Title: Statistical Mechanics (PHYS 602) - Lecture 13

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Abstract:

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Symmetries of 2d Critical Behavior

According to the Extended Singularity Theorem the symmetries of infinite uniform space are crucial for understanding critical phenomena. The question is not so much what is that space like: it is simple and boring. The question is what can happen in that space. We really do not very well know the answer to that question in three dimensions or four or higher, but we have a good start on answering it in two dimensions. That is because two dimensional space has a rich symmetry structure which we, in some measure, understand. We are interested in the topological structure of the space in the presence of the operators $o_\alpha(\mathbf{r})$. Each of these produces a puncture of some sort in the topology at its coordinate. We are interested in understanding this punctured space.

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Operator Product Expansion

short distance expansion

f ***r*** is close to ***s***, one can replace the product $o_\alpha(\mathbf{r})\ o_\beta(\mathbf{s})$ according to

$$o_\alpha(\mathbf{r})\ o_\beta(\mathbf{s}) = \Sigma_\gamma\ C_{\alpha\,\beta\,\gamma}\ \ |\mathbf{r}-\mathbf{s}|^{x_\gamma - x_\alpha - x_\beta}\ \ o_\gamma(\mathbf{s})$$

The idea is that when ***r*** and ***s*** approach one another, the product looks like an operator at ***s.*** ince the number of different operators is quite limited, one must get a sum of the operators n the theory.

Expressions like the one's in operator product expansions provide a sort of algebra for the luctuating operators in the theory. Even before Wilson's work on the renormalization group, t was hoped that algebraic methods would enable a classification of, and perhaps an analytic olution for, d=2 critical phenomena. That was roughly in 1970. Somewhat later, in 1984 Daniel Friedan, Zongan Qiu and Stephen Shenker used algebraic methods related to short listance expansion to find the behavior of all the most familiar problems in two dimensional ritical phenomena. They worked with an extra ingredient which was just starting to be available in 1970, the deep understanding of symmetries provided by conformal invariance. I am going to fill in some pieces of the theory of these symmetries.

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Our generic operator $o_\alpha(r)$ will thus be written as $o_\alpha(z,z^*)$ in this notation. We focus on scalar operators like $\sigma(z,z^*)$ and $\varepsilon(z,z^*)$. In addition we will later devote considerable attention to the rank two tensor, the stress tensor T_{jk}. Instead of using a scaling index x_α for the operators, we use two indices h_α and h^*_α which respective describe the scaling in z and z^*. Thus, $x_\alpha = h_\alpha + h^*_\alpha$. For scalar operators $h_\alpha = h^*_\alpha$ In the Ising model, for the spin operator h=1/16; while for the energy h=1/4. (Christe & Henkel use a notation in which h is replaced by Δ.)

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Symmetries are reflected in correlations

$\sigma(z,z^*)$ = magnetization density
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At criticality, for the d=2 Ising model, $<\sigma(z)>=<\varepsilon(z)>=0$

$<\sigma(z,z^*)\ \sigma(w,w^*)>=1/|z-w|^{1/4}=1/[(z-w)^{1/8}(z^*-w^*)^{1/8}]$ The "1" sets normalization

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The correlations above apply in a continuum limit in which we look at distances large enough so that the lattice structure has disappeared. In this limit, our usual Ising model has translational invariance and rotational invariance, That is the correlation functions are unchanged if we very each coordinate according to z-->z ´=z+a (translation) or z´=λ z with |λ|=1, while holding scalar operators fixed, i.e. by varying the operators according to $o_\alpha(z,z^*)$--> $o_\alpha(z', z^{*\prime})$.

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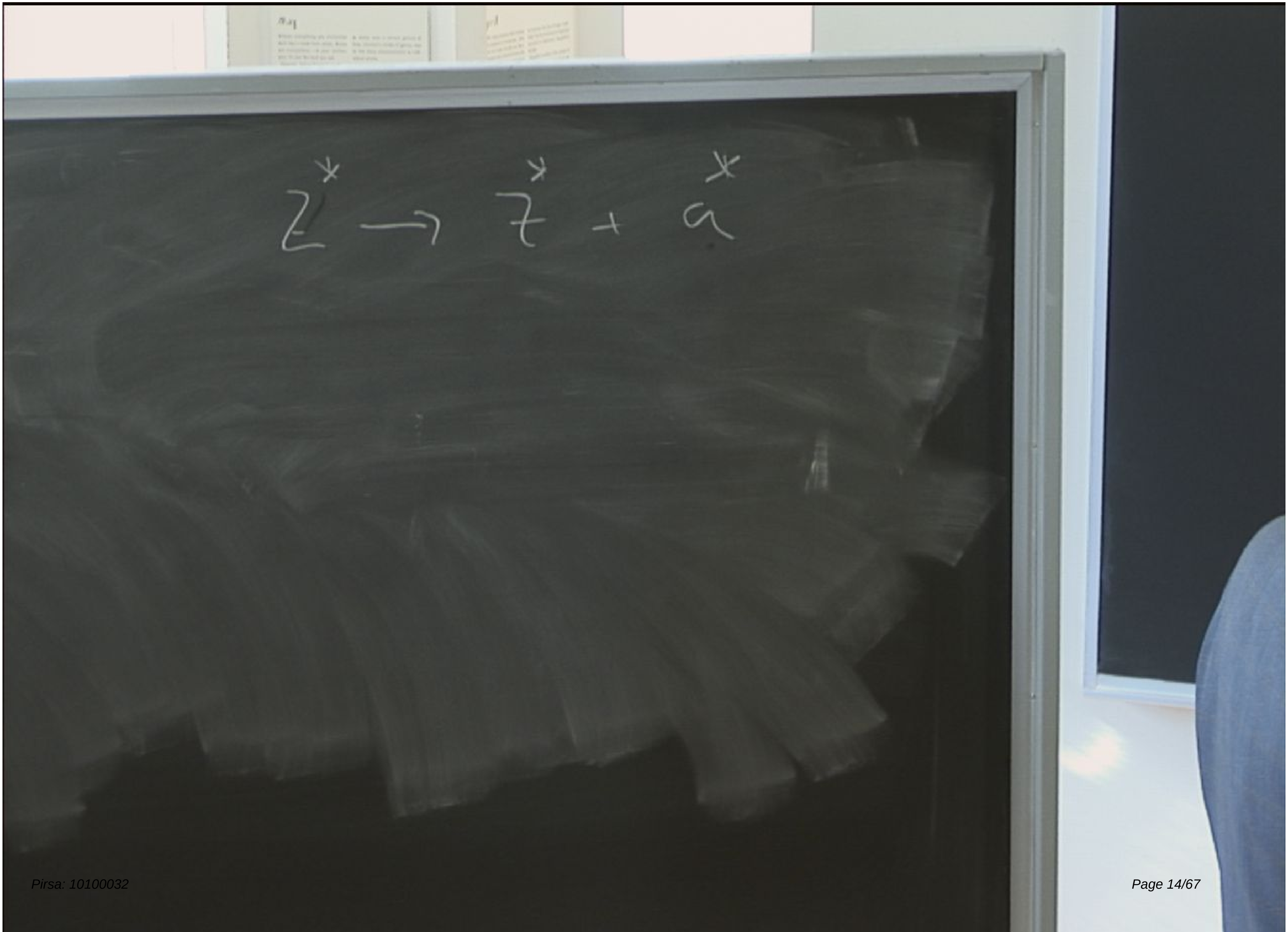
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Application of Complex Analysis

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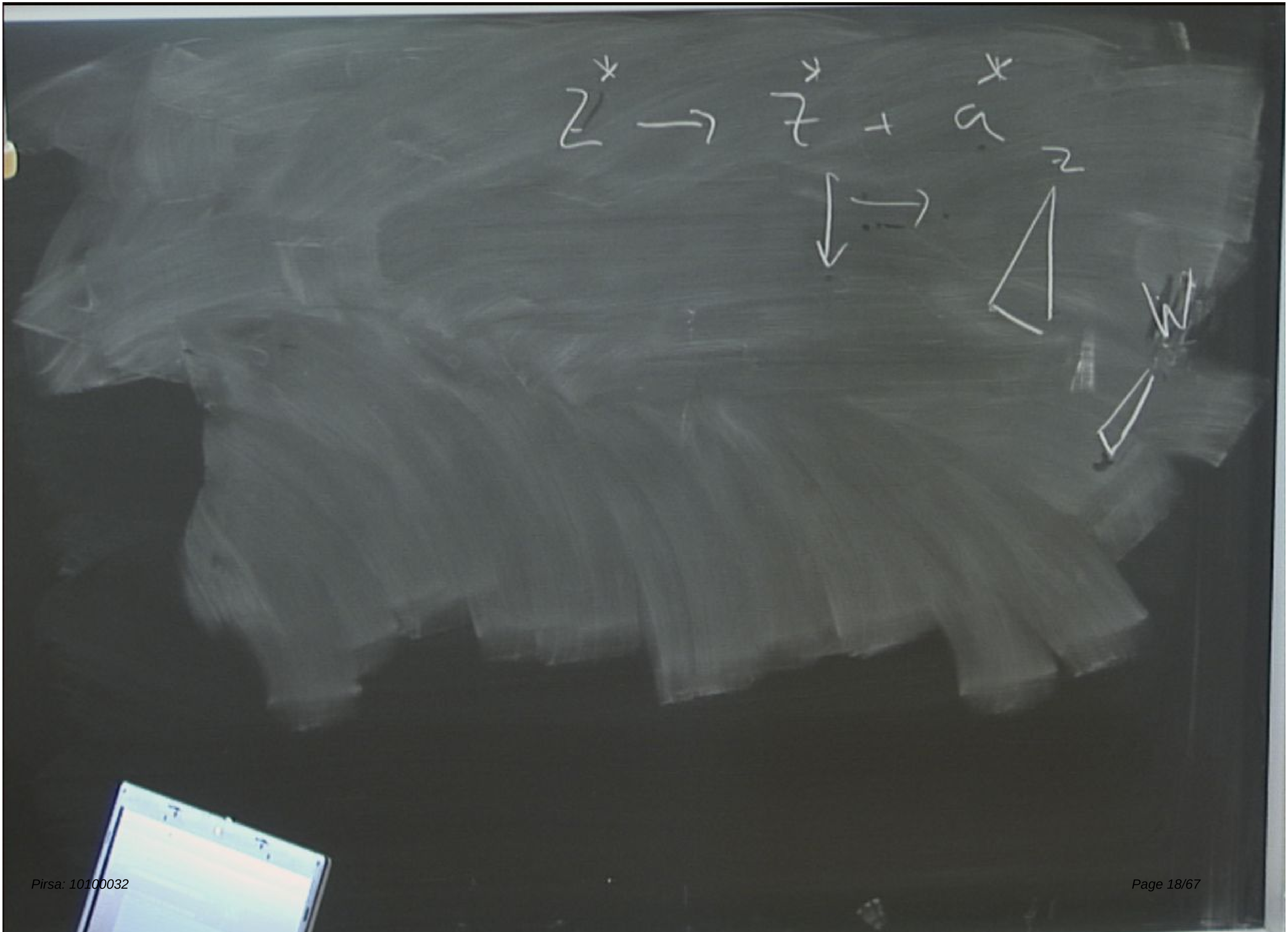
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Transformations like this are called conformal transformations. They were introduced into critical phenomena work in 1970 by A. A. Polyakov. John Cardy showed us how to make use of specific transformations, like the one in the next slide.

Example: $w=(L/2\pi)\ \ln z$,

which then implies $z=\exp(2\pi w/L)$

This function is analytic in z except at z=0. This function takes the entire complex z plane, except for this point into a strip in the complex w-plane $|\mathrm{Im}\ w| \leq L/2$. Now consider the correlation function $<\sigma(z,z^*)\sigma(y,y^*)>=1/|z-y|^{1/4}$ If we make the appropriate substitution, we find

$<o_\alpha(w,w^*)\, o_\alpha(u,u^*)>= 1/[(z-y)^{2h}(z^*-y^*)^{2h}]\,[z'\, z^{*\prime}\, y'\, y^{*\prime}]^h$ with $y=\exp(2\pi u/L)$

$= (2\pi/L)^{4h}\,[z\, z^*\, y\, y^*]^h/[(z-y)^{2h}(z^*-y^*)^{2h}] =|(2\pi/L)^2 (zy)^{1/2}/(z-y)|^{2h}$

$=|(2\pi/L)^2 /|(z/y)^{1/2}-(y/z)^{1/2}|^{2h}$

$= (2\pi/L)^{2x}/\{2\cosh[2\pi(w_1-u_1)/L]- 2\cos[2\pi(w_2-u_2)/L]\}^x$

Notice that the denominator only passes through zero on the unit strip when w=u. That's the only point of singularity. Although it is not obvious, the singularity is proportional to $1/|w-u|^{2x}$, as it should be. Further the result looks exactly as if the behavior was caused by repeated placements of the singularities at positions displaced by $2\pi i$ times any integer, so that we have an infinite number of strips side by side. The solution is rather like the one with image charges in electrostatics.

In fact, two dimensional electrostatics may be analyzed by exactly the same conformal invariance trick.

