

Title: Statistical Mechanics (PHYS 602) - Lecture 7

Date: Oct 13, 2010 10:30 AM

URL: <http://pirsa.org/10100026>

Abstract:

put it all together to find

$$\begin{aligned}
 [\partial_t + (\mathbf{p}/m) \cdot \nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}} U(\mathbf{r},t)) \cdot \nabla_{\mathbf{p}}] f(\mathbf{p}, \mathbf{r}, t) = \\
 - f(\mathbf{p}, \mathbf{r}, t) \int d\mathbf{q} f(\mathbf{q}, \mathbf{r}, t) d\mathbf{p}' d\mathbf{q}' Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') \\
 + \int d\mathbf{q} d\mathbf{p}' f(\mathbf{p}', \mathbf{r}, t) d\mathbf{q}' f(\mathbf{q}', \mathbf{r}, t) Q(\mathbf{p}', \mathbf{q}' \rightarrow \mathbf{p}, \mathbf{q})
 \end{aligned}$$

Since we have far too many symbols here, in the next steps we shall drop the \mathbf{r}, t everywhere. The next step notices that each collision must include conservation of energy and of momentum. That means the Q 's must be proportional to delta functions which enforce conservation of energy and momentum. Specifically,

$$Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') = R(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}'))$$

We then find the equation

$$\begin{aligned}
 [\partial_t + (\mathbf{p}/m) \cdot \nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}} U) \cdot \nabla_{\mathbf{p}}] f(\mathbf{p}) = \\
 - \iiint d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}')) \\
 [R(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') f(\mathbf{p}) f(\mathbf{q}) - R(\mathbf{p}', \mathbf{q}' \rightarrow \mathbf{p}, \mathbf{q}) f(\mathbf{p}') f(\mathbf{q}')]
 \end{aligned}$$

Symmetries of Boltzmann equation

One more statement is needed: This is a statement of time-reversal invariance in which we demand that the inverse process have the same probability. Specifically, the statement, called **detailed balance**, is

$$R(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') = R(\mathbf{p}', \mathbf{q}' \rightarrow \mathbf{p}, \mathbf{q}) \quad \text{detailed balance}$$

This means that if we care to, we can write the Boltzmann equation as

$$\begin{aligned} & [\partial_t + (\mathbf{p}/m) \cdot \nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}} U) \cdot \nabla_{\mathbf{p}}] f(\mathbf{p}) = \\ & - \iiint d\mathbf{q} \, d\mathbf{p}' \, d\mathbf{q}' \, \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \, \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}')) \\ & R(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] \end{aligned}$$

We shall need one more symmetry statement to obtain our last conservation law, namely the statement that \mathbf{p} and \mathbf{q} play symmetrical roles in the scattering event.

$$R(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') = R(\mathbf{q}, \mathbf{p} \rightarrow \mathbf{q}', \mathbf{p}') \quad \text{symmetrical scattering}$$

but we do not need that now.

This symmetry is appropriate because we are thinking that particles of the same kind are involved in the scattering event. If the particles were identical in the quantum sense, we would also have

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$$f(p)f(q) = \exp[\vec{v} \cdot (p + q)]$$

$$= \exp[\vec{v} \cdot (p' + q')]$$

Detailed Balance and Equilibrium

$$[\partial_t + (\mathbf{p}/m) \cdot \nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}} U) \cdot \nabla_{\mathbf{p}}] f(\mathbf{p}) =$$

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$$R(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')]]$$

Thermal equilibrium is the statement that the right hand side of the Boltzmann equation vanishes, specifically that

$$f(\mathbf{p}) f(\mathbf{q}) = f(\mathbf{p}') f(\mathbf{q}')$$

so that relaxation to equilibrium is driven by the relatively slow processes controlled by the gradients on the left hand side of the equation. This provides a mechanism for the system to provide relaxation times much slower than the collision rate of a typical particle. In thermal equilibrium

$$f(\mathbf{p}) = \exp\{-\beta[\epsilon - \mu - \mathbf{p} \cdot \mathbf{v} - v^2/(2m)]\}$$

Notice that the conservation laws for probability, momentum, and energy ensure that this equation is satisfied. Except for the terms in \mathbf{v} , this is just the Maxwell Boltzmann distribution we might have expected to obtain. The chemical potential is just a different way of expressing the fact that we can increase or decrease f to produce a larger or smaller number of particles. The \mathbf{v} -terms come from momentum conservation. They express the fact that the system may be moving past us with speed \mathbf{v} .

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\vec{v} is the velocity
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From local equilibrium to hydrodynamics

What I wrote down is the right solution of the Boltzmann equation for seeing the equilibrium behavior. However, the demand that the right hand side of the equation vanish gives a more general result, names one in which the parameters in the Maxwell-Boltzmann distribution vary in space and time.

$$f(\mathbf{p}, \mathbf{r}, t) = \exp\{-\beta(\mathbf{r}, t)[\varepsilon(\mathbf{p}, \mathbf{r}, t) - \mu(\mathbf{r}, t) - \mathbf{p} \cdot \mathbf{v}(\mathbf{r}, t) - v(\mathbf{r}, t)^2/(2m)]\}$$

This “solution” only makes the right hand side zero. The left hand side remains non-zero. It is described as a local equilibrium solution. The physics makes every portion of the system almost in equilibrium, but has the temperature, local density, and wind velocity each vary in space and time. In fact that is precisely what happens to a gas like the air in this room of the earth’s atmosphere. Collisions among the molecules happen very rapidly. As I recall every molecule collides with another with an interval between collisions of roughly 10^{-12} seconds. In a time of this order, each region of the gas attains local thermodynamic equilibrium. Then slowly, much more slowly, energy diffuses through the gas equalizing the temperature in the different regions. Momentum diffuses and travels in sound waves, equalizing the local density and the local velocities. These slow transports of conserved quantities are called transport process and described by hydrodynamic equations. The latter equations are simply the results of the local forms of conservations laws. As I said earlier, the conservation laws must be supplemented by constitutive equations to get partial differential equations to describe these diffusions and motions. These equations are called hydrodynamic equations.

So, the next thing we are going to do is to derive the conservations laws from the

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Conservation of Particle Number

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$$\partial_t n(\mathbf{r}, t) \quad \text{with } n(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t)$$

The second term on the left is the divergence of the particle current:

$$\nabla_{\mathbf{r}} \cdot \mathbf{j}(\mathbf{r}, t) \quad \text{with } \mathbf{j}(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) \mathbf{p}/m$$

The third term on the left vanishes because it contains a total derivative with respect to momentum and we assume that the momentum integrands drop off fast enough at infinity so that the integral of the total derivative is zero.

When integrated over momentum, the two collision terms are exactly the same except for a sign and thus cancel with one another.

We are left with the differential form of the number conservation law

$$\partial_t n(\mathbf{r}, t) + \nabla_{\mathbf{r}} \cdot \mathbf{j}(\mathbf{r}, t) = 0.$$

$$\left[\frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_p \right] \rho$$

$$= - \int dr' dq' dq R(r, q \rightarrow r', q')$$

$$[\rho(r) f(q) =$$

for Atoms

MOLECULES

$$\left[\frac{1}{2} \nabla_r^2 + \frac{1}{2} \nabla_r^2 - \nabla U \cdot \nabla_r \right] \psi = (E + q) \psi$$

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$$\left[\psi(r) \psi(q) - \psi(r') \psi(q') \right]$$

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$$\left[\frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] f = - \int dr' dq' dq R(r, q \rightarrow r', q') [f(r, q) f(r', q') - f(r', q) f(r, q')]$$

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Evidence
for Atoms

Is A
Molecule?

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Conservation of Particle momentum

To obtain the local law for the conservation of particles, multiply the Boltzmann equation by \mathbf{p} integrate over all momentum, and look at the result term by term, starting from the left. The first term is the time derivative of the j th component of momentum density:

$$\partial_t g_j(\mathbf{r}, t) \quad \text{with } g_j(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) p_j$$

The second term is on the left is the divergence of the momentum current, a tensor T_{jk} :

$$\sum_k \partial_k T_{jk}(\mathbf{r}, t) \quad \text{with } T_{jk}(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) p_j p_k / m$$

T_{jk} describes the current of the j th component of momentum in the direction k .

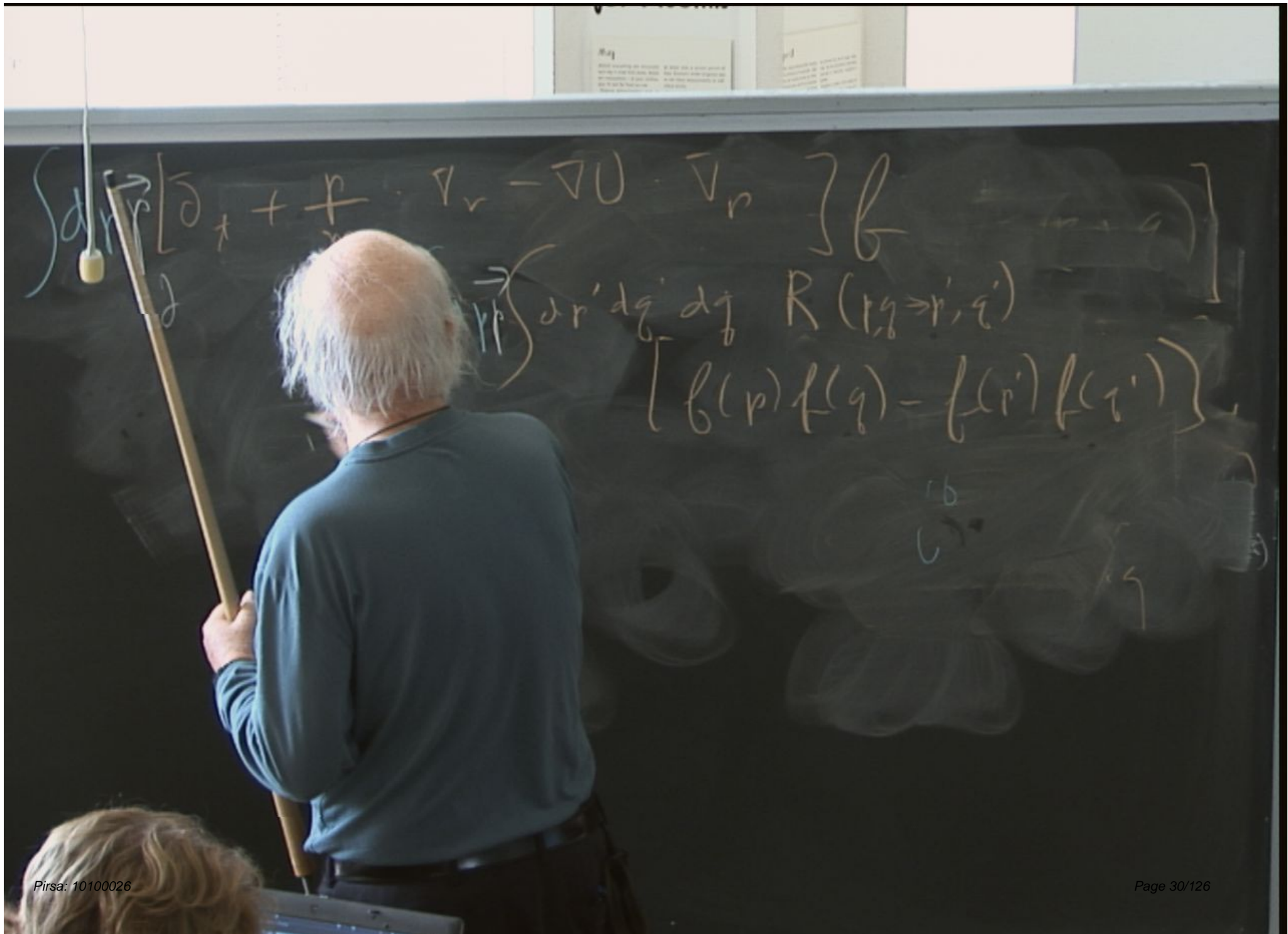
The third term on the left vanishes is a force density $n(\mathbf{r}, t) \partial_j U(\mathbf{r}, t)$.

When integrated over momentum, the two collision terms are exactly the same except for sign and thus cancel with one another.

