Title: Chiral Operators on Hypermultiplet Moduli Spaces

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Abstract: This talk will focus on hypermultiplet moduli spaces of various N=2 supersymmetric gauge theories in (3+1)d. In the first part of the talk, we discuss the moduli space of instantons on C^2. For the classical groups, the ADHM construction of the moduli space can be realised on the Higgs branch of N=2 gauge theories on D3-branes probing D7-branes. No known construction is available for exceptional groups. We go over the computation of Hilbert series for the one instanton moduli space and show that it is possible to count all chiral operators on the moduli space even though a Lagrangian is not known for exceptional gauge groups. In the second part, we discuss a class of N=2 gauge theories on two M5-branes wrapping Riemann surfaces. This talk will go over Hilbert series for the hypermultiplet moduli space of such theories and show that it is possible to count all chiral operators on the hypermultiplet moduli space for any genus and any number of punctures of the Riemann surface.

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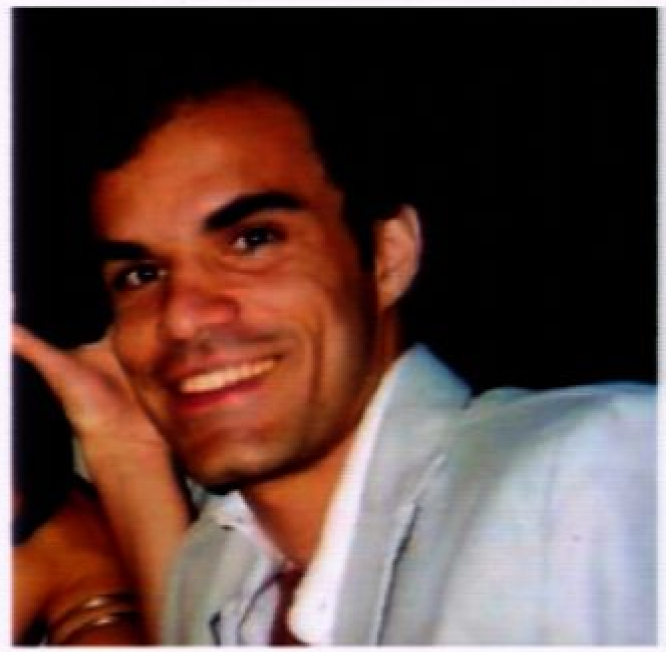

Part I: The Hilbert series

Aim: Introduce a quantity which gives information about the moduli space of supersymmetric gauge theories.

Introduction

- Every SUSY gauge theories has a collection of chiral operators which form a chiral ring.
- The chiral ring contains information on the structure of possible operators in the theory and the relations that they may satisfy.
- The Hilbert series counts chiral operators in the theory.

Example 1: One free chiral field ϕ

- The chiral operators are ϕ^k (where $k = 0, 1, 2, \ldots$)
- There is a conserved $U(1)$ charge
  - Suppose that ϕ carries charge 1. Then, ϕ^k carries charge k
- Introduce a chemical potential for the charge μ. The fugacity $t = e^{-\mu}$.
- The fugacity for ϕ^k is t^k
- The Hilbert series is a partition function

$$H(t) = \sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$$

- Pole at $t = 1$ of order 1 $\Rightarrow$ the moduli space is 1 complex dimensional
- The moduli space is $\mathbb{C}$

Example 2: n free chiral fields $\phi_1, \ldots, \phi_n$

- The chiral operators are $\phi_1^{k_1} \ldots \phi_n^{k_n}$ (where $k_1, \ldots, k_n = 0, 1, 2, \ldots$)
- There is a $U(n)$ global symmetry
  - There is a collection of maximally commuting $U(1)$'s in $U(n)$.
  - Assign the charge fugacity t_i for each $U(1)$ to the field ϕ_i (with $i = 1, \ldots, n$).
- The Hilbert series is

$$H(t_1, \ldots, t_n) = \sum_{k_1, \ldots, k_n = 0}^{\infty} t_1^{k_1} \ldots t_n^{k_2} = \prod_{k=1}^{n} \frac{1}{1 - t_i}$$

- Rewrite the fugacities t_i as

$$t_1 = t x_1,\ t_2 = t\frac{x_2}{x_1},\ t_3 = t\frac{x_3}{x_2},\ \ldots,\ t_{n-1} = t\frac{x_{n-1}}{x_{n-2}},\ t_n = \frac{t}{x_{n-1}}$$

where $x_1, \ldots, x_{n-1}$ are $SU(n)$ fugacities and t is the $U(1)$ fugacity

Example 2: n free chiral fields $\phi_1, \ldots, \phi_n$ (continued)

- Then, we obtain

$$H(t; x_1, x_2, \ldots, x_{n-1}) = 1 + t \underbrace{\left(x_1 + \frac{x_2}{x_1} + \ldots + \frac{x_{n-1}}{x_{n-2}} + \frac{1}{x_{n-1}} \right)}_{\text{the character of the fundamental rep of } SU(n)} + \ldots$$

- **Notation:** $[1, 0, ..., 0]$ = the character of the fundamental rep of $SU(n)$
- In general, we use the Dynkin label $[k_1, \ldots, k_{n-1}]$ to denote the character of an irrep of $SU(n)$
- In fact,

$$H(t; x_1, x_2, \ldots, x_{n-1}) = \sum_{n=0}^{\infty} [n, 0, \ldots, 0] t^n \equiv \mathrm{PE}\left[[1, 0, \ldots, 0] t\right]$$

where $[n, 0, \ldots, 0]$ is the n-th symmetric power of the fundamental rep

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The plethystic exponential

- The word 'plethystic' comes from the word 'plethora'
- For a function of several variables $f(t_1, \dots, t_n)$ vanishing at the origin, $f(0, \dots, 0) = 0$, the plethystic exponential is defined as

$$\mathrm{PE}[f(t_1, \dots, t_n)] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k, \dots, t_n^k)\right)$$

- **Example:** For $f(t) = t$, $\mathrm{PE}[f(t)] = \frac{1}{1-t}$
- **Example:** For $f(t) = \sum_k a_k t^k$, $\mathrm{PE}[f(t)] = \prod_k \frac{1}{(1-t^k)^{a_k}}$
- **Example:** $\sum_{n=0}^{\infty} [n, 0, \dots, 0] t^n = \mathrm{PE}\left[[1, 0, \dots, 0] t\right]$

  The PE generates symmetrisations.

Example 2: n free chiral fields $\phi_1, \ldots, \phi_n$ (revisted)

- $H(t; x_1, x_2, \ldots, x_{n-1}) = \sum_{n=0}^{\infty} [n, 0, \ldots, 0] t^n = \mathrm{PE}\left[[1, 0, \ldots, 0] t\right]$
- Setting $x_1 = \ldots = x_{n-1} = 1$, we obtain

$$H(t; 1, \ldots, 1) = \mathrm{PE}[nt] = \frac{1}{(1-t)^n}$$

- The order of the pole at $t = 1$ is n

  $\Rightarrow$ The moduli space is n complex dimensional

- Indeed, the moduli space is $\mathbb{C}^n$

Part II: Chiral operators on the instanton moduli space

Summary of the $\mathcal{N} = 2$ SUSY

- All $\mathcal{N} = 2$ theories have an $SU(2)$ R-symmetry which acts on the two supercharges of given chirality.
- **The $\mathcal{N} = 2$ vector multiplet:** Gauge fields A_μ, two Weyl fermions λ, ψ, and a scalar φ, all in the adjoint rep.

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \varphi & \end{array}$$

  - The $SU(2)_R$ symmetry acts on the rows
  - In terms of $\mathcal{N} = 1$ SUSY, these fields can be arranged into a vector multiplet $\mathcal{W}_\alpha$ containing (A_μ, λ) and a chiral multiplet Φ containing (φ, ψ)

Summary of the $\mathcal{N} = 2$ SUSY

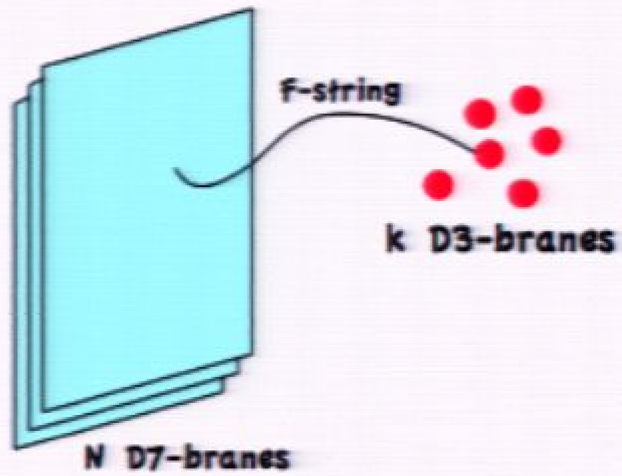
- **The hypermultiplet:** Two Weyl fermions $\psi_q, \psi_{\tilde{q}}^\dagger$, and complex bosons $q, \tilde{q}^\dagger$

$$\begin{array}{ccc} & \psi_q & \\ q & & \tilde{q}^\dagger \\ & \psi_{\tilde{q}}^\dagger & \end{array}$$

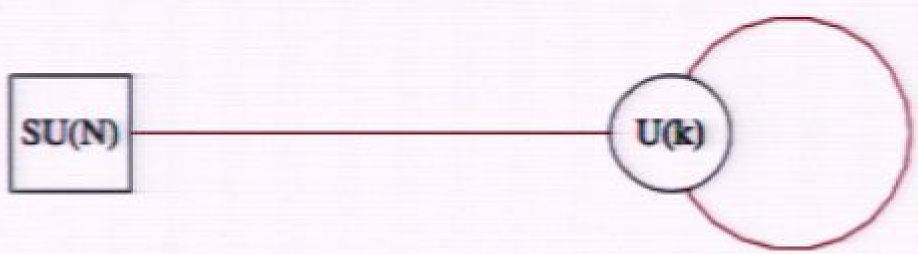
  - The $SU(2)_R$ symmetry acts on the rows
  - In terms of $\mathcal{N} = 1$ SUSY, these fields can be arranged into two chiral multiplet Q containing (q, ψ_q) and $\widetilde{Q}$ containing $(\tilde{q}, \psi_{\tilde{q}})$

k $SU(N)$-instantons on $\mathbb{R}^4$

- Best realised on the worldvolume of D3-branes probing D7-branes



- N D7-branes $\rightarrow$ $SU(N)$ global symmetry on the w.v. of the D3-branes
- $\mathcal{N}=2$ SUSY gauge theory in $(3+1)$d with $U(k)$ gauge group and $SU(N)$ flavour symmetry. Two types of h-plet: adjoint and bifundamental.