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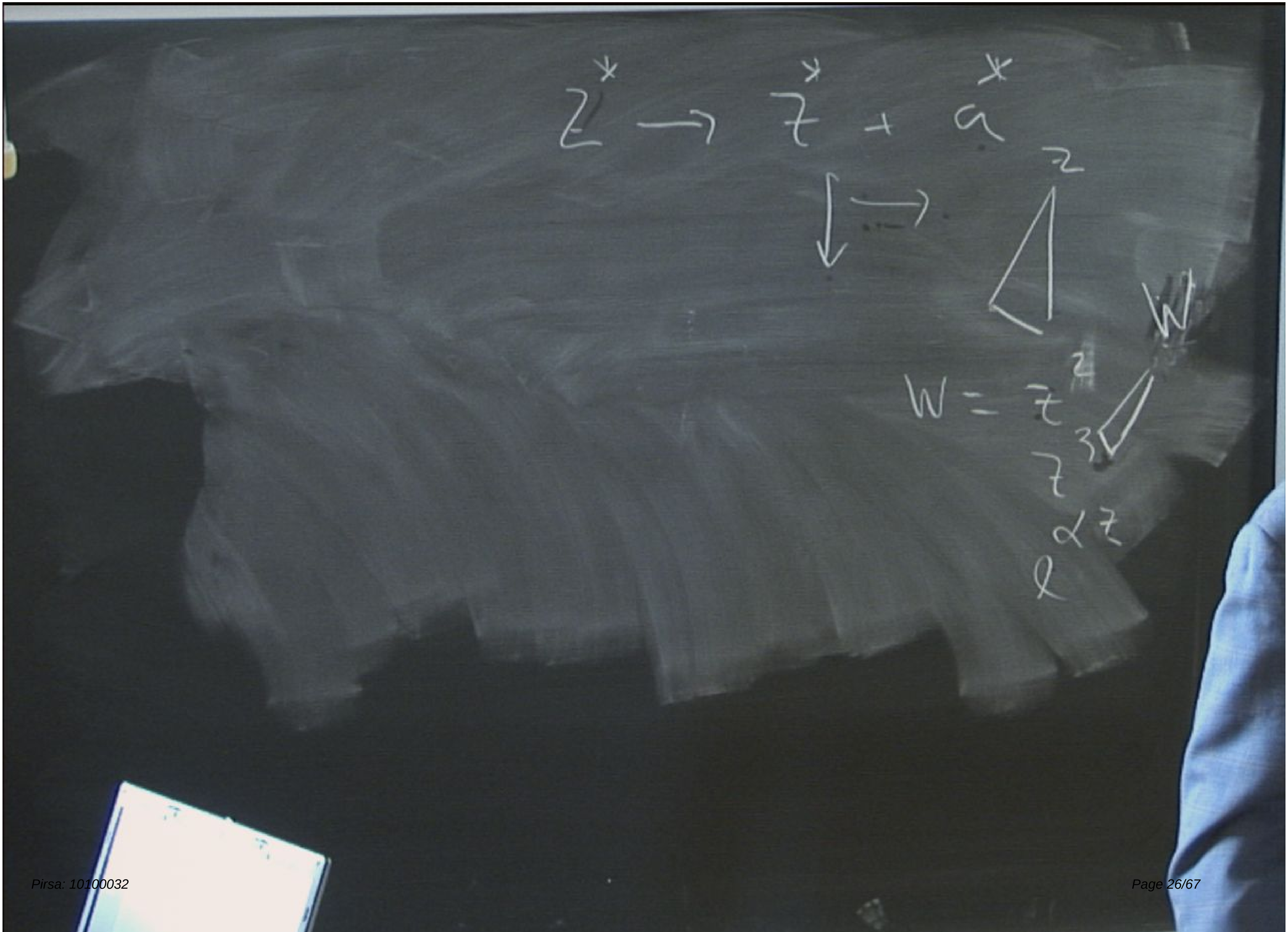
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W
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$<o_\alpha(w,w^*)\, o_\alpha(u,u^*)>= 1/[(z-y)^{2h}(z^*-y^*)^{2h}]\,[z'\, z^{*\prime}\, y'\, y^{*\prime}]^h$ with $y=\exp(2\pi u/L)$

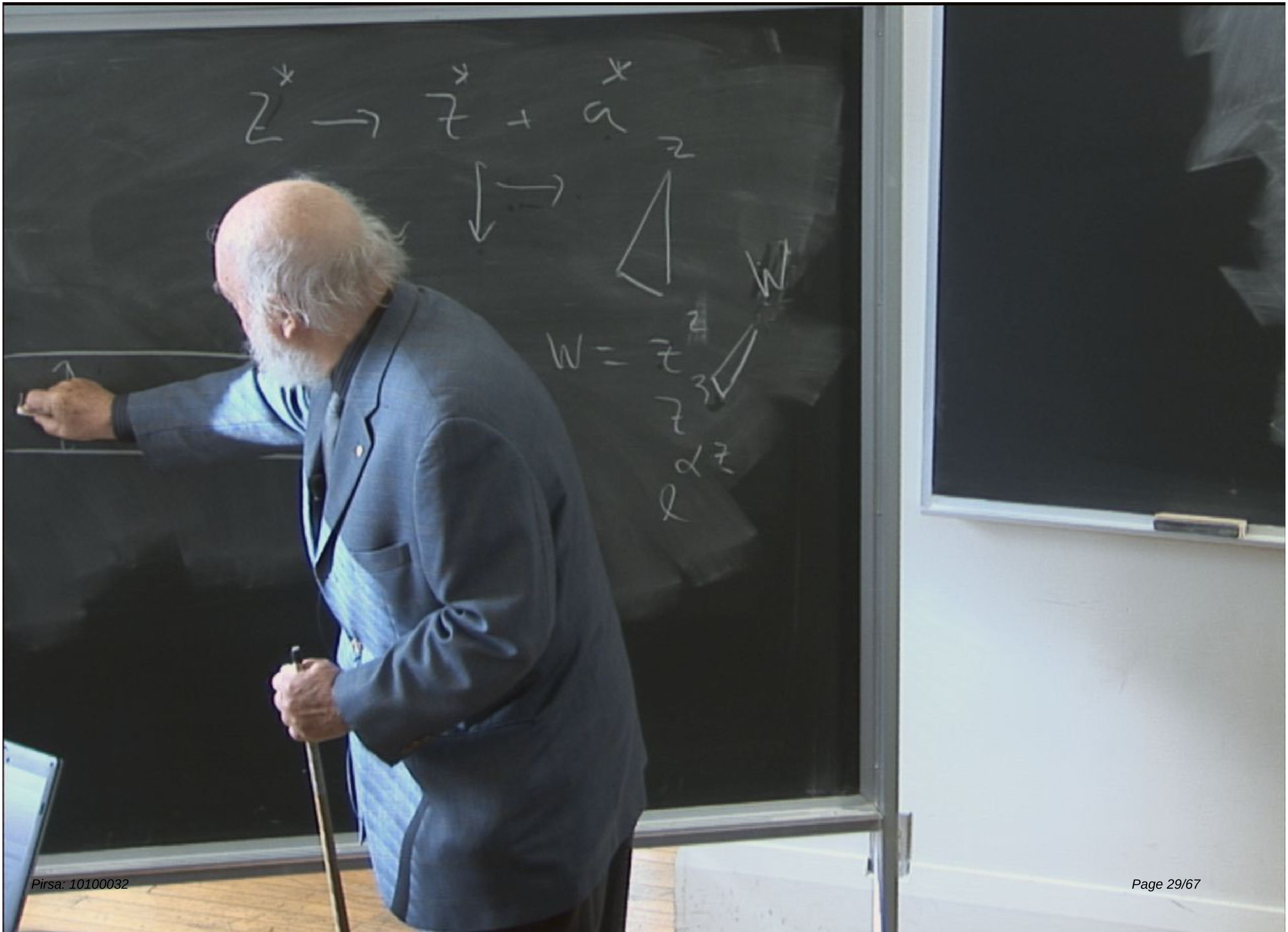
$= (2\pi/L)^{4h}\,[z\, z^*\, y\, y^*]^h/[(z-y)^{2h}(z^*-y^*)^{2h}] =|(2\pi/L)^2 (zy)^{1/2}/(z-y)\,|^{2h}$

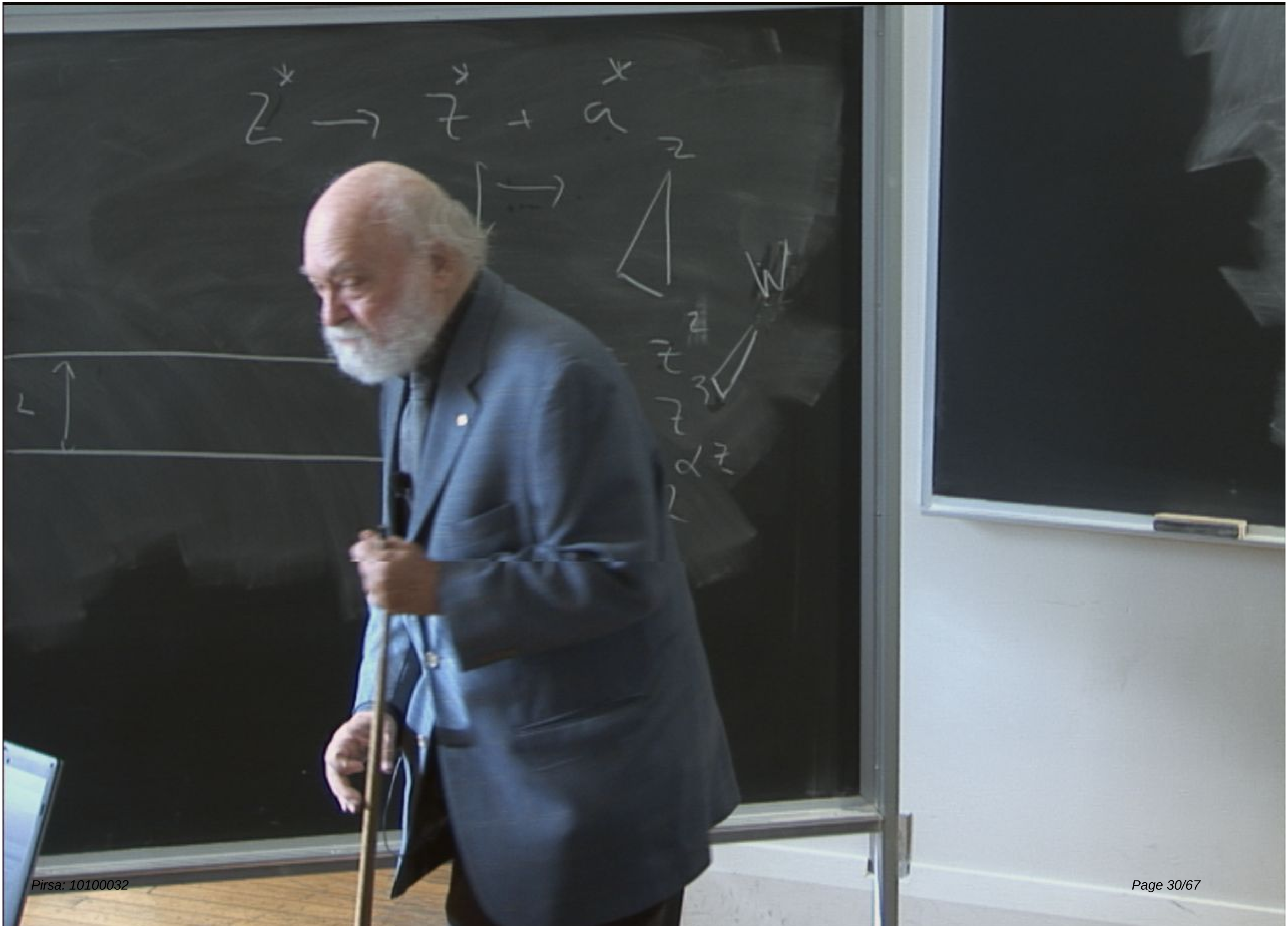
$=|(2\pi/L)^2 /|(z/y)^{1/2}-(y/z)^{1/2}|^{2h}$

$= (2\pi/L)^{2x}/\{2\cosh[2\pi(w_1-u_1)/L]- 2\cos[2\pi(w_2-u_2)/L]\}^x$

Notice that the denominator only passes through zero on the unit strip when w=u. That's the only point of singularity. Although it is not obvious, the singularity is proportional to $1/|w-u|^{2x}$, as it should be. Further the result looks exactly as if the behavior was caused by repeated placements of the singularities at positions displaced by $2\pi i$ times any integer, so that we have an infinite number of strips side by side. The solution is rather like the one with image charges in electrostatics.

In fact, two dimensional electrostatics may be analyzed by exactly the same conformal invariance trick.





