Title: Anyonic entanglement renormalization

Date: Oct 13, 2010 04:00 PM

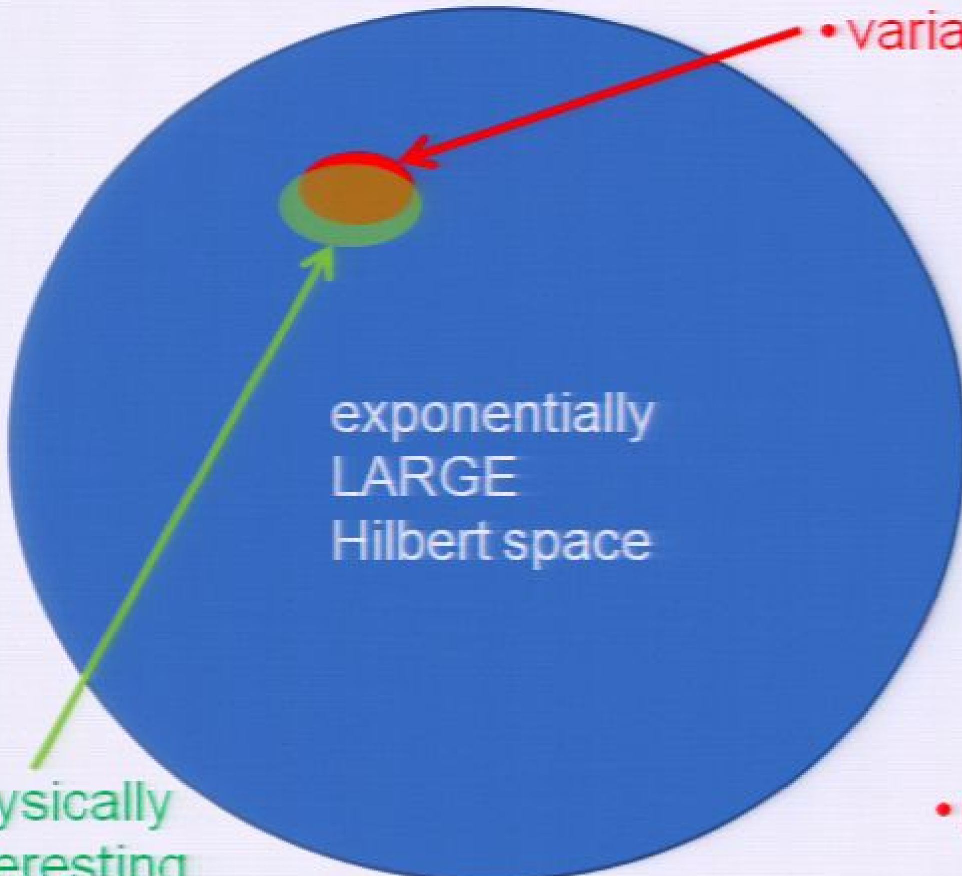
URL: http://pirsa.org/10100045

Abstract: We introduce a family of variational ansatz states for chains of anyons which optimally exploits the structure of the anyonic Hilbert space. This ansatz is the natural analog of the multi-scale entanglement renormalization ansatz for spin chains. In particular, it has the same interpretation as a coarse-graining procedure and is expected to accurately describe critical systems with algebraically decaying correlations. We numerically investigate the validity of this ansatz using the anyonic golden chain and its relatives as a testbed. This demonstrates the power of entanglement renormalization in a setting with non-abelian exchange statistics, extending previous work on qudits, bosons and fermions in two dimensions. This is joint work with Ersen Bilgin.

Many-body quantum physics: approaches

• variational ansatz states

e.g.,
mean-field, Hartree-Fock,
BCS, Laughlin,
matrix product states

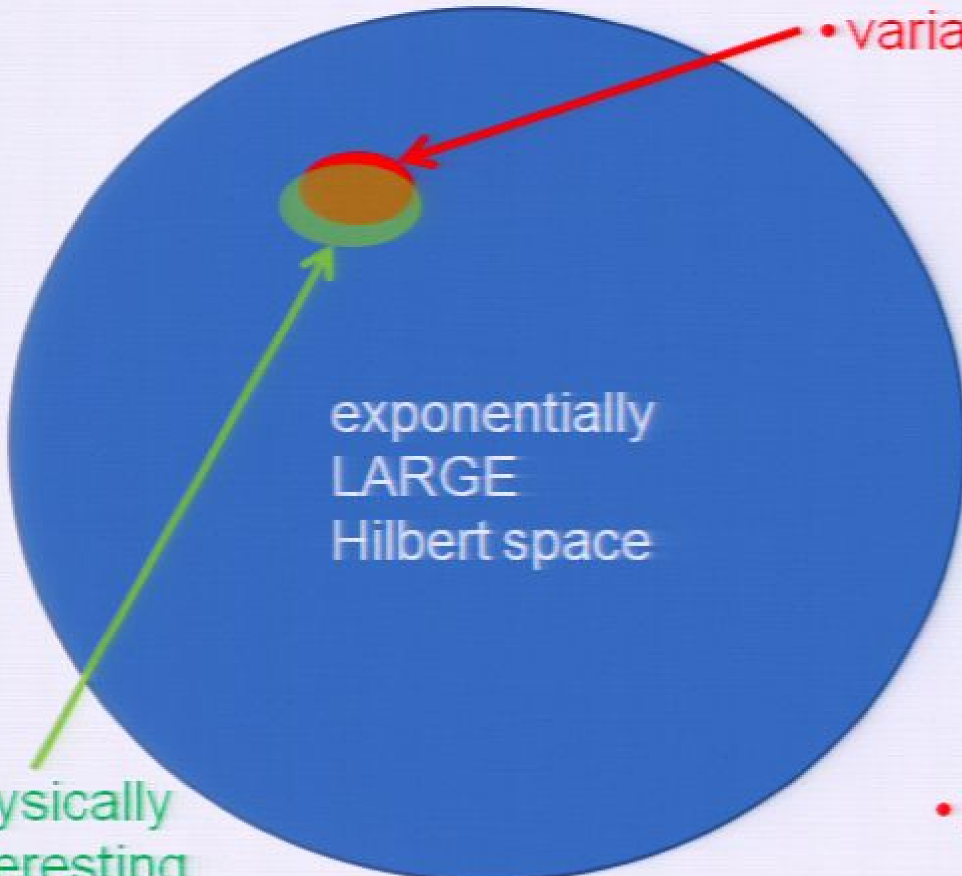


physically
interesting
states

e.g.,
ground/thermal states

• renormalization

Many-body quantum physics: approaches



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- renormalization

Many-body quantum physics: approaches

• variational ansatz states

e.g.,
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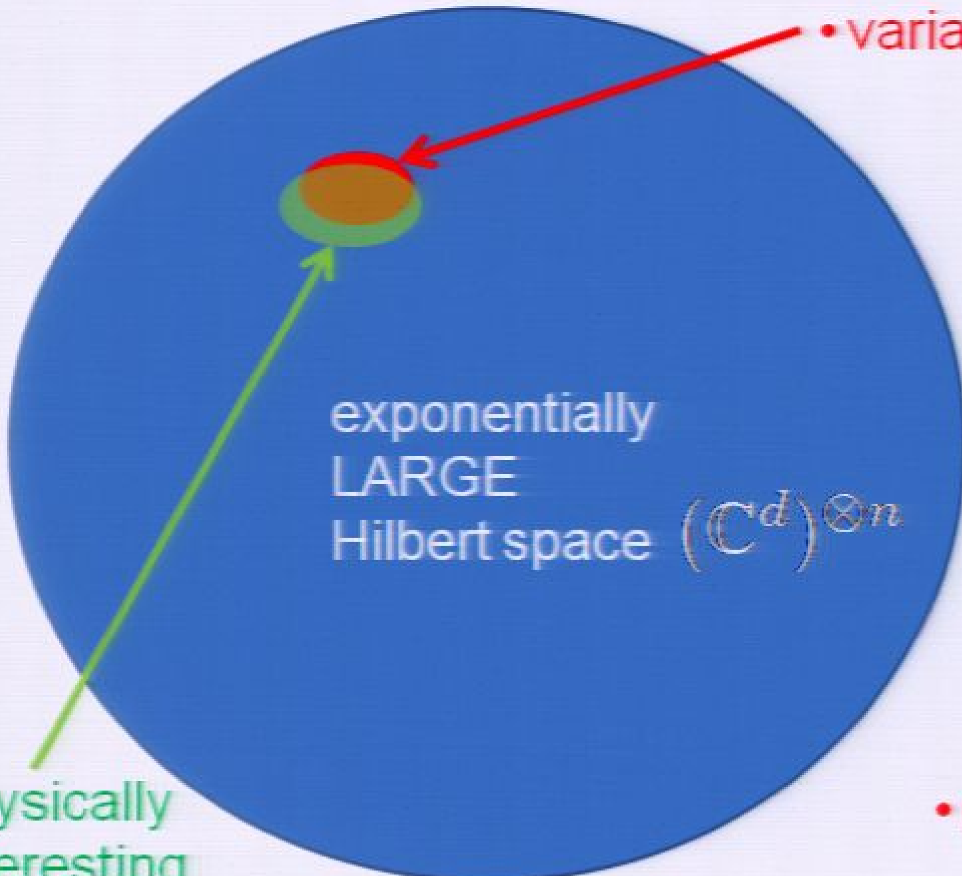
matrix product states

entanglement
renormalization ansatz
(for multiqudit systems)



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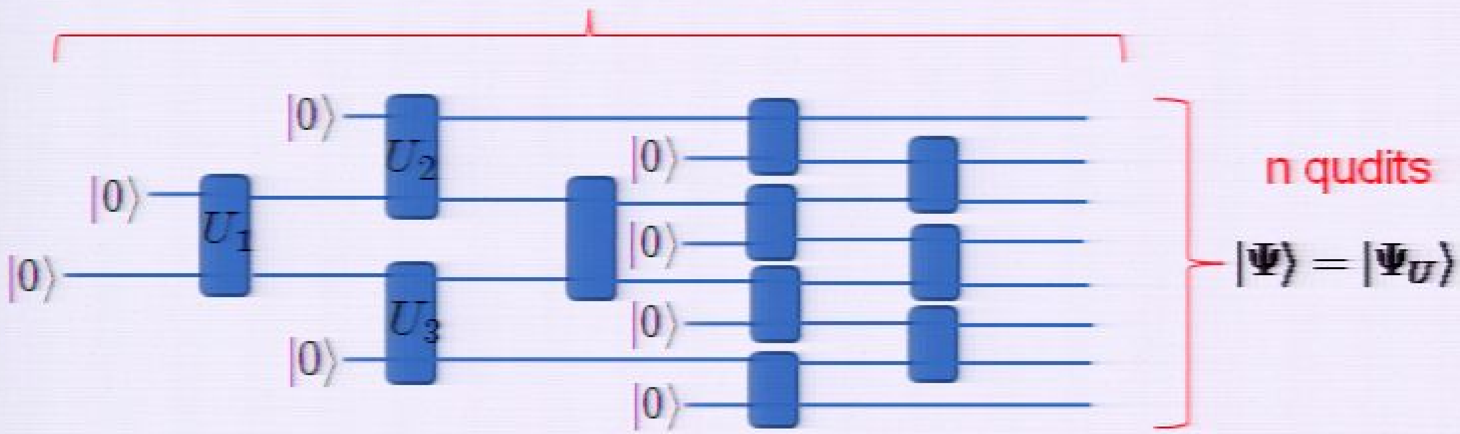
Entanglement renormalization for spins

G. Vidal, Phys. Rev. Lett. 99, 220405 (2007)

efficient $(O(\log n)$-depth) quantum circuit preparing n-qudit state



Varying over gates $U = \{U_i\}_i$ gives a variational family of states

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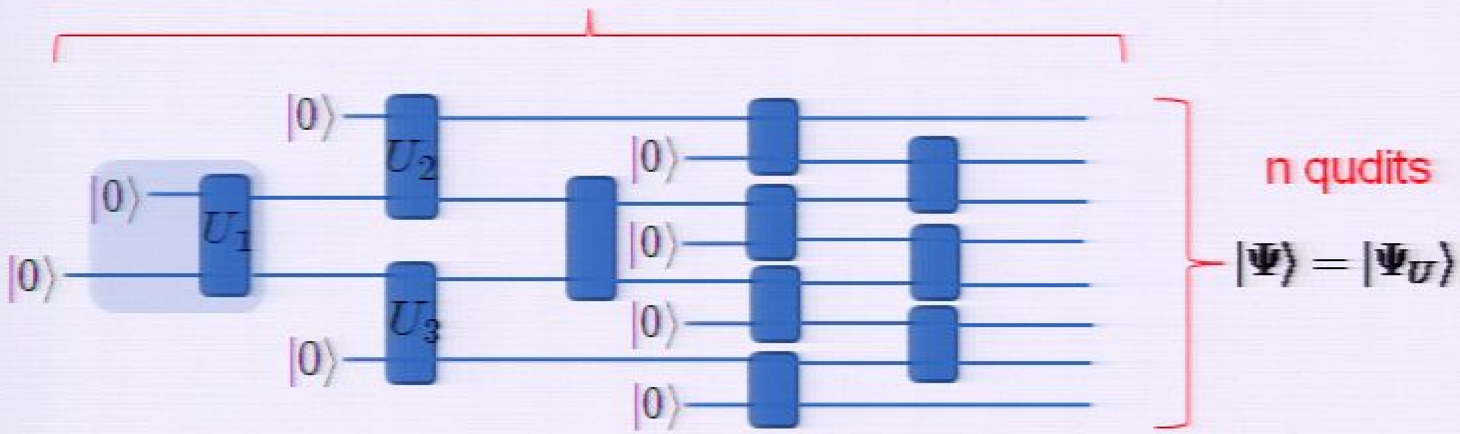

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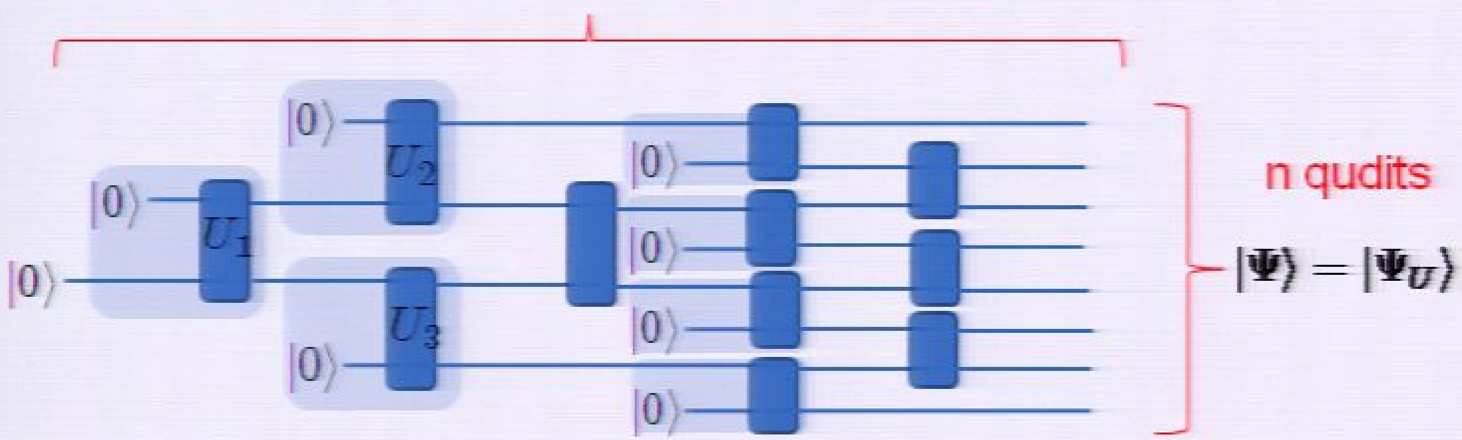

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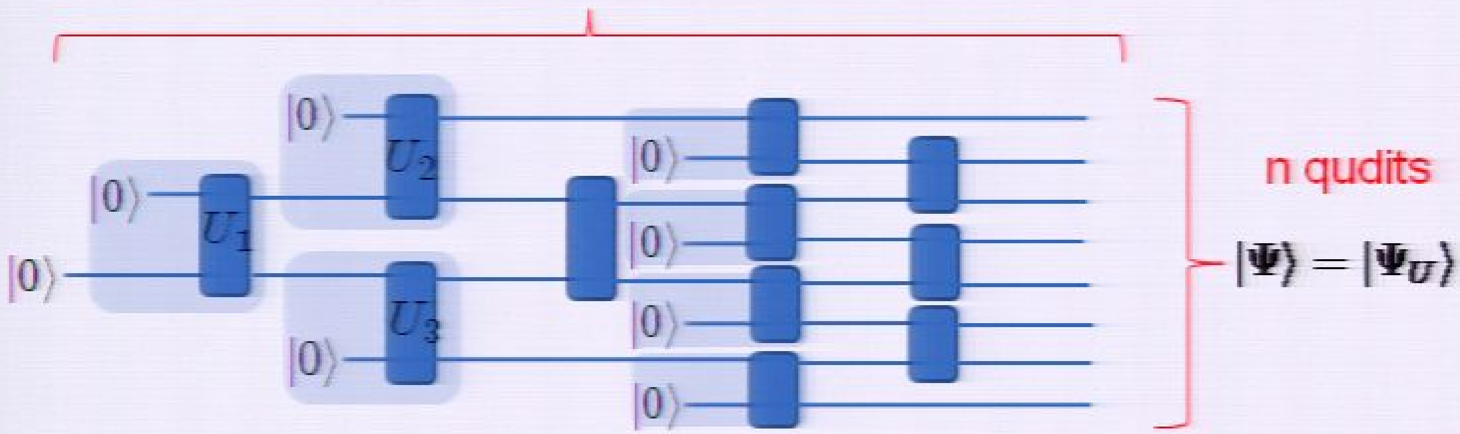

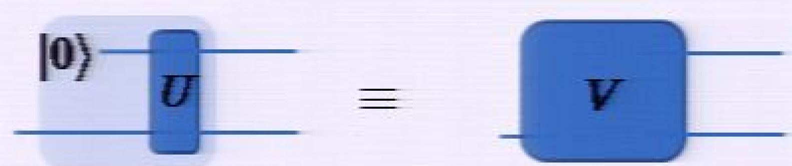

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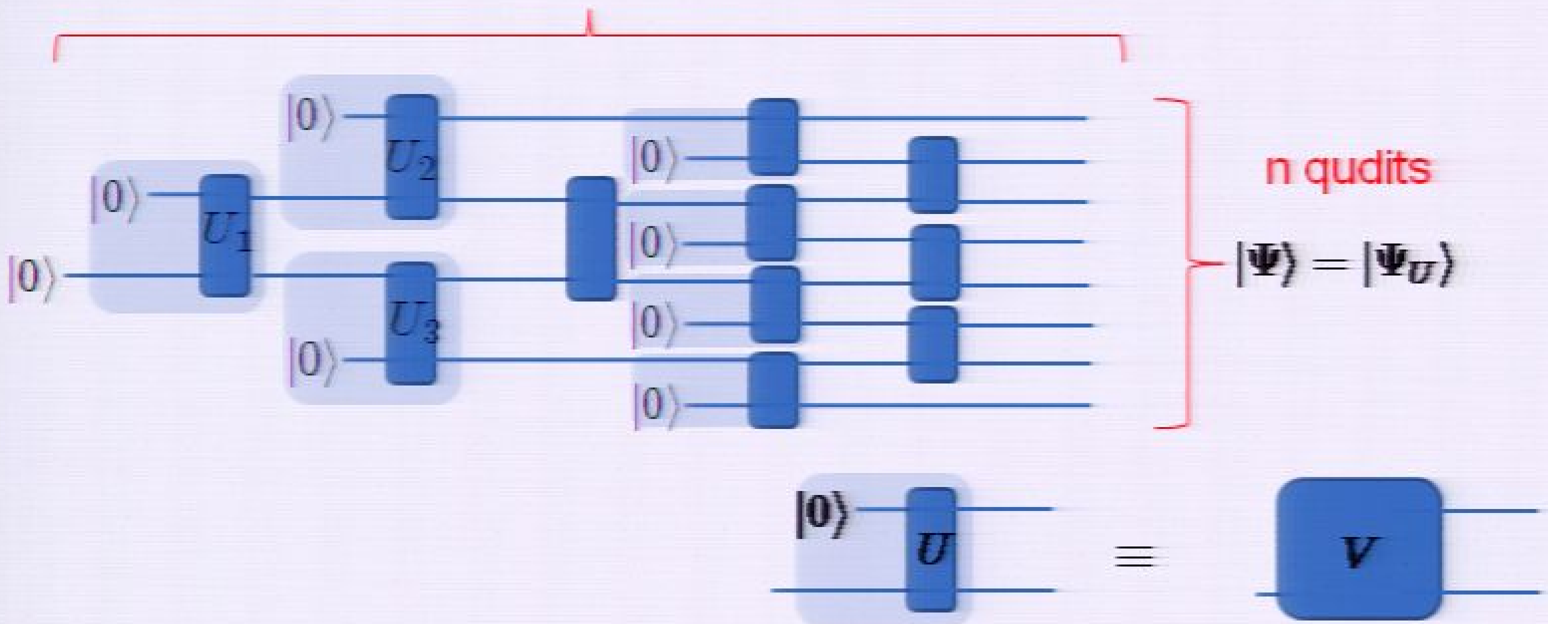

Varying over isometries $V = \{V_j\}_j$ gives a variational family of states

Entanglement renormalization for spins

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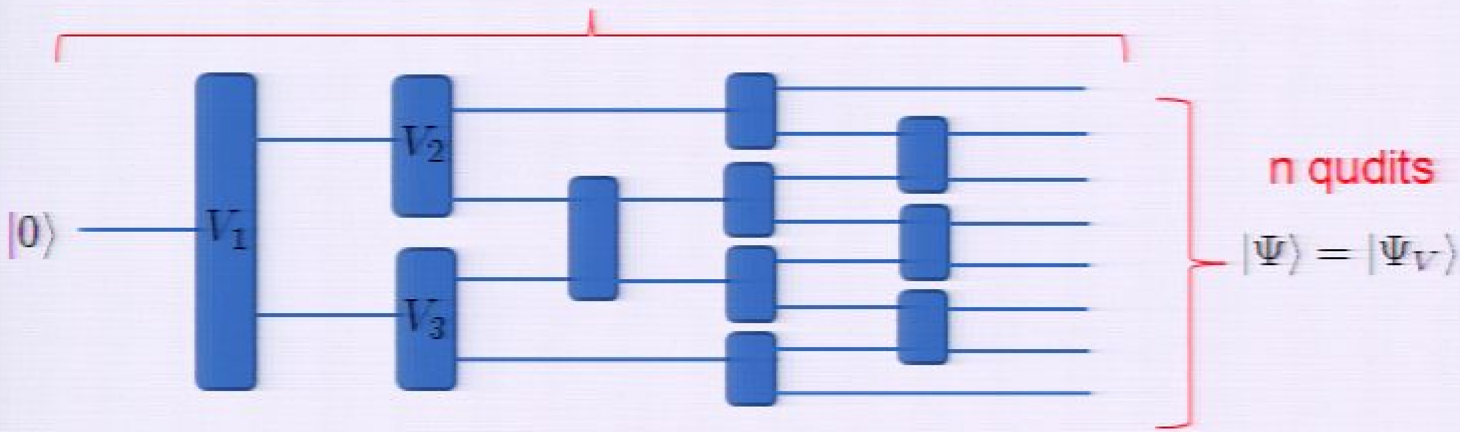