Coulomb and Higgs branches

Two branches of the moduli space:

1. **Coulomb branch:** The D3-branes are outside the D7-branes
   - The vector multiplet acquires a VEV (the position of the D3-branes in the transverse directions to the D7-branes)
   - The $U(k)$ gauge symmetry is broken to $U(1)^k$
   - The hypermultiplets become massive
   - Massless fields charged under $U(1)^k$ experience a Coulomb interaction

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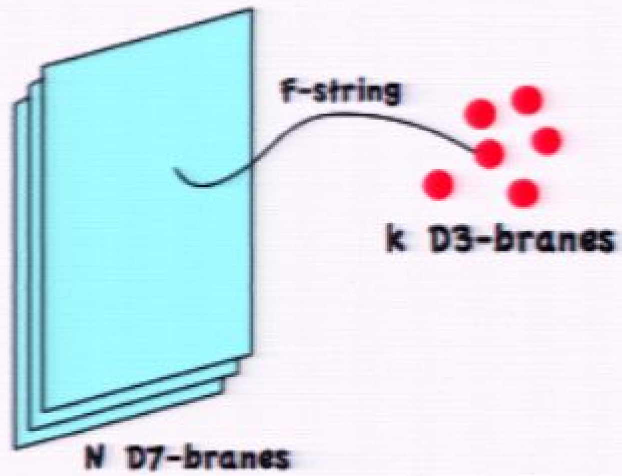

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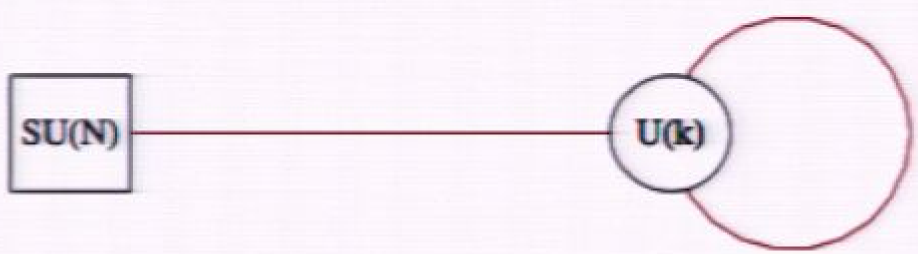

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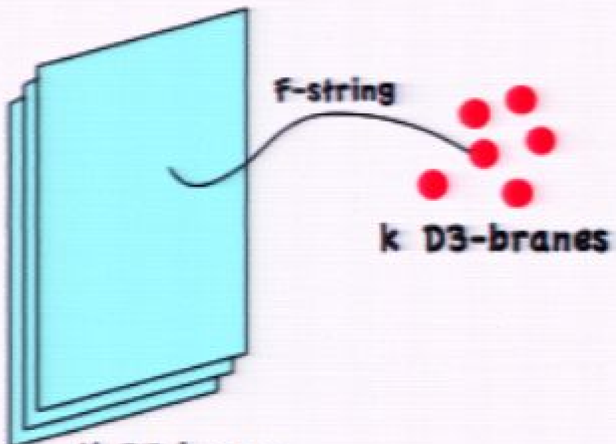

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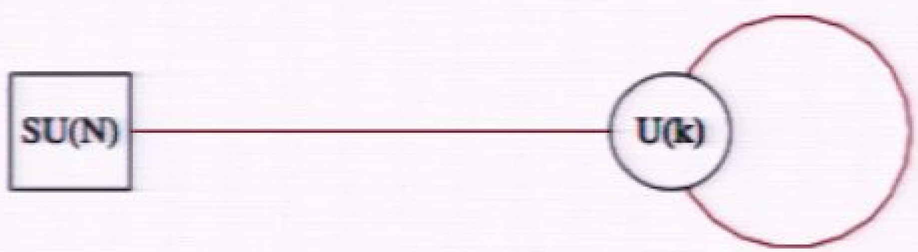

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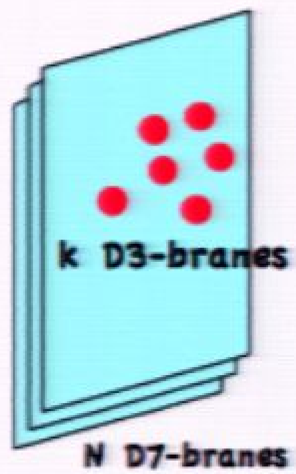
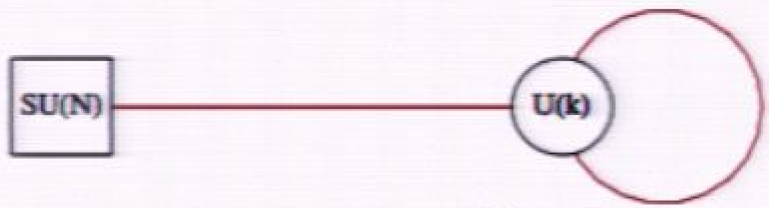


2. **Higgs branch:** The D3-branes are on top the D7-branes

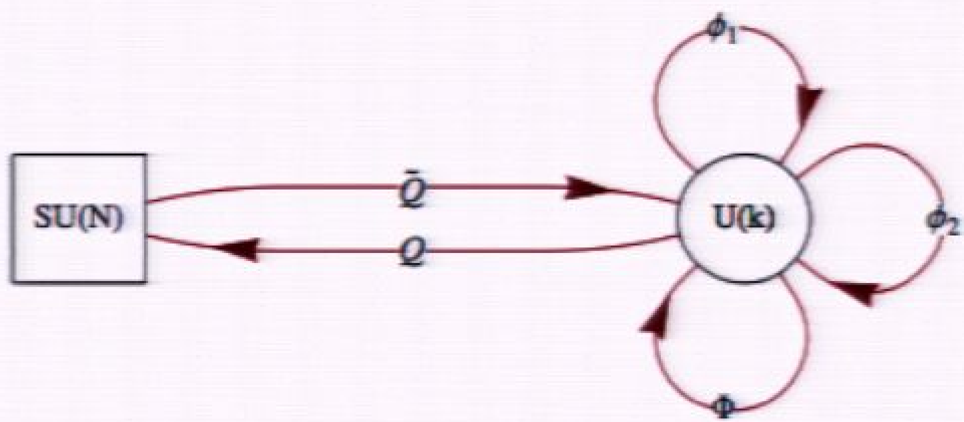
- The h-plets become massless and acquire VEVs
- The $U(k)$ gauge symmetry is generically completely broken
- By the Higgs mechanism, the v-plet becomes massive
- The D3-branes are instantons as viewed from the D7 perspective
- Higgs branch = the moduli space of instantons

The ADHM construction

- Let's translate the $\mathcal{N}=2$ quiver diagram into the $\mathcal{N}=1$ language



- The $\mathcal{N}=1$ quiver diagram with $W = \mathrm{Tr}(\widetilde{Q}\cdot\Phi\cdot Q + \Phi\cdot[\phi_1,\phi_2])$



- **F-terms:** $Q_i\cdot\widetilde{Q}^i + [\phi_1,\phi_2] = 0$ (with $i=1,\ldots,N$)
- **D-terms:** $Q_i\cdot Q^{i\dagger} - \widetilde{Q}_i^\dagger\cdot\widetilde{Q}^i + [\phi_1,\phi_1^\dagger] + [\phi_2,\phi_2^\dagger] = 0$

- These F and D terms are the ADHM equations for k $SU(N)$-instantons

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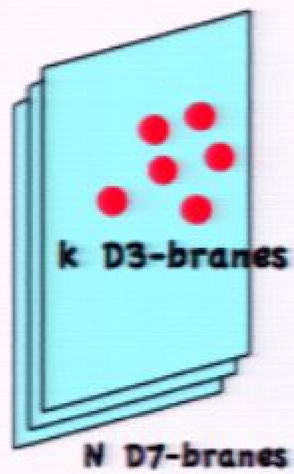

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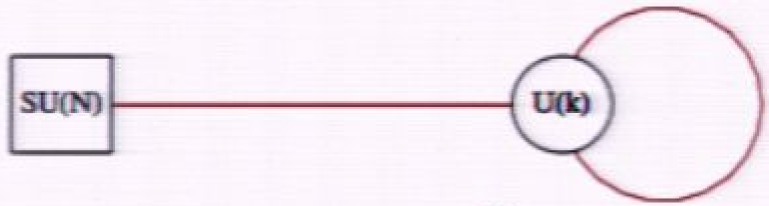

