Title: Statistical Mechanics (PHYS 602) - Lecture 3

Date: Oct 06, 2010 10:30 AM

URL: http://pirsa.org/10100022

Abstract:

Statistical Correlations in a Long Chain

We should be able lots about the statistical mechanics of a the long chain with Ising style interactions. For example, let us calculate the average of the jth spin on a long chain or the correlations among the spins in the chain. Start from

$$Z = \mathrm{Tr}\ \exp[\ \sum_{j=1}^{N} K\sigma_{j+1}\sigma_j]$$

$$< \sigma_k >= (1/Z)\mathrm{Tr}\,\sigma_k\ \exp[\ \sum_{j=1}^{N} K\sigma_{j+1}\sigma_j]$$

$$< \sigma_k\sigma_{k+r} >= (1/Z)\mathrm{Tr}\,\sigma_k\sigma_{k+r}\ \exp[\ \sum_{j=1}^{N} K\sigma_{j+1}\sigma_j]$$

Here Tr means "sum over all the N spin-values". We use periodic boundary conditions. In this equation all the σ's are numbers, and they commute with each other.

We can make the calculation easier by replacing all the couplings by their expressions in terms of Pauli spins matrices giving these three calculations as, first,

$$Z=\mathrm{trace}_\tau \prod_{j=1}^{N} \exp\,(\tilde{K}_0 + \tilde{K}\tau_1) = \mathrm{trace}_\tau \exp[N\,(\tilde{K}_0 + \tilde{K}\tau_1)]$$

$$= (2\cosh K)^N + (2\sinh K)^N \qquad \approx \quad (2\cosh K)^N$$

The ≈ is an approximate equality which holds for large *N*. Note that in this limit the term with eigenvalue of $\tau_1 = 1$ dominates because the dual coupling is positive.

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Correlations...Average magnetization

We know the answer: the system has full symmetry between spin up and spin down so that the average magnetization must be zero. Nonetheless, let's calculate

$$Z\langle\sigma_j\rangle = \mathrm{trace}_T \{\exp[-H\,\tau(j-1)]\,\tau_3\ \exp[-H\,\tau(N-j+1)]\}$$

Since we can rearrange terms under a trace, as trace (ab)=trace (ba), this expression simplifies to

$$= \mathrm{trace}_T \{\tau_3\ \exp[-H\,\tau N]\} = \mathrm{trace}_T \{\tau_3\ \exp[-(\tilde{K}_0+\tilde{K}\,\tau_1)N]\}$$

To evaluate the last expression we must take diagonal matrix elements of τ_3 between eigenstates of τ_1. Both such matrix elements are zero. why? Because τ_3 acts to change the value of τ_1 so that $\tau_3\,|\tau_1=1\rangle = |\tau_1=-1\rangle$ so that $\langle\tau_1=1|\,\tau_3\,|\tau_1=1\rangle = \langle\tau_1=1|\,|\tau_1=-1\rangle=0$. Therefore the entire result is zero and the average has the value zero, as expected.

$$\langle\sigma_j\rangle = 0$$

At zero magnetic field, the magnetization of the one-dimensional Ising model is zero. Thus, this Ising model has no ordered state. In fact no one-dimensional system with finite interactions has one. This model is always in the disordered phase at all finite temperatures.

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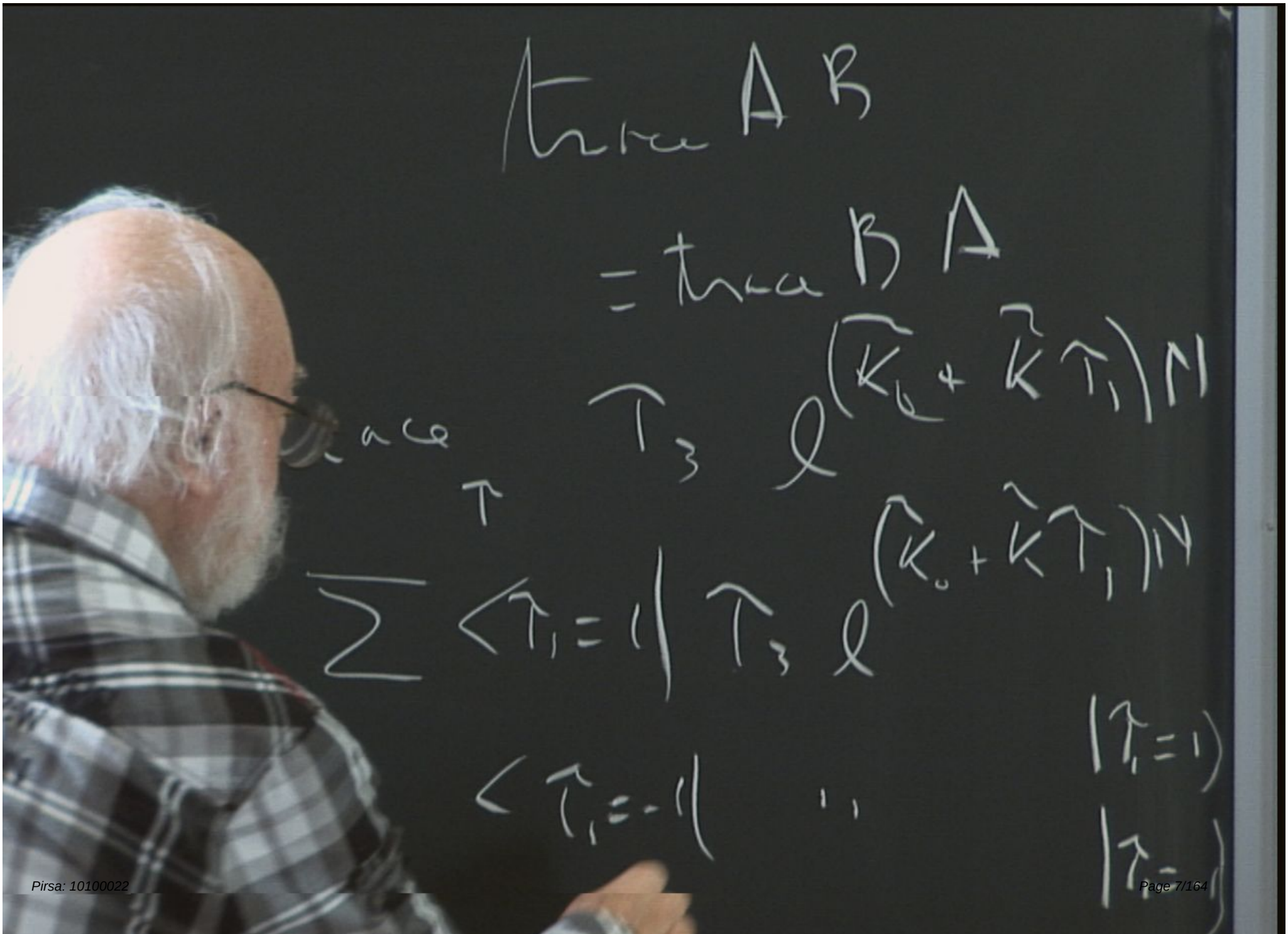
trace A B
= trace B A
trace
T
$\hat{T}_3$ $e^{(\hat{K}_0 + \hat{K}\hat{T}_1)N}$
$\sum \langle \hat{T}_1 = 1 | \hat{T}_3 e^{(\hat{K}_0 + \hat{K}\hat{T}_1)N}$
$\langle \hat{T}_1 = -1 |$,,
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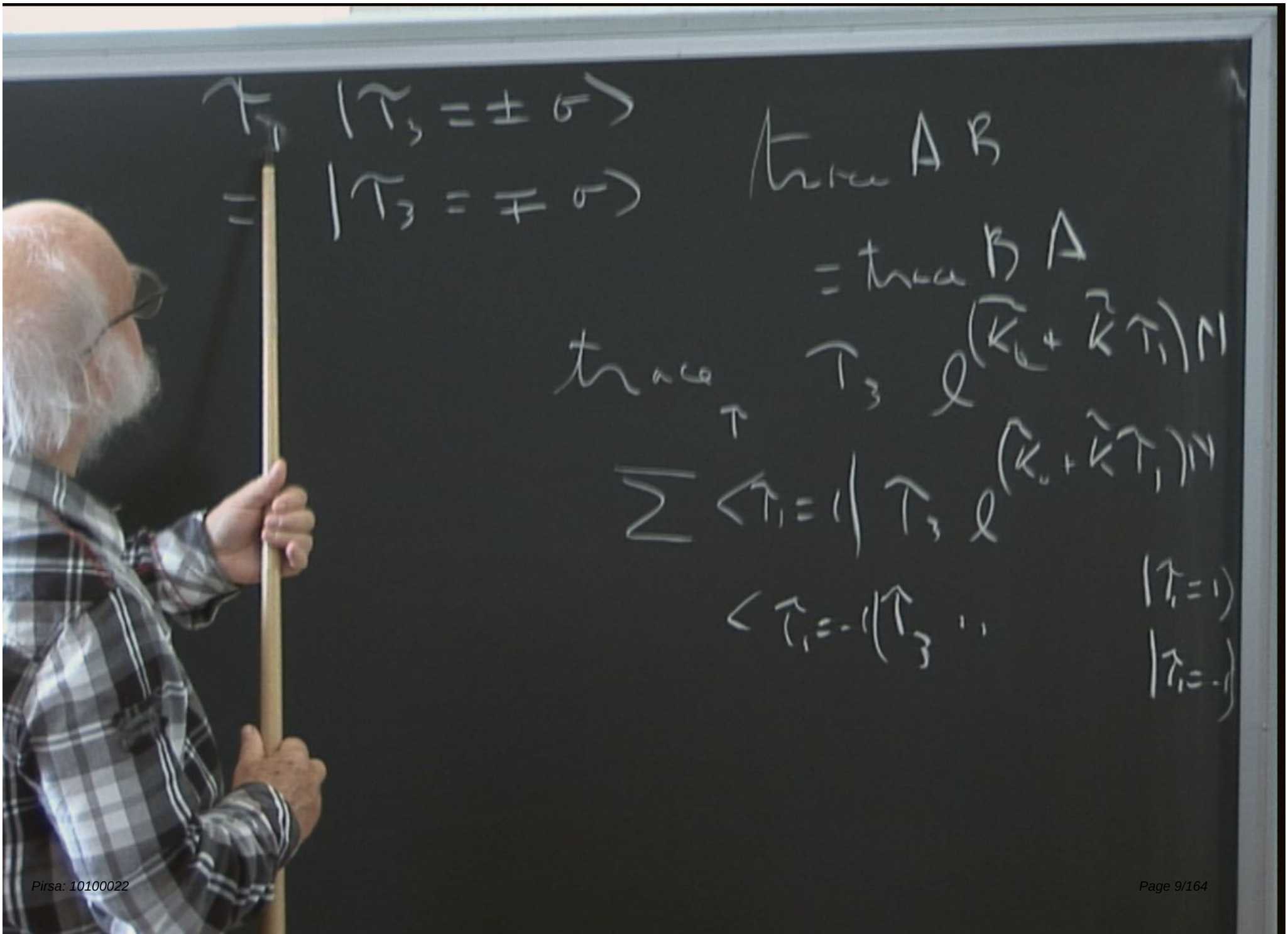
$\hat{\tau}_1 \; |\hat{\tau}_3 = \pm\sigma\rangle$
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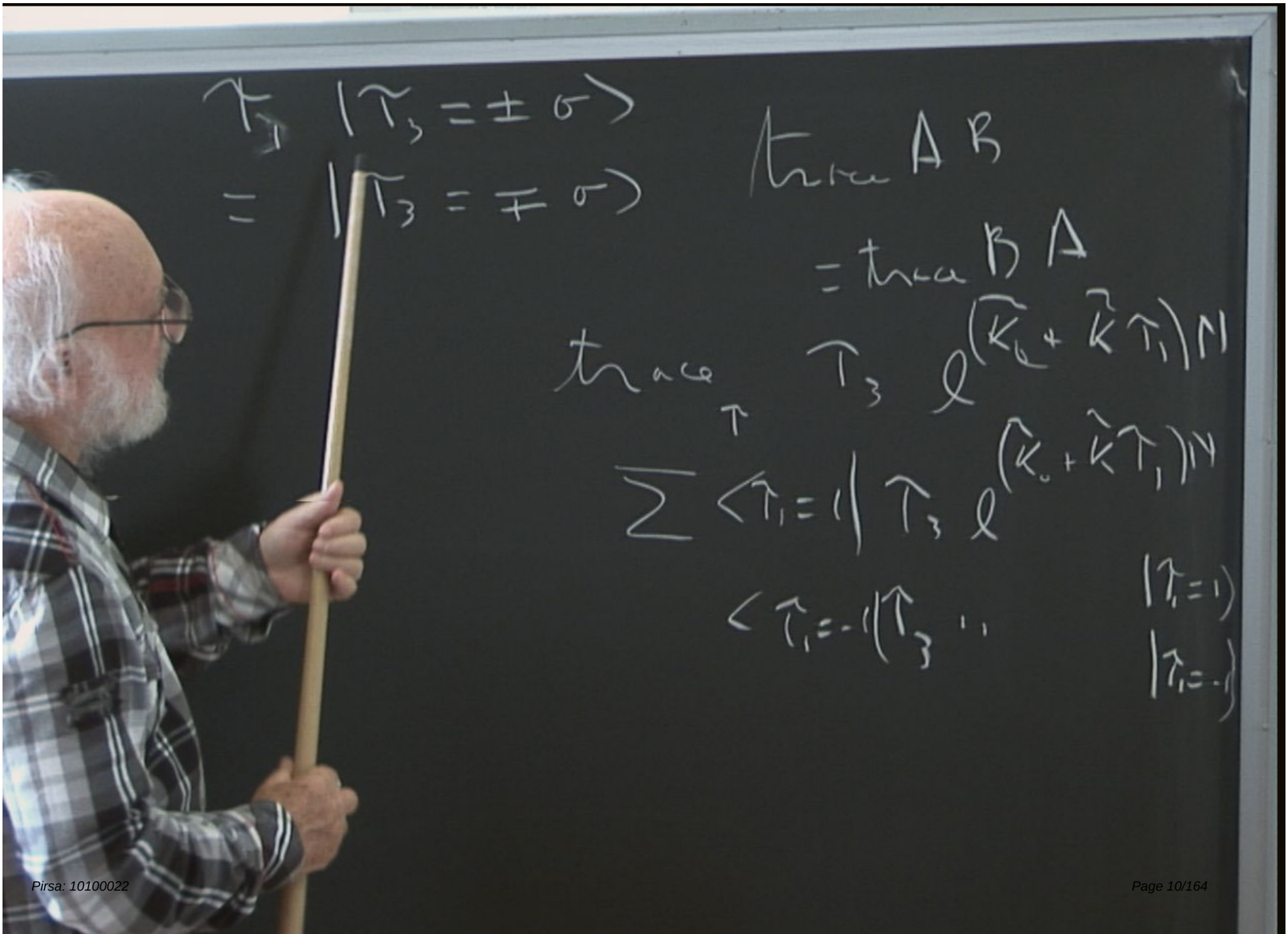
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Correlations of two spins in Large N limit

Let N be large. Z simplifies to $Z = \exp(N\tilde{K}_0 + N\tilde{K})$ since the τ_1=1 term dominates the trace

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The result is that correlations fall off exponentially with distance, with the typical falloff distance, denoted as ξ , being the distance between lattice points (usually called a) times 1/(2D(K))=1/(2$\tilde{K}$).

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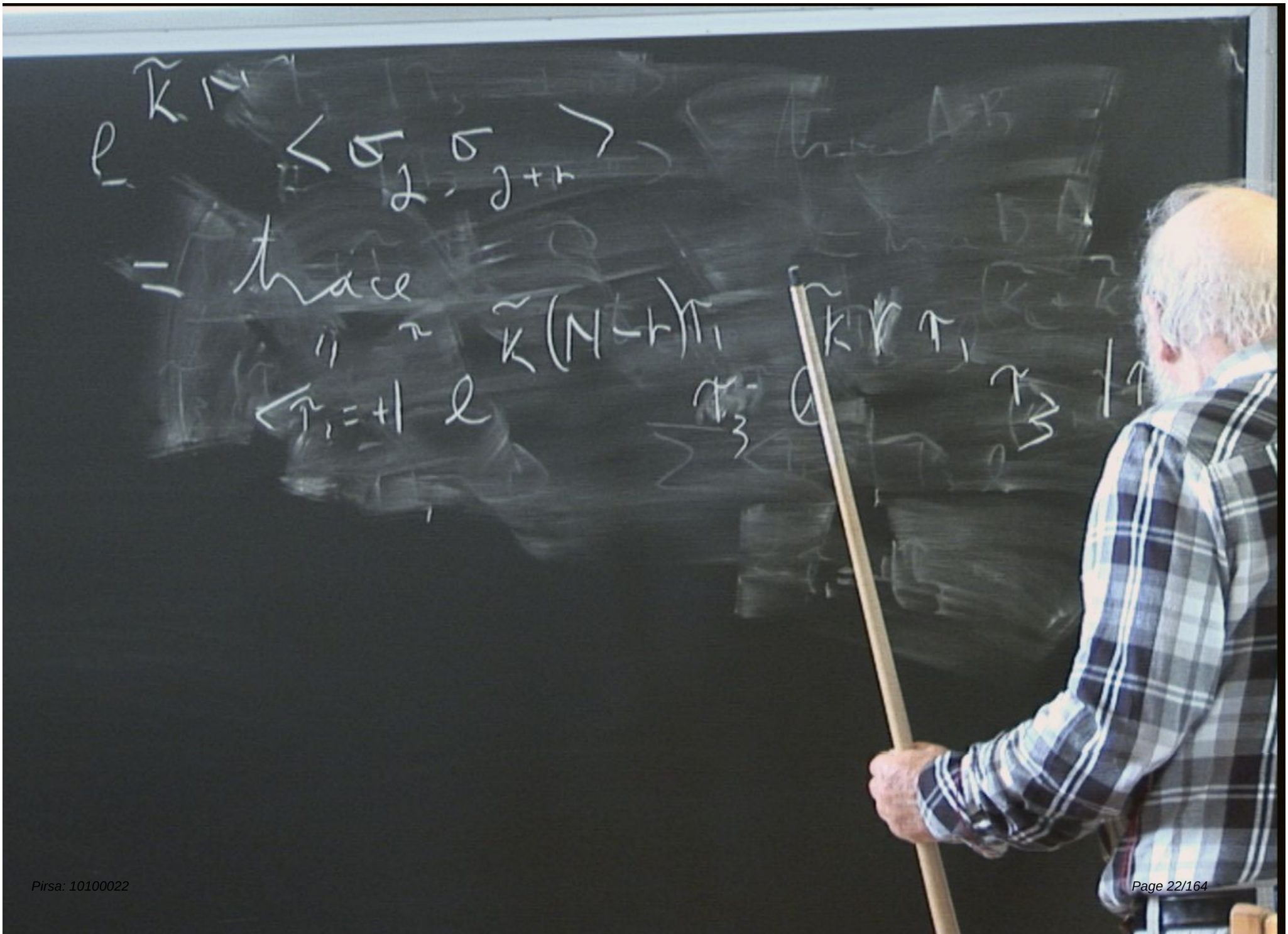
$$= \mathrm{trace}$$

$$\| \quad \pi \quad K$$

$$\langle \pi_1 = +1 | \, e$$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τ_3
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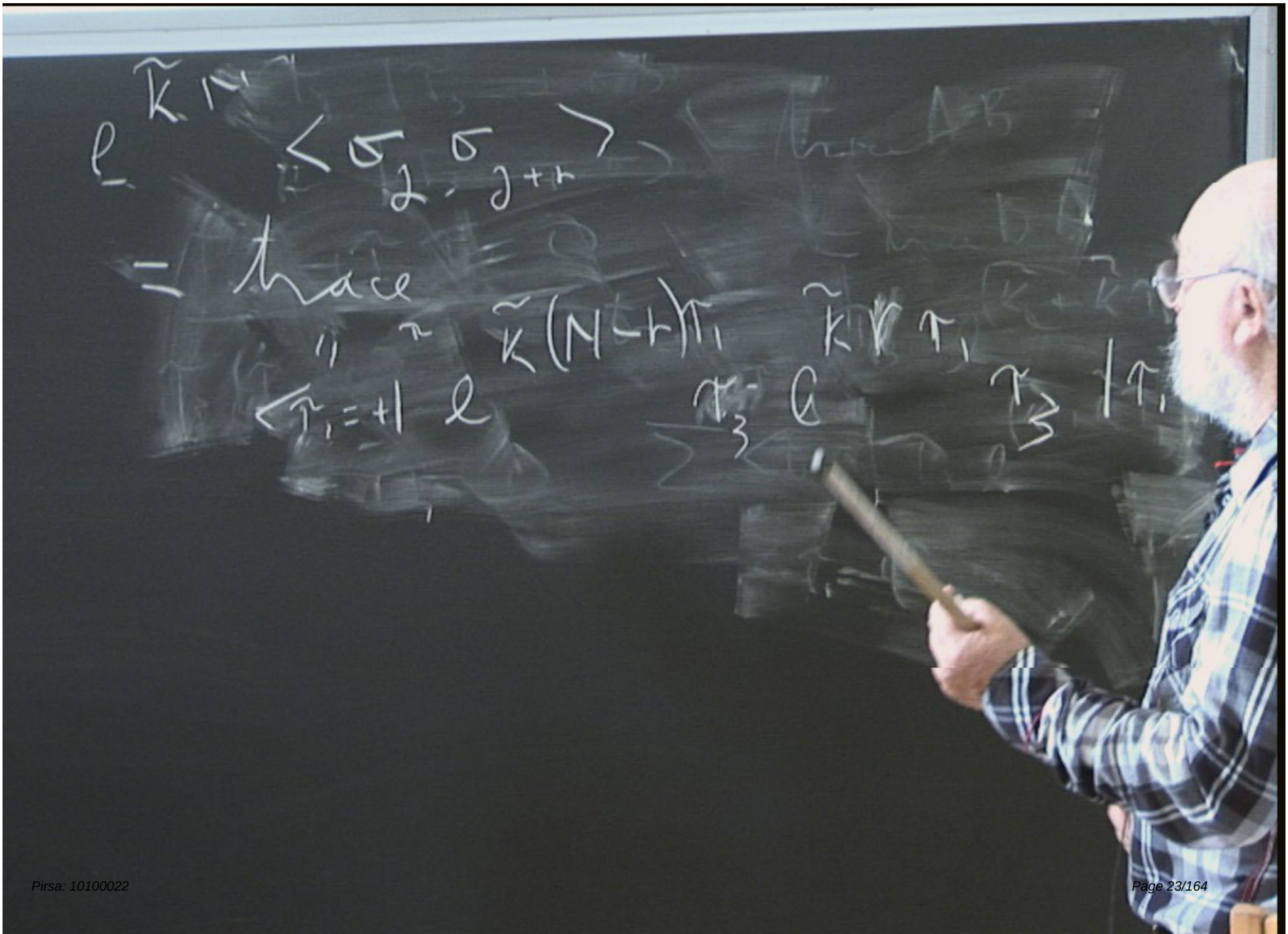
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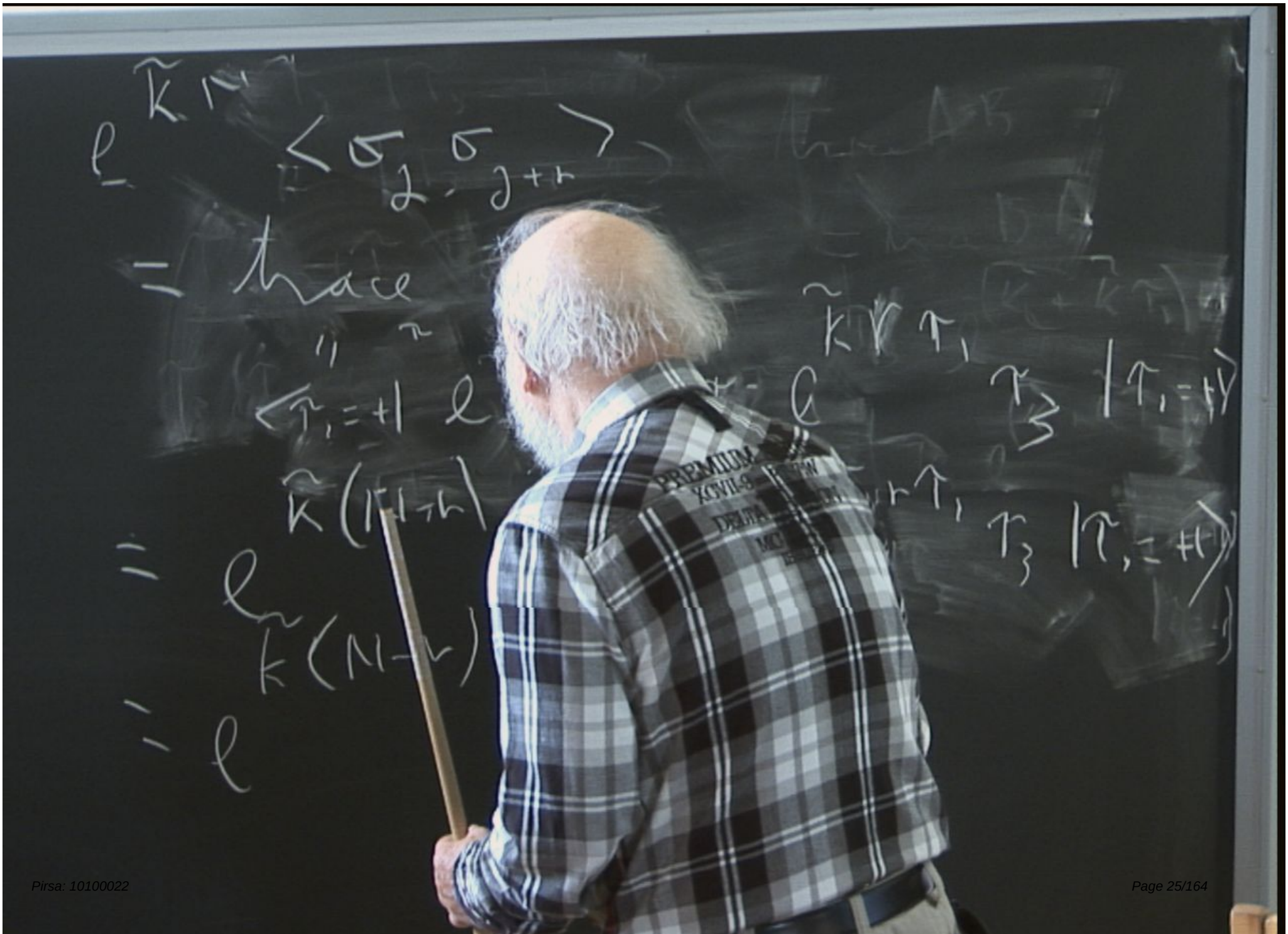
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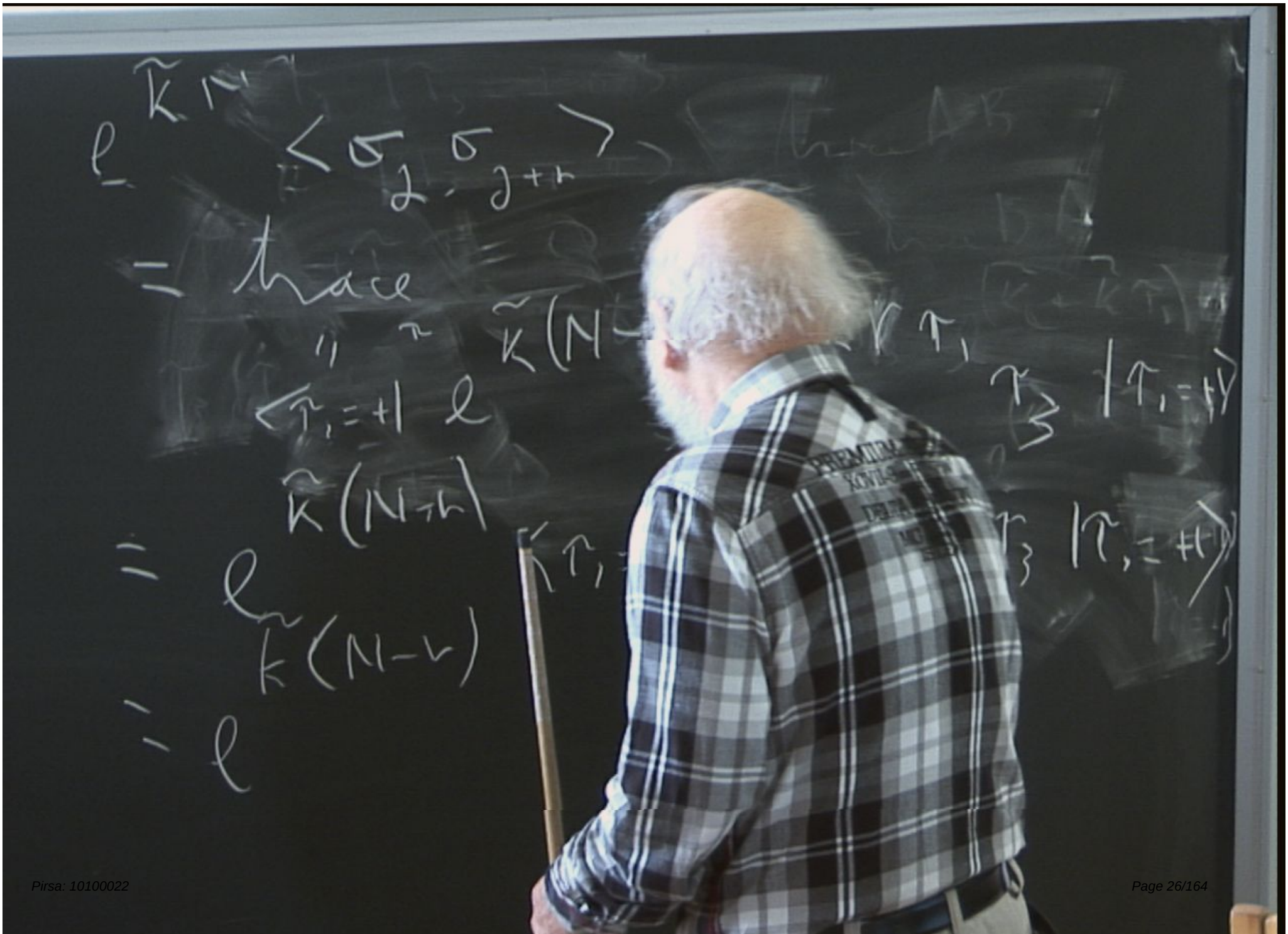
= trace
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Correlation Length

$$< \sigma_j \sigma_{j+r} >= exp(-2r\tilde{K}) \qquad = \exp(-ar/\xi)$$

Here *ar* is distance between the sites of the two spins.