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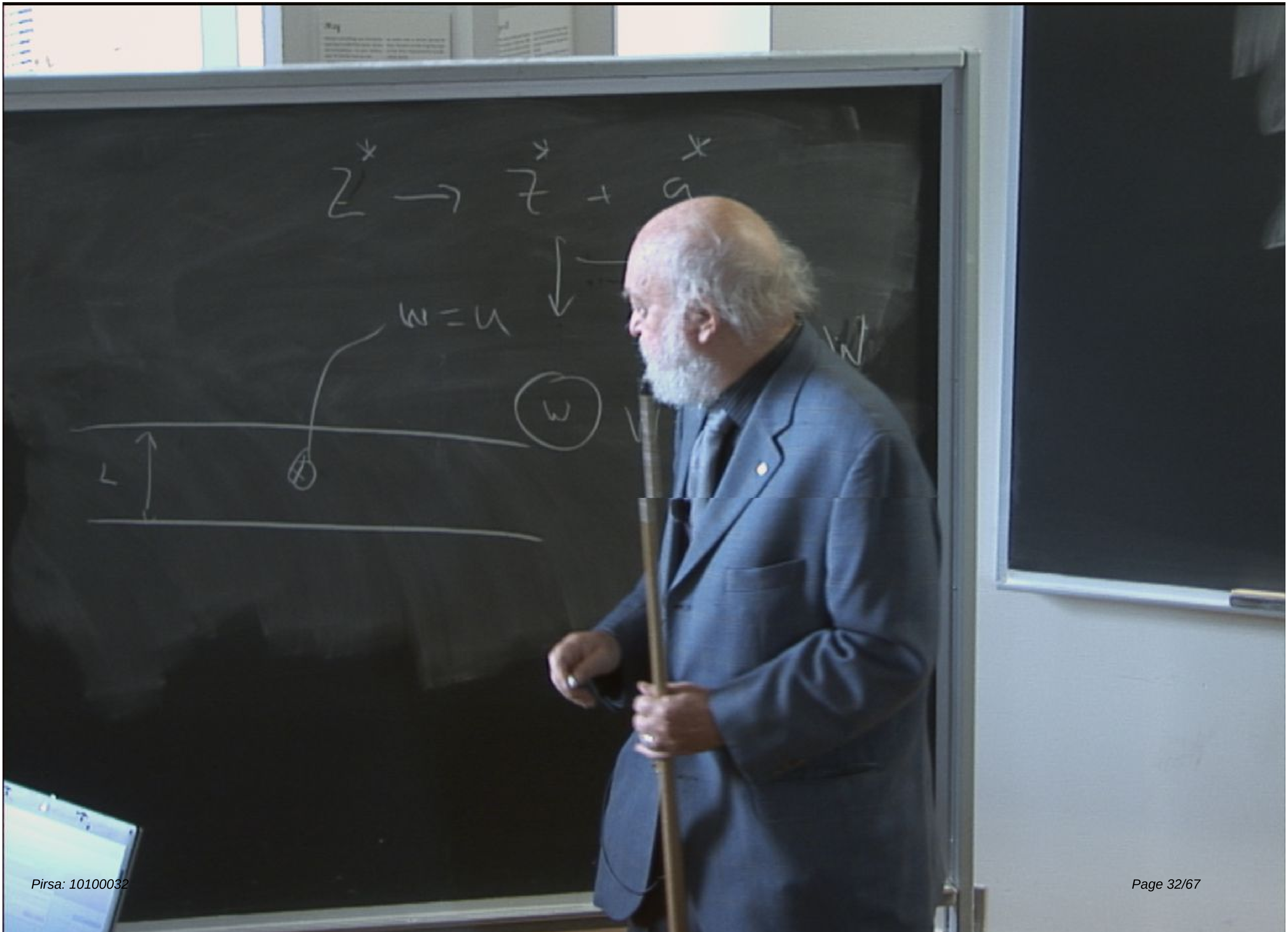
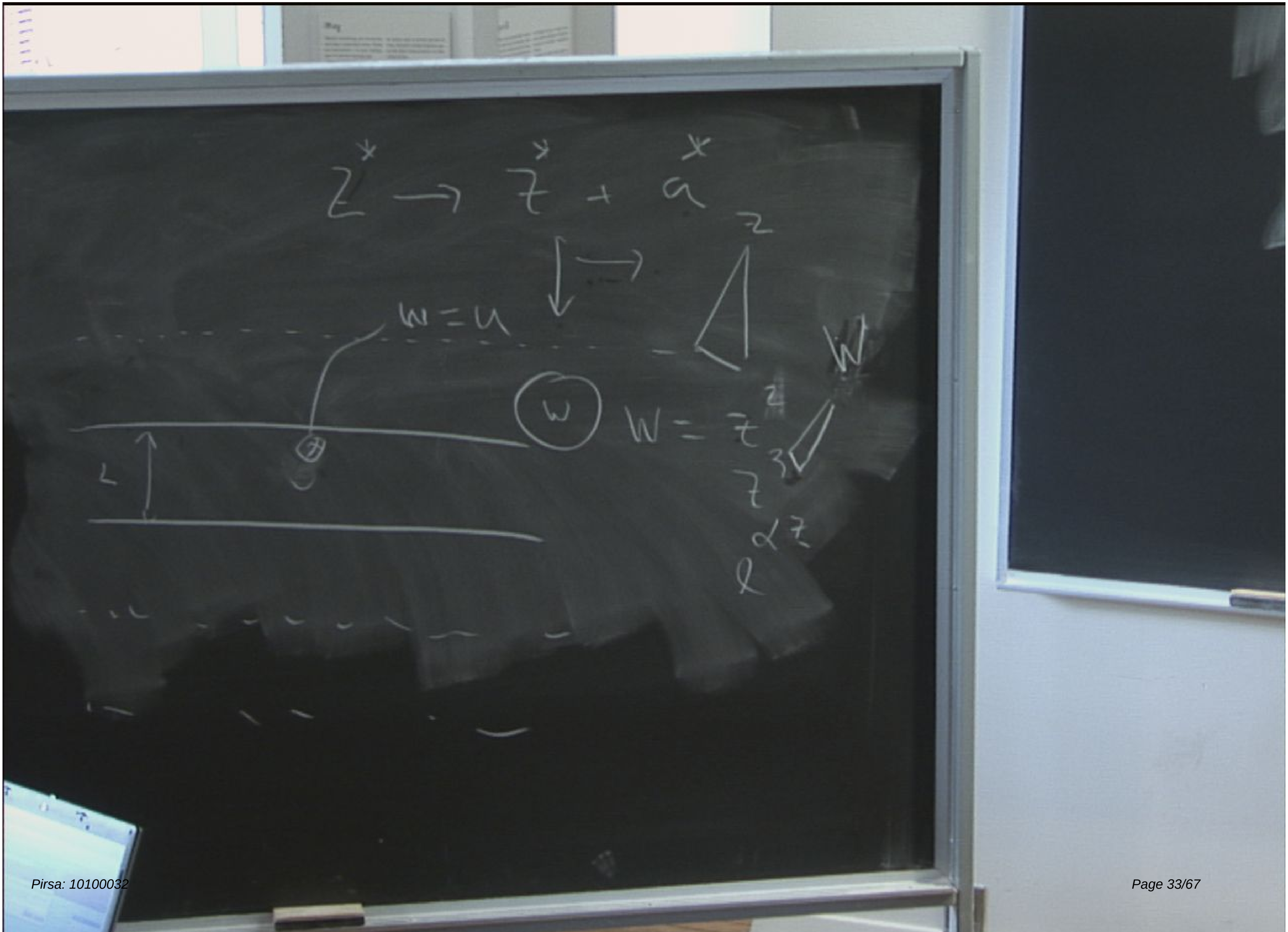

Z* → Z* + g*
w = u
L
w


z* → z* + a*
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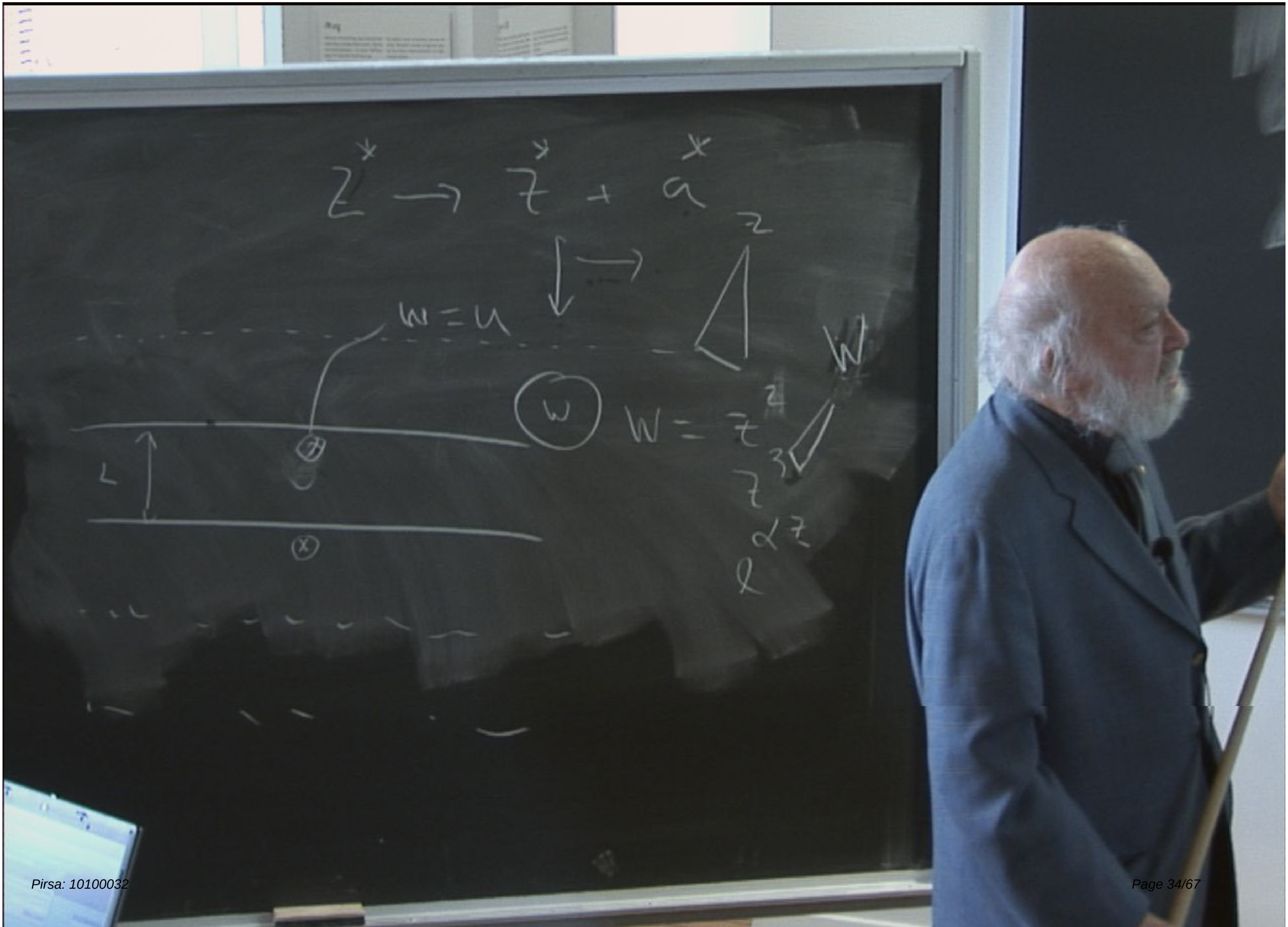
w
W = z
L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Symmetries, Continued: Inversion

$<\sigma(z,z^*)\ \sigma(w,w^*)>=1/[(z-w)^{1/8}\,(z^*-w^*)^{1/8}\,]$

For the two dimensional Ising model, scale invariance would permit

$<\varepsilon(z,z^*)\ \sigma(w,w^*)>=B/|z-w|^{17/16} = B/(z-w)^{h_\sigma+h_\varepsilon}\ (z^*-w^*)^{h^*_\sigma+h^*_\varepsilon}$

However in fact this correlation function is zero at the critical point (B=0).

Inversion invariance: Only holds at the critical point or other massless situations: This is a transformation which takes the point at infinity into the origin of the complex plane. The content of this invariance is that the point at infinity is like every other point. That is true at the critical point for most, but not all, models.

The transform is $\mathbf{r} \to \mathbf{r}/r^2$ $\quad o_\alpha(\mathbf{r}) \to r^{-2x_\alpha}\, o_\alpha(\mathbf{r}/r^2)$. In two dimensions, we have $o_\alpha(z,z^*) \to z^{-2h_\alpha}\, z^{*-2h^*_\alpha} o_\alpha(1/z,1/z^*)$. As pointed out by Polyakov, this invariance produces significant limitations upon the form of correlation functions. For example it permits two-point correlations like $<o_\alpha(z,z^*)\ o_\beta(w,w^*)>$ to be non-zero only if $h_\alpha=h_\beta$ and $h^*_\alpha=h^*_\beta$. I ask you to prove this in a homework exercise

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Correlation Functions

(from conformal symmetry in two dimensions)

Conformal symmetry gives a tremendous amount of information about the structure of critical point correlation functions. A few results are listed here.