We are left with the differential form of the momentum conservation law

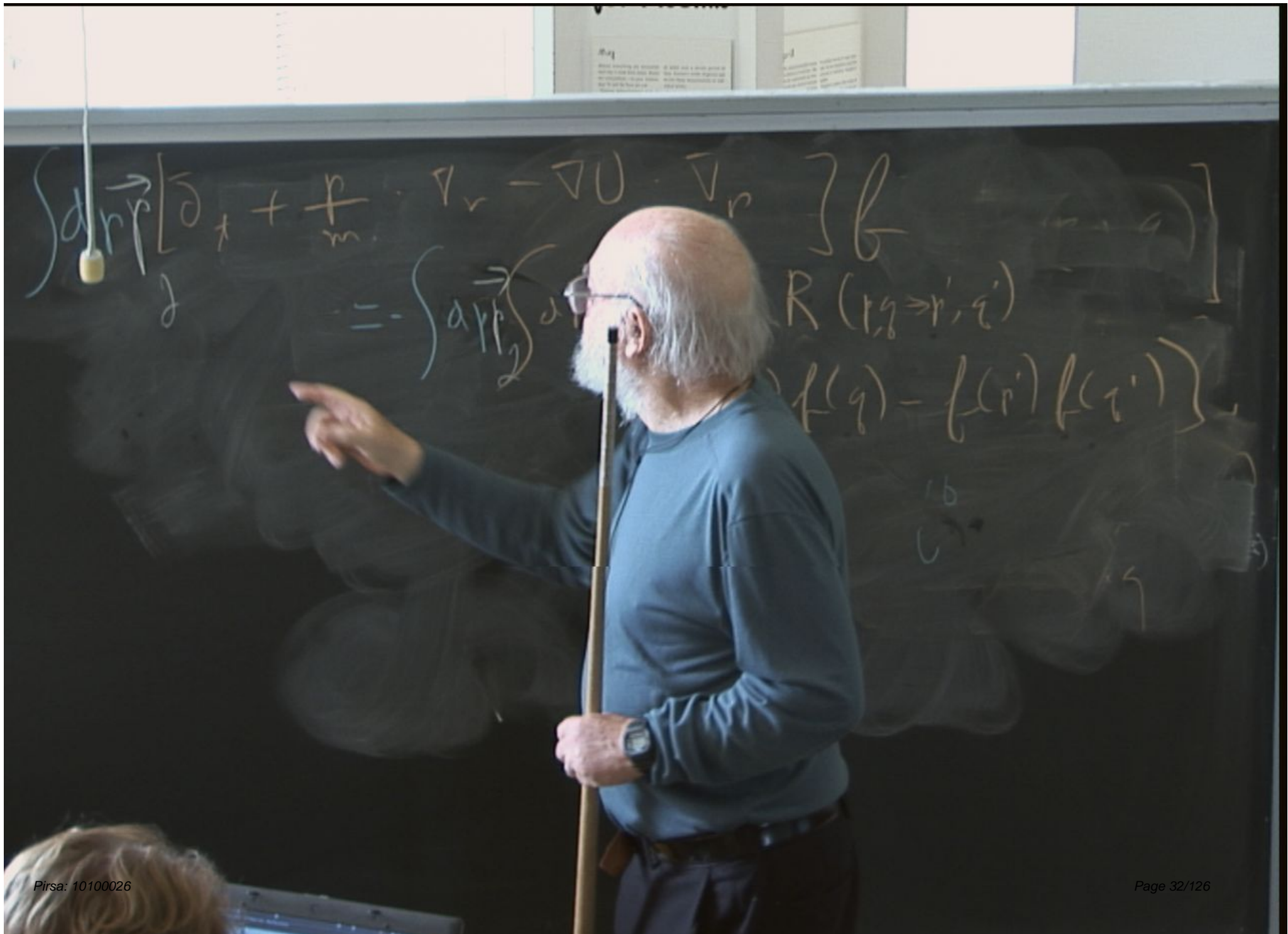
$$\partial_t g_j(\mathbf{r}, t) + \sum_k \partial_k T_{jk}(\mathbf{r}, t) = -n(\mathbf{r}, t) \partial_j U(\mathbf{r}, t) ..$$

$$\begin{aligned}
 & \left(\frac{d\vec{r}}{dt} \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] \right) \\
 & = - \int d\vec{r}' d\vec{q} d\vec{q}' R(r, q \rightarrow r', q') \\
 & \quad [f(r) f(q) - f(r') f(q')]
 \end{aligned}$$



$$\left\{ \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_p \right\} \psi$$
$$\Rightarrow \int \int d\mathbf{r}' d\mathbf{q}' d\mathbf{q} R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \left[f(\mathbf{r}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}') \right]$$

$$\begin{aligned}
 & \int d\vec{r} \vec{r} \left[\frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] \rho \\
 & = - \int d\vec{r} \int d\vec{r}' \int d\vec{q} \, R(\vec{r}, \vec{q} \rightarrow \vec{r}', \vec{q}') \\
 & \quad [f(\vec{r}) f(\vec{q}) - f(\vec{r}') f(\vec{q}')]
 \end{aligned}$$



Conservation of Particle momentum

To obtain the local law for the conservation of particles, multiply the Boltzmann equation by \mathbf{p} integrate over all momentum, and look at the result term by term, starting from the left. The first term is the time derivative of the j th component of momentum density:

$$\partial_t g_j(\mathbf{r}, t) \quad \text{with } g_j(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) p_j$$

The second term is on the left is the divergence of the momentum current, a tensor T_{jk} :

$$\sum_k \partial_k T_{jk}(\mathbf{r}, t) \quad \text{with } T_{jk}(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) p_j p_k / m$$

T_{jk} describes the current of the j th component of momentum in the direction k .

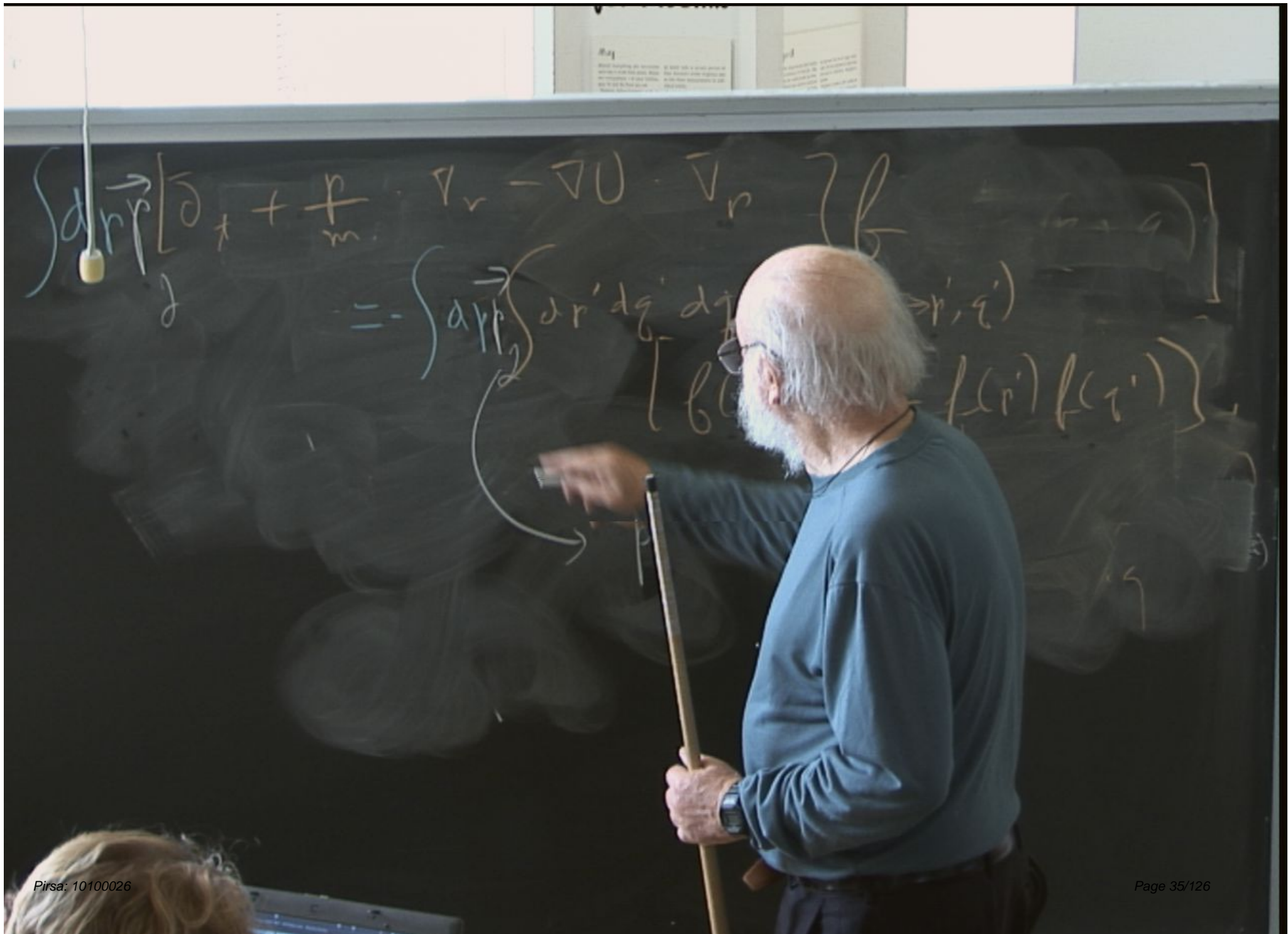
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When integrated over momentum, the two collision terms are exactly the same except for sign and thus cancel with one another.

We are left with the differential form of the momentum conservation law

$$\partial_t g_j(\mathbf{r}, t) + \sum_k \partial_k T_{jk}(\mathbf{r}, t) = -n(\mathbf{r}, t) \partial_j U(\mathbf{r}, t) ..$$

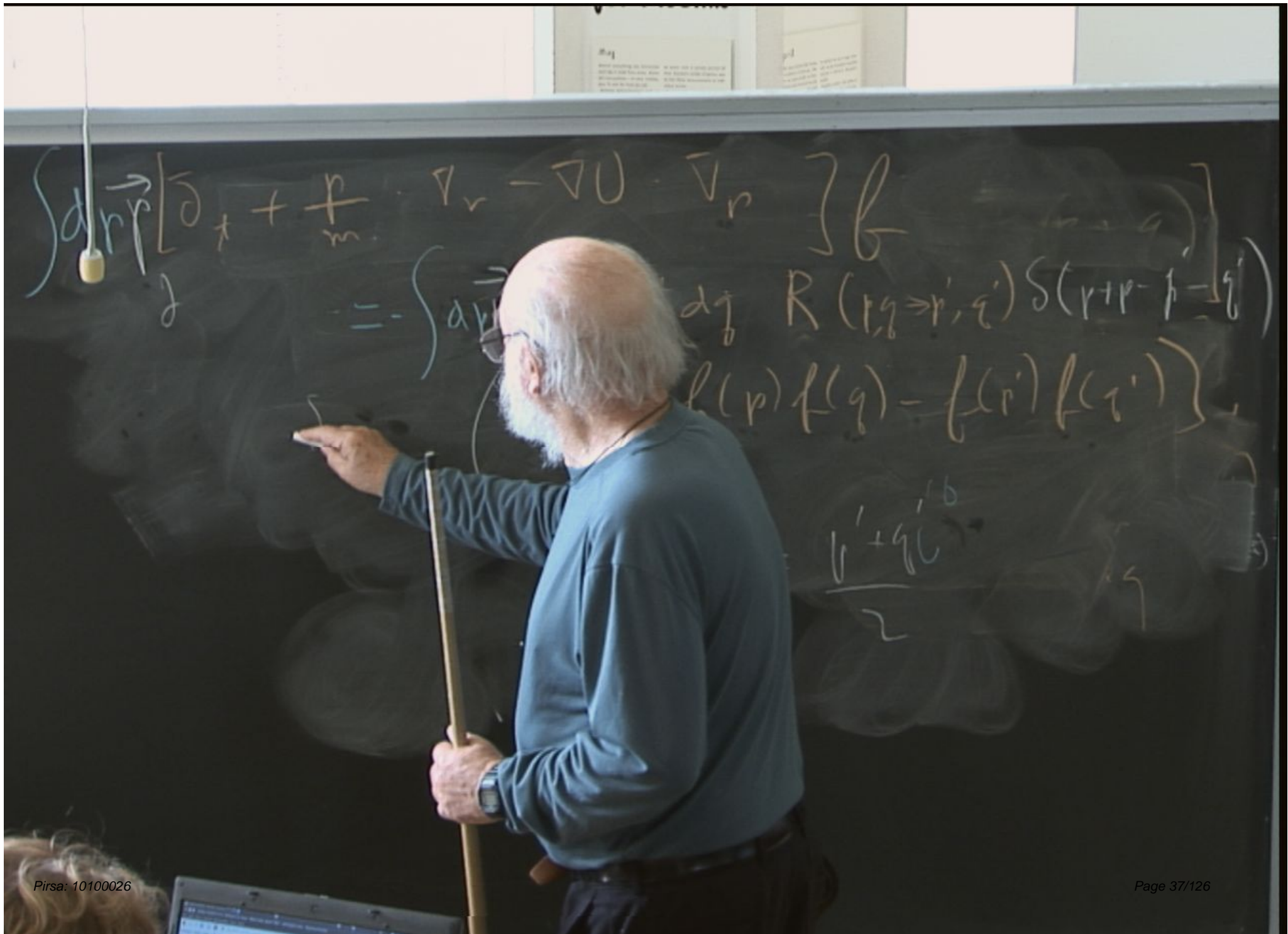
$$\begin{aligned}
 & \int d\mathbf{r} \left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} - \nabla U \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{r}, \mathbf{p}, t) \\
 &= - \int d\mathbf{r} d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \\
 & \quad [f(\mathbf{r}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}')]
 \end{aligned}$$



