Title: Open systems in modal quantum theory

Date: Oct 05, 2010 04:00 PM

URL: http://pirsa.org/10100050

Abstract: In this talk we quickly review the basics of the modal "toy model" of quantum theory described by Schumacher in his September 22 colloquium at PI. We then consider how the theory addresses more general open systems. Because the modal theory has a more primitive mathematical structure than actual quantum mechanics, it lacks density operators, positive operator measurements, and completely positive maps. As we will show, however, modal quantum theory has an elegant description of the states, effects and operations of open modal systems -- a description with close analogies to actual quantum mechanics.

Open Systems in Modal Quantum Theory

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Modal quantum theory

<u>States</u>

A system is described by a vector space V over a field F.

- F might be finite.
- V need have no norm or inner product.
- The state of a system may be any non-zero $|\phi) \in \mathcal{V}$

<u>Basic measurements</u>

$$\{(a|\} \leftrightarrow \{|a)\}$$

A basic measurement is a basis $\{ (a| \}$ for V*.

Each element $(a|$ is an "effect" associated with a result of the measurement

The result a is possible if $(a|\phi) \neq 0$

$$|\phi) = \sum_a \phi_a |a)$$

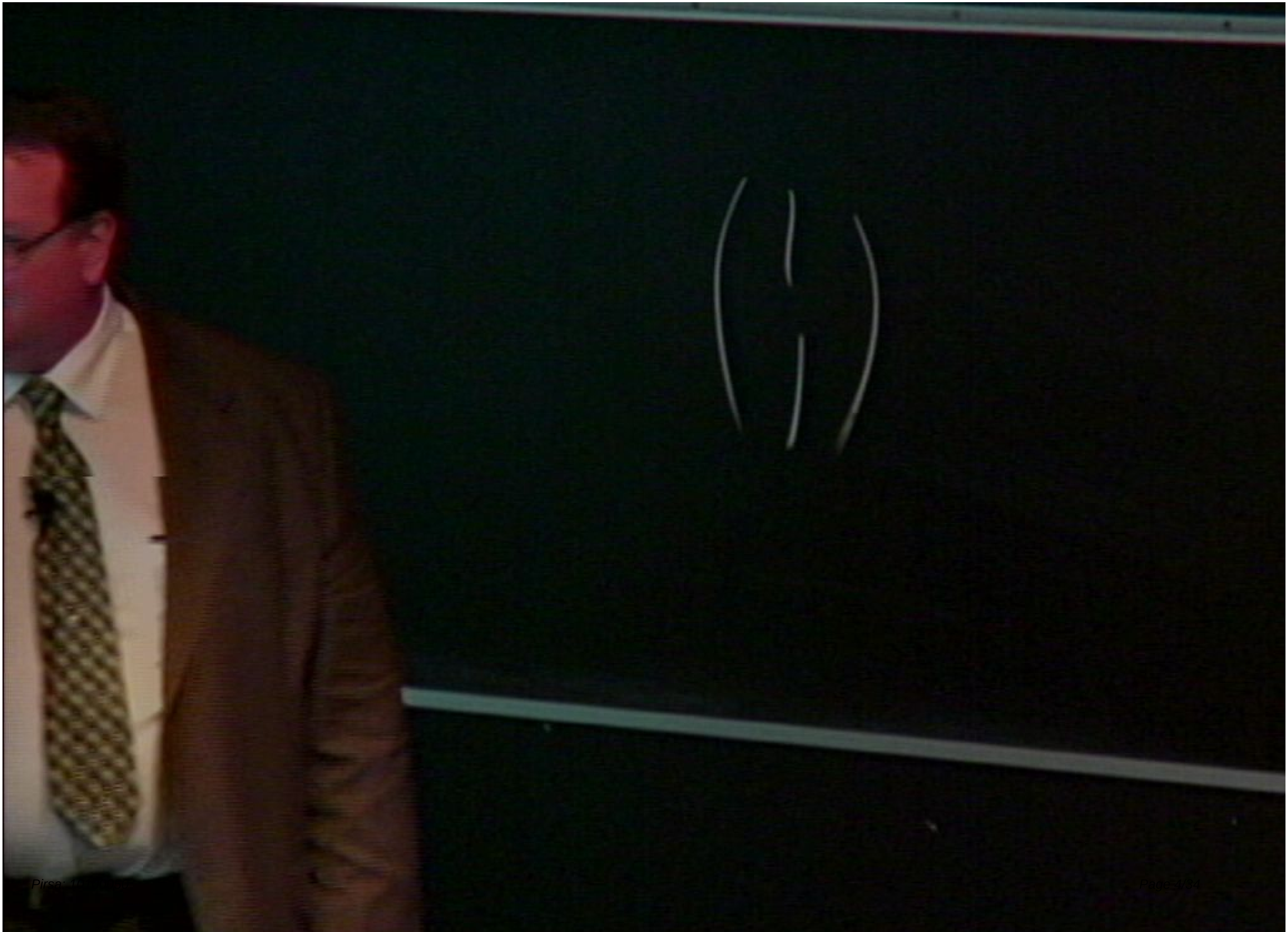
$$\phi_a = (a|\phi)$$

<u>Linear evolution</u>

In the absence of measurement, the state evolves by an invertible linear operator T.

$$|\phi(t)) = T(t,0)\,|\phi(0))$$

<u>Composite systems</u> $\mathcal{V}^{(12)} = \mathcal{V}^{(1)} \otimes \mathcal{V}^{(2)}$



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Annihilators

Given a set $A \subseteq \mathcal{V}$, the **annihilator** is

$$A^\circ = \{(a| \in \mathcal{V}^* : (a|\psi) = 0, \ \forall |\psi) \in A\}$$

If A is a set of states, then A° is the set of effects that are impossible for all states in A.

If A is a set of effects, then A° is the set of states for which all effects in A are impossible.

- A° is a subspace.
- If $A \subseteq B$ then $B^\circ \subseteq A^\circ$.
- A and its span $\langle A \rangle$ have the same annihilator.
- A and B have the same annihilator iff $\langle A \rangle = \langle B \rangle$.
