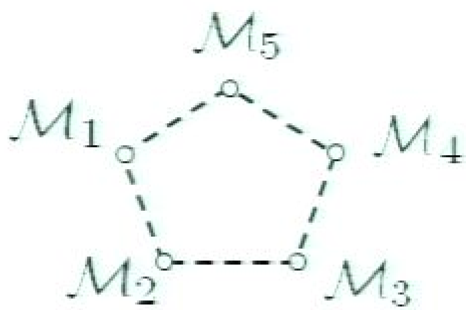

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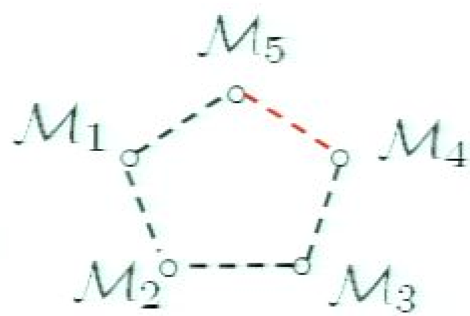

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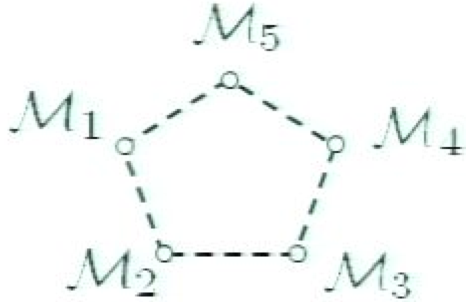

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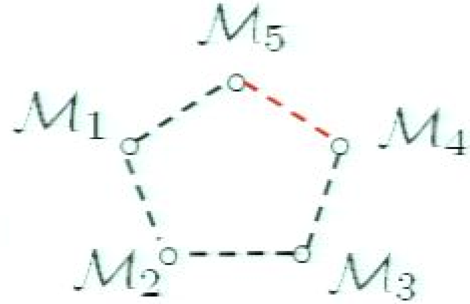
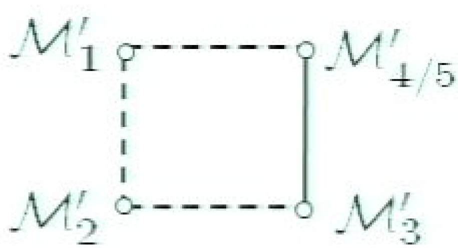


On this set, (a relabelling of) $\mathcal{M}_4$ and $\mathcal{M}_5$ are op'tly equivalent



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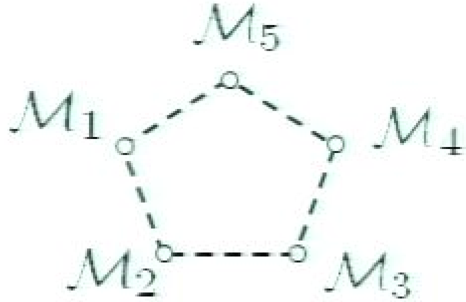

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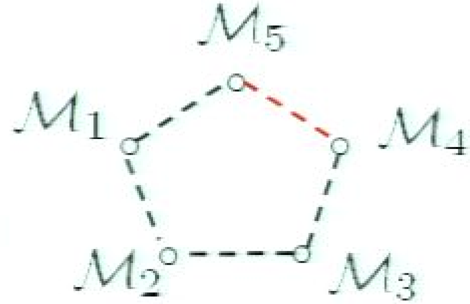