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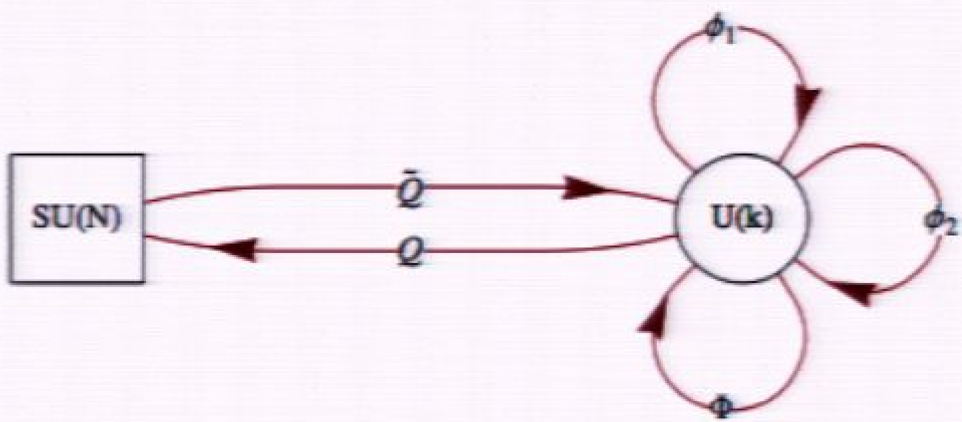

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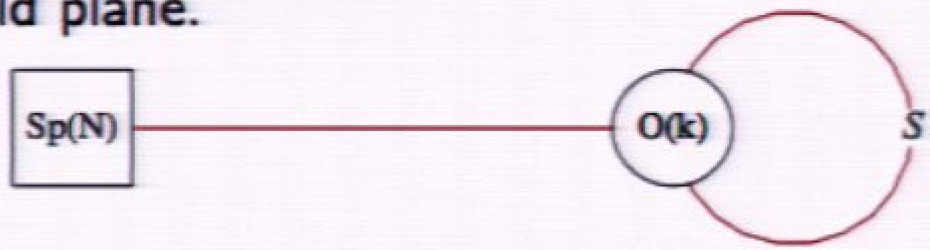
k classical gauge group (G) instantons on $\mathbb{R}^4$

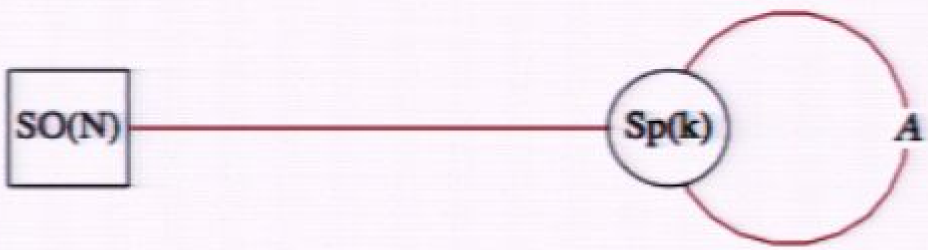
(Douglas '95, Witten '94-'95)

- For $G = Sp(N)$, consider a system of k D3-branes with N D7-branes on top of an O7$^+$ orientifold plane.



S denotes a h-plet in the rank 2 symmetric rep of $O(k)$

- For $G = SO(2N+1),\ SO(2N)$, the orientifold planes are resp. O7$^-$, $\widetilde{\mathrm{O7}}^-$



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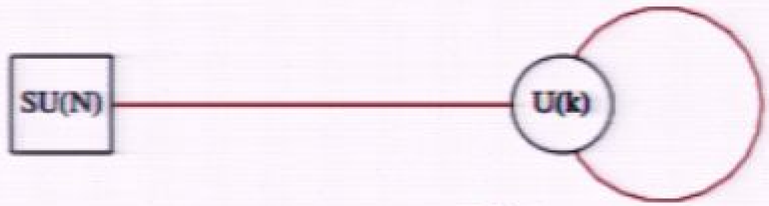

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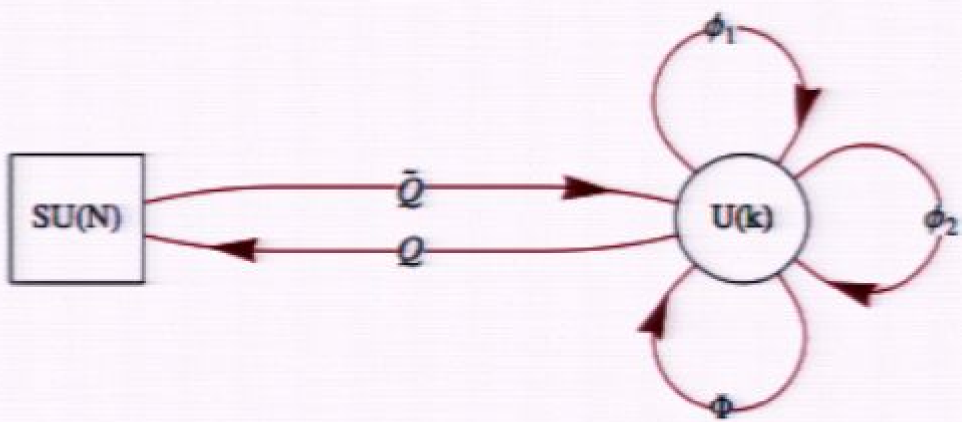

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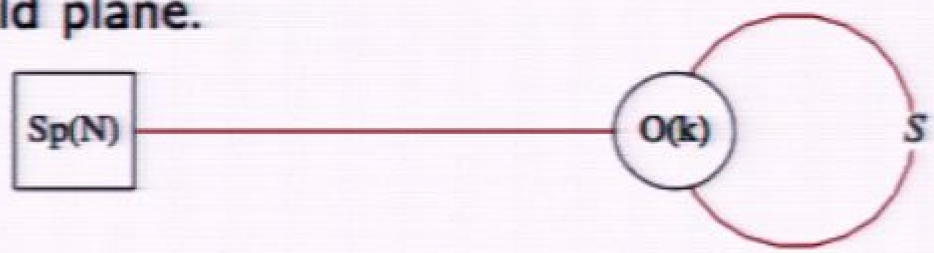

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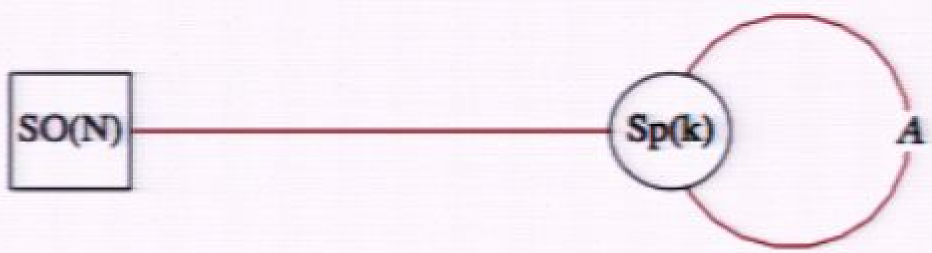

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Exceptional gauge group instantons on $\mathbb{R}^4$

- No known ADHM constructions

- For $G = E_6,\ E_7,\ E_8$, the blob denotes a theory with unknown Lagrangian (Minahan, Nemeschansky '96)
- Can be realised on M5 branes wrapping Riemann surfaces (Gaiotto '09).
    - For example, the E_6 theory arises from 3 M5-branes wrapping a sphere with 3 maximal punctures.
    - Each puncture corresponds to a global $SU(3)$. The $SU(3)^3$ symmetry enhances to E_6.
- No known corresponding theory for F_4 and G_2

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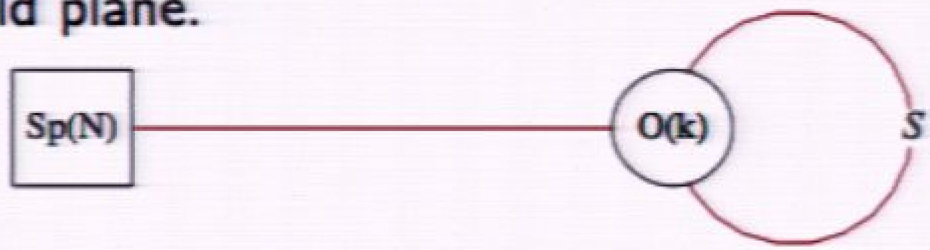

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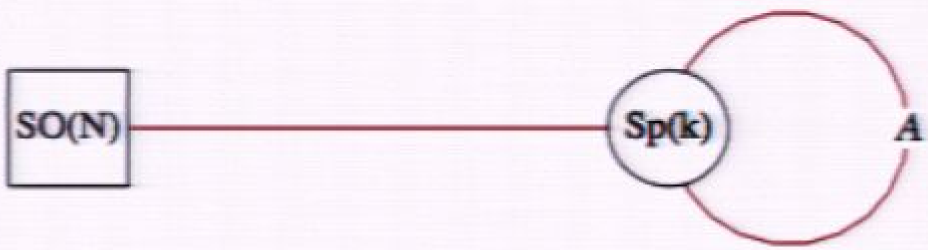

