Title: Statistical Mechanics (PHYS 602) - Lecture 4

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Abstract:

Renormalization for d-2 Ising model

Ben Widom, myself, Kenneth Wilson.

$Z=\mathrm{Trace}_{\{\sigma\}}\exp(W_K\{\sigma\})$

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$\Sigma_\mu\, g(\mu,\{s\})=1$. For example, μ might be efined to be an Ising variable with the ame sign as the sum of σ's in its box. low we are ready. Define

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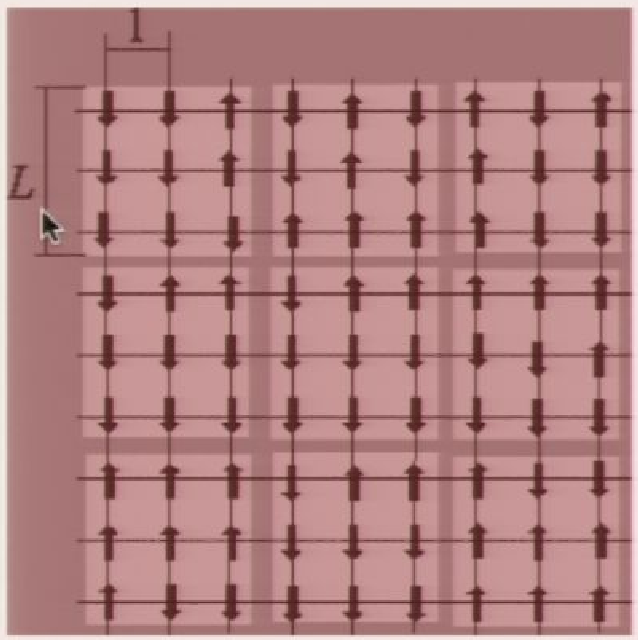

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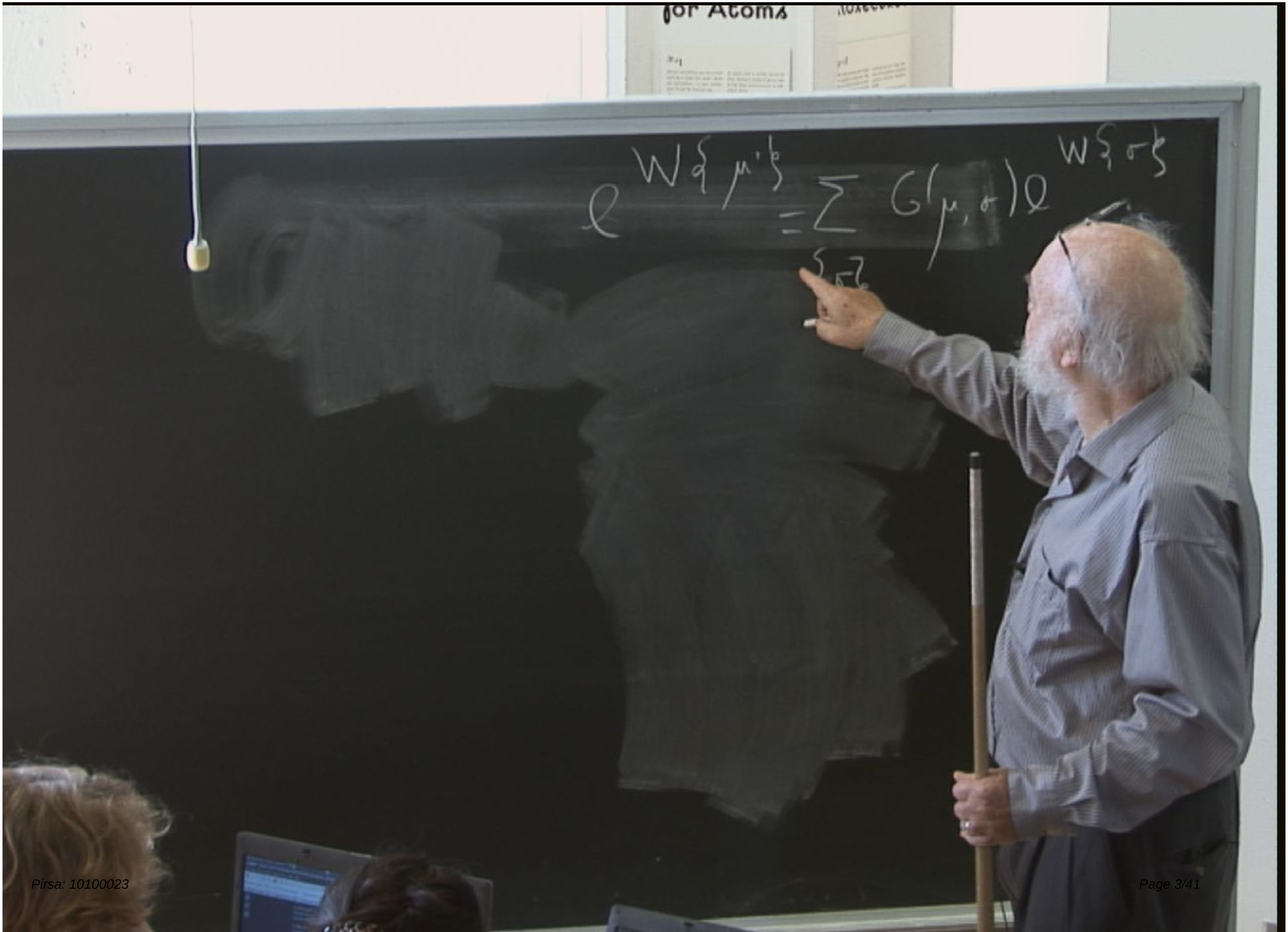
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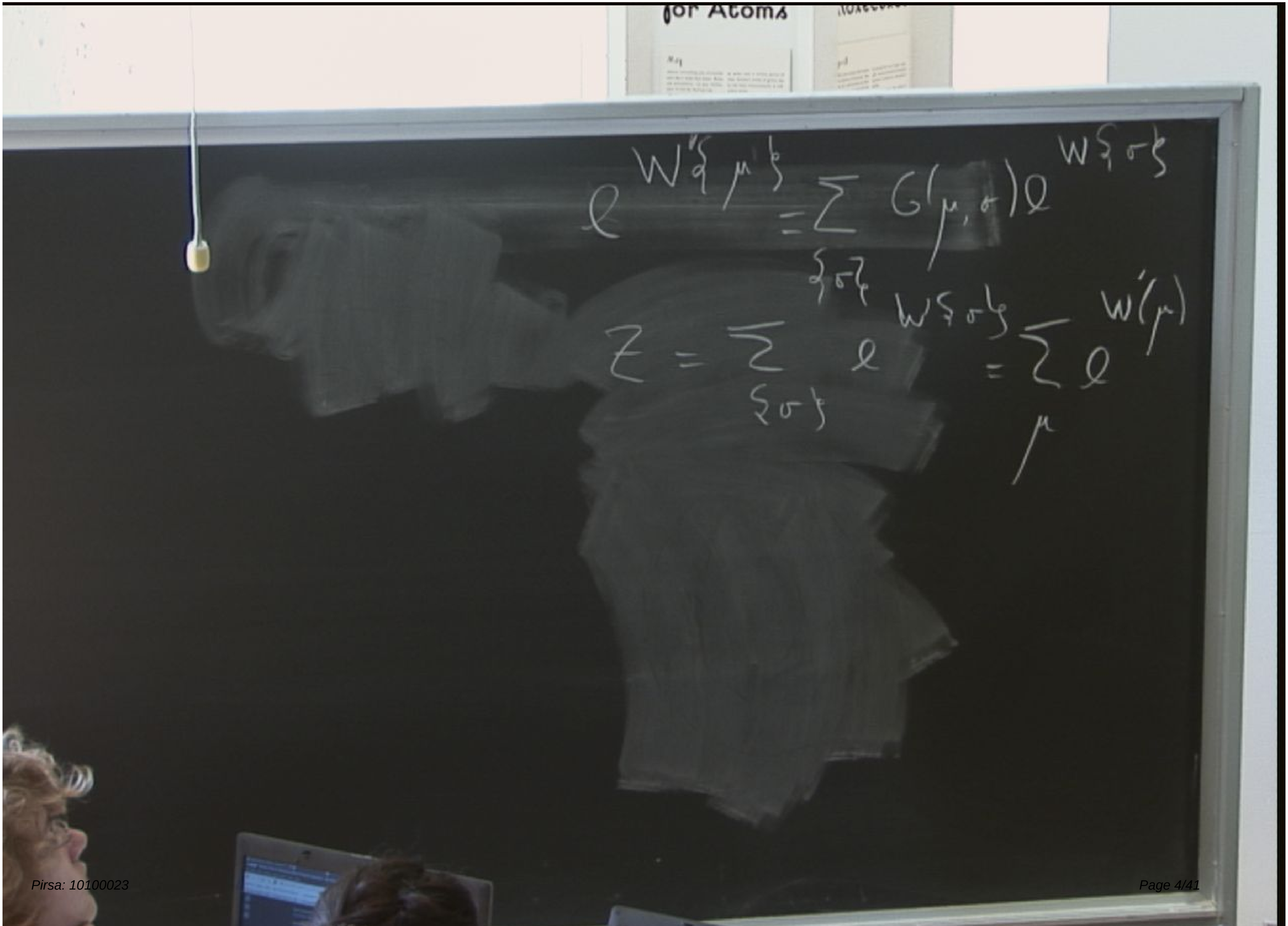
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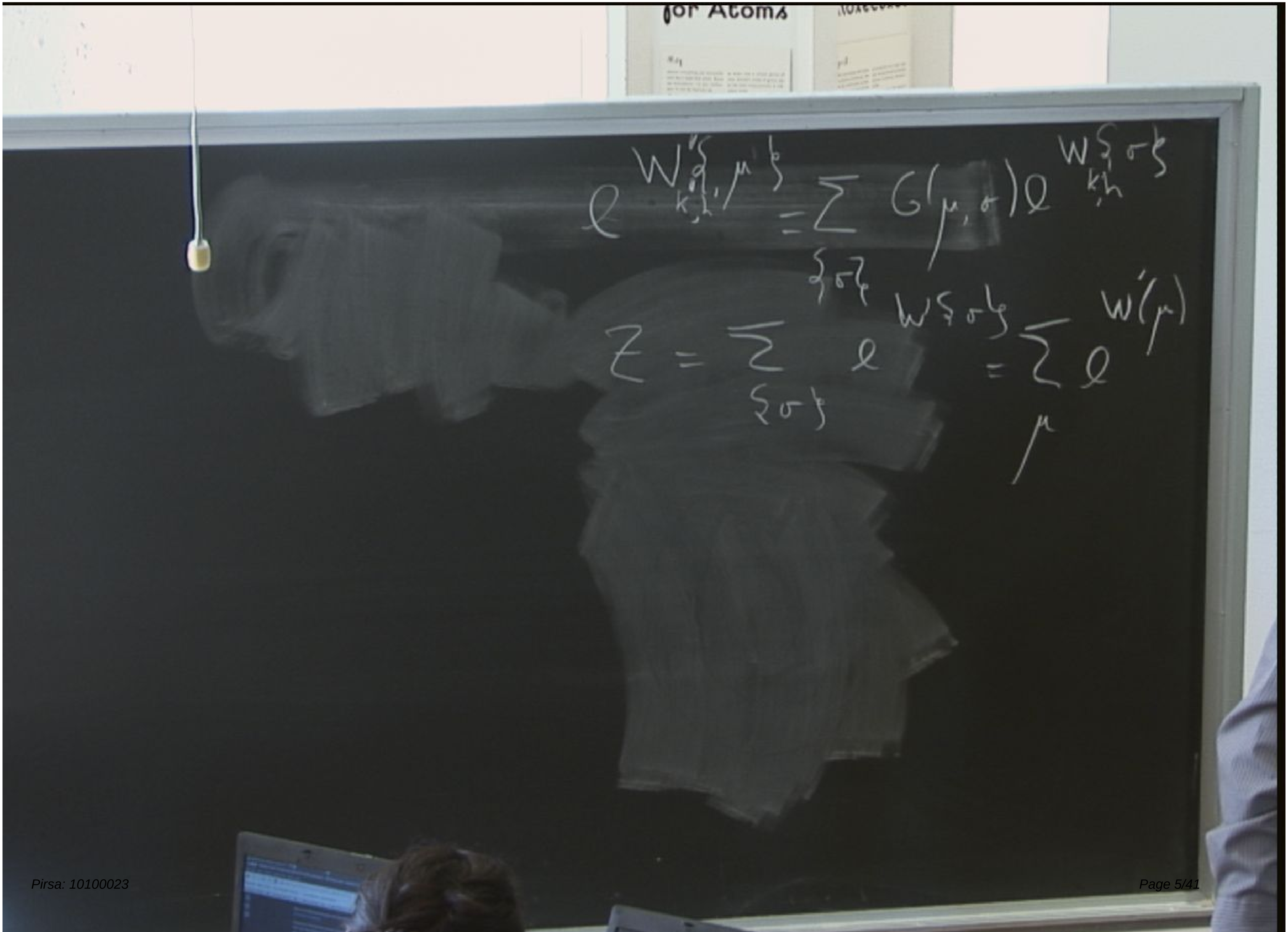
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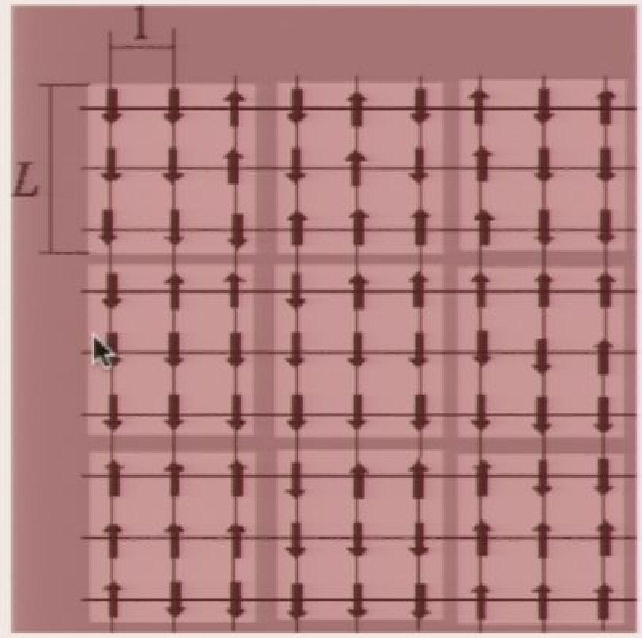