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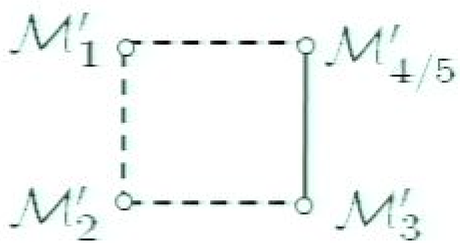
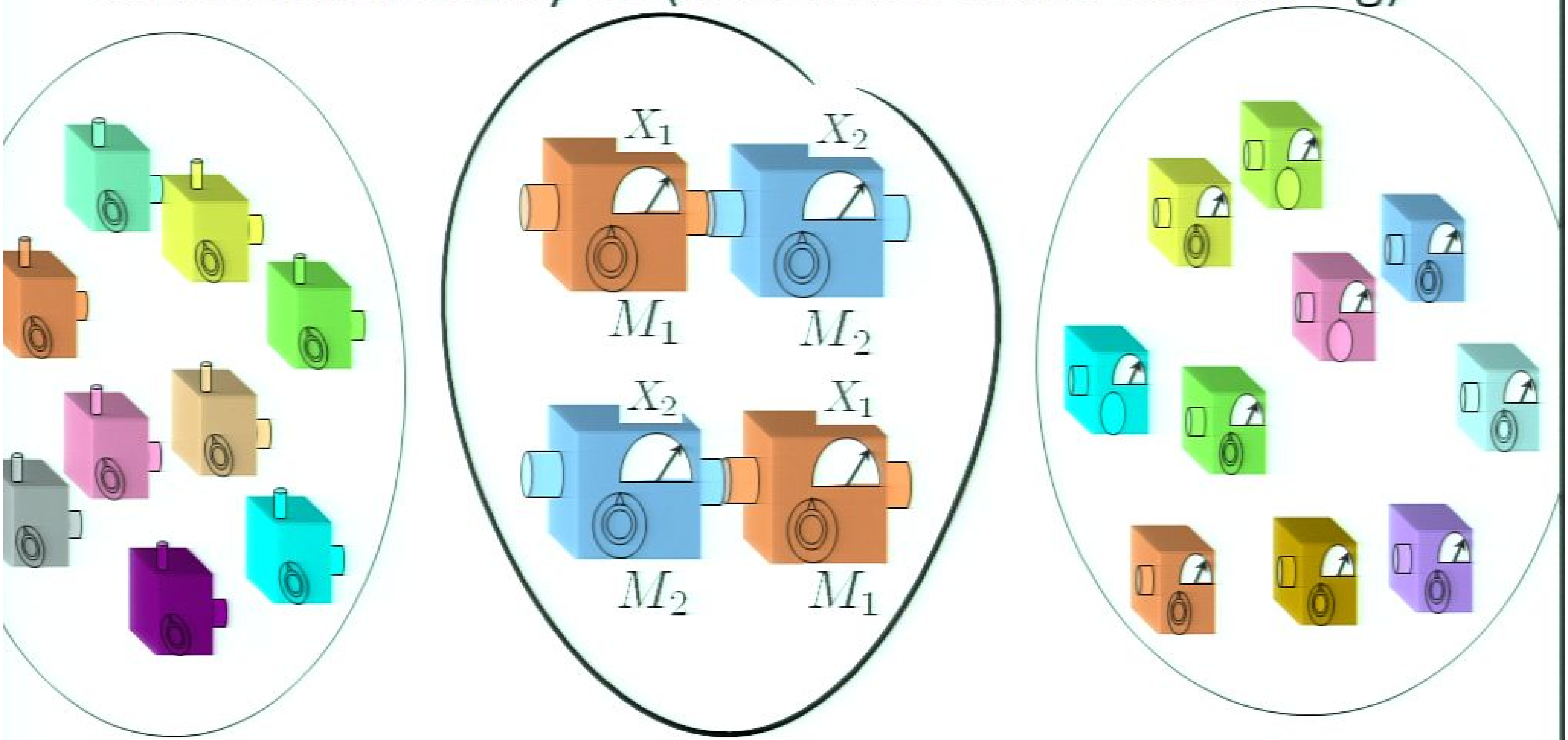


Any principle that rules out the OS correlations also rules out all other extremal correlations for frustrated networks

Joint Measurability #2 (invariance under reordering)



$\mathcal{M}_1$ and $\mathcal{M}_2$ are jointly measurable if $\exists M_1 \in \mathcal{M}_1$ and $\exists M_2 \in \mathcal{M}_2$ such that

$$\forall \mathcal{P} \forall \mathcal{M}_3 : p(X_1, X_2, X_3 | \mathcal{P} \text{ then } M_1 \text{ then } M_2 \text{ then } \mathcal{M}_3)$$
$$= p(X_1, X_2, X_3 | \mathcal{P} \text{ then } M_2 \text{ then } M_1 \text{ then } \mathcal{M}_3)$$

Lemma A: Pairwise JM #2 $\Rightarrow$ Triplewise JM #2

Proof:

1 2 3
1 3 2
2 1 3
2 3 1
3 1 2
3 2 1

Lemma B: JM #2 (invariance under reordering)