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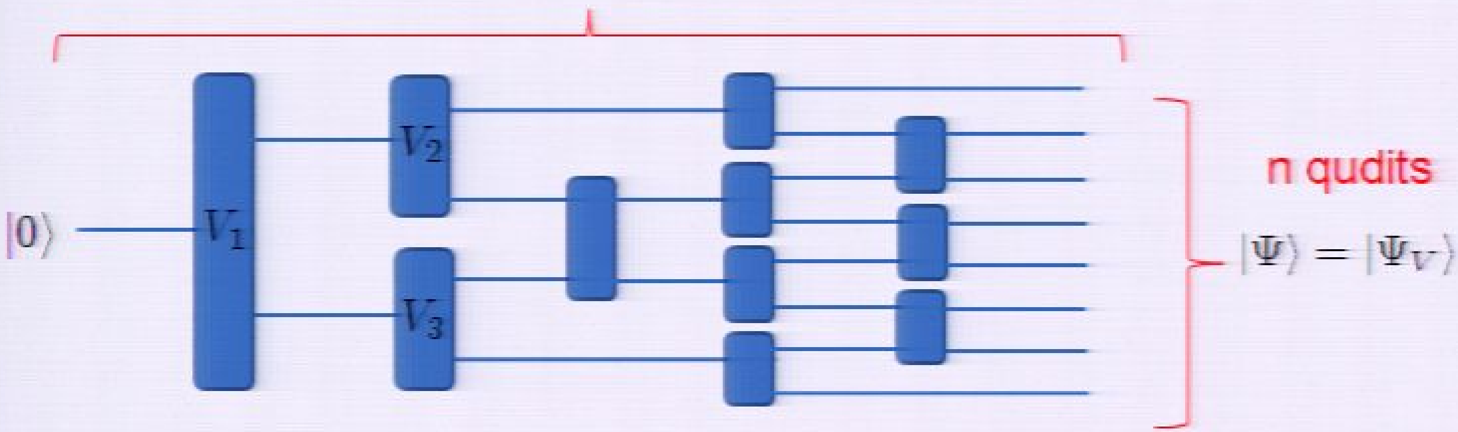
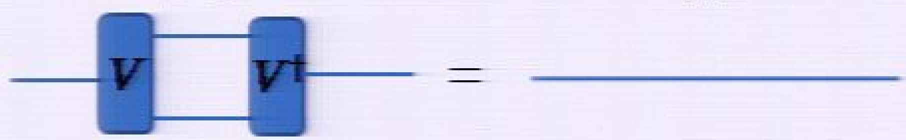


$V^\dagger V \quad = \quad \mathbf{id}$

defining property of isometries:



Varying over isometries $V = \{V_j\}_j$ gives a variational family of states

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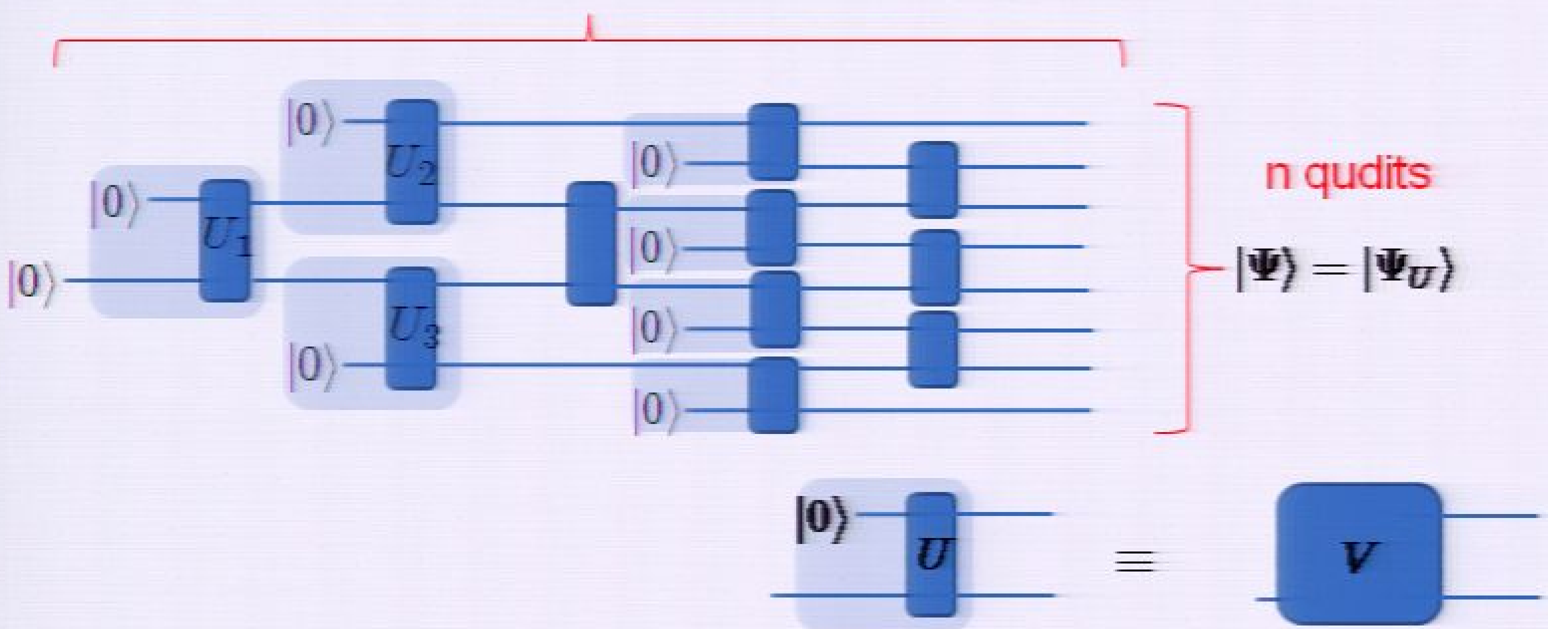
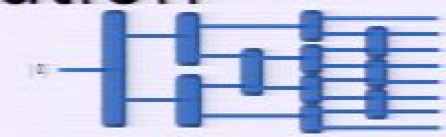


Varying over isometries $V = \{V_j\}_j$ gives a variational family of states

Entanglement renormalization



Evidence that this is a good ansatz:

• good approximations to ground state energies & correlation functions of critical systems (Evenbly, Vidal '09 &Vidal '07)

• describes states with $S(\rho_L) \approx \log L$ and algebraically decaying correlations in 1D

• exact description of interesting topologically ordered states (Aguado/Vidal '07 & K, Reichardt, Vidal '08)

Useful as a numerical method because:

• efficient evaluation of expectation values
correlation function
critical exponents

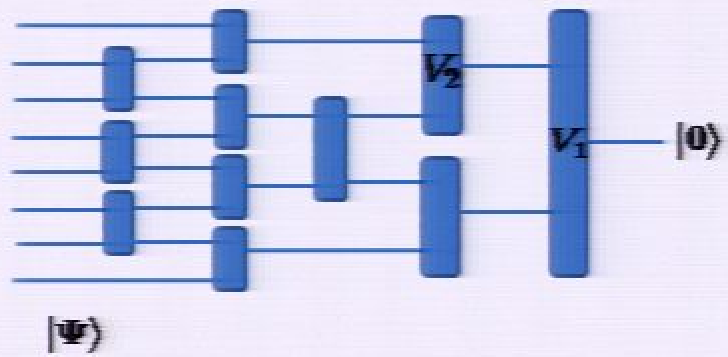
• generalizes to D>1 and to e.g., fermions (Evenbly, Vidal '07, Pineda et al. '09)

•iterative algorithms for minimizing energy

Efficient evaluation of expectation values

The expectation value of a *local observable* can be efficiently computed as only isometries within its "causal cone" contribute.

Efficient evaluation of expectation values



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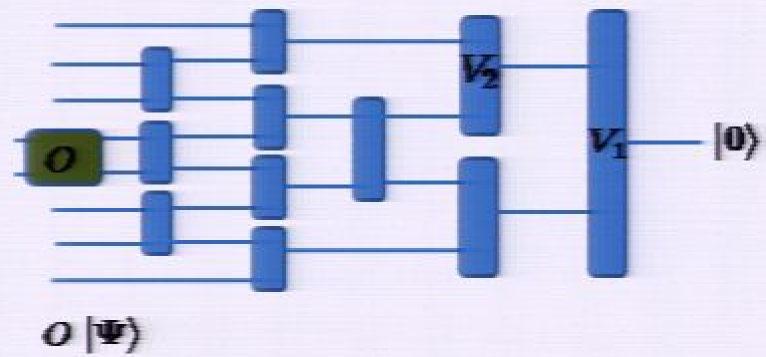

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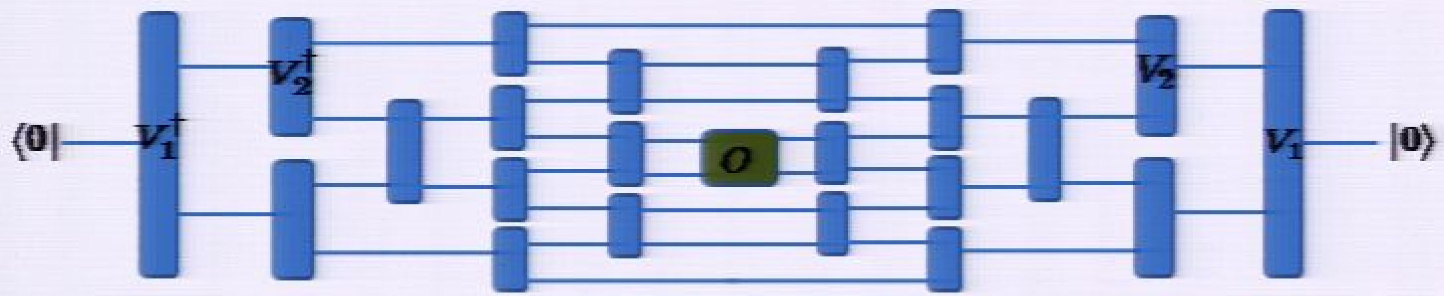

$\langle \Psi | O | \Psi \rangle$

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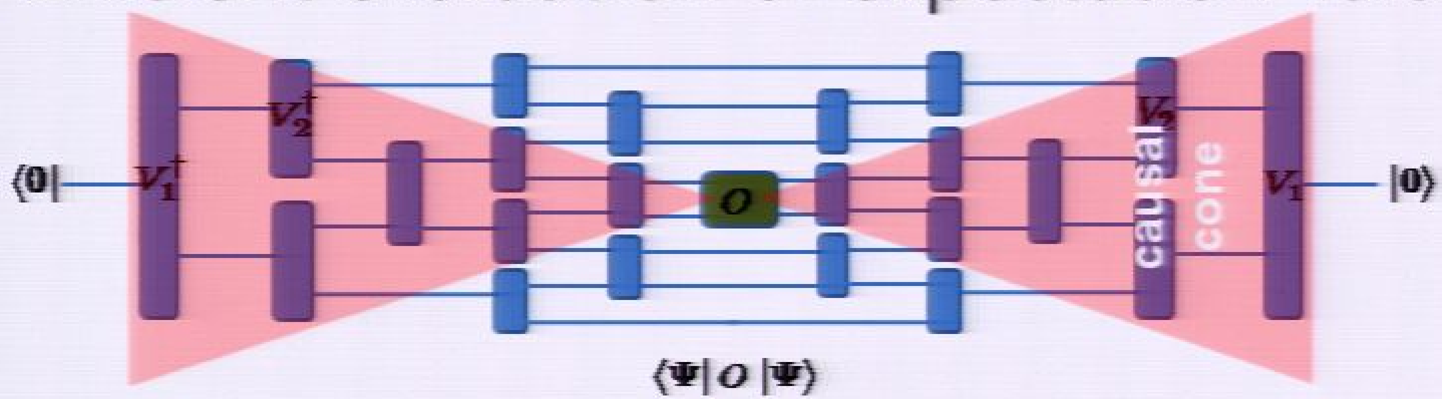

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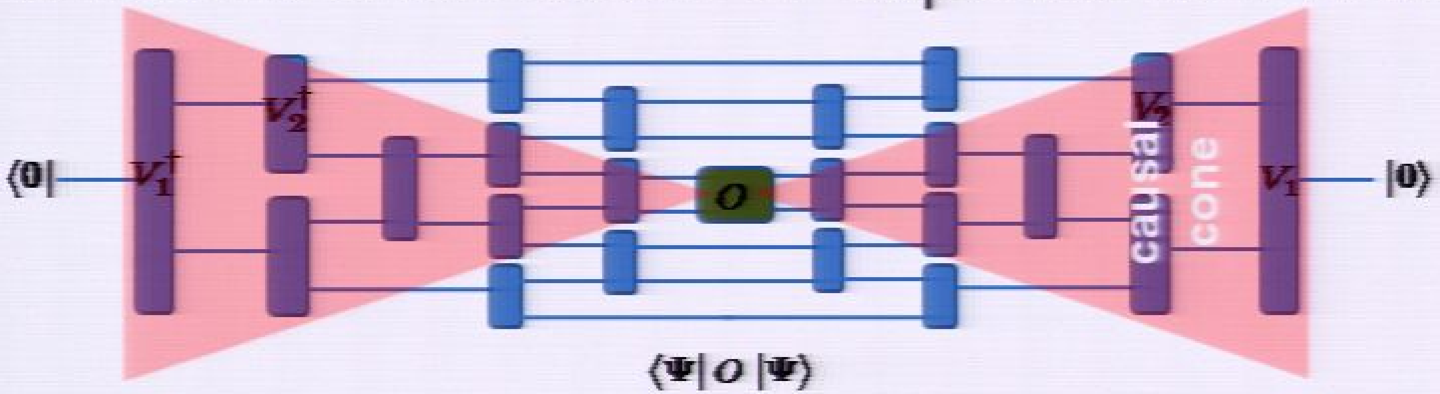
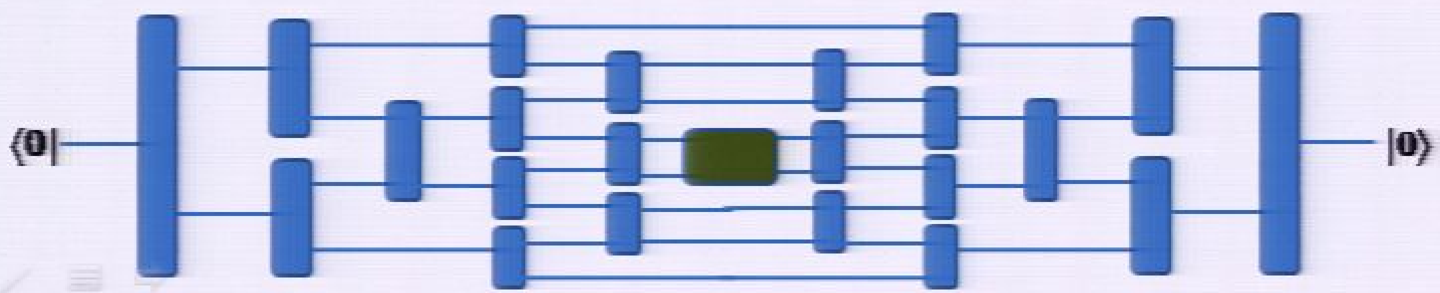


$\langle \Psi | O | \Psi \rangle$

The expectation value of a *local observable* can be efficiently computed as only isometries within its "causal cone" contribute.

This follows from the defining property of isometries (resp. unitaries)



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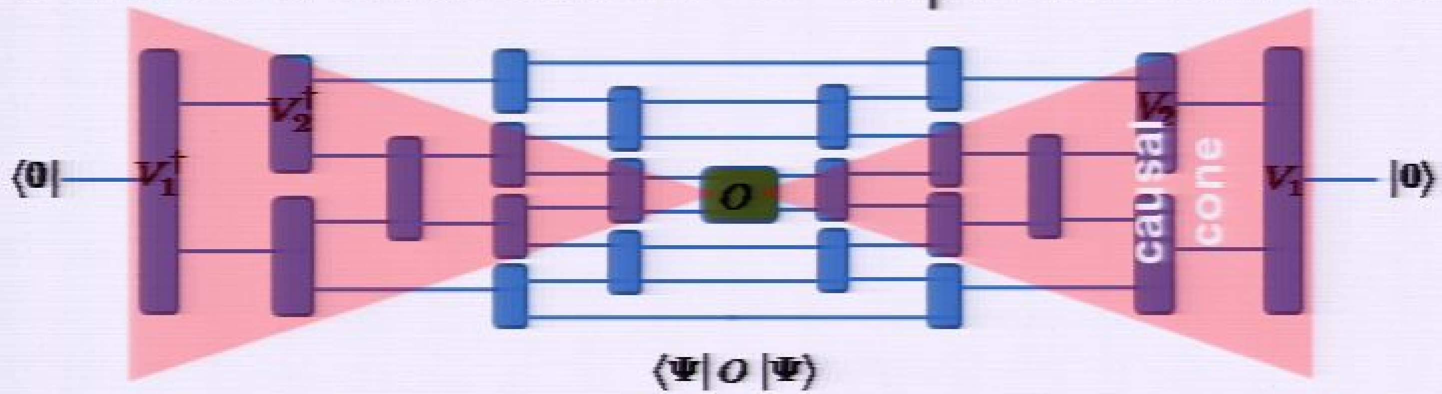

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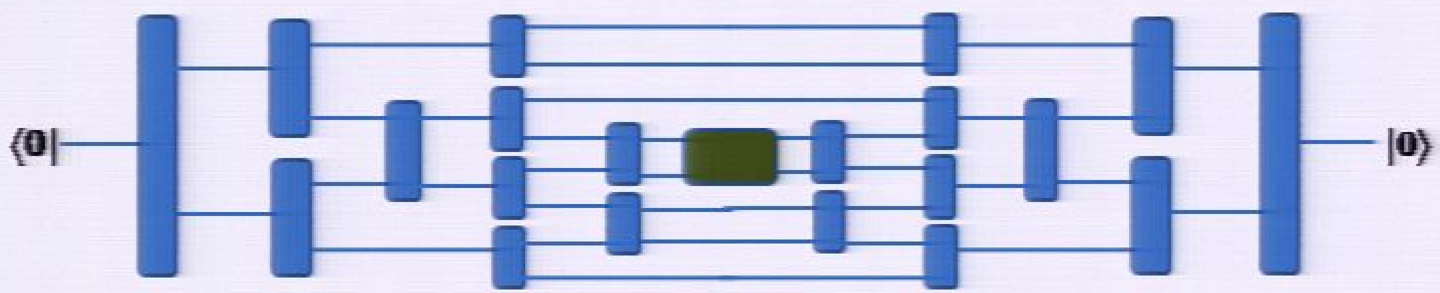

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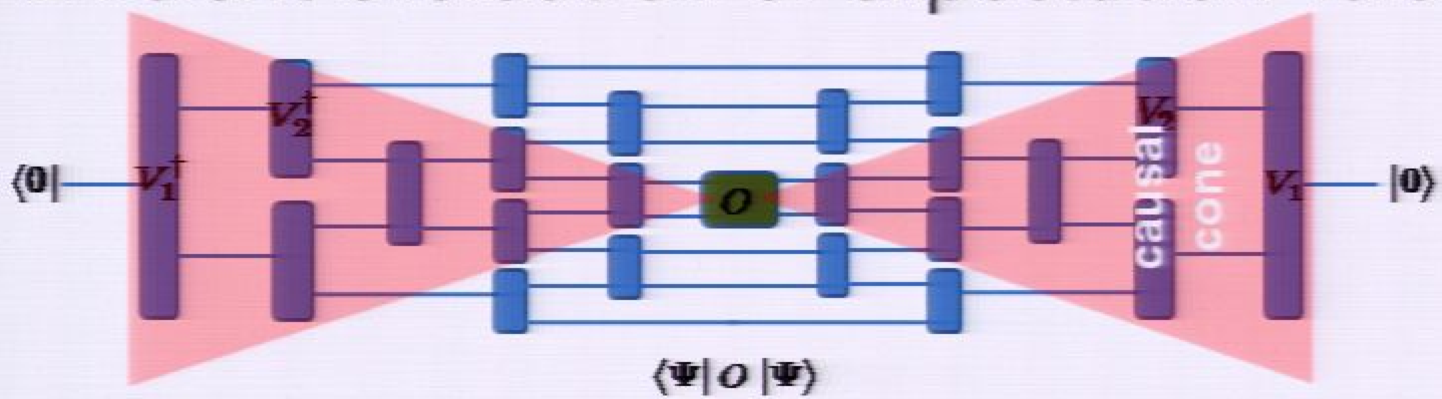

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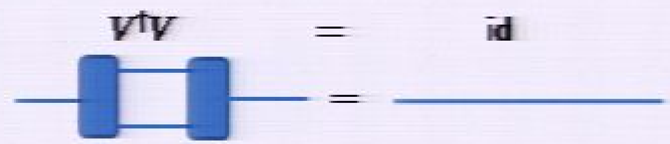
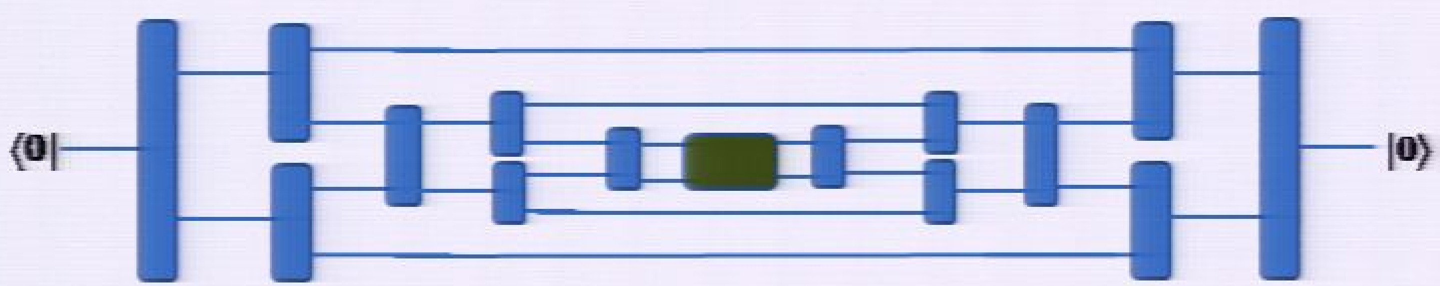
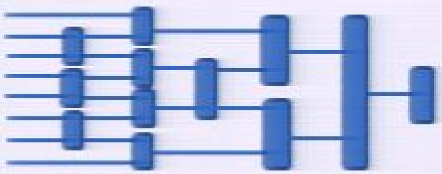




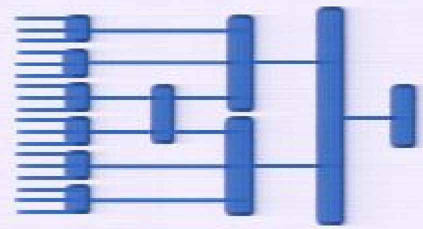
Entanglement renormalization: Summary

1. Choose a suitable tree-like graph

necessary condition: sufficiently "contracting" (causal cone of finite width)



binary

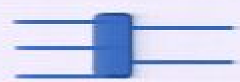


ternary

2. Associate variational parameters

qudit state

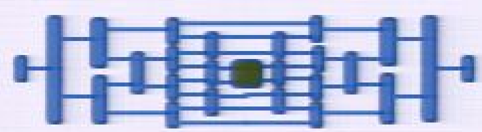
$|0\rangle^{\otimes 3} \in (\mathbb{C}^d)^{\otimes 3}$

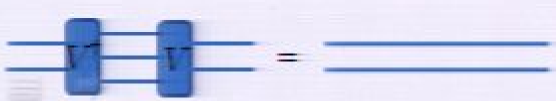


isometry

$V : (\mathbb{C}^d)^{\otimes 2} \to (\mathbb{C}^d)^{\otimes 3}$

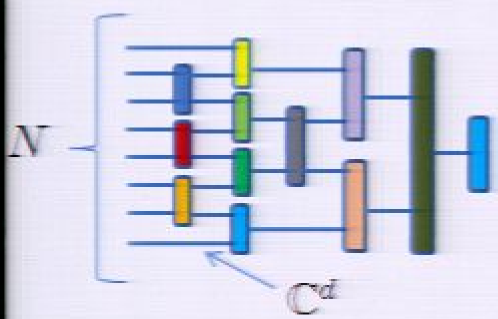
3. Compute expectation values from using substitutions