- $(A \cup B)^\circ = \langle A \cup B \rangle^\circ = A^\circ \cap B^\circ$.

Mixed states

A mixture is a set of possible states $M = \{|a), |b), \ldots\}$

M_1 and M_2 are equivalent mixtures if they lead to the same possible and impossible effects.

Thus, M_1 and M_2 are equivalent if $M_1{}^\circ = M_2{}^\circ$ and thus $\langle M_1 \rangle = \langle M_2 \rangle$

$$\text{Mixed state} \quad \mathcal{M} = \langle M \rangle$$

- Mixtures of mixed states: $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2 = \langle \mathcal{M}_1 \cup \mathcal{M}_2 \rangle$
- Product mixed states: $\mathcal{M}^{(\mathrm{AB})} = \mathcal{M}^{(\mathrm{A})} \otimes \mathcal{M}^{(\mathrm{B})}$
- Time evolution of mixed states: $\mathcal{M} \to T\mathcal{M} = \{T\,|m) : |m) \in \mathcal{M}\}$

Mixed states and entanglement

Conditional states

Joint state: $|\Psi^{(AB)})$

A-effect: $(a^{(A)}|$

$$|\psi_a^{(B)}) = (a^{(A)}\,|\Psi^{(AB)})$$

Conditional state of system B given effect *a* on system A

Why: For any $(b^{(B)}|$

$$(a^{(A)}, b^{(B)}\,|\Psi^{(AB)}) = (b^{(B)}\,|\psi_a^{(B)})$$

The conditional state identifies which B-effects are possible given the A-effect a .

Given joint state $|\Psi^{(AB)})$, system B has the mixed state

$$\mathcal{M}^{(B)} = \{(a^{(A)}\,|\Psi^{(AB)}) : (a^{(A)}| \in \mathcal{V}^{(A)*}\}$$

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$\mathcal{M}^{(B)}$ is always a subspace

Given a basic system A measurement $\{ (k^{(A)}| \}$

$$\mathcal{M}^{(B)} = \langle \{(k^{(A)}|\Psi^{(AB)})\} \rangle \qquad \text{(all } k\text{)}$$

Subsystem state for a mixed state: given $\mathcal{M}^{(AB)}$

$$\mathcal{M}^{(B)} = \mathsf{R}_{(A)}\mathcal{M}^{(AB)} = \langle \{(e^{(A)}|m^{(AB)})\} \rangle$$

$$\text{(all } (e| \in \mathcal{V}^{(A)*}, |m) \in \mathcal{M}^{(AB)})$$

Generalized effects

Generalized effect $\Gamma : \mathcal{M} \to \{\text{possible, impossible}\}$

Postulate: Γ respects mixtures -- that is, if $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2 = \langle \mathcal{M}_1 \cup \mathcal{M}_2 \rangle$

then Γ is possible for $\mathcal{M}$ iff Γ is possible for $\mathcal{M}_1$ or $\mathcal{M}_2$

i.e, Γ is impossible for $\mathcal{M}$ iff Γ is impossible for $\mathcal{M}_1$ and $\mathcal{M}_2$