The result is that correlations fall off exponentially with distance, with the typical falloff distance, denoted as ξ, being the distance between lattice points (usually called *a*) times $1/(2D(K))=1/(2\tilde{K})$. This falloff distance is very important in field theory, particle physics, and phase transition theory. In the latter context it is called the coherence length. It is also called the Yukawa distance because it first came up in Hideki Yukjawa's description of mesons. Here, in the one dimensional Ising model, we have a very large coherence length for large K. Specifically

$$1/(2\tilde{K}) = \xi/a \rightarrow \exp(2K)/2 \text{ as } K \rightarrow \infty$$

while is very small in the opposite limit of small K.

$$\xi/a \rightarrow 1/(-\ln(2K)) \text{ as } K \rightarrow 0$$

Hideki Yukawa

Large correlation lengths, or equivalently small masses, play an important role in statistical and particle physics since they indicate a near-by phase transition or change in behavior.

ξ = correlation length= a/[2D(K)] =a/($2\tilde{K}$)

Correlation Length

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Bloch Walls in 1 d

In the Ising model at large values of the coupling, K, the spins tend to line up.

However, with a cost in probability exp(-2K) a whole region might flip its spins, producing a defect called a Bloch wall

This kind of defect produces the decay of correlations in the Ising model at low temperatures. In any long Ising chain, many such defects will be randomly placed and ruin any possibility of correlations over infinitely long distances.

This is the simplest example of what is called a topological excitation, a defect which breaks the ordering in the system by separating two regions with different kinds of order. Since ordering is crucial in many situations, so are topological excitations.

Notice that, at low temperatures, this kind of excitation is much more likely than a simple flip of a single spin. The wall costs a factor of exp(-2K); the flip costs exp(-4K).

$$\langle \sigma_{j+r}\, \sigma_j \rangle$$

$$= e^{-r/\xi}$$

$$e^{K \sigma \sigma'}$$

$$e^{\tilde{K} N}$$

$$\langle \sigma_j, \sigma_j$$

$$= \text{trace}$$

$$\langle \tilde{\tau}_1 = +1 \; e^{\tilde{K}}$$

$$\tilde{K}(N$$

$$= e^{\tilde{K}(N}$$

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$$\langle \sigma_{j+r}\, \sigma_j \rangle \; e^{\tilde{k} r} \langle \sigma_j\, \sigma_{j+r} \rangle \qquad x = r\,a$$

$$= e^{-r/\xi}$$

$$e^{k \sigma \sigma'}$$

$$= \mathrm{trace}$$

$$\langle \tau_1 = +1 |\, e^{\tilde{k}(N-r)\tau_1}\, \tau_3\, e^{\tilde{k} r \tau_1}\, \tau_3 \,| \tau_1 = +1 \rangle$$

$$= e^{\tilde{k}(N-r)} \langle \tau_1 = +1 |\, \tau_3\, e^{\tilde{k} r \tau_1}\, \tau_3 \,| \tau_1 = +1 \rangle$$

$$= e^{\tilde{k}(N-r)} \langle \tau_1 = -1 |\, e^{-\tilde{k} r} \,| \tau_1 = -1 \rangle$$

$$= e^{\tilde{k}(N-2r)}$$

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Renormalization for 1D ising,

following ideas of Kenneth Wilson, this calculation is due to David Nelson and myself

$$Z = \sum_{\sigma_1,\sigma_2,\cdots} \exp(W_K\{\sigma\}) = \sum_{\sigma_1,\sigma_2,\cdots} \exp(K\sigma_1\sigma_2 + K\sigma_2\sigma_3 + \cdots)$$

Rearrange calculation: Rename spins separated by two lattice sites: let $\mu_1=\sigma_1$; $\mu_2=\sigma_3$, $\mu_3=\sigma_5$, ...; and sum over every other spin, σ_2, σ_4 .....

$$Z = \sum_{\mu_1,\mu_2,\cdots} \sum_{\sigma_2,\sigma_4,\cdots} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2 + \cdots) = \sum_{\mu_1,\mu_2,\cdots} \exp(w'\{\mu\})$$

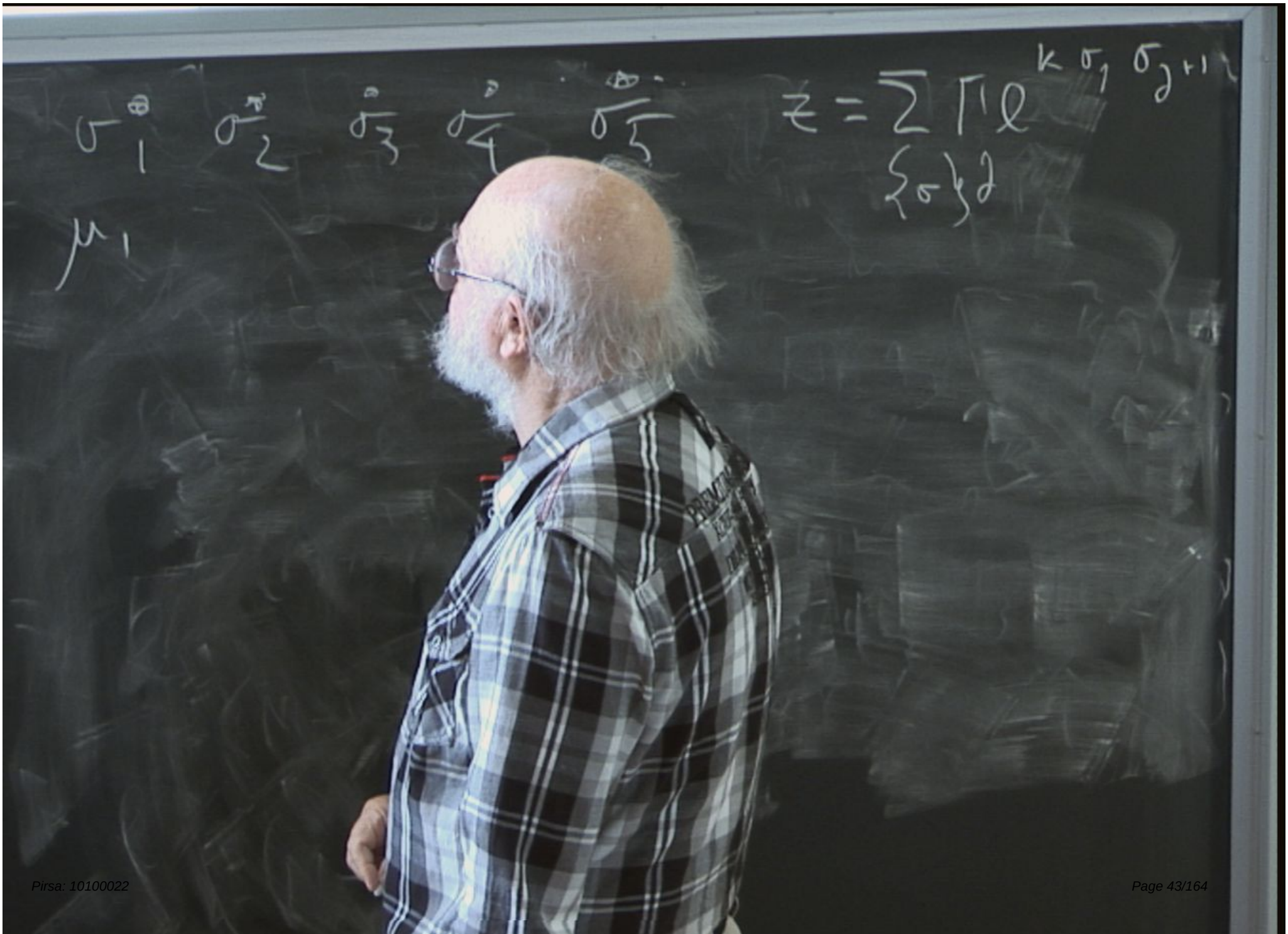
Note that sum over σ_2, σ_4 ..... generates only nearest neighbor interactions for the μ's

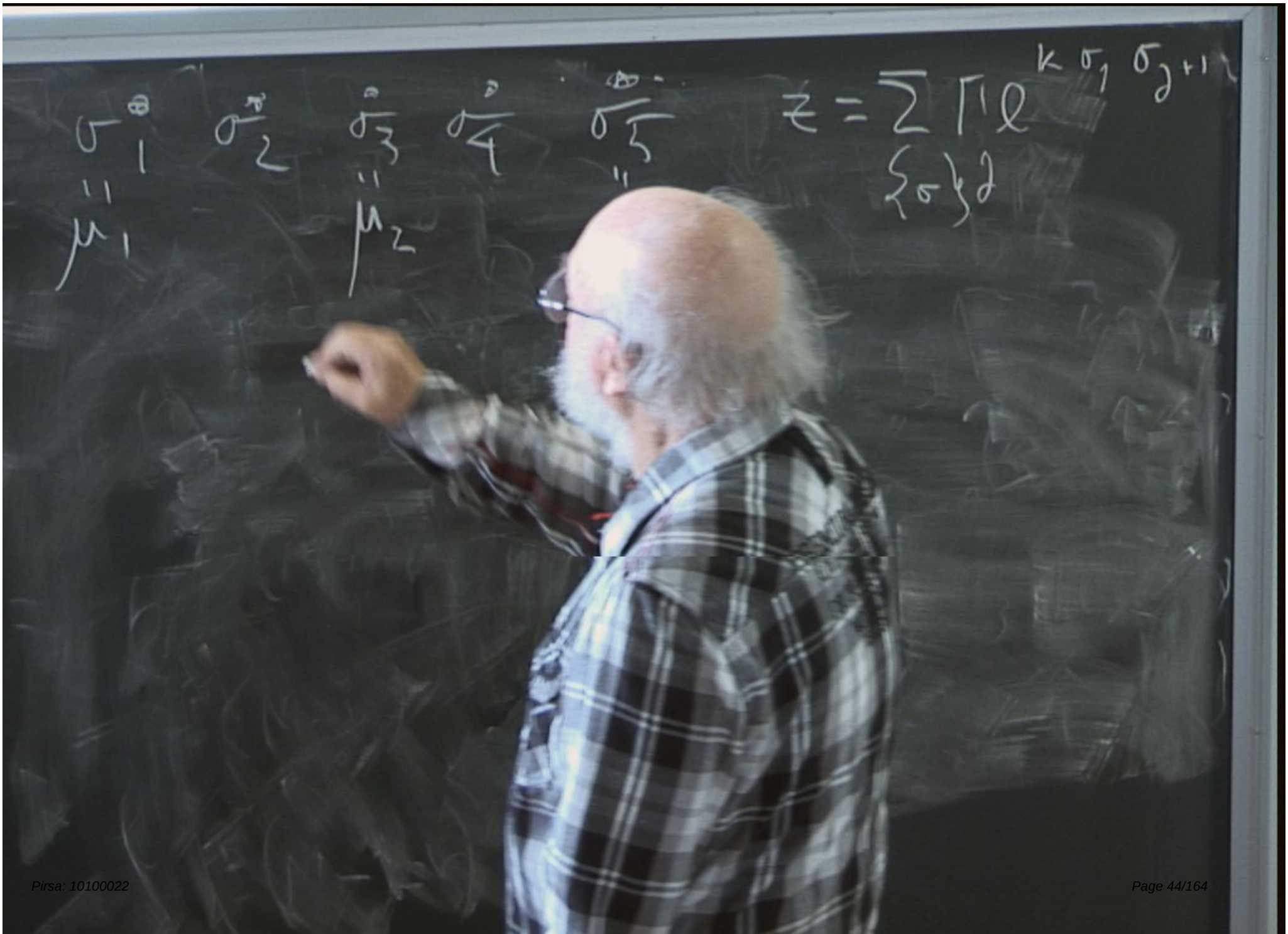
$w'\{\mu\}$= const + $K' \mu_1 \mu_2 + K' \mu_2 \mu_3$ + ...

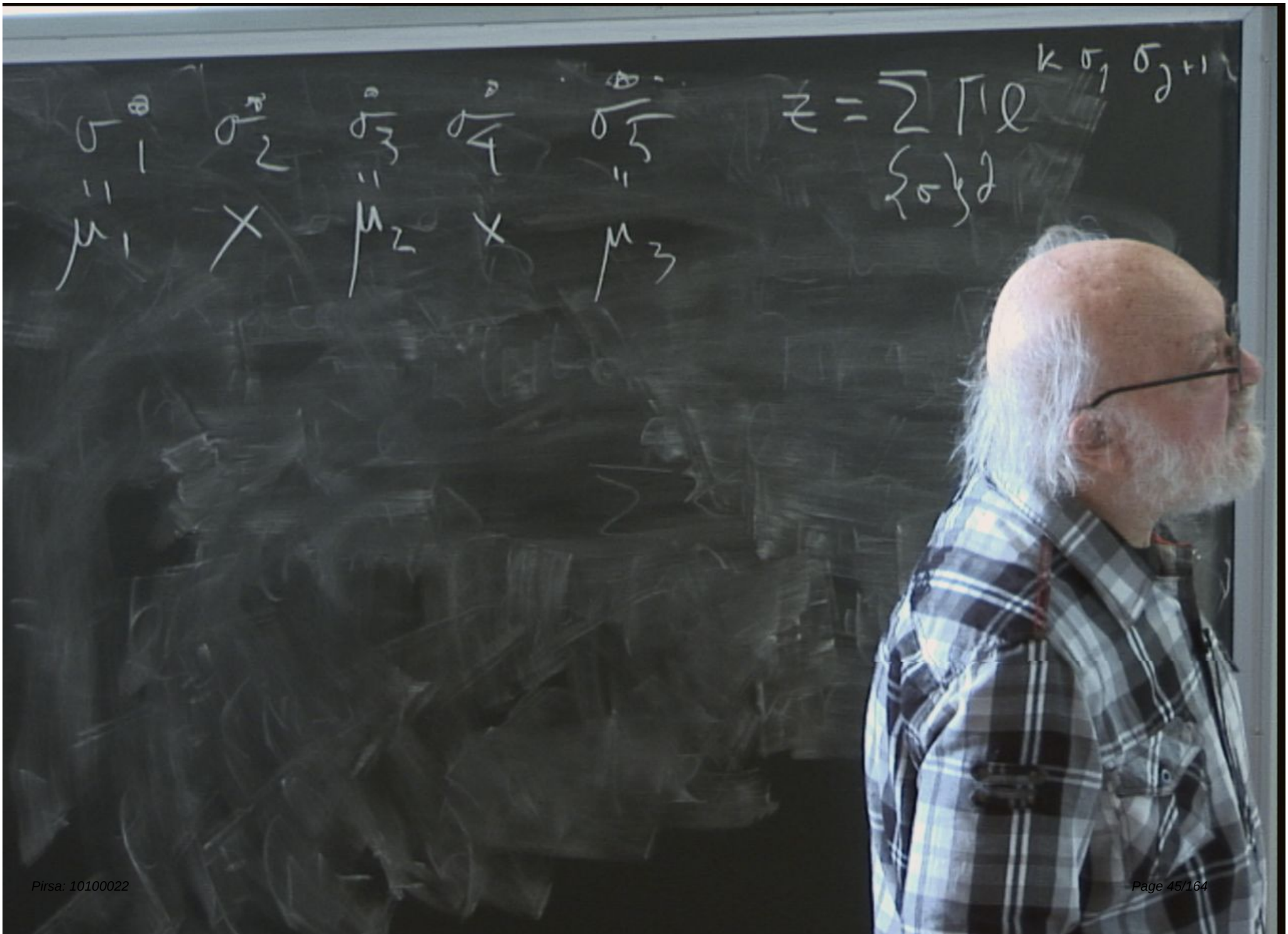
K' describes same system as before, with a new separation between lattice sites, which is twice as big as the old separation. Since the physical system is the same, physical quantities like the correlation length and the entropy are unchanged, but their description in terms of couplings and lattice constants has changed. In particular, the new lattice spacing is $a' = 2a$, but the correlation length is exactly the same $\xi' = \xi$. Since we know that the correlation length is given by

$\xi = a/[2D(K)]$, we know that the new coupling obeys $a/[2D(K)] = a'/[2D(K')]$ we find that the new coupling obeys $D(K') = 2\,D(K)$ before we do any detailed renormalization calculations. Since D is a decreasing function of K we know that the new, renormalized, coupling is smaller than the old one.

$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$


$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$
$Z = \sum_{\{\sigma\}} \prod e^{K \sigma_j \sigma_{j+1}}$
μ_1


$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$
$\mu_1 \quad \mu_2$
$Z = \sum_{\{\sigma\}} \prod e^{K \sigma_j \sigma_{j+1}}$


$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$
$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_3$
$Z = \sum_{\{\sigma\}} \prod e^{K \sigma_j \sigma_{j+1}}$

$$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$$

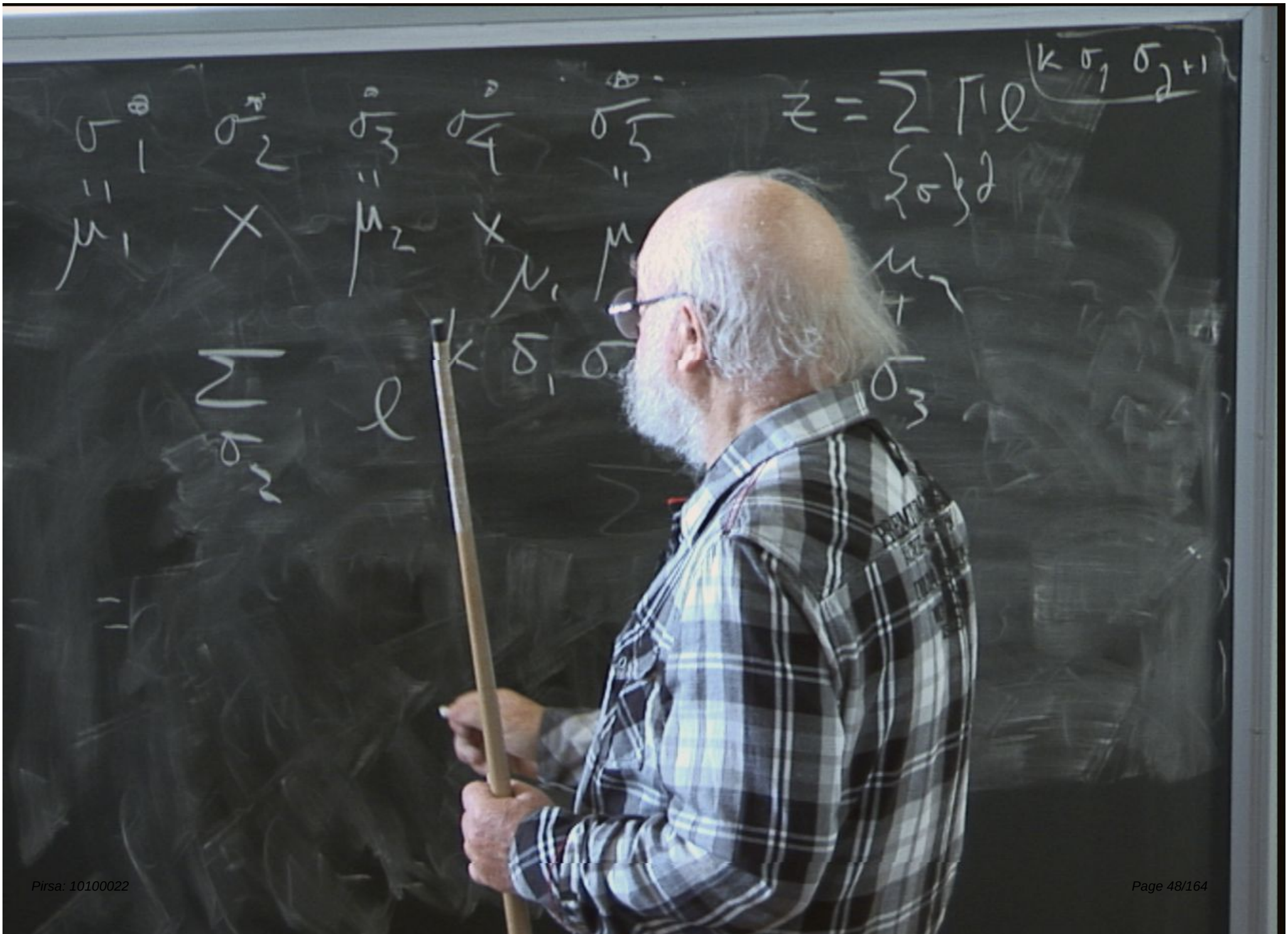
$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_3$$

$$Z = \sum_{\{\sigma\}} \prod_j e^{K \sigma_j \sigma_{j+1}}$$

$$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \qquad Z = \sum_{\{\sigma\}} \prod_j e^{k\sigma_j \sigma_{j+1}}$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_3 \qquad \mu_1 \mu_3 = \mu_n$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$



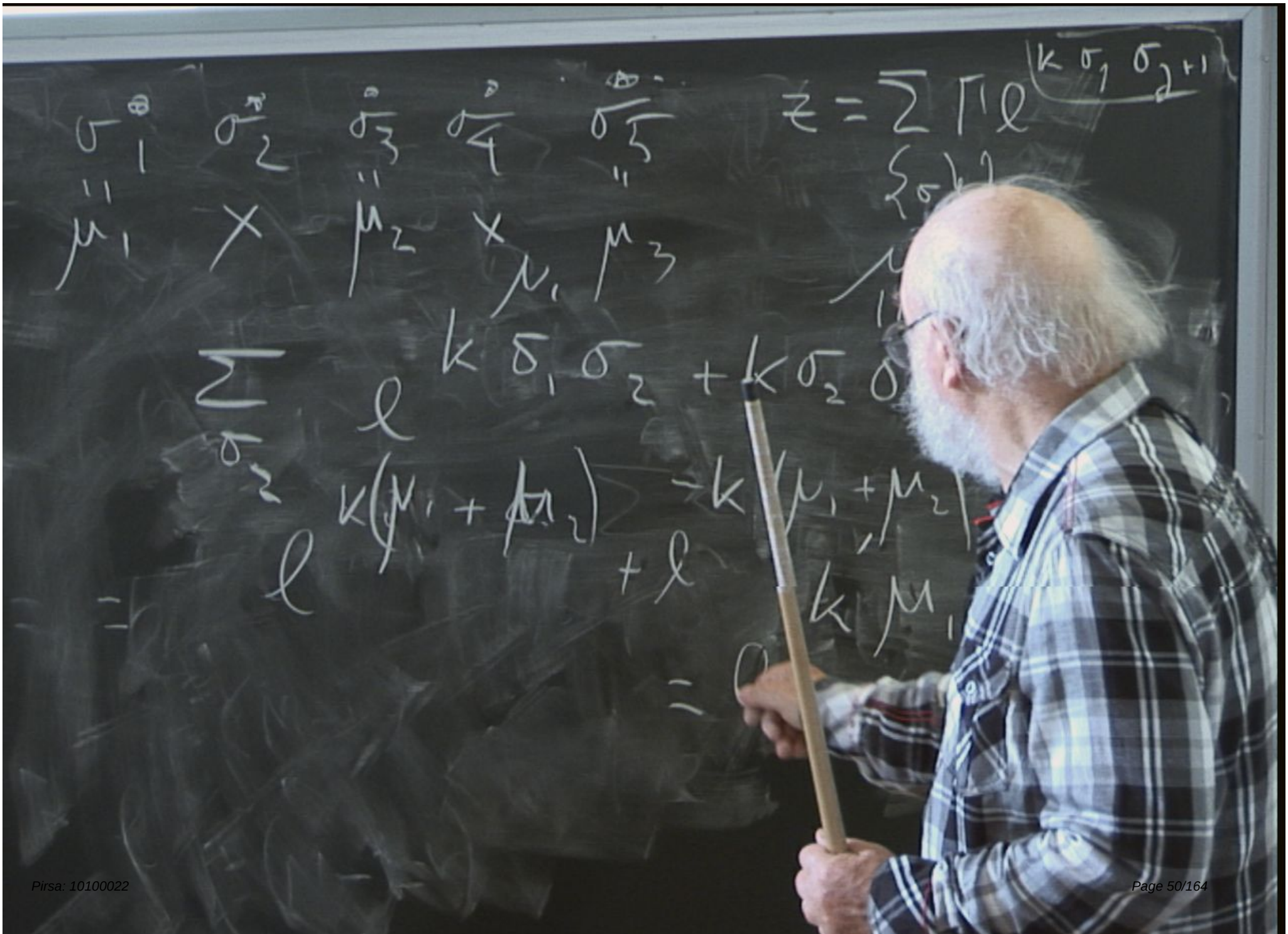
$$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \qquad Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_3$$

$$\mu_1 \quad \mu_3 \qquad \mu_2$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$= e^{k(\mu_1 + \mu_2)} + e^{-k(\mu_1 + \mu_2)}$$


σ1 σ2 σ3 σ4 σ5
μ1 × μ2 × μ3
μ1 μ2 μ3
Z = Σ Π e^{k σ1 σ2 + 1}
Σ_{σ2} e^{k σ1 σ2 + k σ2 σ}
e^{k(μ1 + μ2)} + e^{-k(μ1 + μ2)}
= e^{k μ1}
= 0

$$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_3$$

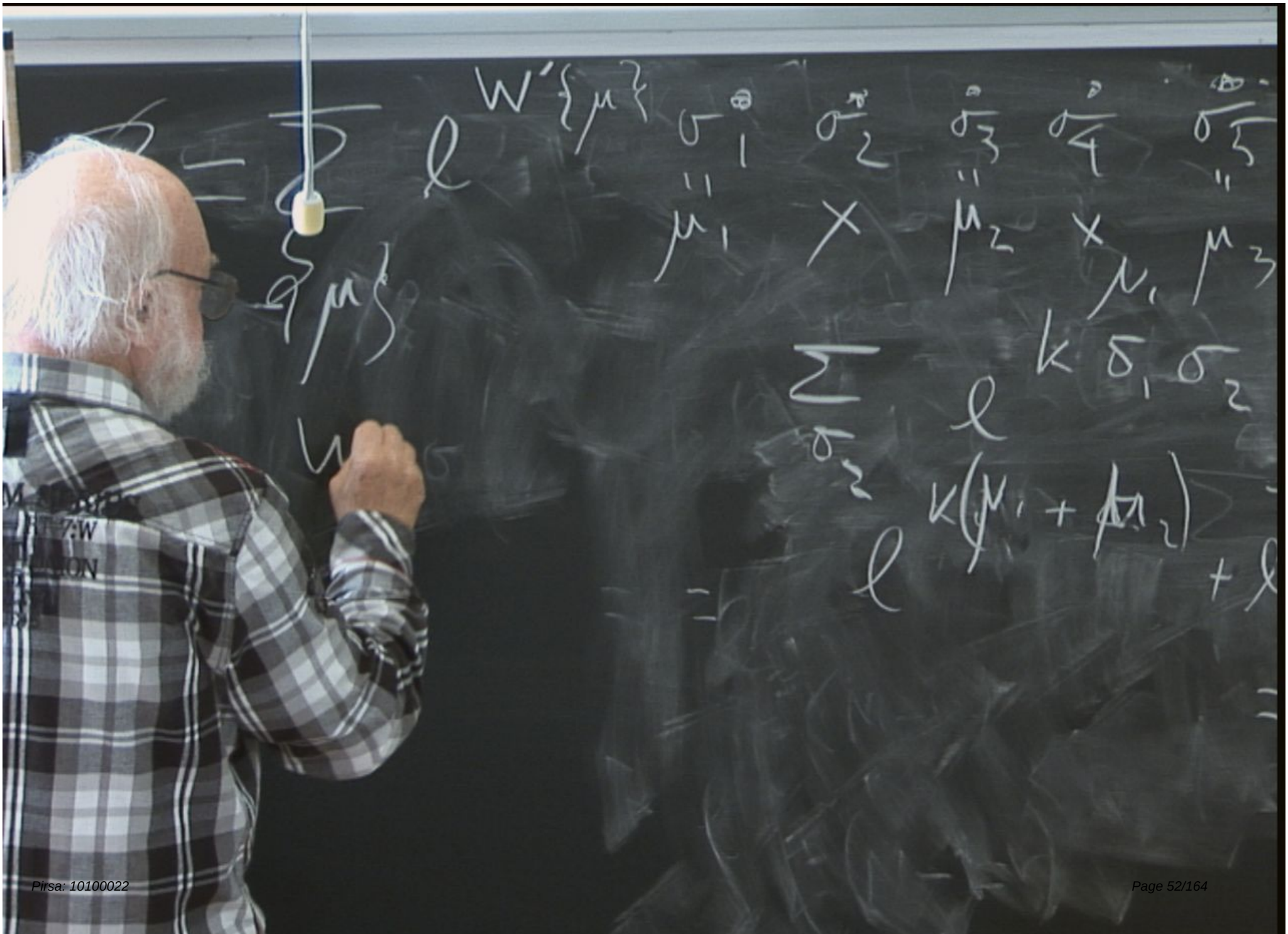
$$Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