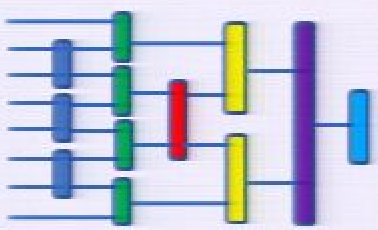


and contracting resulting simplified tensor network

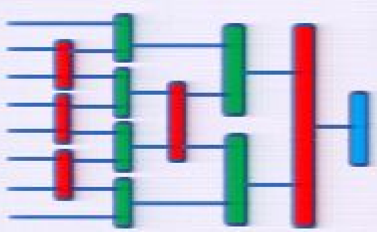
Parameter counting



general $O(poly(d) N \log N)$



translation-invariant $O(poly(d) \log N)$



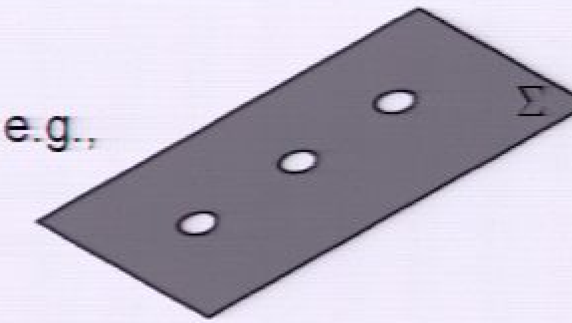
scale-invariant $O(poly(d))$

Introduction to anyons

- Background
- The anyonic Hilbert space
- Conservation of topological charge
- Anyonic Hamiltonians
- Computational rules

Topological order

2D-surface, e.g., plane with punctures



Characteristic properties:

1) Embed "quantum degrees of freedom" into Σ, e.g.,

- qudits placed on edges of a triangulation of Σ
- electrons confined to Σ

• ground space **degeneracy depending on** Σ

• **finite gap**, "stable" under local perturbations

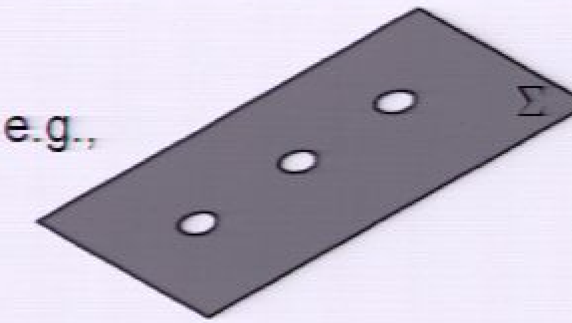
2) Assume certain "topological" Hamiltonian, e.g.,

- Levin-Wen-stringnet or Kitaev-toric code Hamiltonian
- Chern-Simons-action

store quantum info here

Topological order

2D-surface, e.g., plane with punctures



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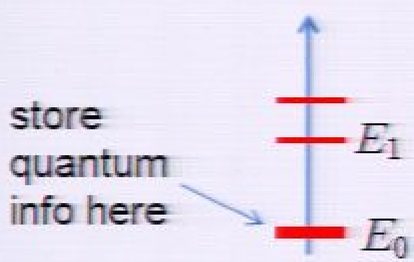
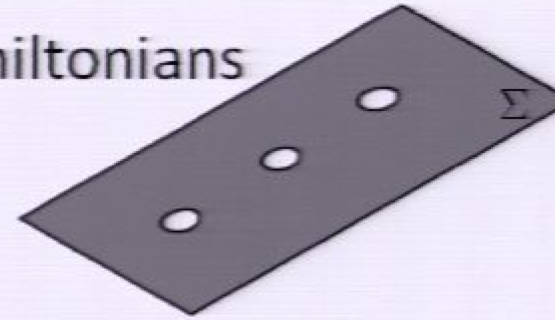


Local perturbations and effective Hamiltonians



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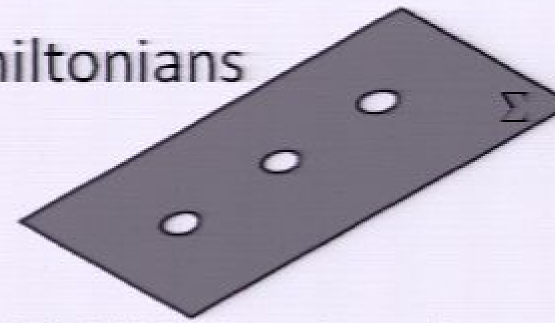

Characteristic properties:

(Bravyi, Hastings & Michalakis '10 have rigorously shown gap stability for a certain class of Hamiltonians. Perturbation theory suggests the same is true more generally.)

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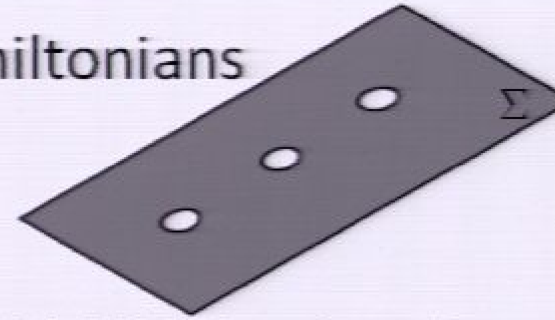

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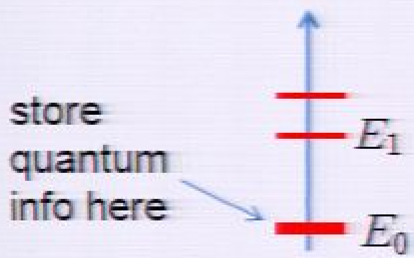
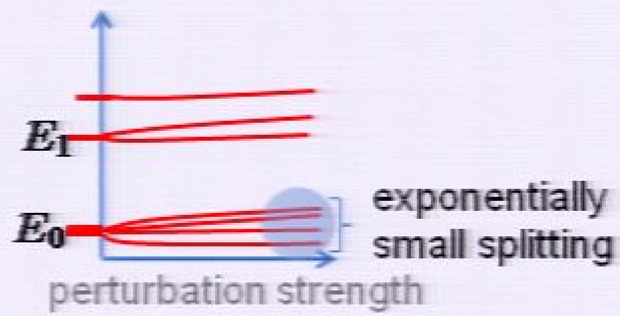
• **finite gap**, "stable" under local perturbations

How do low-energy degrees of freedom behave?

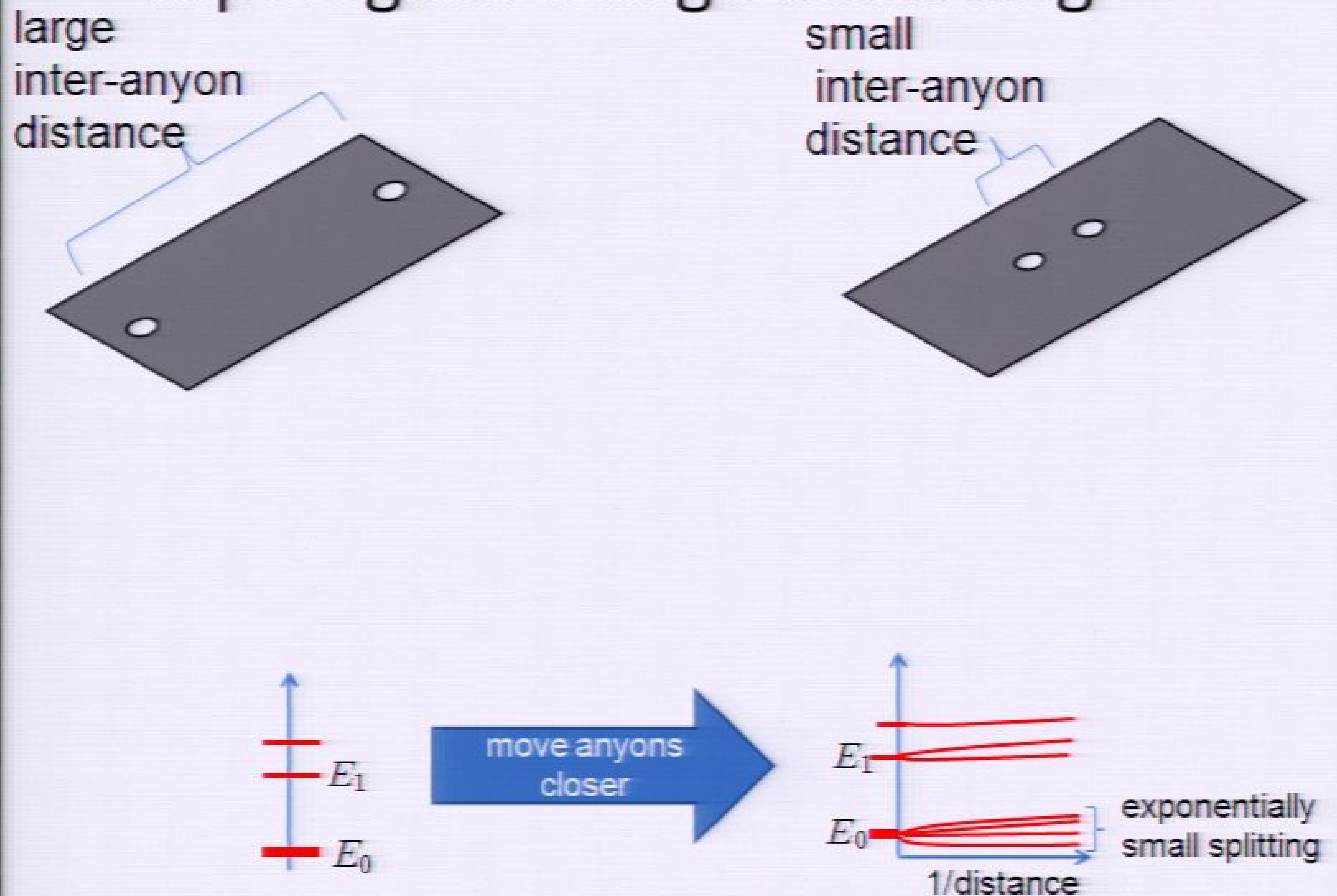
What is a good *effective model* (Hamiltonian)?

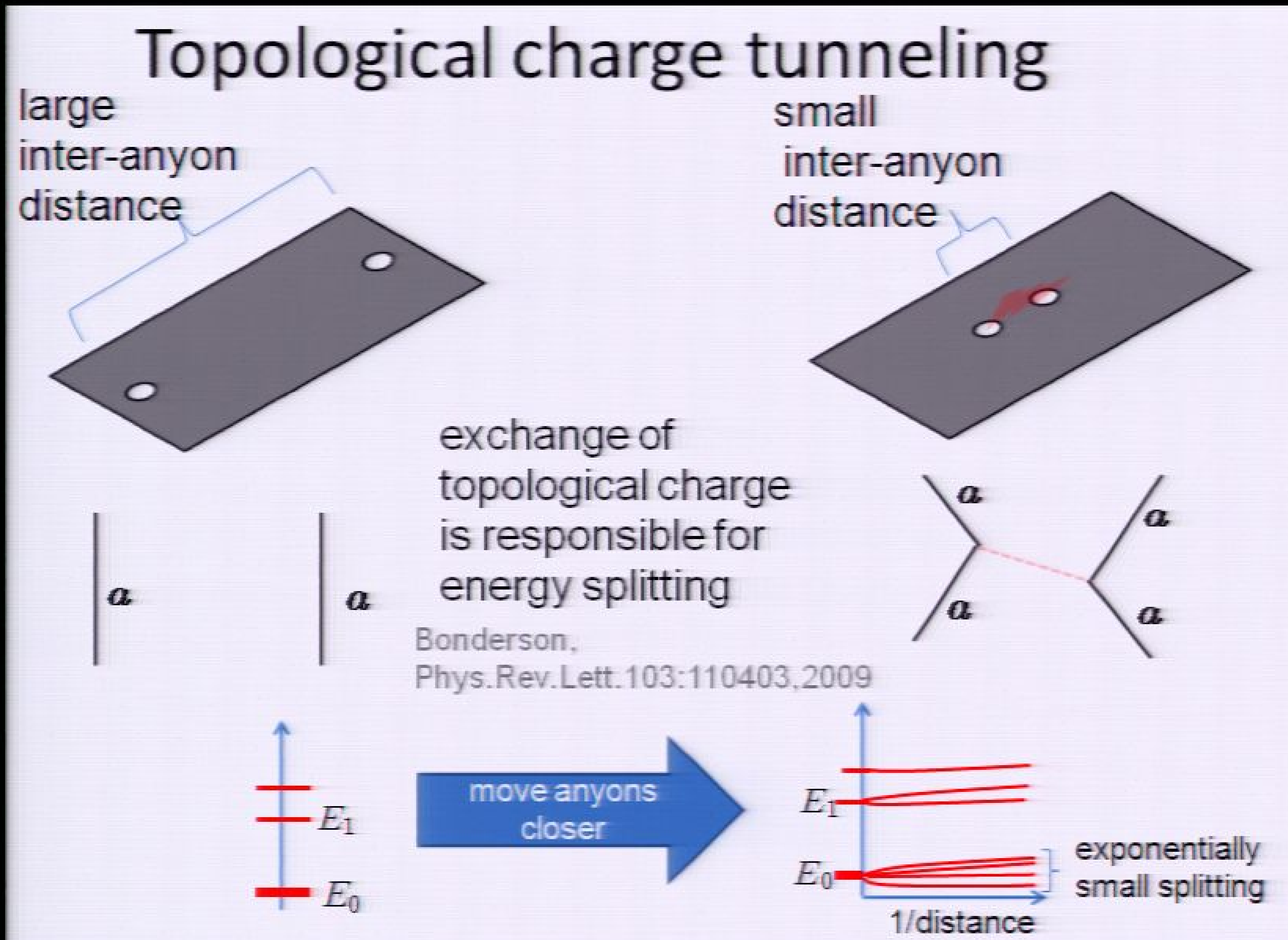


turn on local perturbations



Topological charge tunneling




Topological charge tunneling
large
inter-anyon
distance
small
inter-anyon
distance
a
a
exchange of
topological charge
is responsible for
energy splitting
a
a
a
a
Bonderson,
Phys.Rev.Lett.103:110403,2009
E1
E0
move anyons
closer
E1
E0
exponentially
small splitting
1/distance

Ground space $\mathcal{H}_\Sigma$

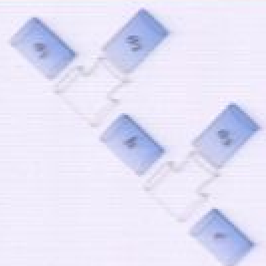
depends on topology of surface Σ

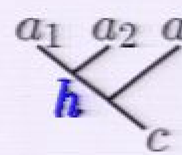
Underlying algebraic data: (UTMF/TQFT)

- set of labels "anyons" $\{a, b, c, \ldots,\}$
- fusion rules: set of triples of labels
- .....

Σ

Basis vectors of $\mathcal{H}_\Sigma$





(a_1, a_2, h)

(h, a_3, c)

fusion-consistent

Ground space $\mathcal{H}_\Sigma$

depends on topology of surface Σ

and a labeling of boundary components

Underlying algebraic data: (UTMF/TQFT)

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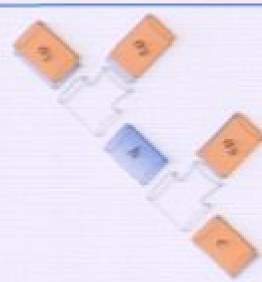
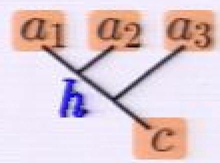
Σ

Basis vectors of $\mathcal{H}_\Sigma$





(a_1, a_2, h)

(h, a_3, c)

fusion-consistent

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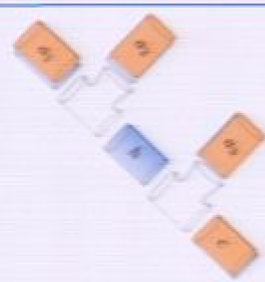
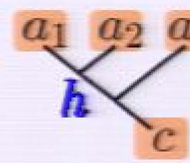
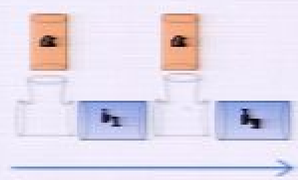
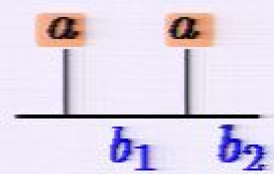
- .....

Σ		Basis vectors of $\mathcal{H}_\Sigma$	
			(a_1, a_2, h) (h, a_3, c) fusion-consistent
	 periodic		(b_2, a, b_1) (b_1, a, b_2) fusion-consistent

Ground space $\mathcal{H}_\Sigma$

depends on topology of surface Σ

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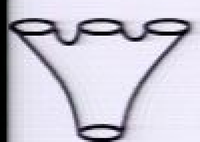
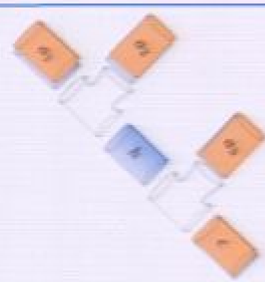
Underlying algebraic data: (UTMF/TQFT)

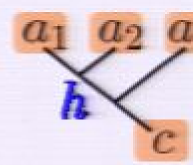
- set of labels "anyons" $\{a, b, c, \ldots,\}$
- fusion rules: set of triples of labels
- .....

Σ

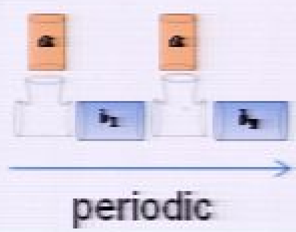
Basis vectors of $\mathcal{H}_\Sigma$

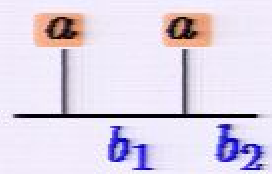




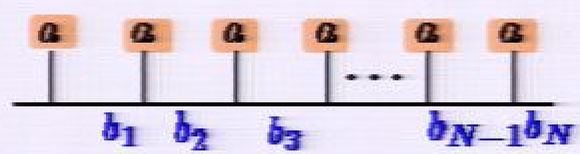


(a_1, a_2, h)
(h, a_3, c)
fusion-consistent





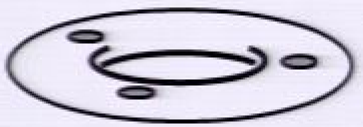
(b_2, a, b_1)
(b_1, a, b_2)
fusion-consistent



(b_j, a, b_{j+1})
fusion-consistent

Example state space of N Fibonacci anyons

torus Σ_N with N holes

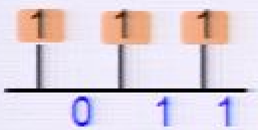


Fibonacci TQFT

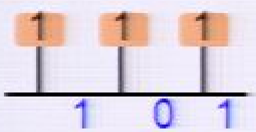
- label set $\{0, 1\}$
- fusion rules

$$(0,0,0), (1,1,0), (1,1,1) + \text{permutations}$$

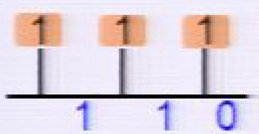
basis for N=3 Fibonacci anyons:



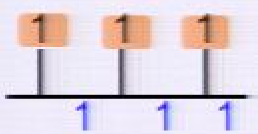
$\hookrightarrow \quad |011\rangle$



$\hookrightarrow \quad |101\rangle$



$\hookrightarrow \quad |110\rangle$

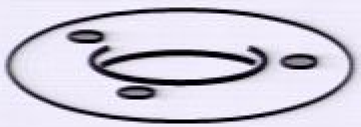


$\hookrightarrow \quad |111\rangle$

Every N-bit string ***without consecutive 0's*** represents a basis state.

Example state space of N Fibonacci anyons

torus Σ_N with N holes



Fibonacci TQFT

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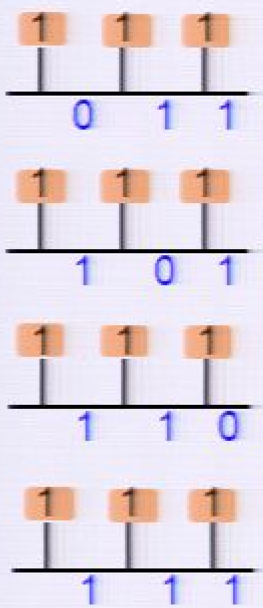


$\hookrightarrow |011\rangle$

$\hookrightarrow |101\rangle$

$\hookrightarrow |110\rangle$

$\hookrightarrow |111\rangle$

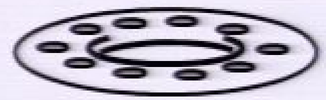
$$\dim \mathcal{H}_{\Sigma_N} \in O\big(\big(\underbrace{\tfrac{\sqrt{5}+1}{2}}_{\approx 1.62}\big)^N\big) << 2^N$$

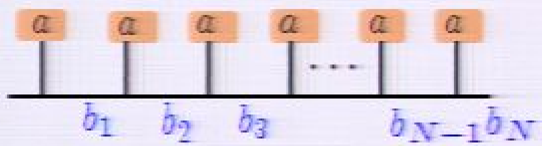
Treating Fibonacci anyons in terms of qubits is computationally disadvantageous!

Every N-bit string ***without consecutive 0's*** represents a basis state.

Example: state space of N anyons of type a

torus Σ_N with N holes each with label a





$\hookrightarrow$ $|b_1, b_2, \ldots, b_N\rangle$

(b_j, a, b_{j+1}) fusion-consistent

$$\mathcal{H}_{\Sigma_N} \subsetneq (\mathbb{C}^d)^{\otimes N}$$

anyonic states

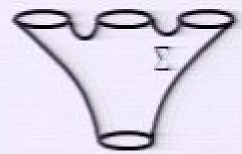
qudit states $d = \#\text{particle types}$

$$\dim \mathcal{H}_{\Sigma_N} \in O({d_a}^N) << d^N$$

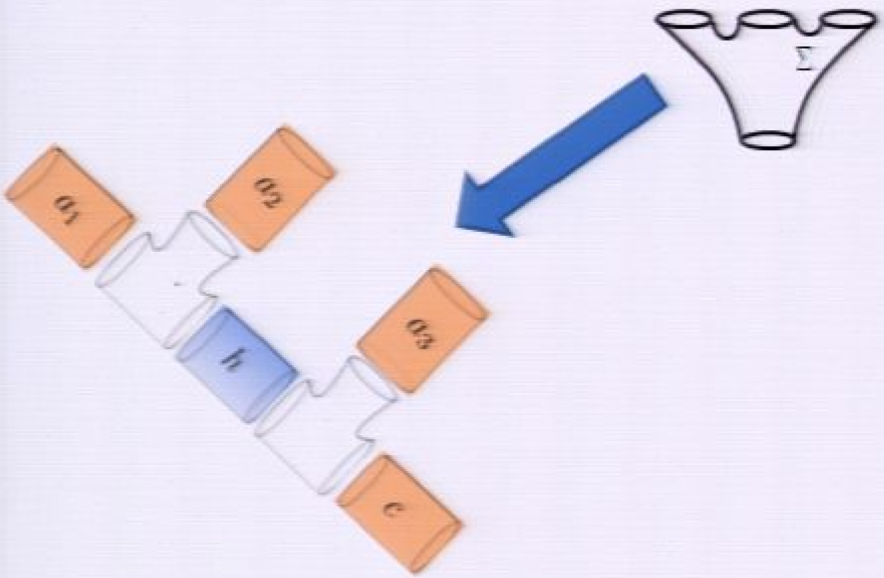
"quantum dimension" of particle a
(not necessarily integer!)

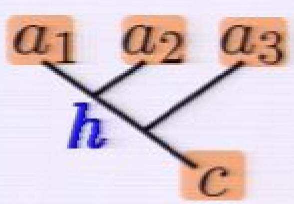
Treating anyons in terms of qudits is computationally disadvantageous!