$\Rightarrow$ JM #1 (joint simulatibility)

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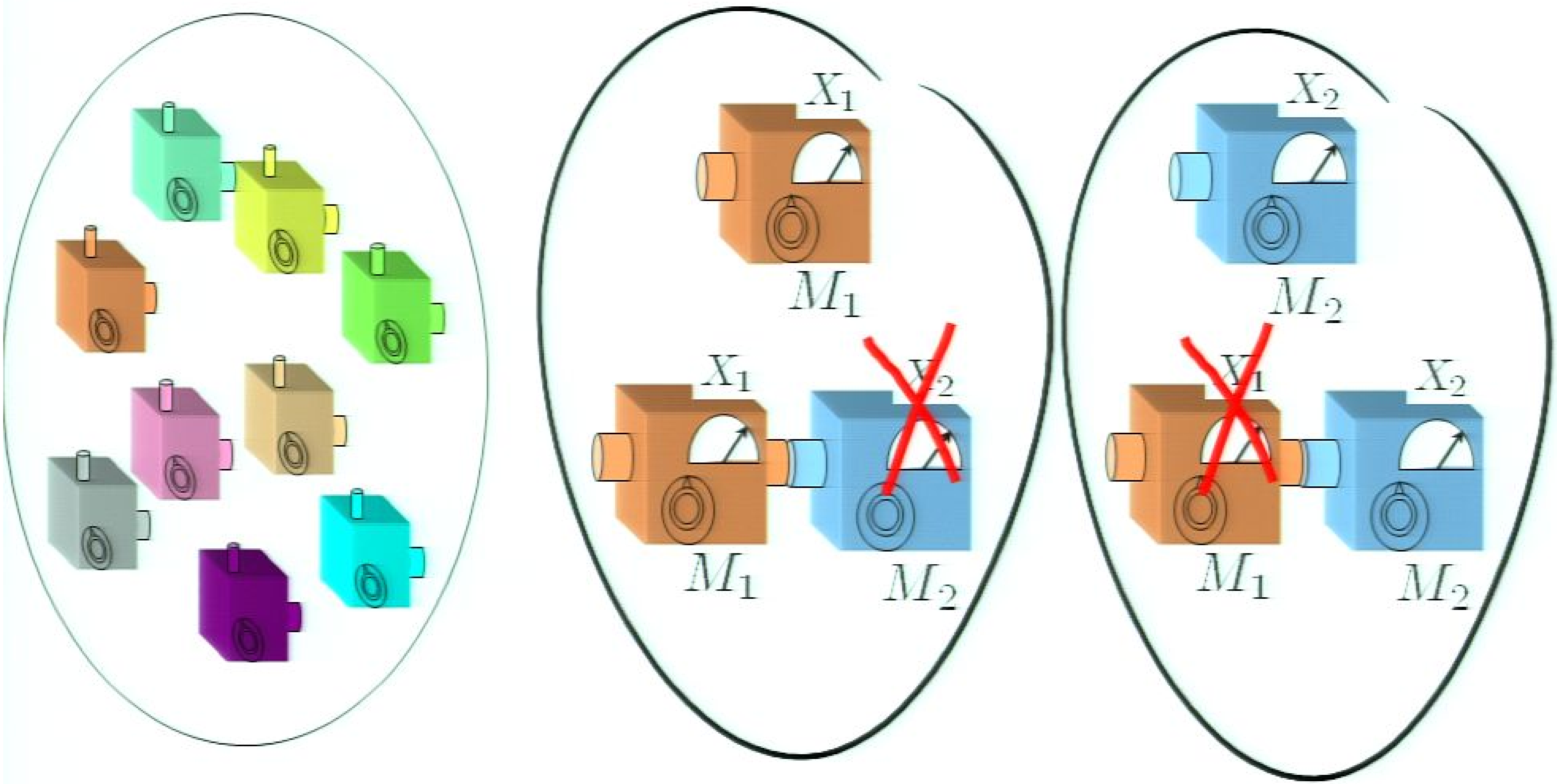