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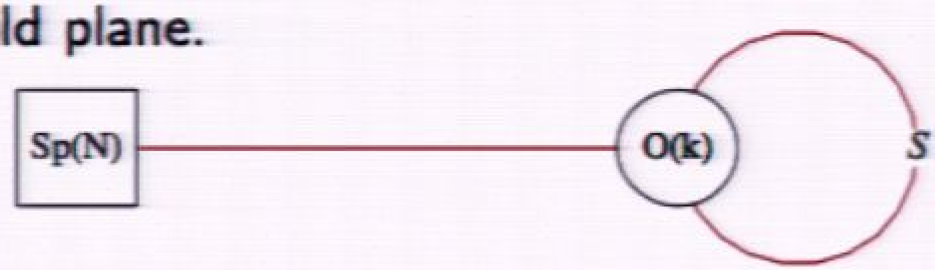

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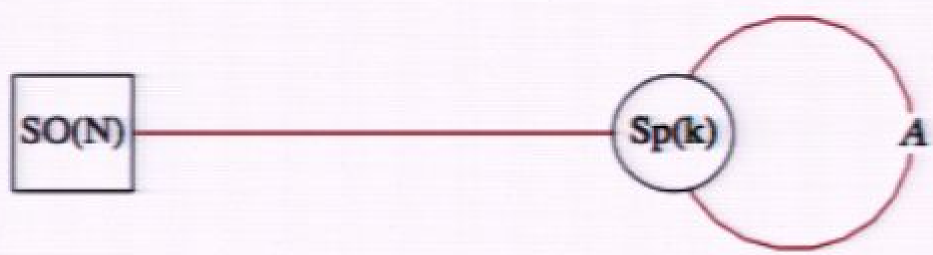

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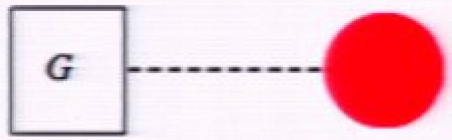

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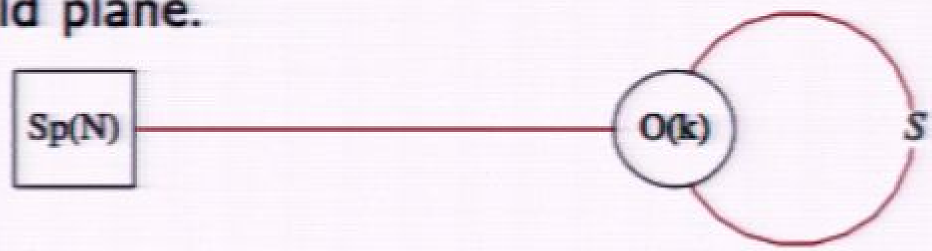

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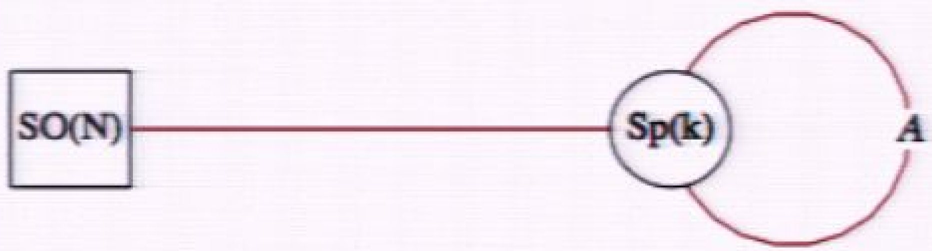


A denotes a h-plet in the rank 2 antisymmetric rep of $Sp(k)$

- These F and D terms are the ADHM equations

Exceptional gauge group instantons on $\mathbb{R}^4$

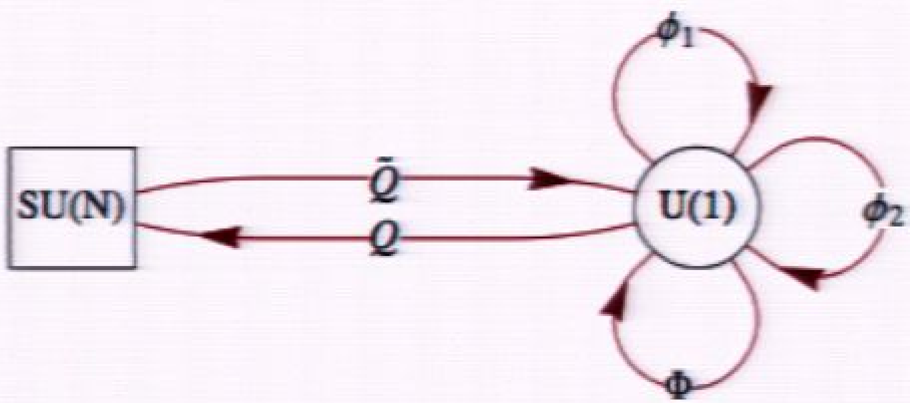
- No known ADHM constructions

- For $G = E_6,\ E_7,\ E_8$, the blob denotes a theory with unknown Lagrangian (Minahan, Nemeschansky '96)
- Can be realised on M5 branes wrapping Riemann surfaces (Gaiotto '09).
  - For example, the E_6 theory arises from 3 M5-branes wrapping a sphere with 3 maximal punctures.
  - Each puncture corresponds to a global $SU(3)$. The $SU(3)^3$ symmetry enhances to E_6.
- No known corresponding theory for F_4 and G_2

The Hilbert series of one instanton moduli space

- **Question:** What's the set of chiral operators on 1 G-instanton moduli space?
- **Aim:** Obtain the spectrum of chiral operators for $G = A_N,\ B_N,\ C_N,\ D_N,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$
- **Method:** First study the Hilbert series of the Higgs branch of gauge theories on D3-D7 brane system and then generalise these results to exceptional gauge group instantons.

The Hilbert series of 1 $SU(N)$-instanton on $\mathbb{R}^4$



$$W = \Phi \widetilde{Q}^i Q_i$$

- Global symmetry $= SU(2) \times U(1) \times SU(N)$
- ϕ_1, ϕ_2 transform as a doublet under the global $SU(2)$

$$H(t; x, x_1, \ldots, x_{n-1}) = \overbrace{\oint_{|z|=1} \frac{dz}{2\pi i z}}^{\text{D-terms and modding out by } U(1)} \times \overbrace{(1-t^2)}^{\text{F-terms}} \times$$

$$\mathrm{PE}\Big[\underbrace{[1]_{SU(2)} t}_{\phi_1, \phi_2} + \underbrace{[1,0,\ldots,0]_{SU(N)} t z^{-1}}_{\widetilde{Q}} + \underbrace{[0,\ldots,1]_{SU(N)} t z}_{Q}\Big]$$

Fugacities: x for $SU(2)$, t for global $U(1)$, x_i for $SU(N)$, and z for gauged $U(1)$

The Hilbert series of $1\ SU(N)$-instanton on $\mathbb{R}^4$

- The Hilbert series of $1\ SU(N)$-instanton on $\mathbb{R}^4$ is

$$H(t; x, x_1, \ldots, x_{N-1}) = \underbrace{\frac{1}{(1-tx)\left(1-\frac{t}{x}\right)}}_{\substack{\text{position of the instanton} \\ \text{on } \mathbb{R}^4 \cong \mathbb{C}^2}} \times \sum_{n=0}^{\infty} [n, 0, \ldots, 0, n] t^{2n} ,$$

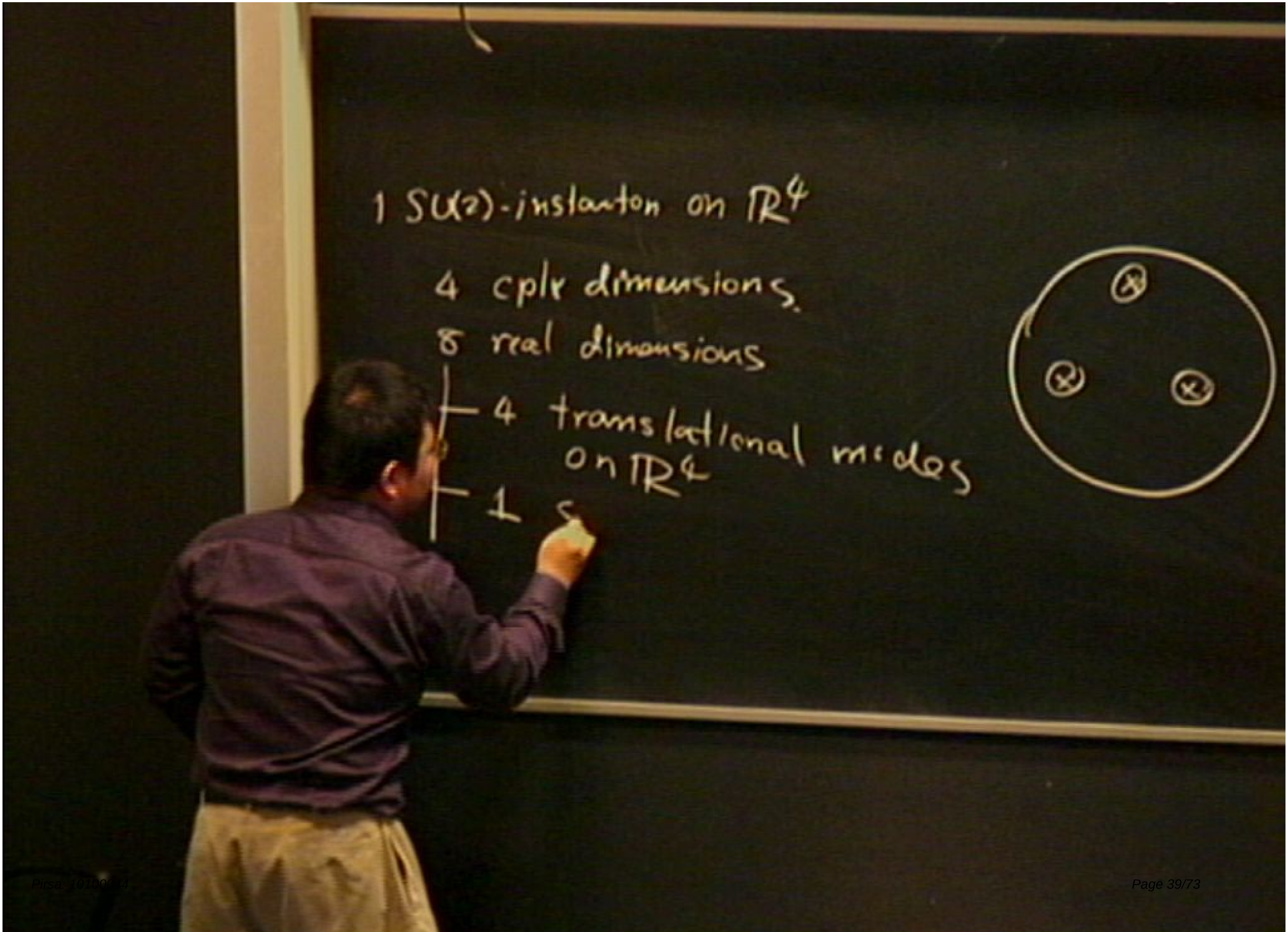
  where $[1, 0, \ldots, 0, 1]$ is the adjoint rep of $SU(N)$.