$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi - V \psi = E \psi$$
$$\langle E \rangle = \int \psi^* \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \psi d\tau$$

$$\begin{aligned}
 & \int d\mathbf{r} \left[\frac{1}{2} \frac{d}{dt} \left(\frac{1}{2} m \mathbf{v}^2 \right) + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] \rho \\
 & = - \int d\mathbf{r} \int d\mathbf{r}' \int d\mathbf{q} \int d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \\
 & \quad \left[f(\mathbf{r}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}') \right]
 \end{aligned}$$

$\xrightarrow{\quad} \frac{p+q}{2}$



$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi - V \psi = E \psi$$

$$\langle V \rangle = \int d^3r \psi^* V \psi$$

$$\langle V \rangle = \int d^3r \int d^3r' \psi^*(\mathbf{r}) V(\mathbf{r}) \psi(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}')$$

$$\langle V \rangle = \int d^3r \psi^*(\mathbf{r}) V(\mathbf{r}) \psi(\mathbf{r})$$

$$\begin{aligned}
 & \int d\mathbf{r} \left[\frac{1}{2} \frac{d}{dt} \left(\mathbf{p} \cdot \nabla_{\mathbf{r}} - \nabla U \cdot \nabla_{\mathbf{r}} \right) \right] \\
 & = - \int d\mathbf{r} d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') S(\mathbf{r} + \mathbf{r}' - \mathbf{p} - \mathbf{q}') \\
 & \quad \times \left(\epsilon_{\mathbf{r}} + \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{r}'} - \epsilon_{\mathbf{q}'} \right) \left(f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}') \right) \\
 & \quad \rightarrow \frac{\mathbf{p} + \mathbf{q}}{2} = \frac{\mathbf{r}' + \mathbf{q}'}{2}
 \end{aligned}$$

$$\begin{aligned}
 & \int d\mathbf{r} \left[\frac{1}{2} \frac{d}{dt} \left(\mathbf{p} \cdot \mathbf{v} \right) + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] b \\
 & = - \int d\mathbf{r} d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') S(\mathbf{r} + \mathbf{r}' - \mathbf{q} - \mathbf{q}') \\
 & \quad \frac{1}{2} (\epsilon_r + \epsilon_q - \epsilon_{r'} - \epsilon_{q'}) [b(\mathbf{r}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}')] \\
 & \quad \rightarrow \frac{p+q}{2} = \frac{p'+q'}{2}
 \end{aligned}$$

Constitutive Equation for T_{jk}

The stress tensor is particularly important in condensed matter physics, particle physics, and relativity. In all cases it describes how momentum is moved around. In relativistic theories it describes the flow of both energy and momentum.

In the fluid context, we have

$$T_{jk} = \delta_{j,k} p \quad \text{pressure comes from disordered flow of momentum} \\
+ \rho v_j v_k \quad \text{flow of momentum due to motion of particles} \\
- \eta [\partial_j v_k + \partial_k v_j - 2 \delta_{j,k} \partial \cdot v] \quad \text{dissipation term from viscosity} \\
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The pressure can be calculated from statistical mechanics. The quantities η and ζ describe non-equilibrium behavior. They cannot come from stat mech but they can be calculated from the Boltzmann equation. In classical mechanics there is a splitting between equilibrium and non-equilibrium calculations. In quantum theory they are mixed together.

$$\int d\vec{r} \vec{p} \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] f = m \frac{d\langle \vec{r} \rangle}{dt}$$

$$= - \int d\vec{r} \left(\int d\vec{r}' d\vec{q} d\vec{q}' R(\vec{r}, \vec{q} \rightarrow \vec{r}', \vec{q}') S(\vec{r} + \vec{r}' - \vec{r} - \vec{r}') \right)$$

$$\frac{\partial}{\partial t} (e_r + e_q) (-e_{r'} - e_{q'}) [f(r) f(q) - f(r') f(q')]$$

$$\frac{p+q}{2} = \frac{p'+q'}{2}$$

$$\frac{\partial}{\partial t} \left(\frac{p+q}{2} \right) = \frac{\partial}{\partial t} \left(\frac{p'+q'}{2} \right)$$

$$\frac{\partial}{\partial t} f(r, p)$$

$$\begin{aligned}
 & \int d\mathbf{r} \left[\frac{1}{2} \left(\frac{\partial \psi}{\partial t} + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right) \psi \right] \\
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 & \quad \times \left(\epsilon_r + \epsilon_{q'} - \epsilon_{p'} - \epsilon_{q'} \right) \left(f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}') \right) \\
 & \quad \rightarrow \frac{\mathbf{p} + \mathbf{q}}{2} = \frac{\mathbf{p}' + \mathbf{q}'}{2}
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$\frac{\partial}{\partial t} n$
 $\frac{\partial}{\partial t} g$
 $\frac{\partial}{\partial t} f(\mathbf{v}, \mathbf{r})$

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$$\int d\mathbf{r} \left[\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} - \nabla U \cdot \nabla_{\mathbf{p}} \right] f$$

$$= - \int d\mathbf{r} d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') S(\mathbf{r} + \mathbf{r}' - \mathbf{p} - \mathbf{q})$$

$$\int d\mathbf{r} d\mathbf{q} d\mathbf{q}' \left[f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{r}') \right]$$

$$\frac{\mathbf{p} + \mathbf{q}}{2} = \frac{\mathbf{p}' + \mathbf{q}'}{2}$$

$$\frac{\partial}{\partial t} n$$

$$\frac{\partial}{\partial t} g$$

$$\frac{\partial}{\partial A} f(v, A)$$

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$$= - \int d\mathbf{r} \int d\mathbf{r}' \int d\mathbf{q} \int d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \delta(\mathbf{r} + \mathbf{p} - \mathbf{r}' - \mathbf{q}')$$

$$\delta(\epsilon_r + \epsilon_q - \epsilon_{r'} - \epsilon_{q'}) \{ b(\mathbf{r}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}') \}$$

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$$\frac{\partial}{\partial t} \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{q}} f(\mathbf{r}, \mathbf{q})$$

$$f(\mathbf{r}, \mathbf{q}) = \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{q}} f(\mathbf{r}, \mathbf{q})$$

$$= v(\mathbf{r}, \mathbf{q}) / m$$

$$\int d\vec{r} \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] \rho = m \cdot g$$

$$= - \int d\vec{r} \int d\vec{r}' \int d\vec{q} R(\vec{r}, \vec{q} \rightarrow \vec{r}', \vec{q}') \delta(\vec{r} + \vec{p} - \vec{r}' - \vec{q}') \delta(\epsilon_r + \epsilon_q - \epsilon_{r'} - \epsilon_{q'}) [f(\vec{r}) f(\vec{q}) - f(\vec{r}') f(\vec{q}')] .$$

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$$\frac{\partial}{\partial t} \rho$$

$$\frac{\partial}{\partial t} \rho$$

$$\frac{\partial}{\partial t} f(\vec{r}, t)$$

$$f(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{m}$$

$$= v(\vec{r}, t) n(\vec{r}, t)$$

$$\int d\vec{r} \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] \rho = - \int d\vec{r} \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right] \rho$$

$$= - \int d\vec{r} \int d\vec{r}' \int d\vec{q} \int d\vec{q}' R(\vec{r}, \vec{q} \rightarrow \vec{r}', \vec{q}') \delta(\vec{r} + \vec{p} - \vec{r}' - \vec{q}') \left[\rho(\vec{r}) \rho(\vec{q}) - \rho(\vec{r}') \rho(\vec{q}') \right]$$

$$\frac{\vec{p} + \vec{q}}{2} = \frac{\vec{r}' + \vec{q}'}{2}$$

$$\frac{\partial}{\partial t} \rho(\vec{r}, t)$$

$$\rho(\vec{r}, t) = \frac{1}{m} \frac{\partial}{\partial t} \rho(\vec{r}, t)$$

$$= v(\vec{r}, t) \rho(\vec{r}, t)$$

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H-Theorem

Boltzmann proved a result called the H-theorem, which is our first representation of a law describing the non-equilibrium behavior of entropy. In fact, we have few other examples! To obtain this take the Boltzmann equation, equation v.14, and multiply by $\ln f$. Note that $(\ln f) df = d[f \ln f] - df = d[f \ln f/e]$. Now, integrate over all momentum. The first term on the left hand side of the Boltzmann equation becomes

$$\partial_t h(\mathbf{r}, t) \quad \text{with } h(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) \ln [f(\mathbf{p}, \mathbf{r}, t)/e]$$

The second term is on the left is the divergence of the particle current:

$$\nabla_{\mathbf{r}} \cdot \mathbf{j}_h(\mathbf{r}, t) \quad \text{with } \mathbf{j}_h(\mathbf{r}, t) = \int d\mathbf{p} \mathbf{p}/m f(\mathbf{p}, \mathbf{r}, t) \ln [f(\mathbf{p}, \mathbf{r}, t)/e]$$

A brief calculation, **involving an integration by parts**, shows that the third term on the left vanishes because it contains a total derivative with respect to momentum and we assume that the momentum integrands drop off fast enough at infinity so that the integral of the total derivative is zero.

When integrated over momentum, the two collision terms on the right become

$$D_H = - \int \int \int \int d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}')) \\ Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] \ln f(\mathbf{p})$$

Because this result, except for the factor in blue, is symmetrical in \mathbf{p} and \mathbf{q} , we can replace $\ln f(\mathbf{p})$ by $\ln f(\mathbf{q})$ or by the averages of these two $[\ln f(\mathbf{p}) + \ln f(\mathbf{q})]/2$ giving

$$\left\{ dr \left[\frac{\partial}{\partial t} + \frac{1}{r} \nabla_r - \nabla U - \nabla_r \right] b \right. \\ \left. = \left(dr' dg' dg R(r, g \rightarrow r', g') S(r+r'-b-b') \right) \right. \\ \left. - \epsilon_i \left[b(r) f(g) - f(r') f(g') \right] \right\}$$

$$\frac{p+q}{2} = \frac{p'+q'}{2}$$

$$f(v, A) = \frac{g(v, A)}{h(v, A)} \\ = v(v, A) / p(v, A)$$

evidence
for Atoms

Molecules?

$$\left(\frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right) \psi = i \hbar \frac{d}{dt} \psi$$

$$= - \frac{\hbar^2}{2m} \nabla^2 \psi + \psi \left(\frac{1}{2} (E_r + E_r - E_p - E_p) \right) \left(f(r) f(r') - f(r') f(r') \right)$$

$$\frac{\partial}{\partial t} \psi = \frac{\partial}{\partial t} \psi = \frac{\partial}{\partial t} \psi$$

$$\psi(r, A) = \frac{g(r, A)}{m}$$

$$= \psi(r, A) / \psi(r, A)$$

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$$\left(\frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right) \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi$$

$$= -\frac{\hbar^2}{2m} \nabla^2 \psi$$

$$2(\epsilon_r + \epsilon_l - \epsilon_{r'}) = f(q) - f(r') f(r')$$

$$\frac{\partial}{\partial t} \psi = \frac{\partial}{\partial t} \psi + \frac{\partial}{\partial t} \psi$$

$$\psi(r, A) = \frac{g(r, A)}{m}$$

$$= \psi(r, A) / \psi(r, A)$$

$$\left(\frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{p}{m} \cdot \nabla_{\mathbf{r}} - \nabla U \cdot \nabla_{\mathbf{r}} \right) \psi$$

$$= - \int d\mathbf{r}' d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \delta(\mathbf{r} + \mathbf{p} - \mathbf{r}' - \mathbf{q}') \delta(\mathbf{r} + \mathbf{p} - \mathbf{r}' - \mathbf{q}') \left[\psi(\mathbf{r}) \psi(\mathbf{q}) - \psi(\mathbf{r}') \psi(\mathbf{q}') \right]$$

$$\frac{p+q}{2} = \frac{p'+q'}{2}$$

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, \mathbf{p})$$

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$$= \mathbf{v}(\mathbf{r}, \mathbf{p}) \psi(\mathbf{r}, \mathbf{p})$$