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Renormalization: $a \dashrightarrow 3a = a'$ $\quad W_K\{\sigma\} \dashrightarrow W_{K'}\{\mu\}$ $\quad Z' = Z$ $\quad K' = R(K)$

Scale Invariance at the critical point: --> $K_c = R(K_c)$

Temperature Deviation: $K = K_c - t$ $\quad K' = K_c - t'$
if $t=0$ then $t'=0$
ordered region ($t<0$) goes into ordered region ($t'<0$)
disordered region goes into disordered region
if t is small, $t' = bt$. $\quad b = (a'/a)^y$ defines y. b can be found through a numerical calculation.

coherence length: $\xi = \xi_0\, a\, t^{-\nu}$ 2d Ising has $\nu=1$; 3d has $\nu \approx 0.64$....
$\xi = \xi'$ $\qquad \xi_0\, a\, t^{-\nu} = \xi_0\, a'\, (t')^{-\nu}$
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Specific heat: $C = d^2F/dt^2 \sim t^{dy-2}$ form of singularity determined by y

One can do many more roughly analogous calculations and compare with experiment and numerical simulation. Everything works!

However notice that this is not a complete theory. It is a *phenomenological* theory. We have no way to find y from theory.

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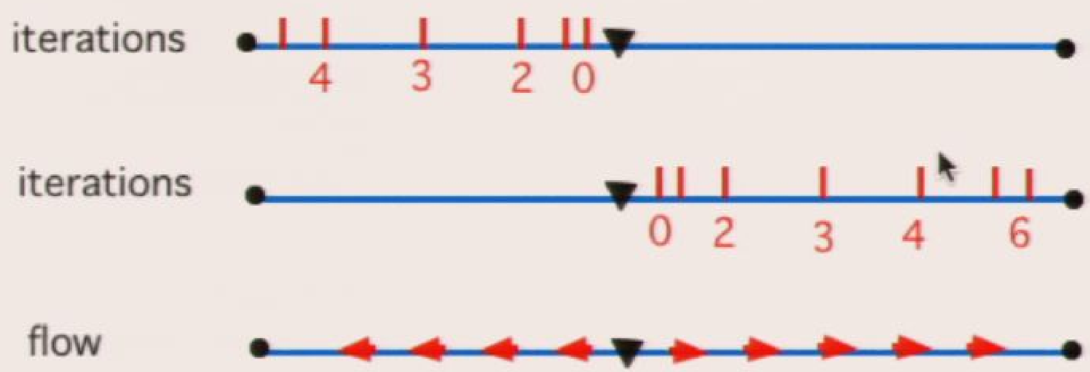

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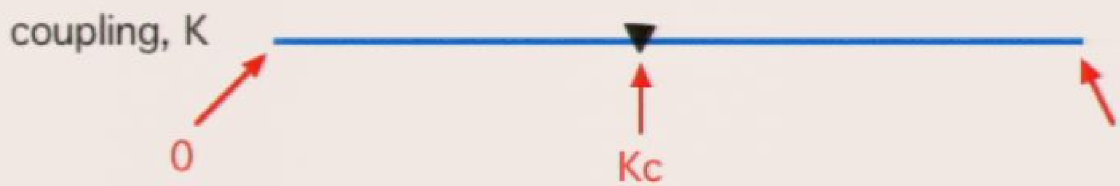

Homework:

Add a term in $\Sigma_j (h\ \sigma_j)$ to the weighting function, W, for the one dimensional Ising Hamiltonian. Find the value of the average spin in the presence of a small magnetic field h. Define the magnetic susceptibility as the derivative of the magnetization with respect to h at fixed K. Show that this susceptibility diverges as K goes to infinity. Shows that it is proportional to a sum of fluctuations in the magnetization.