One point function. In an infinite system $< o_\alpha(z,z^*)>=0$ (from scale invariance)

Two point functions. With only a few reservations and restrictions one can that the two-point function is only non-zero if the two operators have the same h's. Then by taking linear combinations we can ensure that
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Distortions in the Plane

We already know that distortions of our planar system will produce interesting changes in our correlation functions. The coordinate transform in question is of the form $r_j \to r_j' = r_j + \eta_j$ where η_j is infinitesimal and might depend upon position. We can build up all the finite changes we would like from infinitesimal ones except for inversions which can be generated from special transformations of the form $1/z' = 1/z + a$, with an infinitesimal a, (This is an inversion, followed by an infinitesimal translation, followed by an inversion.) Equivalently it is $z' = z - a\, z^2$
We want to include all kinds of small distortions except those which produce shears. We do not expect our correlation functions to be invariant under shearing operations. This no-shear requirement is the statement
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Students of complex analysis will recognize this no-shear requirement as the precise requirement that produce conformal transformations of the sort
$z' = z + \eta(z)$ with $\eta(z)$ being an analytic function in some region of z.

Symmetries, Summary

global conformal group: z-->z+a   z-->λ z   z--> 1/z. These transformation are global analytic transforms of the complex plane. All other transforms have singularities somewhere. This group transforms a function of the form (az+b)/(cz+d) into itself. We can think of it as performing transformations upon the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Another equally good way of thinking about the complex analytic transforms is to think about the infinitesimal version of these transforms. These are generated by the operators: $\ell_n = -z^{n+1}\partial_z$ which then have the commutator structure $[\ell_n, \ell_m] = (n-m)\,\ell_{n+m}$ The algebra of this structure is closed if we let *n* run over all integers and closed if we allow it to run over only n=0,-1,1. The former represents the entire conformal structure, the latter is the global transforms, with *n=-1* corresponding to translations, *n=0* corresponding to rotations and scale transformations, and *n=1* corresponding to the special transformations.

The extension beyond this limited range of *n (n=0,+1,-1)* to all *n* works because the critical problem is local and permits slowly varying local changes in η(z).

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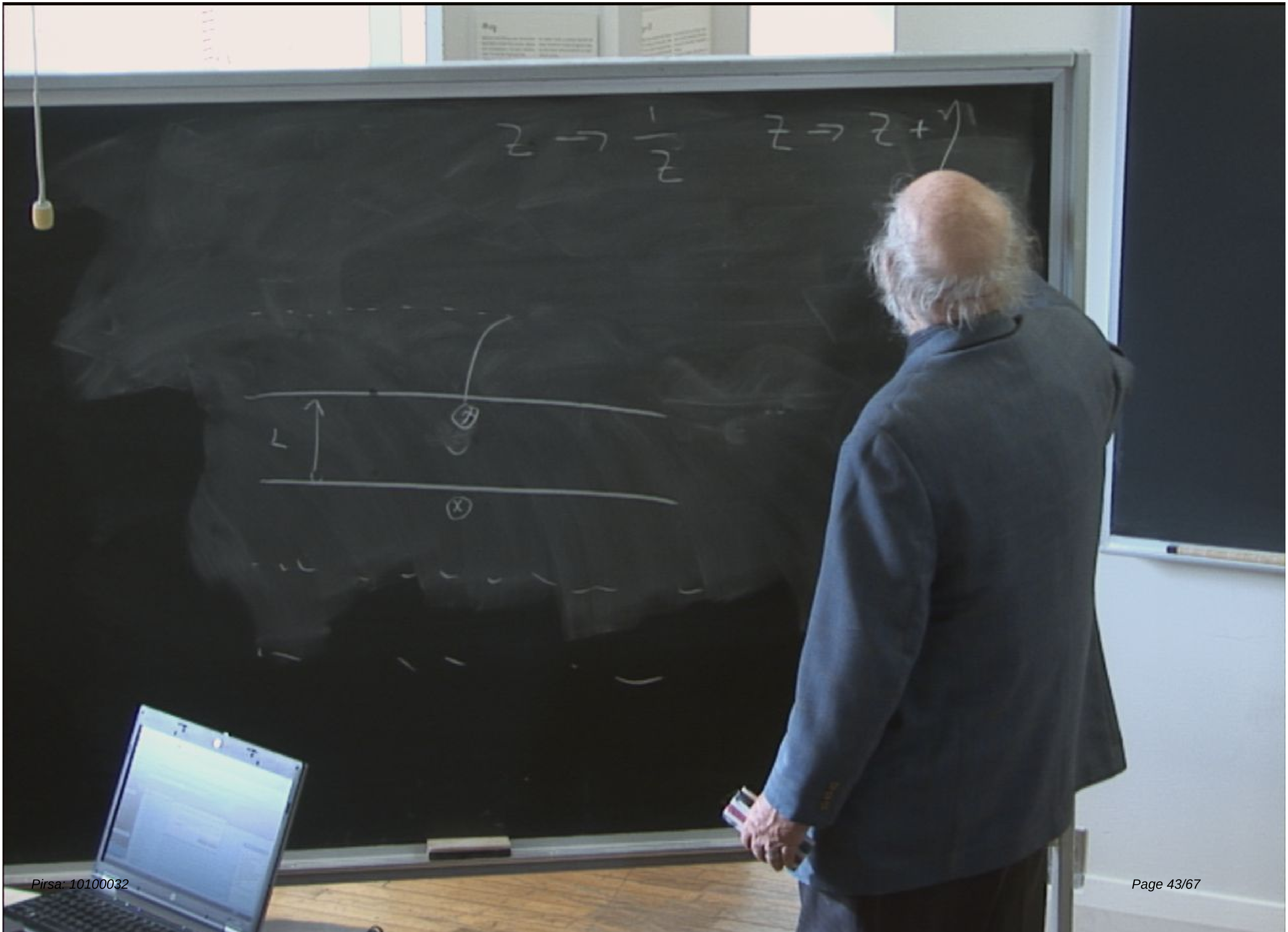
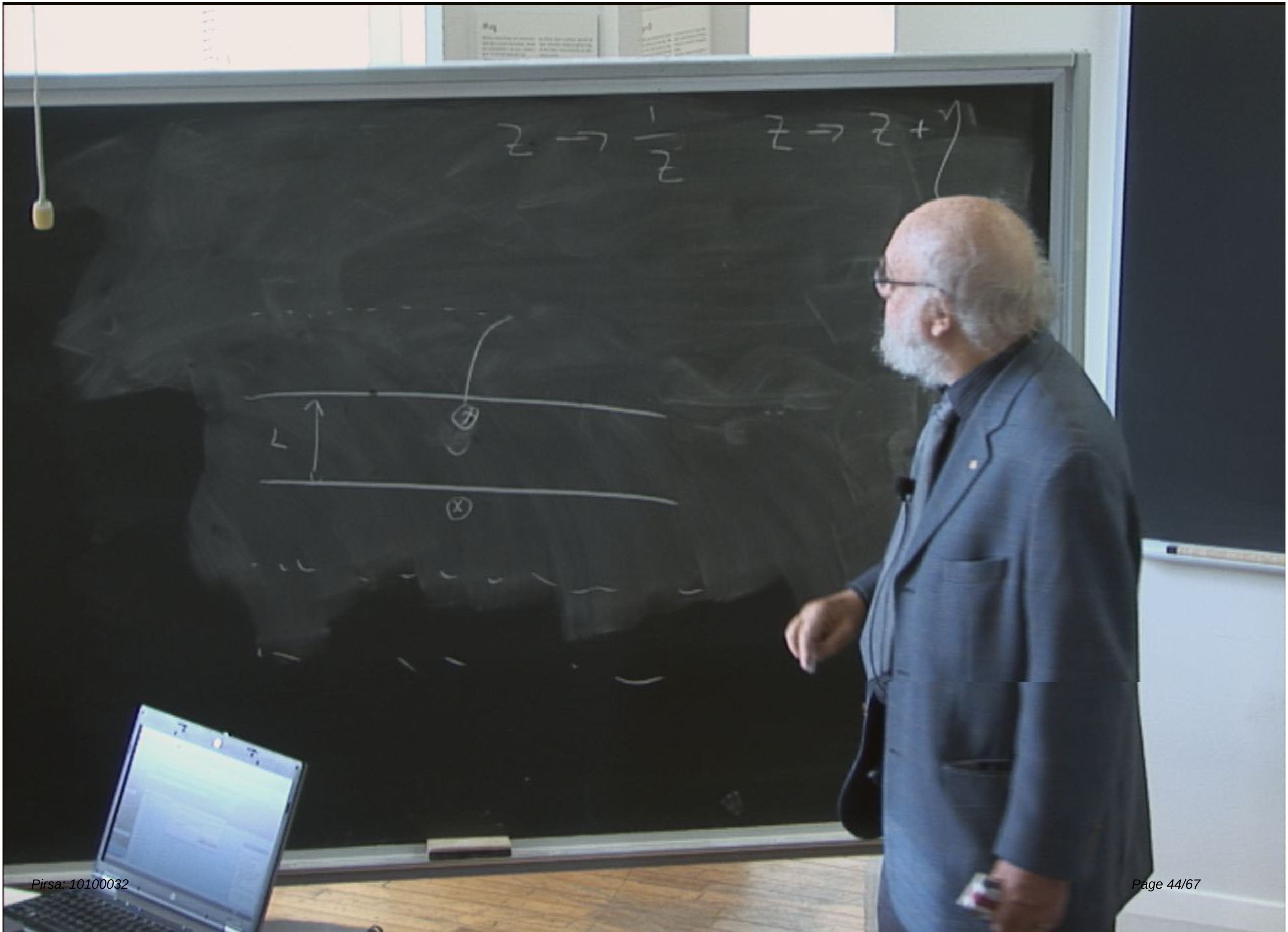
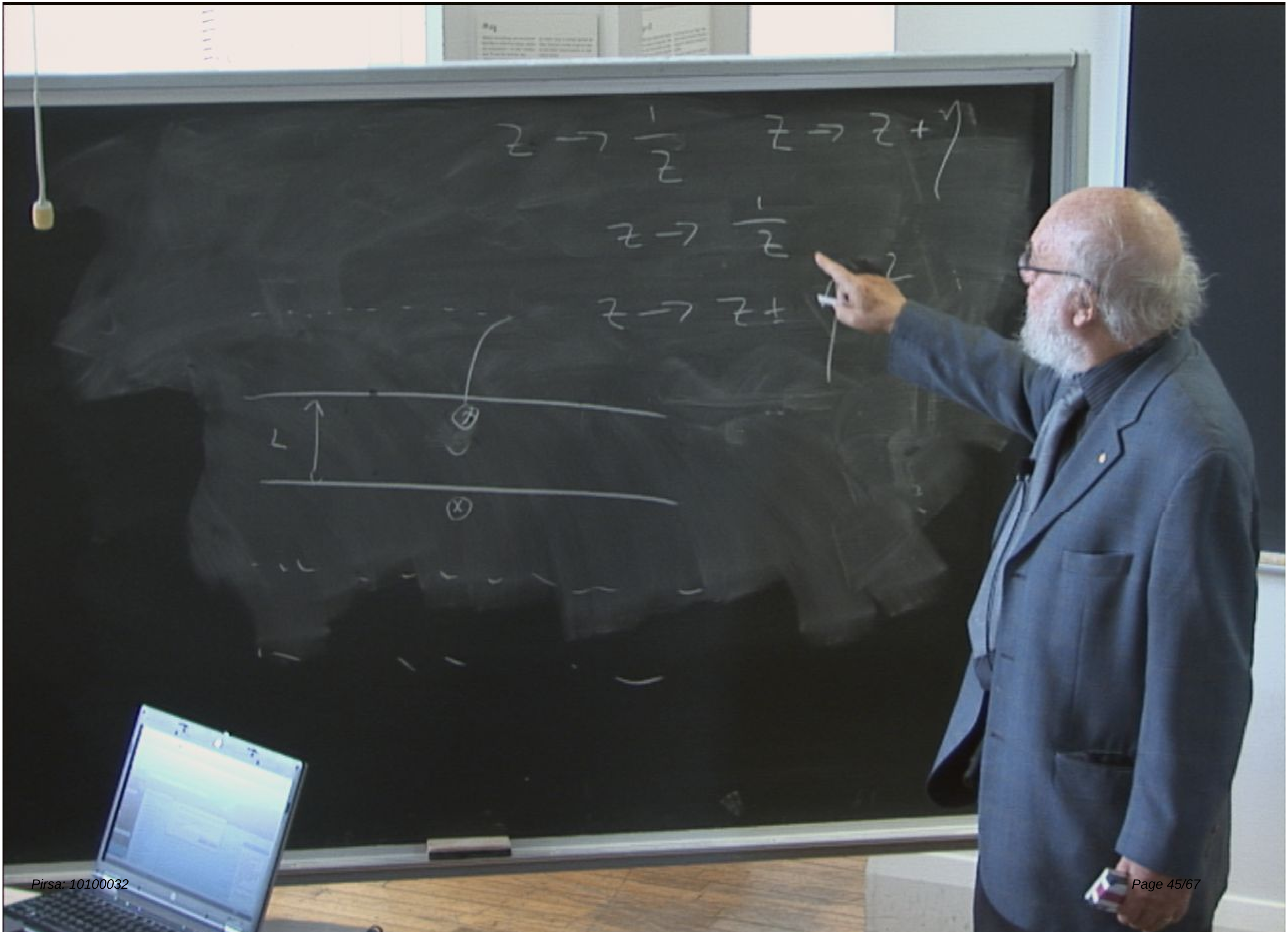