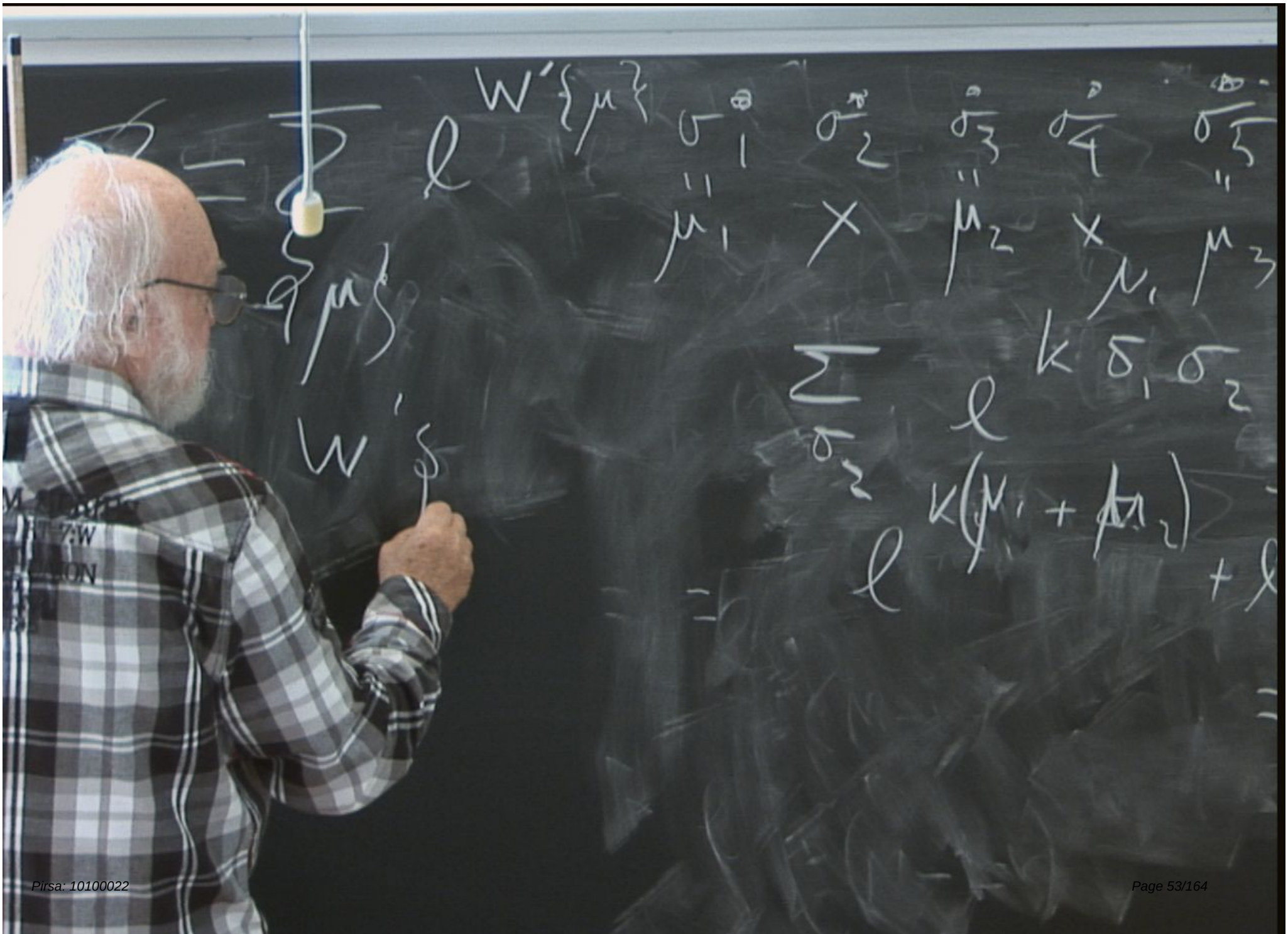
$$\mu_1 \quad \mu_2$$

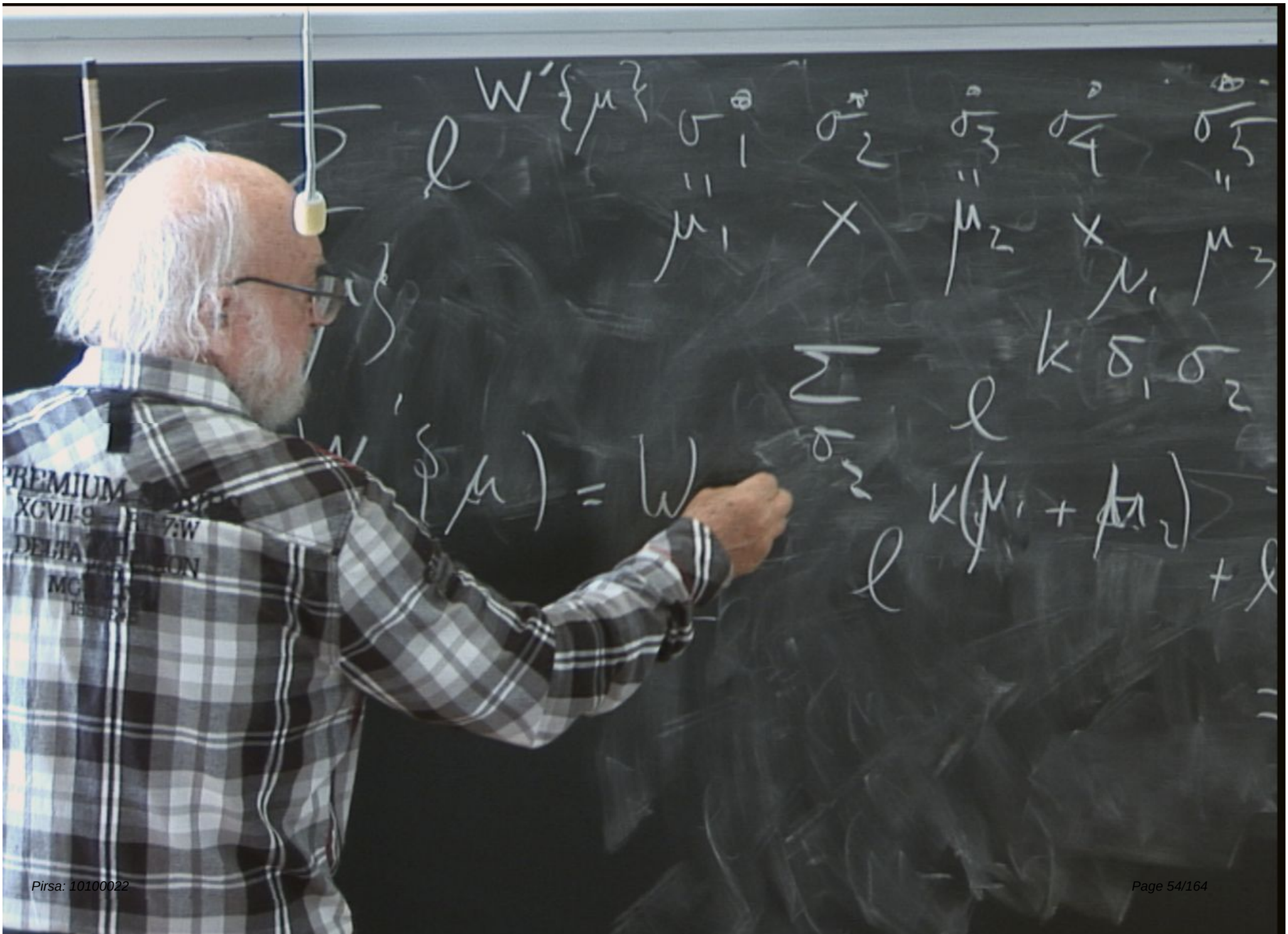
$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

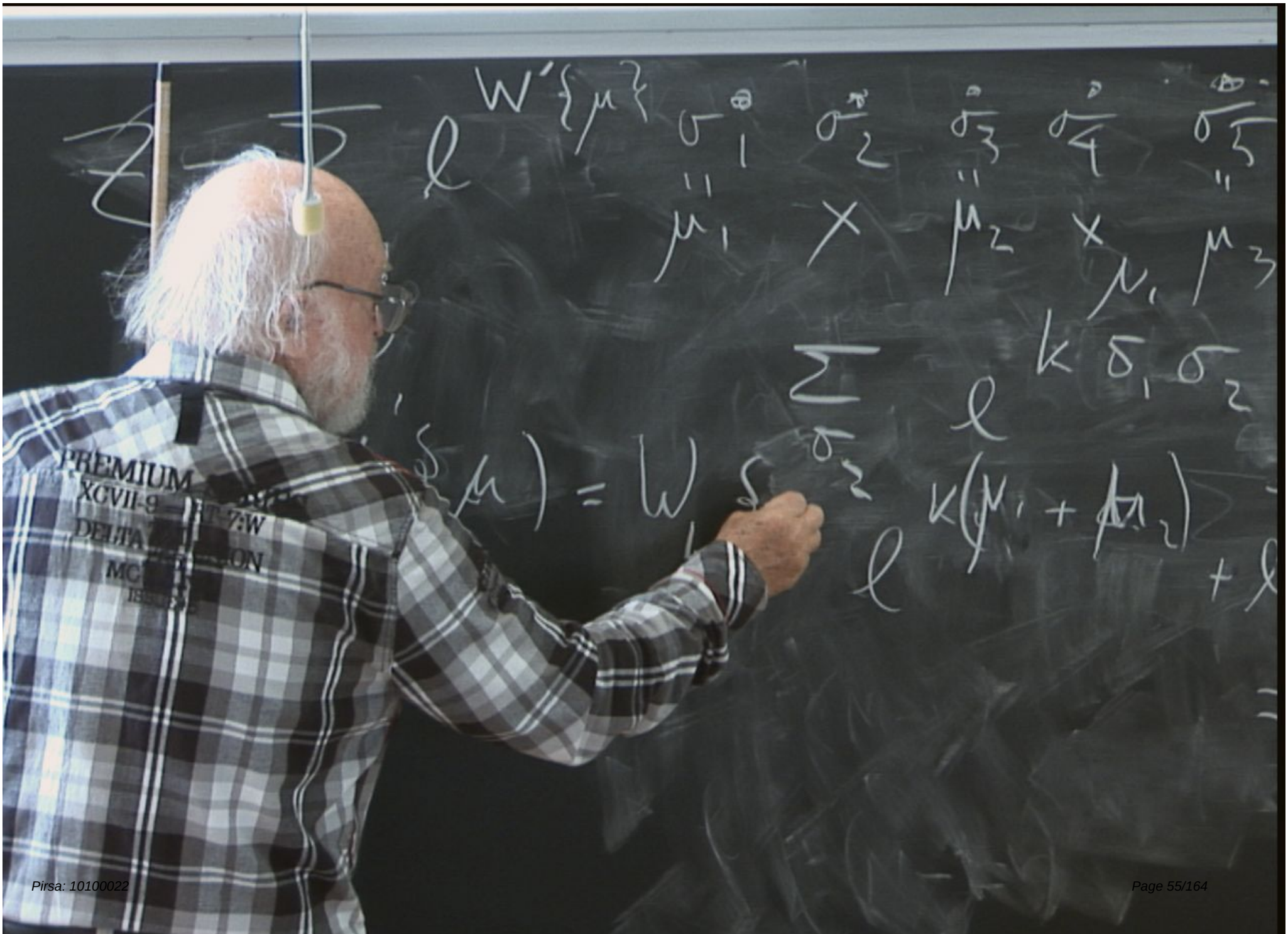
$$= e^{k(\mu_1 + \mu_2)} + e^{-k(\mu_1 + \mu_2)}$$

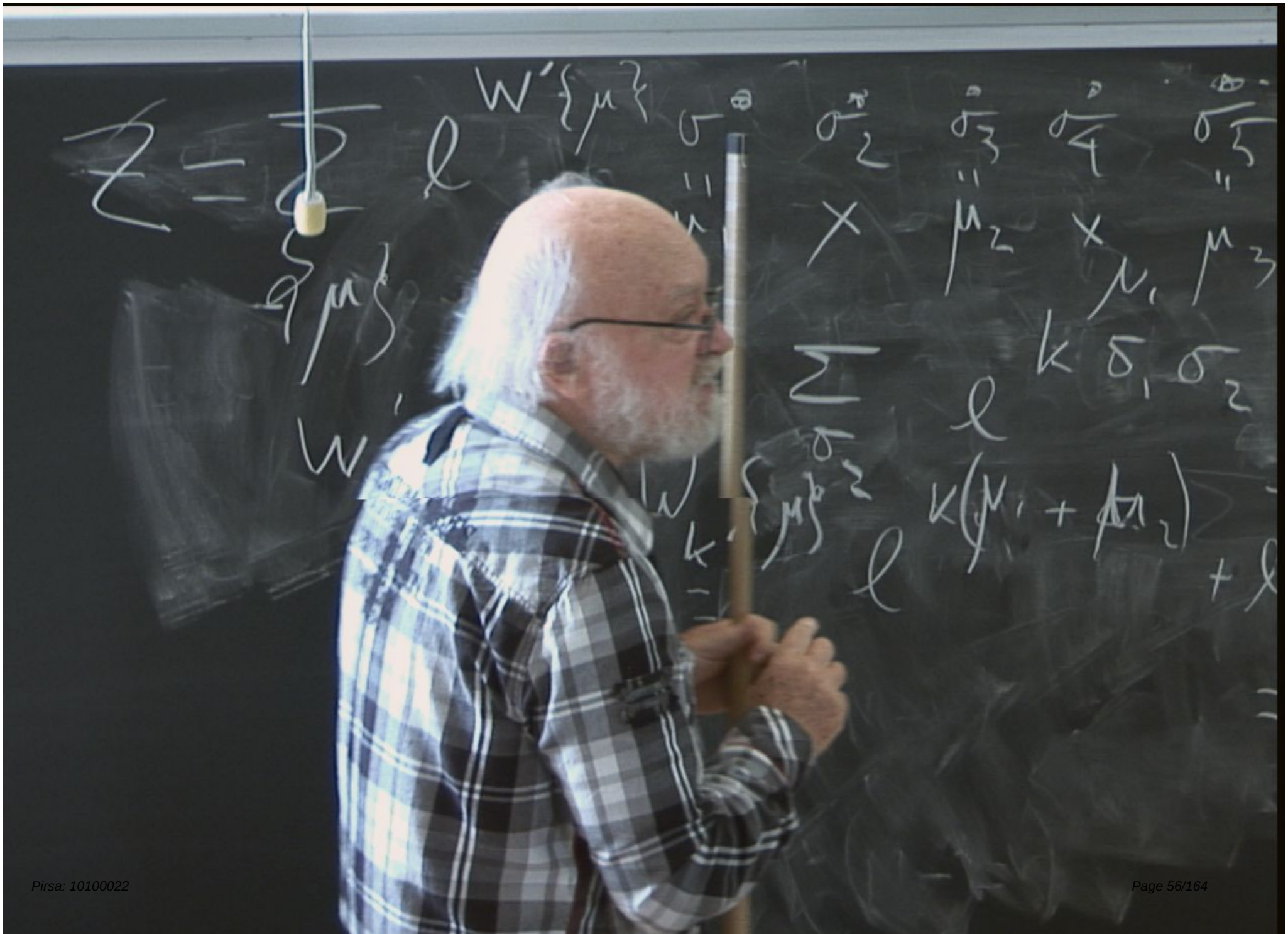
$$= 2e^{k\mu_1\mu_2}$$






W'{μ}
σ1 σ2 σ3 σ4 σ5
μ1 X μ2 X μ1 μ3
k σ1 σ2
W'(μ) = W
k(μ1 + μ2)


PREMIUM



$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_1 \quad \mu_3$$

$$\sum_{\sigma} e^{k\sigma_1\sigma_2}$$

$$W'(\mu) = W_{k'\{\mu\}} = e^{k(\mu_1+\mu_2)} + e$$

$$=$$

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}}$$

$$\sigma_1 = \mu_1 \quad \sigma_2 = \times \quad \sigma_3 = \mu_2 \quad \sigma_4 = \times \quad \sigma_5 = \mu_3$$

$$Z = \sum_{\{\sigma\}} \prod e^{k\sigma}$$

$$\mu_2 = \sigma_3$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$W'\{\mu\} = W_{k'}\{\mu\} = e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)} = e^{k'\mu_1\mu_2}$$

$$k' = 2k$$

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_1 \quad \mu_3$$

$$Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_n$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$W'\{\mu\} = W_{k'}\{\mu\} = e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)}$$

$$= 2e^{k'\mu_1\mu_2}$$

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$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \qquad Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_1 \times \mu_2 \times \mu_1 \quad \mu_3 \qquad \mu_n$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$W'\{\mu\} = W_{k'}\{\mu\} = e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)}$$

$$= 2e^{k\mu_1\mu_2}$$

$$a' = 2a \qquad k' = R(k)$$

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \; \sigma_2 \; \sigma_3 \; \sigma_4 \; \sigma_5$$

$$\mu_1 \times \mu_2 \times \mu_1 \; \mu_3$$

$$Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

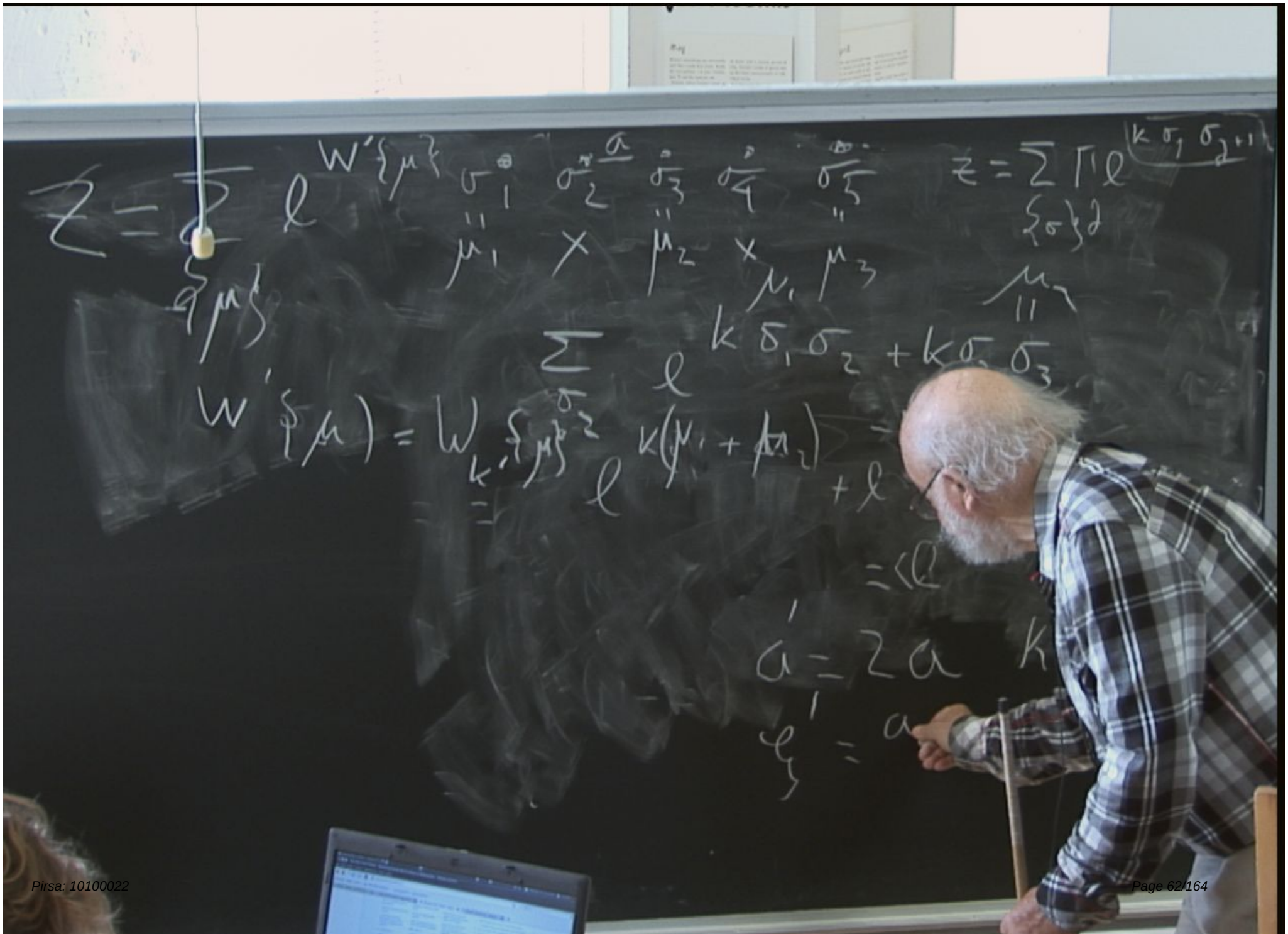
$$\mu_2$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$W'\{\mu\} = W_{k'}\{\mu\} = e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)} + e^{k\mu_1\mu_2}$$

$$= \langle e$$

$$a' = 2a \qquad k' = R(k)$$


Z = Σ_{{μ}} e^{W'{μ}} σ_1 σ_2 σ_3 σ_4 σ_5
μ_1 × μ_2 × μ_1 μ_3
Z = Σ_{{σ}} Π e^{k σ_1 σ_2 + 1}
μ_n
Σ_{σ_2} e^{k σ_1 σ_2 + k σ_2 σ_3}
W'({μ}) = W_k({μ}) e^{k(μ_1 + μ_2)}
a' = 2a
k

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_1 \quad \mu_3$$

$$Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_2$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$W'\{\mu\} = W_k\{\mu\} + e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)} + e^{k\mu_1\mu_2}$$

$$= \langle e$$

$$a' = 2a \qquad k' = R($$

$$\xi' = \frac{a'}{2D(k')} = \xi =$$

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \qquad Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_1 \times \mu_2 \times \mu_1 \mu_3 \qquad \mu_2$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

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$$= 2e$$

$$a' = 2a \qquad k' = R($$

$$\xi' = \frac{a'}{2D(k')} = \xi = \frac{a}{2D}$$

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \qquad Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_1 \quad \mu_3 \qquad \mu_2$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

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$$= 2e^{k\mu_1\mu_2}$$

$$a' = 2a \qquad k' = R(k)$$

$$\xi' = \frac{a'}{D(k')} = \xi = \frac{a}{D(k)}$$

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \; \sigma_2 \; \sigma_3 \; \sigma_4 \; \sigma_5 \qquad Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_1 \; \mu_3 \qquad \mu_n$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$W'\{\mu\} = W_k\{\mu\} \quad e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)} + e^{k\mu_1\mu_2}$$

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$$Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_2$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

$$W'\{\mu\} = W_k\{\mu\} = e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)}$$

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$$a' = 2a \qquad k' = R(k)$$

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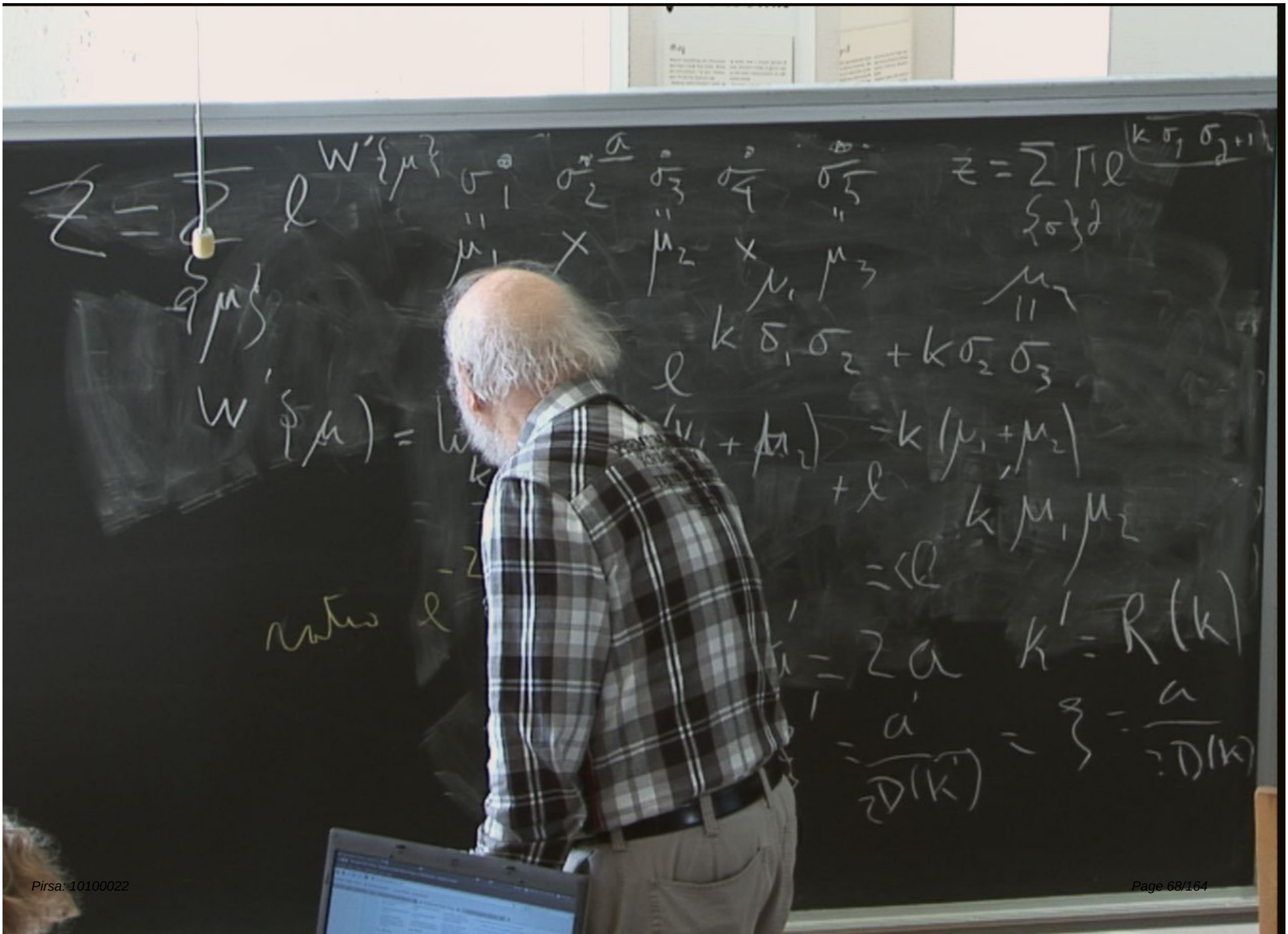
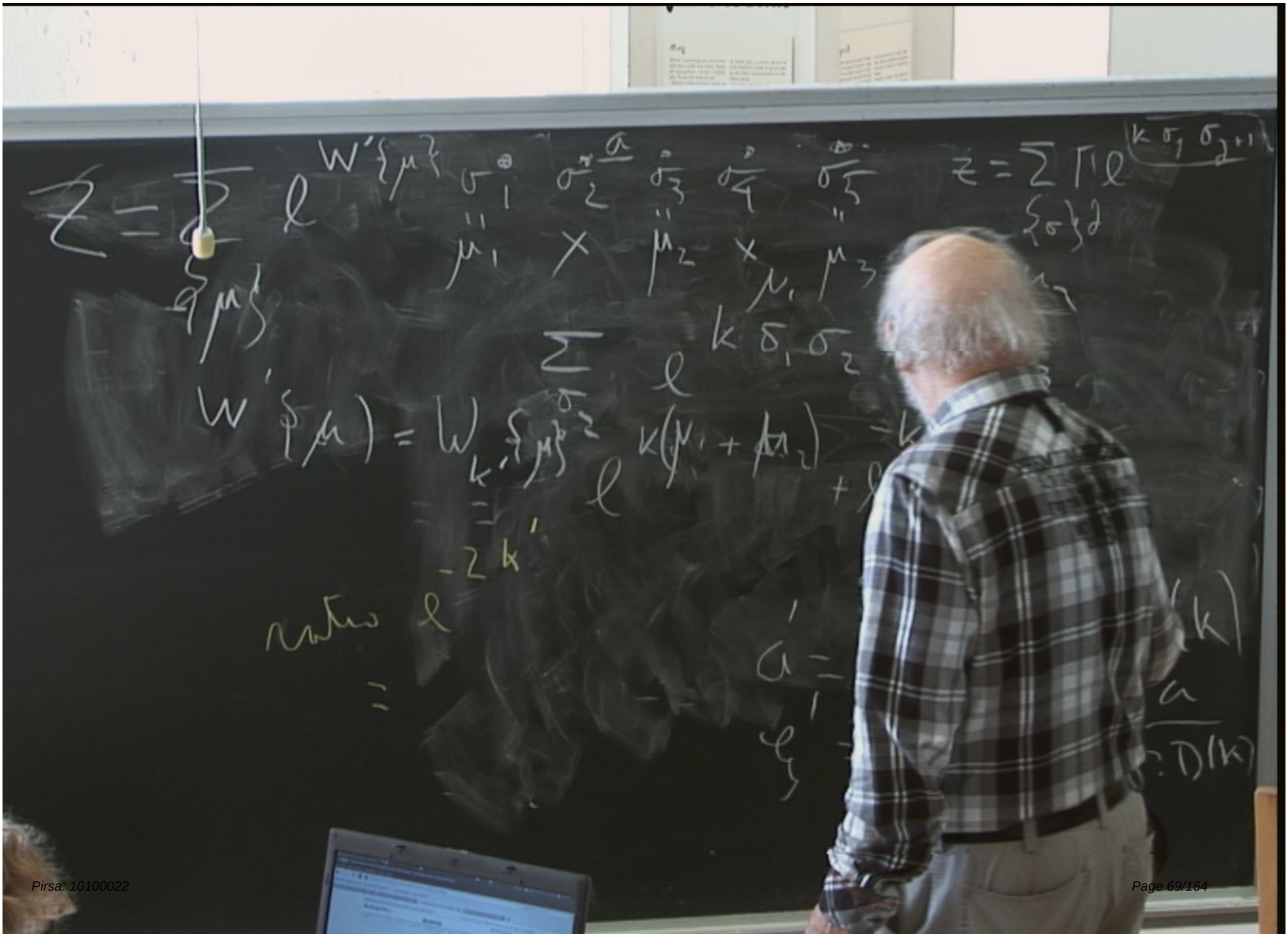
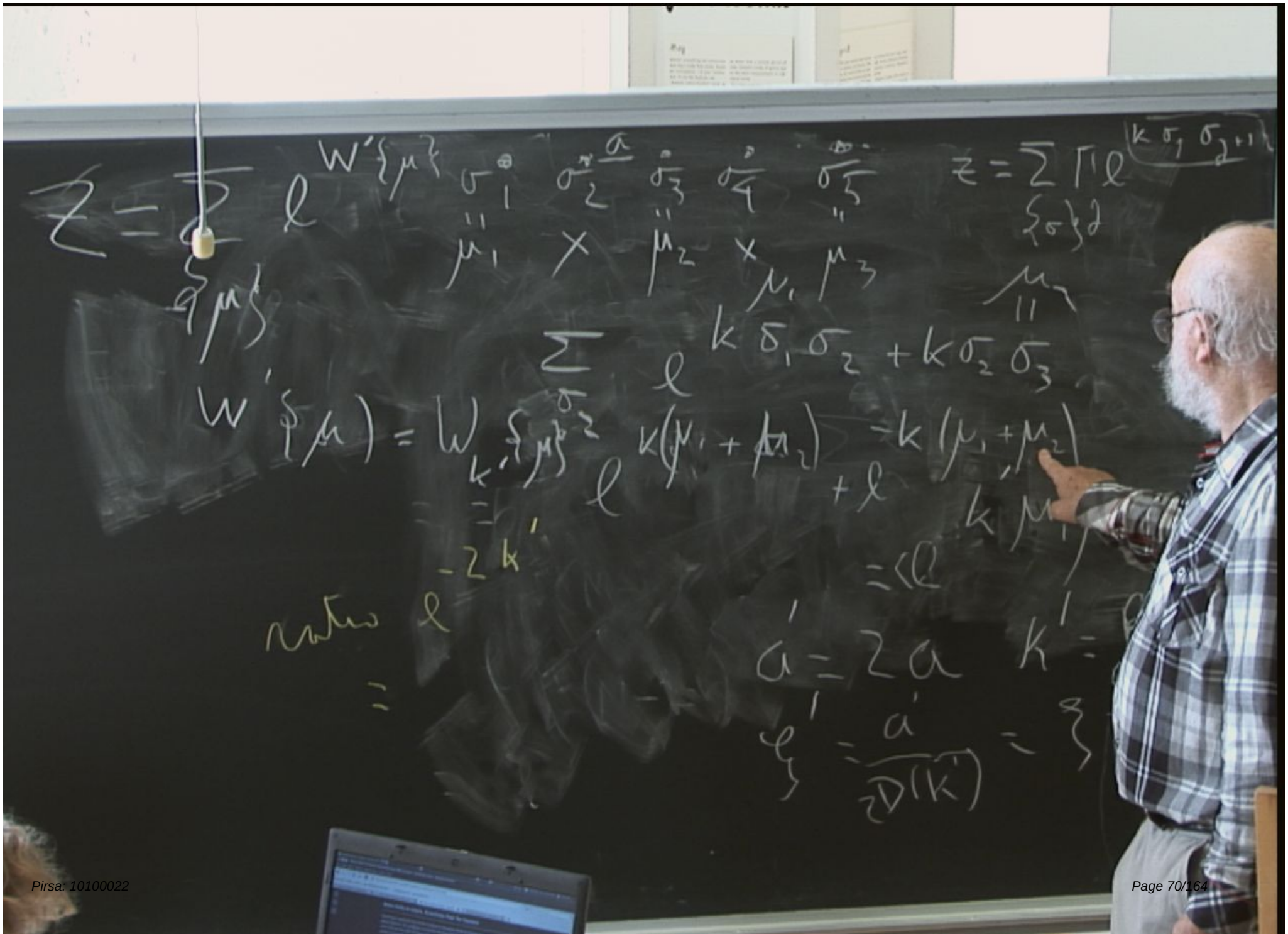