The three-state Potts model is just like the Ising model except that its "spin" variable σ_j can take on three values $= -1,0,1$. It has $w(\sigma_j, \sigma_{j+1}) = K$ if the two variables are the same and zero otherwise. Find the partition function and coherence length of the one dimensional model. How does the renormalization work for this case.

What is the critical temperature of the three-state Potts model on the square lattice in two dimensions?

Other Variables

Ising variable takes on 2 values coupling between variables depends on whether they are the same or different. Z_2. Ferromaqgnetic, antiferromagnetic coupling, even mixed. Phase transition in dimensions greater than one.

q-state Potts model, variable takes on q-values, simplectic coupling.

interaction ***s.s'*** ***s*** is a q-component vector. q=2 XY equivalent to U1 q=3 called (classical) heisenberg model, higher q's as well

SU_2 and any symmetry group you can think of
phase transitions explored in all dimensions

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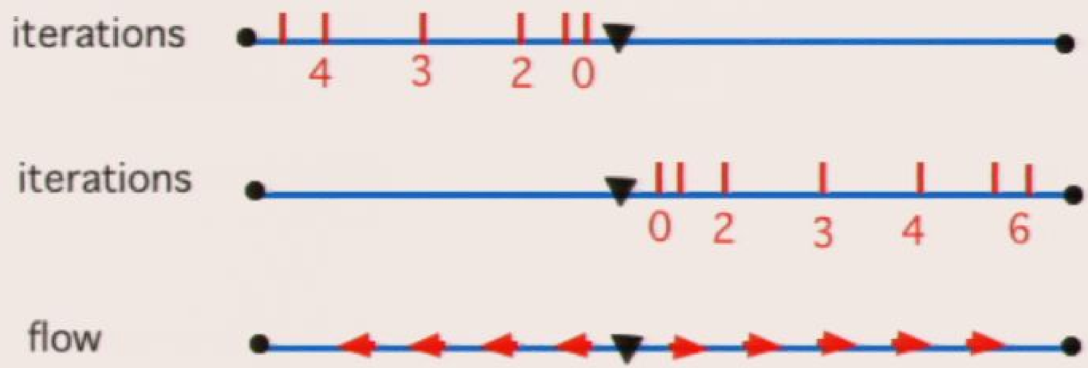

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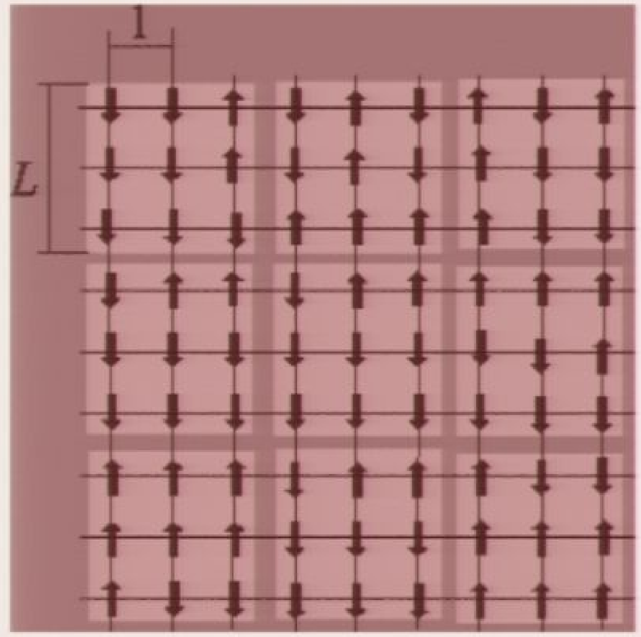

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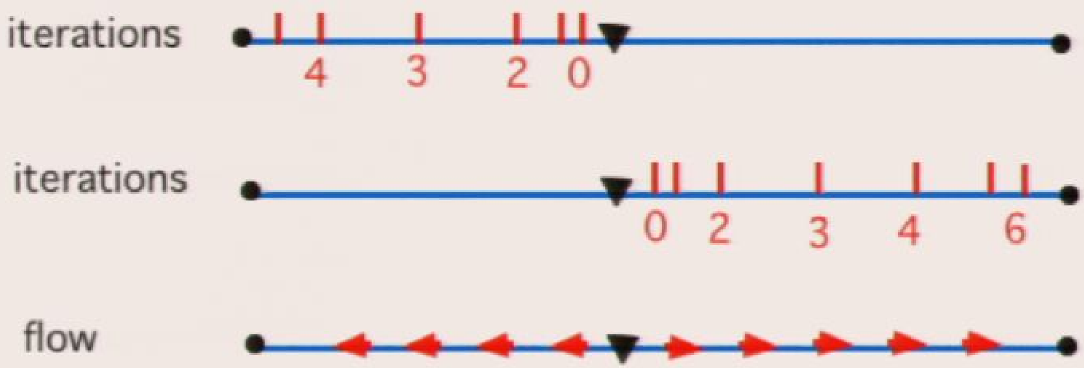

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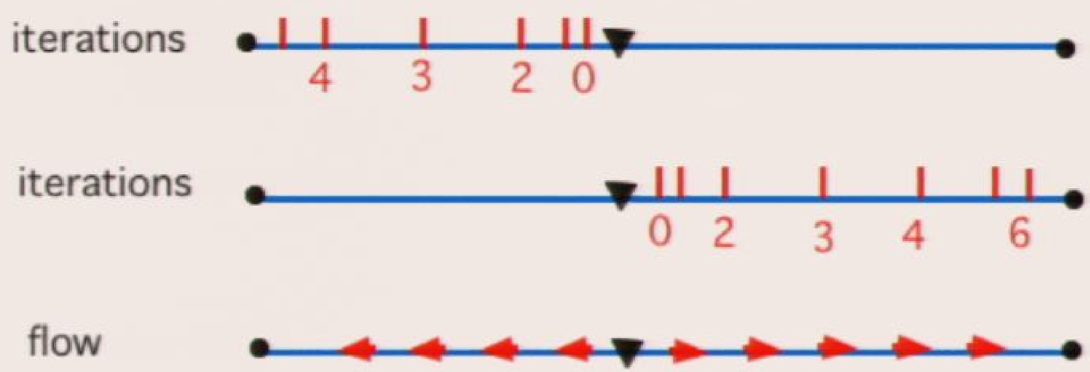

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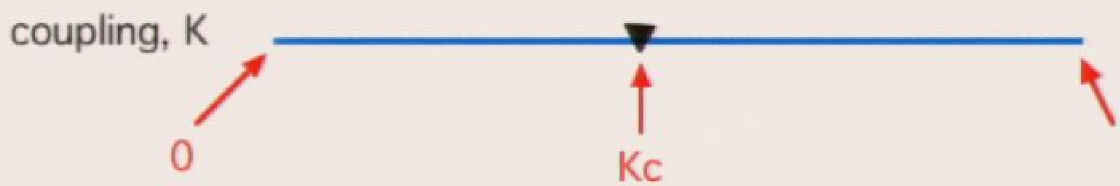