"Locally related" pants decompositions



“Locally related” pants decompositions





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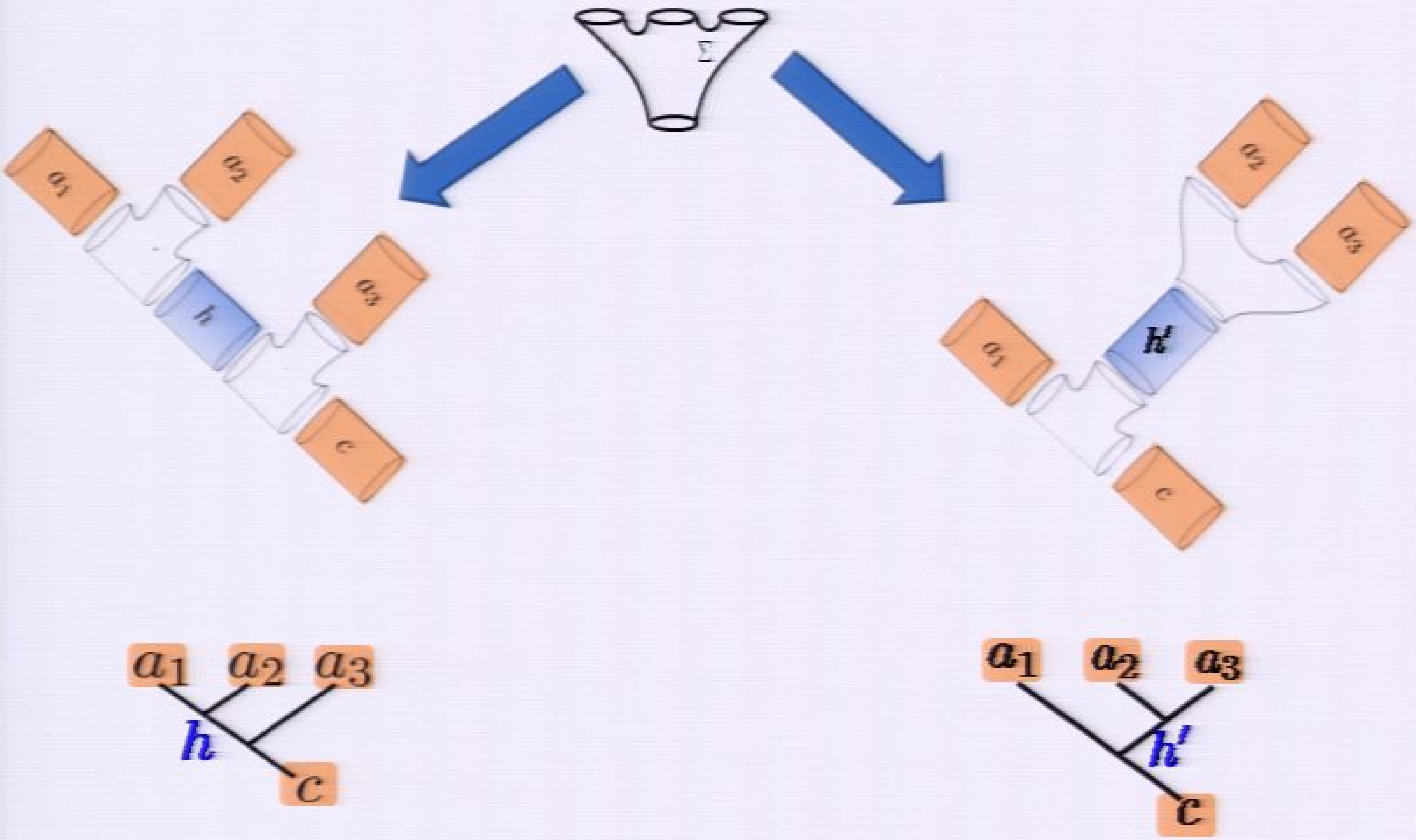
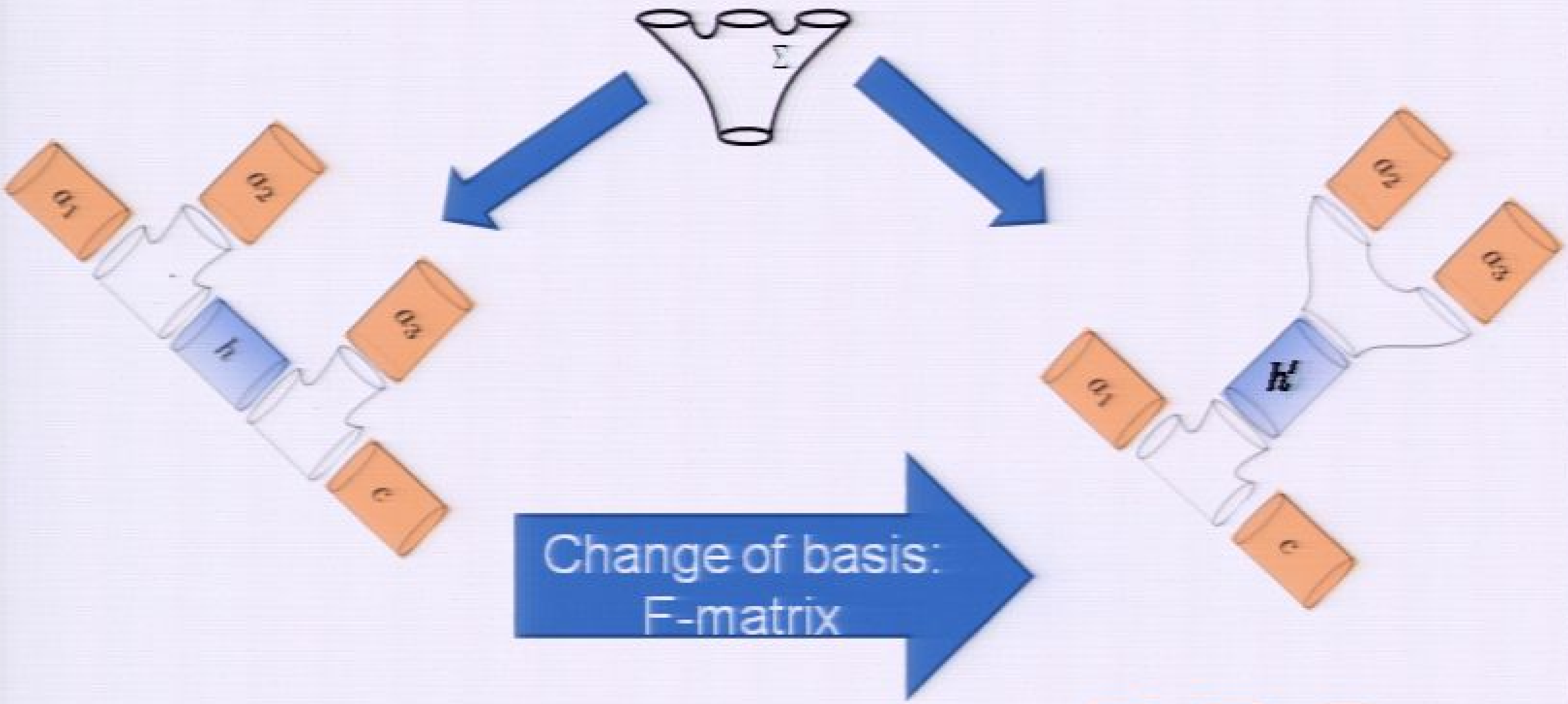
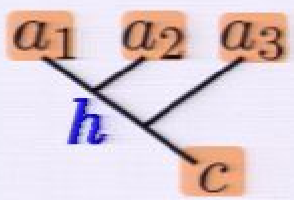
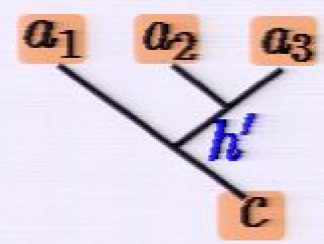


“Locally related” pants decompositions



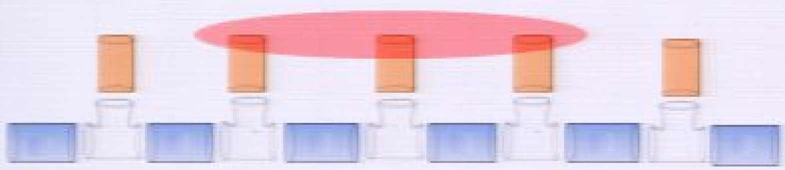


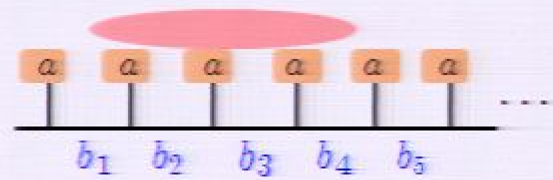
$$= \sum_{h'} F^{a_2 a_1 h}_{c a_3 h'}$$



Topological charge conservation by local operators

A local operator O acting on a region $\mathcal{A}$ (e.g., with 3-anyons) should only affect topological variables "inside" $\mathcal{A}$.





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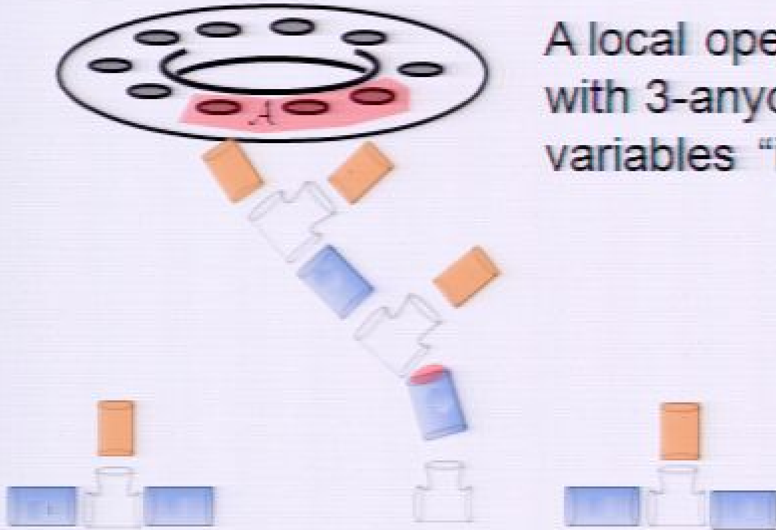

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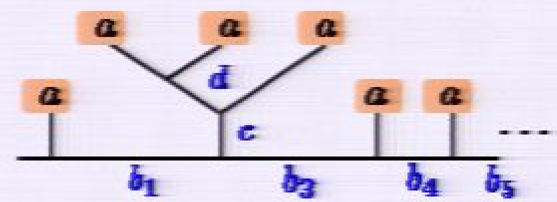
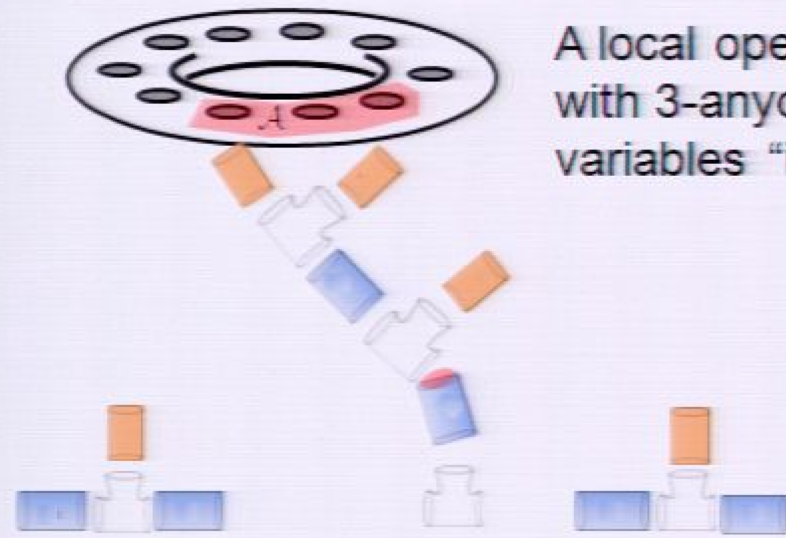


Consider a pants decomposition where the boundary ∂A is the boundary of a pant segment.

Topological charge conservation by local operators



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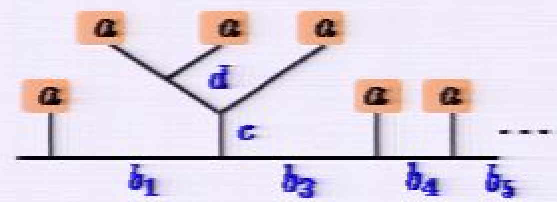
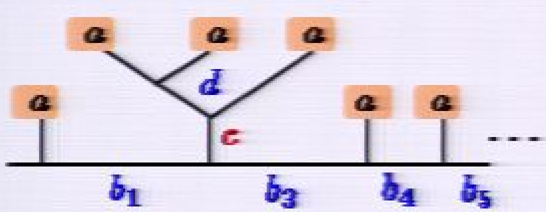
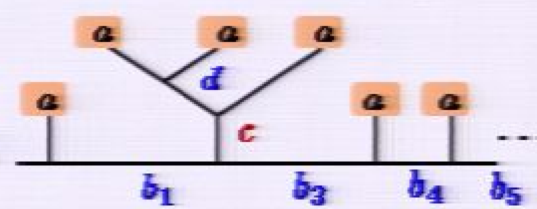


Consider a pants decomposition where the boundary $\partial\mathcal{A}$ is the boundary of a pant segment.

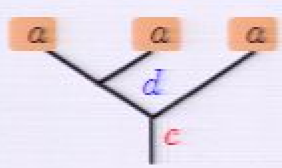


$$\mapsto \sum_{d'} (O_c)_{d',d}$$



The operator preserves only affects/depends on the subtree, and **preserves its total charge** c (->block-diagonal).

Ribbon graph representation of local operators



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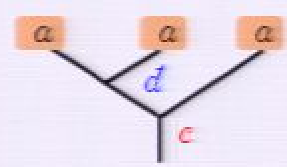

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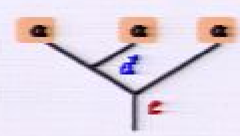
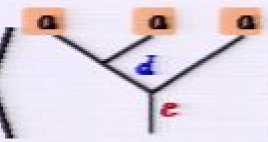
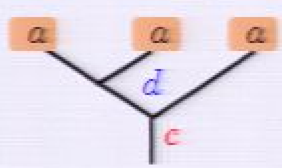
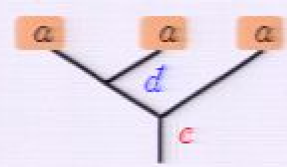
$O = \bigoplus_c \Big(\sum_{d,d'} (O_c)_{d',d} \Big|$ 

$\Big\rangle\Big\langle$ 

$\Big| \Big)$



$\rightarrow \sum_{d'} (O_c)_{d',d}$



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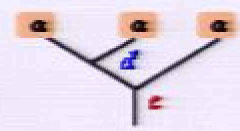
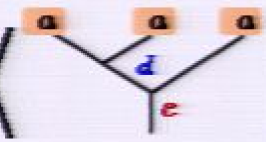
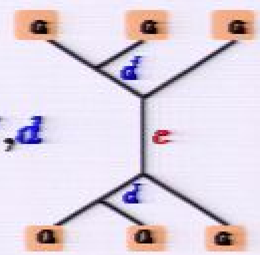
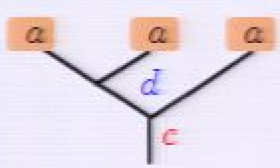
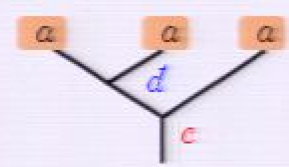
$O = \bigoplus_c \Big(\sum_{d,d'} (O_c)_{d',d} \Big|$  $\Big\rangle\Big\langle$  $\Big| \Big)$

$=: \sum_{c,d,d'} (O_c)_{d',d}$ 

This graphical representation of operators incorporates charge conservation and obeys simple rules for taking adjoints, multiplication,...



$\to \sum_{d'} (O_c)_{d',d}$ 

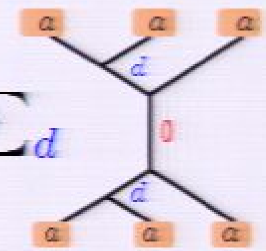
The operator preserves only affects/depends on the subtree, and **preserves its total charge** c (->block-diagonal).

Projection onto trivial charge and the golden chain

$$\Pi_{1,2} =$$

pair-interaction

$$\Pi_{1,2,3} = \sum_d$$



3-anyon interaction

Projection onto trivial charge and the golden chain

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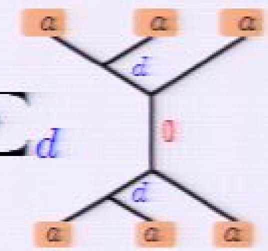

3-anyon interaction

translation-invariant effective "anyonic chain" model

$$H = -J_1 \sum_i \Pi_{i,i+1} - J_2 \sum_i \Pi_{i,i+1,i+2}$$

Projection onto trivial charge and the golden chain

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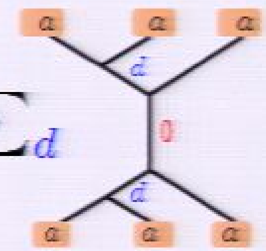


3-anyon interaction

translation-invariant effective "anyonic chain" model

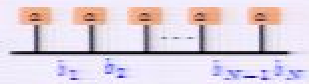
$$H = -J_1 \sum_i \Pi_{i,i+1} - J_2 \sum_i \Pi_{i,i+1,i+2}$$

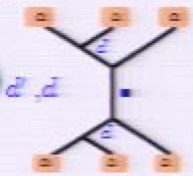
For Fibonacci-anyons:
- exact solution known for $J_2 = 0$
  Feiguin et al., Phys. Rev. Lett. 98, 160409 (2007)
- phase diagram (numerical)
  Trebst et al., Phys. Rev. Lett. 101, 050401

Rest of this talk: (1) variational ansatz
(2) numerical test of validity

various generalizations:
- su(2)$_k$, 2D arrangements of anyons
  Gils et al., Phys. Rev. Lett. 103, 070401
  Ludwig et al., arxiv:1003.3453
- random couplings
  Fidkowski et al., Phys. Rev. B 78, 22204
  Fidkowski et al., Phys. Rev. B 79,155120

Anyonic formalism: Computational rules

$|\Psi\rangle = \sum_{b_1,\ldots,b_N} \Psi_{b_1,\ldots,b_N}$ 

$O = \sum_{c,d,d'} (O_c)_{d',d}$ 

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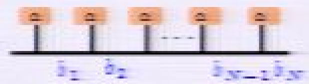
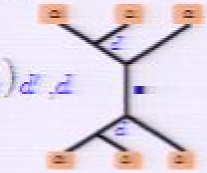
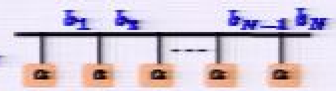
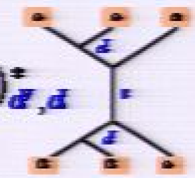
$|\Psi\rangle = \sum_{b_1,\ldots,b_N} \Psi_{b_1,\ldots,b_N}$ 

$O = \sum_{c,d,d'} (O_c)_{d',d}$ 

- **adjoints**: invert & conjugate

$\langle\Psi| = \sum_{b_1,\ldots,b_N} \Psi^*_{b_1,\ldots,b_N}$ 

$O^\dagger = \sum_{c,d,d'} (O_c)^*_{d',d}$ 

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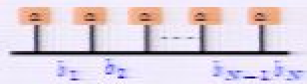
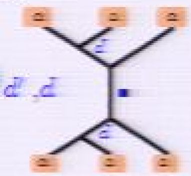
$|\Psi\rangle = \sum_{b_1,\dots,b_N} \Psi_{b_1,\dots,b_N}$ 

$O = \sum_{c,d,d'} (O_c)_{d',d}$ 

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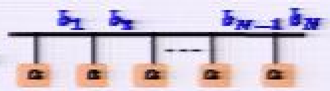
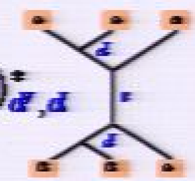
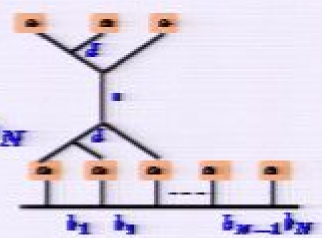
$\langle\Psi| = \sum_{b_1,\dots,b_N} \Psi^*_{b_1,\dots,b_N}$ 

$O^\dagger = \sum_{c,d,d'} (O_c)^*_{d',d}$ 

- **multiplication**: stack

$O|\Psi\rangle = \sum_{\substack{b_1,\dots,b_N \\ c,d,d'}} (O_c)_{d',d} \Psi_{b_1,\dots,b_N}$ 

Anyonic formalism: Computational rules

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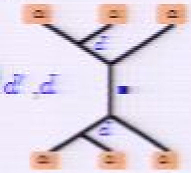
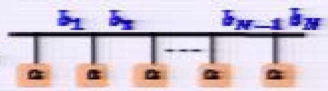
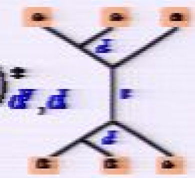
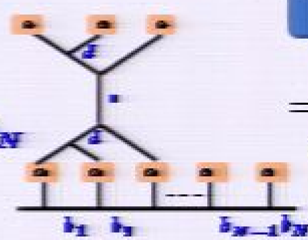
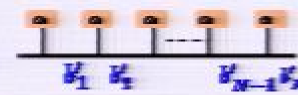
$O = \sum_{c,d,d'} (O_c)_{d',d}$ 

- **adjoints**: invert & conjugate

$\langle\Psi| = \sum_{b_1,\dots,b_N} \Psi^*_{b_1,\dots,b_N}$ 

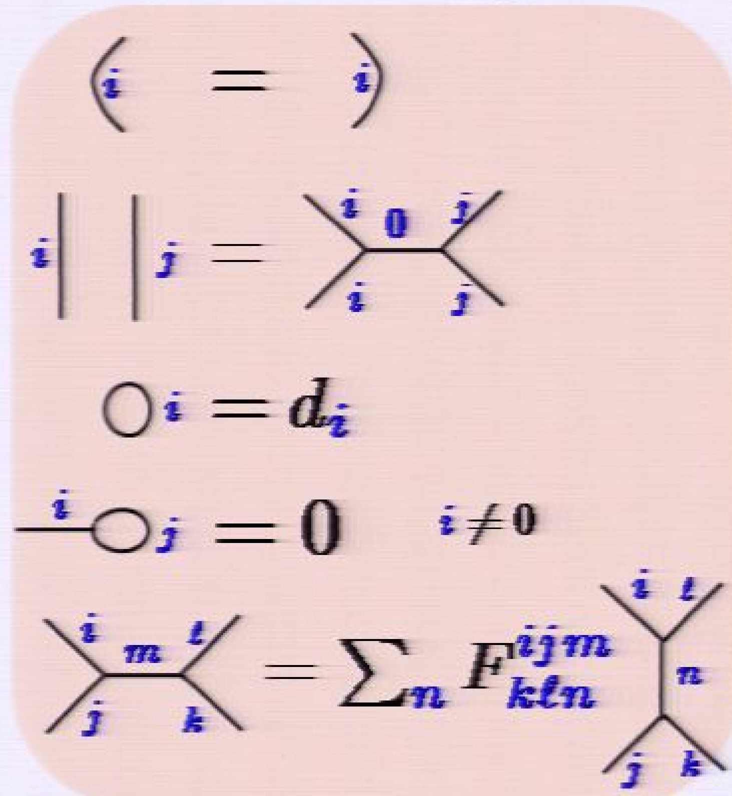
$O^\dagger = \sum_{c,d,d'} (O_c)^*_{d',d}$ 

- **multiplication**: stack & simplify

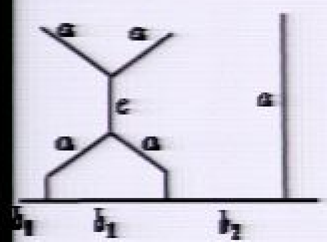
$O|\Psi\rangle = \sum_{\substack{b_1,\dots,b_N\\ c,d,d'}} (O_c)_{d',d} \Psi_{b_1,\dots,b_N}$  $= \sum_{b'_1,\dots,b'_N} \Psi'_{b'_1,\dots,b'_N}$ 