z → 1/z
z ⇒ z +
L


$z \to \frac{1}{z}$
$z \to z + \eta$
L


z → 1/z
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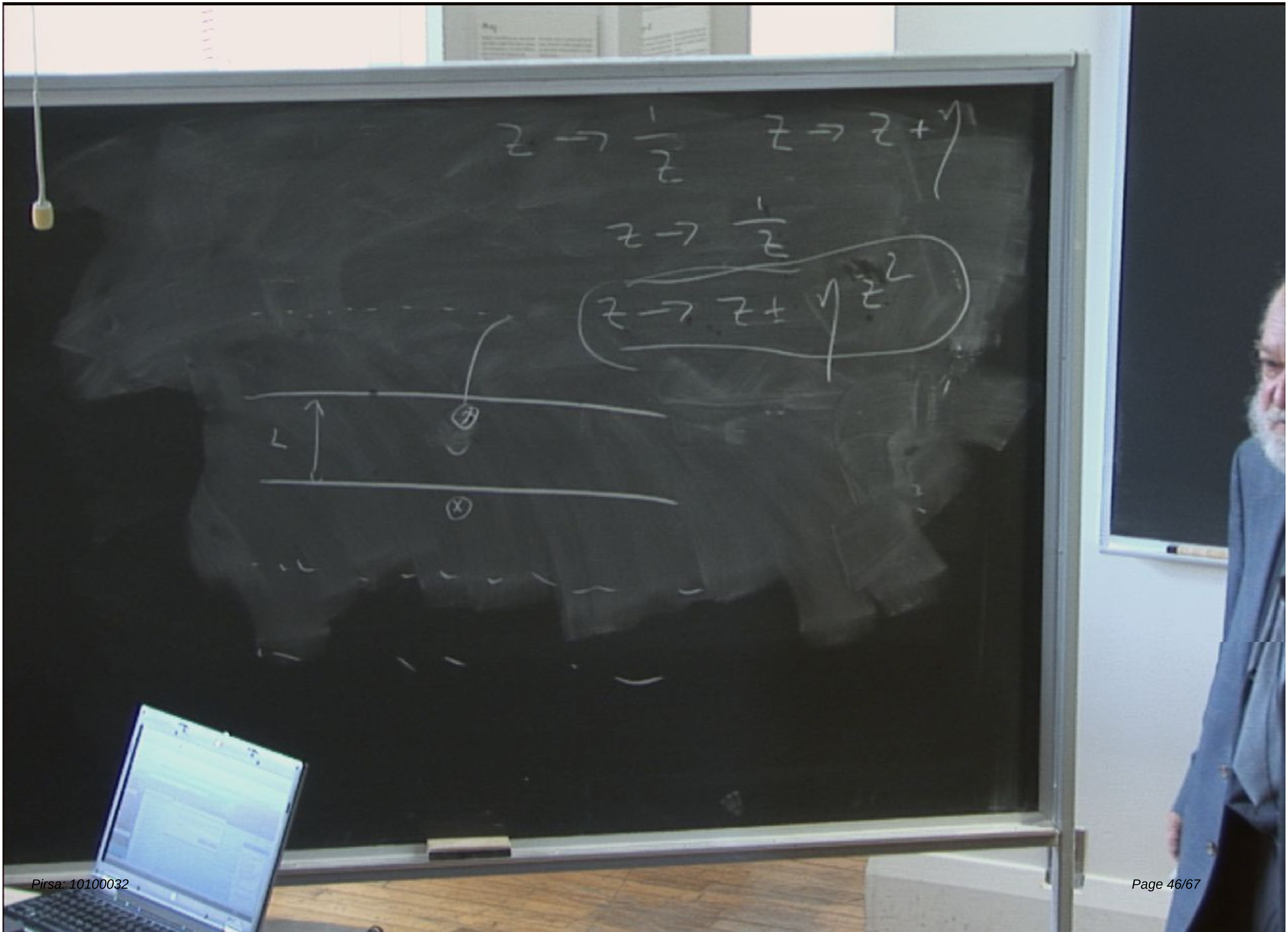
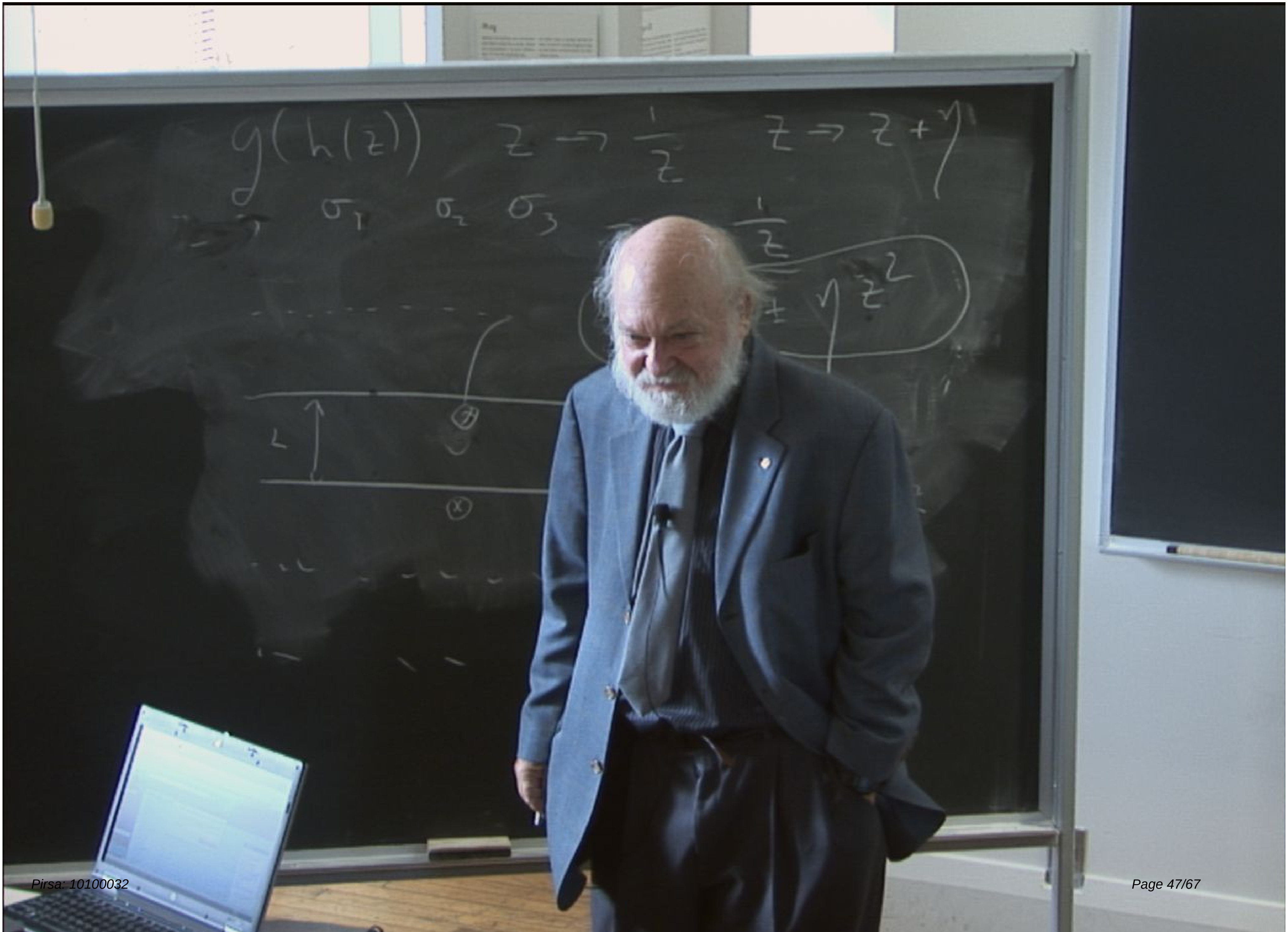
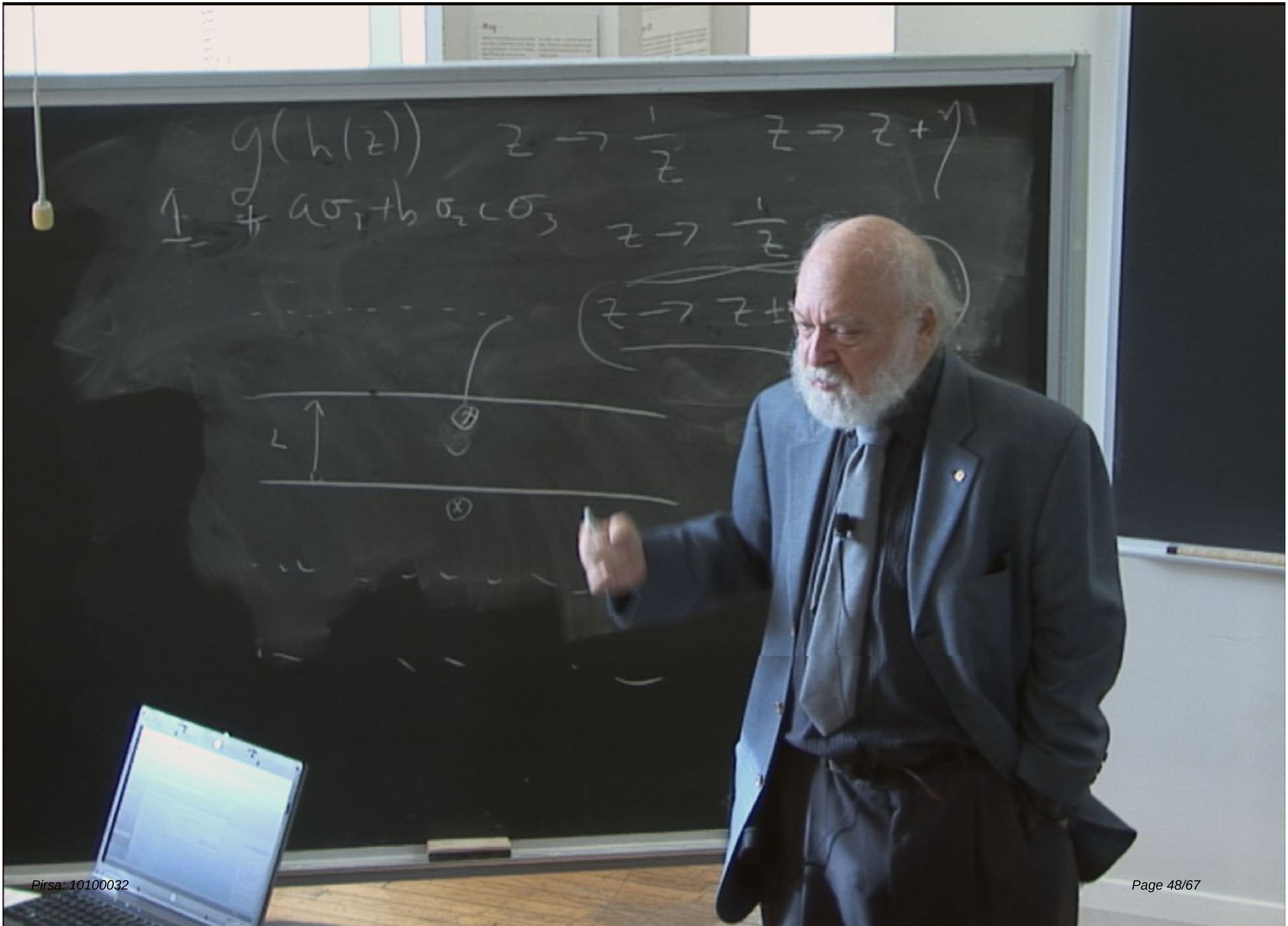
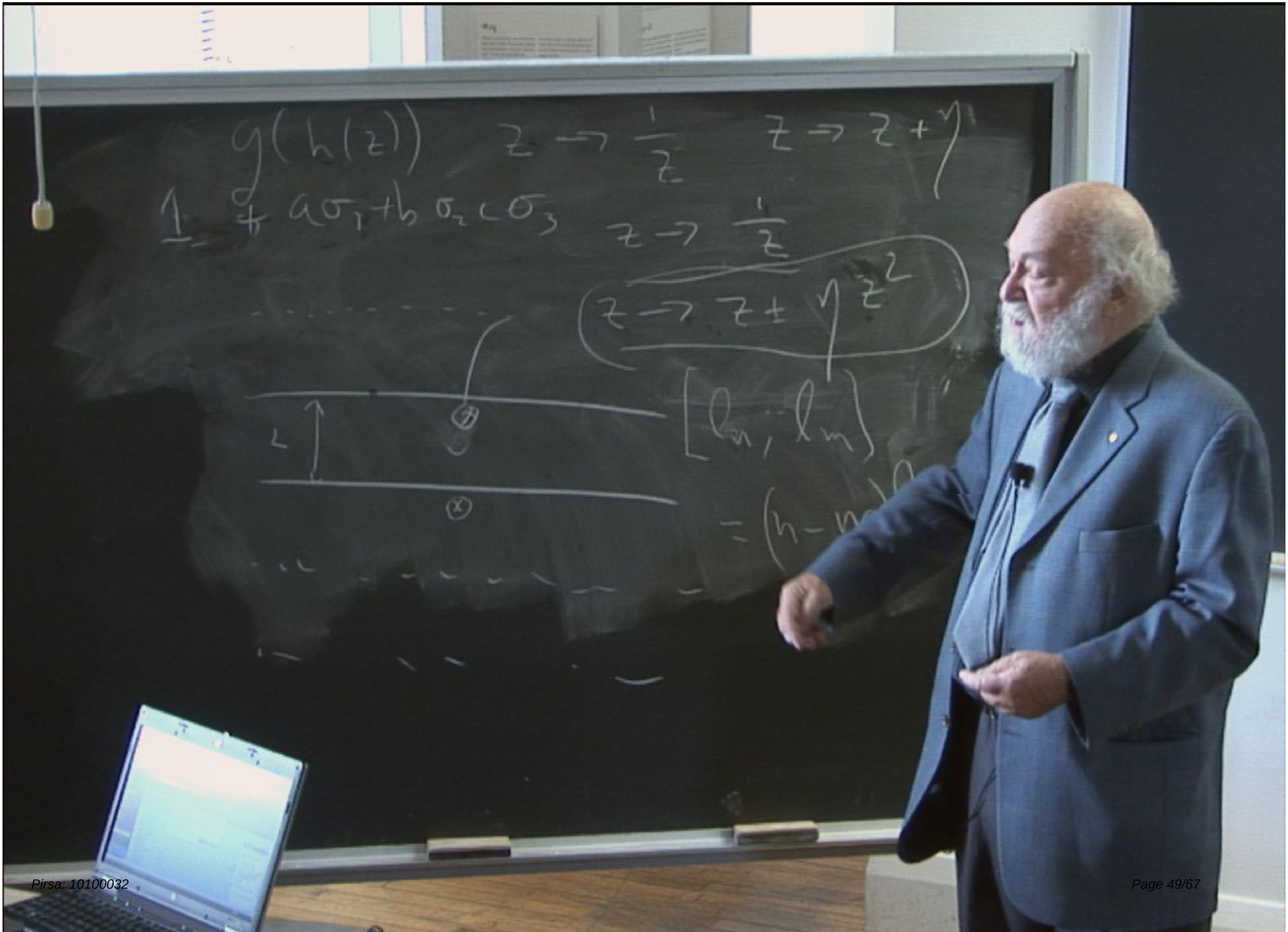
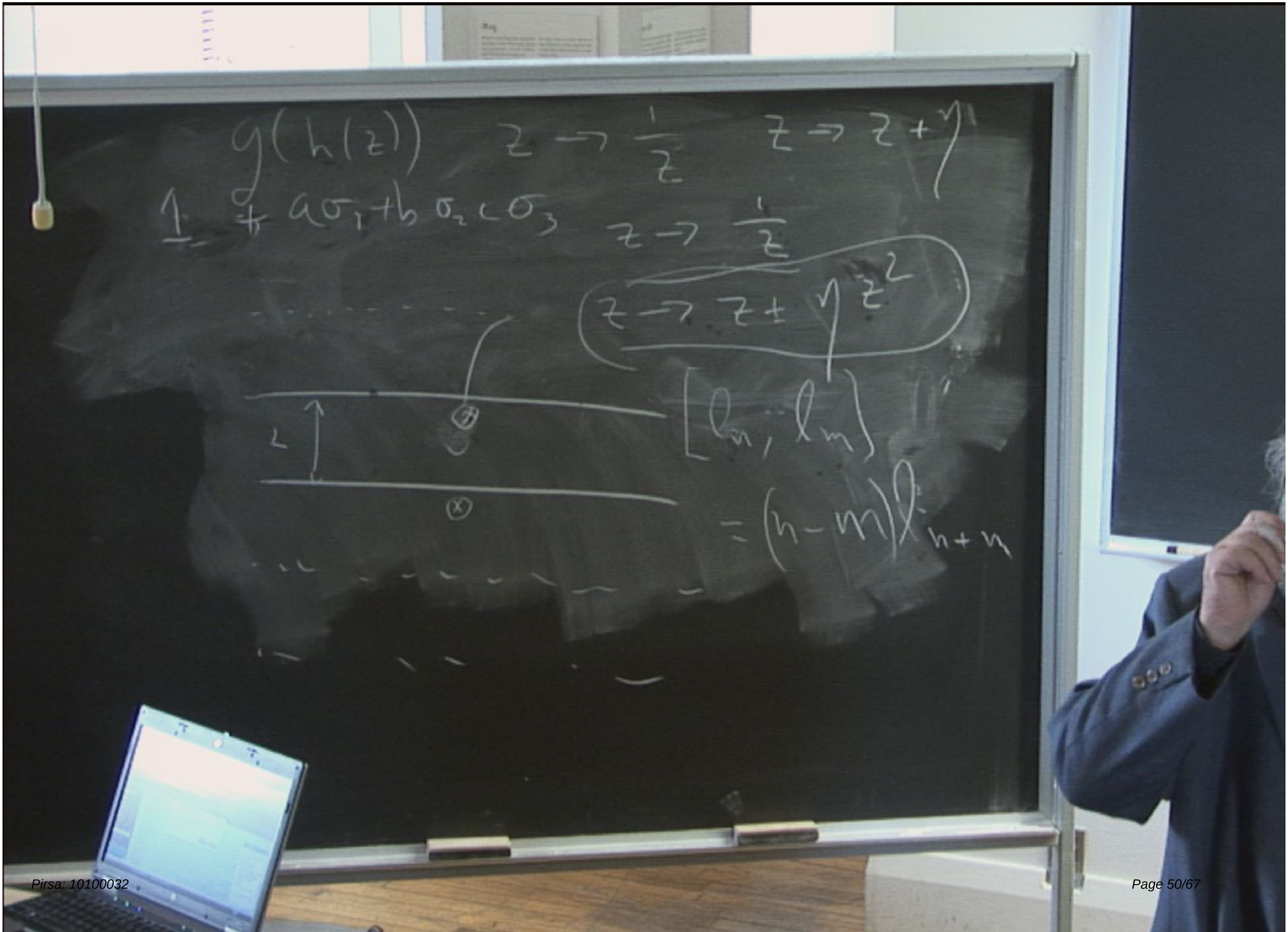
L