Z = Σ_{{μ}} e^{W'{μ}} σ_1 σ_2 σ_3 σ_4 σ_5
Z = Σ_{{σ}} Π e^{k σ_1 σ_2 + 1}
μ_1 × μ_2 × μ_1 μ_3
μ_2
e^{k σ_1 σ_2 + k σ_2 σ_3}
W'{μ} = W
(μ_1 + μ_2) = k (μ_1 + μ_2)
+ e^{k μ_1 μ_2}
= 2e
k' = R(k)
= 2a
= a'/2D(k') = ξ = a/2D(k)




$Z = \sum_{\{\mu\}} e^{W'\{\mu\}}$
$\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5$
$\mu_1 \quad \times \quad \mu_2 \quad \times \quad \mu_1 \quad \mu_3$
$Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$
μ_2
$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$
$W'\{\mu\} = W_{k'}\{\mu\}$
$e^{k(\mu_1+\mu_2)} + e^{k(\mu_1+\mu_2)}$
$= 2k'$
$= 2e^{k\mu_1}$
$a' = 2a$
$e' = \frac{a'}{2D(k')}$

$$Z = \sum_{\{\mu\}} e^{W'\{\mu\}} \quad \sigma_1 \; \sigma_2 \; \sigma_3 \; \sigma_4 \; \sigma_5 \qquad Z = \sum_{\{\sigma\}} \prod e^{k\sigma_1\sigma_2 + 1}$$

$$\mu_1 \times \mu_2 \times \mu_1 \; \mu_3 \qquad \mu_2$$

$$\sum_{\sigma_2} e^{k\sigma_1\sigma_2 + k\sigma_2\sigma_3}$$

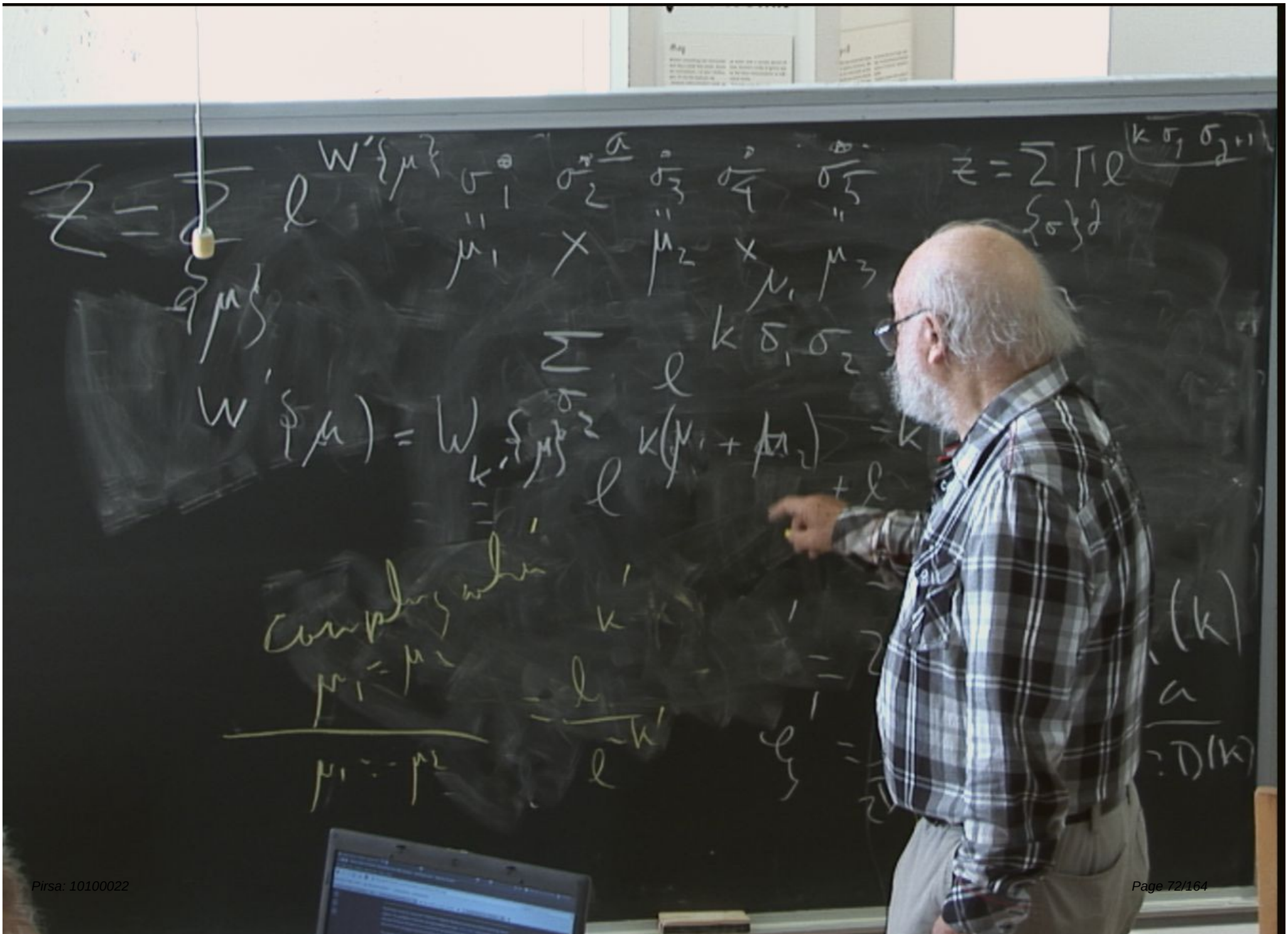
$$W'\{\mu\} = W_{k'}\{\mu\} \quad e^{k(\mu_1+\mu_2)} + e^{-k(\mu_1+\mu_2)} = 2\cosh k\mu_1\mu_2$$

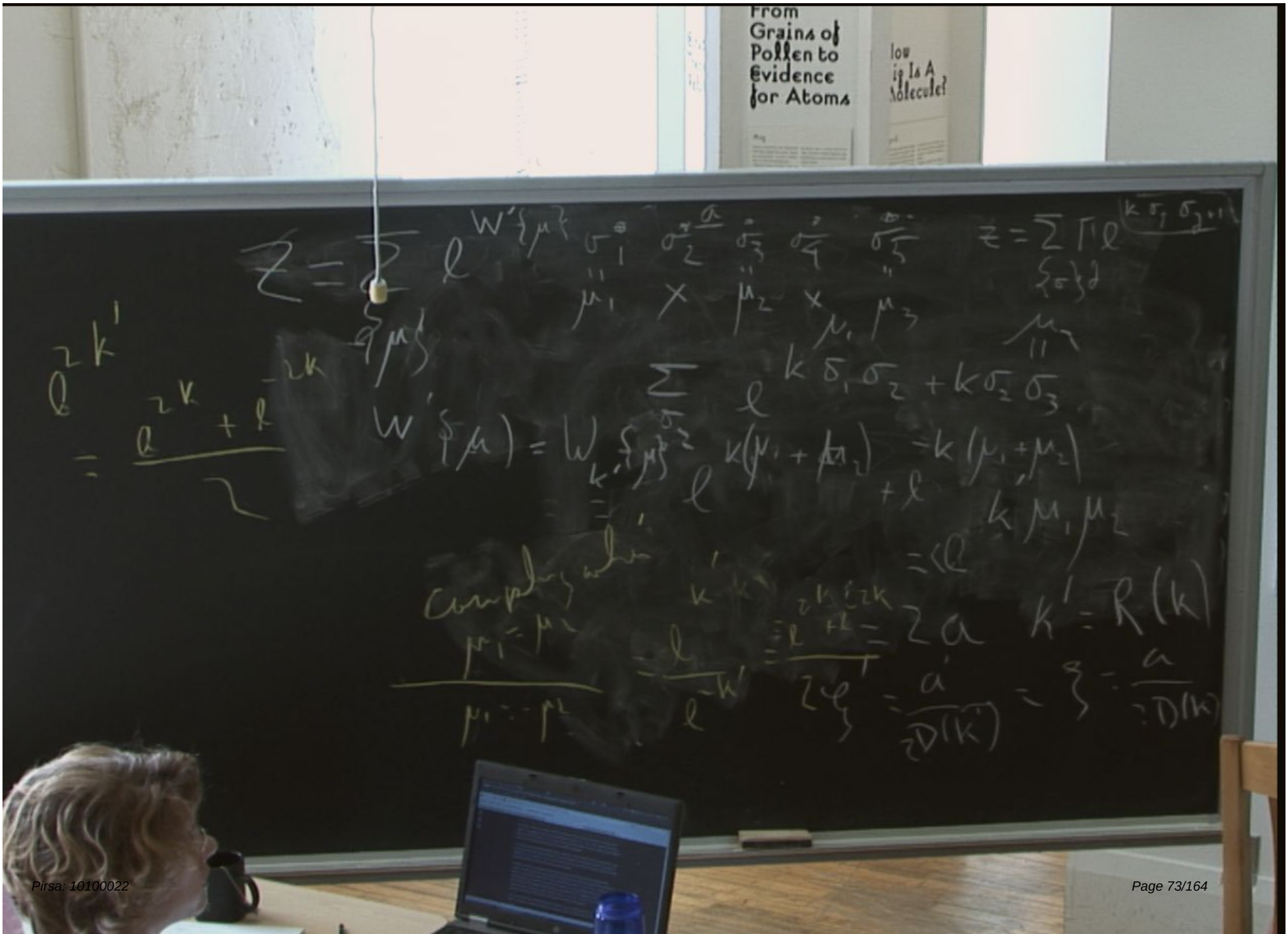
$$= e^{-2k'}$$

$$\text{notes } e^{2k} + e^{-k^2} \qquad a' = 2a \qquad k' = R(k)$$

$$= \frac{e^{2k} + e^{-k^2}}{}$$

$$\xi' = \frac{a'}{2D(k')} = \xi = \frac{a}{2D(k)}$$


Z = Σ_{{μ}} e^{W'{μ}} σ_1 σ_2 σ_3 σ_4 σ_5
μ_1 × μ_2 × μ_1 μ_3
Z = Σ_{{σ}} Π e^{k σ_1 σ_2 + 1}
Σ_σ e^{k σ_1 σ_2}
W'{μ} = W_k {μ}
e^{k(μ_1 + μ_2)}
coupling when
μ_1 = μ_2
μ_1 ≠ μ_2
= e / e^{-k'}
(k)
a
D(k)


From
Grains of
Pollen to
Evidence
for Atoms
How
Big Is A
Molecule?

Renormalization Calculation

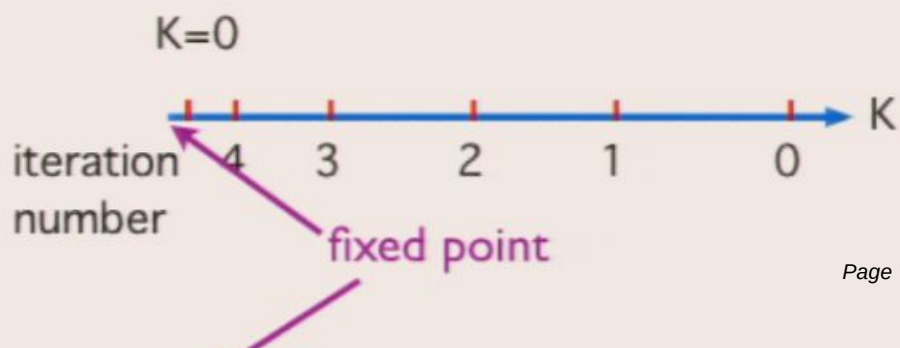
$$Z = \sum_{\mu_1,\mu_2,\cdots} \sum_{\sigma_2,\sigma_4,\cdots} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2 + \cdots) = \sum_{\mu_1,\mu_2,\cdots} \exp(w'\{\mu\})$$

So the new nearest neighbor coupling term is given by

$$\exp(K_0' + K'\mu_1\mu_2) = \sum_{\sigma_2} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2)$$

which then gives us $\exp(K_0' + K') = e^{2K} + e^{-2K}$ and $\exp(K_0' - K') = 2$
so that $e^{2K'} = \cosh 2K$. [Note $K' < K$]

The renormalization calculation tells us what we know already, namely that the one-dimensional model has no phase transition. A phase transition is a change in the long-ranged structure of correlations in a system. Here the couplings gradually weaken as you renormalize to longer and longer distances. All possible values of the coupling reduce to weak couplings at long distances. The system is always in the weak coupling phase. So there is no phase transition.



After many iterations coupling approaches fixed point at $K=0$

Renormalization Calculation

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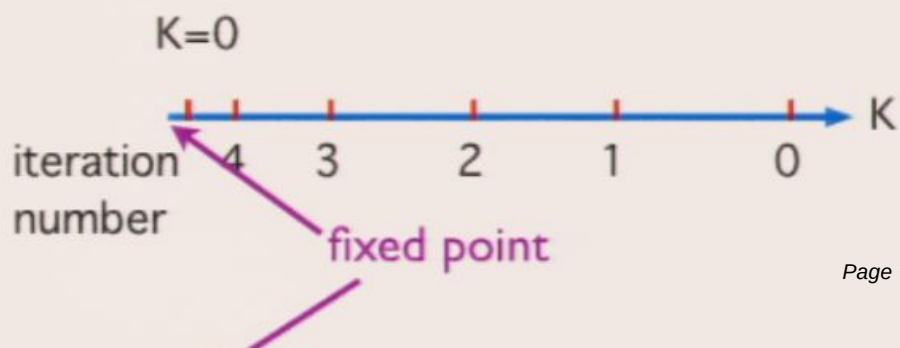


After many iterations coupling approaches fixed point at $K=0$

Ising Model in d=2

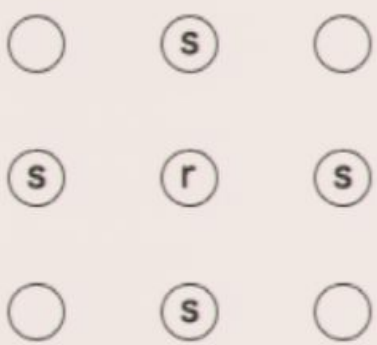
$$-H/(kT) = K\sum_{nn}\sigma_r\sigma_s + h\sum_r \sigma_r$$

$\sigma_r = \pm 1$

nn indicates a sum over nearest neighboirs

square lattice

Onsager calculated partition function and phase transition for this situation



Nearest neighbor structure
s's are nearest neighbors to r
Bonds=exp($K\sigma\sigma'$) connect nearest neighbors

Renormalization Calculation

$$Z = \sum_{\mu_1,\mu_2,\cdots} \sum_{\sigma_2,\sigma_4,\cdots} \exp(K\mu_1\sigma_2 + K\sigma_2\mu_2 + \cdots) = \sum_{\mu_1,\mu_2,\cdots} \exp(w'\{\mu\})$$

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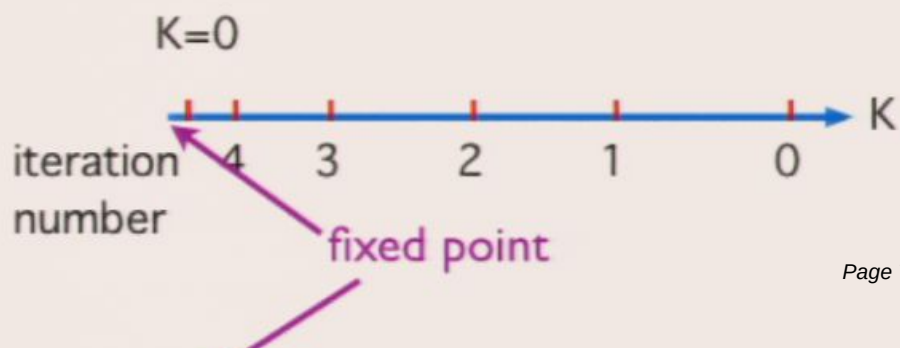

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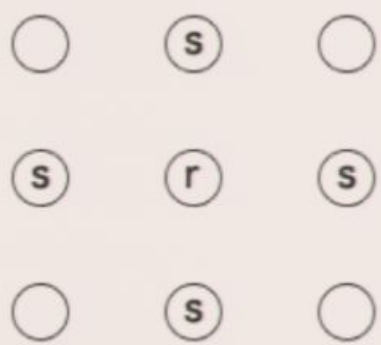


Nearest neighbor structure
s's are nearest neighbors to r
Bonds=exp($K\sigma\sigma'$) connect nearest neighbors

High Temperature Expansion

Nearest neighbor structure

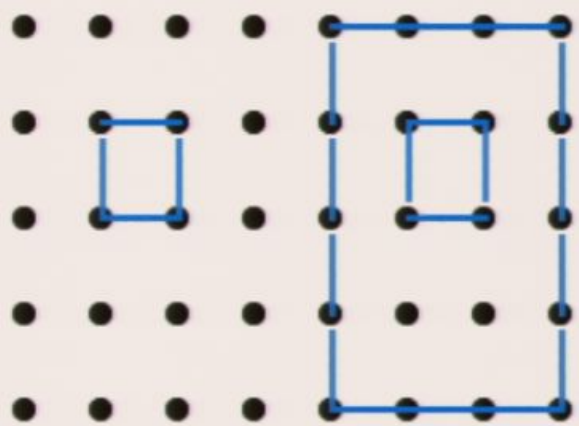
A bond=exp($K\sigma\sigma'$) connect nearest neighbors

Bond=cosh K +$\sigma\sigma'$ sinh K =cosh K[1+ $\sigma\sigma'$ tanh K]

$Z=(2 \cosh K \cosh K)^N$ < products on all bonds of [1+ $\sigma\sigma'$ tanh K] >

$= (2 \cosh K \cosh K)^N$ sum < products of $(\tanh K)^M$ >

for nonzero terms, when there are N sites



To get a non-zero value each spin must appear on a even number of bonds. You then get the lattice covered by closed polygons.

With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

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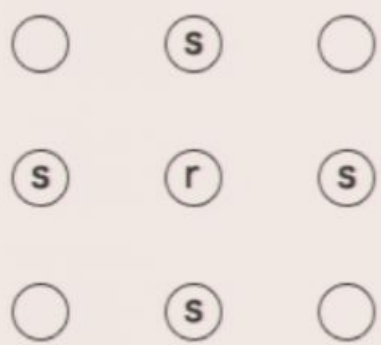

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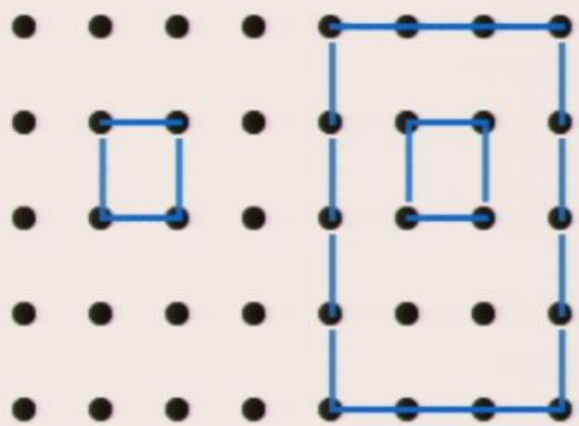

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High Temperature Expansion

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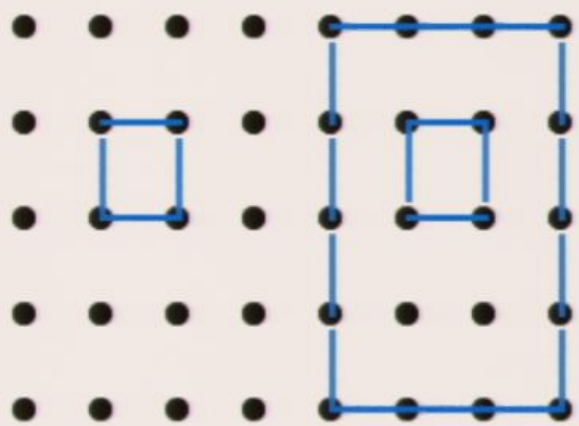

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With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

$$\varphi(\mu) =$$

$$m \sum_{n=0}^{10} a_n k^n$$

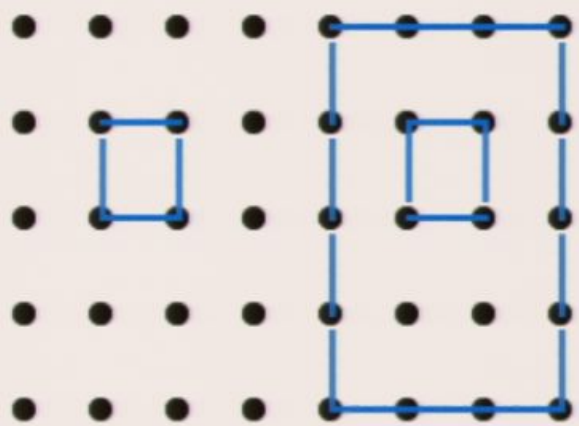
$$\mu_1 = \mu$$

$$\mu_1 = \mu_2$$

High Temperature Expansion

Nearest neighbor structure
A bond=exp($K\sigma\sigma'$) connect nearest neighbors
Bond=cosh K +$\sigma\sigma'$ sinh K =cosh K[1+ $\sigma\sigma'$ tanh K]

Z=(2 cosh K cosh K)^N < products on all bonds of [1+ $\sigma\sigma'$ tanh K] >
= (2 cosh K cosh K)^N sum < products of (tanh K)^M >
for nonzero terms, when there are N sites



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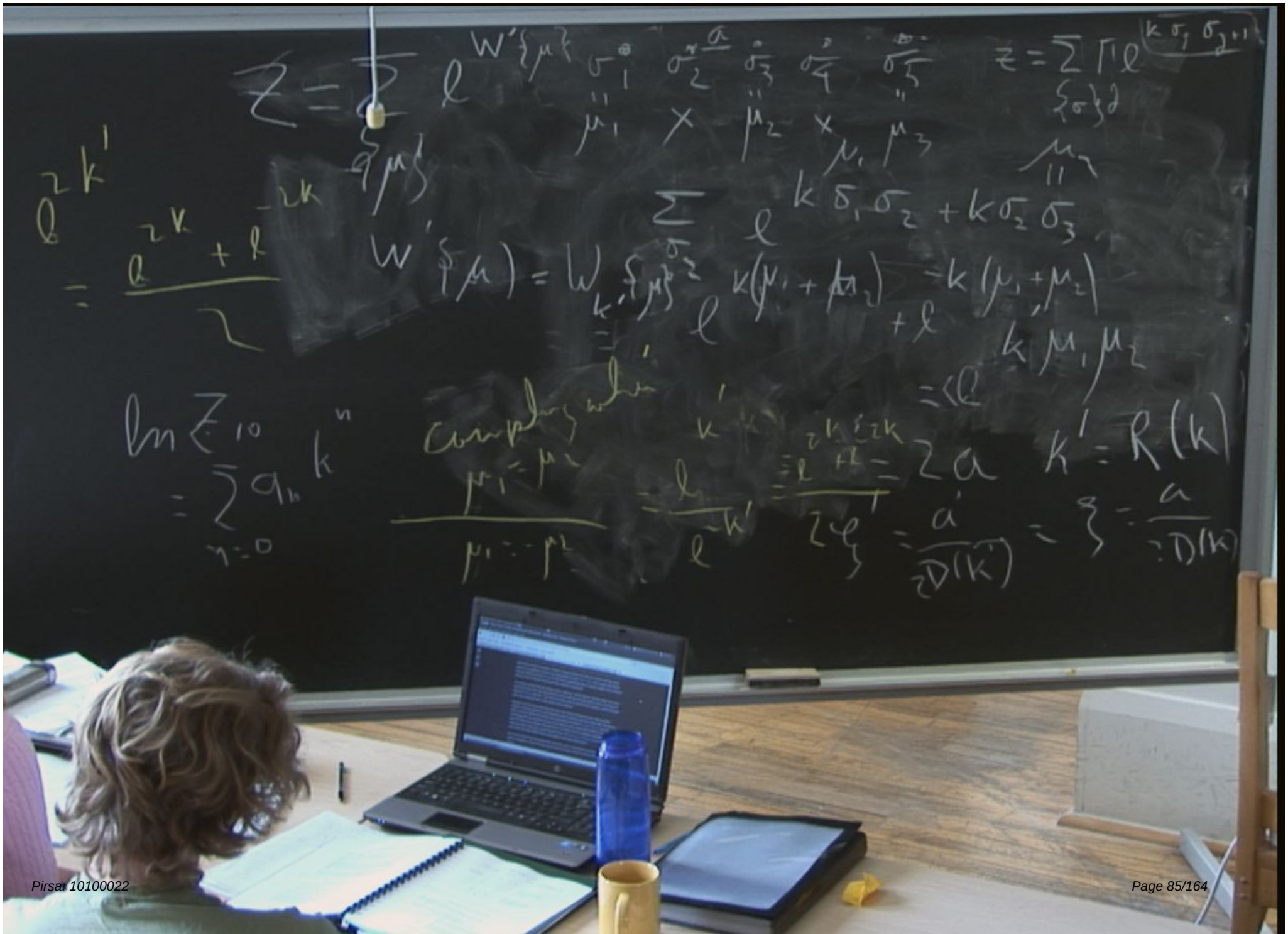

High Temperature Expansion

Nearest neighbor structure

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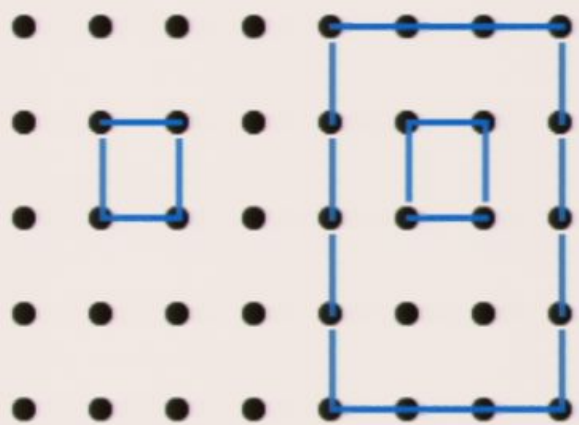

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With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

$$\ln Z = \sum_{n=0}^{\infty} a_n \, \beta^n$$

High Temperature Expansion

Nearest neighbor structure

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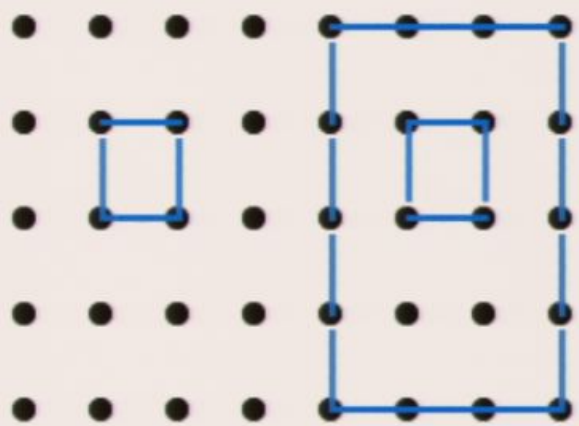

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$$\left(\frac{\sinh k}{k}\right)^{N} \sum_{n=0}^{\infty} a_n (\tanh k)^n =$$