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$$= -\frac{\hbar^2}{2m} \int d\mathbf{r}' d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \delta(\mathbf{r} + \mathbf{r}' - \mathbf{r} - \mathbf{q}') \delta(\mathbf{r} + \mathbf{q} - \mathbf{r}' - \mathbf{q})$$

$$\delta(\mathbf{r} + \mathbf{q} - \mathbf{r}' - \mathbf{q}') \{ \psi(\mathbf{r}) \psi(\mathbf{q}) - \psi(\mathbf{r}') \psi(\mathbf{q}') \}$$

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$$\frac{\partial}{\partial t} \psi(\mathbf{r}, t)$$

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, t)$$

$$\frac{\partial}{\partial \mathbf{p}} \psi(\mathbf{r}, t)$$

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$$D_H = - \int \int \int \int d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}')) \\ Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] \ln f(\mathbf{p})$$

H-Theorem

Boltzmann proved a result called the H-theorem, which is our first representation of a law describing the non-equilibrium behavior of entropy. In fact, we have few other examples! To obtain this take the Boltzmann equation, equation v.14, and multiply by $\ln f$. Note that $(\ln f) df = d[f \ln f] - df = d[f \ln f/e]$. Now, integrate over all momentum. The first term on the left hand side of the Boltzmann equation becomes

$$\partial_t h(\mathbf{r}, t) \quad \text{with } h(\mathbf{r}, t) = \int d\mathbf{p} f(\mathbf{p}, \mathbf{r}, t) \ln [f(\mathbf{p}, \mathbf{r}, t)/e]$$

The second term is on the left is the divergence of the particle current:

$$\nabla_{\mathbf{r}} \cdot \mathbf{j}_h(\mathbf{r}, t) \quad \text{with } \mathbf{j}_h(\mathbf{r}, t) = \int d\mathbf{p} \mathbf{p}/m f(\mathbf{p}, \mathbf{r}, t) \ln [f(\mathbf{p}, \mathbf{r}, t)/e]$$

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$$DH = - \int \int \int \int d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}'))$$

$$Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] \ln f(\mathbf{p})$$

Because this result, except for the factor in blue, is symmetrical in \mathbf{p} and \mathbf{q} , replace $\ln f(\mathbf{p})$ by $\ln f(\mathbf{q})$ or by the averages of these two, $[\ln f(\mathbf{p}) + \ln f(\mathbf{q})]/2$

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$$R(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] \ln f(\mathbf{p})$$

$$\left(\frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{1}{m} \cdot \nabla_{\mathbf{r}} \cdot \nabla U \cdot \nabla_{\mathbf{r}} \right) \psi$$

$$= - \int d\mathbf{r}' d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \delta(\mathbf{r} + \mathbf{r}' - \mathbf{r} - \mathbf{q}') \delta(\mathbf{r} + \mathbf{q} - \mathbf{r}' - \mathbf{q})$$

$$\delta(\mathbf{r} + \mathbf{q} - \mathbf{r}' - \mathbf{q}') \left[\psi(\mathbf{r}) \psi(\mathbf{q}) - \psi(\mathbf{r}') \psi(\mathbf{q}') \right]$$

$$\frac{p+q}{2} = \frac{p'+q'}{2}$$

$$\frac{\partial}{\partial \mathbf{r}} \psi(\mathbf{r}, \mathbf{A})$$

$$\psi(\mathbf{r}, \mathbf{A}) = \frac{1}{\sqrt{m}} \psi(\mathbf{r}, \mathbf{A})$$

$$\ln \psi(\mathbf{r}) \rightarrow \frac{1}{2} \left(\ln \psi(\mathbf{r}) \psi(\mathbf{q}) \right) \psi(\mathbf{r}, \mathbf{A}) \psi(\mathbf{r}')$$

$$DH = - \int \int \int \int d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}'))$$

$$Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] [\ln f(\mathbf{p}) + \ln f(\mathbf{q})]/2$$

Now the whole integral, except for the factor in blue, is anti-symmetrical in the replacement of unprimed variables by primed ones. For this reason we can make the replacements

$$[\ln f(\mathbf{p}) + \ln f(\mathbf{q})]/2 \rightarrow -\ln f(\mathbf{p}') + \ln f(\mathbf{q}')]/2 \rightarrow [\ln f(\mathbf{p}) + \ln f(\mathbf{q}) - \ln f(\mathbf{p}') - \ln f(\mathbf{q}')]/4$$

Since $\ln a + \ln b = \ln(ab)$ we can rewrite our entire result as

$$DH = - \int \int \int \int d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}'))$$

$$Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] \{\ln [f(\mathbf{p}) f(\mathbf{q})] - \ln [f(\mathbf{p}') f(\mathbf{q}')] \} / 4$$

The entire integral is negative, except perhaps for the factor in red. However, this factor is of the form $[X - Y] [\ln X - \ln Y]$. This factor is positive if $X > Y$, equally positive if $Y > X$, and only zero when $X = Y$. In that case, we are in local equilibrium!

Put it all together, our result is that

$$\partial_t h(\mathbf{r}, t) + \nabla_{\mathbf{r}} \cdot \mathbf{j}_h(\mathbf{r}, t) = DH$$

$$DH = - \int \int \int \int d\mathbf{p} d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}'))$$

$$Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] [\ln f(\mathbf{p}) + \ln f(\mathbf{q})]/2$$

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$$Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{p}') f(\mathbf{q}')] \{\ln [f(\mathbf{p}) f(\mathbf{q})] - \ln [f(\mathbf{p}') f(\mathbf{q}')] \} / 4$$

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$$\partial_t h(\mathbf{r}, t) + \nabla_{\mathbf{r}} \cdot \mathbf{j}_h(\mathbf{r}, t) = DH$$

Grains of
Pollen to
Evidence
for Atoms

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$$\begin{aligned}
 & \left(\frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right) \psi \\
 & = - \int d\mathbf{r}' d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') S(\mathbf{r} + \mathbf{r}' - \mathbf{r} - \mathbf{r}') \\
 & (X - Y) \left[\ln X - \ln Y \right] \frac{1}{2} (\epsilon_r + \epsilon_r' - \epsilon_r - \epsilon_r') \left[f(\mathbf{r}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}') \right] \\
 & \begin{aligned}
 & \frac{\partial}{\partial \mathbf{r}} \frac{\partial}{\partial \mathbf{q}} \frac{\partial}{\partial \mathbf{A}} f(\mathbf{v}, \mathbf{A}) \\
 & \frac{p+q}{2} = \frac{v' + q'}{2} \\
 & f(\mathbf{v}, \mathbf{A}) = g(\mathbf{v}, \mathbf{A}) \\
 & \ln f(\mathbf{v}) \rightarrow \frac{1}{2} \left[\ln f(\mathbf{v}) f(\mathbf{q}) \right] / v(\mathbf{v}, \mathbf{A}) p(\mathbf{v})
 \end{aligned}
 \end{aligned}$$

From
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$$\left\{ \frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{1}{m} \nabla \cdot \nabla U \cdot \nabla \right\} \psi$$

$$dh = (x - y) \left[\ln x - \ln y \right] = 2 \left(\frac{x+y}{2} \right) \left[\ln \left(\frac{x+y}{2} \right) - \ln \left(\frac{x-y}{2} \right) \right]$$

$$dh = 0 \quad x = y$$

$$\frac{+q}{2} = \frac{v' + q}{2}$$

$$g(v, t) = g(v, t)$$

$$b(v) \rightarrow \frac{1}{2} \left[\ln \left(\frac{v}{v_0} \right) \right] / \ln \left(\frac{v}{v_0} \right) \ln \left(\frac{v}{v_0} \right)$$

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$$\left(\frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] \right) \mathbf{r} = - \nabla U \quad \nabla_r = \nabla_{\mathbf{r}}$$

$$d h = (x - y) \left[\frac{1}{2} m \dot{\mathbf{r}}^2 - \frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \epsilon \left(- \epsilon_r - \epsilon_i \right) \left[b(r) f(q) - f(r) f(r') \right]$$

$$d h = 0$$

$$d h > 0$$

$$x = y$$

$$x = y$$

$$\frac{p+q}{2} = \frac{v'+q'}{2}$$

$$g(v, \pi) = g(v, \pi)$$

$$b(r) \rightarrow \frac{1}{2} \left[\ln f(r) / f(r') \right] / v(r, \pi) p(r, \pi)$$

From
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$$\left\{ \frac{d}{dt} \left[\frac{1}{2} m \dot{\mathbf{r}}^2 \right] + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right\} \psi$$

$$= - \int d\mathbf{r}' d\mathbf{q} d\mathbf{g} R(\mathbf{r} \rightarrow \mathbf{r}', \mathbf{q}) S(\mathbf{r} + \mathbf{r}' - \mathbf{q})$$

$$d h = (X - Y) \left[\ln X - \ln Y \right] \left\{ \frac{1}{2} (\epsilon_r + \epsilon_g - \epsilon_r - \epsilon_g) [f(r) f(q) - f(r') f(q')] \right\}$$

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$x=y$
 $x > y \quad \frac{\partial}{\partial x}$
 $x < y \quad \frac{\partial}{\partial x} f(v, \pi)$

$dh = 0$
 $dh > 0$
 $dh < 0$

$$\left(d p \left[\frac{h_f(r)}{2} \right] \vec{0}_x + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right) = - \left(d p \left[\frac{h_f(r)}{2} \right] \right) \int d r' d g' d g R ($$

$$d h = (X - Y) \left[\ln X - \ln Y \right] \left\{ 2 (\epsilon_r + \epsilon_g - \epsilon_{r'} - \epsilon_{g'}) \right\} b(r) f(r)$$

$$\begin{aligned} d h &= 0 \\ d h &> 0 \\ d h &> 0 \end{aligned}$$

$$\begin{aligned} X &= Y \\ X &> Y & \frac{\partial h}{\partial r} < 0 \\ X &< Y & \frac{\partial h}{\partial r} > 0 \end{aligned}$$

$$\frac{p+q}{2} = \frac{1}{2}$$

$$\ln b(r) \rightarrow \frac{1}{2} \ln$$

$$+ \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \} b$$

$$= - \int d\mathbf{r}' d\mathbf{q}' d\mathbf{q} R(r, q \rightarrow r', q') \delta(r+p-r'-q')$$

$$\int d\mathbf{r} d\mathbf{q} \delta(\epsilon_r + \epsilon_q - \epsilon_{r'} - \epsilon_{q'}) [b(r) f(q) - f(r') f(q')]$$

$$\frac{p+q}{2} = \frac{r'+q'}{2}$$

$$\frac{\partial}{\partial r} \frac{\partial}{\partial q} f(r, A)$$

$$f(r, A) = \frac{g(r, A)}{m}$$

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$$\partial_t h(\mathbf{r},t) + \nabla_{\mathbf{r}} \cdot \mathbf{j}_h(\mathbf{r},t) = D H$$

and that $D H < 0$,

except in local equilibrium when $D H = 0$.

Thus $-D H$ is the rate of entropy production in the system.

$$S(t)/k = -\int d\mathbf{r} h(\mathbf{r},t) + \text{conserved things}$$

S is the entropy for a weakly coupled system and $dS/dt > 0$.



Homework:

Assume our Brownian particle, as described by equation v.12, is charged? How can I include electric and magnetic fields in this equation? Does the system go to equilibrium in the presence of space and time-independent fields? What happens when the field depends upon time?

How can we be sure that equation conserves the total probability of finding the Brownian particle? Should it conserve the momentum or energy of that particle? What are the equations for the time dependence of the particle's energy and momentum? What about its angular momentum?

Find the laws of conservation of energy and momentum from the Boltzmann equation, equation v.14.