z → 1/z
z → z + η
z → 1/z
z → z ± η z²
L


g(h(z))
z → 1/z
z ⇒ z + η
σ1 σ2 σ3
L




g(h(z))
z → 1/z
z → z + η
1 + aσ1 + bσ2 cσ3
z → 1/z
z → z ± η z²
L
[ln, lm]
= (n − m)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z → 1/z
z → z + η
z → 1/z
z → z ± η z²
[l_n, l_m]
= (n − m) l_{n+m}

The Stress Tensor, $t_{jk}(\mathbf{r})$, is the jth component of the momentum current defined by the momentum in the direction k. The momentum is conserved so $t_{jk}(\mathbf{r})$ has divergence zero. The momentum acts like a spatial gradient. Its current density, t, is important because that induces deformations in the plane. The structure of the plane and its deformations defines and describes all the usual models of critical phenomena. In fact, it is doubly important: its correlation functions with all the basic operators in the theory describe how they fit into the space, and its correlation functions with stress tensors at other points in space describes how the space responds to things acting on it.

From a study of mechanics or field theory we learn that the effect of stressing a system is studied by adding to its Hamiltonian (or free energy functional) a term of the form

H/T ---> $H/T - (1/2\pi)\sum_{\mathbf{r}} t_{j,k}(\mathbf{r})\,[\partial_j \eta_k(\mathbf{r})]$

where $\boldsymbol{\eta}(\mathbf{r})$ is an infinitesimal displacement of the coordinate at r. This change then produces a small distortion of the system. The stress tensor itself is, at the critical point, a traceless tensor with $t_{11}=-t_{22}$ and $t_{12}=t_{21}$. It can be split into two parts via $t_{zz} = 2(t_{11} + i\, t_{12})$ $\quad t_{z^*z^*} = 2(t_{11} - i\, t_{12})$

We are going to calculate the change induced by this variation of the Hamiltonian upon a product of operators

$< o_{\alpha_1}(z_1,z_1^*)\, o_{\alpha_2}(z_2,z_2^*) \ldots\, o_{\alpha_n}(z_n,z_n^*) >$

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The Stress Tensor

One can form these tensors, plus things which are smaller in a scaling sense by using

$t_{11}(j,k) = \text{const}\ \sigma_{j,k}\,[\sigma_{j,k+1} - \sigma_{j+1,k} + \sigma_{j,k-1} - \sigma_{j-1,k}\,]$

$t_{12}(j,k) = \text{const}\ \sigma_{j,k}\,[\sigma_{j+1,k+1} + \sigma_{j-1,k-1} - \sigma_{j+1,k-1} - \sigma_{j-1,k+1}\,]$

These two tensors have the effect of distorting the system by pulling it in one direction and compressing it in another. For example

$$\Sigma_{j,k} < t_{11}(j,k)\, \sigma_{\mathbf{r}}\, \sigma_{\mathbf{s}} > = -[r_1\, \partial/\partial r_1 - r_2\, \partial/\partial r_2 + s_1\, \partial/\partial s_1 - s_2\, \partial/\partial s_2\,] < \sigma_{\mathbf{r}}\, \sigma_{\mathbf{s}} >$$

Using coordinates z and z*, T decomposes into two components T_{zz} and $T_{z^*z^*}$. It follows from the condition for momentum conservation*

$$\Sigma_j\, \partial_j\, t_{jk} = 0$$

that these are respectively holomorphic, $\partial_{z^*} t_{zz} = 0$ so that $t_{zz} = t(z)$,

and antiholomorphic, $t_{z^*z^*} = t^*(z^*)$. Summed over all space, they respectively have the effect

$$\Sigma_{j,k} < t(j+ik)\, \sigma(z,z^*)\, \varepsilon(y,y^*) \ldots > = -(z\, \partial/\partial z + y\, \partial/\partial y) < \sigma(z,z^*)\, \varepsilon(y,y^*) \ldots >$$

$$\Sigma_{j,k} < t^*(j-ik)\, \sigma(z,z^*)\, \varepsilon(y,y^*) \ldots > = -(z^*\, \partial/\partial z^* + y^*\, \partial/\partial y^*) < \sigma(z,z^*)\, \varepsilon(y,y^*) \ldots >$$

*Remember that T is the current for momentum, and in statistical equilibrium the momentum is not changing in time.

The stress tensor as an eigenvalue operator

$\Sigma_z < t(z)\, o_{\alpha 1}(z_1,z_1^*)\, o_{\alpha 1}(z_2,z_2^*) \ldots > = -(z_1\, \partial/\partial z_1 + z_2\, \partial/\partial z_2) < \Pi_{j=1}^{n}\ o_{\alpha j}(z_j,z_j^*) >$

$= -(z_1\, \partial/\partial z_1 + z_2\, \partial/\partial z_{2+})\, [\Pi_{j=1}^{n} (z_j - z_n)^{hj}]\, f[(z_1-z_2)(z_3-z_4)/(z_1-z_3)(z_3-z_2), \ldots >$

$= (h_1+h_2+ \ldots h_n)\, [\Pi_{j=1}^{n} (z_j - z_n)^{hj}]\, f[(z_1-z_2)(z_3-z_4)/(z_1-z_3)(z_3-z_2), \ldots >$