Given Γ, let $\mathcal{Z} = \left\langle \bigcup \{\mathcal{M} : \Gamma \text{ is impossible}\} \right\rangle$

- Γ is impossible for $\mathcal{Z}$
- Γ is impossible for $\mathcal{M}$ iff $\mathcal{M} \subseteq \mathcal{Z}$

Identify: $\Gamma = \mathcal{Z}^\circ$

- Γ is impossible for $\mathcal{M}$ iff $\mathcal{M} \subseteq \Gamma^\circ = \mathcal{Z}$
- Generalized effects are <u>subspaces</u> of $\mathcal{V}^*$

Generalized measurements

A generalized measurement is a set $\{\Gamma_a\}$ of generalized effects.

Requirement: For any state, at least one effect is possible.

$$\bigcap_a \Gamma_a{}^\circ = \langle 0 \rangle \Longrightarrow \bigvee_a \Gamma_a = \left\langle \bigcup_a \Gamma_a \right\rangle = \mathcal{V}^*$$

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A generalized measurement is a spanning set $\{\Gamma_a\}$ of subspaces of $\mathcal{V}^*$.

Generalized conditional states

Now that we have generalized (mixed) states and generalized effects, what about the general concept of a conditional state?

Given: Joint AB state $\mathcal{M}^{(AB)}$ and an A-effect $\Gamma^{(A)}$

Only defined if $\Gamma^{(A)}$ is possible on $\mathcal{M}^{(AB)}$

$$\mathcal{M}_{\Gamma}^{(B)} = \mathsf{C}\left(\mathcal{M}^{(AB)}|\Gamma^{(A)}\right) = \langle\, \{(e^{(A)}\,|m^{(AB)}\,)\}\,\rangle$$

$$(\text{all } \ (e| \in \Gamma^{(A)}, |m) \in \mathcal{M}^{(AB)})$$

Note that $\mathsf{C}(\,\cdot\,|\,\cdot)$ respects mixtures in both places:

$$\begin{aligned}
\mathsf{C}\left(\mathcal{M}_1^{(AB)} \vee \mathcal{M}_2^{(AB)}|\Gamma^{(A)}\right) &= \mathsf{C}\left(\mathcal{M}_1^{(AB)}|\Gamma^{(A)}\right) \vee \mathsf{C}\left(\mathcal{M}_2^{(AB)}|\Gamma^{(A)}\right) \\
\mathsf{C}\left(\mathcal{M}^{(AB)}|\Gamma_1^{(A)} \vee \Gamma_2^{(A)}\right) &= \mathsf{C}\left(\mathcal{M}^{(AB)}|\Gamma_1^{(A)}\right) \vee \mathsf{C}\left(\mathcal{M}^{(AB)}|\Gamma_2^{(A)}\right)
\end{aligned}$$

For an A-measurement $\{\Gamma_a\}$: $\quad \mathsf{R}_{(A)}\mathcal{M}^{(AB)} = \mathsf{C}\left(\mathcal{M}^{(AB)}|\mathcal{V}^{(A)*}\right)$

$$= \bigvee \mathsf{C}\left(\mathcal{M}^{(AB)}|\Gamma_a^{(A)}\right)$$

Generalized evolution

A generalized evolution would be a map on subspaces of $\mathcal{V}$

Such a map ought to have two properties:

$$\mathcal{E}\left(\mathcal{M}_1 \vee \mathcal{M}_2\right) = \mathcal{E}(\mathcal{M}_1) \vee \mathcal{E}(\mathcal{M}_2)$$

$$\mathcal{M} \neq \langle 0 \rangle \implies \mathcal{E}(\mathcal{M}) \neq \langle 0 \rangle$$

That is, the map ought to "respect mixtures" and never map a state to the zero subspace.

A **Type M** map on subspaces is one that satisfies these requirements.

Any reasonable generalized evolution must be Type M.