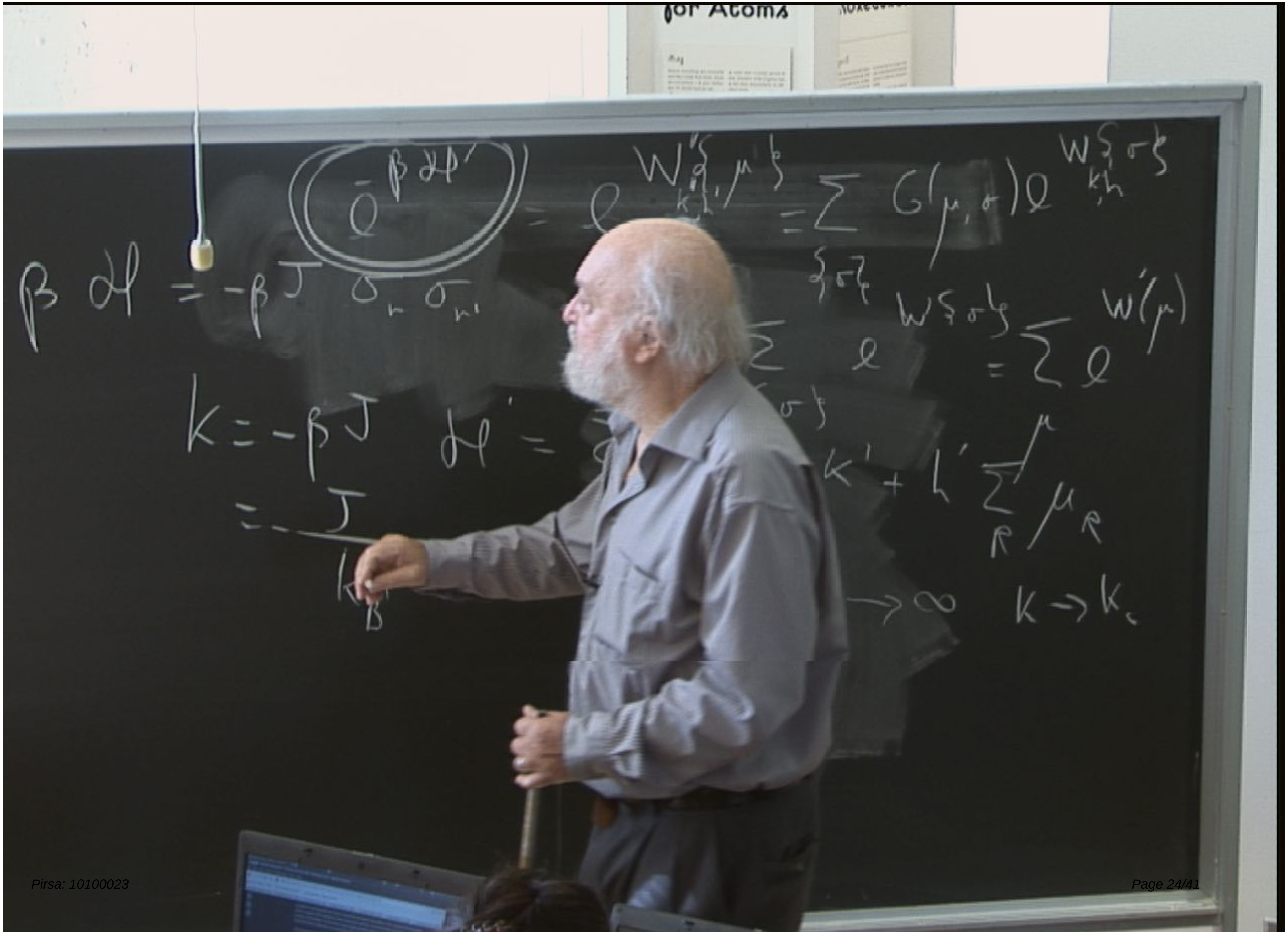
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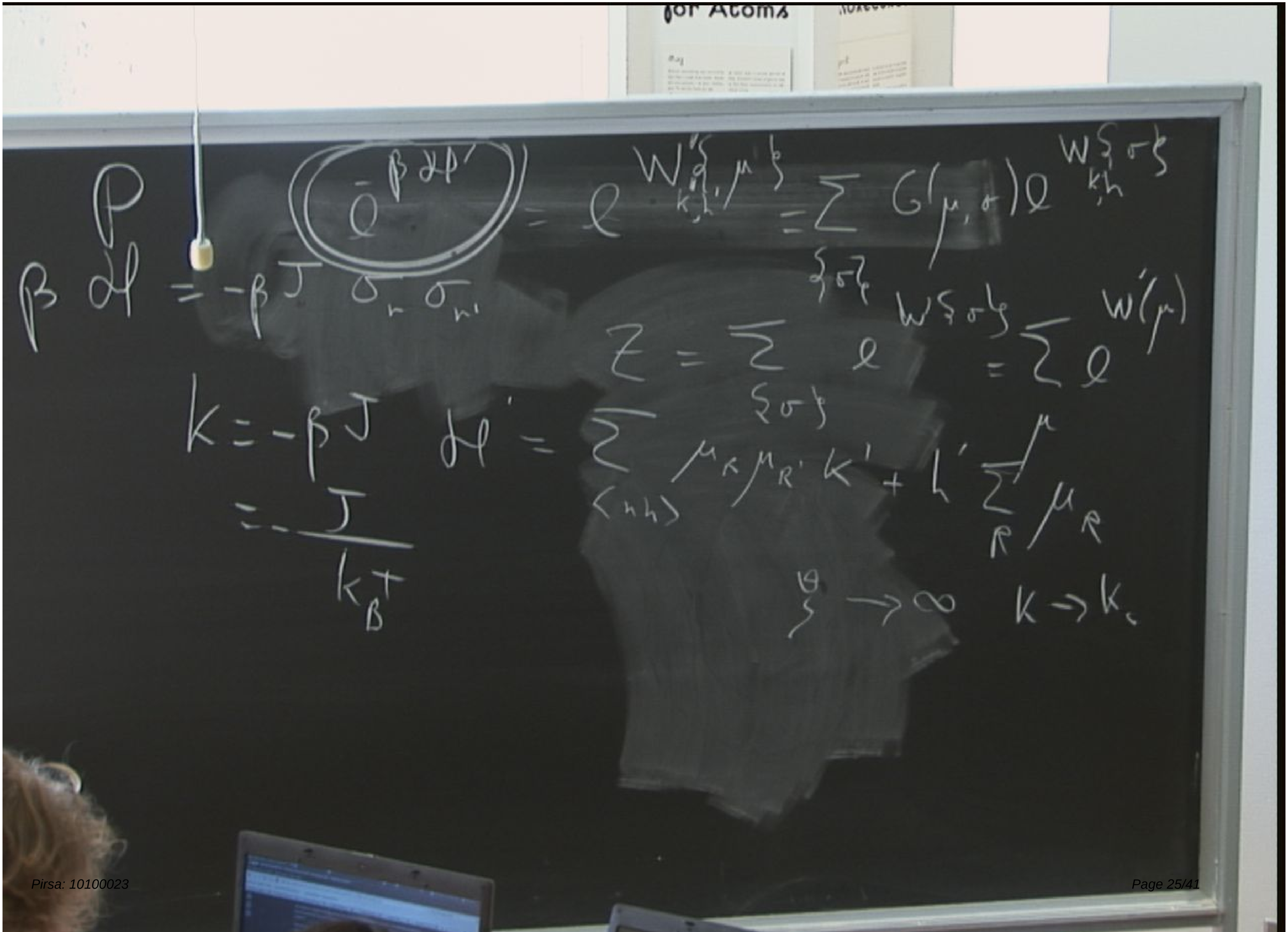
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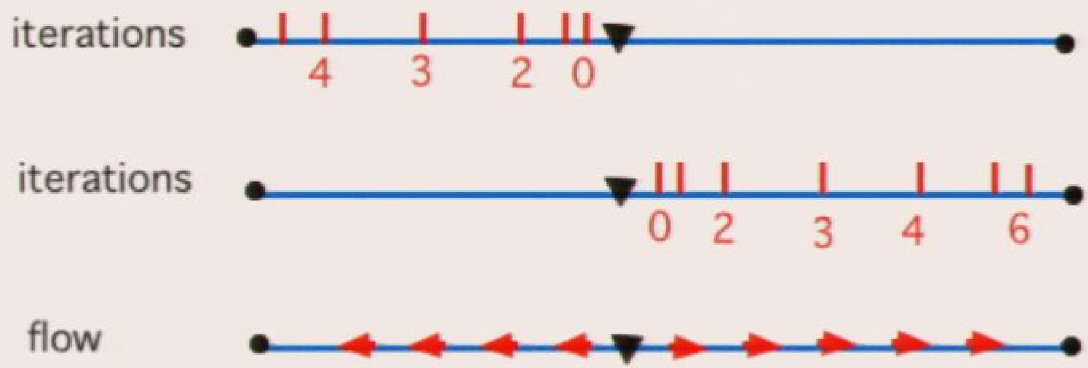