Lemma B: JM #2 (invariance under reordering)
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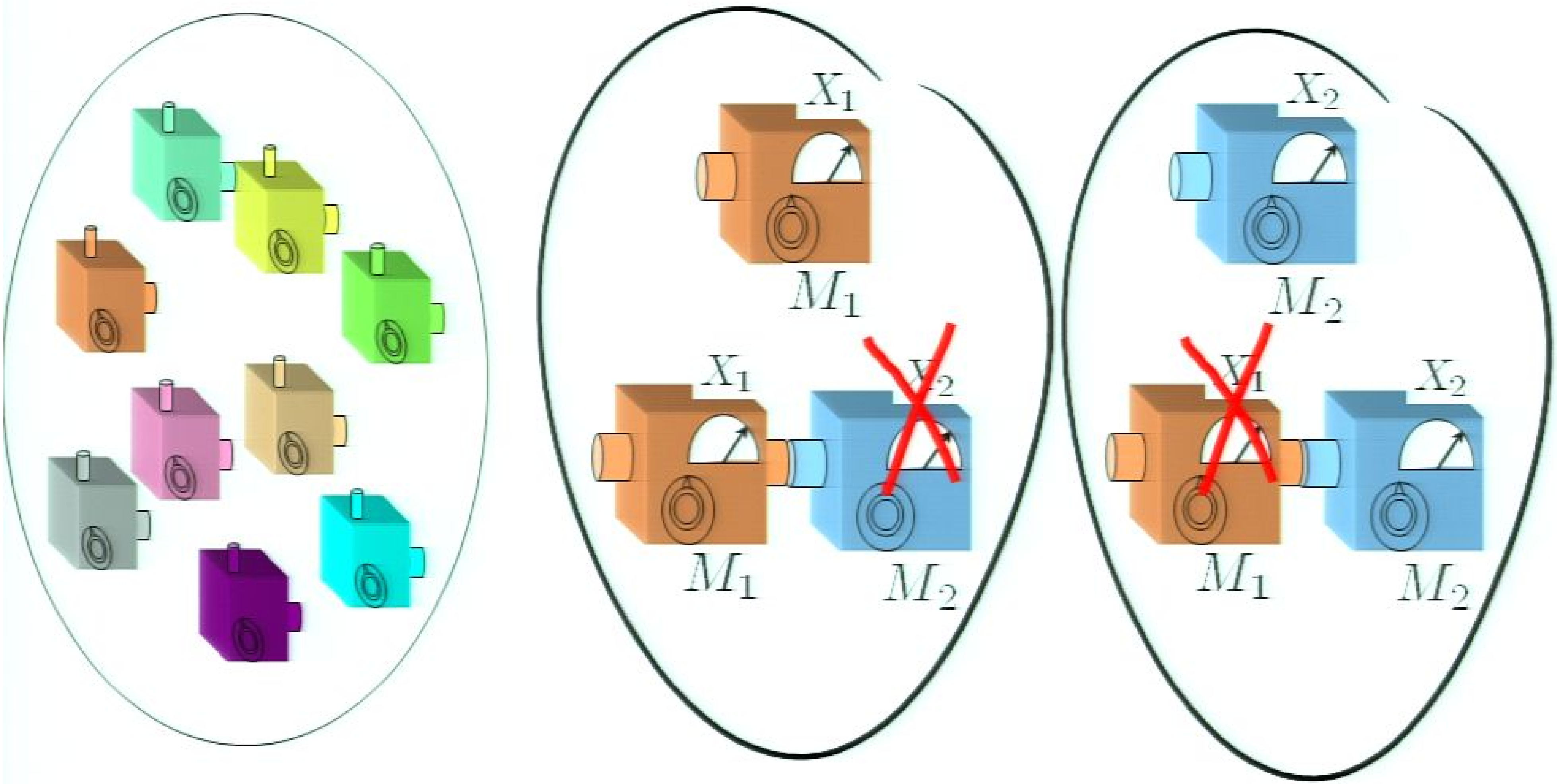

Open question: Are there cases in which JM #1 $\not\Rightarrow$ JM #2 ?

A possible argument against OS correlations:

Pairwise JM #1

$\Downarrow$????????

Pairwise JM #2

$\Downarrow$ lemma A

Triplewise JM #2

$\Downarrow$ lemma B

Triplewise JM #1

Research Directions

- In a general theory, for which measurements is it justified to assume deterministic outcomes in the ontological model?
- Can one find principles which disqualify the OS correlations?
- Counterfactual retrodiction (failure of joint measurability) vs. counterfactual prediction (standard uncertainty relations). Can we derive the precise amount of contextuality in quantum theory from time symmetry?