Type I, Type L

Some possible types of maps

A **Type I** map on S can be realized as invertible linear evolution on a larger system SE:

$$\mathcal{E}^{(\mathrm{S})}(\mathcal{M}^{(\mathrm{S})}) = \mathsf{R}_{(\mathrm{E})}\left(T^{(\mathrm{SE})}\left(\mathcal{M}^{(\mathrm{S})} \otimes \mathcal{M}_0^{(\mathrm{E})}\right)\right)$$

$\mathcal{M}_0^{(\mathrm{E})} = \langle |0^{(\mathrm{E})}\rangle\rangle$

$T^{(\mathrm{SE})}$ invertible

A **Type L** map on S can be represented as a mixture of linear evolutions (not necessarily invertible):

$$\mathcal{E}(\mathcal{M}) = \bigvee_k A_k \mathcal{M}$$

Note: The A_k operators must satisfy $\bigcap_k \ker A_k = \langle 0 \rangle$

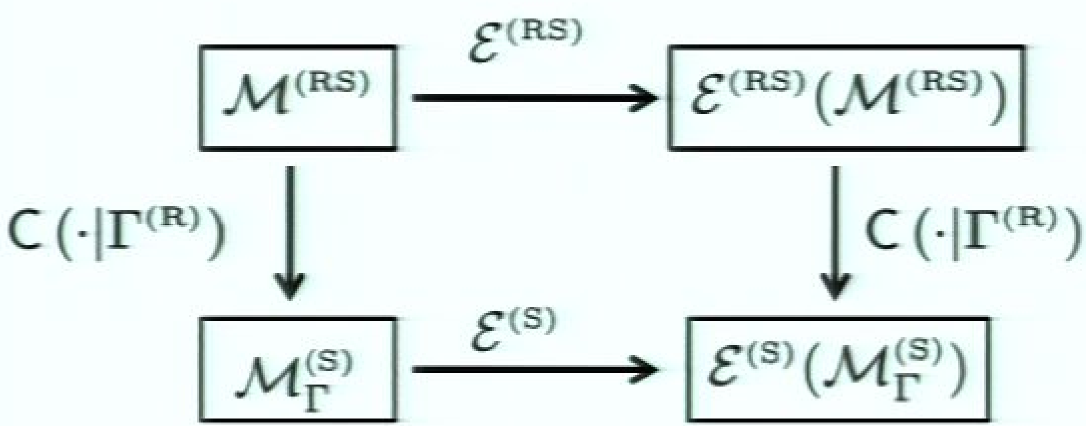
Type E

A map on S is Type E if it can be extended in a way that "commutes" with deriving conditional states.

That is, $\mathcal{E}^{(\mathrm{S})}$ is Type E if for any R there exists $\mathcal{E}^{(\mathrm{RS})}$ such that

$$\mathcal{E}^{(\mathrm{S})}\left(\mathsf{C}\left(\mathcal{M}^{(\mathrm{RS})}|\Gamma^{(\mathrm{R})}\right)\right) = \mathsf{C}\left(\mathcal{E}^{(\mathrm{RS})}\left(\mathcal{M}^{(\mathrm{RS})}\right)|\Gamma^{(\mathrm{R})}\right).$$

for any R-effect $\Gamma^{(\mathrm{R})}$ and RS-state $\mathcal{M}^{(\mathrm{RS})}$



For a Type E map on S, we can find an RS map for which this diagram commutes.

$\mathrm{L} \Rightarrow \mathrm{I} \Rightarrow \mathrm{E}$

Type L ⇒ Type I

Given the A_k operators for a Type L map on S, introduce system E and define $T^{(\mathrm{SE})}$ by

$$T^{(\mathrm{SE})}\,|m^{(\mathrm{S})}, 0^{(\mathrm{E})}\rangle = \sum_k \left(A_k\,|m^{(\mathrm{S})}\rangle\right) \otimes |k^{(\mathrm{E})}\rangle$$

Type I ⇒ Type E

Given the $T^{(\mathrm{SE})}$ representation for a Type I map on S, for any added system R define

$$\mathcal{E}^{(\mathrm{RS})}(\mathcal{M}^{(\mathrm{RS})}) = \mathsf{R}_{(\mathrm{E})}\left[\left(\mathbf{1}^{(\mathrm{R})} \otimes T^{(\mathrm{SE})}\right)\left(\mathcal{M}^{(\mathrm{RS})} \otimes \mathcal{M}_0^{(\mathrm{E})}\right)\right]$$

Key fact: $\mathsf{R}_{(\mathrm{E})}$ commutes with $\mathsf{C}(\,\cdot \mid \Gamma^{(\mathrm{R})})$