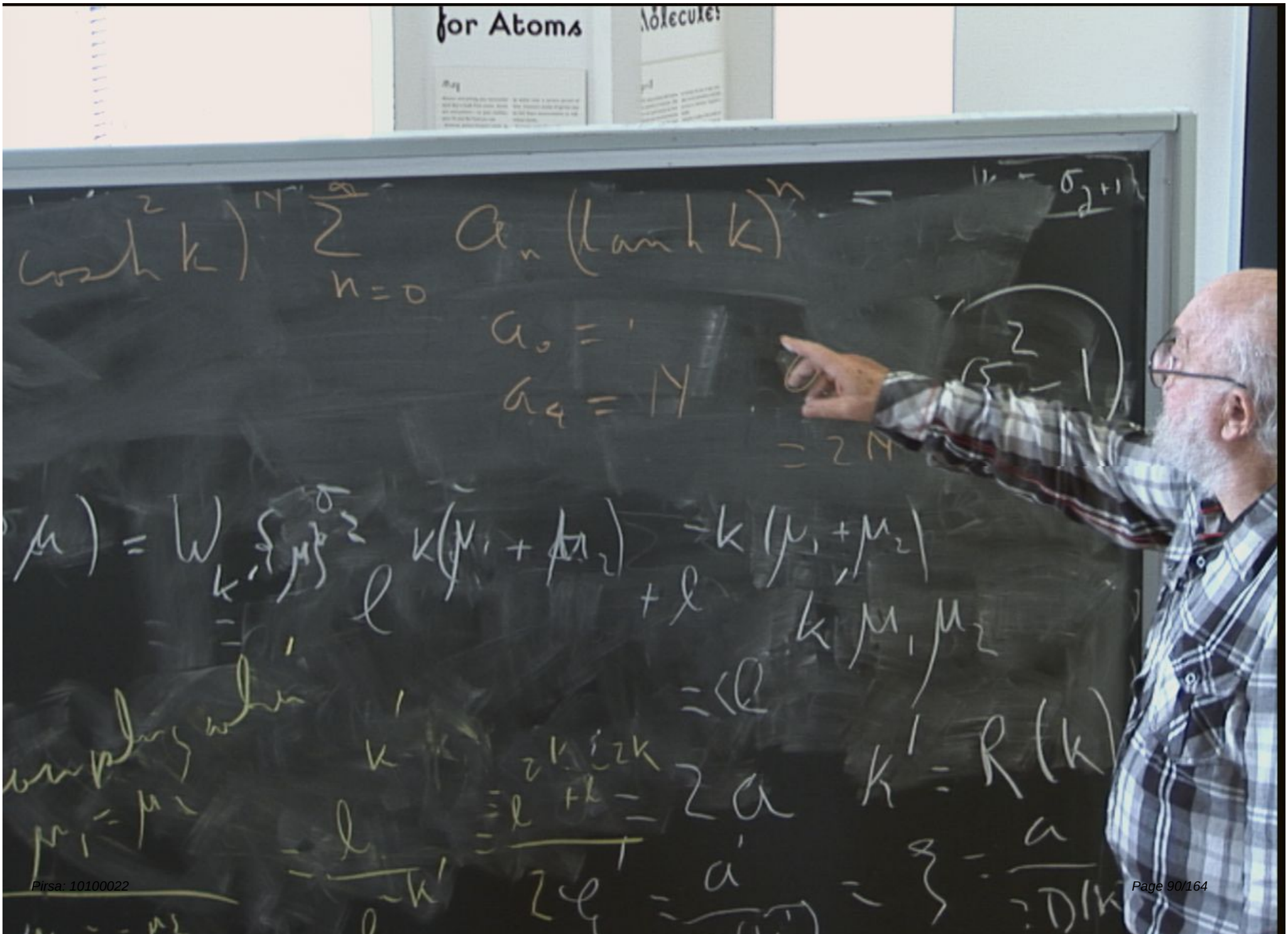
$$a_0 = 1$$

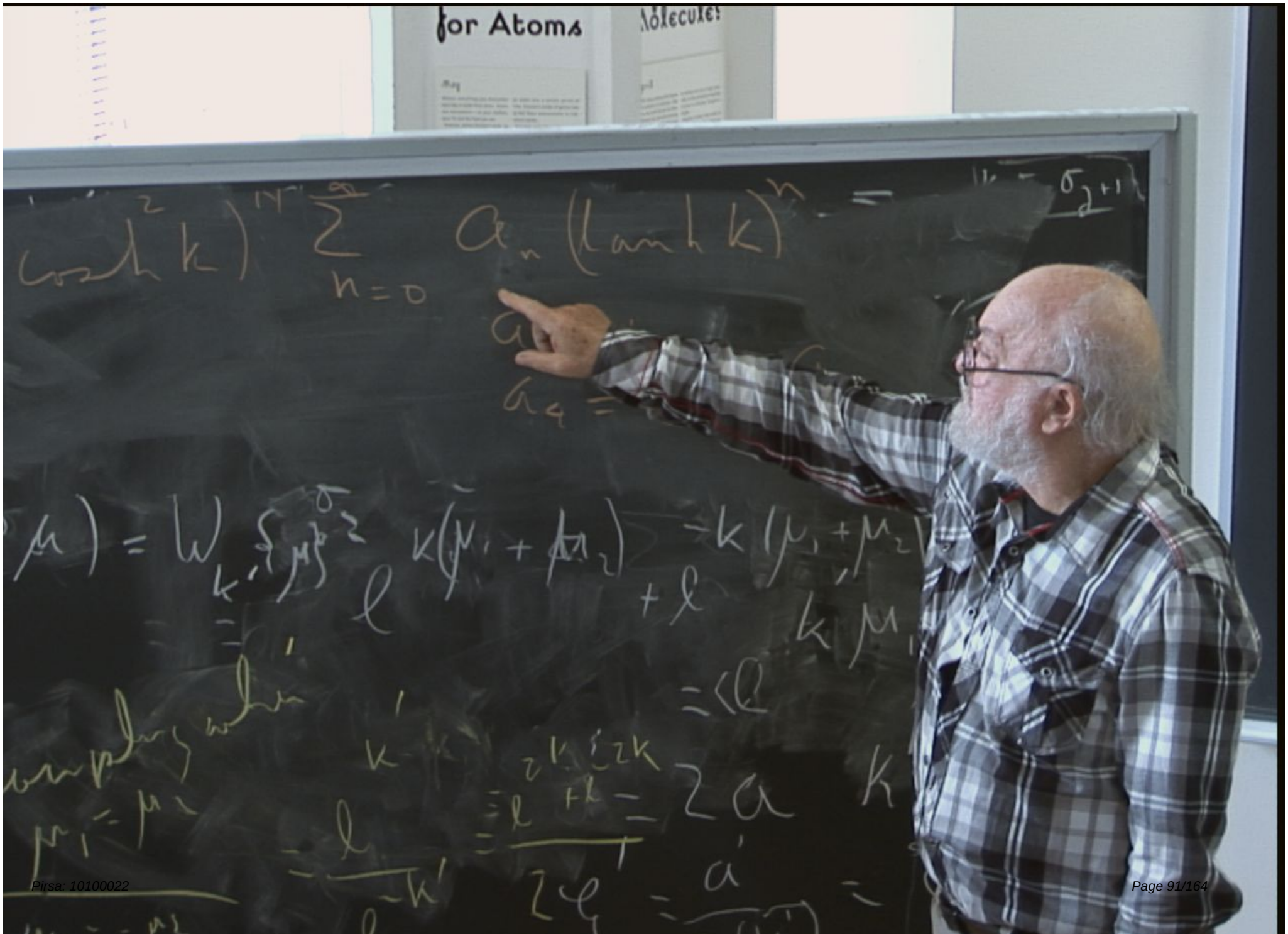
$$a_4 = N \quad a_6 = 2N$$

$$= W_k \, e^{k(\mu_1 + \mu_2)} = e^{k(\mu_1 + \mu_2)} = e^{k\mu_1\mu_2}$$

$$= e^{k\mu_1\mu_2}$$

$$k \quad 2k \quad 2k$$


for Atoms
Molecules
$a_0 = 1$
$a_4 = N$
$= 2N$
$k' = R(k)$


for Atoms

High Temperature Expansion

Nearest neighbor structure

A bond=exp($K\sigma\sigma'$) connect nearest neighbors

Bond=cosh K +$\sigma\sigma'$ sinh K =cosh K[1+ $\sigma\sigma'$ tanh K]

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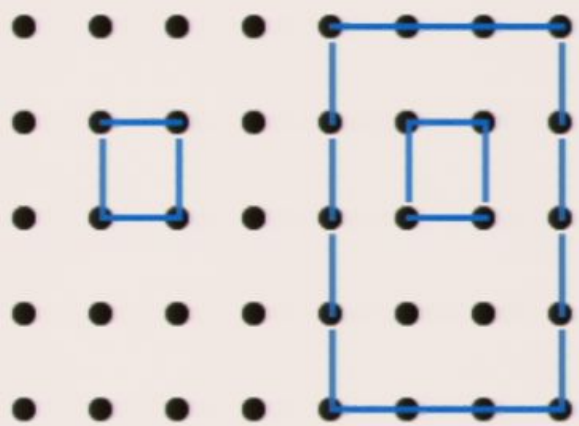

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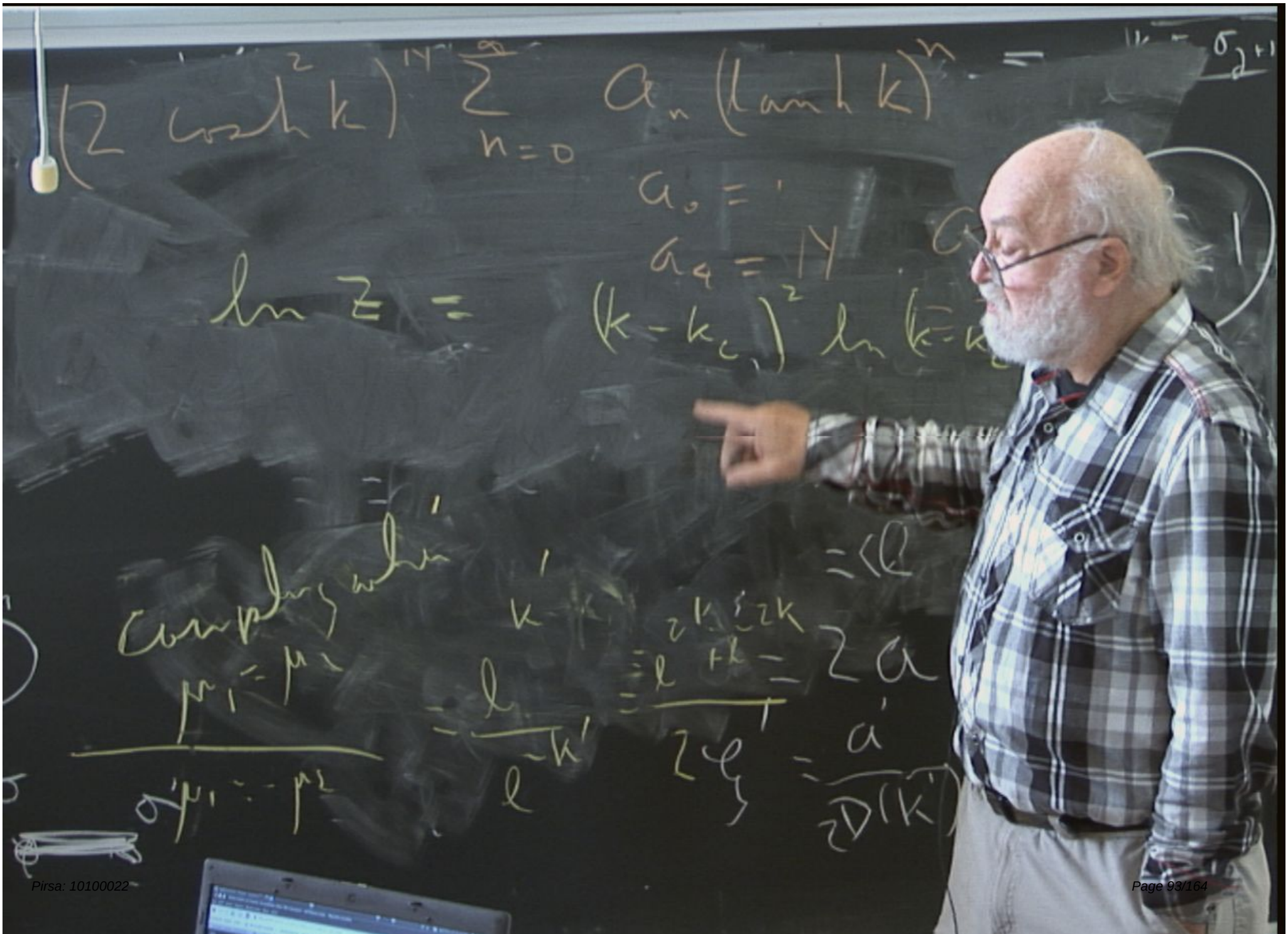
$(2\cosh k)^{14} \sum_{n=0}^{\infty} a_n (\tanh k)^n =$
$a_0 = 1$
$a_4 = 14$
$\ln Z = (k-k_c)^2 \ln(k-k_c)$

High Temperature Expansion

Nearest neighbor structure

A bond=exp($K\sigma\sigma'$) connect nearest neighbors

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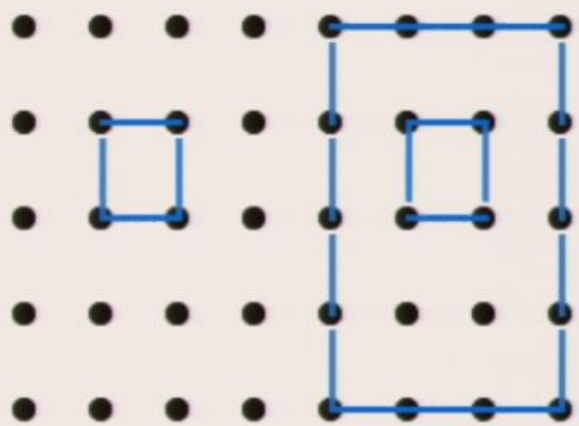

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With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

$$(2\cosh^2 k)^N \sum_{n=0}^{\infty} a_n (\tanh k)^n = \quad \sigma_{2+1}$$

$$a_0 = 1$$

$$a_4 = N \quad a_6 \qquad \sigma^2 = 1$$

$$\ln Z = (k-k_c)^2 \ln (k-k_c)^N$$

coupling when

$$\frac{\mu_1 = \mu_2}{\sigma\mu_1 = \mu_2} = \frac{l}{l'} = \frac{2k \quad 2k}{l \quad l'} = 2a \qquad k' = R(k)$$

$$2\xi = \frac{a'}{D(k')} = \xi = \frac{a}{D(k)}$$

High Temperature Expansion

Nearest neighbor structure

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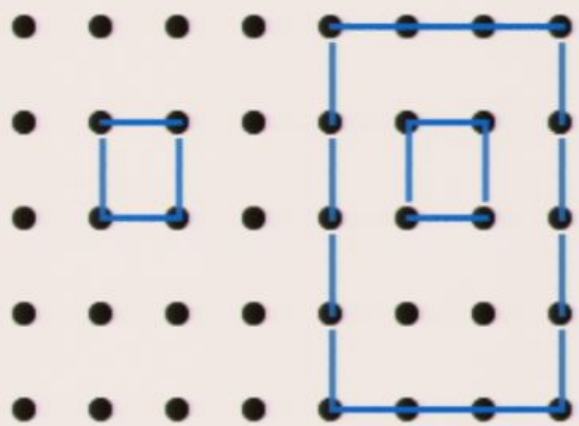

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With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

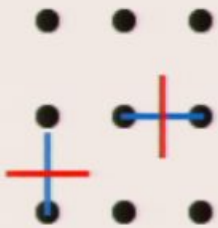
Low Temperature Expansion

Nearest neighbor structure

A bond$=\exp(K\sigma\sigma') = e^{K}\delta_{\sigma,\sigma'} + e^{-K}\delta_{\sigma,-\sigma'}$

Bond $= e^{K}[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}]$

We draw these bonds differently from the high T bonds. We draw them rotated 90 degrees in comparison to the other bonds.

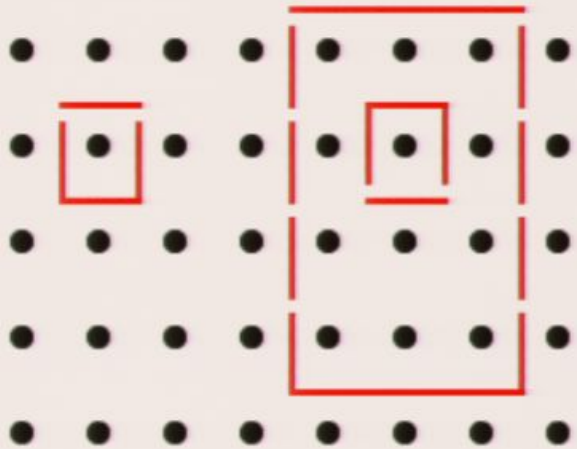


note $e^{-2K} = \tanh \tilde{K}$

$Z = 2(e^{K})^{N}(e^{K})^{N} <$ products of $[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}] >$

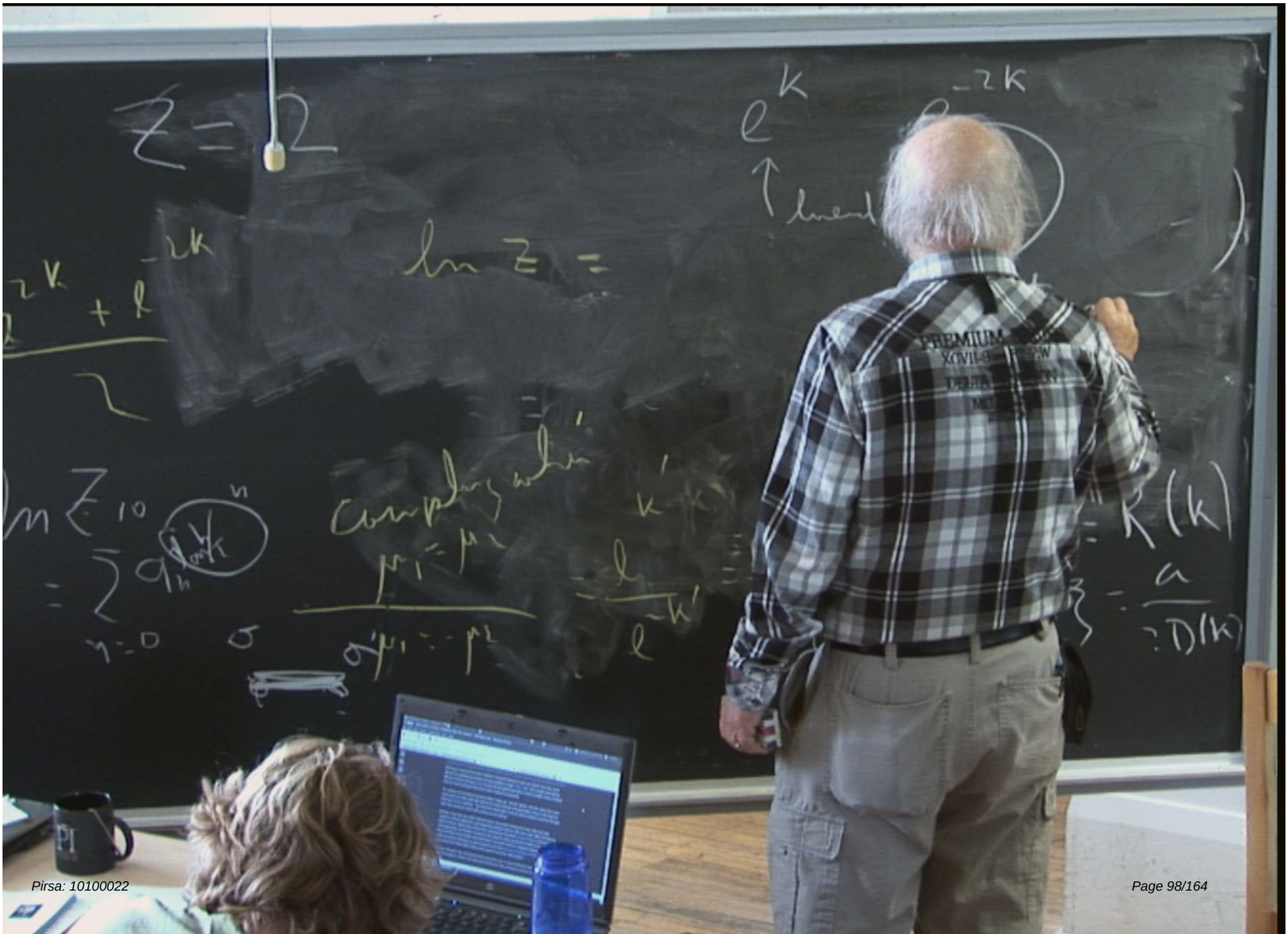
$= 2e^{2NK}$ sum $<$ products of $(e^{-2K})^{M} >$

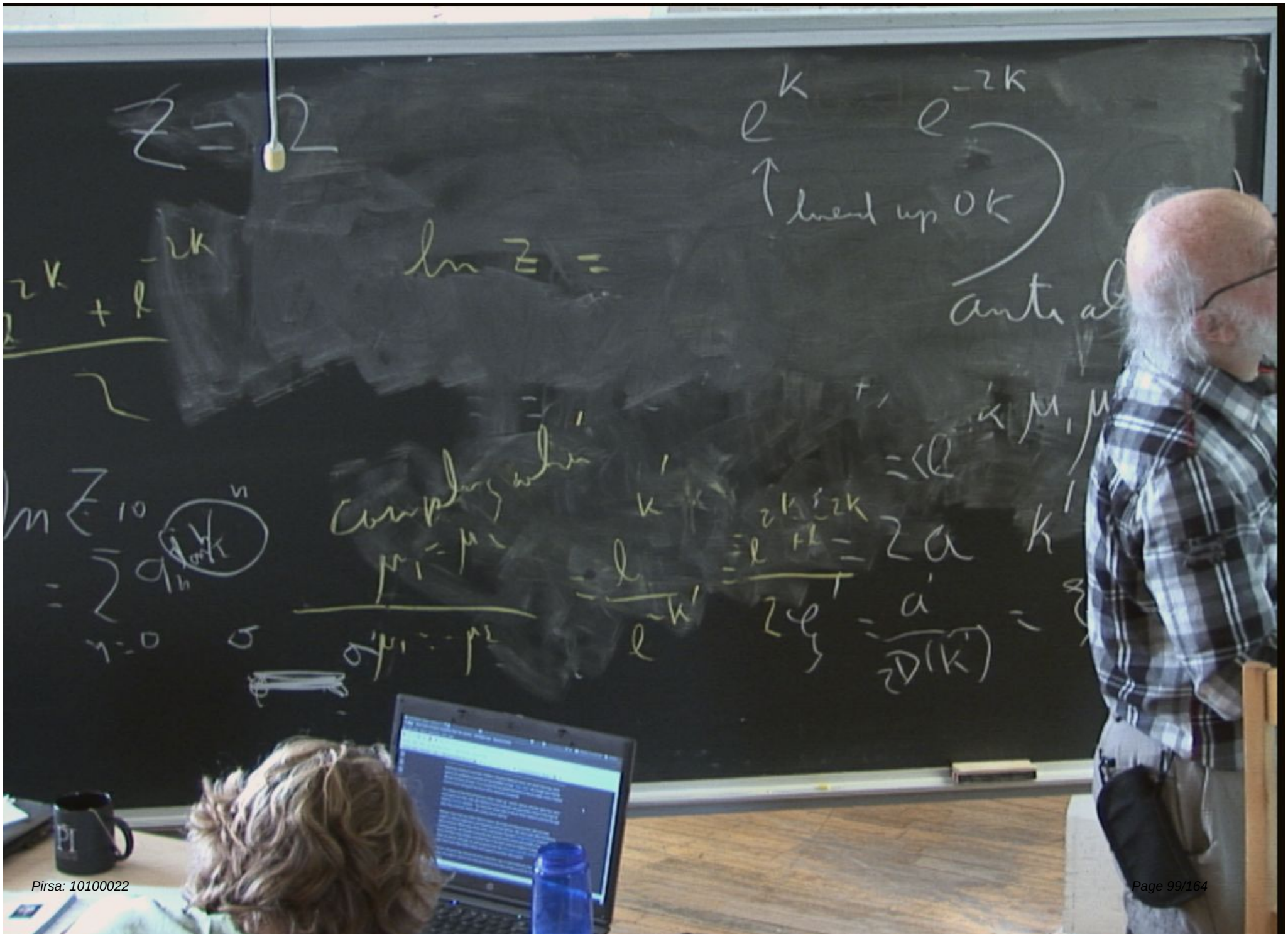
for nonzero terms

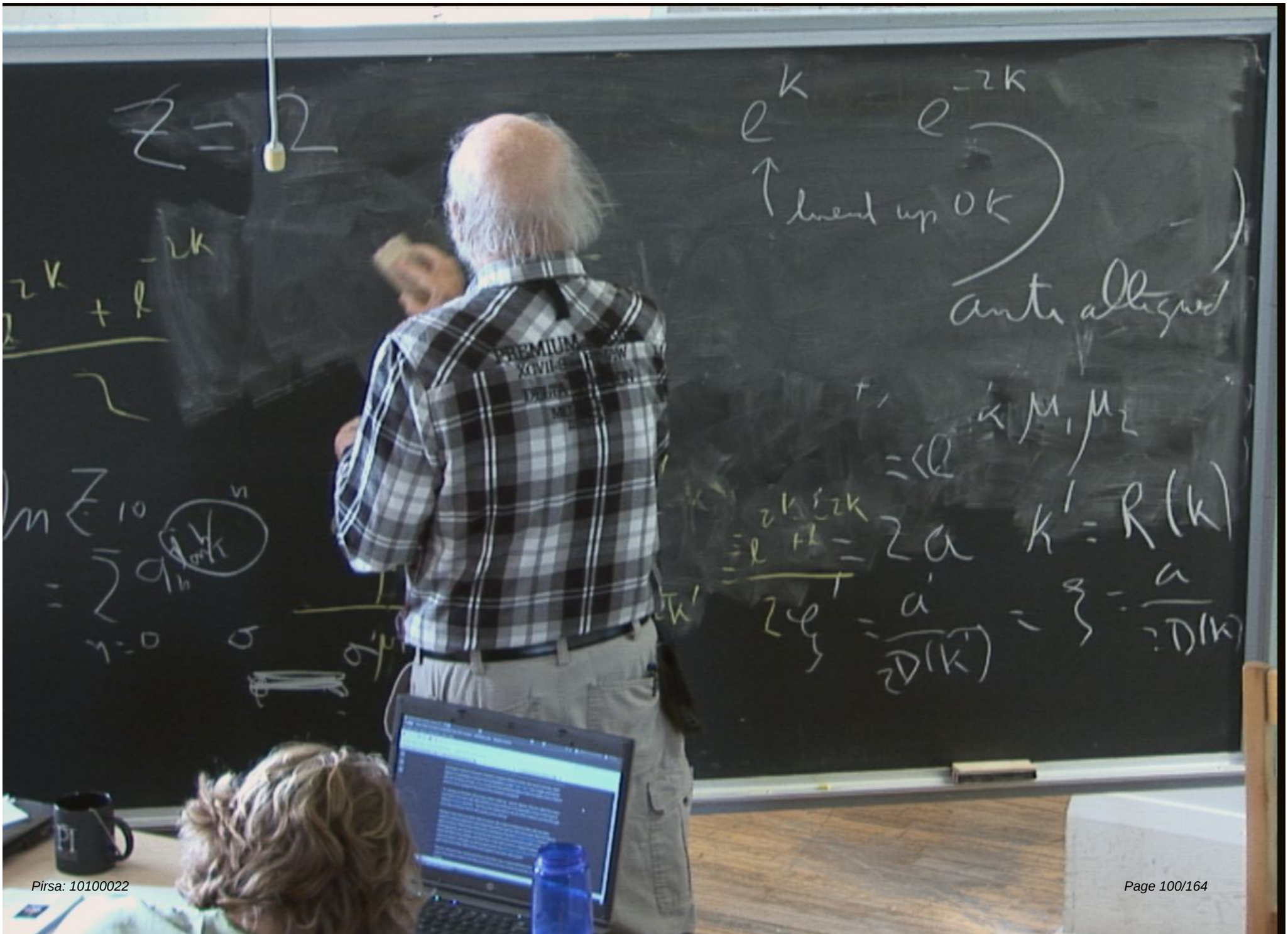


To get a non-zero term, assign a value to one spin. Then every time you cross a red line, change the spin-value to the opposite. Your valid pictures become a series of closed red polygons.

With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres


Z = 2
ln Z =


Z = 2
ln Z =
blend up OK
anti al


e^{k}
e^{-2k}
lined up OK
anti aligned
$k' = R(k)$

Low Temperature Expansion

Nearest neighbor structure

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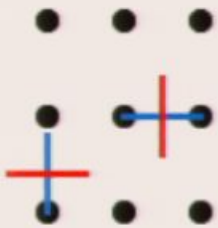
Bond $= e^{K}[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}]$

We draw these bonds differently from the high T bonds. We draw them rotated 90 degrees in comparison to the other bonds.

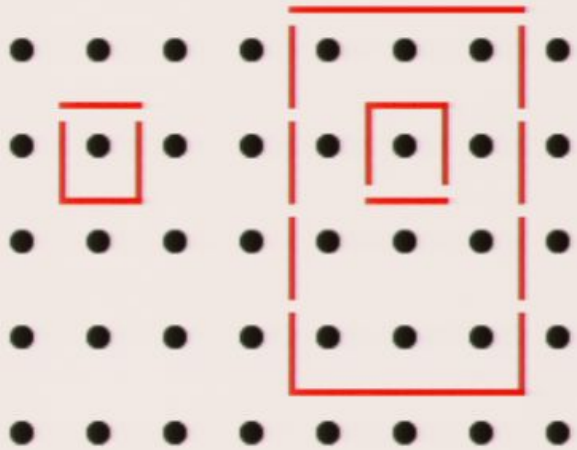


note $e^{-2K} = \tanh \tilde{K}$

$Z = 2(e^{K})^{N}(e^{K})^{N} <$ products of $[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}] >$

$= 2e^{2NK}$ sum $<$ products of $(e^{-2K})^{M} >$

for nonzero terms



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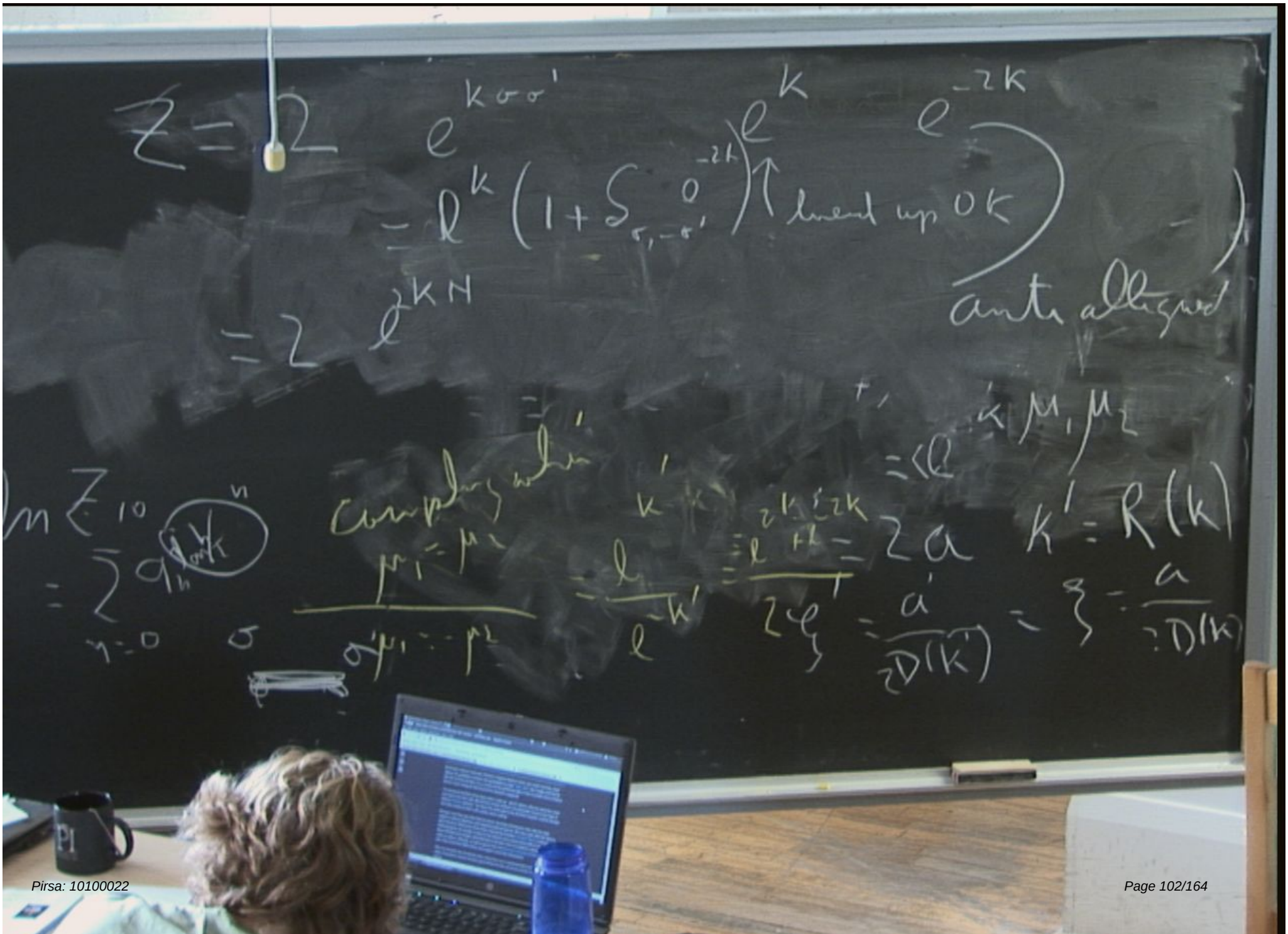
Z = Σ e^{Kσσ'}
= e^K (1 + δ_{σ,-σ'}) e^K e^{-2K}
= 2 e^{KN}
lowest up OK
antialligned
coupling
μ1 = μ2
K' = R(K)

High Temperature Expansion

Nearest neighbor structure

A bond=exp($K\sigma\sigma'$) connect nearest neighbors

Bond=cosh K +$\sigma\sigma'$ sinh K =cosh K[1+ $\sigma\sigma'$ tanh K]

$Z=(2 \cosh K \cosh K)^N$ < products on all bonds of [1+ $\sigma\sigma'$ tanh K] >