- Neglecting the $\mathbb{C}^2$ part, the chiral operators at order t^{2n} transform in the $[n, 0, \ldots, 0, n]$ rep of $SU(N)$

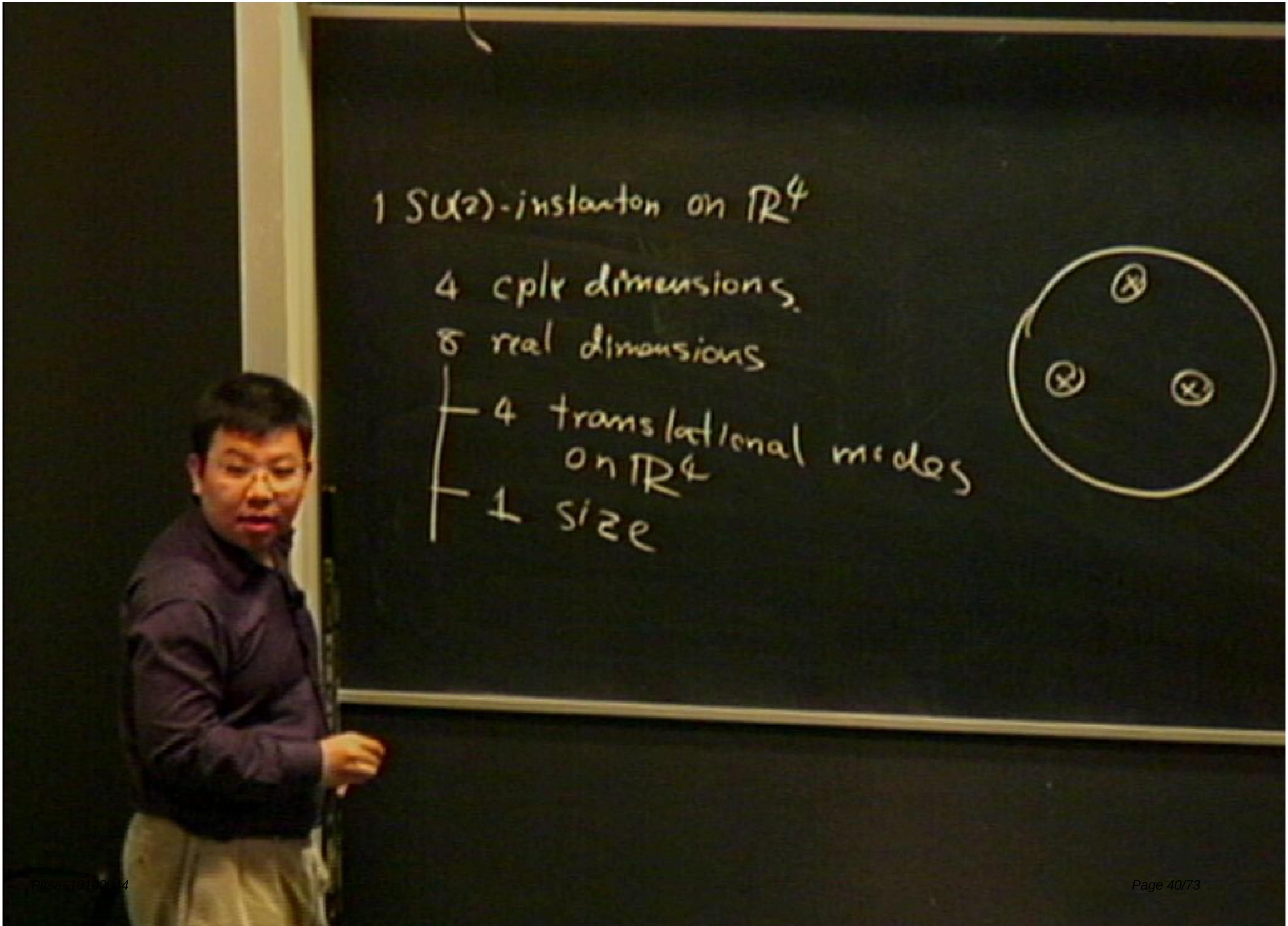
- **Dimension:**

$$H(t, 1, 1, \ldots, 1) \sim \frac{c}{(1-t)^{2N}} \quad \text{as} \quad t \to 1 ,$$