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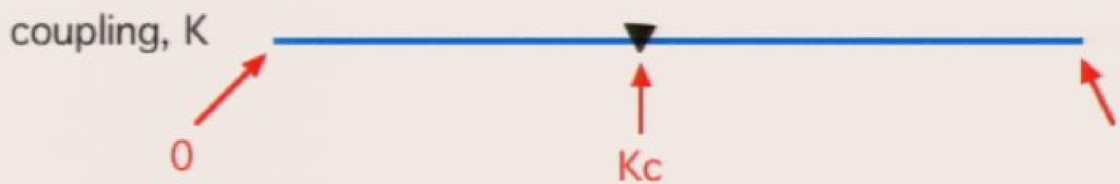

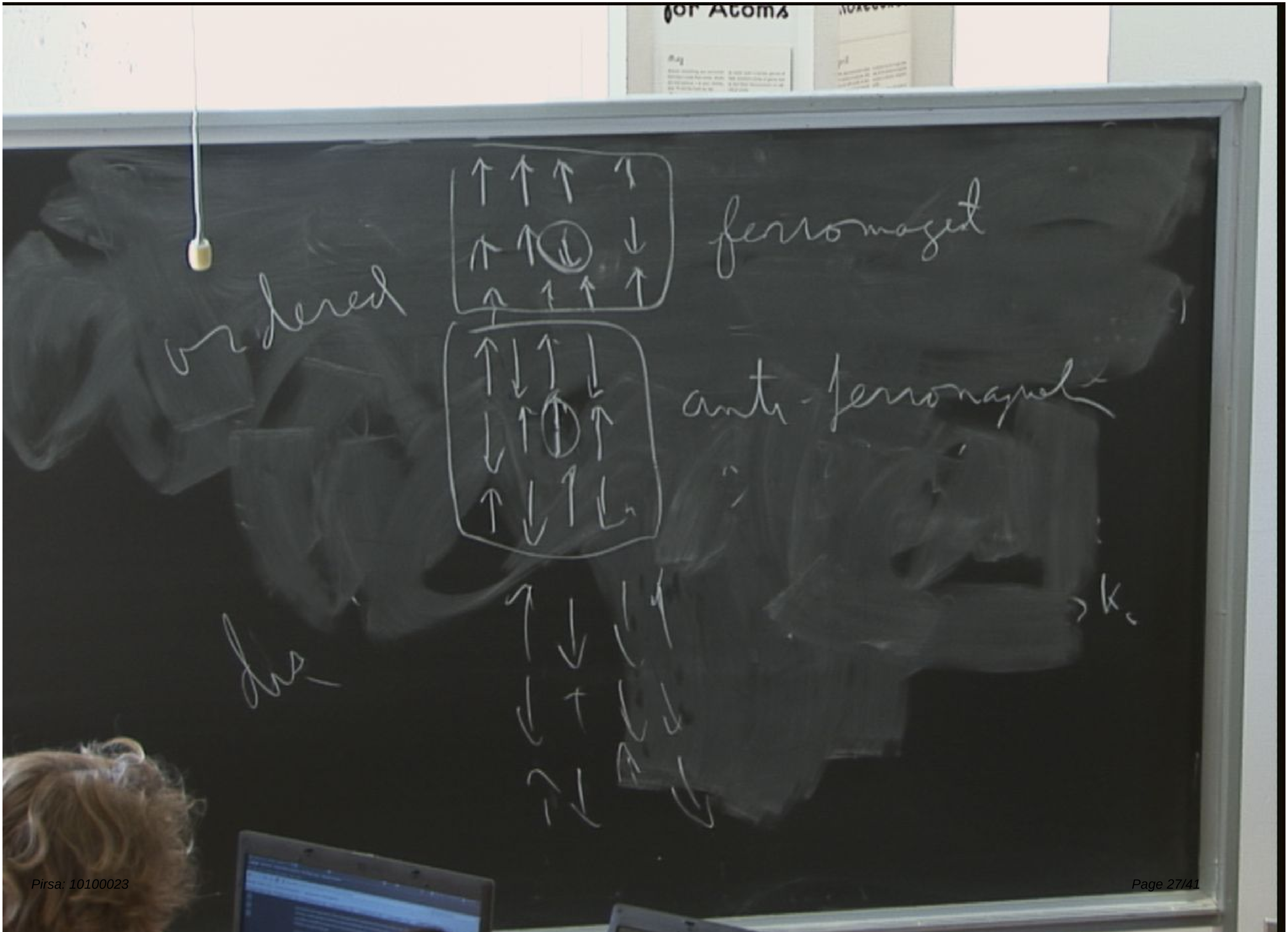
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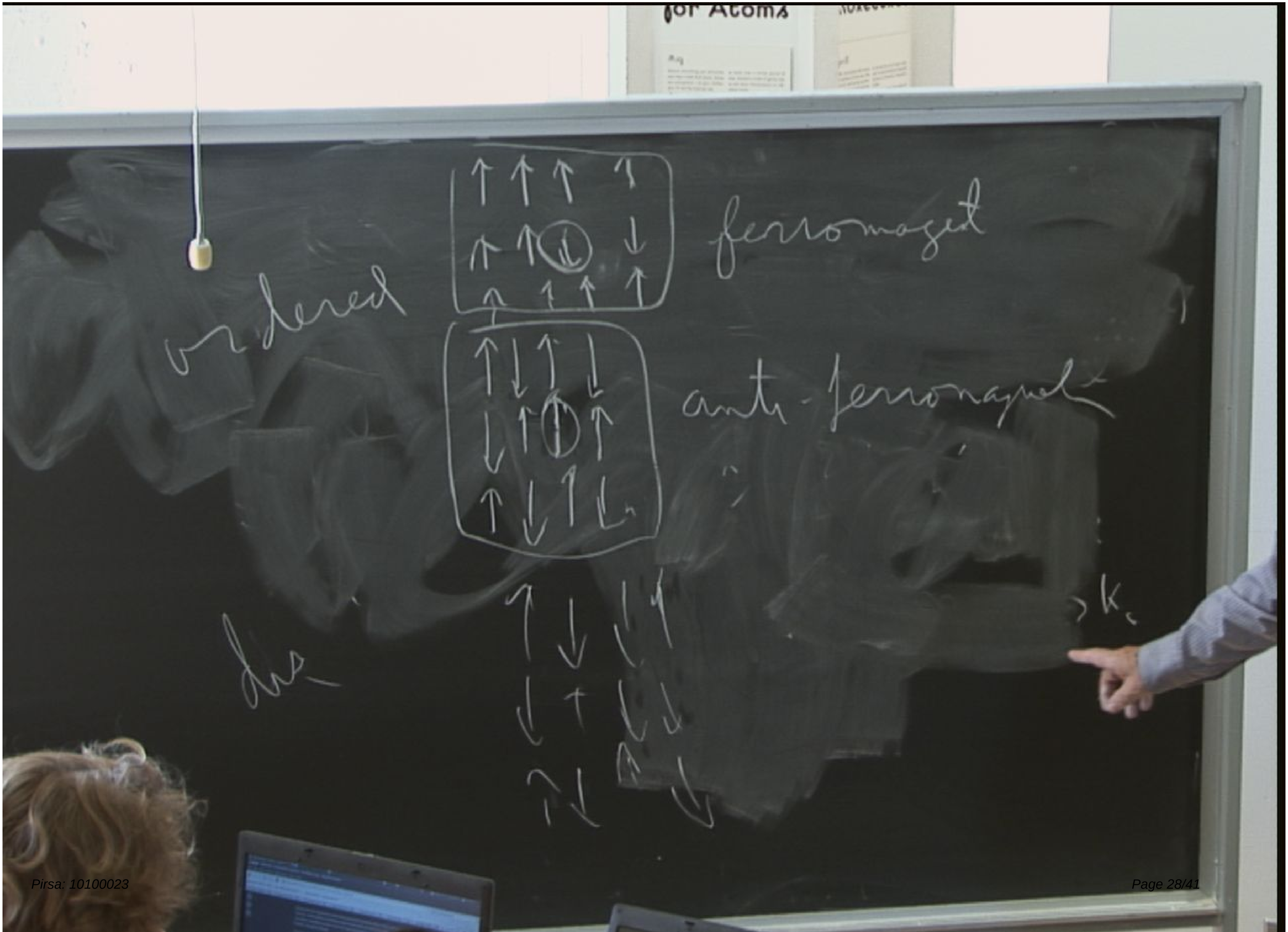
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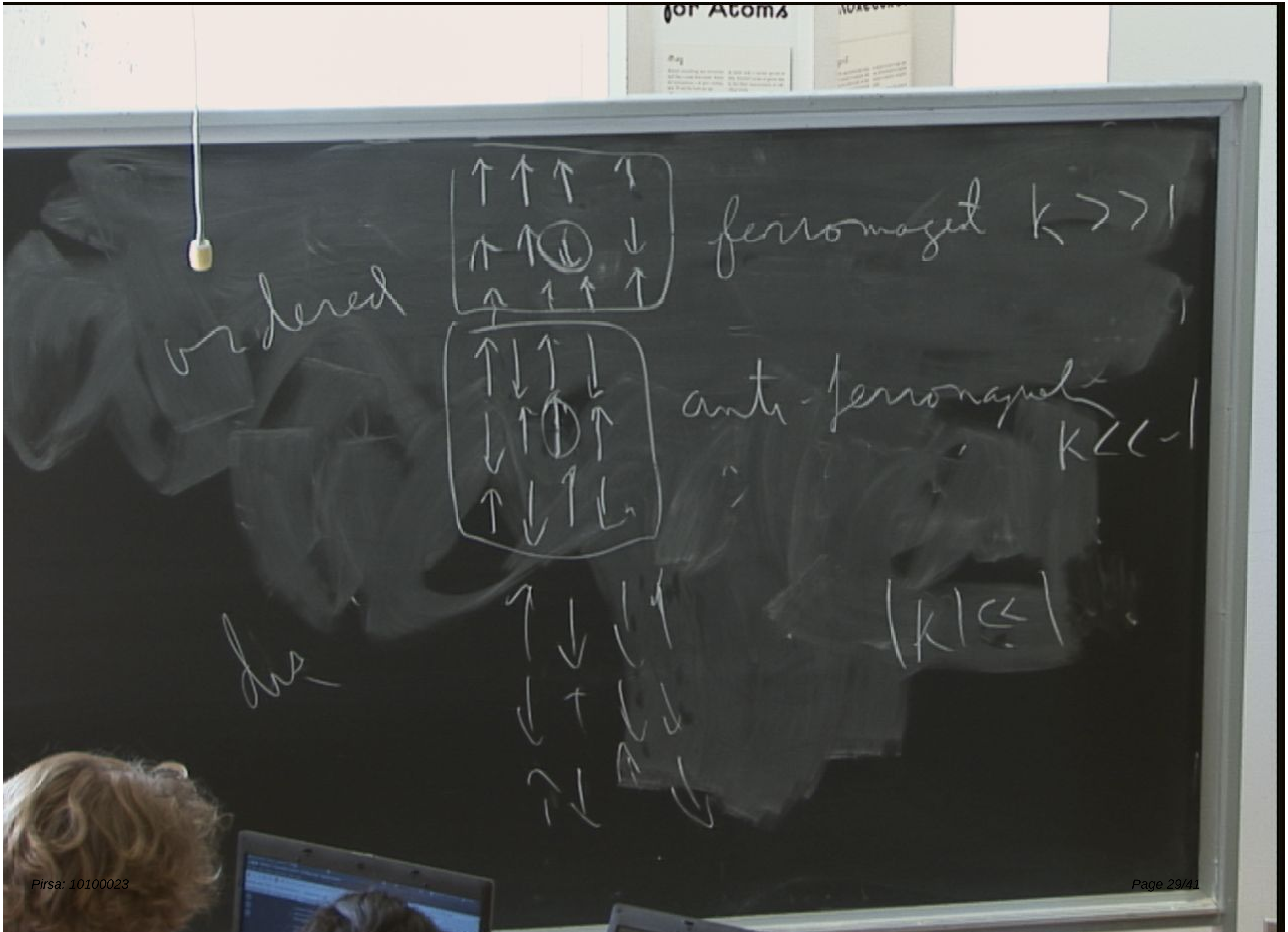
for Atoms
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anti-ferromagnet
k << -1
|k| ≤ 1