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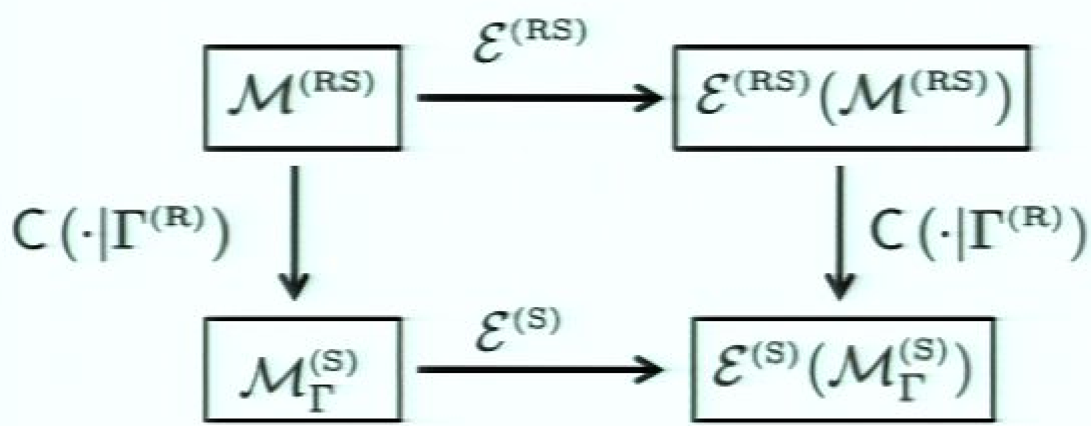

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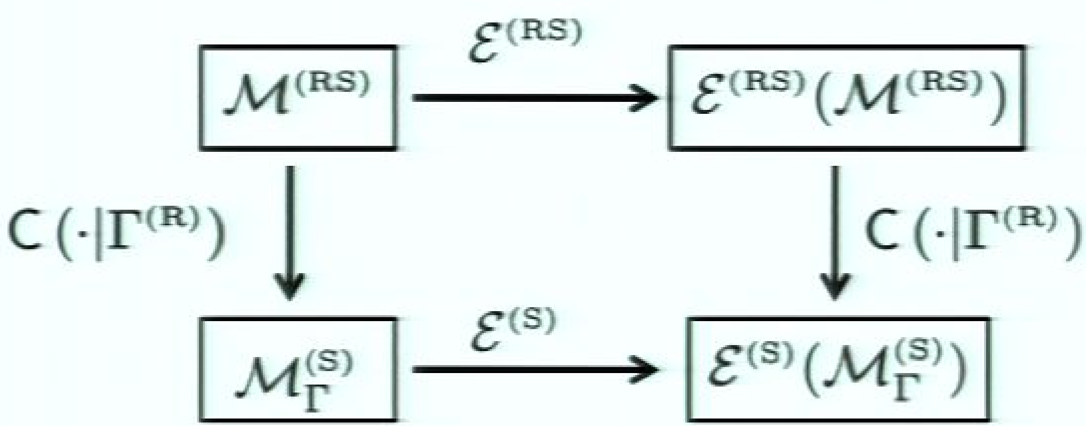