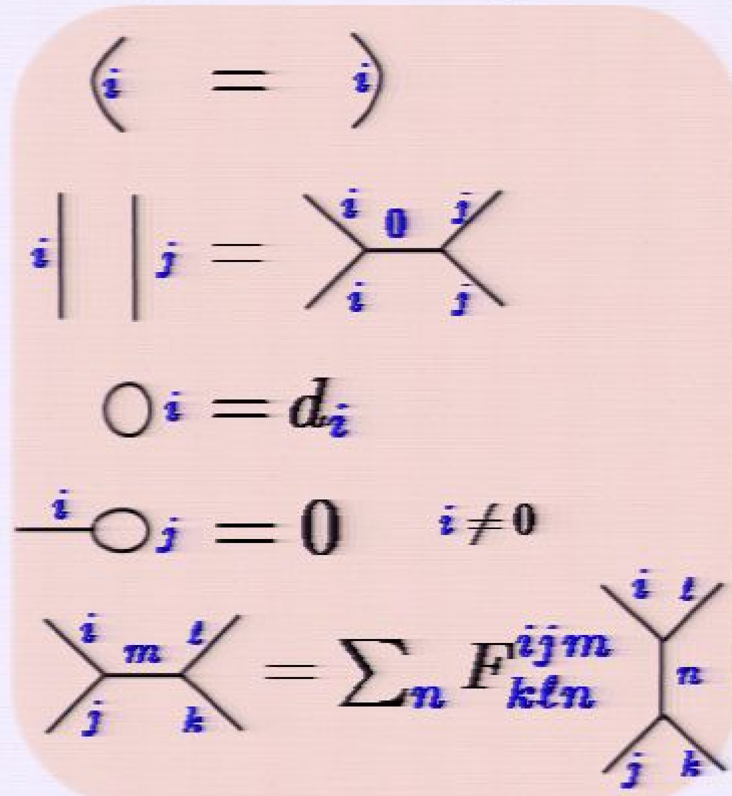
some sequence of subgraph deformations/substitutions

Simplification rules for anyonic diagrams

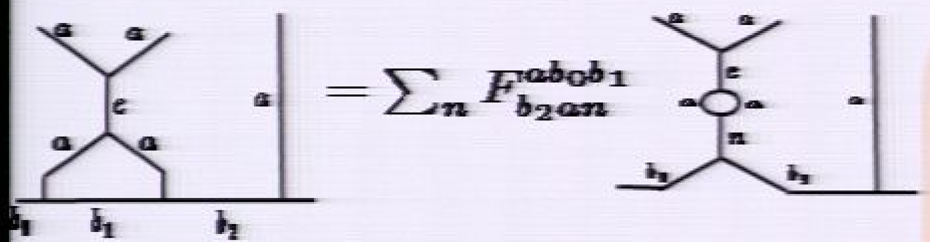


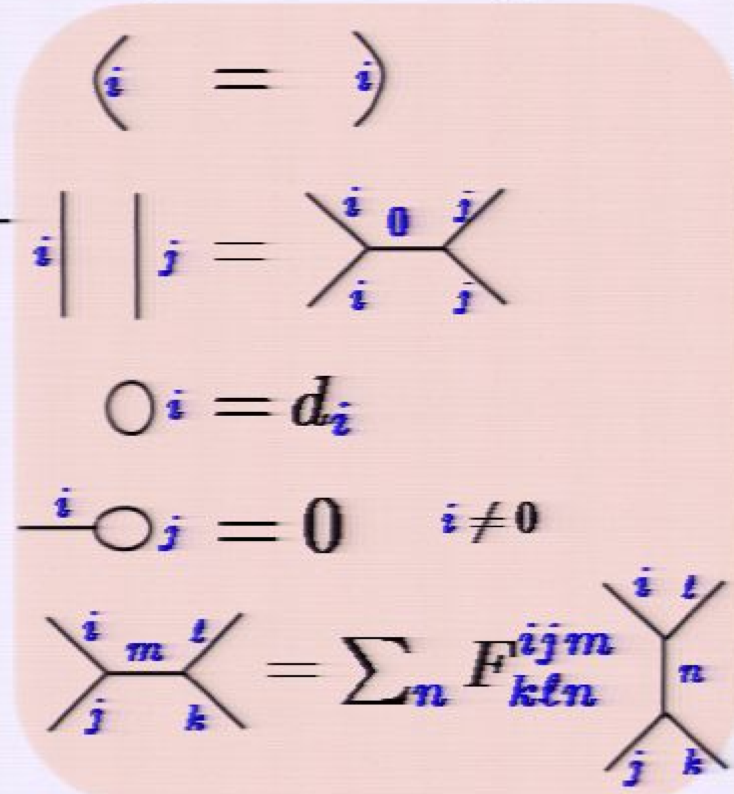
Simplification rules for anyonic diagrams



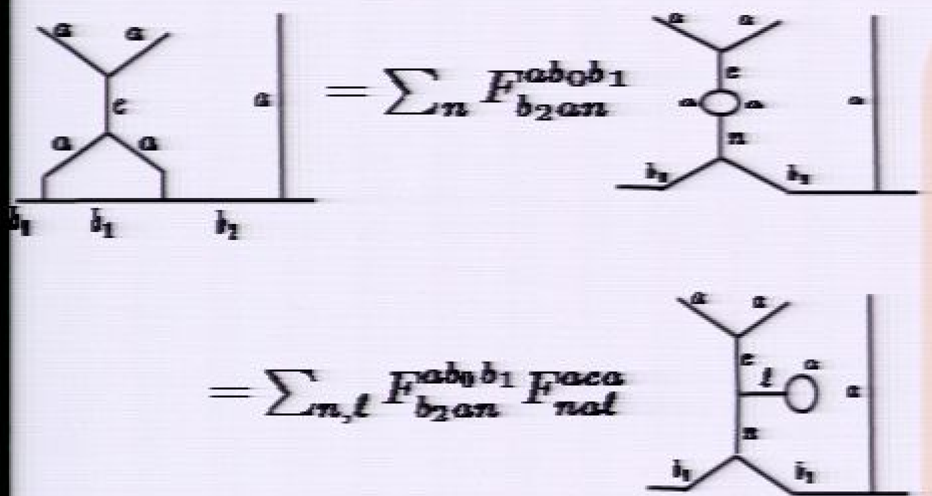


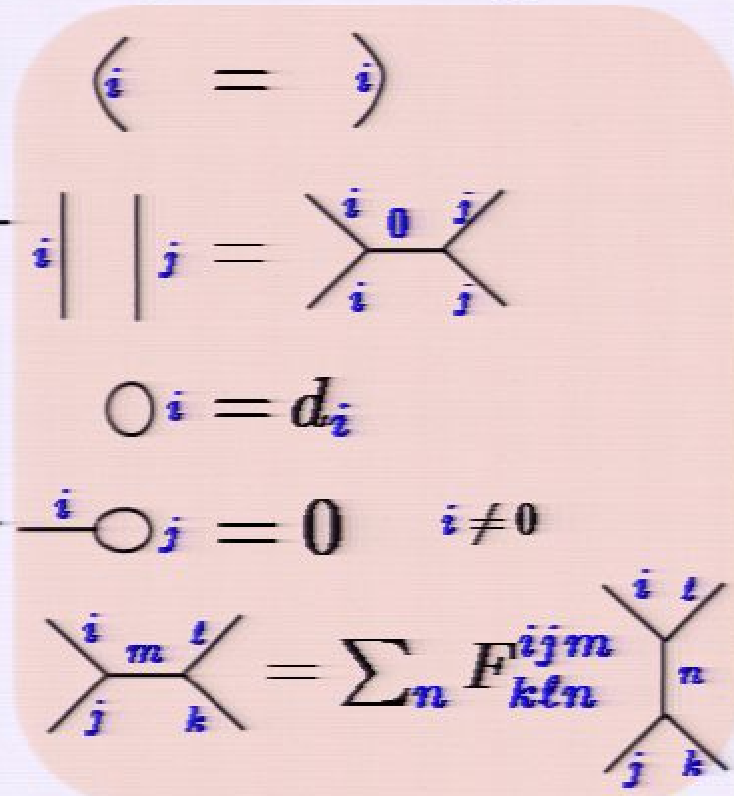
Simplification rules for anyonic diagrams



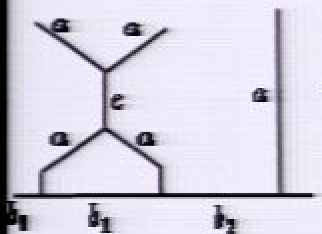


Simplification rules for anyonic diagrams

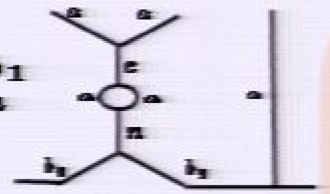




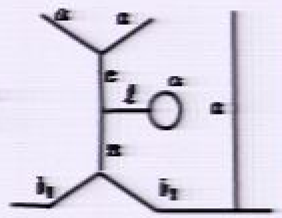
Simplification rules for anyonic diagrams



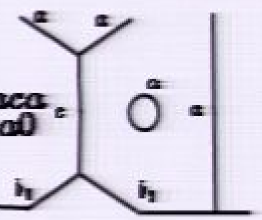
$= \sum_n F^{ab_0b_1}_{b_2an}$



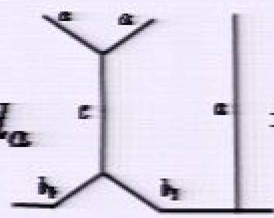
$= \sum_{n,\ell} F^{ab_0b_1}_{b_2an} F^{aca}_{na\ell}$



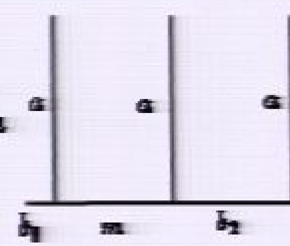
$= F^{ab_0b_1}_{b_2ac} F^{aca}_{ca0}$

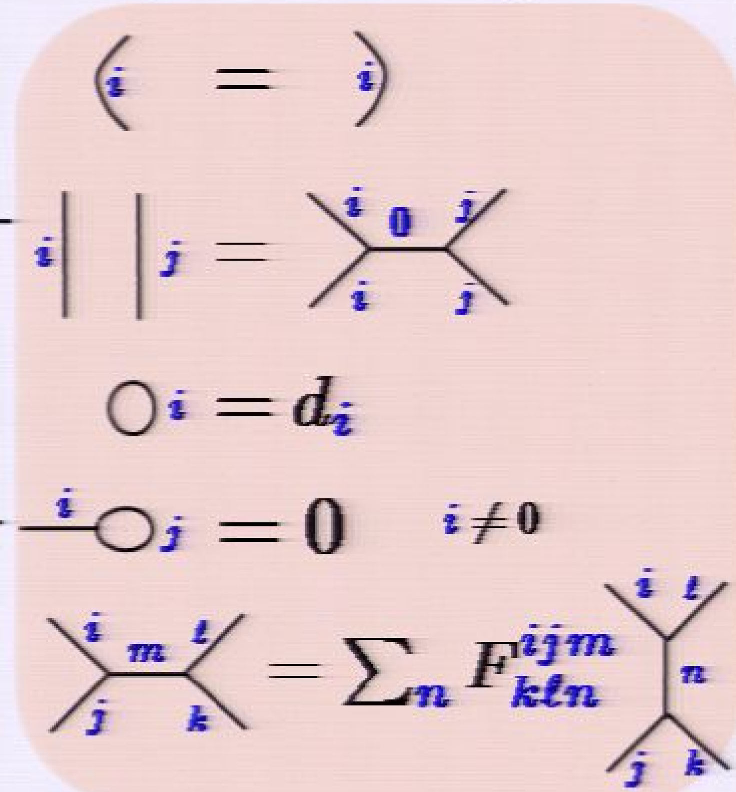


$= F^{ab_0b_1}_{b_2ac} F^{aca}_{ca0} d_a$

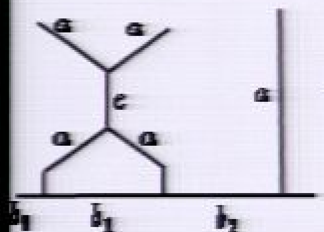
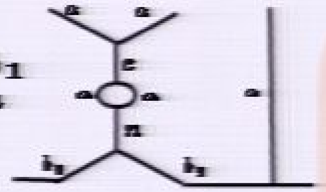


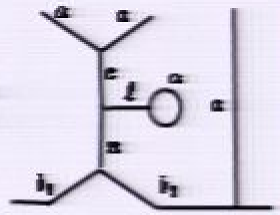
$= \sum_m F^{ab_0b_1}_{b_2ac} F^{aca}_{ca0} d_a F^{aac}_{b_0b_2m}$

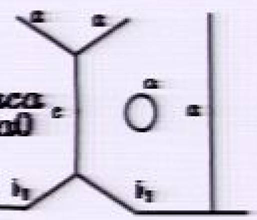


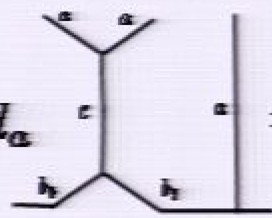
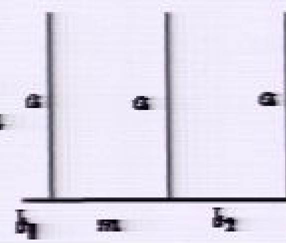


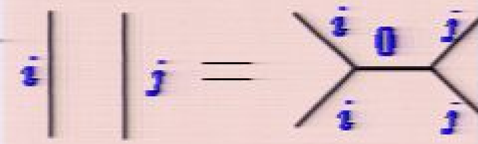
Simplification rules for anyonic diagrams

 $= \sum_n F^{ab_0b_1}_{b_2an}$ 

$= \sum_{n,\ell} F^{ab_0b_1}_{b_2an} F^{aca}_{na\ell}$ 

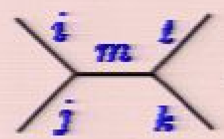
$= F^{ab_0b_1}_{b_2ac} F^{aca}_{ca0}$ 

$= F^{ab_0b_1}_{b_2ac} F^{aca}_{ca0} d_a$  $= \sum_m F^{ab_0b_1}_{b_2ac} F^{aca}_{ca0} d_a F^{aac}_{b_0b_2m}$ 



 $= d_i$

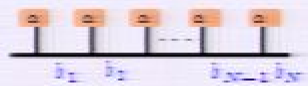
 $= 0 \qquad i \neq 0$

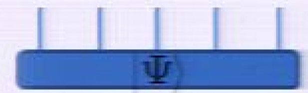
 $= \sum_n F^{ijm}_{k\ell n}$ 

Anyonic formalism *versus* tensor networks

state

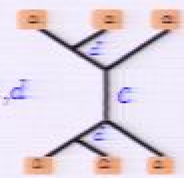
$$|\Psi\rangle = \sum_{\vec{b}} \Psi_{\vec{b}}$$

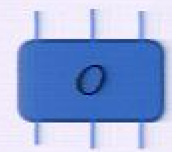




local operator

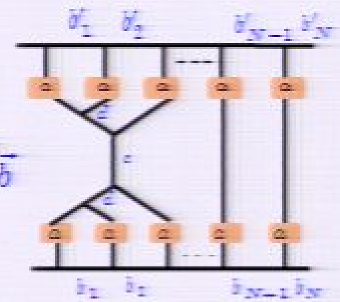
$$O = \sum_{c,d,d'} (O_c)_{d',d}$$

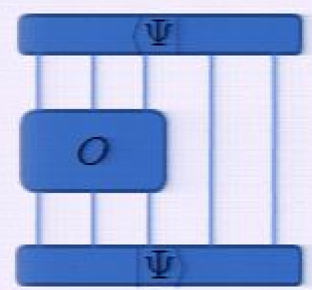




expectation value

$$\langle\Psi|O|\Psi\rangle = \sum_{\substack{\vec{b},\vec{b}' \\ c,d,d'}} \Psi^*_{\vec{b}'} \cdot (O_c)_{d',d} \cdot \Psi_{\vec{b}}$$





evaluated by **local rules**

evaluated by **tensor contraction**

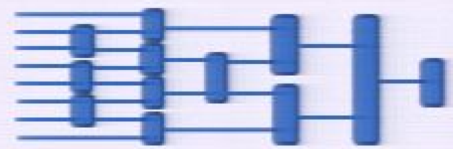
Anyonic entanglement renormalization:

a variational ansatz which

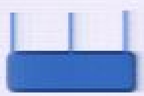
- exploits structure of Hilbert space
- allows straightforward generalization to 2D
- has operational meaning

Anyonic entanglement renormalization: Definition

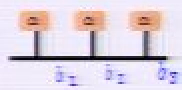
1. Choose a suitable tree-like graph



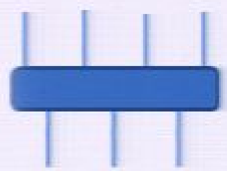
2. Associate (variational) parameters



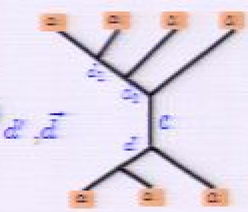
$|\varphi\rangle = \sum_{\vec{b}} \varphi_{\vec{b}}$



(3-)anyon state



$V = \sum_{c,\vec{d},\vec{d}'} (V_c)_{\vec{d}',\vec{d}}$



topological charge-preserving isometry

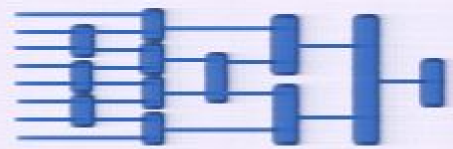
3. Substitute boxes by graphical representation

Result: formal linear combination,

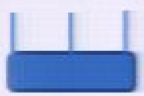
represents variational state $|\Psi\rangle$

Anyonic entanglement renormalization: Definition

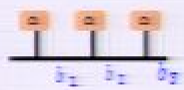
1. Choose a suitable tree-like graph



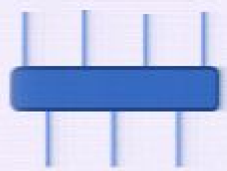
2. Associate (variational) parameters



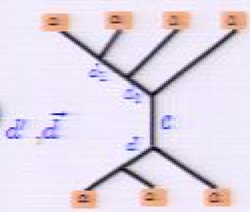
$|\varphi\rangle = \sum_{\vec{b}} \varphi_{\vec{b}}$



(3-)anyon state



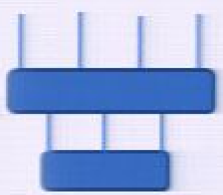
$V = \sum_{c,\vec{d},\vec{d}'} (V_c)_{\vec{d}',\vec{d}}$



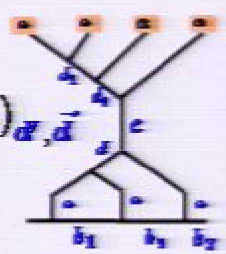
topological charge-preserving isometry

3. Substitute boxes by graphical representation

example:



$|\Psi\rangle = \sum_{\vec{b},c,\vec{d},\vec{d}'} \varphi_{\vec{b}} \cdot (V_c)_{\vec{d}',\vec{d}}$

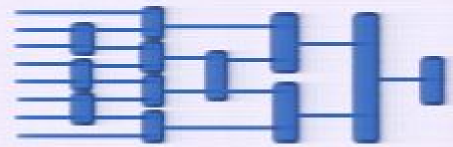


Result: formal linear combination,

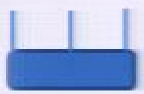
represents variational state $|\Psi\rangle$

Anyonic entanglement renormalization: Definition

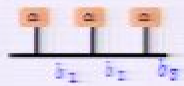
1. Choose a suitable tree-like graph



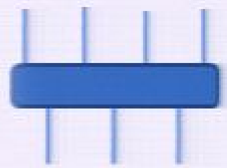
2. Associate (variational) parameters



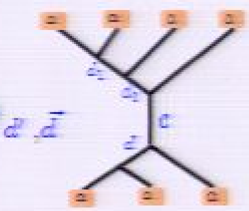
$|\varphi\rangle = \sum_{\vec{b}} \varphi_{\vec{b}}$



(3-)anyon state



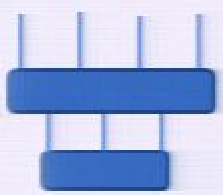
$V = \sum_{c,\vec{d},\vec{d}'} (V_c)_{\vec{d}',\vec{d}}$



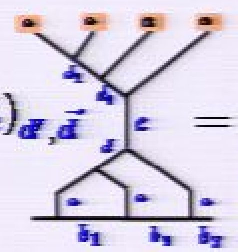
topological charge-preserving isometry

3. Substitute boxes by graphical representation

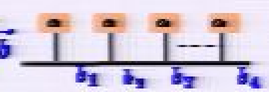
example:



$|\Psi\rangle = \sum_{\vec{b},c,\vec{d},\vec{d}'} \varphi_{\vec{b}} \cdot (V_c)_{\vec{d}',\vec{d}}$



$= \sum_{\vec{b}} \Psi_{\vec{b}}$



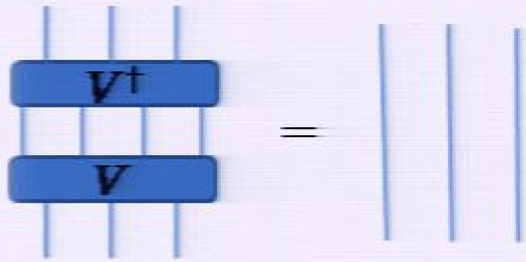
Computing this coefficient is generally hard, but we don't need to!

Result: formal linear combination,

represents variational state $|\Psi\rangle$

Topological charge-preserving isometries

The familiar rule

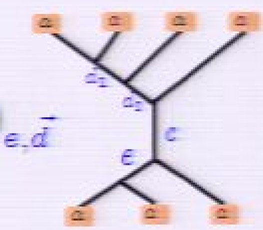


applies in the anyonic context

if $\boldsymbol{V}$ is a

charge-preserving isometry:

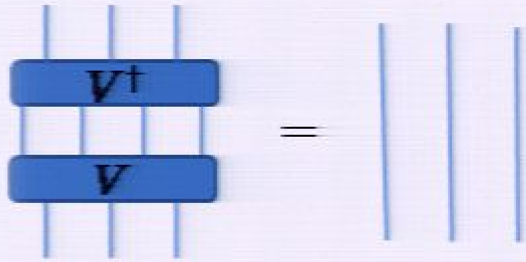
$$V = \bigoplus_c \sum_{\vec{d}, e} (V_c)_{e,\vec{d}}$$



defined by a family $\{V_c\}_c$ of isometries, i.e., $V_c^\dagger V_c = \mathrm{id}_c$

Topological charge-preserving isometries

The familiar rule

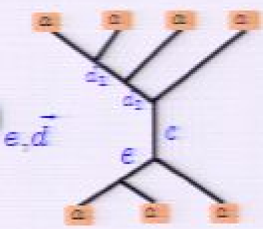


applies in the anyonic context

if V is a

charge-preserving isometry:

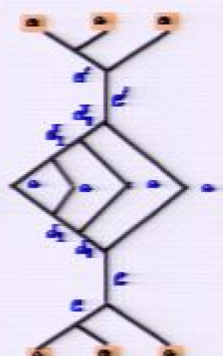
$$V = \bigoplus_{c} \sum_{\vec{d},e} (V_c)_{e,\vec{d}}$$



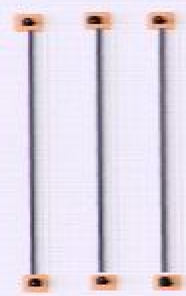
defined by a family $\{V_c\}_c$ of isometries, i.e., $V_c^\dagger V_c = \mathrm{id}_c$

This is because such operators satisfy the diagrammatic identity

$$V^\dagger V = \bigoplus_{c,c'} \sum_{\vec{d},\vec{d}',e,e'} (V_{c'})_{e',\vec{d}'} \cdot (V_c)_{e,\vec{d}}$$

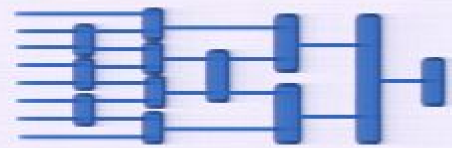


$=$

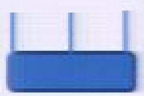


Anyonic entanglement renormalization

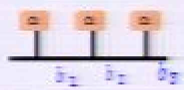
1. Choose a suitable tree-like graph



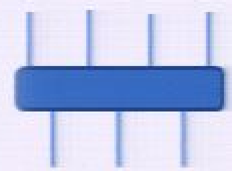
2. Associate (variational) parameters



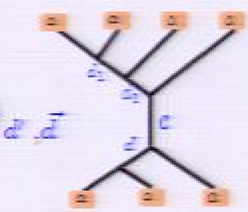
$|\varphi\rangle = \sum_{\vec{b}} \varphi_{\vec{b}}$



(3-)anyon state

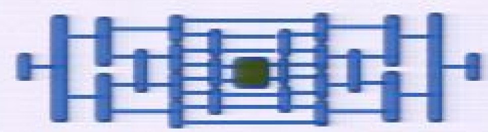


$V = \sum_{c,\vec{d'},\vec{d}} (V_c)_{\vec{d'},\vec{d}}$



topological charge-preserving isometry

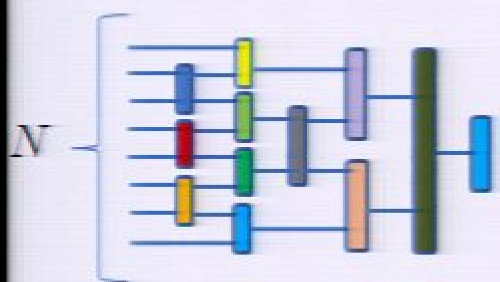
3. Compute expectation values from using substitutions



and evaluating resulting simplified anyon diagram

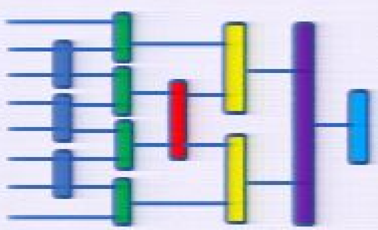
With these modifications, **techniques** developed **for spin chains** (e.g., for numerically varying over isometries, extracting critical exponents/CFT data) **can be applied to anyons**.