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Pairwise JM #2

$\Downarrow$ lemma A

Triplewise JM #2

$\Downarrow$ lemma B

Triplewise JM #1

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Ruling out extremal frustrated network correlations by ruling out OS correlations

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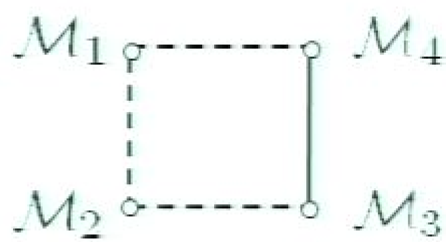
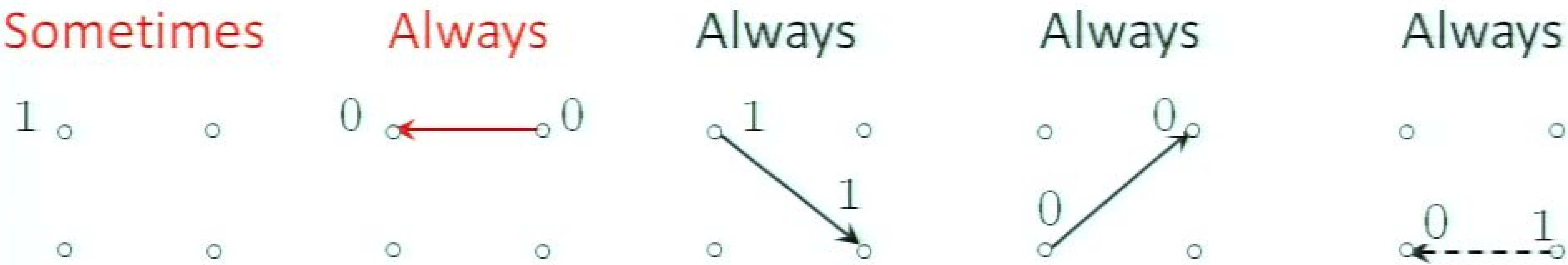


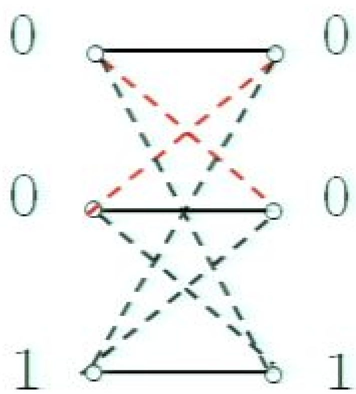
How the implications could be even more striking

Question: Can we find A_1, A_2, B_1 and B_2 and a state such that



Bell inequality

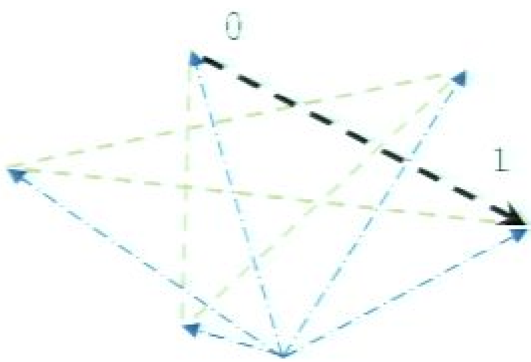
Recall: Bell locality + perfect correlation
$\Rightarrow$ determinstic noncontextual values



$$R \leq \frac{7}{9}$$

Local bound

A novel proof of the Kochen-Specker theorem based on the failure of transitivity of implication