Put it all together, our result is that

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Header

$\sqrt{c} \approx$

$\sigma_z^3 = \pm 1 \quad \tau_3$

$\hbar \pm \mp \sqrt{\int \Sigma \approx \nabla \partial}$

$\hbar \pm \mp \sqrt{\int \Sigma \approx \nabla \partial}$

\Rightarrow \rightarrow

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Header

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$$\sigma_z^3 = \pm 1 \quad \tau_3$$

$$\hbar \pm \sqrt{\int \Sigma} \approx \nabla \partial$$

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Daniel Kleppner, "Rereading Einstein on Radiation", *Physics Today*, (February 2005).
A Einstein, *Phys. Z.* **18** 121 (1917). English translation, D. ter Haar, *The Old Quantum Theory*, Pergamon Press, New York, p. 167 (1967).

The screenshot shows a Beamer presentation navigation pane. The pane is titled 'Master Slides' at the top. It contains a list of slides with thumbnails. The slide titled 'Master #14 Slides' is highlighted. The navigation pane includes buttons for 'Previous', 'Next', 'Table of Contents', and 'Index'.

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Header

 $\sqrt{c} \approx$
$$\sigma_z^3 = \pm 1 \quad \tau_3$$
$$\hbar \pm \mp \sqrt{f \Sigma} \approx \nabla \partial$$
$$\hbar \pm \sqrt{\int \Sigma} \approx \nabla \partial$$

[illegible]

Albert Einstein (1905) explained this dancing by many, many collisions with molecules.

$$p = (p_x, p_y, p_z)$$

v.1

$$\langle \eta_j(t) \eta_k(s) \rangle = \Gamma \delta(t-s) \delta_{j,k}$$

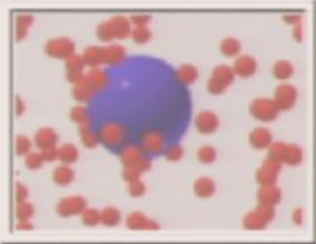
$$\langle \eta_j(t) \eta_k(s) \rangle = \Gamma \delta(t-s) \delta_{j,k}$$

v.2

Perimeter Institute statistical physics Lecture Notes part V: Hopping in Momentum Version 1.52 10/12

Brownian motion:

Robert Brown (1773-1858) saw particles of pollen “dance around” in fluid under microscope. This motion was caused by many tiny particles hitting the grains of pollen.



The many moving tiny particles are of course **molecules of the liquid**. They were too small to see under a microscope when Brownian motion was discovered, but it was obvious they were there. You can see the molecules of liquid hitting the bigger particle in the animation on the left. (The size of the molecules has been dramatically *increased* in order to make them visible).

<http://www.worsleyschool.net/science/files/brownian/motion.html>

Albert Einstein (1905) explained this dancing by many, many collisions with molecules in fluid

$$dp/dt = \dots + \eta(t) - p/\tau$$

$$p = (p_x, p_y, p_z)$$

$$\eta = (\eta_x, \eta_y, \eta_z)$$

v.1

$\eta(t)$ is a **Gaussian random variable** resulting from random kicks produced by collisions. Since the kicks have random directions $\langle \eta(t) \rangle = 0$. Different collisions are assumed to be statistically independent

$$\langle \eta_j(t) \eta_k(s) \rangle = \Gamma \delta(t-s) \delta_{j,k}$$

v.2

The relaxation time, τ , describes friction slowing down as the particles moves through the medium. In contrast Γ describes the extra momentum picked up via the collisions. Both represent the same physical effect, little particles hitting our big one. However, they operate in a somewhat different fashion. The individual kicks point in every which direction and only in

in the animation on the left. (The size of the molecules has been dramatically *increased* in order to make them visible).

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Albert Einstein (1905) explained this dancing by many, many collisions with molecules

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v.1

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$$\langle \eta_j(t) \eta_k(s) \rangle = \Gamma \delta(t-s) \delta_{j,k}$$

v.2

The relaxation time, τ , describes friction slowing down as the particles move through the medium. In contrast Γ describes the extra momentum picked up via the collisions. They represent the same physical effect, little particles hitting our big one. However, they do so in a somewhat different fashion. The individual kicks point in every which direction and in the long run produce any concerted change in momentum. On the other hand there is a friction tending to continually push our particle toward smaller speeds relative to the medium.

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Part 5 Momentum Hops.key Part 6 Boson-Fermion.key

in the animation dramatically increased in order to make them visible).

<http://www.worsleyschool.net/science/files/brownian/motion.html>

Albert Einstein (1905) explained this dancing by many, many collisions with molecules

$$\frac{dp}{dt} = \dots + \eta(t) - p/\tau$$

$p = (p_x, p_y, p_z)$ $\eta = (\eta_x, \eta_y, \eta_z)$ v.1

$\eta(t)$ is a Gaussian random variable resulting from random kicks produced by collisions. The kicks have random directions $\langle \eta(t) \rangle = 0$. Different collisions are assumed statistically independent

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Bosons and fermions

Second Quantization

Second Quantization vs Classical

Quantum Description

One Mode

Independent Excitations

Extreme limit for fermions

Extreme limit for bosons

Waves

Waves= Special Bosons

photons in cavity

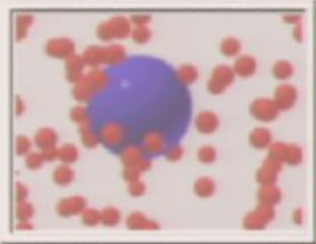
waves as independent excitations

Conserved particles

conserved fermions in a box

Brownian motion:

Robert Brown (1773-1858) saw particles of pollen “dance around” in fluid under microscope. This motion was caused by many tiny particles hitting the grains of pollen.



The many moving tiny particles are of course **molecules of the liquid**. They were too small to see under a microscope when Brownian motion was discovered, but it was obvious they were there. You can see the molecules of liquid hitting the bigger particle in the animation on the left. (The size of the molecules has been dramatically *increased* in order to make them visible).

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v.2

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Part 6 Boson-Fermion.key

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Bosons and fermions

- Second Quantization
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From
Grains of
Pollen to
Evidence
for Atoms

low
is a
molecule?

$$\left\{ \frac{d}{dt} \left[\frac{1}{2} m \dot{r}^2 \right] + \frac{p}{m} \cdot \nabla_r - \nabla U \cdot \nabla_r \right\} \psi = - \frac{\hbar^2}{2m} \nabla^2 \psi$$

$$h = (X - Y) \left[\ln X - \ln Y \right] \left\{ \frac{1}{2} (e_r + e_g - e_r - e_g) [f(r) f(g) - f(r') f(g')] \right\}$$

$$\begin{aligned} dh &= 0 & X=Y \\ dh &> 0 & X>Y \quad \frac{\partial h}{\partial r} > 0 \\ dh &> 0 & X<Y \quad \frac{\partial h}{\partial g} < 0 \end{aligned}$$

$$\frac{p+q}{2} = \frac{p'+q'}{2}$$

$$f(v, A) = g(v, A)$$

$$\ln f(v) \rightarrow \frac{1}{2} \left[\ln f(v) + \ln f(v) \right] \frac{1}{v(v, A) p(v)}$$

Second Quantized versus Classical Description

In a classical description, or even in using an ordinary wave function in a quantum description, we base everything on the particle. Particle 7 is sitting right in front of me; particle 23 is in the upper left hand corner, etc. Or, particle 7 has a wave function ψ ; particle 23 is described by the wave function φ . A degenerate quantum system is one composed of identical particles sufficiently squeezed so that their wave functions overlap. To describe such a system, we cannot talk about the behavior of individual particles. We can only specify how many particles are doing this or that. Thus we start with a description of possible modes of the system and talk about their occupation. In this kind of description, we would say that there are seven particles in mode 3 and none in mode 2.

To discuss independent excitations in degenerate quantum theory, we use a formulation in which we allow the number of excitations to vary. Hence we are varying the number of particles. So instead of using $\exp(-\beta H)$, and keeping the number of particles fixed, we use as our weight function $\exp[-\beta(H-\mu N)]$ and we are allowing the number of particles to vary. The former approach is called using the canonical ensemble, and is what we have done up to now. The latter approach uses the grand canonical ensemble and it is the one we shall follow for this chapter.

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Quantum Description

To describe a degenerate quantum situation, we first specify the modes. We imagine placing everything in a box of side L . One neat formulation has periodic boundary conditions. The different modes of excitation are described by wave functions which are of the form $(1/L)^{3/2} \exp(i\mathbf{k}\cdot\mathbf{r})$. The wave number $\mathbf{k}=(k_x, k_y, k_z)$ must be of the form $(2\pi/L)\mathbf{m} = (2\pi/L)(m_x, m_y, m_z)$ where the m 's are integers. This gives periodic wave functions, and what is more important, a complete set of wave functions. Corresponding to these k 's are momenta, p 's, of the form $\mathbf{p}=\hbar \mathbf{k}$

In the quantum mechanics of non-interacting particles, each mode is dynamically and statistically independent of the others. That is to say, the Hamiltonian is a sum of terms each referring to a different mode. We have a **discrete infinity** of modes, labelled by the m 's.

A sum over the independent modes in quantum theory can be written as $\sum_{\mathbf{m}}$

Usually, but not always, many terms contribute from such a sum so that it can be written as an integral over wave vectors or momenta in the form

$$\sum_{\mathbf{m}} \rightarrow \left(\frac{L}{2\pi}\right)^3 \int d\mathbf{k} = \left(\frac{L}{2\pi\hbar}\right)^3 \int d\mathbf{p} \quad \text{vi.1}$$

This result is interpreted by saying that the quantum sum over m goes into a sum over phase space in discrete pieces of size h^3 . Of course, there is no h^3 in any sensible formulation of classical mechanics. So something funny will have to be done to patch together classical mechanics and quantum theory.

One mode

In the grand canonical formulation, the only difference between bosons and fermions is the possible values of the excitation number of a given type, n_i . For bosons this n can be any non-negative integer 0, 1, 2, ... For fermions the excitation or **mode** can either be empty or occupied, corresponding to $n=0$ or 1. In either the bose or the fermi cases, the probabilities are given by $\rho(n) = (1/\xi) \exp[-\beta(\epsilon - \mu)n]$, with $\xi = \sum \exp[-\beta(\epsilon - \mu)n]$. In either case, $\langle n \rangle = \partial_{\beta\mu} \ln \xi$

We next look to a single mode of excitation. For the **fermion**, the normalizing factor is $\xi = 1 + \exp[-\beta(\epsilon - \mu)]$

The probability for finding the state full is

$$\langle n \rangle = 1 / \{1 + \exp[\beta(\epsilon - \mu)]\} \quad \text{vi.2a}$$

The probability for finding the state empty is

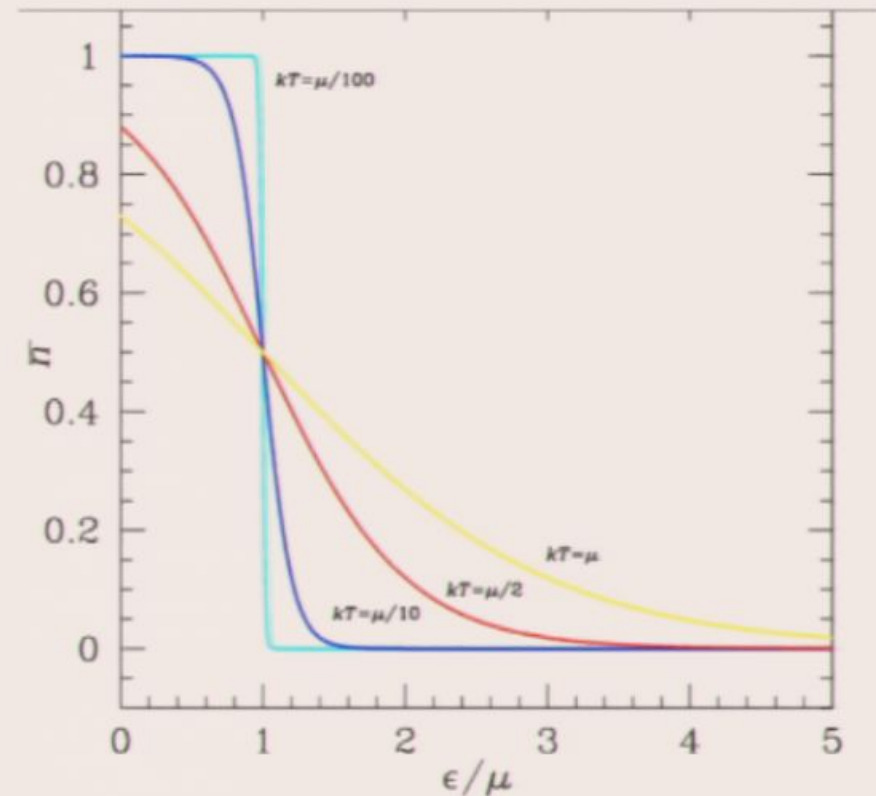
$$1 - \langle n \rangle = 1 / \{1 + \exp[-\beta(\epsilon - \mu)]\} \quad \text{vi.2b}$$



Extreme Limits for fermions

The extreme quantum limit is the one with large values of $\beta \mu = \mu/(kT)$. In that limit the mode is always completely full (empty) depending on whether $(\epsilon - \mu)$ is negative (positive).