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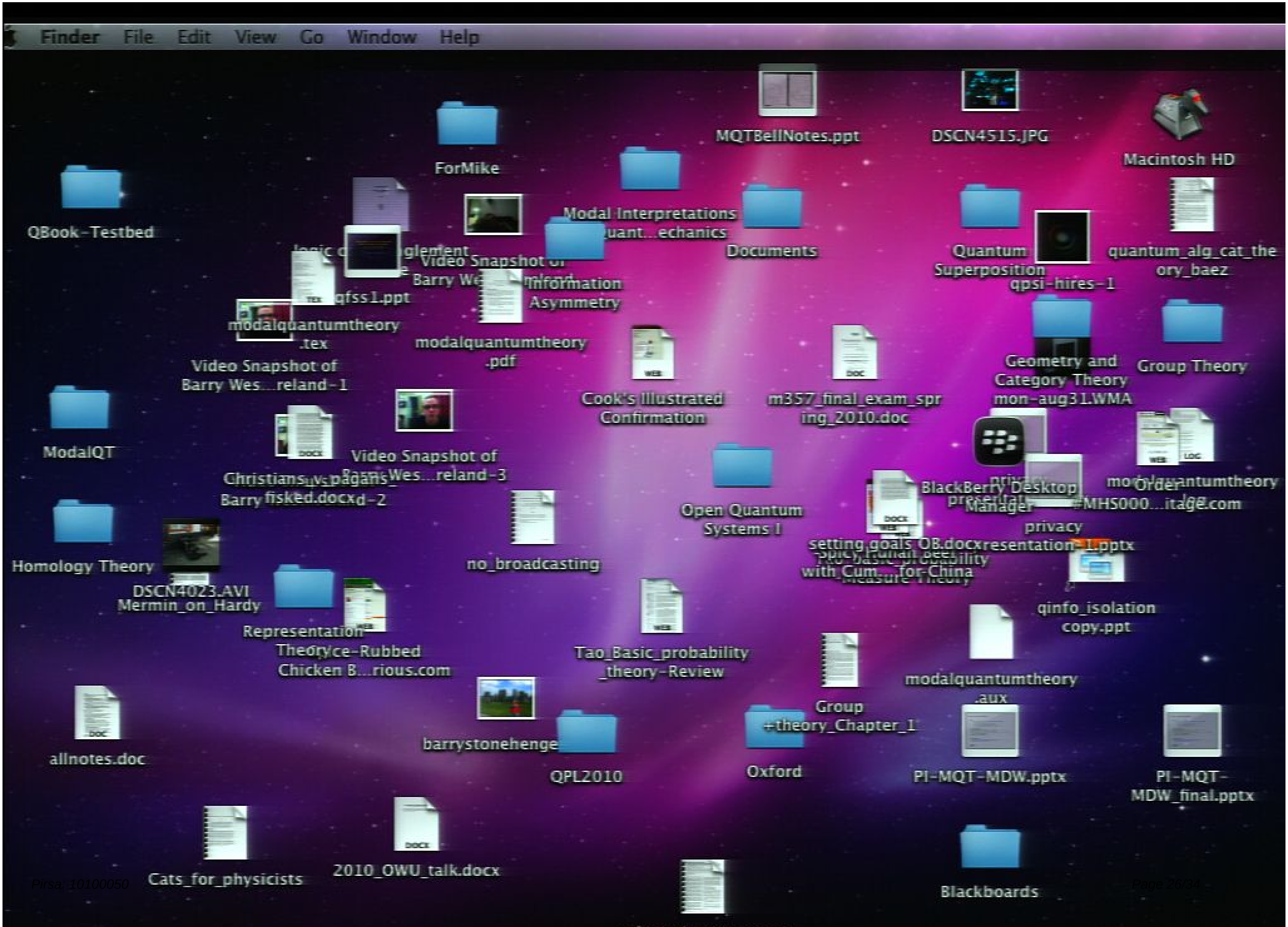
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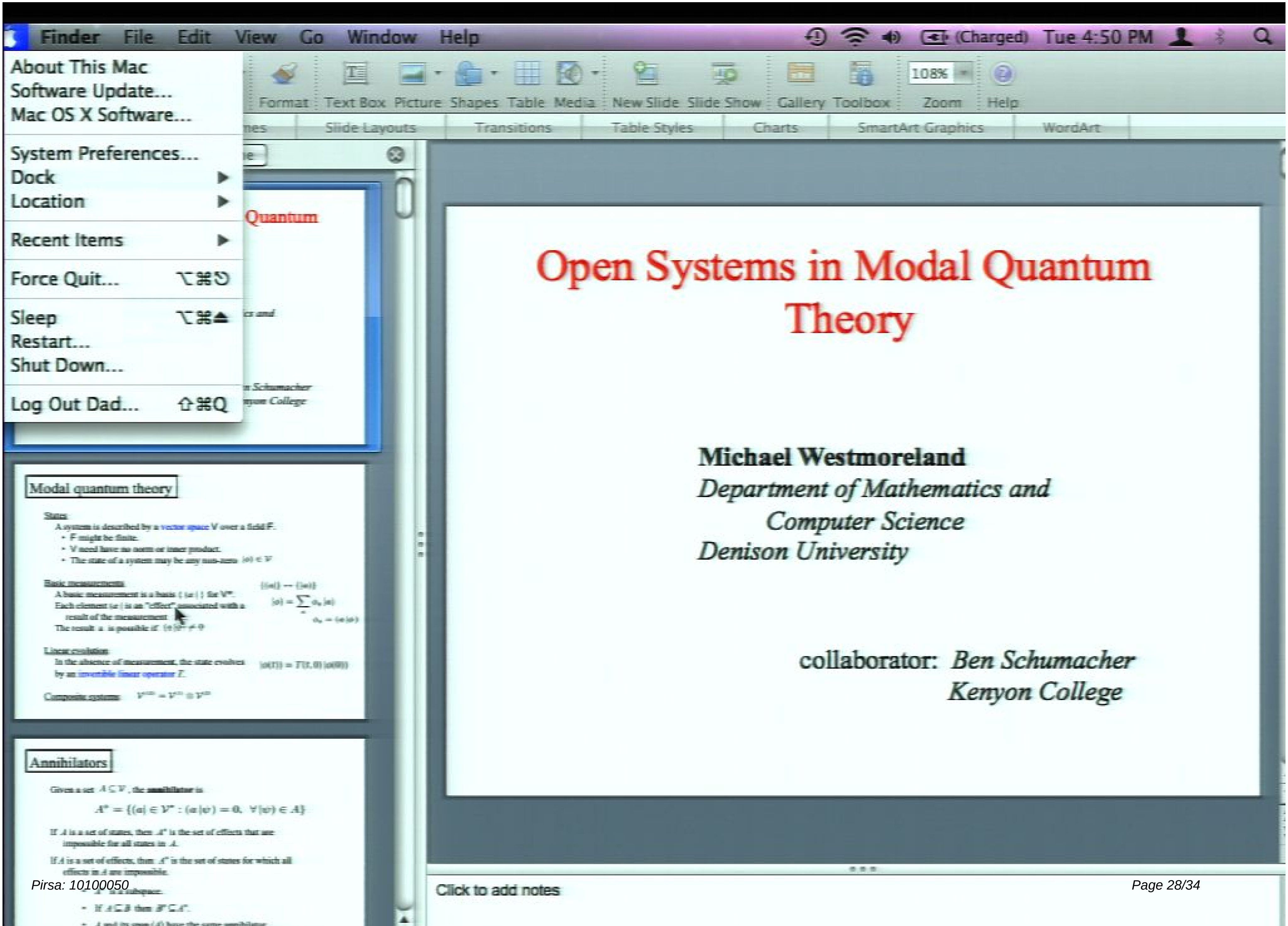