Hopping: From Discrete to Continuous

We are going to be spending some time talking about the physics of a particle moving in a solid. Often this motion occurs as a set of discrete hops. The particle gets stuck someplace, sits for a while, acquires some energy from around it, hops free, gets caught in some trap, and then sits for a while. I'm going to describe two mathematical idealizations of this motion: discrete hopping on a lattice and continuous random motion.

One point is to see the difference between the two different topologies represented by a continuous and a discrete system. One often approximates one by the other and lots of modern physics and math is devoted to figuring out what is gained and lost by going up and back.

There is a fine tradition to this. Boltzmann, one of the inventors of statistical mechanics, liked to do discrete calculations. So he often represented things which are quite continuous, like the energy of a classical particle by discrete approximations,

A little later, Planck and Einstein had to figure out the quantum theory of radiation, which had been thought to be continuous, in terms of discrete photons. So we shall compare continuous and discrete theories of hopping.

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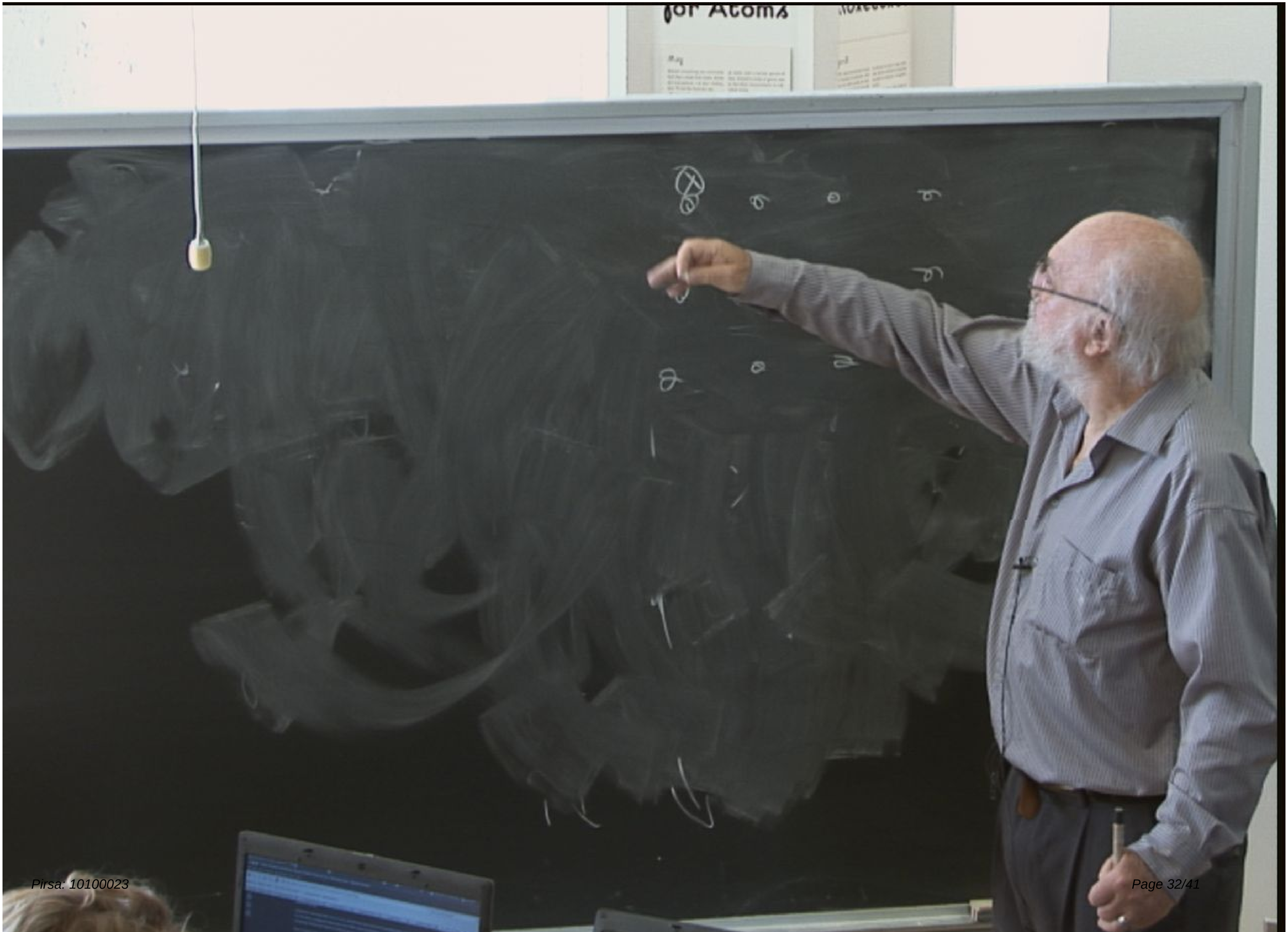
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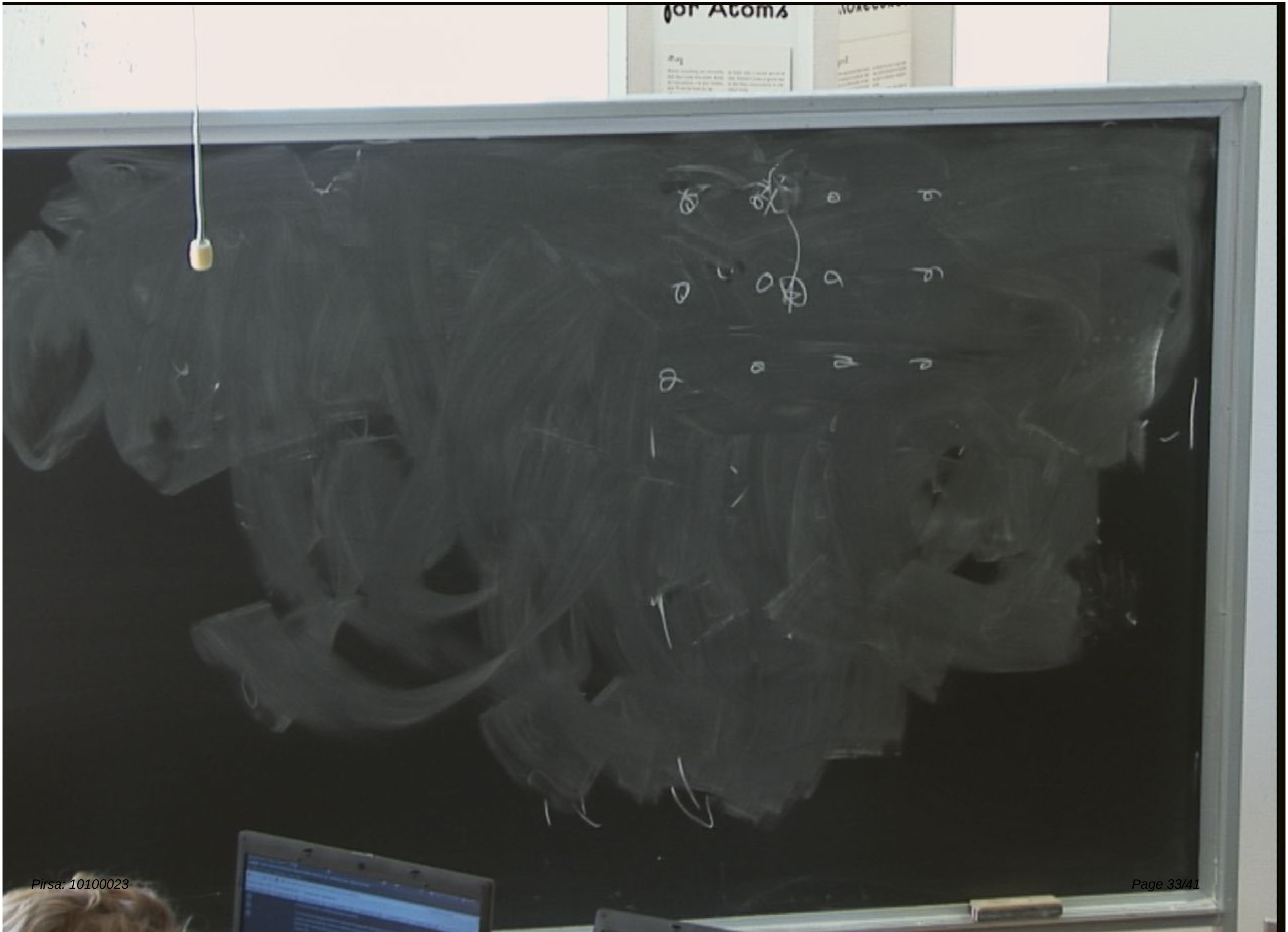
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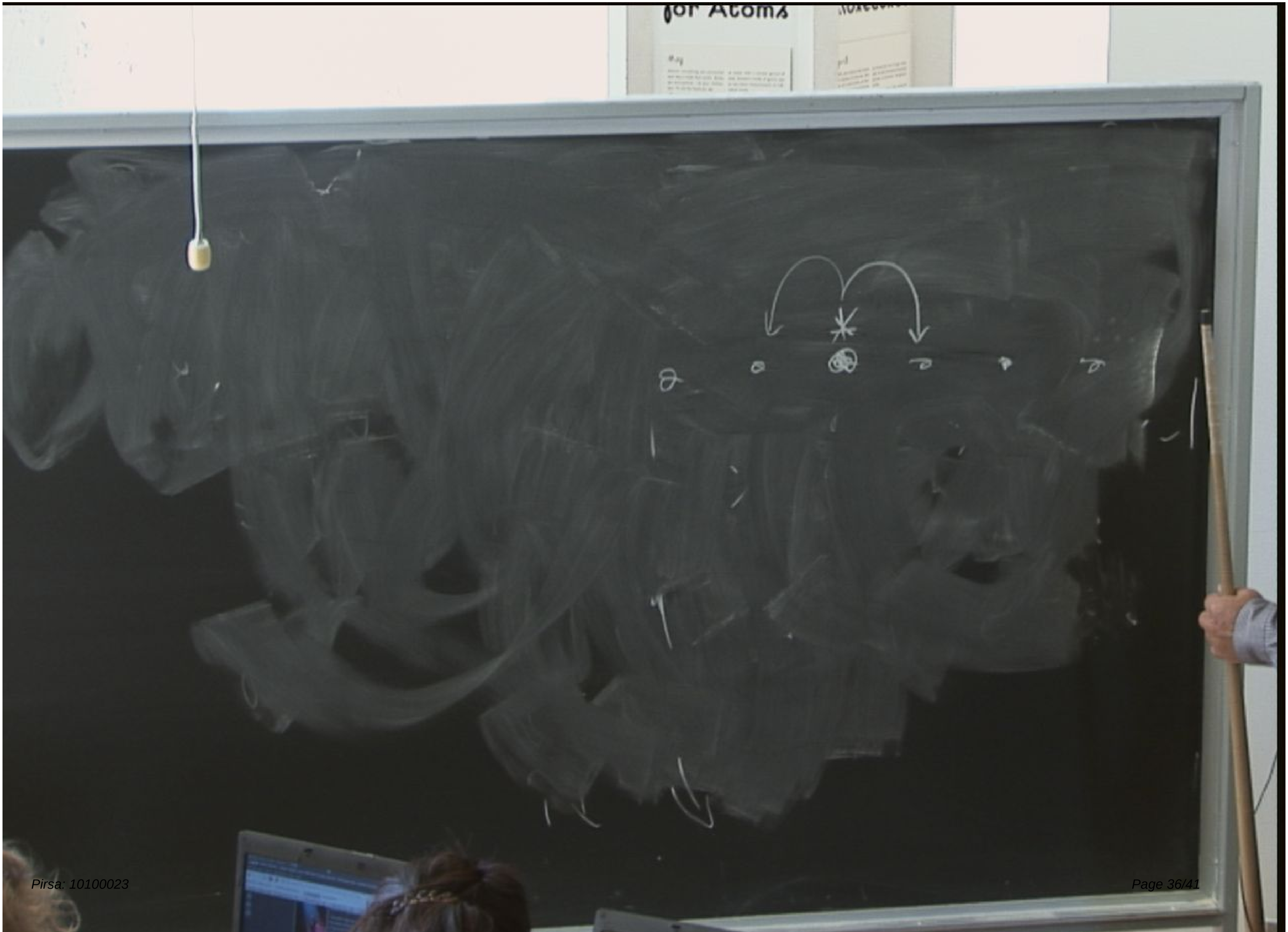
Hopping On a Lattice

A lattice is a group of sites arranged in a regular pattern. One way of doing this can be labeled by giving the position $\boldsymbol{r}=(n_1,n_2,....)a$ where the n's are integers. If we include all possible values of these integers, the particular lattice generated is called is called the simple hypercubic lattice. We show a picture of this lattice in two dimensions.