The extreme classical limit is the one with large values of $-\beta \mu$. In that limit, in equilibrium, all modes have a very low probability of being occupied and

$$\langle n \rangle \approx \exp[-\beta(\epsilon - \mu)]$$


This picture gives plots of $\langle n \rangle$ versus ϵ/μ for various values of $1/(\beta \mu)$. The large numbers indicate highly degenerate situations, while the smaller ones are closer to the classical limit.

for the boson

the equilibrium probability distribution for occupation of the single mode is $\rho(n) = (1/\xi) \exp[-\beta(\epsilon - \mu)n]$. All integer values of n between zero and infinity are permitted.

The normalizing factor is

$$= 1 + \exp[-\beta(\varepsilon - \mu)] + \exp[-2\beta(\varepsilon - \mu)] + \exp[-3\beta(\varepsilon - \mu)]$$

...

$$\xi = 1 / \{ 1 - \exp[-\beta(\varepsilon - \mu)] \}$$

note that $\varepsilon - \mu$ must be positive.

the average occupation is $\langle n \rangle = 1 / \{ \exp[\beta(\epsilon - \mu)] - 1 \}$ vi.3

In extreme quantum limit is the one with very small positive values of $\beta(\epsilon - \mu)$. In that limit, the mode can have lots and lots of quanta in it. You can even have **macroscopic occupation of a single mode**, in which a finite fraction of the entire number of particles is in a single mode. This is also called **Bose-Einstein** condensation after the discoverers of this effect.

The extreme classical limit is once more a very large value of $-\beta \mu$ and a small average occupation of the state. Once more $\langle n \rangle \approx \exp[-\beta(\epsilon - \mu)]$ in this

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Satyendra Nath Bose

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Satyendra Nath Bose

Independent Excitations: waves

One example of a boson excitation is provided by a set of waves. There are two major examples: light waves and sound waves. In these two cases, the quanta are called respectively photons and phonons. In the simplest situation, the Hamiltonian for the system is a sum over terms corresponding to the different excitations in the system

$$\mathcal{H} = \sum_j \epsilon_j n_j \quad \text{vi.4}$$

and the statistical mechanics is given by the usual formula

$$\rho\{n\} = (1/\Xi) \exp(-\beta H\{n\})$$

where the normalizer, Ξ , is called the grand partition function.

Here, ϵ_j is the energy of a single excitation of type j and n_j is the number of excitations of that type. These quanta have the property that they are not conserved. When the basic objects under consideration are conserved quantities, e.g. atoms or molecules, and they don't interact, the Hamiltonian is of exactly the same form, but it is convenient to use a statistical theory in which we allow the total number of particles to vary, and use a probability function of the form

$$\rho\{n\} = (1/\Xi) \exp(-\beta[H\{n\} - \mu N\{n\}]) \quad \text{where } N \text{ is the total particle number}$$

$$N = \sum_j n_j$$

Here μ is called the **chemical potential**. The density of particles increases as μ increases.

Waves=Special bosons

$\varepsilon = \hbar\omega$, so in the classical limit the energy of a photon goes to zero.

the probability distribution for the single mode is

$$\rho(n) = (1/\xi) \exp[-\beta \varepsilon n]$$

The normalizing factor is

$\xi = 1 + \exp[-\beta \varepsilon] + \exp[-2\beta \varepsilon] + \exp[-3\beta \varepsilon] + \dots$ so that

$$\xi = \frac{1}{1 - \exp[-\beta \varepsilon]}$$

Note that ε must be positive or zero. The average energy in the mode is

$$\langle n \rangle \varepsilon = \varepsilon / \{\exp[\beta \varepsilon] - 1\} = \hbar\omega / \{\exp[\beta \hbar\omega] - 1\}$$

Classical limit = high temperature $\langle n \rangle \varepsilon = 1/\beta = kT$

Therefore classical physics gives kT per mode. A cavity has an infinite number of electromagnetic modes. Therefore, a cavity has infinite energy!!?

In quantum theory high frequency modes are cut off because they must have small average occupations numbers, $\langle n \rangle$. Therefore the classical result of kT per mode is simply wrong. So there is no infinity.

In this way, Planck helped us get the right answer by introducing photons and starting off the talk about occupation numbers!

photons in Cubic Cavity

$\mathbf{k}=(k_x,k_y,k_z)=2\pi(m_x,m_y,m_z)/L$ where the m 's are integers describing the cavity's modes
Here $\omega=ck$ (There are two modes for each frequency)

$$\mathcal{H} = \sum_{\mathbf{m},\sigma} n_{\mathbf{m},\sigma} \hbar ck(\mathbf{m})$$

We can then find the average energy in the form

$$\langle \mathcal{H} \rangle = 2 \sum_{\mathbf{m}} n_{\mathbf{m},\sigma} \hbar ck(\mathbf{m}) \frac{1}{\exp(\beta \hbar ck(\mathbf{m})) - 1}$$

If the box is big enough, the sum over m can be converted into an integral over k .

$$\langle \mathcal{H} \rangle = 2 \left(\frac{L}{2\pi} \right)^3 \int d^3k \hbar ck \frac{1}{\exp(\beta \hbar ck) - 1}$$

The integration variable can then be made dimensionless

$$\langle \mathcal{H} \rangle = \frac{2}{\beta} \left(\frac{L}{2\pi \beta \hbar ck} \right)^3 \int d^3q q \frac{1}{\exp(q) - 1}$$

and the integral rewritten in a form which converges nicely at zero and infinity.

$$\langle \mathcal{H} \rangle = 2(kT)^4 \left(\frac{L}{2\pi \hbar ck} \right)^3 \int_0^\infty dq 4\pi q^3 \frac{1}{\exp(q) - 1}$$



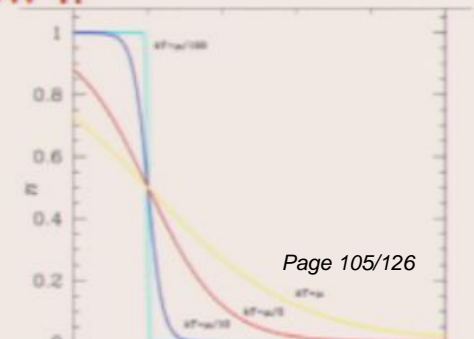
Paul Dirac has a beautiful quantum mechanics book which I enjoyed reading when I was a grad student.

Conserved Fermions in Box

In a metal electrons act as independent quasiparticles with energy and momentum relation ϵ_p . For modes with energy near the value of the chemical potential, these modes behave very much like non-interacting particles with a changed energy-momentum relation. For example they move with a velocity $\mathbf{v} = \nabla_p \epsilon_p$. Only the electrons with energies close to the chemical potential, called in this context the Fermi energy, play an important role in moving heat and particles through the system. The electrons near the Fermi energy are said to be close to the top of the Fermi sea. For lesser energies, within that sea, the electrons are mostly frozen into their momentum states and cannot do much.

For some materials, like Aluminum, $\epsilon_p \approx p^2/(2m)$. For these the Fermi sea forms a ball with radius p_F .

Calculate the $T=0$ energy density, particle density, and pressure of these electrons in terms of p_F . You may use the free particle energy-momentum relation. Harder: Calculate the entropy density as a function of T at low T .



Dirac, Paul A. M. (1926). "Sulla quantizzazione del gas perfetto monoatomico" (in Italian). Rend. Istit. Lomb. 60: 145-9., translated as On the Quantization of the Monoatomic Ideal Gas. doi:arXiv:cond-mat/9912229v1.

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Dirac, Paul A. M. (1926). "On the Theory of Quantum Mechanics". Proceedings of the Royal Society Series A. 112: 661-77. ISTR: 84682.

Conserved Bosons in Box

At low temperatures fluid Helium⁴, that is bosonic helium, undergoes a phase transition into a superfluid state in which it can move without viscosity. This is believed to arise because a finite fraction of the entire number of atoms falls into a single quantum mode, described by a single wave function. The basic theory of how this occurs is due to Nikolay Nikolaevich Bogolyubov.



Nikolay Nikolaevich Bogolyubov

The Einstein-Bose theory of a phase transition in a non-interaction Bose liquid is a pale reflection of the real superfluid transition. However, it is quite interesting both in its own right and also because the recent development of low temperature-low density Bose atomic or molecular gases may make it possible to observe this weak-interaction-effect.

I shall outline the three dimensional theory. The theory in two dimensions is more complex.



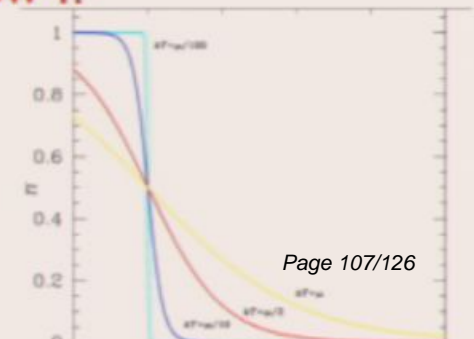
Paul Dirac has a beautiful quantum mechanics book which I enjoyed reading when I was a grad student.

Conserved Fermions in Box

In a metal electrons act as independent quasiparticles with energy and momentum relation $\epsilon_{\mathbf{p}}$. For modes with energy near the value of the chemical potential, these modes behave very much like non-interacting particles with a changed energy-momentum relation. For example they move with a velocity $\mathbf{v} = \nabla_{\mathbf{p}} \epsilon_{\mathbf{p}}$. Only the electrons with energies close to the chemical potential, called in this context the Fermi energy, play an important role in moving heat and particles through the system. The electrons near the Fermi energy are said to be close to the top of the Fermi sea. For lesser energies, within that sea, the electrons are mostly frozen into their momentum states and cannot do much.

For some materials, like Aluminum, $\epsilon_{\mathbf{p}} \approx p^2/(2m)$. For these the Fermi sea forms a ball with radius p_F .

Calculate the $T=0$ energy density, particle density, and pressure of these electrons in terms of p_F . You may use the free particle energy-momentum relation. Harder: Calculate the entropy density as a function of T at low T .



Dirac, Paul A. M. (1926). "Sulla quantizzazione del gas perfetto monoatomico" (in Italian). Rend. Semin. Mat. Univ. Padova 27: 339-427, translated as On the Quantization of the Monoatomic Ideal Gas. doi:10.1016/S0370-1545(26)80122-9.

Pirsa: 10100026

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Conserved Bosons in Box

At low temperatures fluid Helium⁴, that is bosonic helium, undergoes a phase transition into a superfluid state in which it can move without viscosity. This is believed to arise because a finite fraction of the entire number of atoms falls into a single quantum mode, described by a single wave function. The basic theory of how this occurs is due to Nikolay Nikolaevich Bogolyubov.



Nikolay Nikolaevich Bogolyubov

The Einstein-Bose theory of a phase transition in a non-interaction Bose liquid is a pale reflection of the real superfluid transition. However, it is quite interesting both in its own right and also because the recent development of low temperature-low density Bose atomic or molecular gases may make it possible to observe this weak-interaction-effect.

I shall outline the three dimensional theory. The theory in two dimensions is more complex.

Bose Transition

$$n = \text{number of particles per unit volume} = \frac{1}{L^3} \sum_{\mathbf{m}} \frac{1}{1 + \exp\{\beta[\varepsilon(\mathbf{m}) - \mu]\}}$$

Here the sum is over a vector of integers of length three, and the energy is $\varepsilon(\mathbf{m}) = \mathbf{m}^2 \hbar^2 / (2ML^2)$, M being the mass of the particle. For a sufficiently large box, there are two qualitatively different contributions to the sum. The term in which $\mathbf{m} = \mathbf{0}$ can be arbitrarily large because μ can be arbitrarily small. The remaining terms contribute to an integral which remains bounded as μ goes to zero. The result is

$$n = \frac{1}{-L^3 \beta \mu} + \int \frac{d\mathbf{p}}{h^3} \frac{1}{1 + \exp[\beta \frac{p^2}{2M} - \beta \mu]}$$

The integration has a result that goes to zero as T^3 as the temperature goes to zero. If this system is to maintain a non-zero density as T goes to zero, which we believe it can, it can only do so by having the first term on the right become large enough so that a finite proportion of the entire number of particles in the system will fall into the lowest mode. This is believed to be the basic source of superfluidity.