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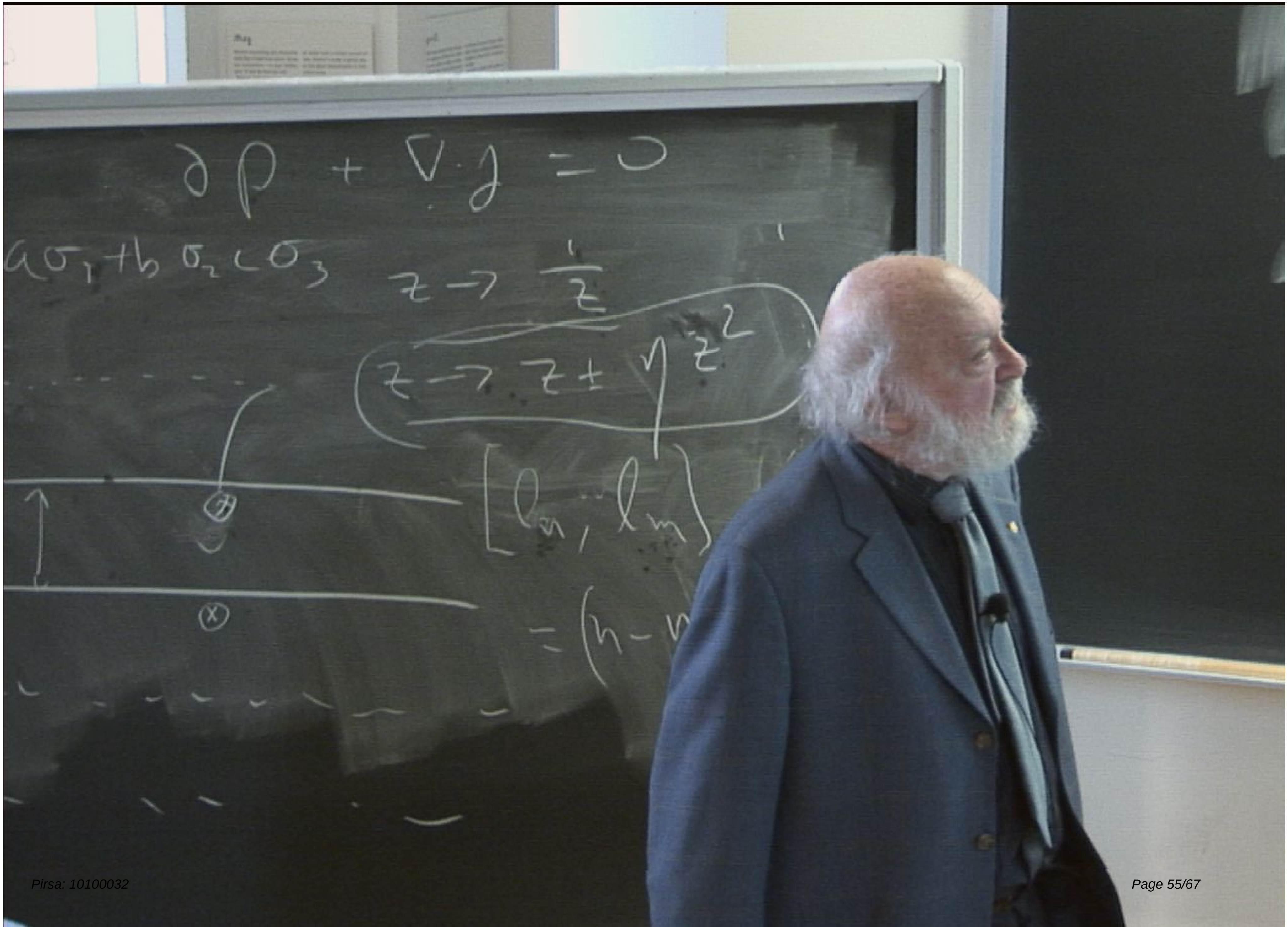
$t(z)$ when multiplying $o_{\alpha j}(z_j,z_j^*)$ and summed over z has eigenvalue $h_j + h^*_j$

This result provides a solid starting point that serves as a base for an algebric approach to two dimensional critical phenomena. The result is of course based upon the scale invariance of the theory, and the eigenvalue is precisely the scaling index.

We expect $t(z)$ to itself have simple scaling properties. Since it is holomorphic it should have $h^* = 0$. On dimensional grounds, we expect operators which change the scaling behavior of other operators to be marginal, i.e. to have $x=2$. Therefore for $t(z)$, we expect to find the eigenvalue $h=2$.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$t(z)$ when multiplying $o_{\alpha j}(z_j,z_j^*)$ and summed over z has eigenvalue $h_j + h^*_j$

This result provides a solid starting point that serves as a base for an algebric approach to two dimensional critical phenomena. The result is of course based upon the scale invariance of the theory, and the eigenvalue is precisely the scaling index.

We expect $t(z)$ to itself have simple scaling properties. Since it is holomorphic it should have $h^* = 0$. On dimensional grounds, we expect operators which change the scaling behavior of other operators to be marginal, i.e. to have $x=2$. Therefore for $t(z)$, we expect to find the eigenvalue $h=2$.

Put it together

One can put the change in the free energy functional induced by the strain η in better form by starting with

$H/T \dashrightarrow H/T - (1/2\pi)\sum_{\mathbf{r}} t_{j,k}(\mathbf{r})\, [\partial_j \eta_k(\mathbf{r})]$

Now, imagine that we wished to calculate the effect of this kind of distortion upon a correlation function of operators at points, z_k, in a certain compact bounded region, R, of the complex plane, with $\eta(z)$ being analytic in that region. If we are in the continuum limit, so that sums over lattice points can be replaced by integrals, Stokes law gives us a result

$H/T \dashrightarrow H/T + (1/2\pi i)\int dz\, t(z)\, \eta(z) - (1/2\pi i)\int dz^*\, t^*(z^*)\eta^*(z^*)$

after we split the stress tensor into its component parts and make use of its divergence-free character. The integrals go around the boundary of the region R in a positive sense. We now have a change in the correlation function which is constructed in the form

$\delta < \Pi_{j=1}^{n}\, o_{\alpha_j}(z_j,z_j^*) > = (1/2\pi i)\int dz\;\; \eta(z)\;\; < t(z)\, \Pi_{j=1}^{n}\, o_{\alpha_j}(z_j,z_j^*) > +$ complex conjugate

Now we shall evaluate this integral, using the fact that t(z) is holomorphic and our expectation that it will have a simple short distance expansion with the $o_{\alpha j}$

Short distance expansion for basic operators o_{α_j} (called primary operators)

$t(z)\, o_{\alpha_j}(w,w^*) \;\; = h\,(z-w)^{-2}\, o_{\alpha_j}(w,w^*) + \;(z-w)^{-1}\;\; \partial_w\, o_{\alpha_j}(w,w^*)$ + higher powers of (z-w)

for z in the neighborhood of w. Shrink the contour to surround the singularities at z_j

change in correlator of primary operators

$\delta <\Pi_{j=1}^{n} o_{\alpha_j}(z_j,z_j^*)> = \Sigma_{k=1}^{n} [h_k (\partial_k \eta(z_k)) + (\eta(z_k) \partial_k)] < \Pi_{j=1}^{n} o_{\alpha_j}(z_j,z_j^*)>$ + complex conjugate

This is exactly the result we expect from our discussion of scaling properties of primary operators. What we previously called b is, for infinitesimal η, $1+ h_k (\partial_k \eta(z_k))$ while the term $(\eta(z_k) \partial_k$ describes a shift in coordinates caused by the deformation. This formula does not exhibit the change in the shape of the boundaries produced by the coordinate change.

One very important case not encompassed among the primary operators is the change in

t(z) itself. One kind of information about this can be obtained by studying the correlation functions in the solvable models that have been studied. These studies show that the correlations of t(z) contain an extra term not encompassed in the primary operators, specifically $<t(z)\, t(w)> = \text{const}/ (w-z)^4$. The fourth power is expected from scaling the "const" differs from model to model. In addition, one can find that the short distance expansion for t(z) contains terms just like the ones with the primary operators so that

$t(z)\, t(w) = c/2 \;(w-z)^4 + 2\, (z-w)^{-2}\, t(w) + (z-w)^{-1}\; \partial_w\, t(w)$ + higher powers of z-w

Here, c is a constant which varies from model to model. It is called the conformal charge. It measures the non-linearity of the distortions of the space caused b y the stress tensor. It is for example, ½ in the two-dimensional Ising model.

Short distance expansions provide us with what we need to know about the algebraic structure of the primary operators and of t(z). In fact they will provide us with an algebra which will tell us

Algebraic Structure

One can analyze the structure of the stress tensor by using an analog of Fourier Transformation. One writes the tensor as a sum of terms

$t(z)=\sum_n L_n(w)\,(z-w)^{-(n+2)}$

This summation structure is a reflection of the holomorphic structure if t(z) which does not allow it to have singularities except is very special places and keeps the rest of the z-dependence to be akin to a power law. The series expansion enables one to analyze t(z) by looking in the Heisenberg (quantum) representation a a time equal to the imaginary part of w. This point of view enables us to look at the short-distance expansion

$t(z)\, t(w) = c/(z-w)^4 + (2/(z-w)^2)\, t(w) + 1/(z-w)\, \partial_w t(w) +$

and think about the relatively simple algebra of commutators, as produced by the equal time expression

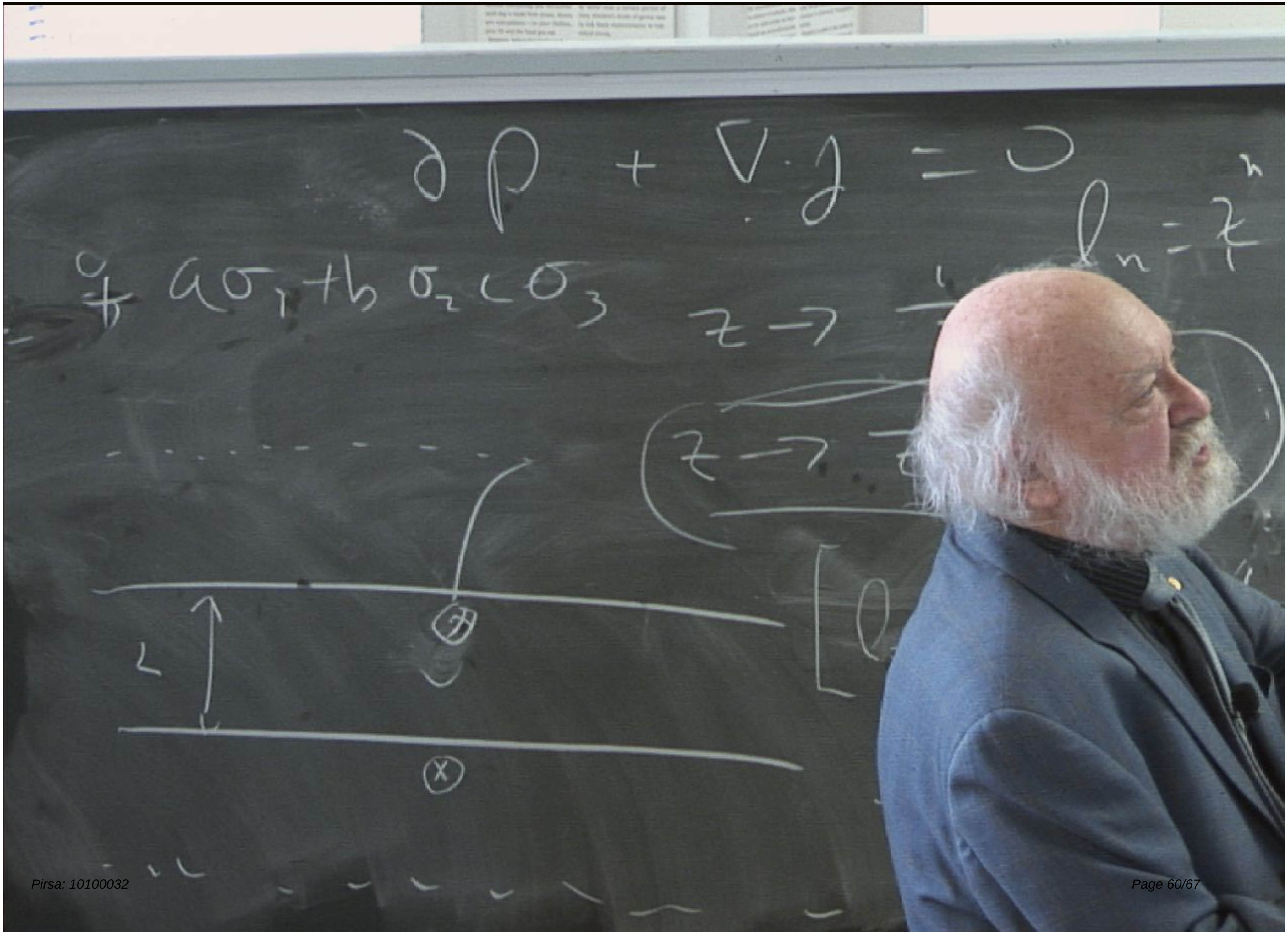
$1/(x-u \pm i\varepsilon)$ with Re $z=x$, Re $w=u$ and ε being infinitesimal. Here

$1/(x-u \pm i\varepsilon) = \text{Pr}\ 1/(x-u) \mp \pi i\, \delta(x-u)$

After a little fancy footwork, which I am talking about but not doing, this analysis gives equal-time commutation relations for the basic operators

$[L_n, L_m] = c\delta_{n,-m}\,(n^2-1)n/12 + (n-m)L_{n+m}$

This statement defines the Virasoro algebra. Taken together with algebraic relations between o_α and the L_n we get enough information to find allowed values of c and h_α.


∂ρ + ∇·j = 0

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Application of Complex Analysis

in two dimensions

In two dimensions we can study the effect of analytic function maps upon correlation functions. In general a transformation z-->w=w(z) maps a portion of the space defined by z plane into some portion of the space defined by w. This transformation provides no local shears except at points of non-analyticity. Local angles are preserved at all points of analyticity. The general rule is that this change transforms o_α according to

$o_\alpha(z,z^*) \to b(w)b(w^*)o_\alpha(w(z),w^*(z^*))$

with $b(w) = [dw/dz]^{h_\alpha}$ and $b(w^*) = [dw^*/dz^*]^{h^*_\alpha}$

This is particularly simple for the global transformations described so far. It is trivial for the translation w=z+a, easy for $w=\lambda z$, which is a pure dilation for λ real, together with a rotation through the phase of λ for complex λ.