This section is devoted to developing the concept of a random walk. We could do this in any number of dimensions. However, we shall approach it is the simplest possible way by first working it all out in one dimension and then stating results for higher dimensions A random walk is a stepping through space in which the successive steps occur at times $t=M\,\tau$. At any given time, the position is $X(t)$, which lies on one of the a lattice sites, $x=an$, where n is an integer. In one step of motion one progress from $X(t)$ to $X(t+\tau) = X(t)+ a\sigma_j$, where σ_j is picked at random from among the two possible nearest neighbor hops along the lattice, σ_j $=1$ or $\sigma_j = -1$. Thus, $< \sigma_j >=0$, but of course the average its square is non-zero and is given by $< \sigma_j^2 > =1$. We assume that we start at zero, so that our times $t =j\,\tau$. It is not accidental that we express the random walk in the same language as the Ising model. We do this to emphasize that geometric problems can often be expressed in algebraic form and vice versa.


for Atoms

Hopping On a Lattice

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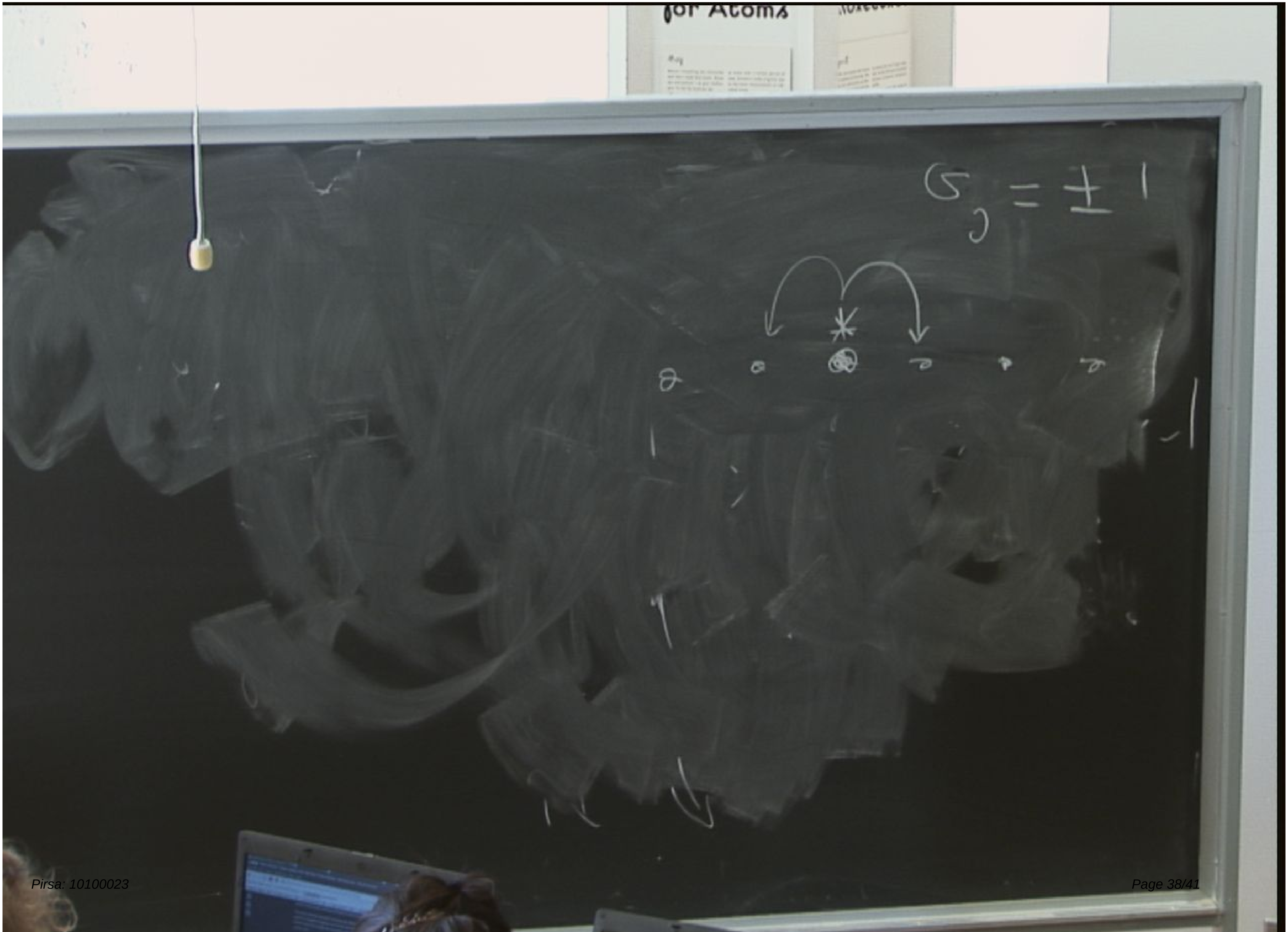
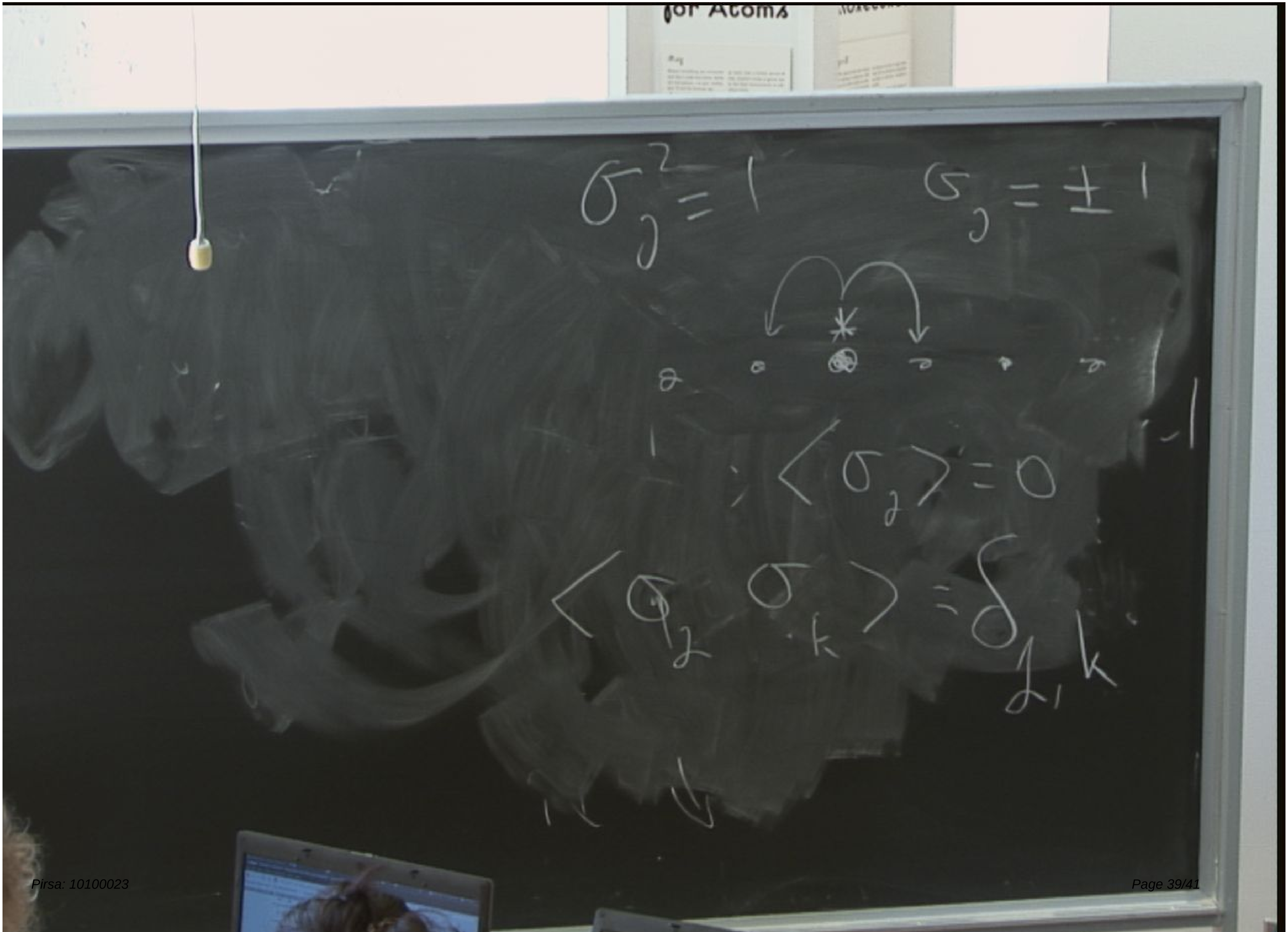

for Atoms
$\sigma_j = \pm 1$


for Atoms
$\sigma_j^2 = 1$
$\sigma_j = \pm 1$
$\langle \sigma_j \rangle = 0$
$\langle \sigma_j \sigma_k \rangle = \delta_{j,k}$

Notation: Even and Odd

We represent the walk by two integers, M, the number of steps taken and n, the displacement from the origin after M steps. We start from n=0 at M=0. The general formula is

$$n = \sum_{k=1}^{M} \sigma_k$$

Notice that n is even if M is even and odd if M is odd. We shall have to keep track of this property in our later, detailed, calculation.

We shall also use the dimensional variables for time $t=M\tau$ and for space $X(t)=na$. We use a capital X to remind ourselves that it is a random variable. When we need a non-random spacial variable, we shall use a lower case letter, usually x.

$$X(\cdot) = a \sum_{j=1}^{n} \sigma_j \qquad \sigma_j^2 = 1 \qquad \sigma_j = \pm 1$$

$$\langle \sigma_j \rangle = 0$$

$$\langle \sigma_j \sigma_k \rangle = \delta_{j,k}$$