$\Rightarrow$ The instanton moduli space is $2N$ complex dimensional



1 SU(2)-instanton on $\mathbb{R}^4$
4 cplx dimensions
8 real dimensions
4 translational modes on $\mathbb{R}^4$
1


1 SU(2)-instanton on R^4
4 cplx dimensions
8 real dimensions
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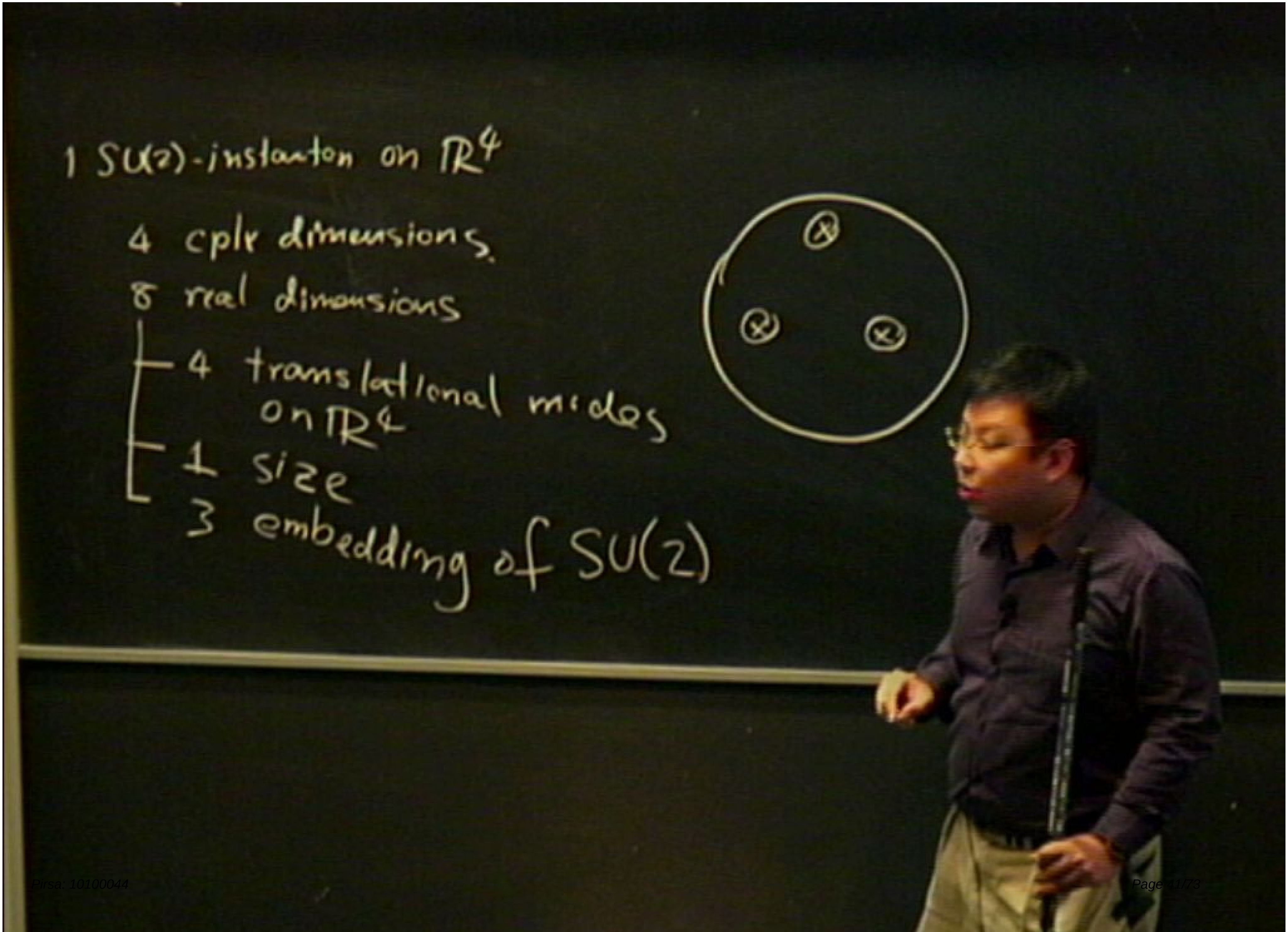
1 size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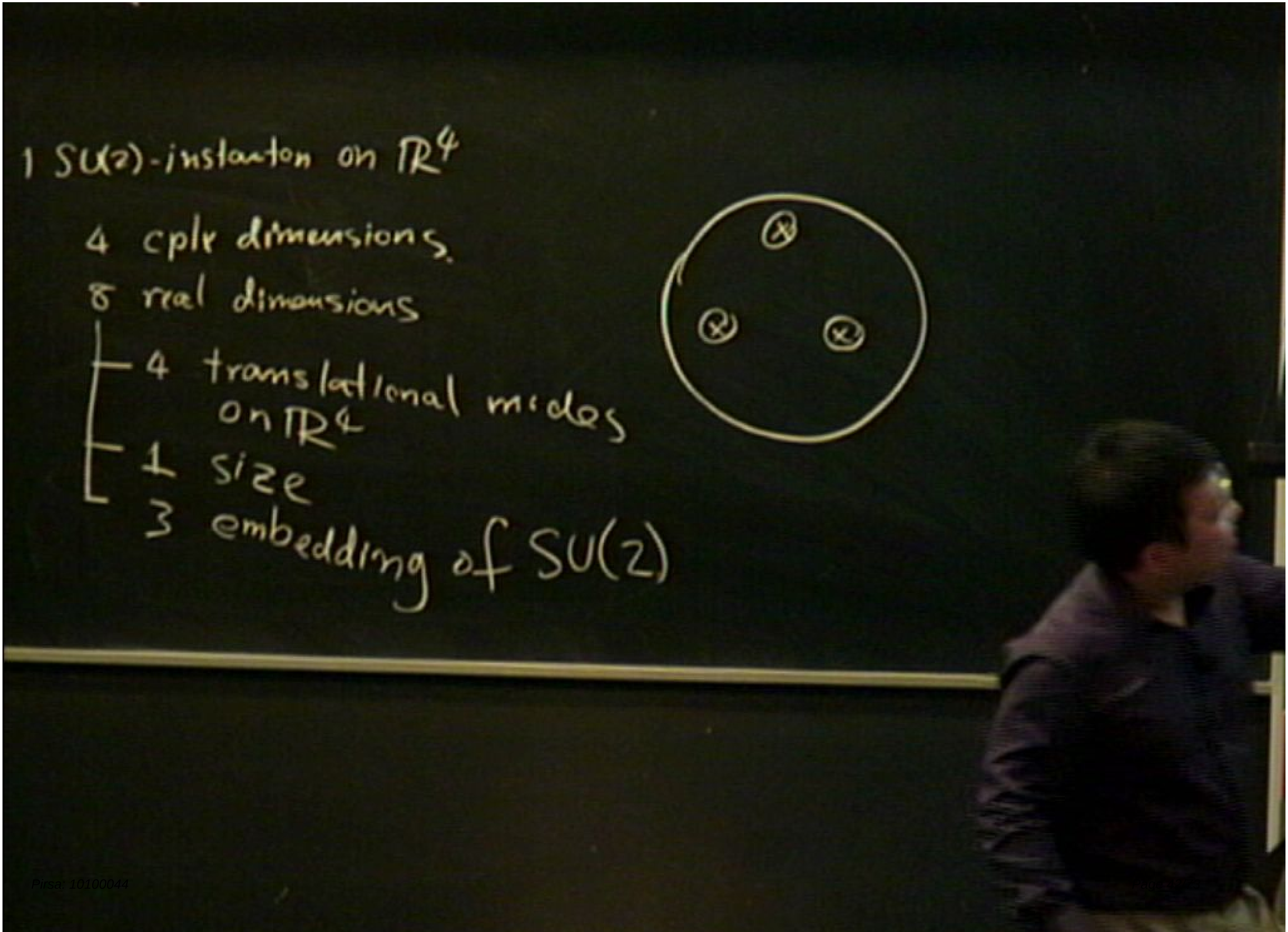
3 embedding of SU(2)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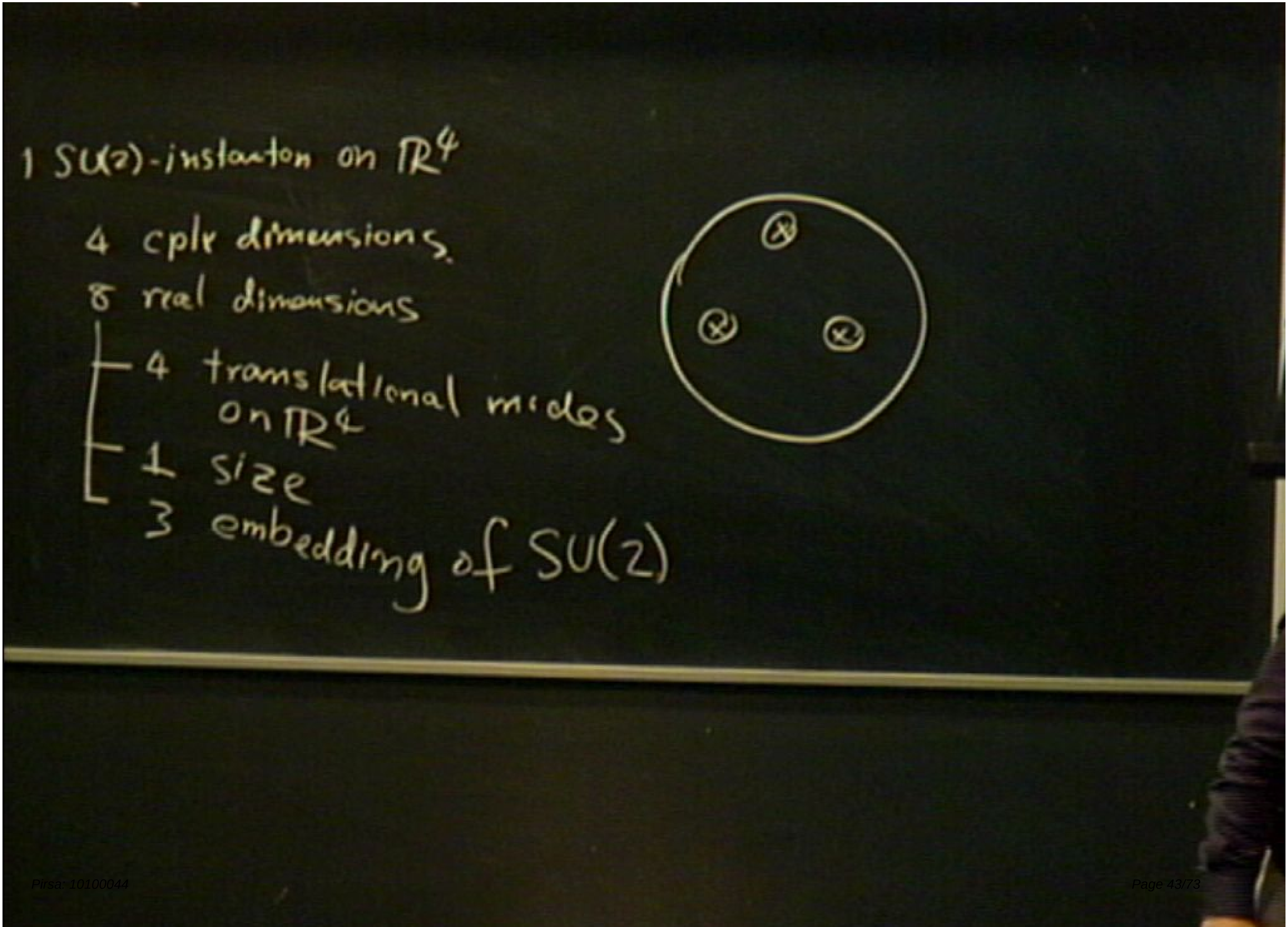
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The Hilbert series of 1 G-instanton on $\mathbb{R}^4$

- The Hilbert series of 1 G-instanton on $\mathbb{R}^4$ is

$$H(t; x, x_1, \ldots, x_r) = \frac{1}{(1-tx)\left(1-\frac{t}{x}\right)} \sum_{n=0}^{\infty} [0, n, 0 \ldots, 0] t^{2n} ,$$

  where $[0, 1, 0 \ldots, 0]$ is the adjoint rep of G, and r is the rank of G.

- This result is true for a classical group G and is also conjectured to be true for any exceptional group G

- **Dimension:**

$$H(t, 1, 1, \ldots, 1) \sim \frac{C}{(1-t)^{2h_G}} \quad \text{as} \quad t \to 1 ,$$

  where h_G is the dual coxeter number for G

$\Rightarrow$ The moduli space of 1 G-instanton on $\mathbb{R}^4$ is $2h_G$ complex dimensional

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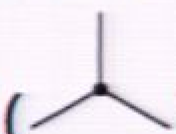
Part III: Tri-vertices and $SU(2)$'s

A class of $\mathcal{N} = 2$ SUSY theories in $(3 + 1)$d

Study a class of $\mathcal{N} = 2$ theories consisting of 2 ingredients (Giaotto '09)

1. A line (———) represents an $SU(2)$ gauge group with gauge coupling $1/g^2 \sim L$, the length of the line.

   Note: A line with an infinite length gives rise to an $SU(2)$ global symmetry.