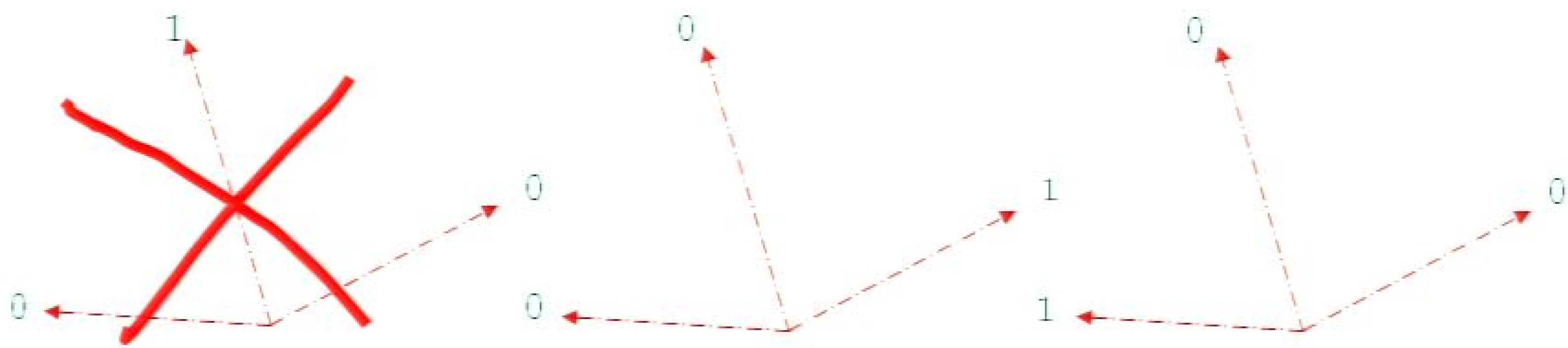
NONETHELESS

for all states $|\psi\rangle \in \mathrm{span}(|l_1\rangle, |l_2\rangle)$

If $|l_1\rangle \mid |l_2\rangle$

Then $v(|l_1\rangle\langle l_1|) = 0 \implies v(|l_2\rangle\langle l_2|) = 1$

Because:



How the seer can achieve his ends in a quantum world

	Kochen-Specker bound	Quantum bound
n odd	$R \leq \frac{n-1}{n}$ $= 1 - \frac{1}{n}$	$R \leq \frac{2\cos(\frac{\pi}{n})}{1+\cos(\frac{\pi}{n})}$ $\simeq 1 - \frac{\pi^2}{4n^2}$

The seer's challenge to the suitor:
identify a correlated pair of boxes

Suitor's expected prob. of winning: $\frac{1}{n}$

Suitor's actual prob. of winning: $\simeq O(\frac{1}{n^2})$

Suppose $n \ll \text{no. of suitors} \ll n^2$

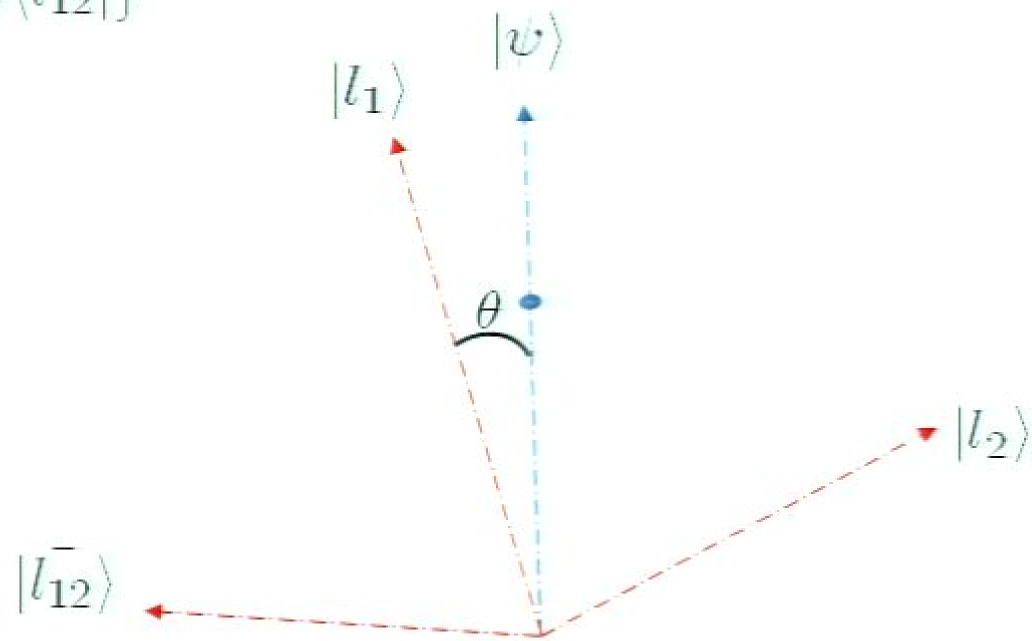
Suitors believe it is very likely that one of them will win