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Restart...
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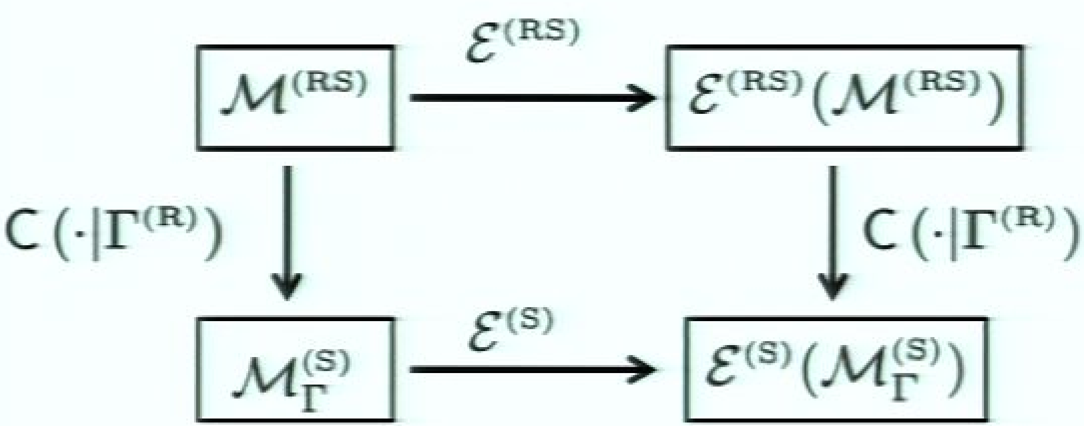