Parameter counting: N anyons of type a

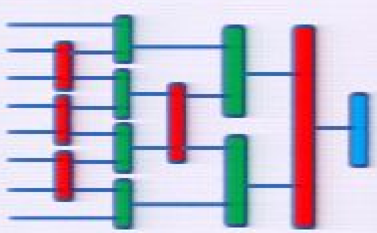


general $O(poly(d_a) N \log N)$

quantum dimension



translation-invariant $O(poly(d_a) \log N)$



scale-invariant $O(poly(d_a))$

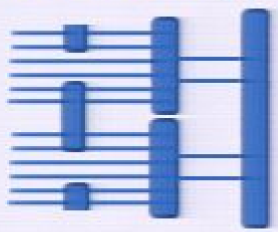
Numerical test

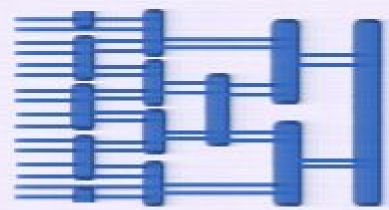
Application to the golden chain

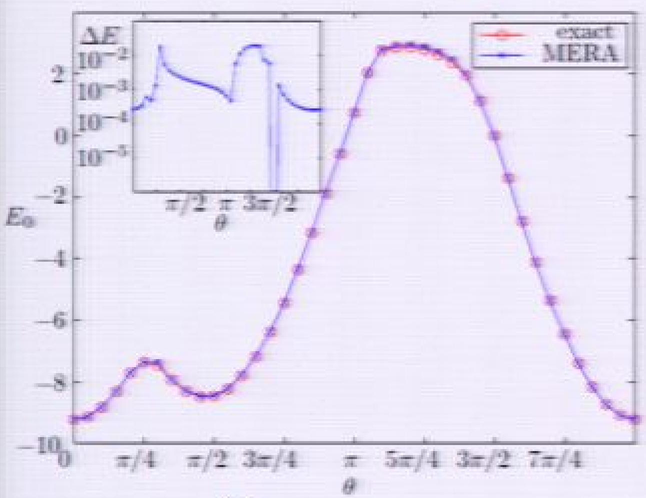
$$H = -J_1 \sum_i \Pi_{i,i+1} - J_2 \sum_i \Pi_{i,i+1,i+2}$$

$J_1 = \cos\theta,\ J_2 = \sin\theta$

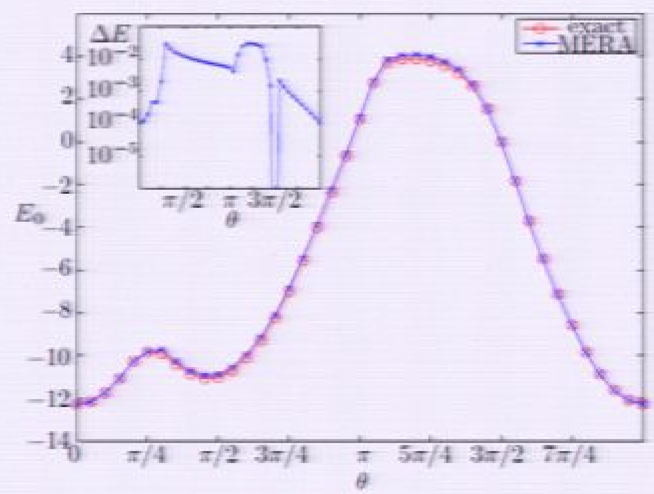
Ground state energy computed by anyonic MERA:







12 anyons

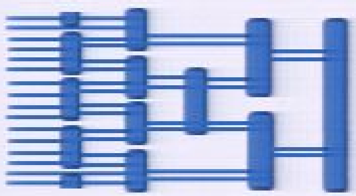


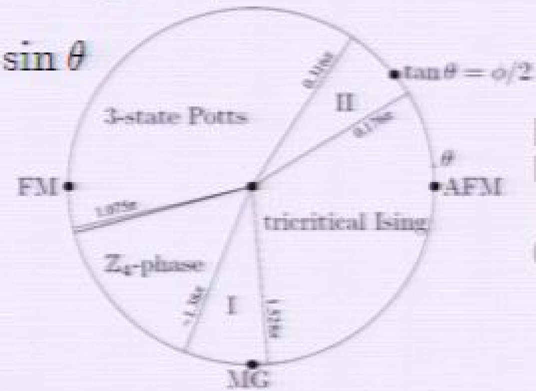
16 anyons

Application to the golden chain

$$H = -J_1 \sum_i \Pi_{i,i+1} - J_2 \sum_i \Pi_{i,i+1,i+2}$$

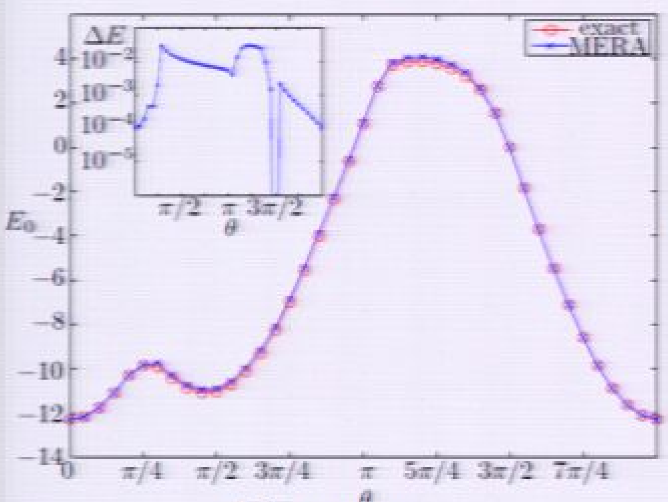
$J_1 = \cos\theta, J_2 = \sin\theta$



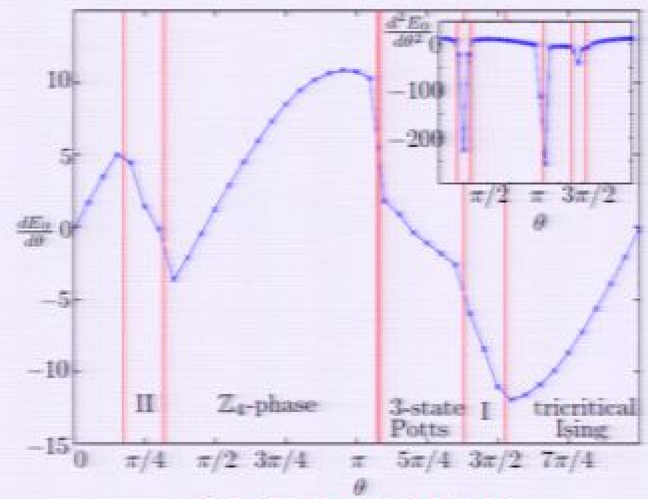


phase diagram obtained by Trebst et al., Phys. Rev. Lett. 101, 050401

ground state energy computed by anyonic MERA



16 anyons

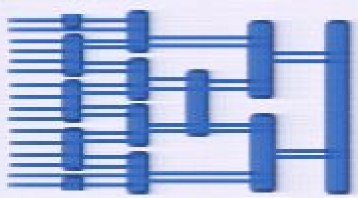


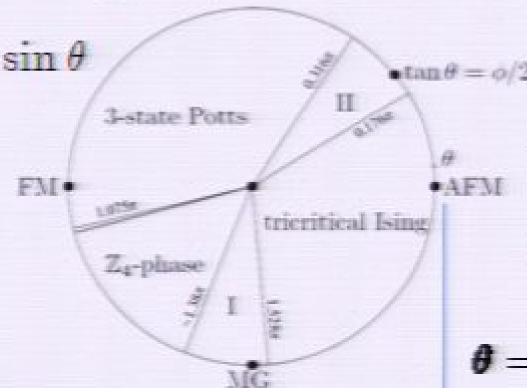
phase boundaries

Application to the golden chain

$$H = -J_1 \sum_i \Pi_{i,i+1} - J_2 \sum_i \Pi_{i,i+1,i+2}$$

$J_1 = \cos\theta,\ J_2 = \sin\theta$



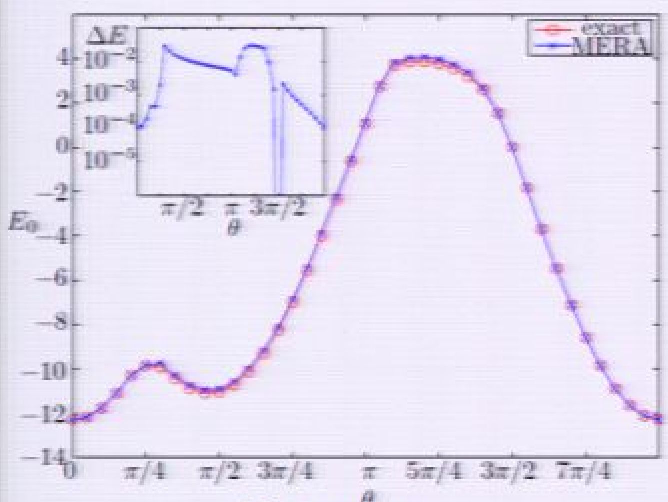


phase diagram obtained by Trebst et al., Phys. Rev. Lett. 101, 050401

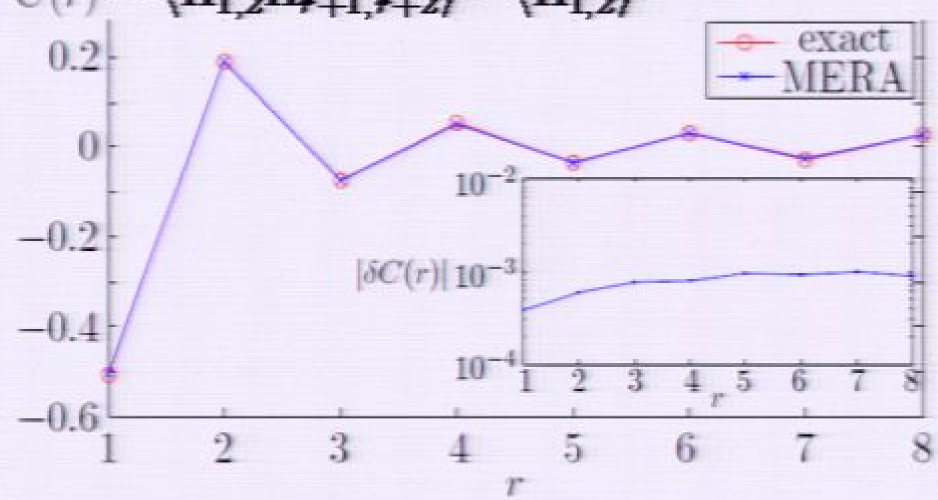
ground state energy computed by anyonic MERA

$\theta = 0$

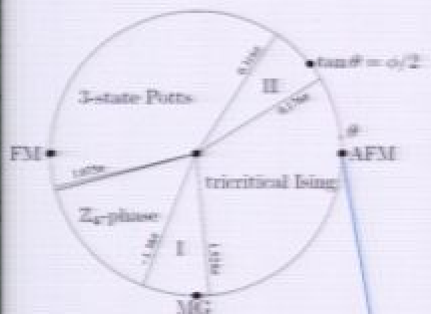
$$C(r) = \langle \Pi_{1,2}\Pi_{r+1,r+2}\rangle - \langle \Pi_{1,2}\rangle^2$$



16 anyons



Large systems and CFT



Feiguin et al.,
Phys. Rev. Lett. 98, 160409 (2007)

CFT identified: $c = 7/10$

spectrum of periodic 1D critical system of length N

$$E_{h_L,h_R} = \varepsilon N + \frac{2\pi v}{N}\left(h_L + h_R - \frac{c}{12}\right)$$

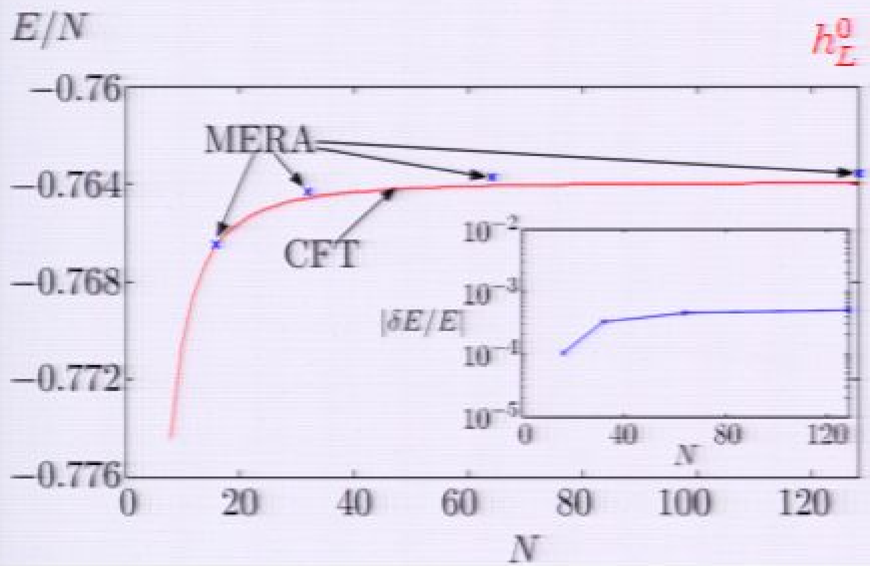
$h_L = h_L^0 + m_L$

$h_R = h_R^0 + m_R$

$m_L, m_R \in \mathbb{N} \cup \{0\}$

$$h_L^0 = h_R^0 \in \{\underbrace{0, 3/80}, 1/10, 7/16, 3/5, 3/2\}$$

use these to estimate non-universal constants ε, v



agrees with CFT in thermodynamic limit

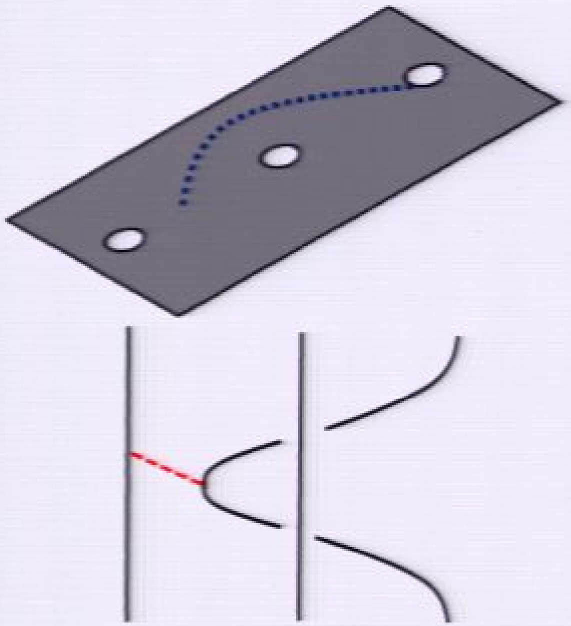
(see also recent related work by Pfeifer et al., arXiv:1006.3532)

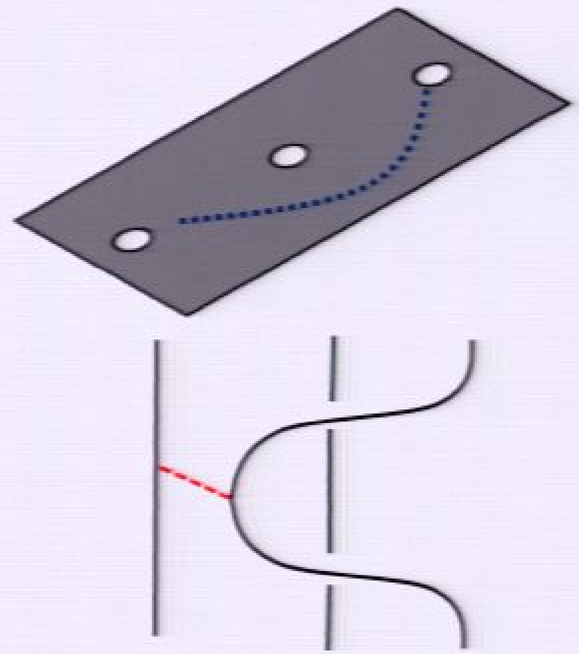
2D arrangements of anyons

- Hamiltonians
- evaluation of expectation values by braiding

Non-abelian exchange statistics

Next-to-nearest neighbor interactions:





$B^{-1}\color{red}{H}\color{black}{B}$

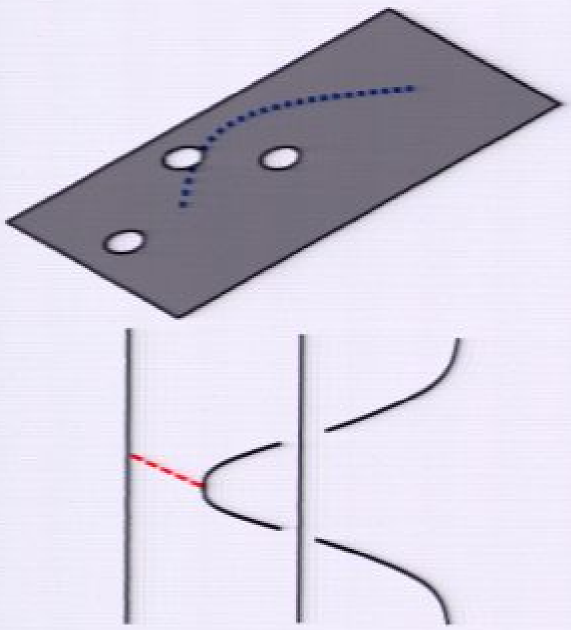
$\tilde{B}^{-1}\color{red}{H}\color{black}{\tilde{B}}$

$B =$ 

$\tilde{B} =$ 

Non-abelian exchange statistics

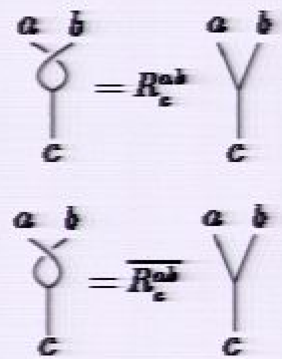
Next-to-nearest neighbor interactions:

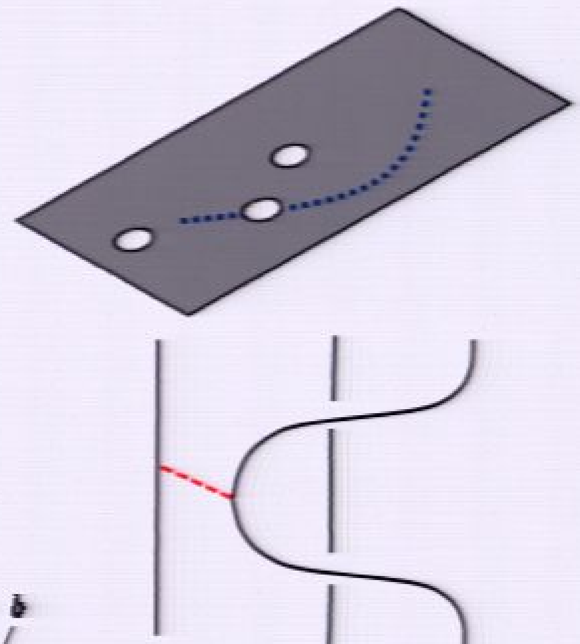


$B^{-1}\mathbf{H}B$

$B=$ 

additional rule:



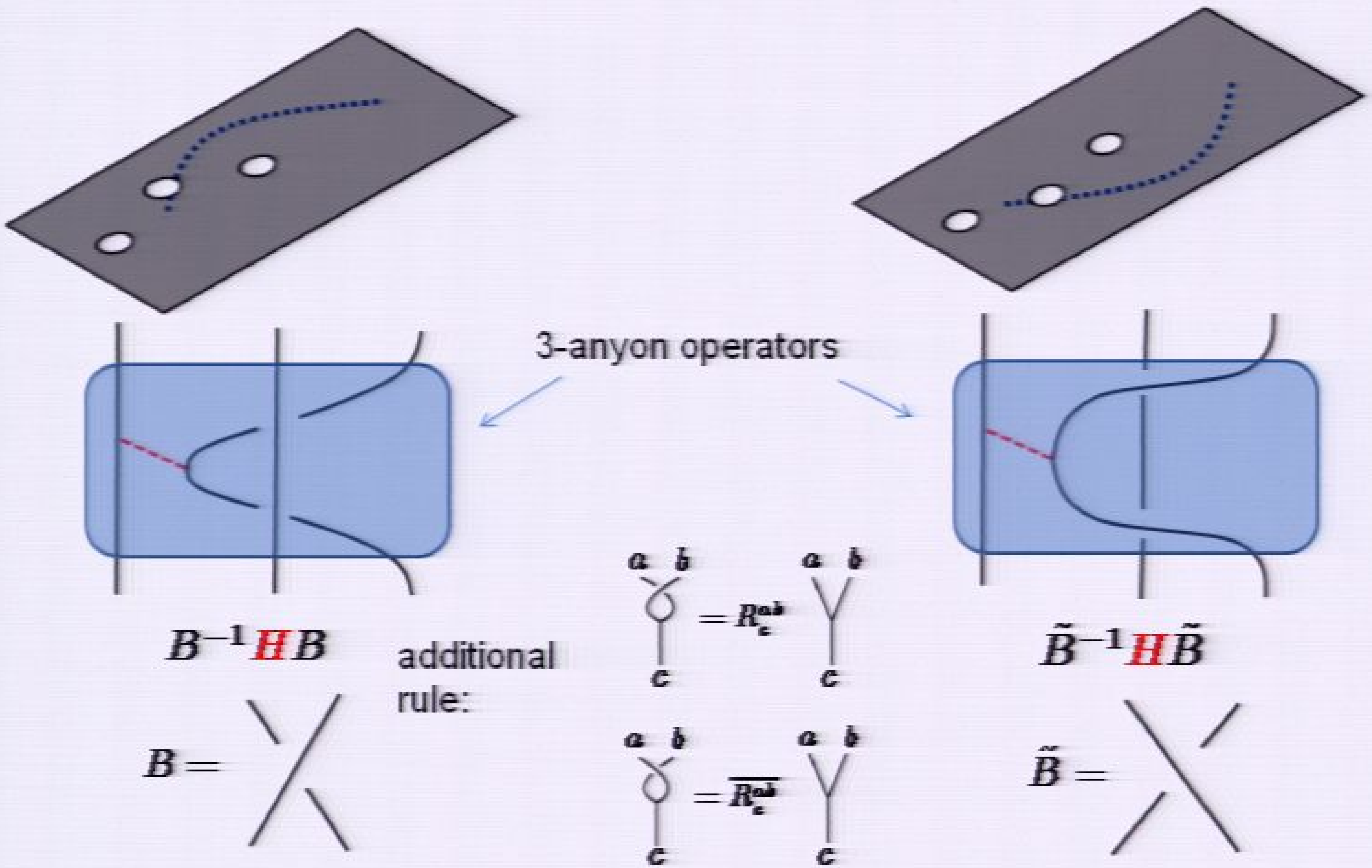


$\tilde{B}^{-1}\mathbf{H}\tilde{B}$

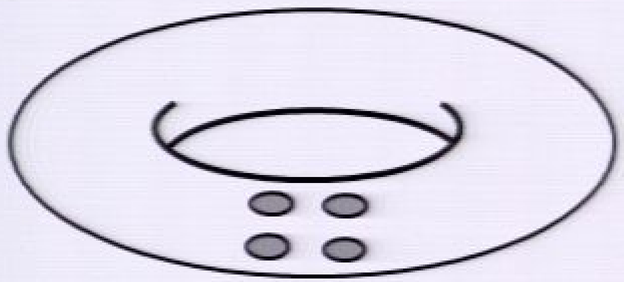
$\tilde{B}=$ 

Non-abelian exchange statistics

Next-to-nearest neighbor interactions:

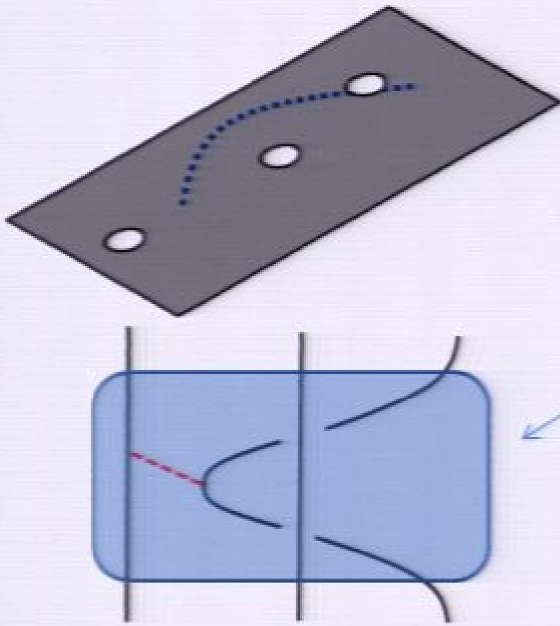


Example: anyonic array

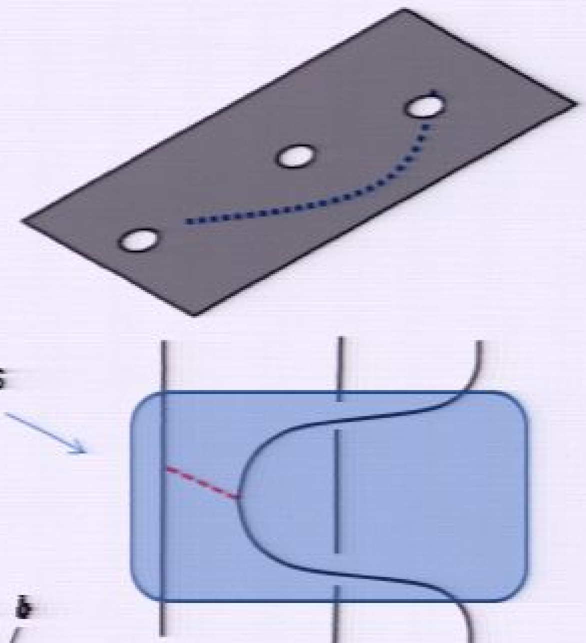


Non-abelian exchange statistics

Next-to-nearest neighbor interactions:



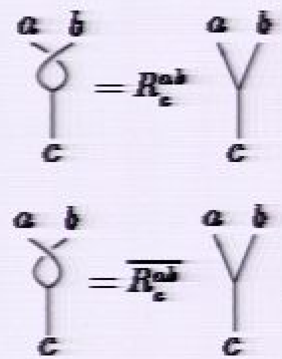
3-anyon operators



$B^{-1}\color{red}{H}\color{black}B$

$\tilde{B}^{-1}\color{red}{H}\color{black}\tilde{B}$

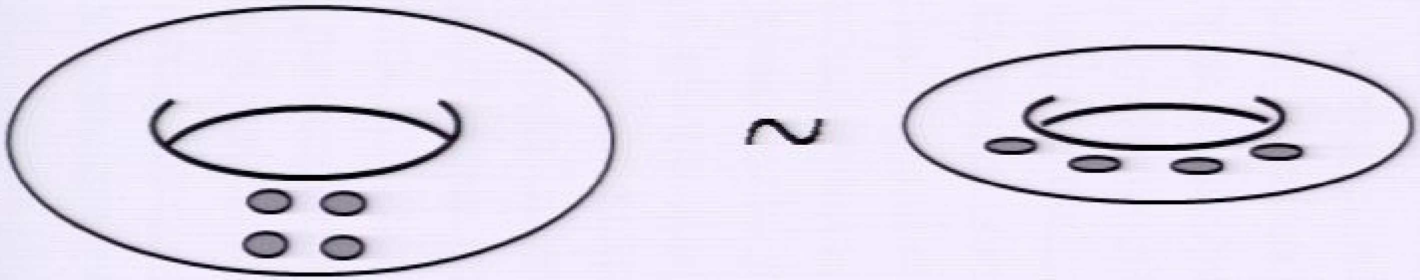
additional rule:



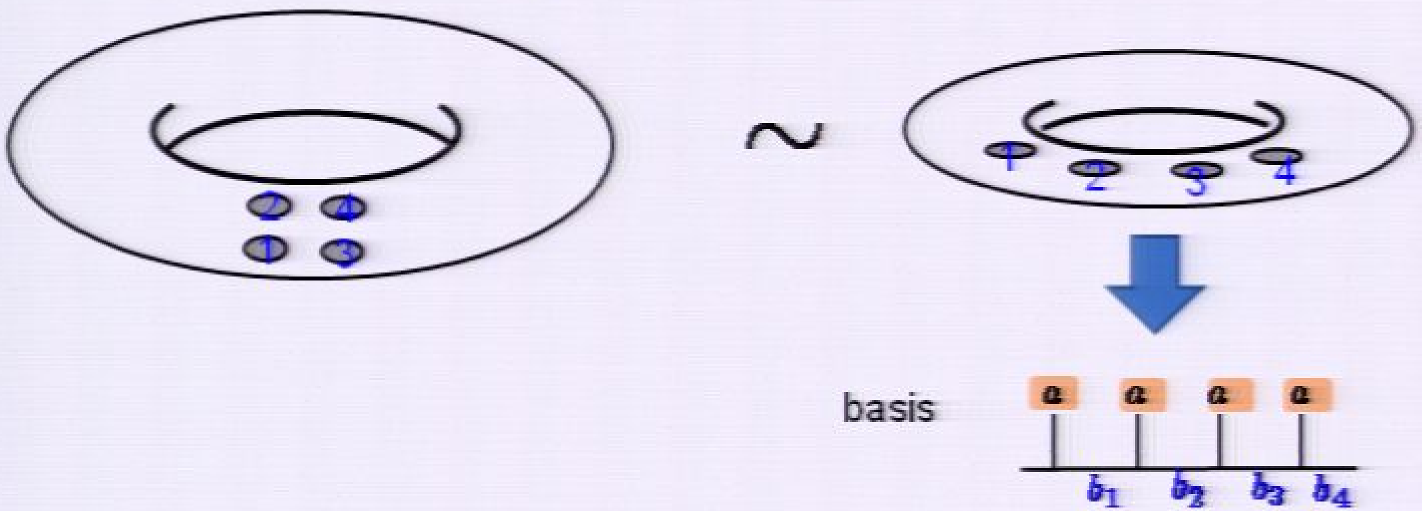
$B =$ 

$\tilde{B} =$ 

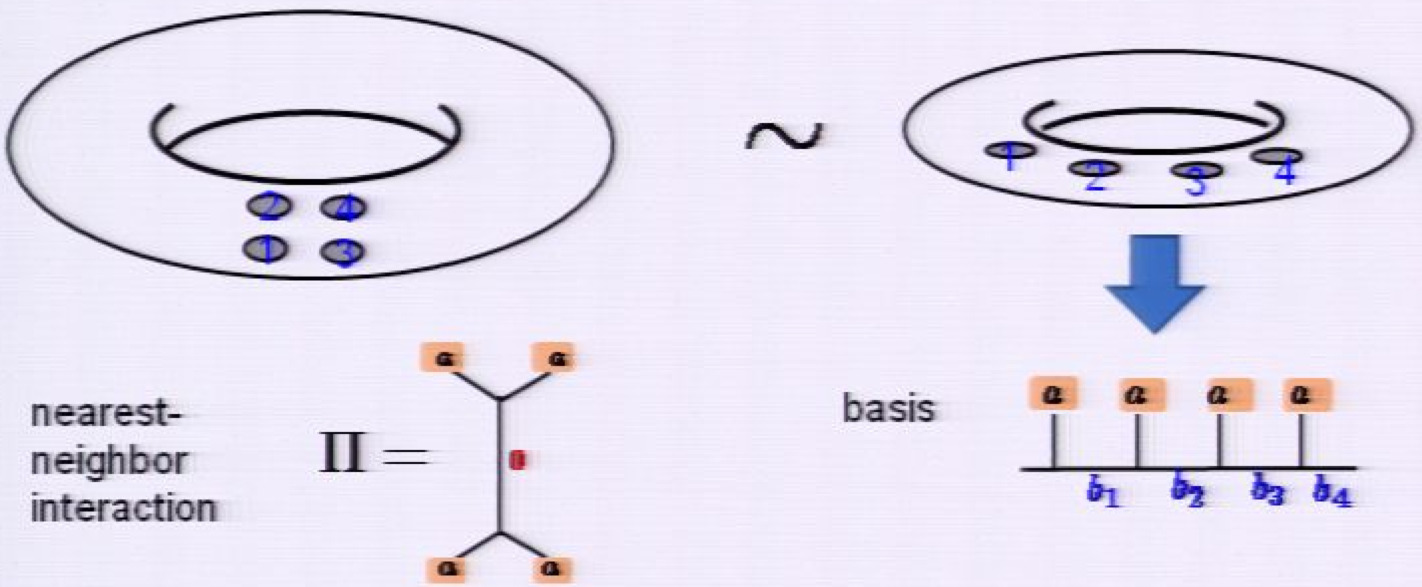
Example: anyonic array



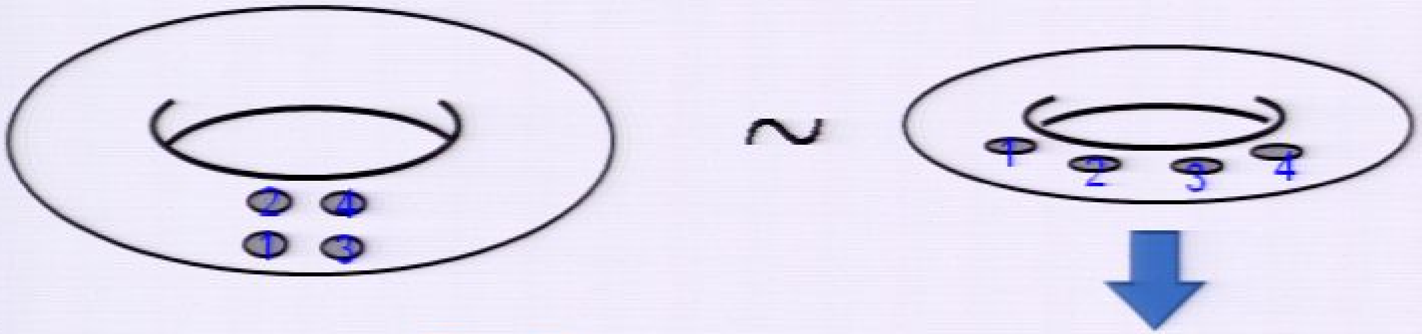
Example: anyonic array



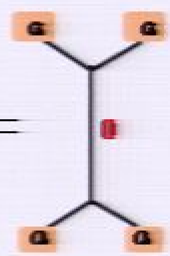
Example: anyonic array

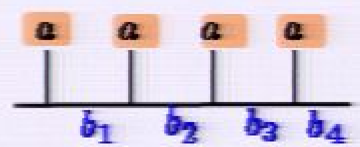


Example: anyonic array

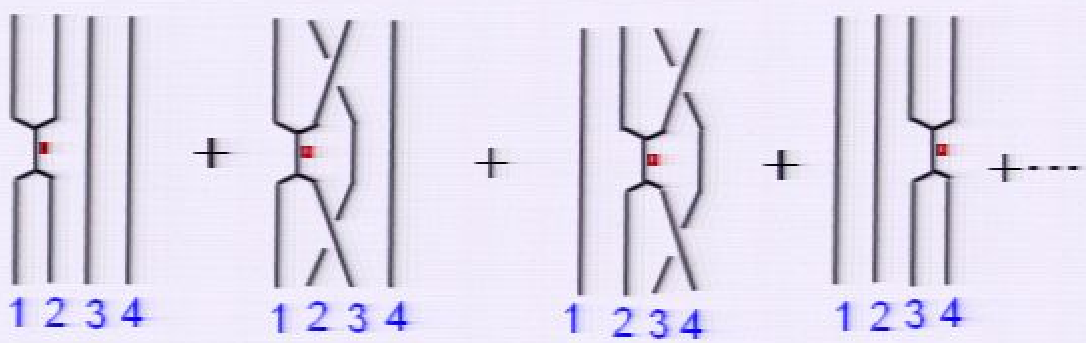


nearest-neighbor interaction

$\Pi =$ 

basis 

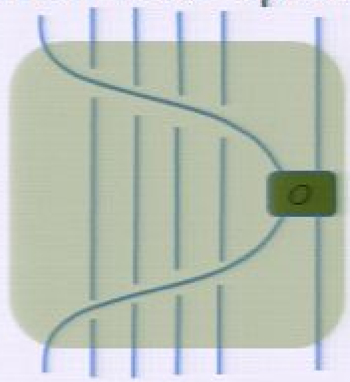
Hamiltonian:



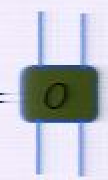
Braiding-type non-local operators

$\langle\Psi_{\{V\}}|\hat{O}|\Psi_{\{V\}}\rangle \quad = \quad \langle\Psi_{\{W\}}|O|\Psi_{\{W\}}\rangle$

for the non-local operator

$\hat{O} =$ 

for the “local” operator

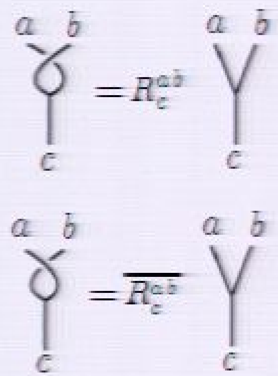
$O =$ 

isometries: $\{V\} \mapsto \{W\}$

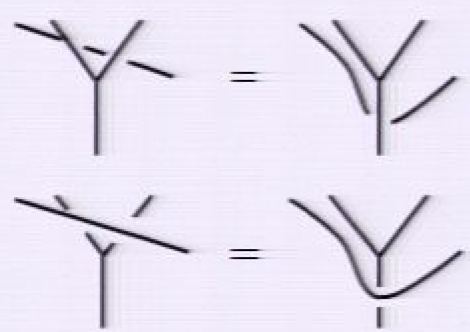
computable with complexity of order $O(poly(\#crossings))$
(on causal cone)

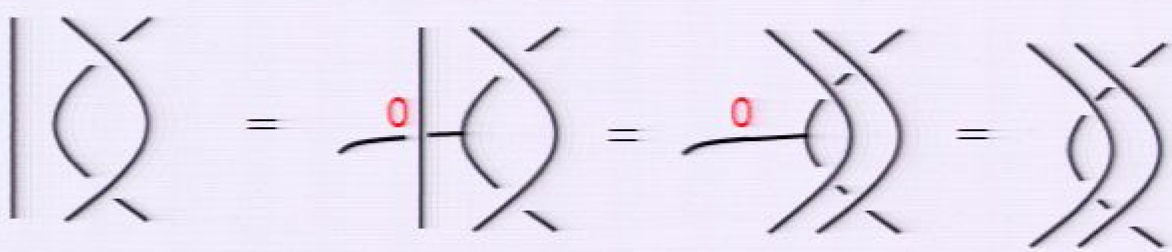
This provides a technique for treating 2D anyon-arrangements (similar to fermionic tensor networks).

Rules for resolving crossings…

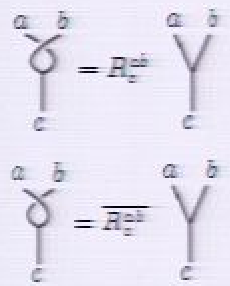


compatibility with F-matrix implies the identites

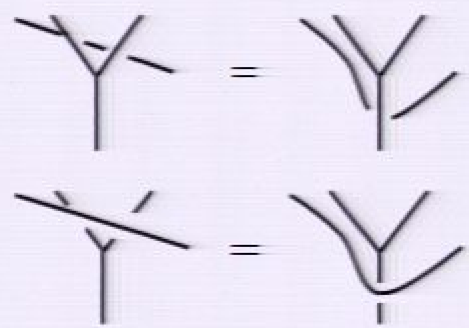


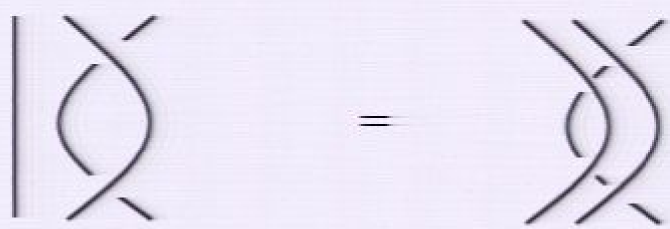


Rules for resolving crossings…

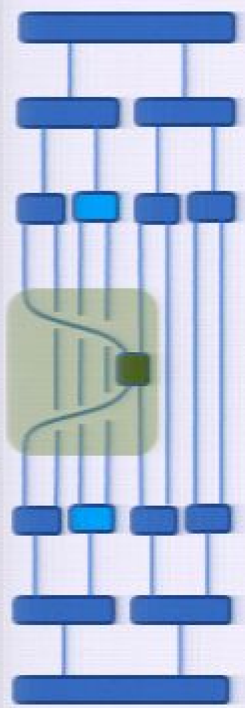


compatibility with F-matrix implies the identites



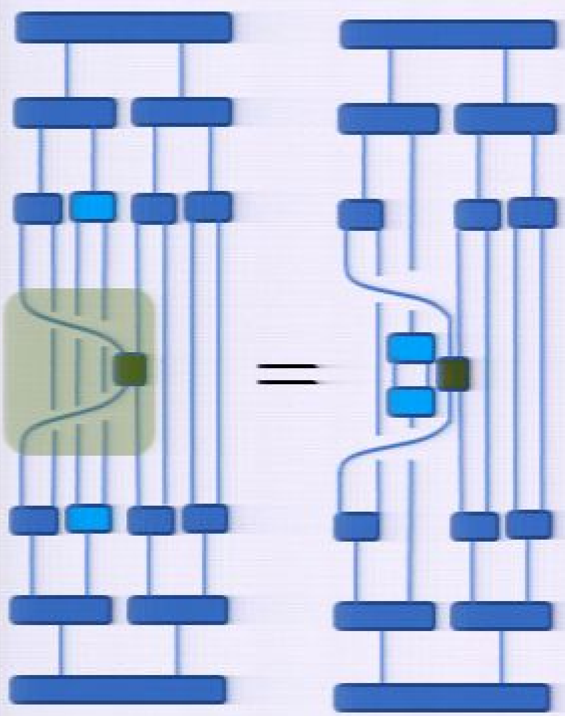


Example



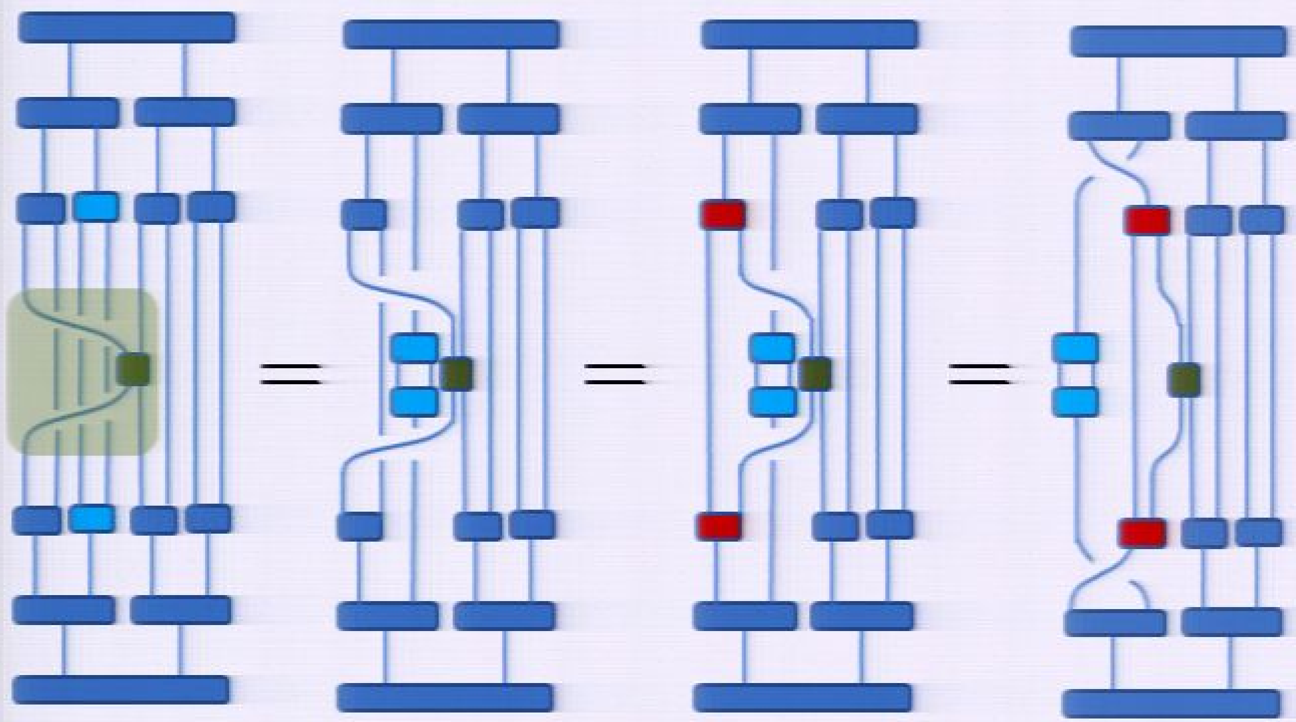
$\langle\Psi_{\{v\}}|\hat{O}|\Psi_{\{v\}}\rangle$

Example



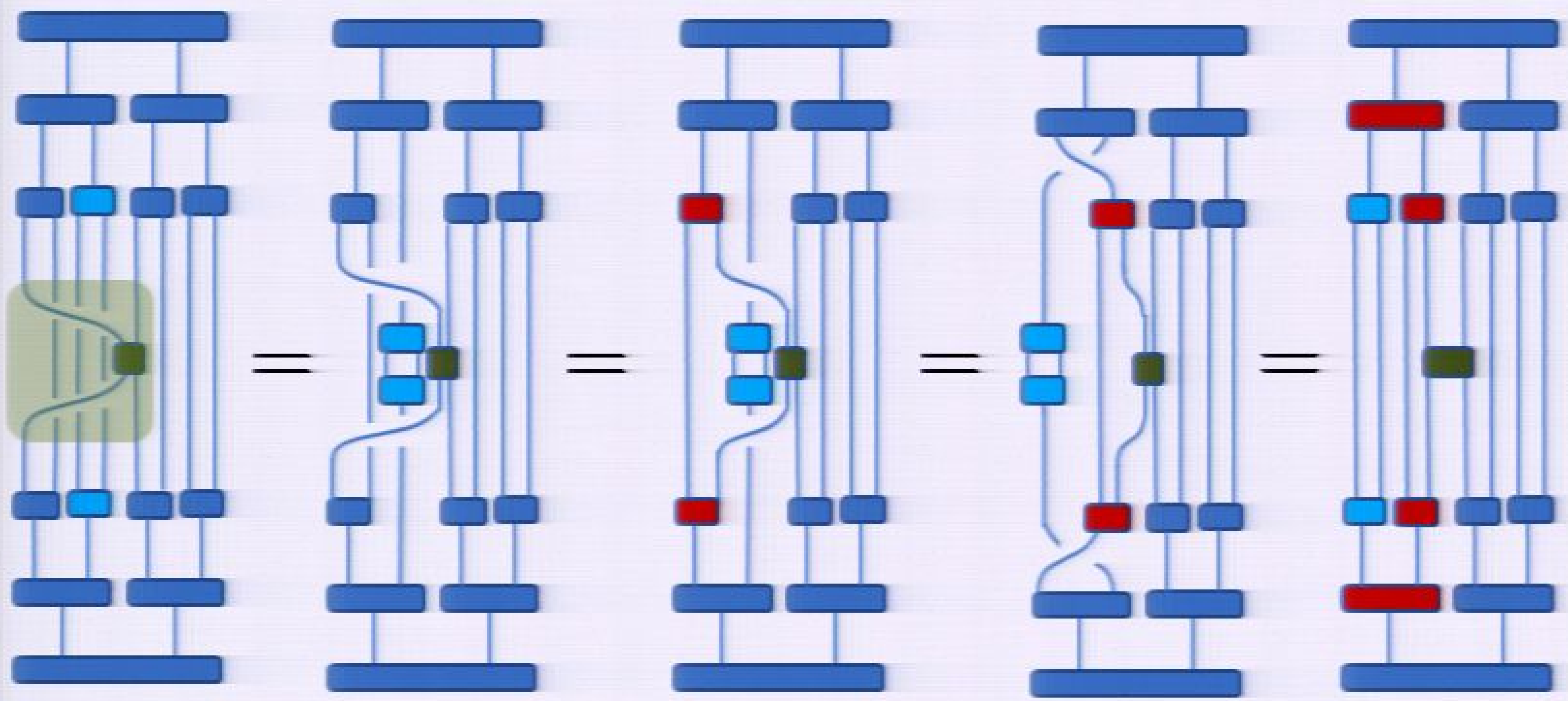
$\langle \Psi_{\{v\}} | \hat{O} | \Psi_{\{v\}} \rangle$