In fact it is very likely that none of them will win

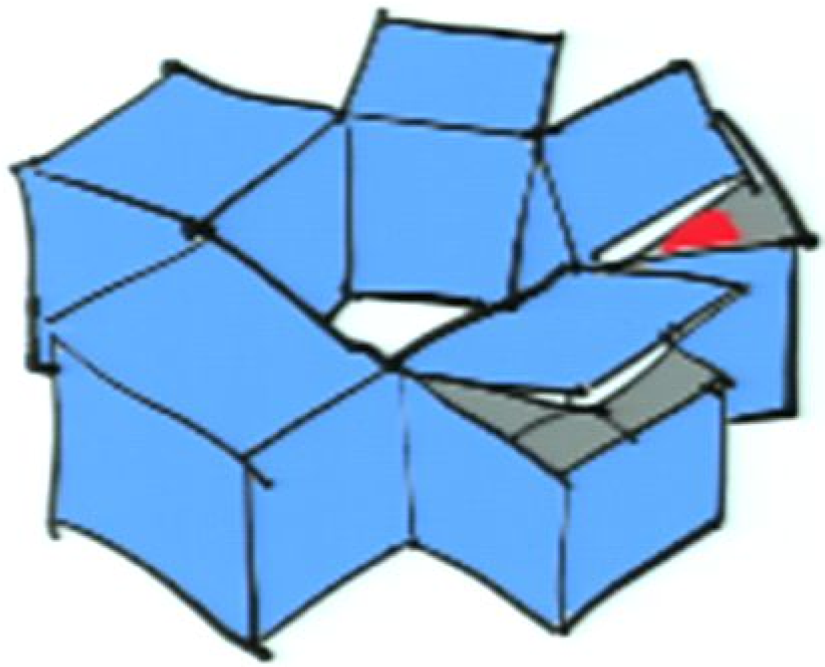
Klyachko's proof of the Kochen-Specker theorem

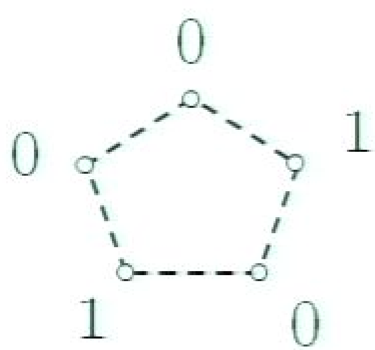
$$\cos^2 \theta = \frac{1}{\sqrt{5}}$$

Consider: $\{|l_1\rangle\langle l_1| \ , \ |l_2\rangle\langle l_2| \ , \ |\bar{l_{12}}\rangle\langle \bar{l_{12}}|\}$



Double-query 5-box system allowing only **adjacent** queries





$$R \leq \frac{4}{5}$$

Frustrated Networks

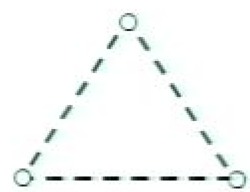
Nodes are binary variables

Edges imply joint measurability

 Perfect correlation

 Perfect anti-correlation

E.g. Correlations in the parable of the overprotective seer:



Frustration = no valuation satisfying all correlations

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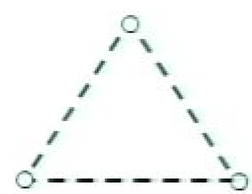