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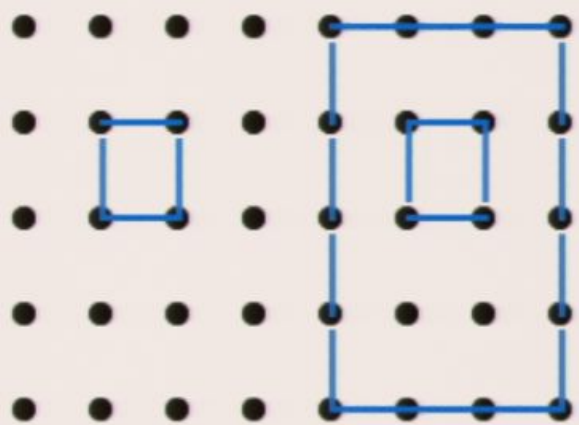
for nonzero terms, when there are N sites



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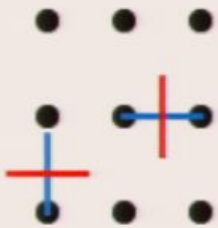

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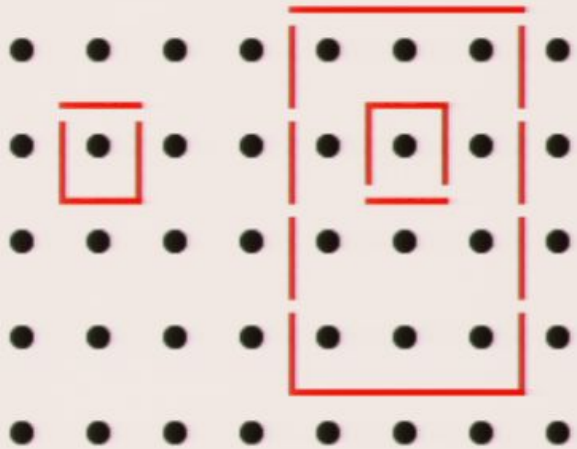

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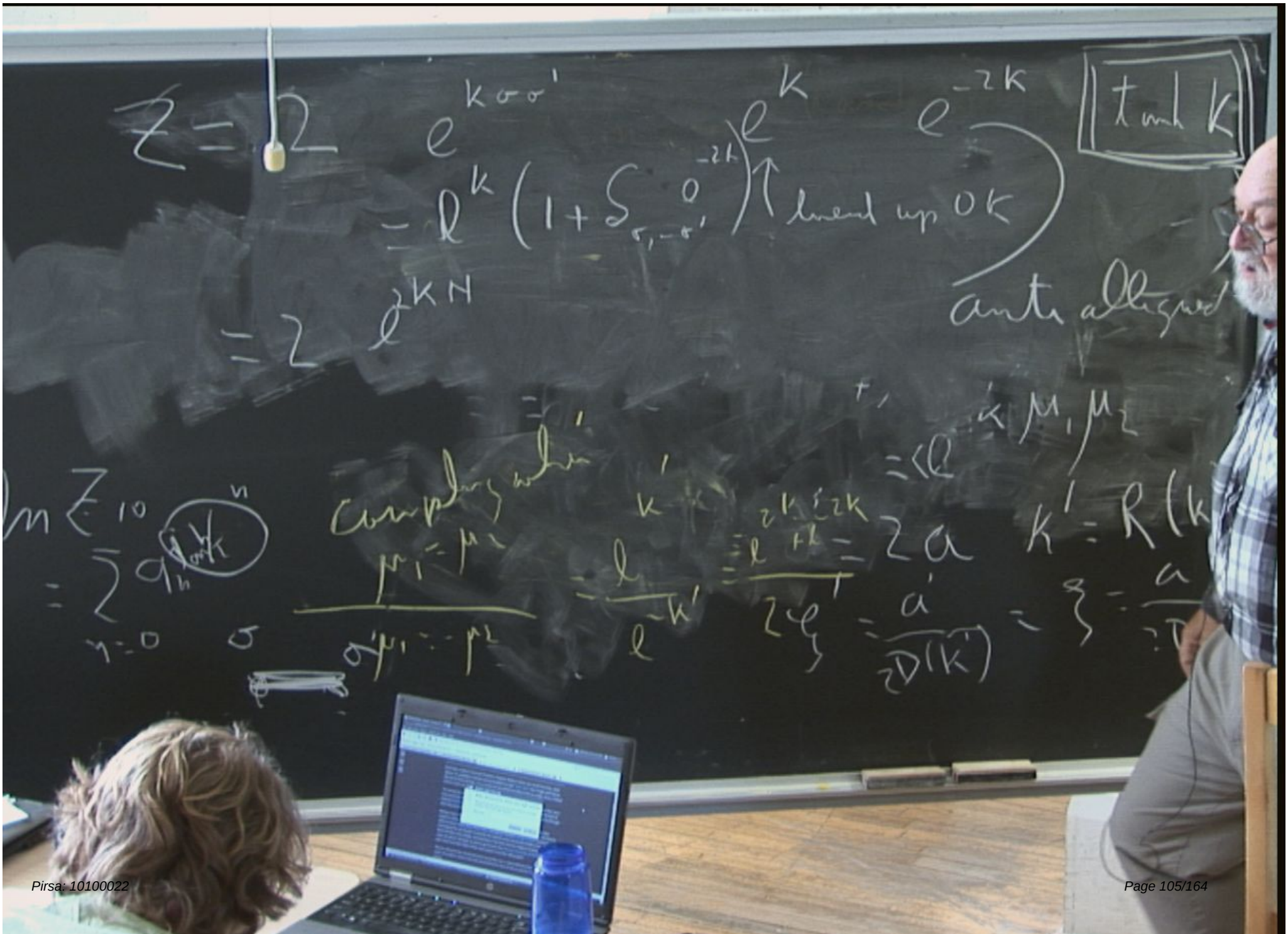
tanh K
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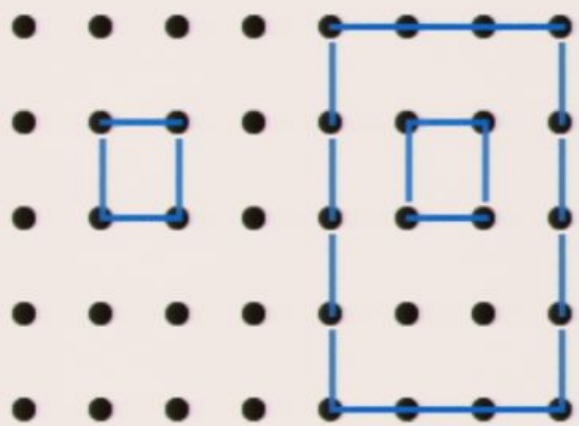

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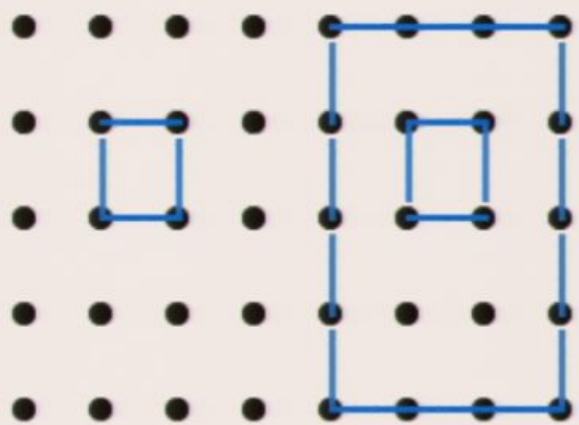

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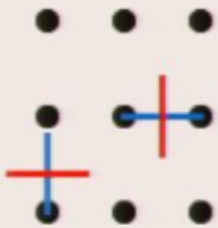

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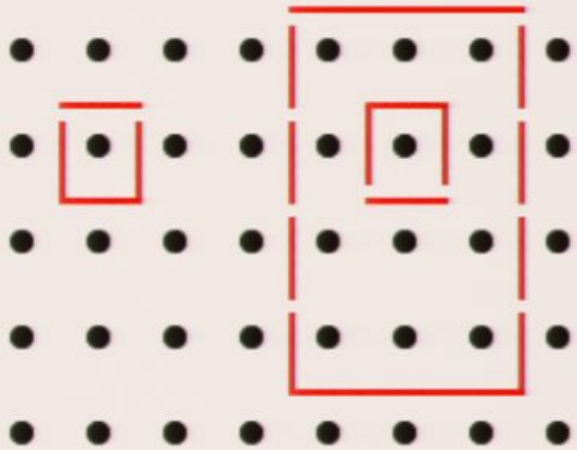

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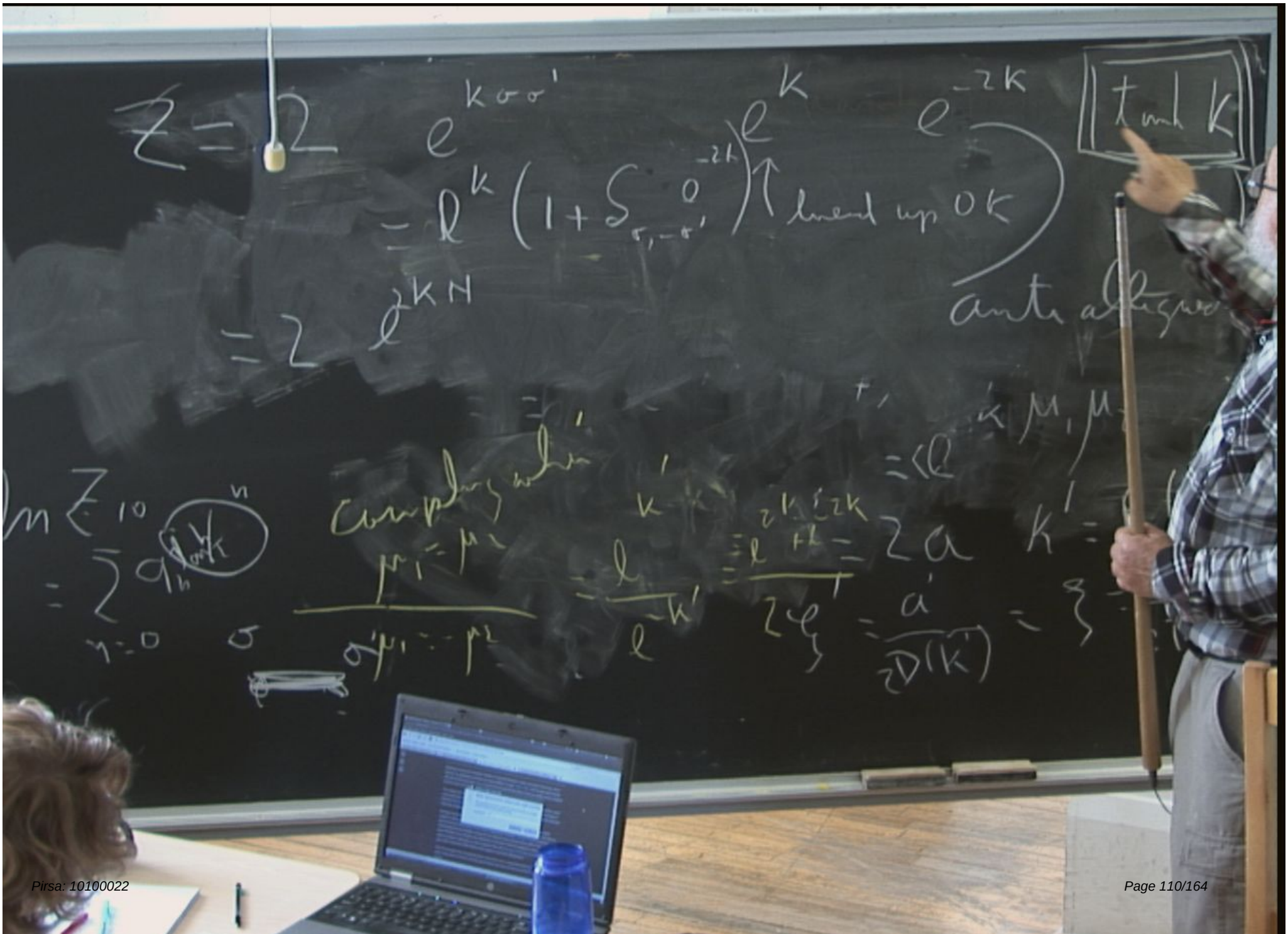
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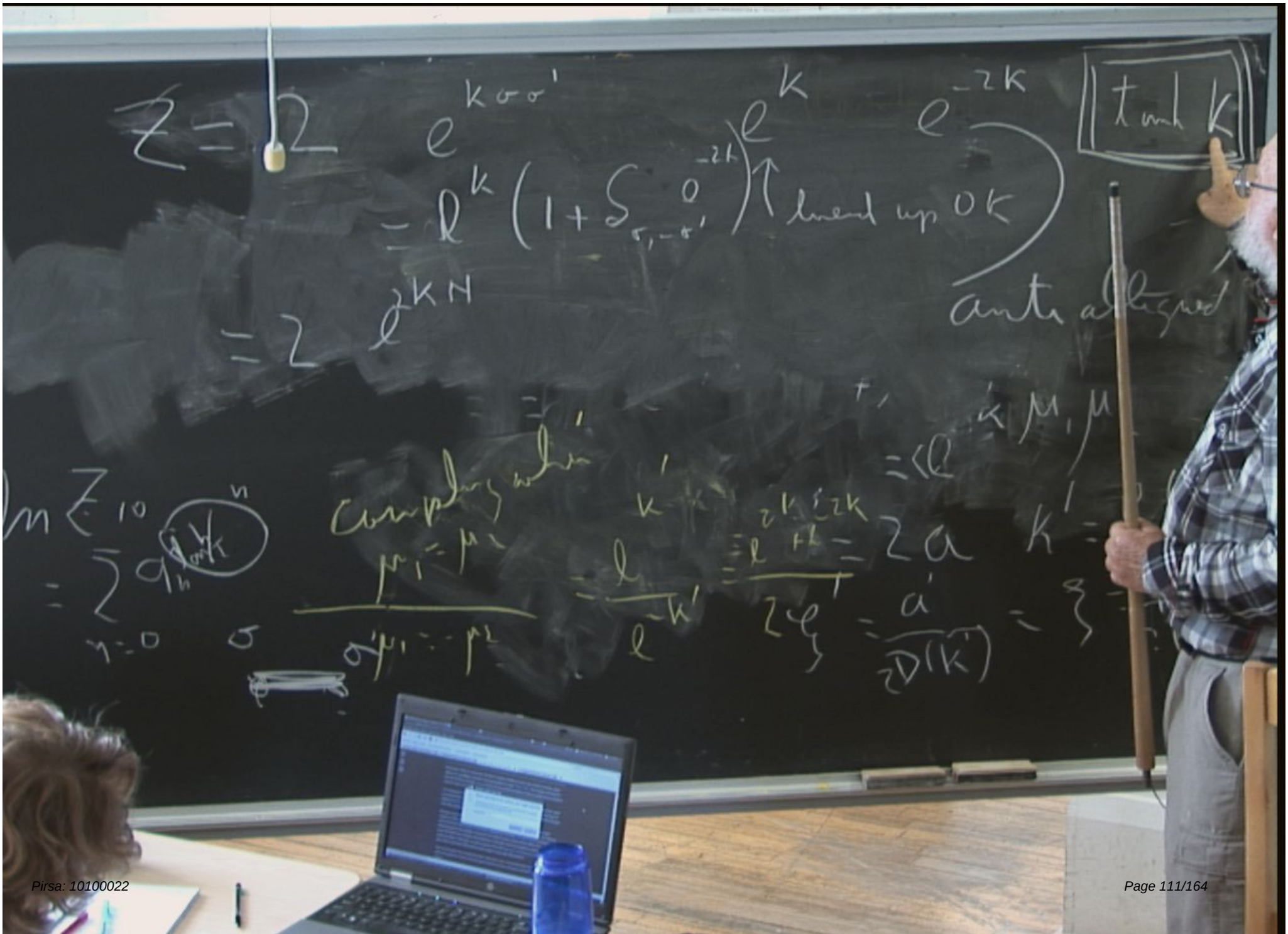
bend up OK
anti alligned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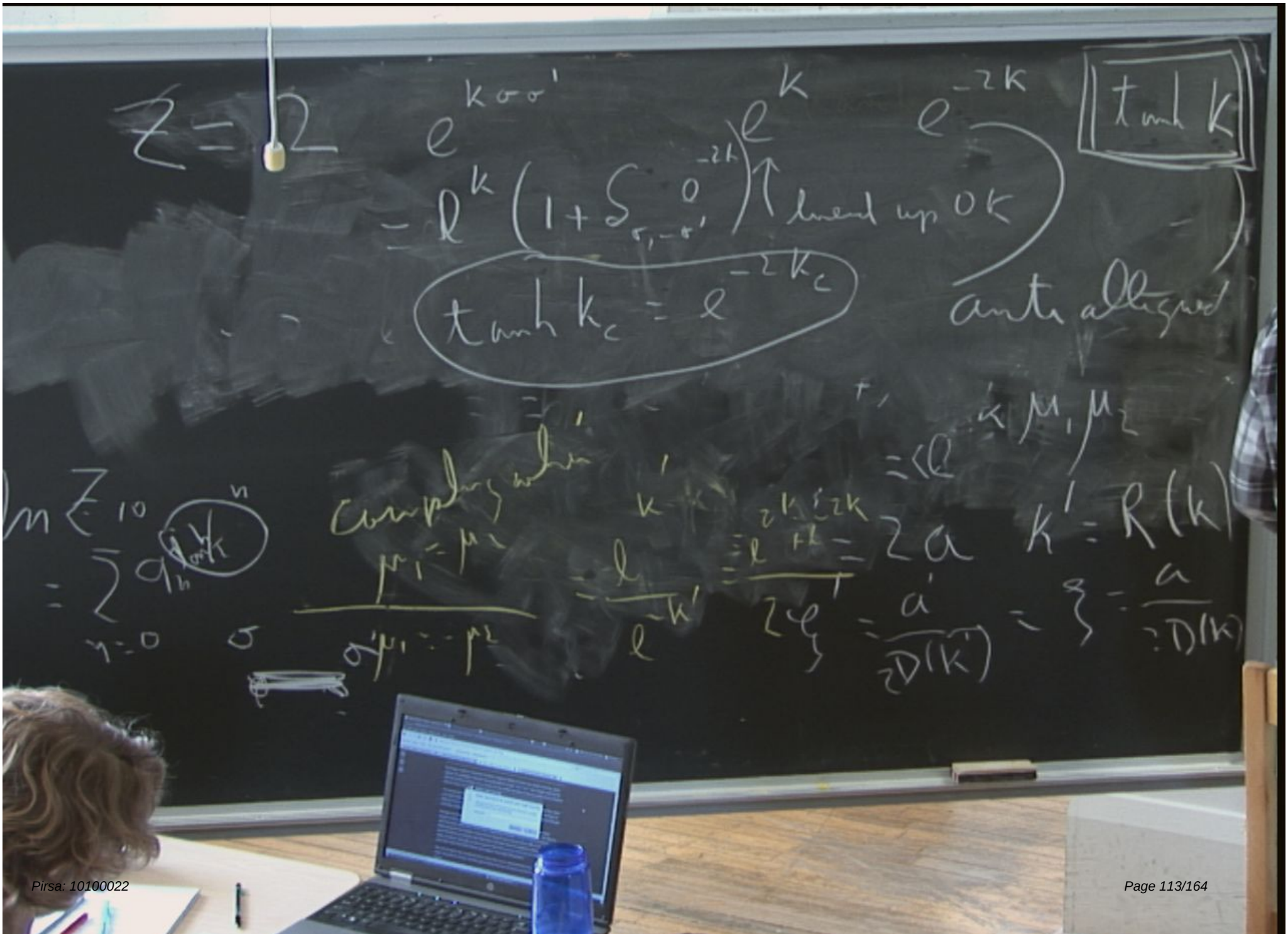
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$$\boxed{\tanh K}$$

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bend up OK

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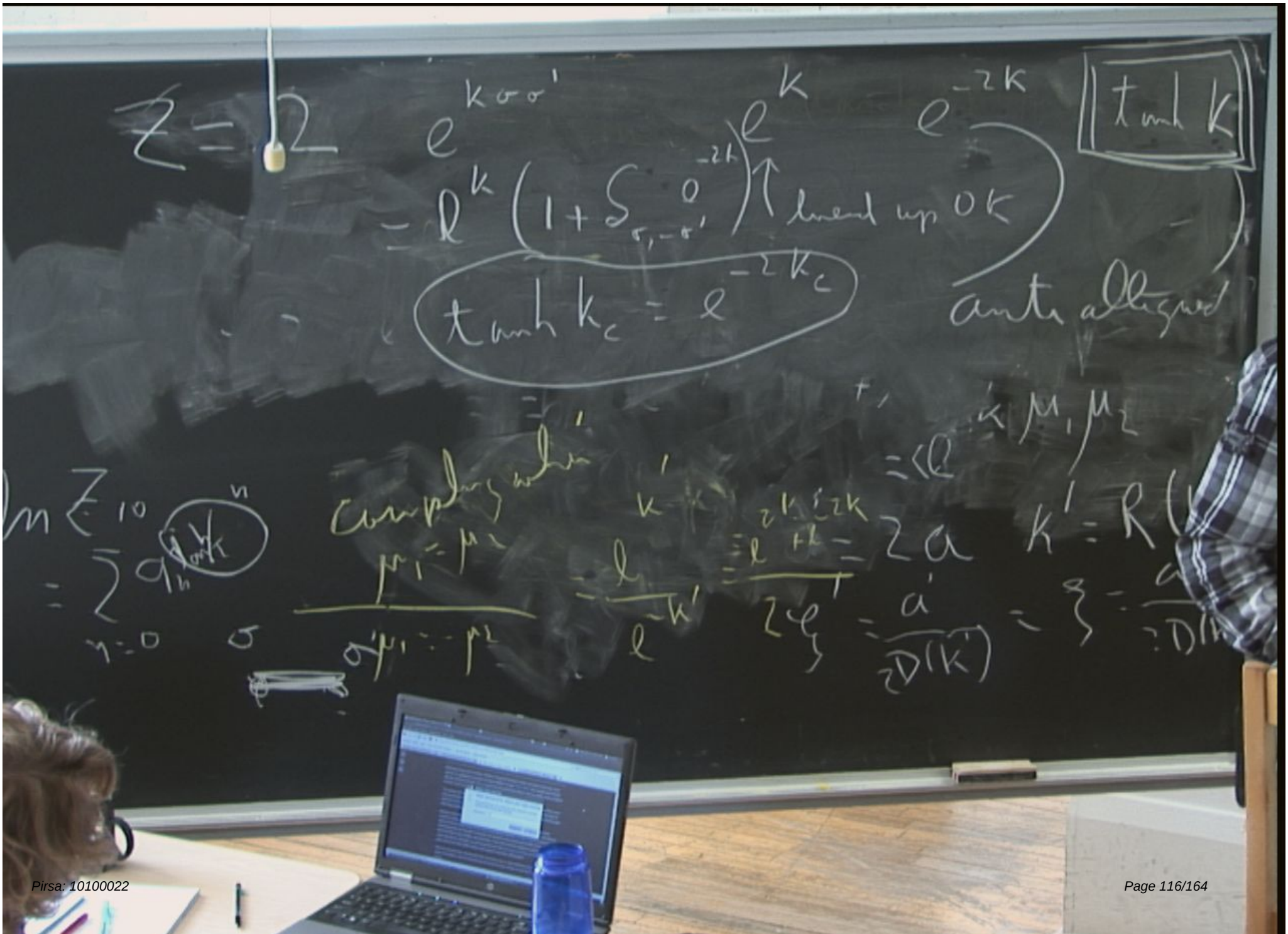
$$= e^{K\mu_1\mu_2}$$

$$\ln Z = \sum_{n=0}^{\infty} a_n \quad \sigma$$

coupling when

$$\mu_1 = \mu_2 \qquad K' \qquad = e^{2K} + e^{-2K} = 2a \qquad K' = R(K)$$

$$\frac{\mu_1 = \mu_2}{\sigma\mu_1 = -\mu_2} = \frac{e^{K}}{e^{-K'}} = \frac{e^{2K}+1}{2e^{K}} = \frac{a}{2D(K)} = \xi = \frac{a}{2D(K)}$$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Specific Heat = $d^2 \ln Z / d T^2$

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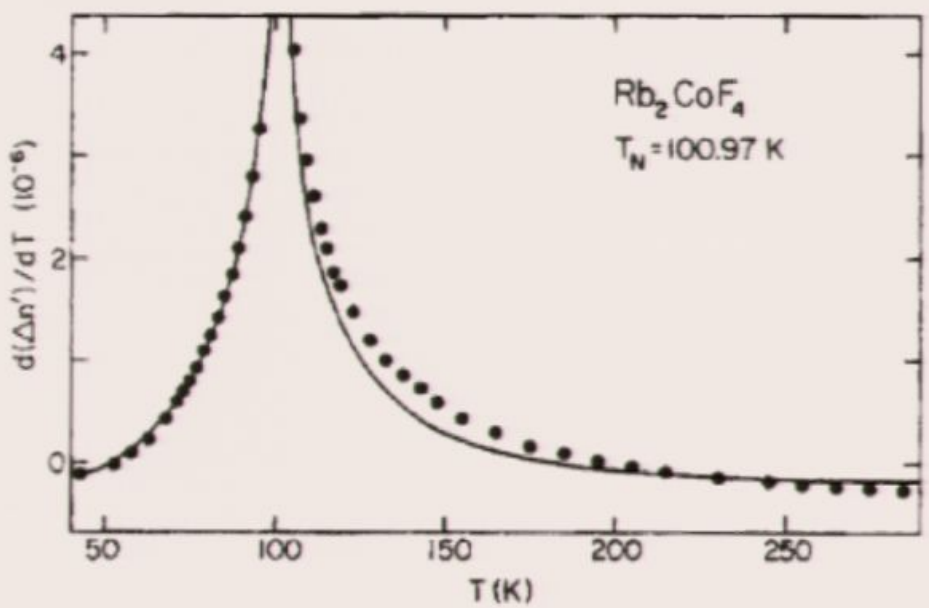

Figure 7. Variation of the magnetic specific heat, as a function of temperature for Rb_2CoF_4. The solid points (·) are experimental results of optical birefringence measurements shown previously to be proportional to the magnetic specific heat. The solid line is the exact Onsager solution for the two-dimensional Ising model with amplitude and critical temperature adjusted to fit the data, and a small constant background term subtracted. After Ref. 22.

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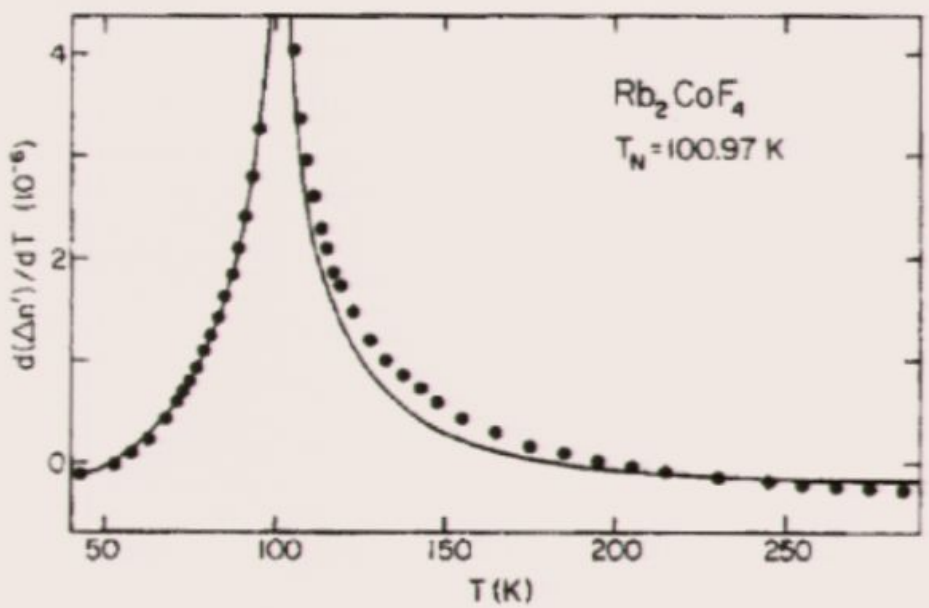

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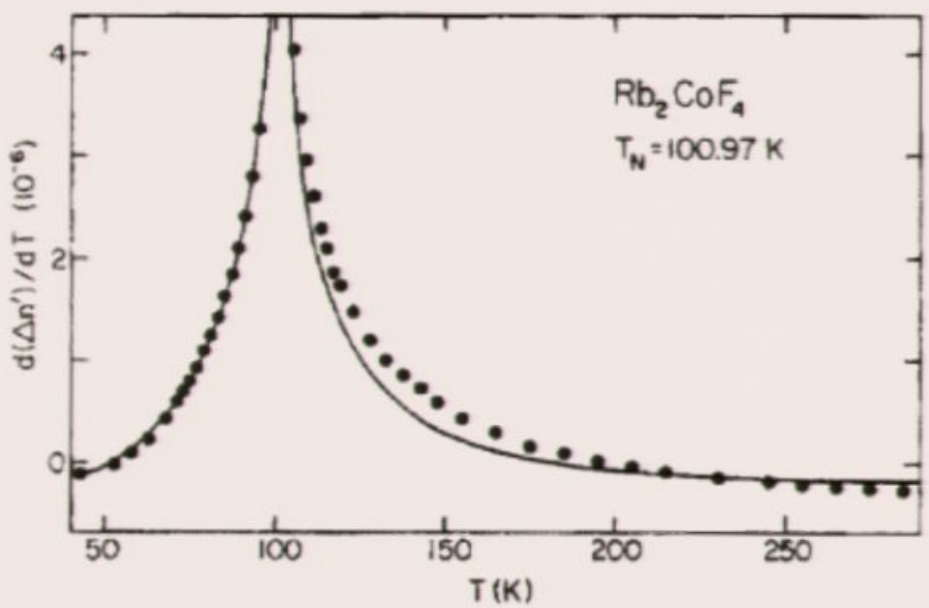

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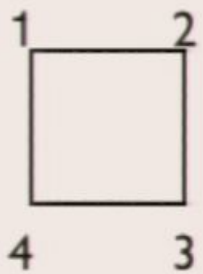
Duality Behavior

The structure of duality behavior depends upon both the lattice structure and the symmetry group of the interactions on that lattice. Many of the deepest results of string theory, gauge theories, and modern mathematics similarly deal with the simultaneous effect of internal symmetries, and the symmetries of space, or of space-time. Once again the condensed matter application gives us a chance to work out things which show up in a more complicated form in other situations.

For example one can consider interactions on plaquettes like



Here the basic variables live on the bonds connecting nearest neighbor lattice sites, as for example, Ising variables σ_{23} and σ_{34}. The basic interaction is on a look which goes around a unit square in the form $K\,\sigma_{12}\,\sigma_{23}\,\sigma_{34}\,\sigma_{41}$. On a three dimensional lattice a set of interactions like this

a. has a fundamental gauge symmetry (symmetry operation at every point) of the form σ_{34} goes into $\mu_3\,\sigma_{34}\,\mu_4$. This is a symmetry operation since the μ's cancel in each plaquette interaction.

b. has a phase transition at sufficiently high value of K in three dimensions

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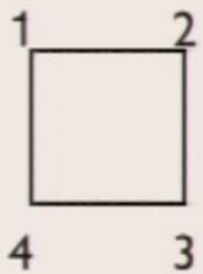

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How does one know?