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$L \Rightarrow I \Rightarrow E$

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Given the A_k operators for a Type L map on S, introduce system E and define $T^{(SE)}$ by

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$$\mathcal{E}^{(\mathrm{RS})}(\mathcal{M}^{(\mathrm{RS})}) = \mathsf{R}_{(\mathrm{E})}\left[\left(\mathbf{1}^{(\mathrm{R})}\otimes T^{(\mathrm{SE})}\right)\left(\mathcal{M}^{(\mathrm{RS})}\otimes\mathcal{M}_0^{(\mathrm{E})}\right)\right]$$

Key fact: $\mathsf{R}_{(\mathrm{E})}$ commutes with $\mathsf{C}(\,\cdot\,|\,\Gamma^{(\mathrm{R})}\,)$

E ⇒ L

Suppose our map is Type E. Let R and S have the same dimension *d*.

Let $|M^{(\mathrm{RS})}) = \sum_k |k^{(\mathrm{R})}, k^{(\mathrm{S})})$ ←------- $\mathcal{M}^{(\mathrm{RS})} = \langle |M^{(\mathrm{RS})})\rangle$

Consider the S-state $\mathcal{G}^{(\mathrm{S})} = \left\{ |g^{(\mathrm{S})}) = \sum_k g_k \, |k^{(\mathrm{S})}) \right\}$

$$G = \{d\text{-tuples } (g_k) \text{ for } \mathcal{G}^{(\mathrm{S})}\}$$

Define the R-effect $\Gamma^{(\mathrm{R})} = \left\{ (g^{(\mathrm{R})}| = \sum_k g_k \, (k^{(\mathrm{R})}| : (g_k) \in G \right\}$

Then $\mathcal{G}^{(\mathrm{S})} = \mathsf{C}\,(\mathcal{M}^{(\mathrm{RS})} | \Gamma^{(\mathrm{R})})$

Why: $|g^{(\mathrm{S})}) = (g^{(\mathrm{R})}\, |M^{(\mathrm{RS})})$

$\mathrm{E} \Rightarrow \mathrm{L}$

$\mathcal{E}^{(\mathbf{RS})}(\mathcal{M}^{(\mathbf{RS})})$ is a subspace, so $\mathcal{E}^{(\mathbf{RS})}(\mathcal{M}^{(\mathbf{RS})}) = \langle\, \{|M_\alpha^{(\mathbf{RS})})\}\,\rangle$

(all α)

Definition of A_α: operators Given $|g^{(\mathbf{S})})$, $(g^{(\mathbf{R})}|$ — same (g_k)

$$A_\alpha^{(\mathbf{S})}\,|g^{(\mathbf{S})}) = (g^{(\mathbf{R})}\,|M_\alpha^{(\mathbf{RS})}\,)$$

Then
$$
\begin{aligned}
\mathcal{E}^{(\mathbf{S})}(\mathcal{G}^{(\mathbf{S})}) &= \mathcal{E}^{(\mathbf{S})}\,(\mathsf{C}\,(\mathcal{M}^{(\mathbf{RS})}|\Gamma^{(\mathbf{R})})) \\
&= \mathsf{C}\,(\mathcal{E}^{(\mathbf{RS})}(\mathcal{M}^{(\mathbf{RS})})|\Gamma^{(\mathbf{R})}) \\
&= \langle\,\{(g^{(\mathbf{R})}\,|M_\alpha^{(\mathbf{RS})}\,)\}\,\rangle \leftarrow\text{---} \ (\text{all } \alpha, (g| \in \Gamma^{(\mathbf{R})}) \\
&= \langle\,\{A_\alpha^{(\mathbf{S})}\,|g^{(\mathbf{S})})\}\,\rangle \leftarrow\text{----} \ (\text{all } \alpha, |g) \in \mathcal{G}^{(\mathbf{S})}) \\
&= \bigvee_\alpha A_\alpha^{(\mathbf{S})}\mathcal{G}^{(\mathbf{S})}
\end{aligned}
$$

Type L!

Equivalence!

Among Type M maps, Types L, I and E are equivalent.

Every map of Type E can be represented as invertible linear evolution on a larger system (Type I).

Every map of Type E can be represented as a mixture of linear evolutions (Type L).

Type M $\Rightarrow$ Type L?

Modal quantum system: $\mathcal{F} = Z_3$

$\dim \mathcal{V} = 2$

States
$$\begin{aligned}
\mathcal{D}_0 &= \langle |0) \rangle \\
\mathcal{D}_1 &= \langle |1) \rangle \\
\mathcal{D}_+ &= \langle |0) + |1) \rangle \\
\mathcal{D}_- &= \langle |0) - |1) \rangle \\
\mathcal{V} &= \langle \{ |0), |1) \} \rangle
\end{aligned}$$

Proposed map
$$\begin{aligned}
\mathcal{E}(\mathcal{D}_0) &= \mathcal{D}_0 \\
\mathcal{E}(\mathcal{D}_1) &= \mathcal{D}_1 \\
\mathcal{E}(\mathcal{D}_+) &= \mathcal{D}_+ \\
\mathcal{E}(\mathcal{D}_-) &= \mathcal{V} \\
\mathcal{E}(\mathcal{V}) &= \mathcal{V}
\end{aligned}$$

This satisfies the Type M requirements

Properties of potential A_k operators

$$\begin{aligned}
A_k |0) &= +|0), -|0), \text{ or } 0 \\
A_k |1) &= +|1), -|1), \text{ or } 0 \\
A_k |+) &= +|+), -|+), \text{ or } 0
\end{aligned}$$

$\Longrightarrow A_k = \mathbf{1}, -\mathbf{1}, 0$

$$\mathcal{E}(\mathcal{D}_-) = \bigvee_k A_k \mathcal{D}_- = \mathcal{D}_- \neq \mathcal{V}$$

Not all Type M maps are also Type L (or I or E)