Example



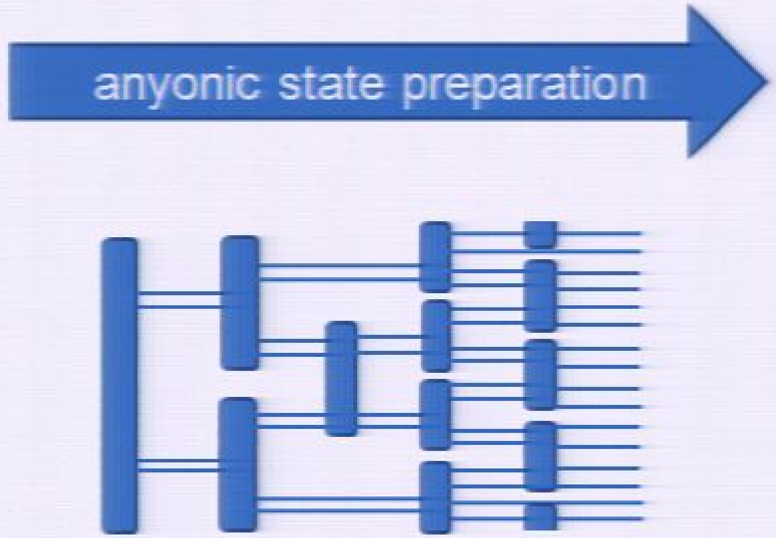
$\langle \Psi_{\{v\}} | \hat{O} | \Psi_{\{v\}} \rangle$

Example

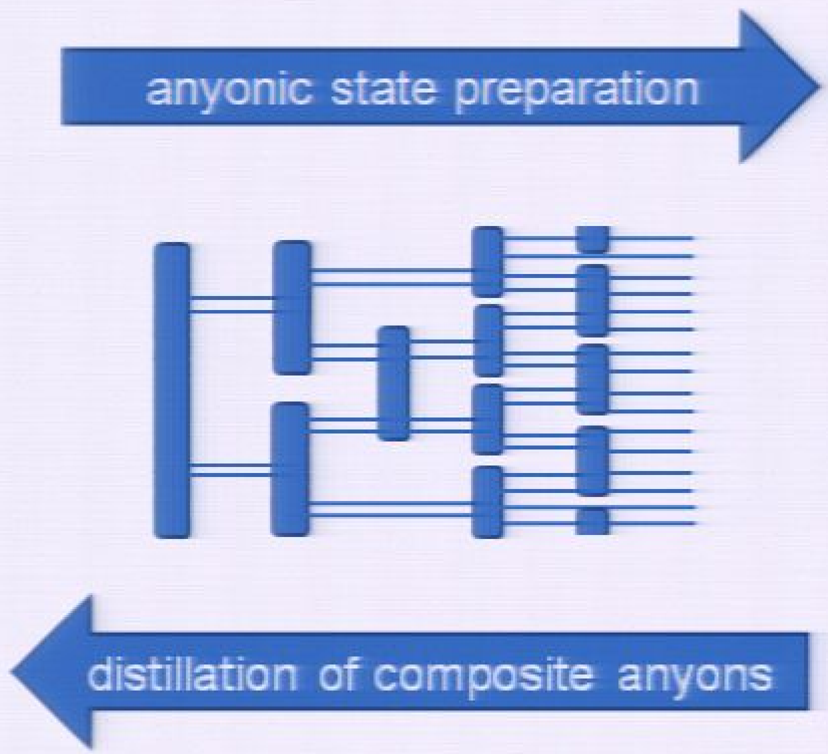


$$\langle \Psi_{\{v\}} | \hat{O} | \Psi_{\{v\}} \rangle = \langle \Psi_{\{w\}} | O | \Psi_{\{w\}} \rangle$$

A new operational interpretation:

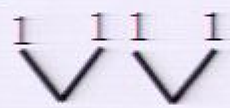


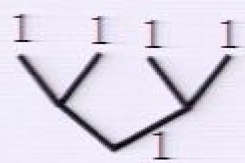
A new operational interpretation:



Universal computation with Fibonacci anyons

Typically used encoding:

$|0\rangle \mapsto$ 

$|1\rangle \mapsto$ 

Need to be able to:

- prepare $|0\rangle^{\otimes n} =$ 

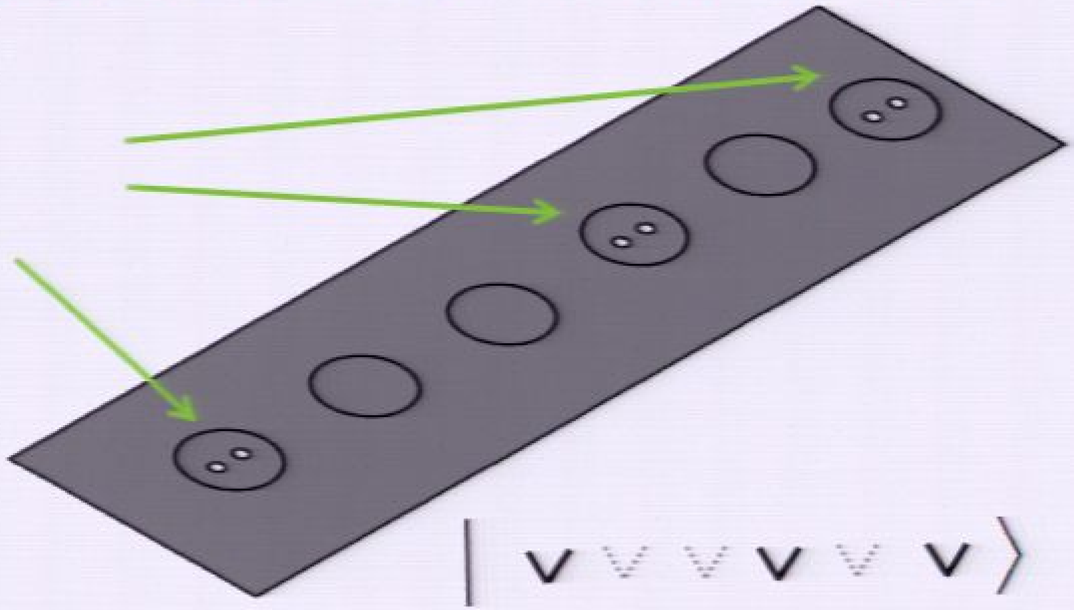
- execute braids/twists

- measure anyonic charge
(repeat)

Dealing with noisy initial states

idealized scenario: Suppose **pair creation is probabilistic**

pair creation only
successful in these
(unknown) locations



How to deal with this situation?

Proposal 1: make ***measurements*** to determine presence/absence of pairs

Proposal 2: concentrate entropy into a certain sector by ***reversible gates***

Sweeping dirt into the bin in a closed house

$\mathcal{H}_{\text{pairs}}$: subspace spanned by states of ***arbitrary*** (non-zero) ***number of particle pairs*** distributed among specified ("potential defect") locations

Example state: $\big| \vee \; \vee \; \vee \; \vee \vee \;\; \vee \;\; \vee \; \vee \big\rangle$

Assumption ρ (**mixed state**) on $\mathcal{H}_{\text{pairs}} \subset \mathcal{H}$

Sweeping dirt into the bin in a closed house

$\mathcal{H}_{\text{pairs}}$: subspace spanned by states of ***arbitrary*** (non-zero) ***number of particle pairs*** distributed among specified ("potential defect") locations

Example state: $| \vee \; \vee \; \vee \; \vee \vee \; \vee \; \vee \; \vee \rangle$

Assumption ρ (**mixed state**) on $\mathcal{H}_{\text{pairs}} \subset \mathcal{H}$

anyonic distillation circuit: unitary U

after distillation: $U \rho U^\dagger = |\vee\rangle\langle\vee|$

pure state of one pair

Sweeping dirt into the bin in a closed house

$\mathcal{H}_{\mathrm{pairs}}$: subspace spanned by states of ***arbitrary*** (non-zero) ***number of particle pairs*** distributed among specified ("potential defect") locations

Example state: $\big|\ \vee\ \vee\ \vee\ \vee\vee\ \ \vee\ \ \vee\ \vee\ \big\rangle$

Assumption ρ (**mixed state**) on $\mathcal{H}_{\mathrm{pairs}} \subset \mathcal{H} \simeq \mathcal{H}_{\mathbf{encoded}} \otimes \mathcal{H}_{\mathbf{bin}}$

anyonic distillation circuit: unitary U

after distillation: $U\rho U^{\dagger} = \underbrace{|\vee\rangle\langle\vee|}_{\text{pure state of one encoded pair}} \otimes \underbrace{\sigma(\rho)_{\mathbf{bin}}}_{\text{mixed state on "bin"}}$

Sweeping dirt into the bin in a closed house

$\mathcal{H}_{\text{pairs}}$: subspace spanned by states of ***arbitrary*** (non-zero) ***number of particle pairs*** distributed among specified ("potential defect") locations

Example state: $\left| \vee \; \vee \; \vee \; \vee \vee \; \vee \; \vee \; \vee \right\rangle$

Assumption ρ (**mixed state**) on $\mathcal{H}_{\text{pairs}} \subset \mathcal{H} \simeq \mathcal{H}_{\textbf{encoded}} \otimes \mathcal{H}_{\textbf{bin}}$

anyonic distillation circuit: unitary U

after distillation: $U\rho U^{\dagger} = |\vee\rangle\langle\vee| \otimes \sigma(\rho)_{\textbf{bin}}$

Main property (subsystem code): $U\mathcal{H}_{\textbf{pairs}} \subset |\vee\rangle \otimes \mathcal{H}_{\textbf{bin}}$

Concatenated codes

physical (qu)bits

logical bits (repetition) code $0_L = 000$ $1_L = 111$

logical bits of concatenated (repetition) code

$$0_{L'} = 0_L 0_L 0_L$$
$$1_{L'} = 1_L 1_L 1_L$$

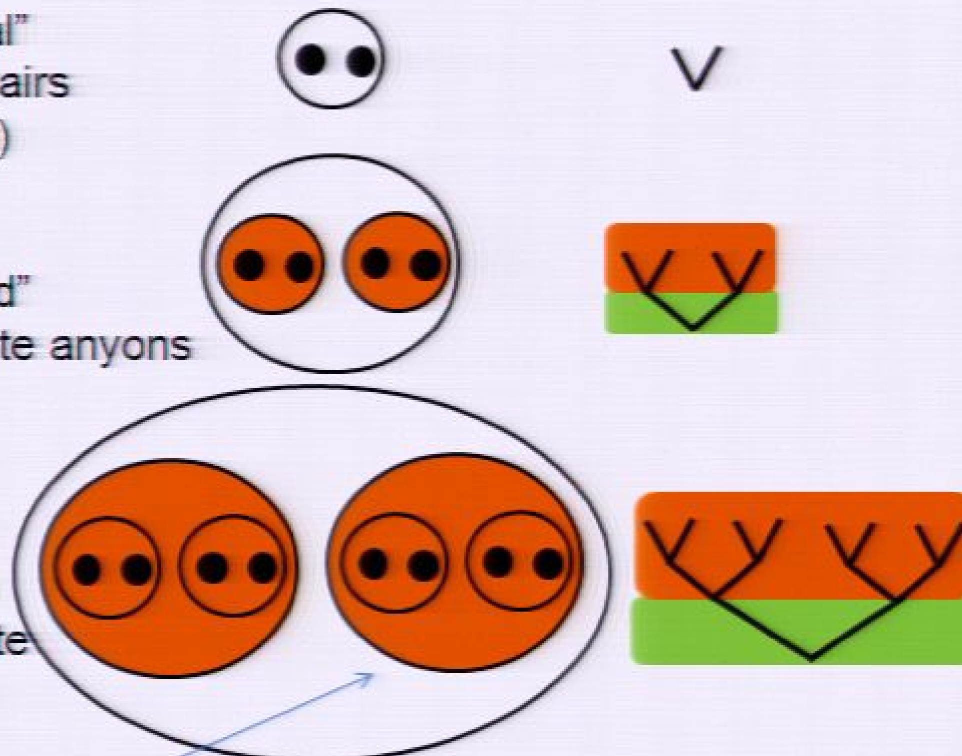
Composite anyons

$$\mathcal{H} \simeq \mathcal{H}_{\text{encoded}} \otimes \mathcal{H}_{\text{bin}}$$

"physical" anyon pairs (defects)

pair of "encoded" composite anyons

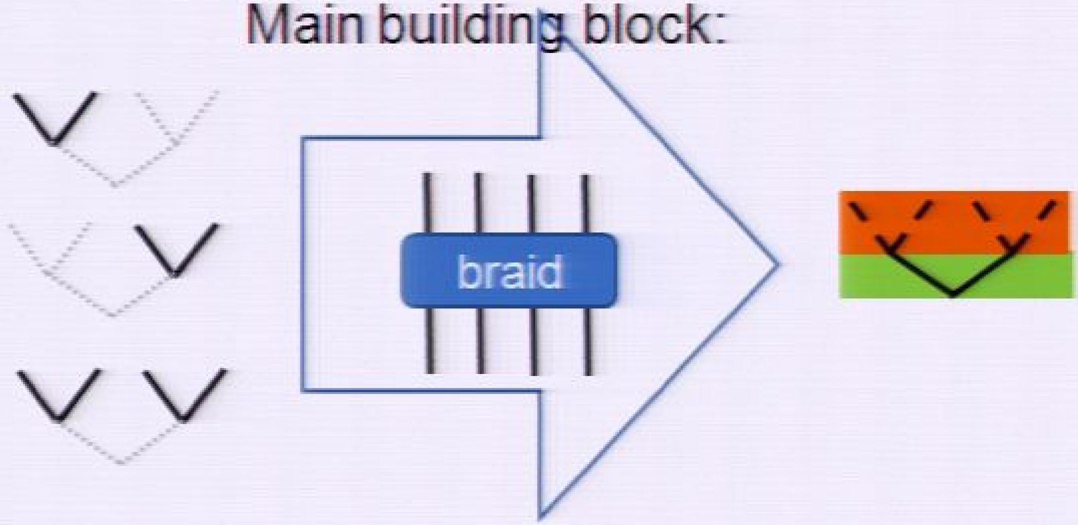
pairs of (doubly) composite anyons



composite anyons are equivalent to bare anyons wrt. computation
– complexity increase only quadratically in circumference

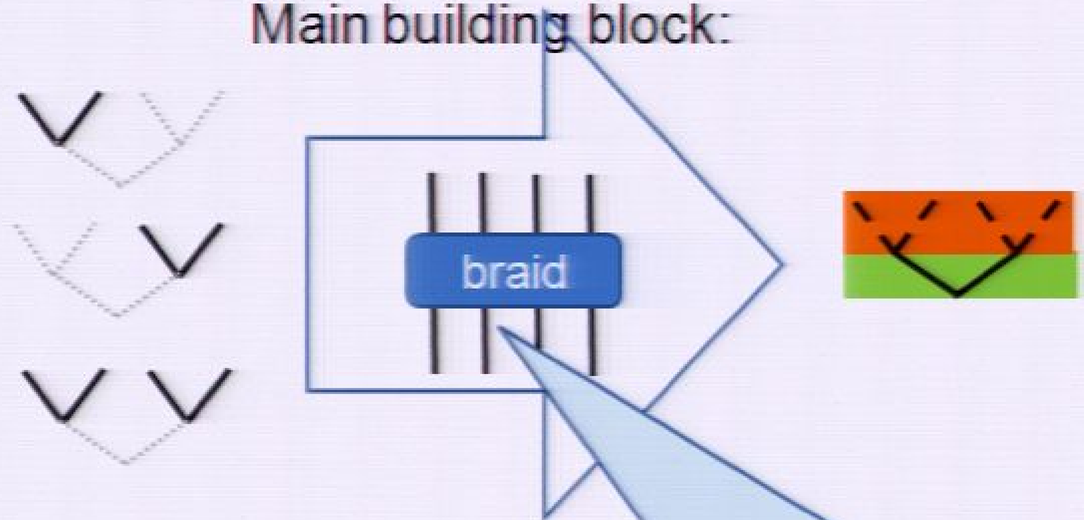
Constructing an anyonic distillation circuit

Main building block:

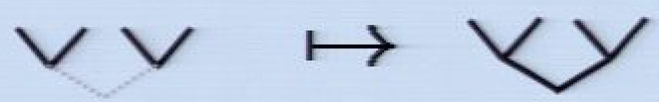


Constructing an anyonic distillation circuit

Main building block:

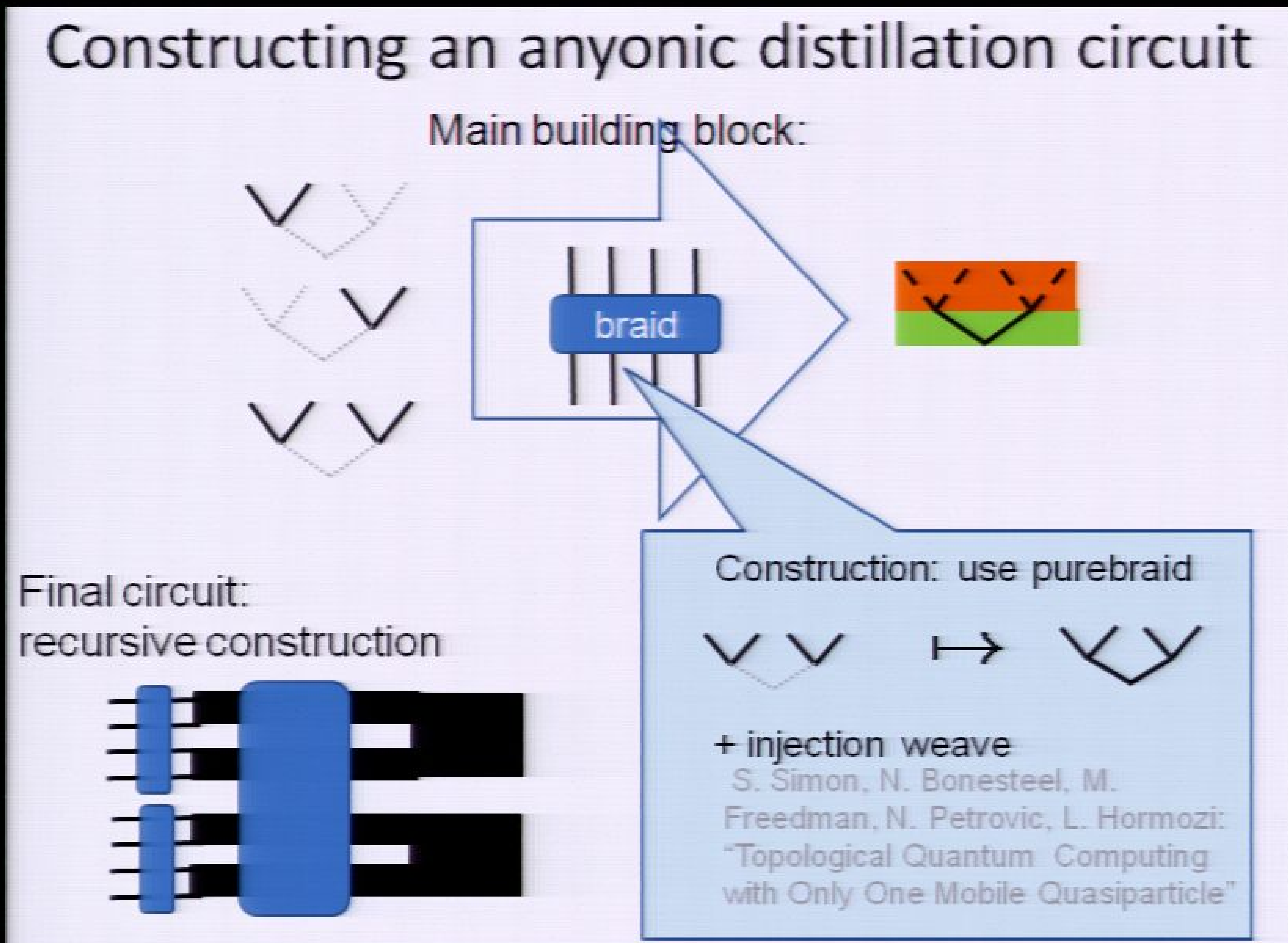


Construction: use purebraid

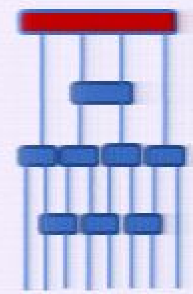


+ injection weave

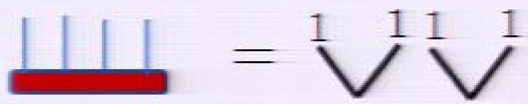
S. Simon, N. Bonesteel, M. Freedman, N. Petrovic, L. Hormozi: "Topological Quantum Computing with Only One Mobile Quasiparticle"


Constructing an anyonic distillation circuit
Main building block:
braid
Final circuit:
recursive construction
Construction: use purebraid
+ injection weave
S. Simon, N. Bonesteel, M. Freedman, N. Petrovic, L. Hormozi: "Topological Quantum Computing with Only One Mobile Quasiparticle"

Relation to anyonic entanglement renormalization



Consider an anyonic entanglement RG with a fixed toplevel state:

e.g., 

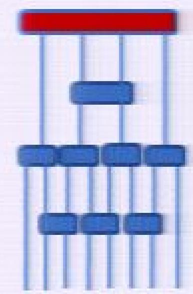
This is the “target” state of composite anyons

anyonic entanglement RG = composite anyon distillation scheme

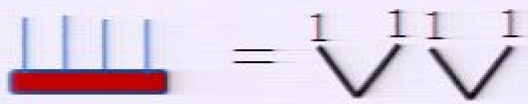
-> characterization of “distillable” states

Relation to anyonic entanglement renormalization

Consider an anyonic entanglement RG with a fixed toplevel state:



e.g.,



This is the "target" state of composite anyons

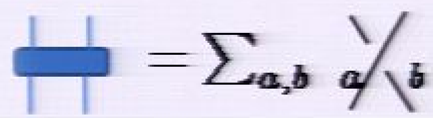
anyonic entanglement RG = composite anyon distillation scheme

-> characterization of "distillable" states

Assume that every isometry/unitary is composed of *physically realizable* processes

fusion

braiding



(+inverse)

Anyonic entanglement renormalization

Variational family for anyons with operational interpretation:

- state preparation circuit
- renormalization group scheme, connects to composite anyon distillation

Evidence for its descriptive power:

- exact description of multi-anyon chains at certain coupling strengths
- accurate numerical estimation of ground state energies & correlation functions for finite-size systems
- good agreement with CFT predictions in thermodynamic limit

.........compare to successes of non-anyonic version!

Provides a new numerical toolbox:

- extends to 2D-arrangements of anyons
- inherits various optimization/evaluation algorithms

Computational savings achieved by exploiting Hilbert space structure