Because this plaquette model is dual to the three dimensional Ising model and has a critical coupling, $K_c = D(K_c^{Ising})$.
The plaquette construction and gauge symmetry is due to K. Wilson.
The duality argument is due to F. Wegner

electromagnetic argument: basic variable is link $=\exp[ie \int dx_\mu A_\mu]$ where the integral goes from one lattice site to its neighbor. Then, a gauge transform has A_μ goes into $A_\mu + \partial_\mu \Lambda$ and *link* goes into $\exp[-ie\Lambda]\,link\,\exp[ie\Lambda]$ where the two Λ's are evaluated at the two ends of the *link*. As in the case described above, a product of four link variables has a gauge symmetry, in this case the gauge symmetry of electromagnetism.

But let's get back to the Ising model.

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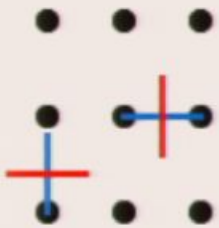

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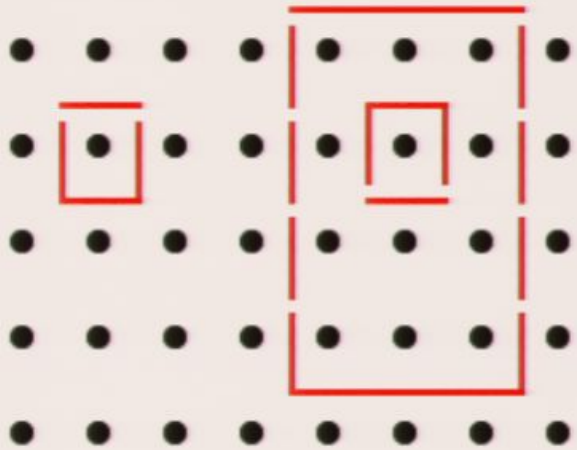


To get a non-zero term, assign a value to one spin. Then every time you cross a red line, change the spin-value to the opposite. Your valid pictures become a series of closed red polygons.

With a lot of hard work one can calculate a series up to ten or even twenty terms long and estimate behavior of thermodynamic functions from these seres

High Temperature Expansion

Nearest neighbor structure

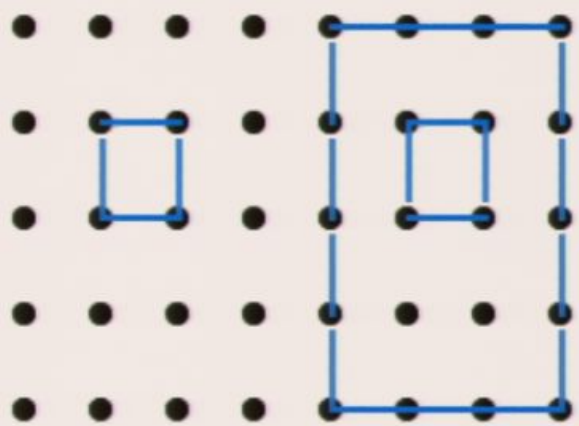
A bond=exp($K\sigma\sigma'$) connect nearest neighbors

Bond=cosh K +$\sigma\sigma'$ sinh K =cosh K[1+ $\sigma\sigma'$ tanh K]

$Z=(2 \cosh K \cosh K)^N$ < products on all bonds of [1+ $\sigma\sigma'$ tanh K] >

$= (2 \cosh K \cosh K)^N$ sum < products of $(\tanh K)^M$ >

for nonzero terms, when there are N sites



To get a non-zero value each spin must appear on a even number of bonds. You then get the lattice covered by closed polygons.

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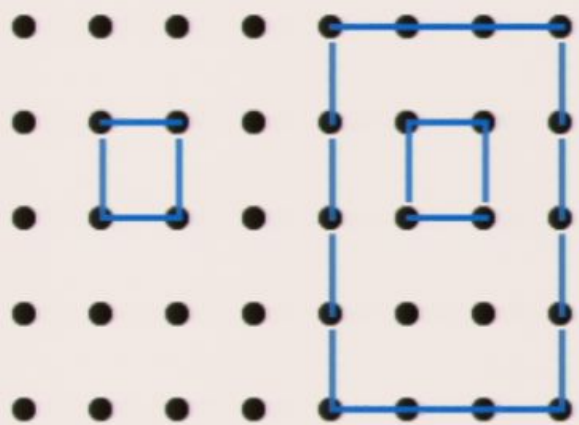

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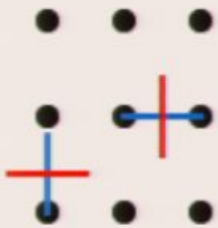
Low Temperature Expansion

Nearest neighbor structure

A bond$=\exp(K\sigma\sigma') = e^{K}\delta_{\sigma,\sigma'} + e^{-K}\delta_{\sigma,-\sigma'}$

Bond $= e^{K}[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}]$

We draw these bonds differently from the high T bonds. We draw them rotated 90 degrees in comparison to the other bonds.



note e^{-2K} $= \tanh \tilde{K}$

$Z = 2(e^{K})^{N}(e^{K})^{N} <$ products of $[\delta_{\sigma,\sigma'} + e^{-2K}\delta_{\sigma,-\sigma'}] >$

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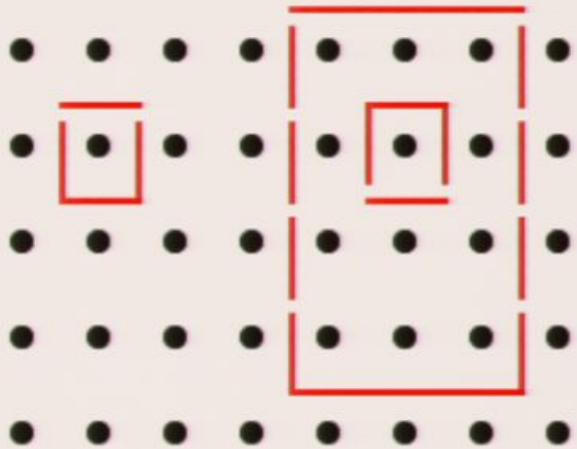

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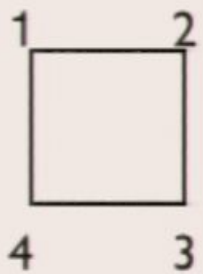
For example one can consider interactions on plaquettes like



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b. has a phase transition at sufficiently high value of K in three dimensions

How does one know?

Because this plaquette model is dual to the three dimensional Ising model and has a critical coupling, $K_c = D(K_c^{Ising})$.
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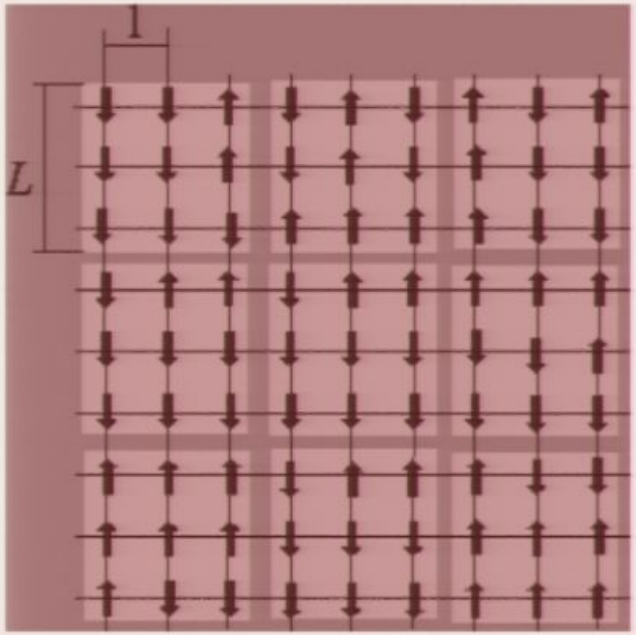
Renormalization for d-2 Ising model

Ben Widom, myself, Kenneth Wilson.

$Z = \mathrm{Trace}_{\{\sigma\}} \exp(W_K\{\sigma\})$



nagine that each box in the picture has in it a
riable called $\mu_{\mathbf{R}}$, where the **R**'s are a set of new
ttice sites with nearest neighbor separation 3*a*. Each
ew variable is tied to an old ones via a
normalization matrix $G\{\mu, \sigma\} = \prod_{\mathbf{R}} g(\mu_{\mathbf{R}}, \{\sigma\})$
here g couples the $\mu_{\mathbf{R}}$ to the

's in the corresponding box. We take each $\mu_{\mathbf{R}}$ to
e ±1 and define g so that,
$\sum_{\mu} g(\mu, \{s\}) = 1$. For example, μ might be
efined to be an Ising variable with the
ame sign as the sum of σ's in its box.
Now we are ready. Define

fewer degrees of freedom
produces "block renormalization"

$\exp(W'\{\mu\}) = \mathrm{Trace}_{\{\sigma\}}\, G\{\mu, \sigma\} \exp(W_K\{\sigma\})$

$Z = \mathrm{Trace}_{\{\mu\}} \exp(W'\{\mu\})$

f we could ask our fairy god-mother what we wished for now it would be that we
ame back to the same problem as we had at the beginning: $W'\{\mu\} = W_{K'}\{\mu\}$

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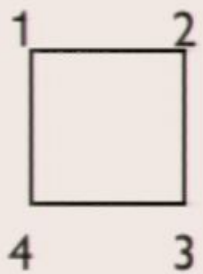

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Specific Heat = $d^2 \ln Z / d T^2$

Further we might notice that in two dimensions ln Z must have a form of singularity which is even about the critical value of the coupling.

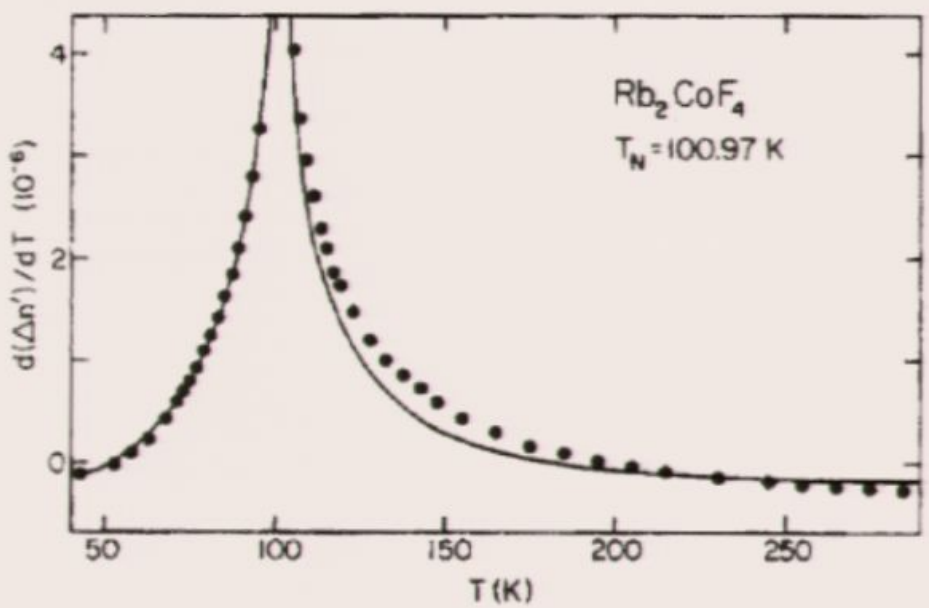

Figure 7. Variation of the magnetic specific heat, as a function of temperature for Rb_2CoF_4. The solid points (·) are experimental results of optical birefringence measurements shown previously to be proportional to the magnetic specific heat. The solid line is the exact Onsager solution for the two-dimensional Ising model with amplitude and critical temperature adjusted to fit the data, and a small constant background term subtracted. After Ref. 22.

22. P. Nordblad, D.P. Belanger, A.R. King, V. Jaccarino and H. Ikeda., Phys. Rev. B 28. 278 (1983).

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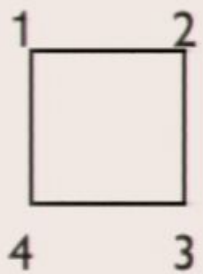

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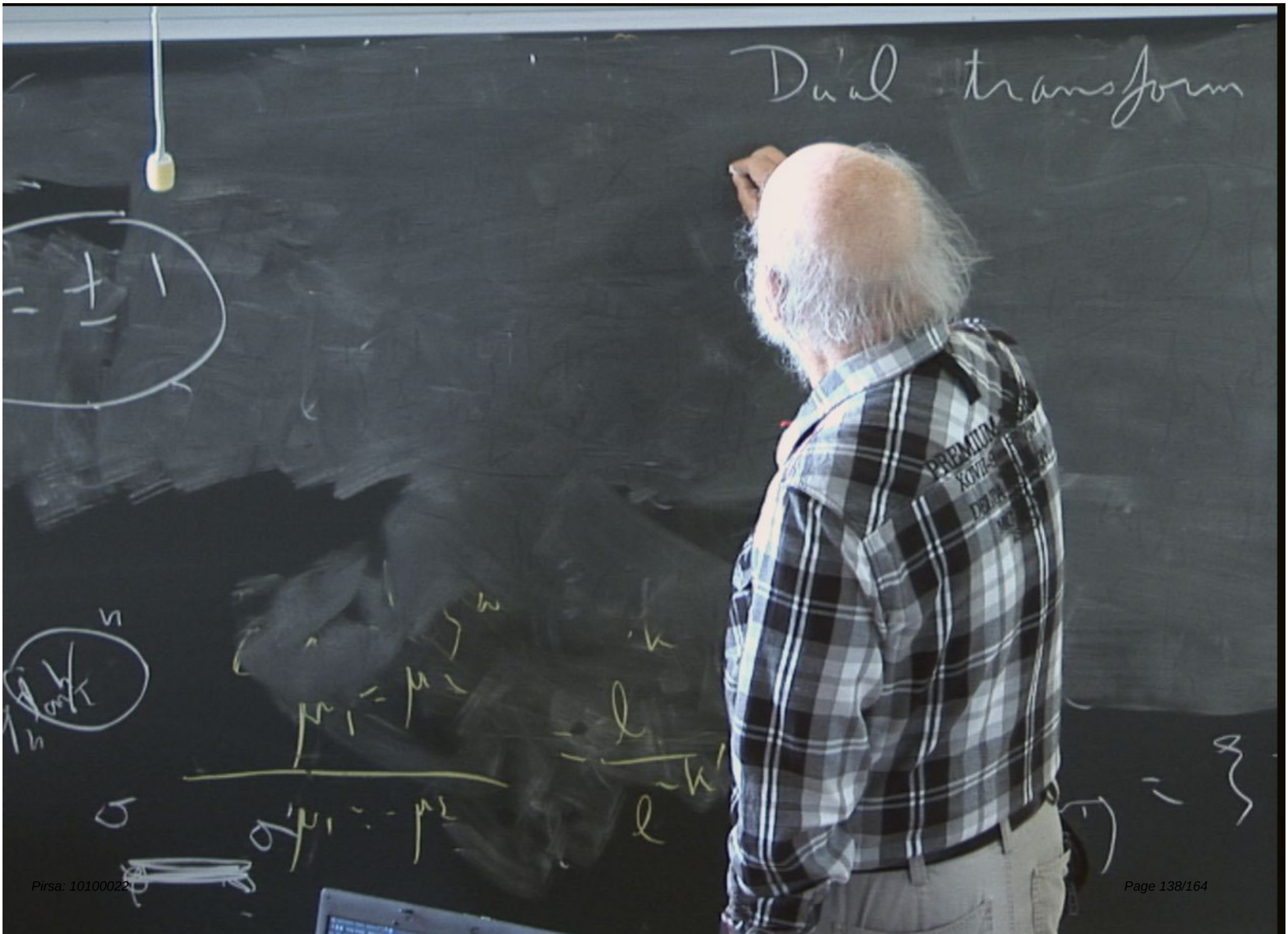
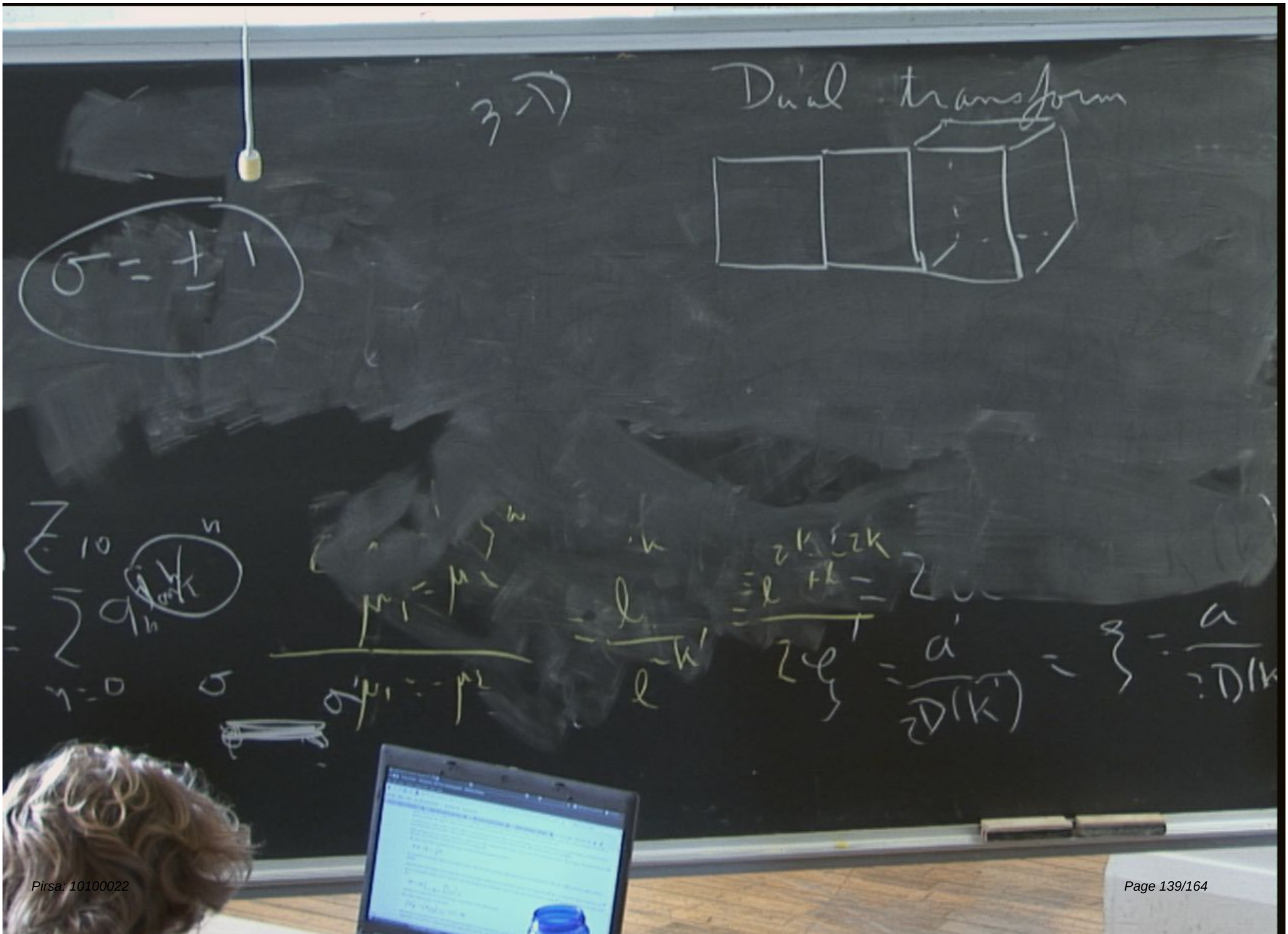
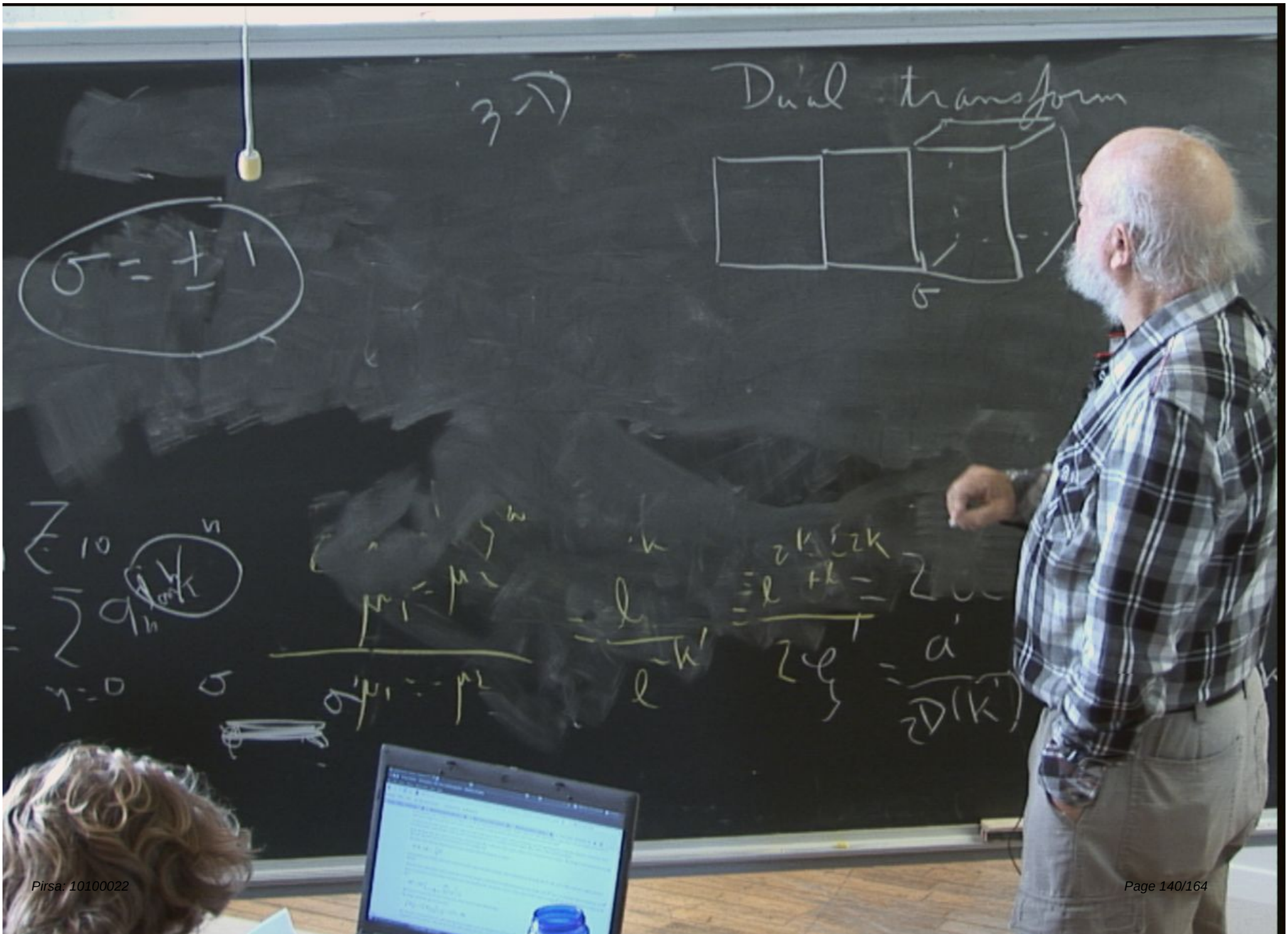

Dual transform


Dual transform
σ = ± 1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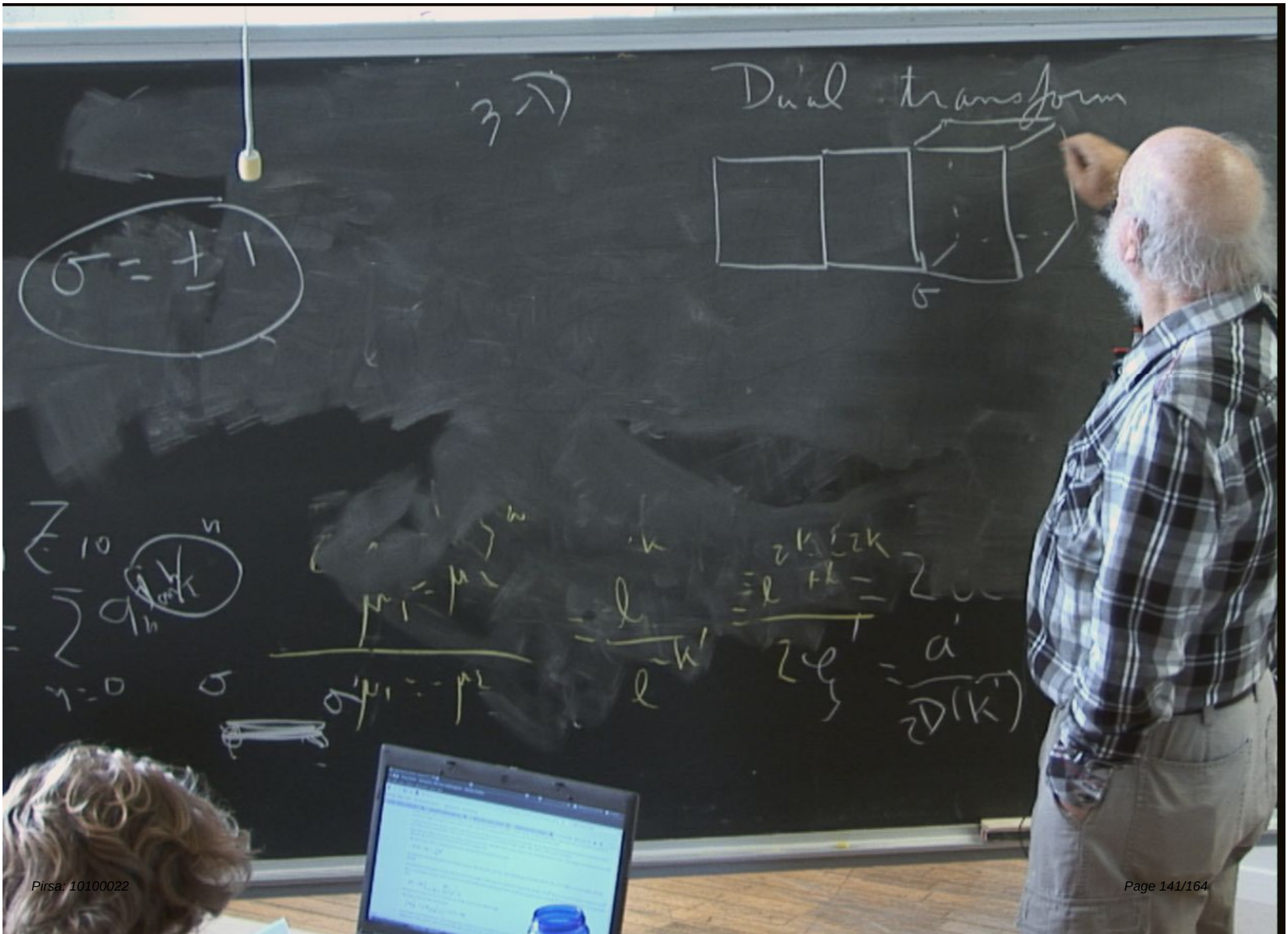
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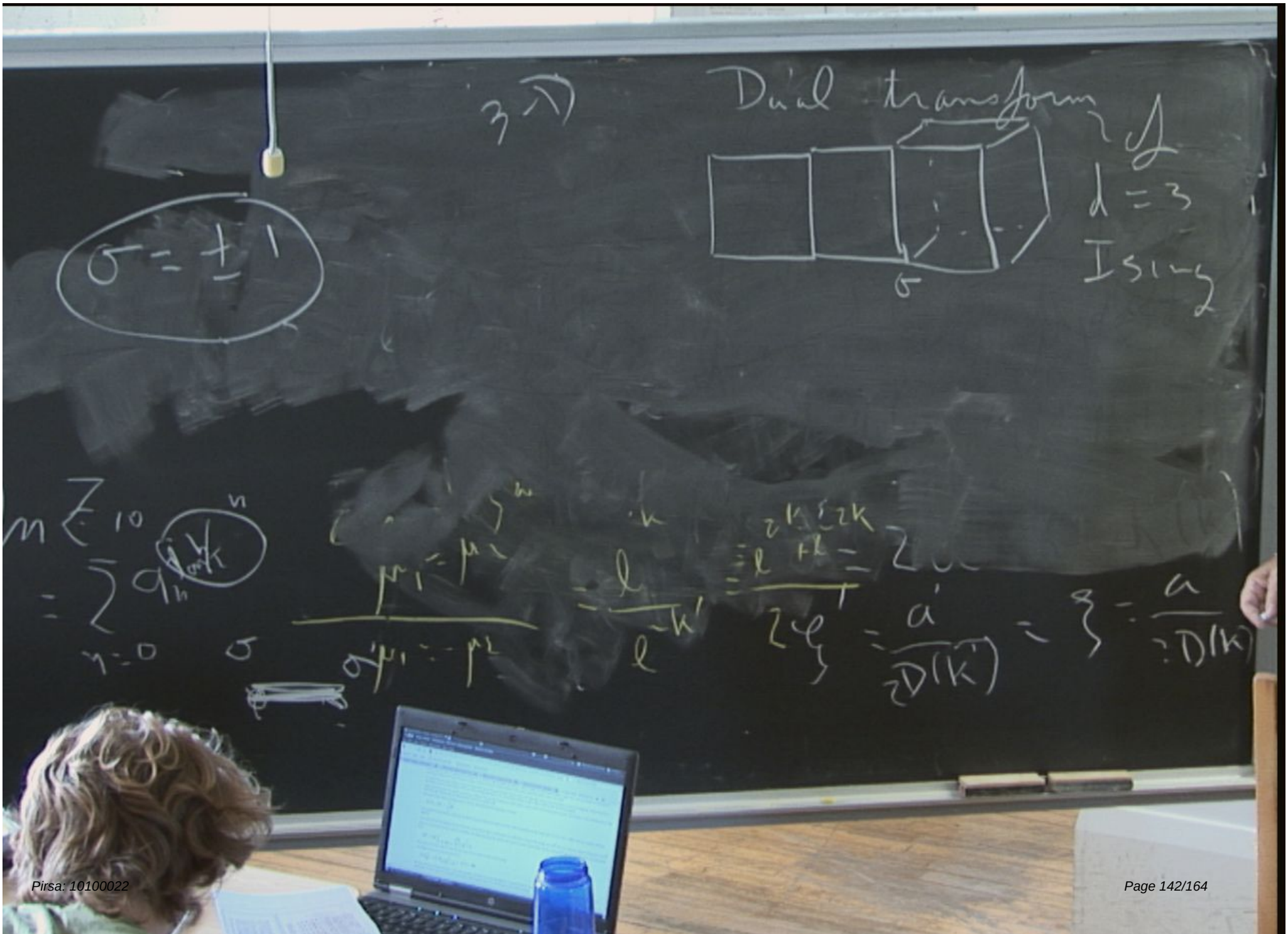
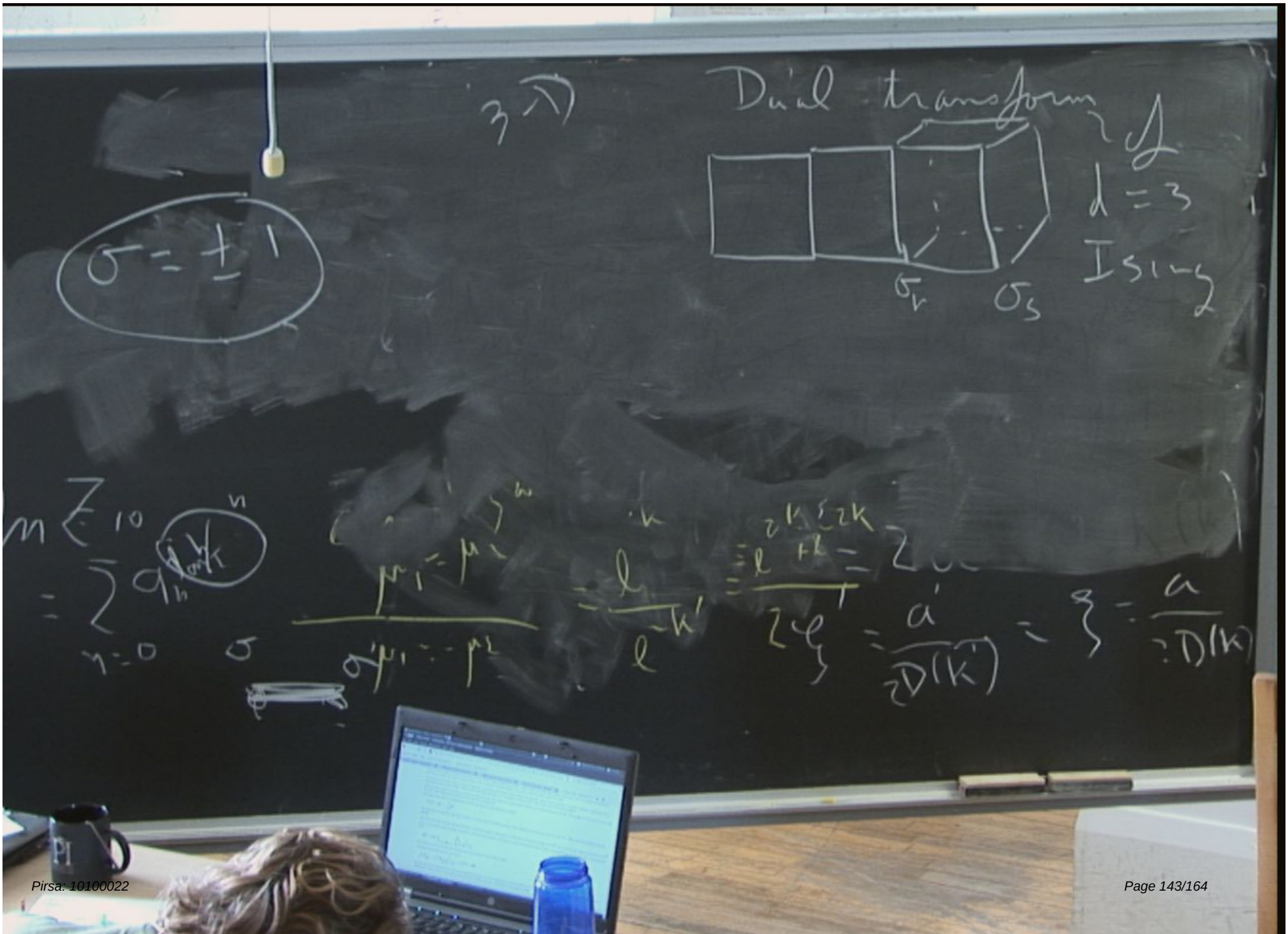