2. A tri-valent vertex or a node () represents 8 half-hypermultiplets transforming in the tri-fundamental $[1; 1; 1]$ representation of $SU(2)^3$.

Note: A class of graphs made out of tri-valent vertices and lines give rise to $\mathcal{N} = 2$ gauge theories in $(3 + 1)$d with Lagrangians.

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- This result is true for a classical group G and is also conjectured to be true for any exceptional group G

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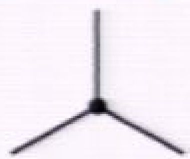
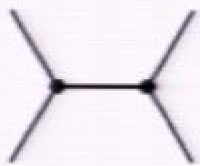
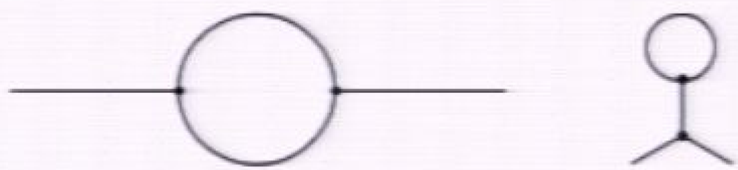
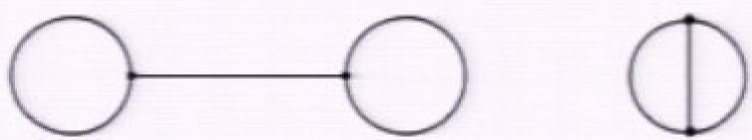
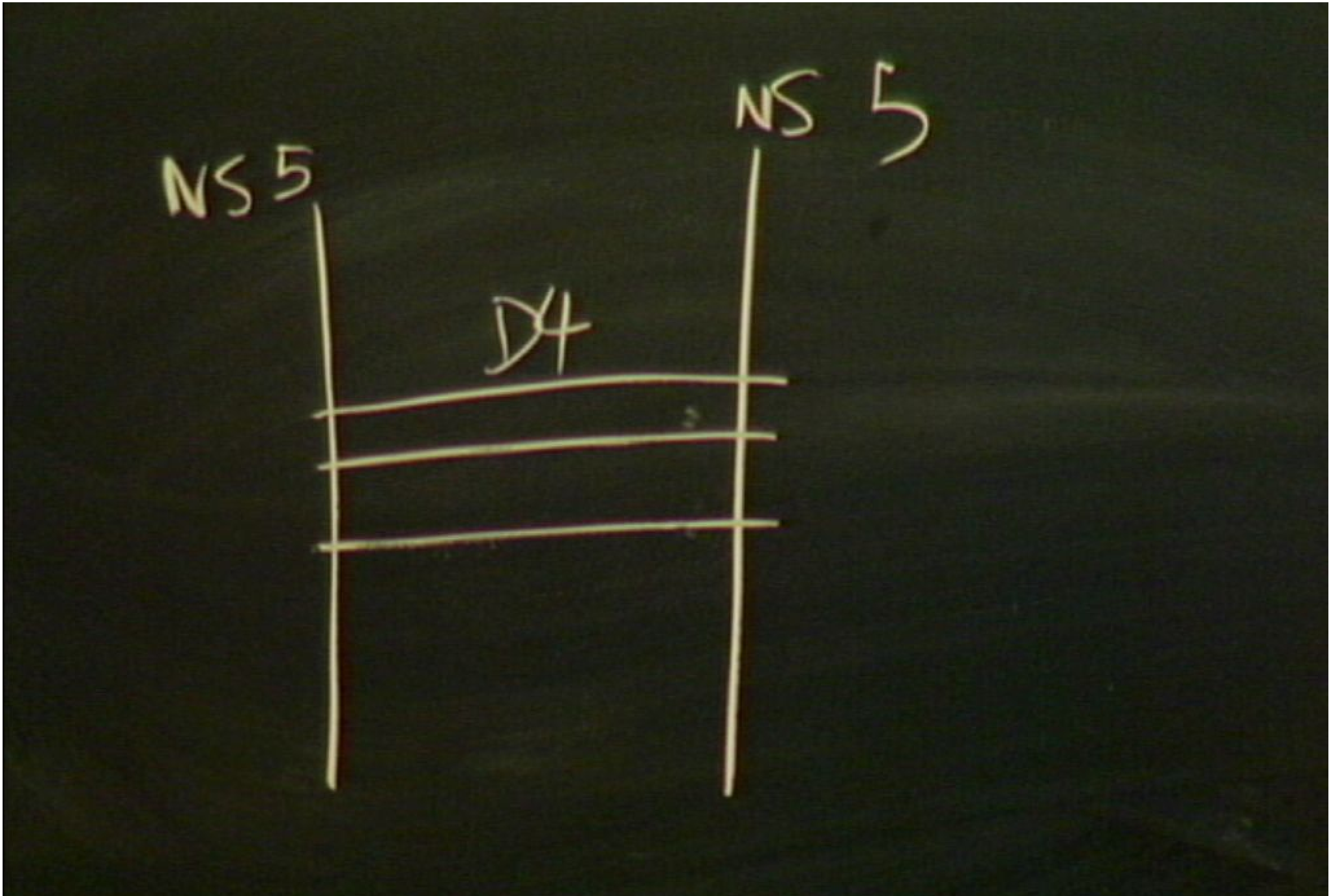
Genus and external legs

One can classify the graphs using genus g and the number of external legs e

- $(g=0, e=3)$: 
- $(g=0, e=4)$: 
- $(g=1, e=2)$: 
- $(g=2, e=0)$: 

Note: A theory with genus g and e external legs can be realised on the w.v. of two M5-branes wrapping a Riemann surface with genus g and e punctures.

(Giaotto '09)


NS5
NS5
D4

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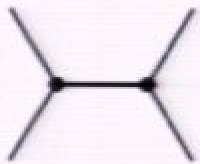
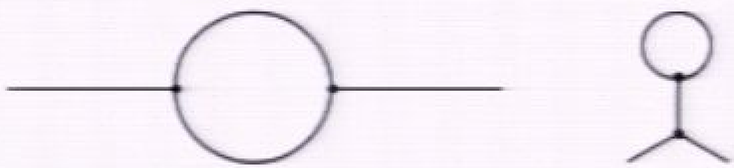
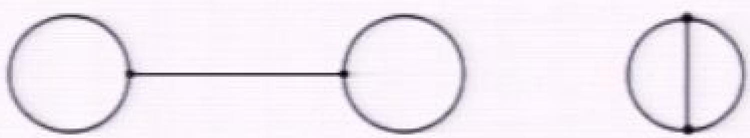
- $(g = 0, e = 4)$: 
- $(g = 1, e = 2)$: 
- $(g = 2, e = 0)$: 

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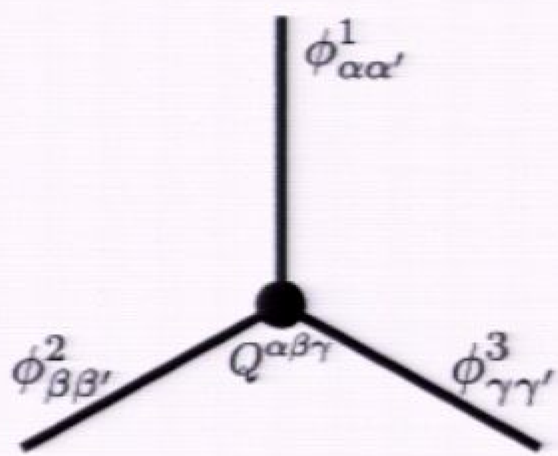
Some comments

- For a theory with genus g and e external legs, the Lagrangian has
  - $3g-3+e$ $SU(2)$ gauge groups
  - $2g-2+e$ matter fields $\times$(8 half-hypers)
- In the $\mathcal{N}=1$ language, the superpotential takes the the form of a sum over all nodes where a contribution of each node is

$$Q^{\alpha\beta\gamma}Q^{\alpha'\beta'\gamma'}(\phi^1_{\alpha\alpha'}\epsilon_{\beta\beta'}\epsilon_{\gamma\gamma'}+\epsilon_{\alpha\alpha'}\phi^2_{\beta\beta'}\epsilon_{\gamma\gamma'}+\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\phi^3_{\gamma\gamma'})$$



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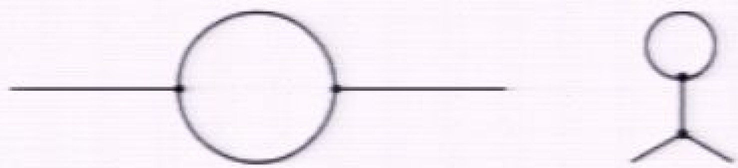
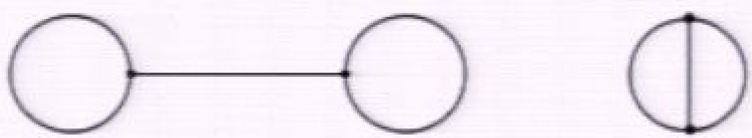
- $(g = 1, e = 2)$: 
- $(g = 2, e = 0)$: 

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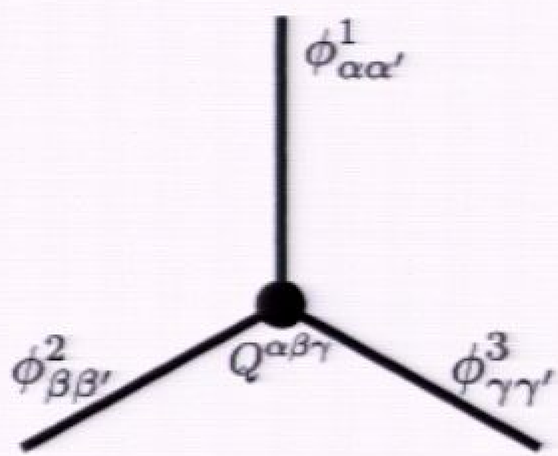

The hypermultiplet moduli space

Consider a theory with genus g and e external legs.