Dynamics of fermions at low temperature

Landau described fermions at low temperature by saying that they had a free energy which depended upon, $f(\mathbf{p}, \mathbf{r}, t)$ the occupations of the fermion modes with momentum in the neighborhood of \mathbf{p} and position in the neighborhood of \mathbf{r} at time t . As the occupations changed the free energy would change by

$$\delta F = \int \frac{d\mathbf{p} d\mathbf{r}}{h^3} \epsilon(\mathbf{p}, \mathbf{r}, t) \delta f(\mathbf{p}, \mathbf{r}, t)$$

Then, using the usual Poisson bracket dynamics the distribution function would obey, as in equation v.13.

$$\partial_t f(\mathbf{p}, \mathbf{r}, t) + (\nabla_{\mathbf{p}} \epsilon(\mathbf{p}, \mathbf{r}, t)) \cdot \nabla_{\mathbf{r}} f(\mathbf{p}, \mathbf{r}, t) - (\nabla_{\mathbf{r}} \epsilon(\mathbf{p}, \mathbf{r}, t)) \cdot \nabla_{\mathbf{p}} f(\mathbf{p}, \mathbf{r}, t) \\ = \text{collision term}$$

The collision term will be the same as in the classical Boltzmann equation with one important difference: Since fermions cannot enter an occupied state, the probabilities of entering a final state will be multiplied by a factor of $(1-f)$. Thus, Landau proposed a "Boltzmann equation" for degenerate fermions of the form below, **with the new terms in red**

$$[\partial_t + (\nabla_{\mathbf{p}} \epsilon) \cdot \nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}} \epsilon) \cdot \nabla_{\mathbf{p}}] f(\mathbf{p}) =$$

$$- \iiint d\mathbf{q} \, d\mathbf{p}' \, d\mathbf{q}' \, \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \, \delta(\epsilon(\mathbf{p}) + \epsilon(\mathbf{q}) - \epsilon(\mathbf{p}') - \epsilon(\mathbf{q}'))$$

Dynamics of bosons

Some part of the story of bosons is much the same. A low temperature conserved boson system could be expected to obey the same sort of equation, under circumstances in which the bosons were conserved, and also the emission and absorption of phonons were not too significant.

Specifically, the equation would look like

$$[\partial_t + (\nabla_{\mathbf{p}} \varepsilon) \cdot \nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}} \varepsilon) \cdot \nabla_{\mathbf{p}}] f(\mathbf{p}) =$$

$$- \iiint d\mathbf{q} d\mathbf{p}' d\mathbf{q}' \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \delta(\varepsilon(\mathbf{p}) + \varepsilon(\mathbf{q}) - \varepsilon(\mathbf{p}') - \varepsilon(\mathbf{q}'))$$

$$Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q})(1 + f(\mathbf{p}')) (1 + f(\mathbf{q}')) - f(\mathbf{p}') f(\mathbf{q}') (1 + f(\mathbf{p})) (1 + f(\mathbf{q}))] \quad \text{vi.9}$$

Once again the new feature is shown in red. In the scattering events there are, for bosons, **more** scattering when the final single particle states are occupied than when they are empty. One says that fermions are unfriendly but bosons are gregarious (or at least attractive to their own tribe.). The f in the $1+f$ term was known in the 19th century in terms of the simulated emission of light, which is a kind of boson. The 20th Century brought **Planck**, and particularly **Einstein**, who first saw the need for the "1" in the $1+f$ term. This extra piece was introduced to make the bose dynamical equation have the right local equilibrium behavior. The logic used by Einstein includes the fact that for local equilibrium via equation vi.9, we must have $f/(1+f)$ be, as in the fermion case, an exponential in conserved

References

Daniel Kleppner., “Rereading Einstein on Radiation”, *Physics Today*, (February 2005).
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$$\begin{aligned}
 & [\partial_t + (\nabla_{\mathbf{p}} \varepsilon) \cdot \nabla_{\mathbf{r}} - (\nabla_{\mathbf{r}} \varepsilon) \cdot \nabla_{\mathbf{p}}] f(\mathbf{p}) = \\
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 & Q(\mathbf{p}, \mathbf{q} \rightarrow \mathbf{p}', \mathbf{q}') [f(\mathbf{p}) f(\mathbf{q})(1 + f(\mathbf{p}')) (1 + f(\mathbf{q}')) - f(\mathbf{p}') f(\mathbf{q}') (1 + f(\mathbf{p})) (1 + f(\mathbf{q}))] \quad \text{vi.9}
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$$f(p) = \frac{1}{O(\epsilon - \mu) - 1}$$

$$- \int_{p+\epsilon_q}^{p+\epsilon_q - \epsilon_{p'} - \epsilon_{q'}} \frac{dr'}{dr} dg dg R(p, q \rightarrow r, q') \partial(r+p-h)$$

$$[f(p)f(q) - f(r')f(q')]$$

$$\frac{p+q}{2} = \frac{p'+q'}{2}$$

$$+ \frac{p}{m}$$

$$f(p) = \frac{1}{\Omega^{\beta(\epsilon - \mu)} - 1}$$

$$= - \int \frac{d\mathbf{r}}{V} \int d\mathbf{r}' d\mathbf{q} d\mathbf{q}' R(\mathbf{r}, \mathbf{q} \rightarrow \mathbf{r}', \mathbf{q}') \delta(\mathbf{r} + \mathbf{p} - \mathbf{r}' - \mathbf{q}') \\ \times \left[f(\epsilon_{\mathbf{r}} + \epsilon_{\mathbf{q}} - \epsilon_{\mathbf{r}'} - \epsilon_{\mathbf{q}'}) [f(\mathbf{p}) f(\mathbf{q}) - f(\mathbf{r}') f(\mathbf{q}')] \right]$$

$$\frac{\mathbf{p} + \mathbf{q}}{2} = \frac{\mathbf{r}' + \mathbf{q}'}{2}$$

$$\frac{\partial}{\partial \mathbf{r}} f(\mathbf{r}, \mathbf{A})$$

$$f(\mathbf{r}, \mathbf{A}) = g(\mathbf{r}, \mathbf{A})$$

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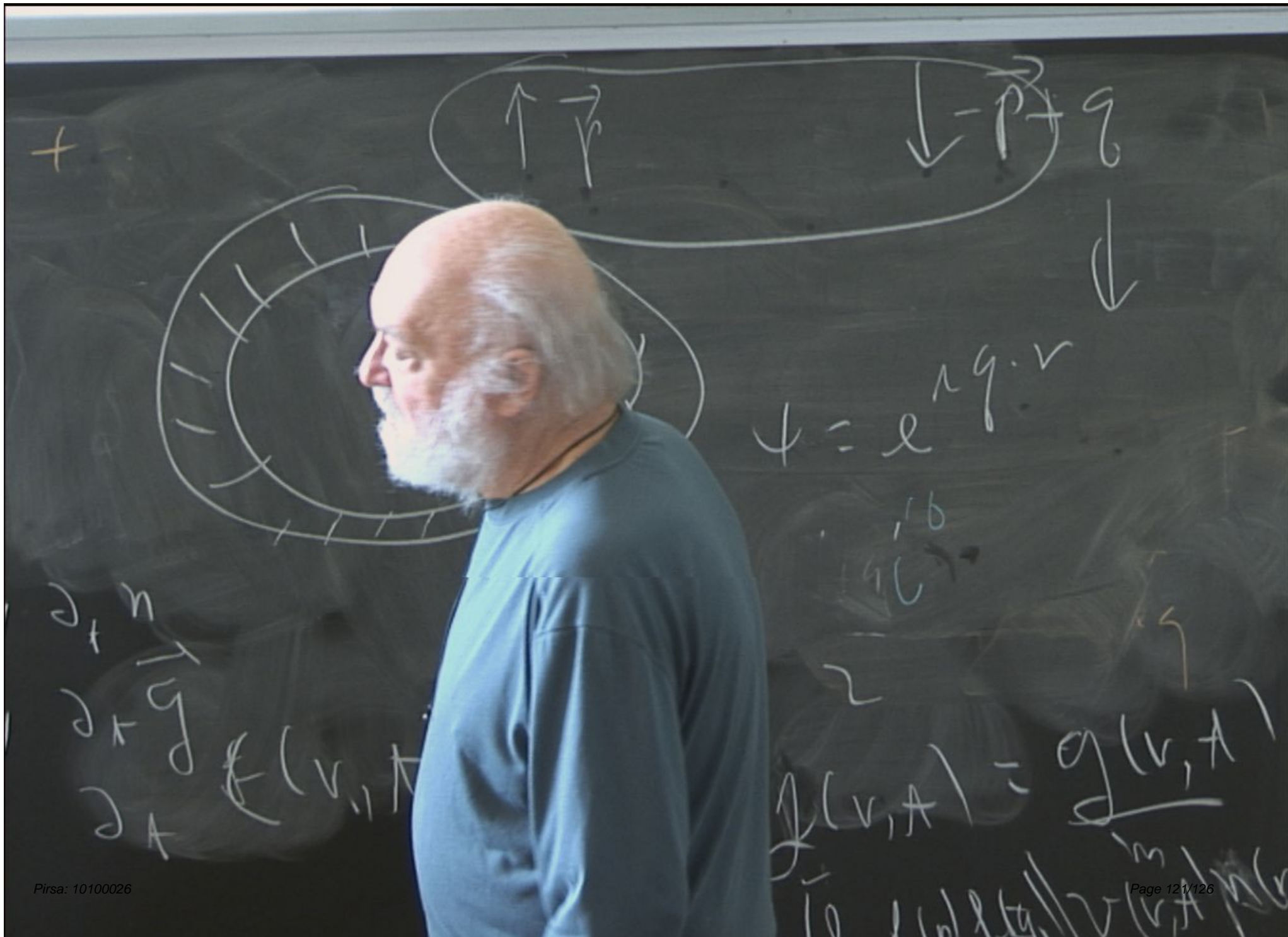
$$\uparrow \vec{p} \quad \downarrow -\vec{p}$$

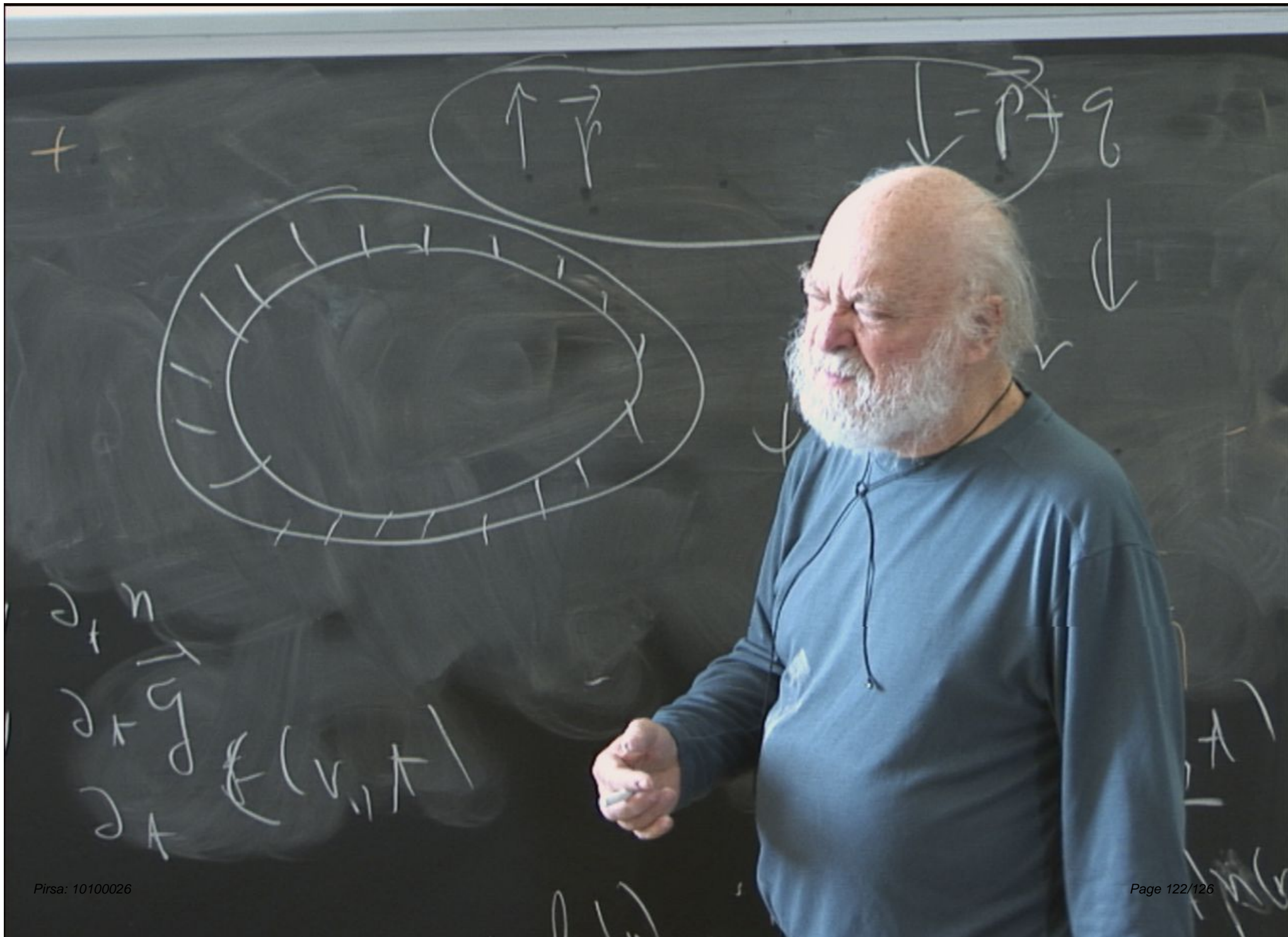
$$e_p + e_q (-e_{p'} - e_{q'}) \{ f(p) f(q) - f(p') f(q') \}$$