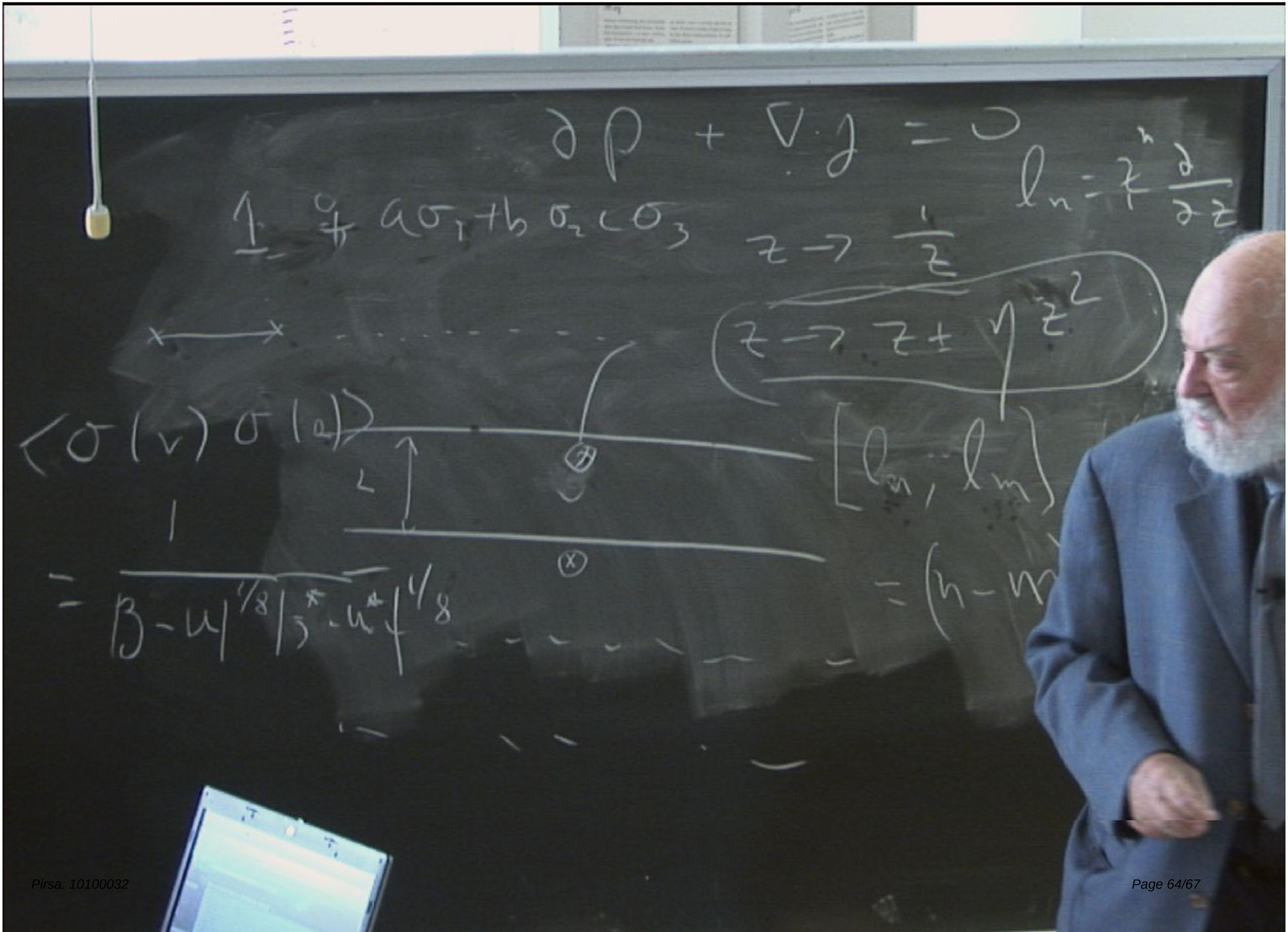
The analogous calculation for any other analytic function requires additional thought. No other function can smoothly (analytically) map the plane into itself. So any other function will change the region under consideration. (Special attention will have to be given to the "point" at infinity.)

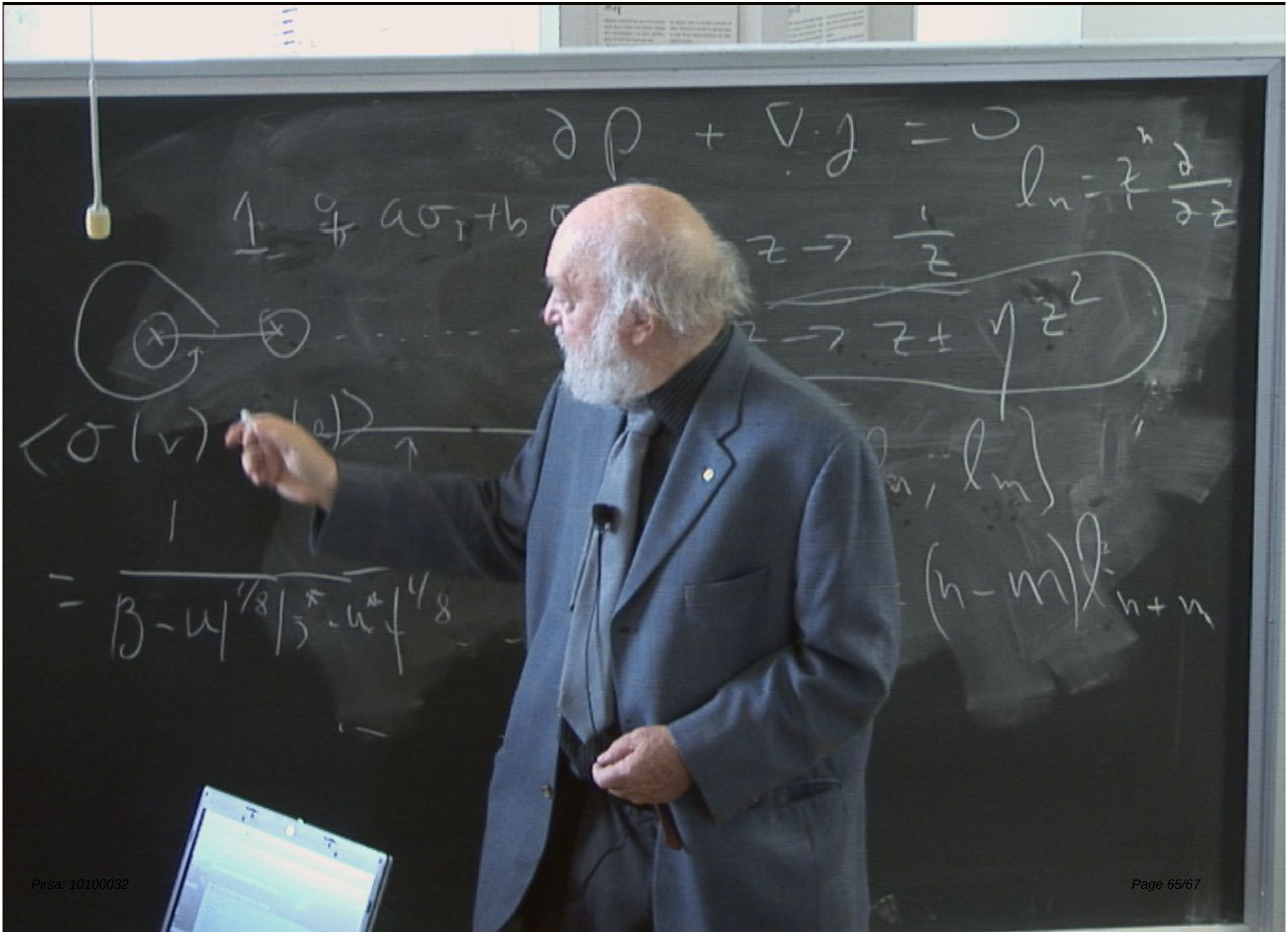
Transformations like this are called conformal transformations. They were introduced into critical phenomena work in 1970 by A. A. Polyakov. John Cardy showed us how to make use of specific transformations, like the one in the next slide.

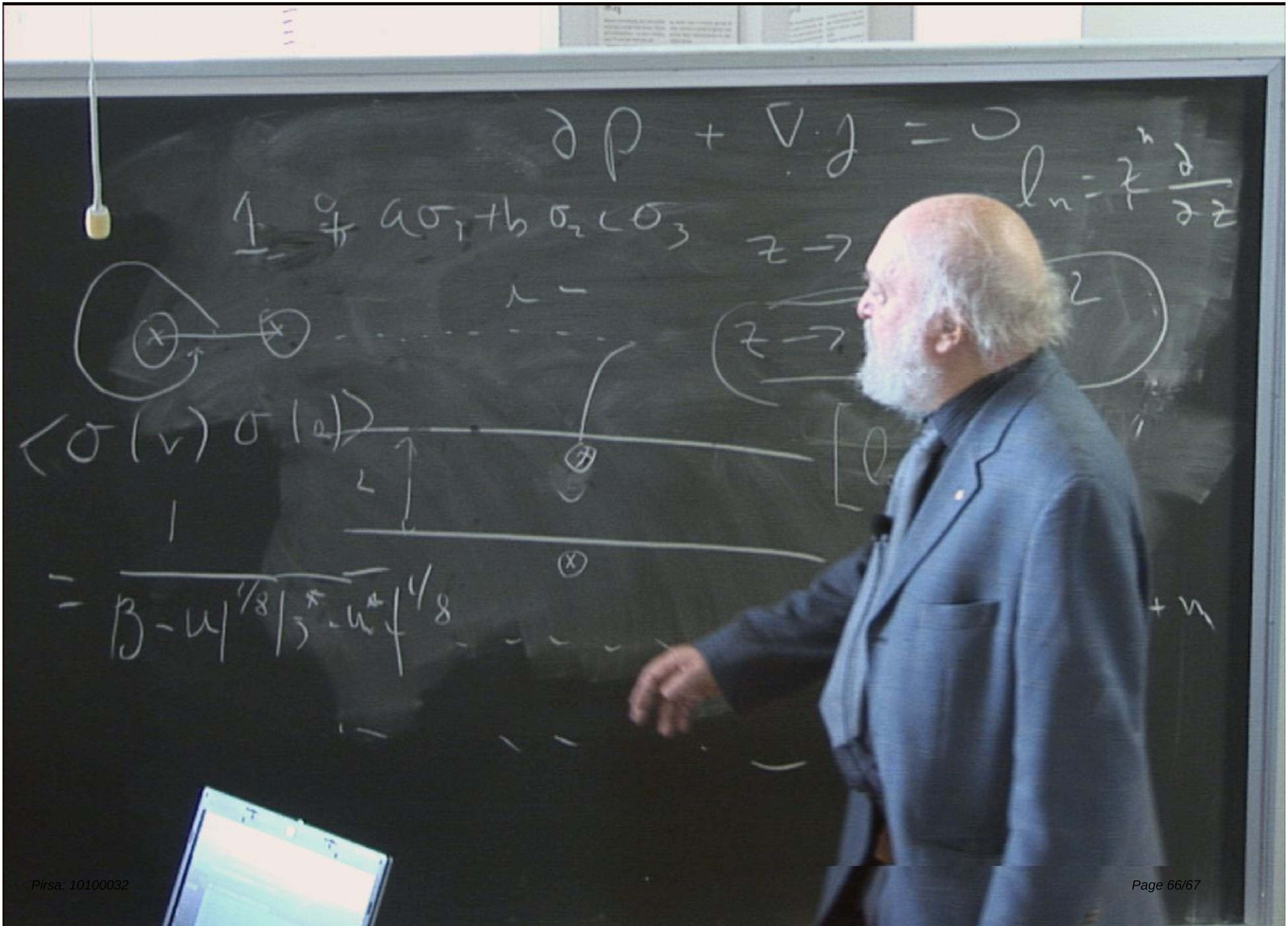
Part 9: Critical Phenomena and 2d Space

I follow Cardy chapter 11

see also Di Francesco, Mathieu, Sénéchal


z → 1/z
[l_n, l_m]





Operator Product Expansion

short distance expansion

f $\mathbf{r}$ is close to $\mathbf{s}$, one can replace the product $o_\alpha(\mathbf{r})\, o_\beta(\mathbf{s})$ according to

$$o_\alpha(\mathbf{r})\, o_\beta(\mathbf{s}) = \Sigma_\gamma\ C_{\alpha\,\beta\,\gamma}\ |\mathbf{r}-\mathbf{s}|^{x_\gamma - x_\alpha - x_\beta}\ o_\gamma(\mathbf{s})$$

he idea is that when $\mathbf{r}$ and $\mathbf{s}$ approach one another, the product looks like an operator at **s.** ince the number of different operators is quite limited, one must get a sum of the operators n the theory.

xpressions like the one's in operator product expansions provide a sort of algebra for the luctuating operators in the theory. Even before Wilson's work on the renormalization group, t was hoped that algebraic methods would enable a classification of, and perhaps an analytic olution for, d=2 critical phenomena. That was roughly in 1970. Somewhat later, in 1984 aniel Friedan, Zongan Qiu and Stephen Shenker used algebraic methods related to short istance expansion to find the behavior of all the most familiar problems in two dimensional ritical phenomena. They worked with an extra ingredient which was just starting to be vailable in 1970, the deep understanding of symmetries provided by conformal invariance. I am going to fill in some pieces of the theory of these symmetries.