- Focus on the branch on which the h-plets are massless. There are in total $\frac{1}{2}8(2g-2+e)$ quarternionic DOFs (QDOFs).
- There are g massless $U(1)$ v-plets at a generic point on the h-plet space.
- The gauge symmetry $SU(2)^{3g-3+e}$ is generically broken to $U(1)^g$.

  (The gauge symmetry is not completely broken - avoid the term 'Higgs branch'.)
- Higgs mechanism: the $3(3g-3+e)-g$ QDOFs of v-plet become massive.
- Hence, $\frac{1}{2}8(2g-2+e)-[3(3g-3+e)-g]=e+1$ QDOFs are left massless.
- The h-plet moduli space is $e+1$ quarternionic dimensional.

(In agreement with Benini, Benvenuti, Tachikawa '09)

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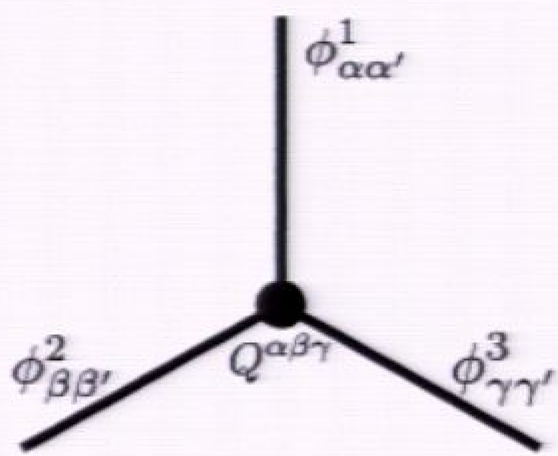

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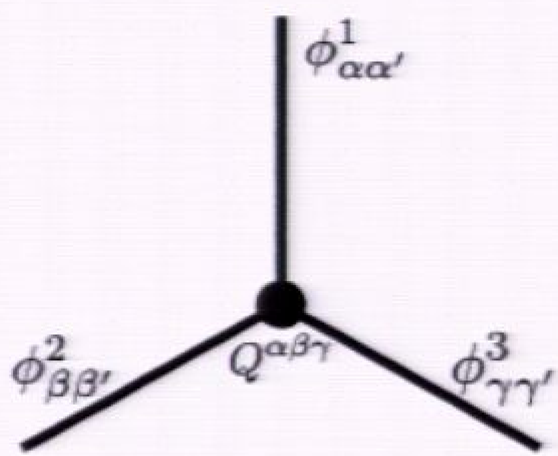

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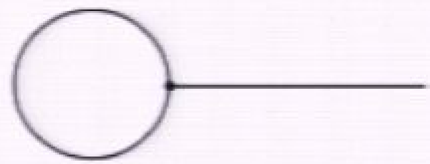
Example: The tadpole theory



- $SU(2)$ gauge group (indices a, b), $SU(2)$ global symmetry (indices α, β)
- The half-hypers $Q_{ab,\alpha}$ can be split into 2 parts:
  1. The two $SU(2)$ singlets: $X_\alpha = \epsilon^{ab} Q_{ab,\alpha}$
  2. The two $SU(2)$ adjoints: $\phi_{ab,\alpha} = Q_{ab,\alpha} - \frac{1}{2} X_\alpha \epsilon_{ab}$
- The F-terms impose the conditions: $[\phi_\alpha, \phi_\beta] = 0$
- This theory is in fact the $SU(2)$ $\mathcal{N} = 4$ gauge theory with a free h-plet X (written in the $\mathcal{N} = 2$ notation)

Example: The tadpole theory (continued)

- **Chiral gauge invariant operators:**
  1. X_α (parametrised $\mathbb{C}^2$)
  2. $M_{\alpha\beta} = \mathrm{Tr}(\phi_\alpha \phi_\beta)$ subject to $\det M = 0$ (parametrised $\mathbb{C}^2/\mathbb{Z}_2$)
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- **The Hilbert series:**

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\begin{aligned}
H(t,x) &= \frac{1-t^4}{(1-tx)\left(1-\frac{t}{x}\right)(1-t^2x^2)\left(1-\frac{t^2}{x^2}\right)} \\
&= (1-t^4)\mathrm{PE}\left[[1]_{SU(2)}t + [2]_{SU(2)}t^2\right]
\end{aligned}
$$

Generalise to any genus g and e external legs

- **Aim:** Obtain the Hilbert series for a theory with any (g, e)
- **Example:** A theory with 8 free half-hypers ()
  - No gauge groups, $SU(2)^3$ global symmetry
  - The Hilbert series:

$$H(t, x_1, x_2, x_3) = \mathrm{PE}\,[[1;1;1]t] = \prod_{\epsilon_i = \pm 1} \left(1 - t\, x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3}\right)^{-1}$$

  - Trick: Expand this in a power series

$$H(t, x_1, x_2, x_3) = 1 + \underbrace{[1;1;1]}_{Q^{\alpha\beta\gamma}}\, t + \underbrace{([2;2;2] + [2;0;0] + [0;2;0] + [0;0;2])}_{Q^{\alpha\beta\gamma}Q^{\alpha'\beta'\gamma'}}\, t^2 + \ldots$$

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General form of the Hilbert series for any (g, e)

$$(t, x_1, \ldots, x_e) = \frac{1}{1-t^4} \sum_{n_1=0}^{\infty} \cdots \sum_{n_e=0}^{\infty} \sum_{m=0}^{\infty} \Big([2n_1 + m, \ldots, 2n_e + m]\, t^{2n_1+\ldots+2n_e+\chi m}$$

$$+ [2n_1 + m + 1, \ldots, 2n_e + m + 1]\, t^{2n_1+\ldots+2n_e+\chi m+\chi+2} \Big) \ ,$$

where $\chi = 2g - 2 + e$.

Special case: zero external leg $(e = 0)$

- **Examples:** 
- The h-plet moduli space is 1 quarternionic dimensional
- The Hilbert series is

$$H(t) = \frac{1 + t^{\chi+2}}{(1-t^4)(1-t^{\chi})} = \frac{1-t^{4g}}{(1-t^4)(1-t^{2g-2})(1-t^{2g})}$$

- The h-plet moduli is a complete intersection:

  Generators at orders $4,\ 2g-2,\ 2g$ subject to one relation at order $4g$

- There is an ADE classification of the hyperKähler manifold of dimension 1
- Indeed, the h-plet moduli space is $\mathbb{C}^2/\hat{D}_{g+1}$

General form of the Hilbert series for any (g, e)

$$(t, x_1, \ldots, x_e) = \frac{1}{1-t^4} \sum_{n_1=0}^{\infty} \cdots \sum_{n_e=0}^{\infty} \sum_{m=0}^{\infty} \left([2n_1 + m, \ldots, 2n_e + m]\, t^{2n_1 + \ldots + 2n_e + \chi m} \right.$$

$$\left. + [2n_1 + m + 1, \ldots, 2n_e + m + 1]\, t^{2n_1 + \ldots + 2n_e + \chi m + \chi + 2} \right) ,$$

where $\chi = 2g - 2 + e$.

Special case: zero external leg ($e = 0$)

- **Examples:**

- The h-plet moduli space is 1 quarternionic dimensional
- The Hilbert series is

$$H(t) = \frac{1 + t^{x+2}}{(1-t^4)(1-t^x)} = \frac{1 - t^{4g}}{(1-t^4)(1-t^{2g-2})(1-t^{2g})}$$

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  Generators at orders $4,\ 2g-2,\ 2g$ subject to one relation at order $4g$

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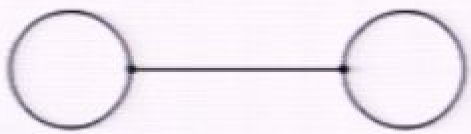
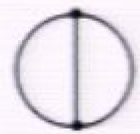
- **Examples:**  
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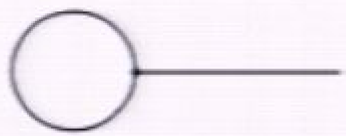
- The h-plet moduli is a complete intersection:

  Generators at orders $4,\ 2g-2,\ 2g$ subject to one relation at order $4g$
- There is an ADE classification of the hyperKähler manifold of dimension 1
- Indeed, the h-plet moduli space is $\mathbb{C}^2/\hat{D}_{g+1}$

Special case: one external leg ($e = 1$)

- **Examples:** 

- The h-plet moduli space is 2 quarternionic dimensional
- The Hilbert series is

$$H(t) = (1 - t^{4g})\mathrm{PE}\left[[2]t^2 + [1]t^{2g-1}\right] \ .$$

- The h-plet moduli is a complete intersection:
  - The generators at order t^2 transform in the rep $[2]$ of the global $SU(2)$
  - The generators at order t^{2g-1} transform in the rep $[1]$ of the global $SU(2)$
  - There is one relation at t^{4g}