Joint measurability of POVMs

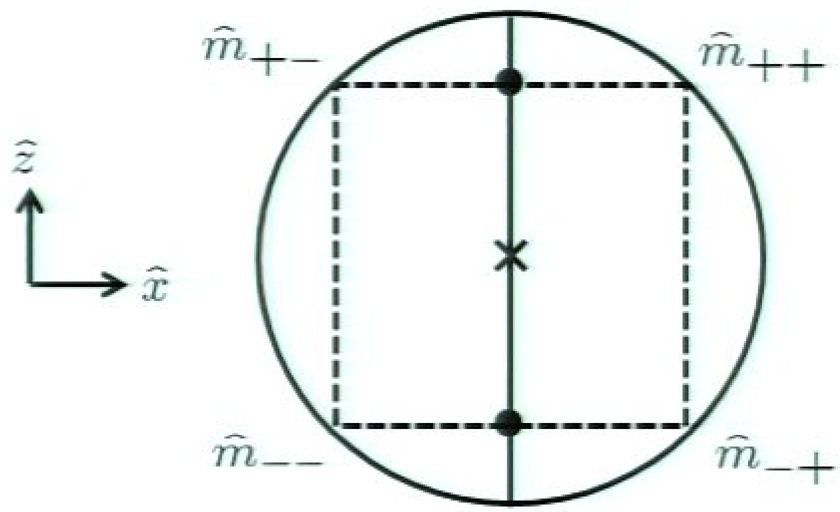
Noisy z-spin $\{E^z_+, E^z_-\}$

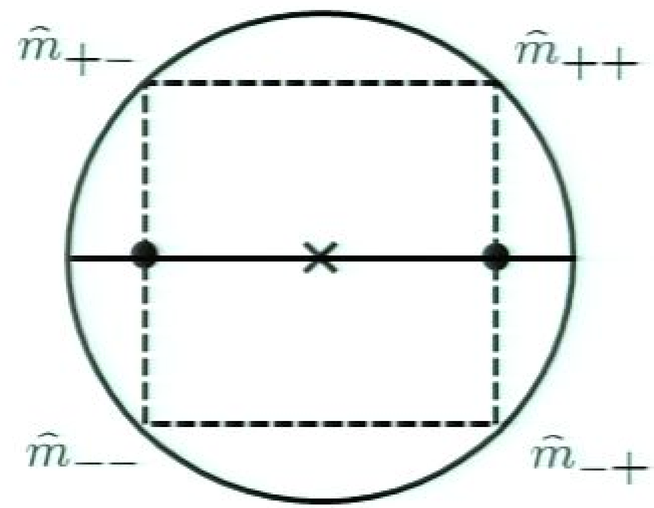
Noisy x-spin $\{E^x_+, E^x_-\}$

Noisy y-spin $\{E^y_+, E^y_-\}$

$$E^z_\pm = \eta | \pm z \rangle \langle \pm z | + (1-\eta)\frac{I}{2}$$

η = sharpness factor





Pairwise JM iff $\quad \eta \leq \frac{1}{\sqrt{2}}$

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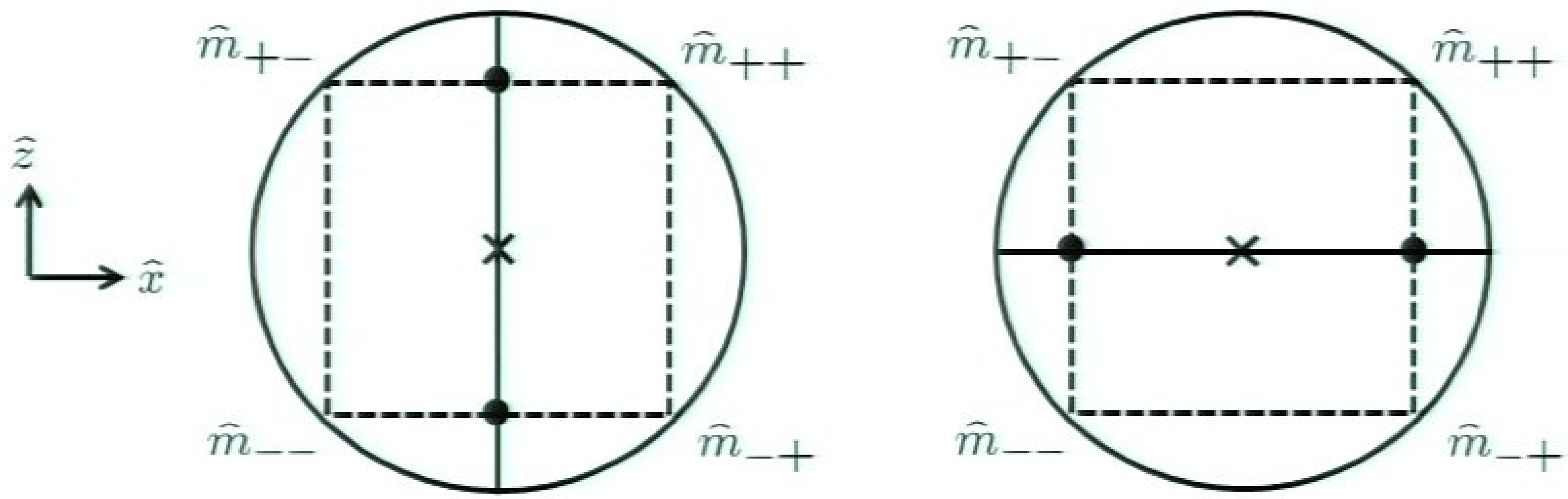

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