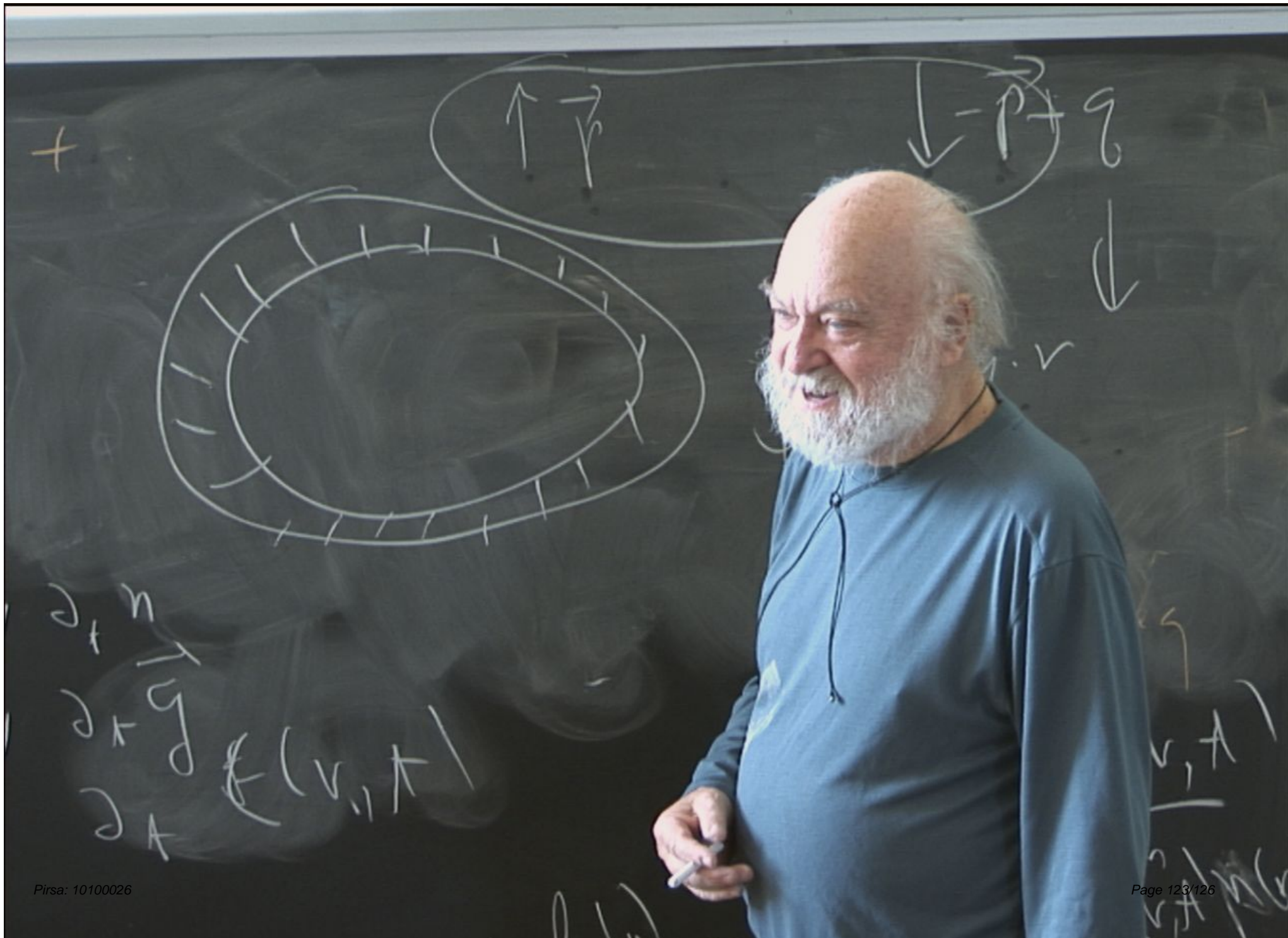
$$\frac{p+q}{2} = \frac{p'+q'}{2}$$

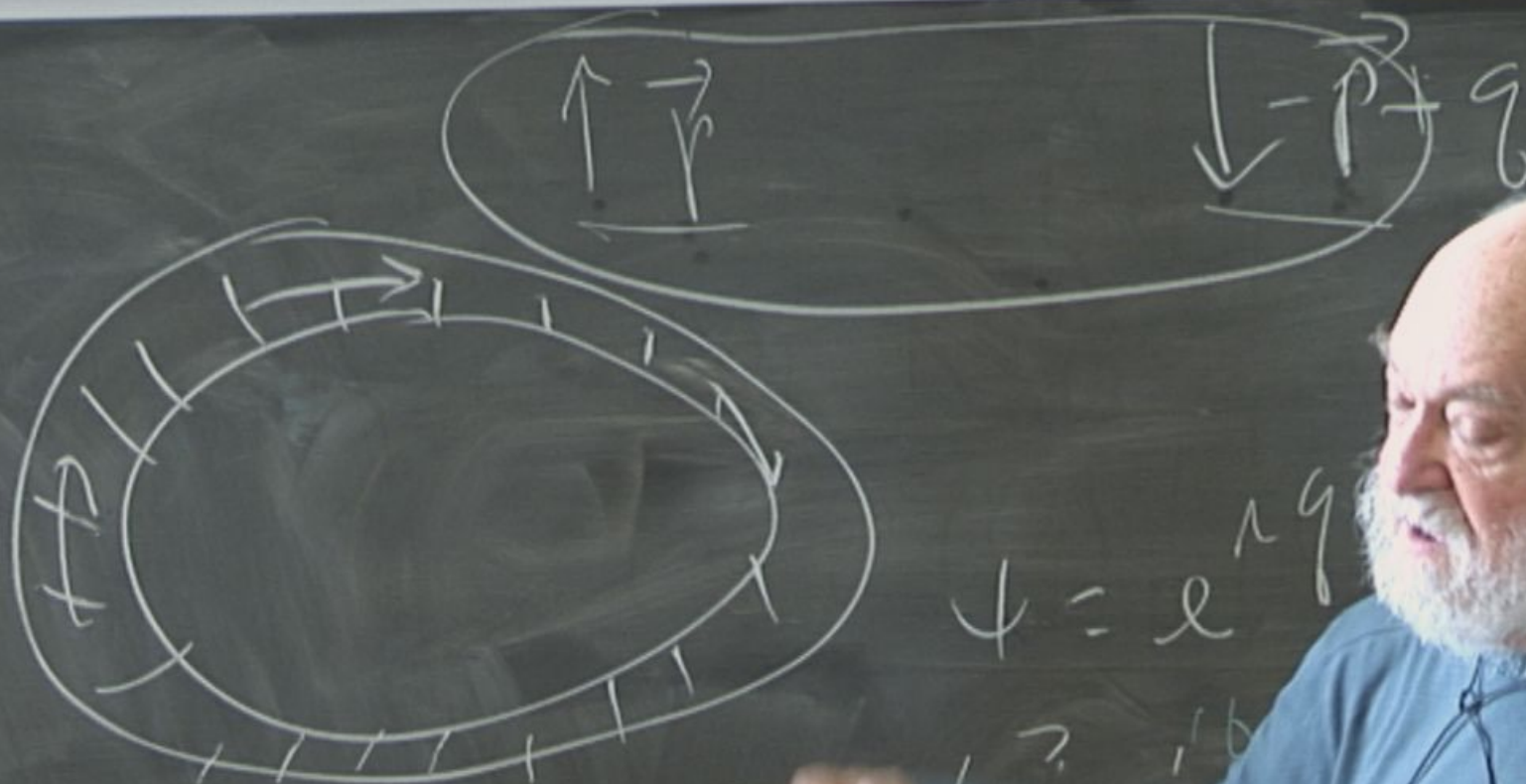
$$f(v, A)$$

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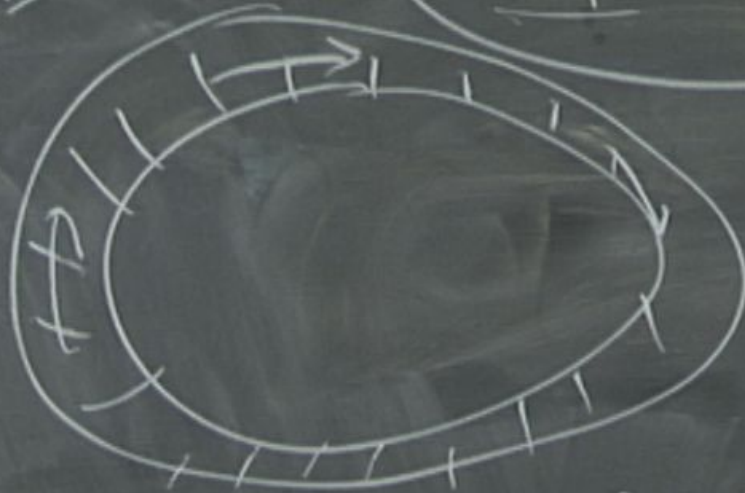








A, B



$x = h$

$$\psi = e^{i g \cdot v}$$

$$x = y$$

$$x > y \quad \partial_t n$$

$$1 \text{ year} = 10^7 \text{ sec}$$

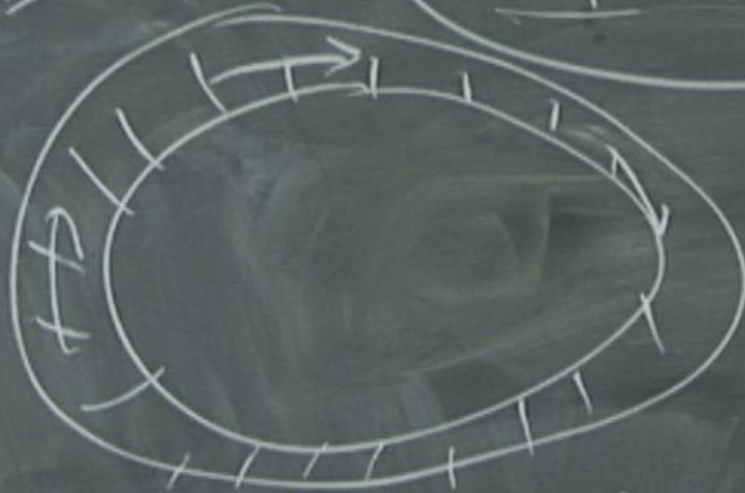
$$10^{-15} \text{ sec}$$

$$x < y \quad \partial_t g \quad f(v, A)$$

$$f(v, A)$$

$$\ln f(v) \rightarrow \frac{1}{2} (\ln f(v) + \ln f(v))$$

A, B



$x = h$

$$\psi = e^{i g \cdot v}$$

$x = y$

$x > y$ at n

1 year = 10^7 ¹⁰ ₄₀

10^{-15} sec

$x < y$ at g $f(v, t)$

$$f(v, t) = g(v, t)$$

$$\ln f(v) \rightarrow \frac{1}{2} [\ln f(v) / f(v)] / v(v, t) / p(v, t)$$