Dual transform
d = 3
Ising
σ = ± 1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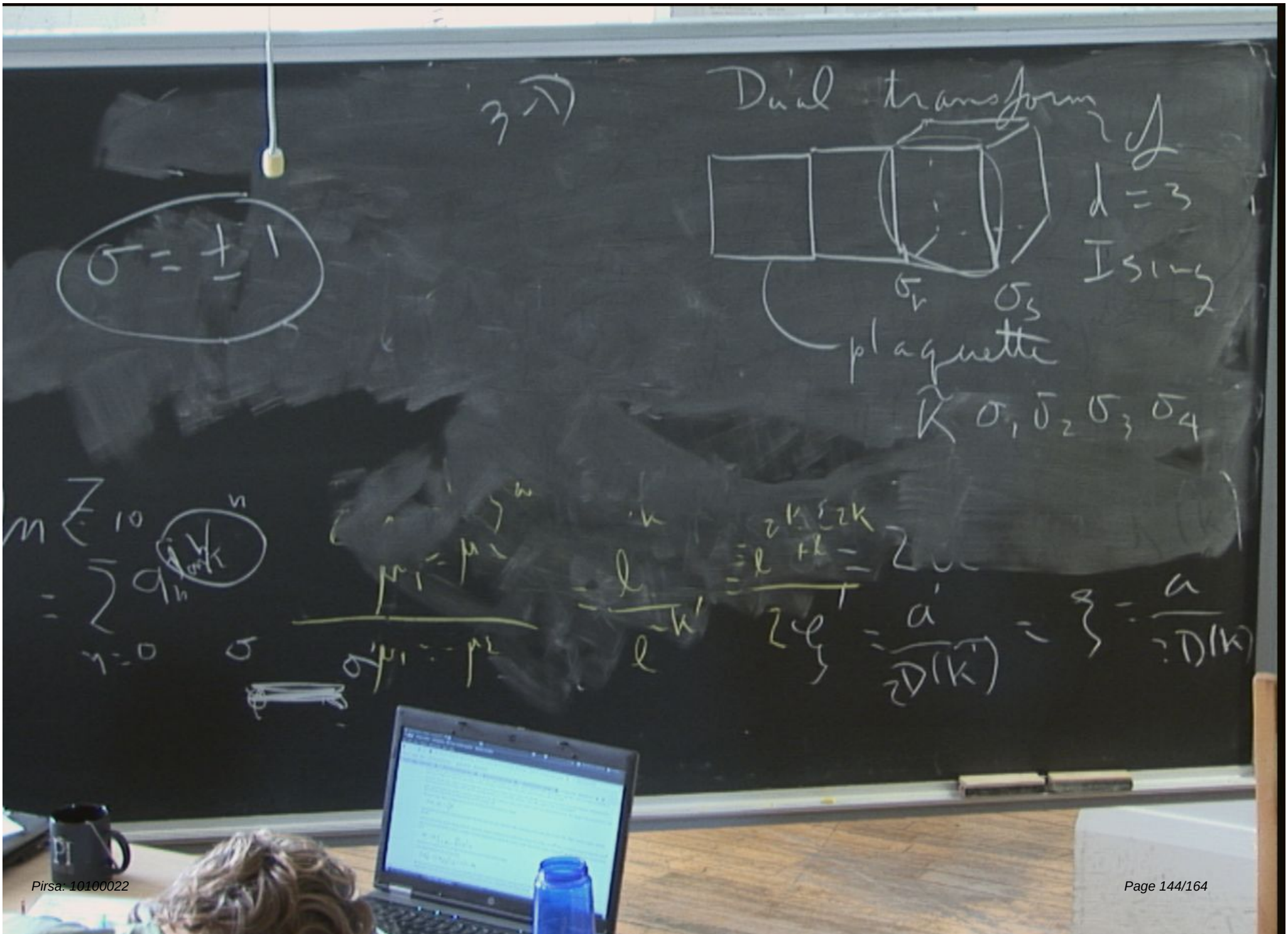
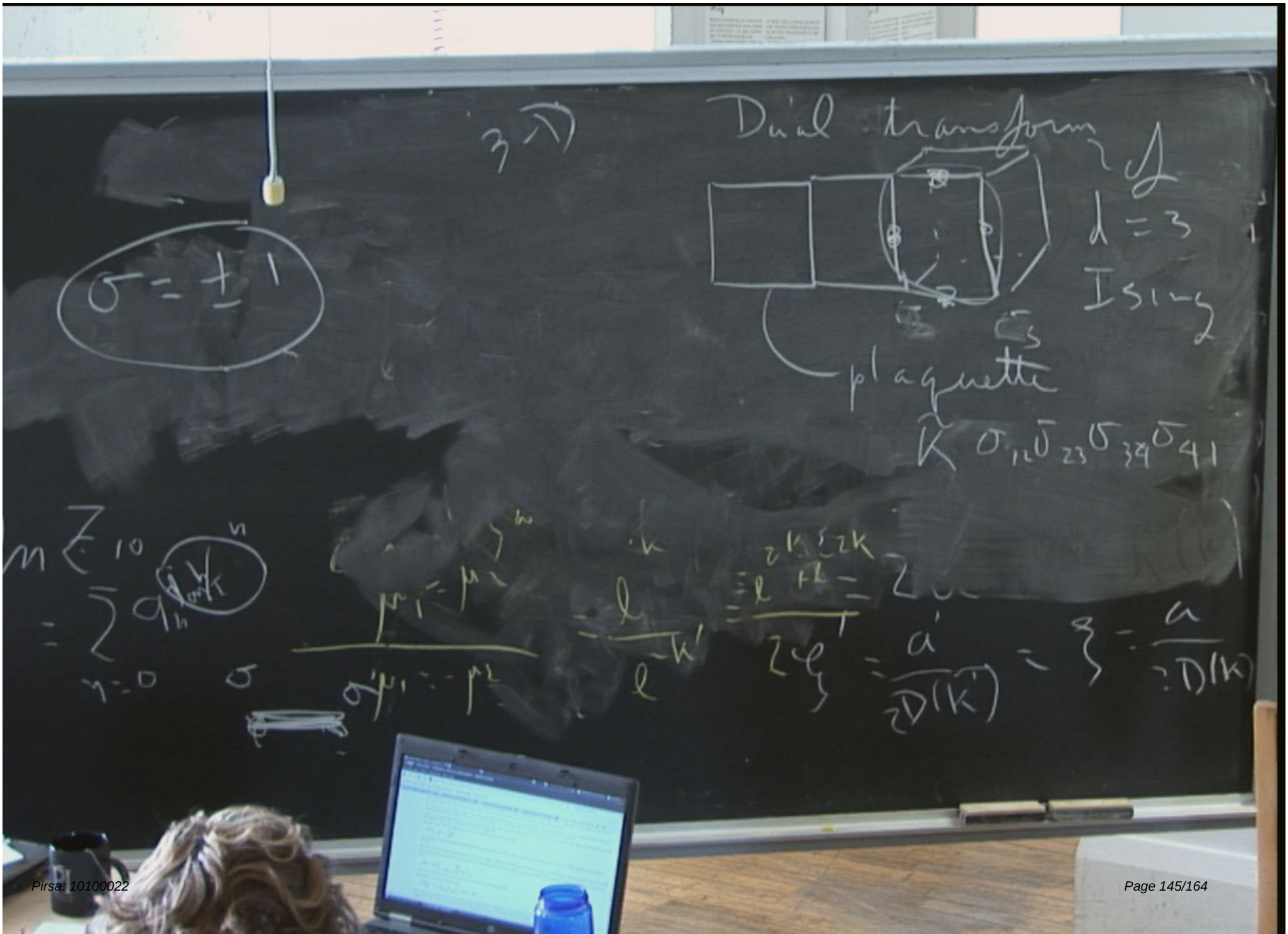

Dual transform
d = 3
Ising
plaquette
σ = ± 1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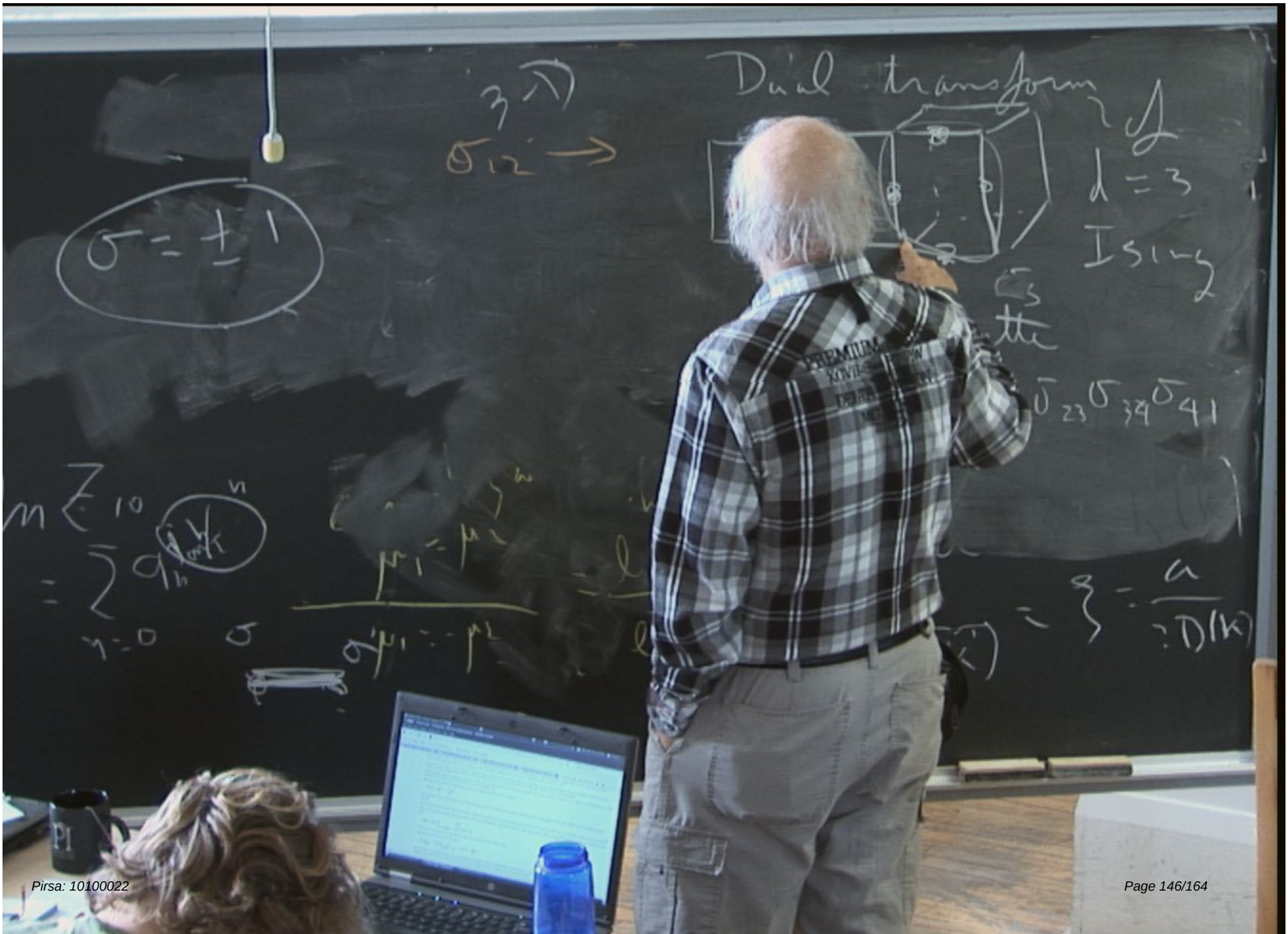
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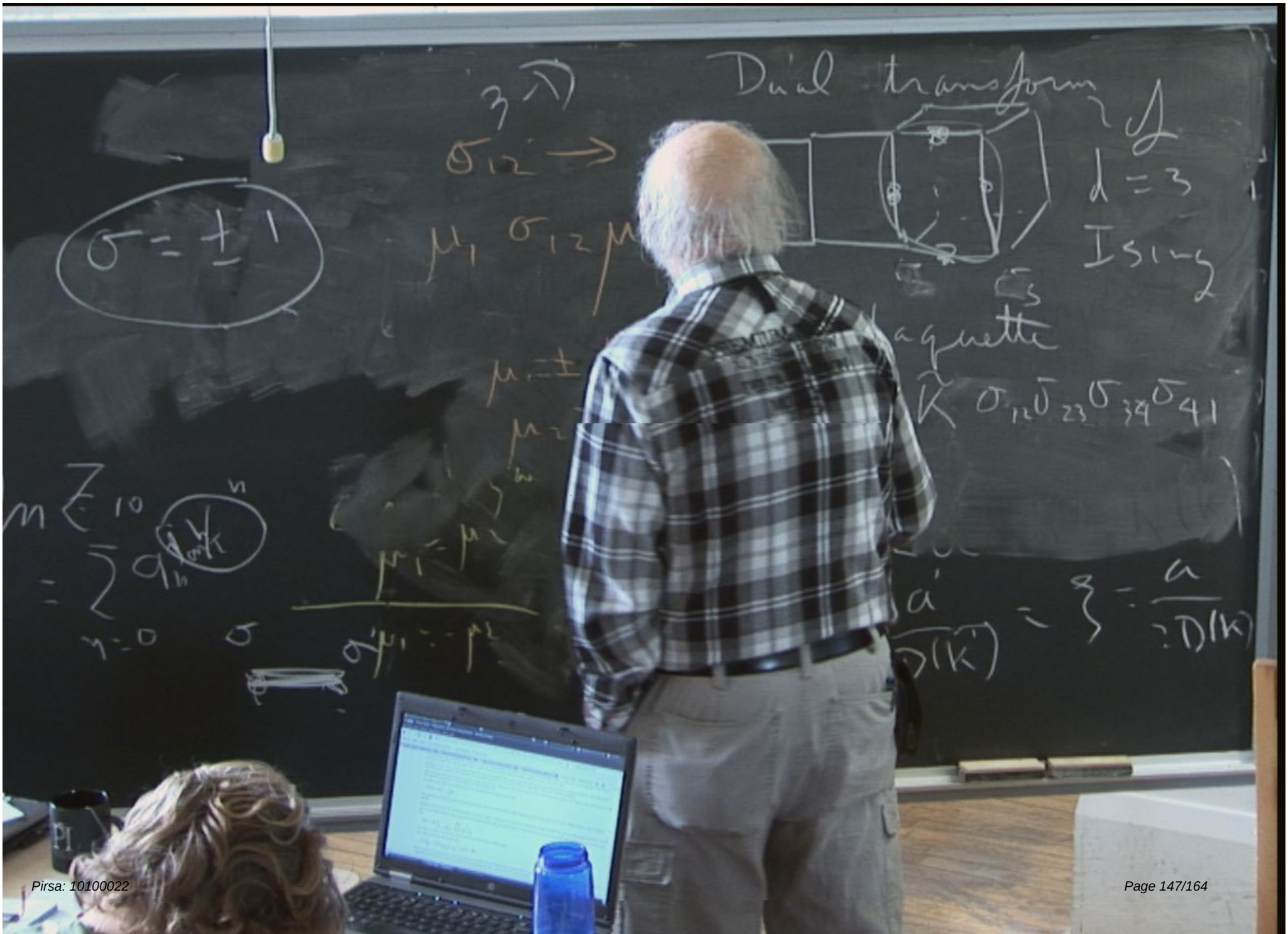
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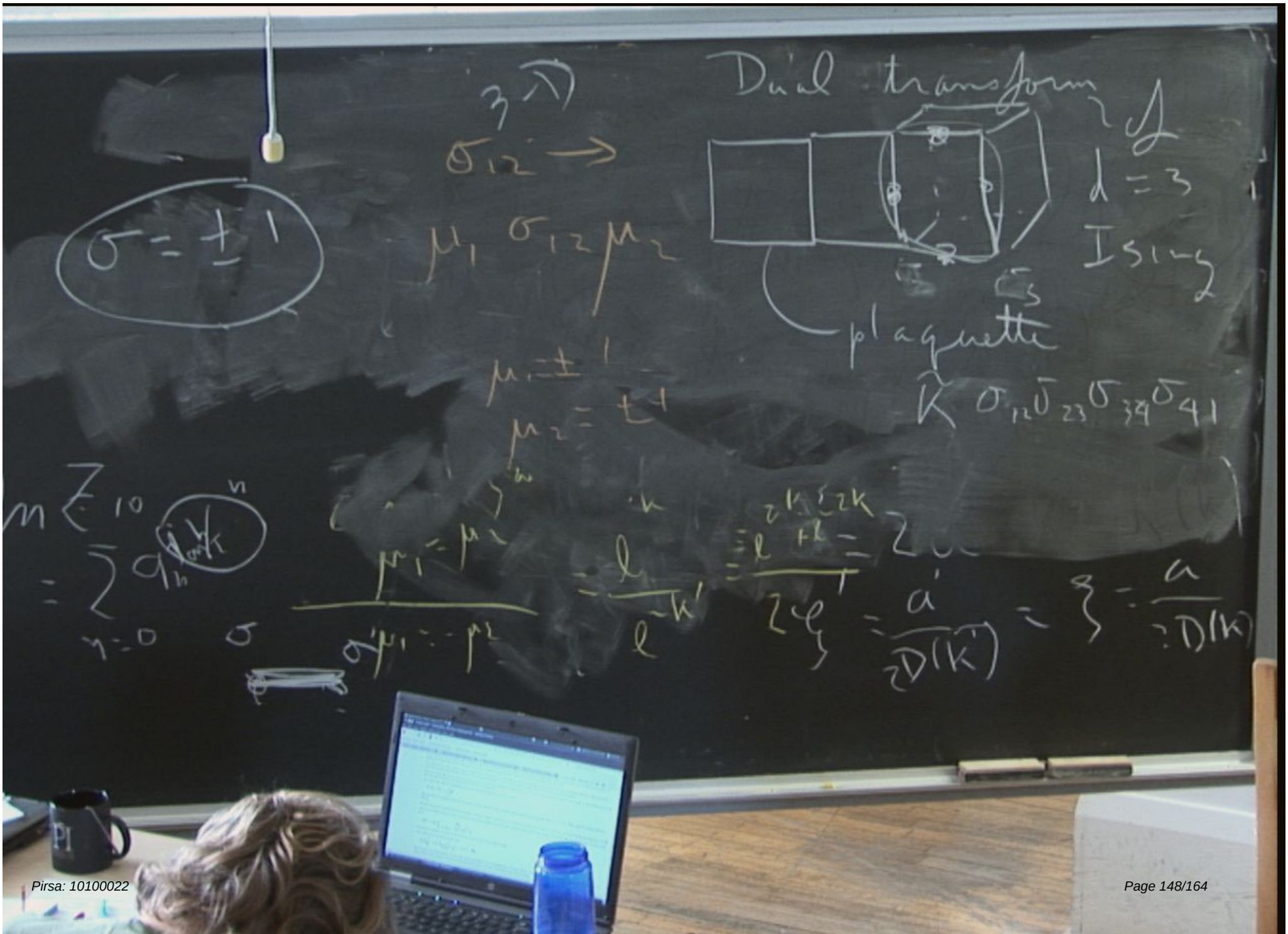
Dual transform
$\sigma = \pm 1$
$\sigma_{12} \rightarrow$
$\mu_1 \sigma_{12} \mu_2$
$d = 3$
Ising
plaquette
$K \sigma_{12}\sigma_{23}\sigma_{34}\sigma_{41}$
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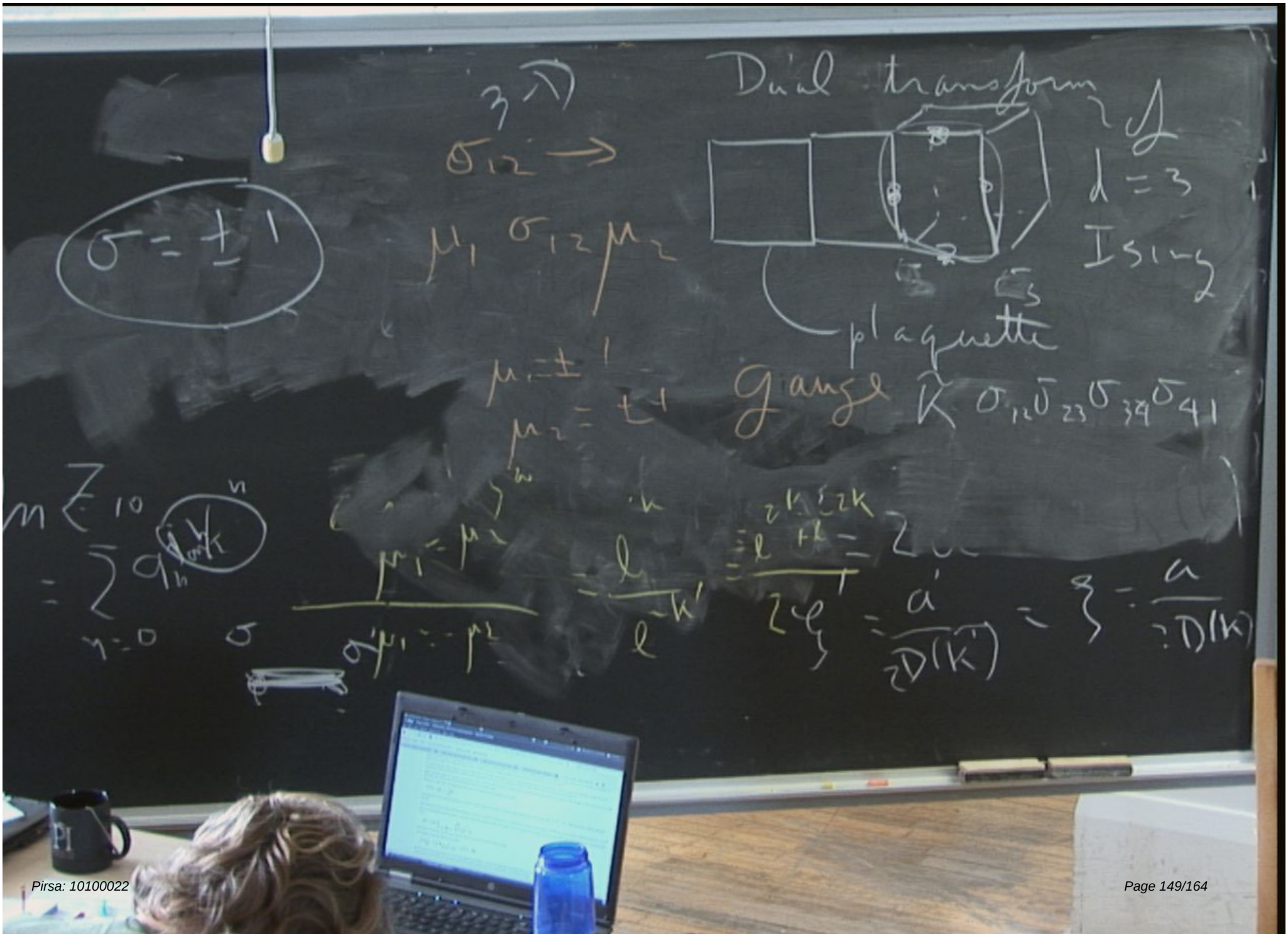
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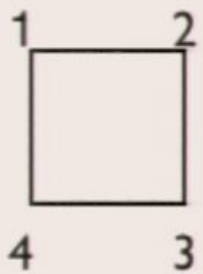

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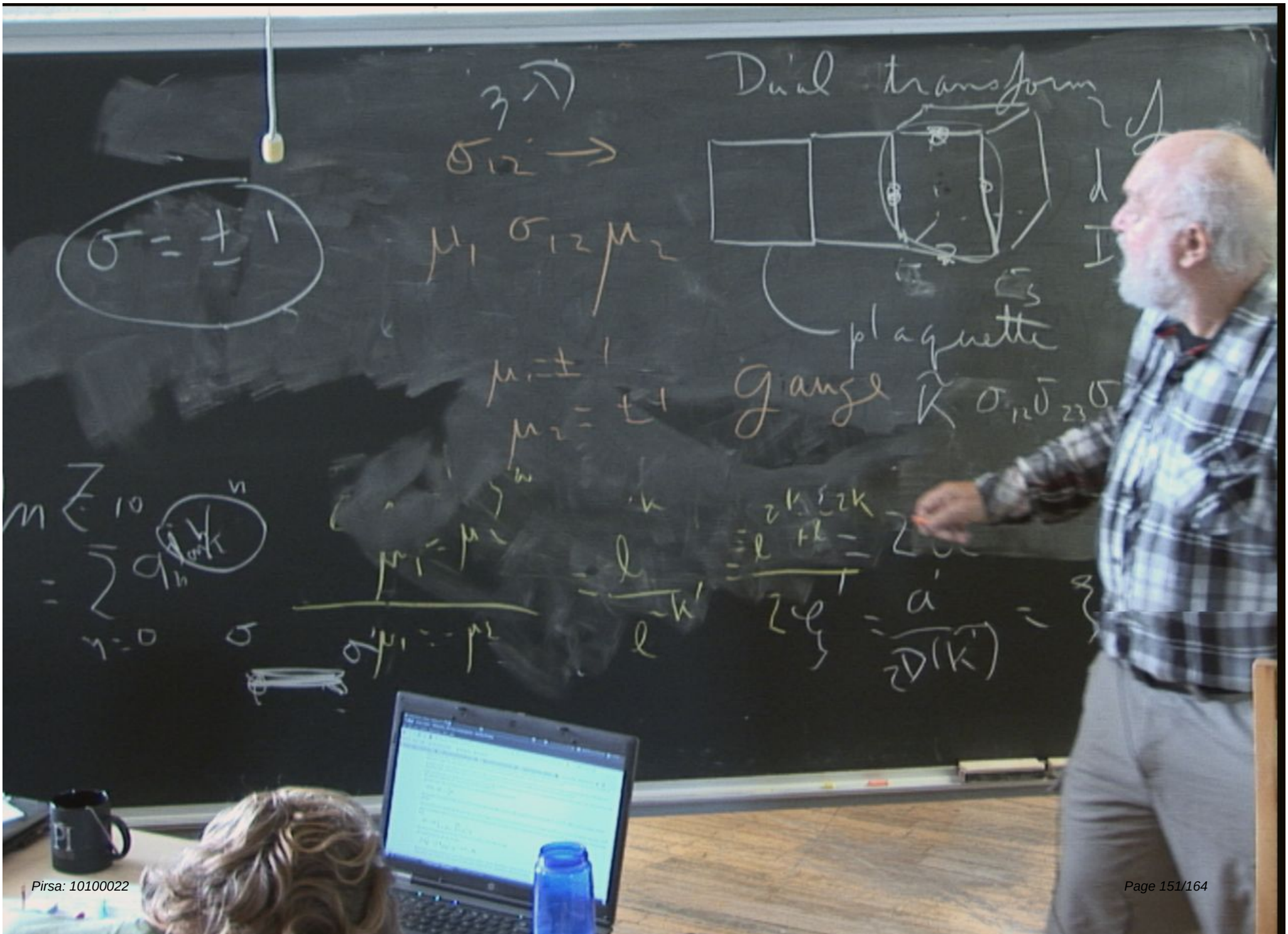
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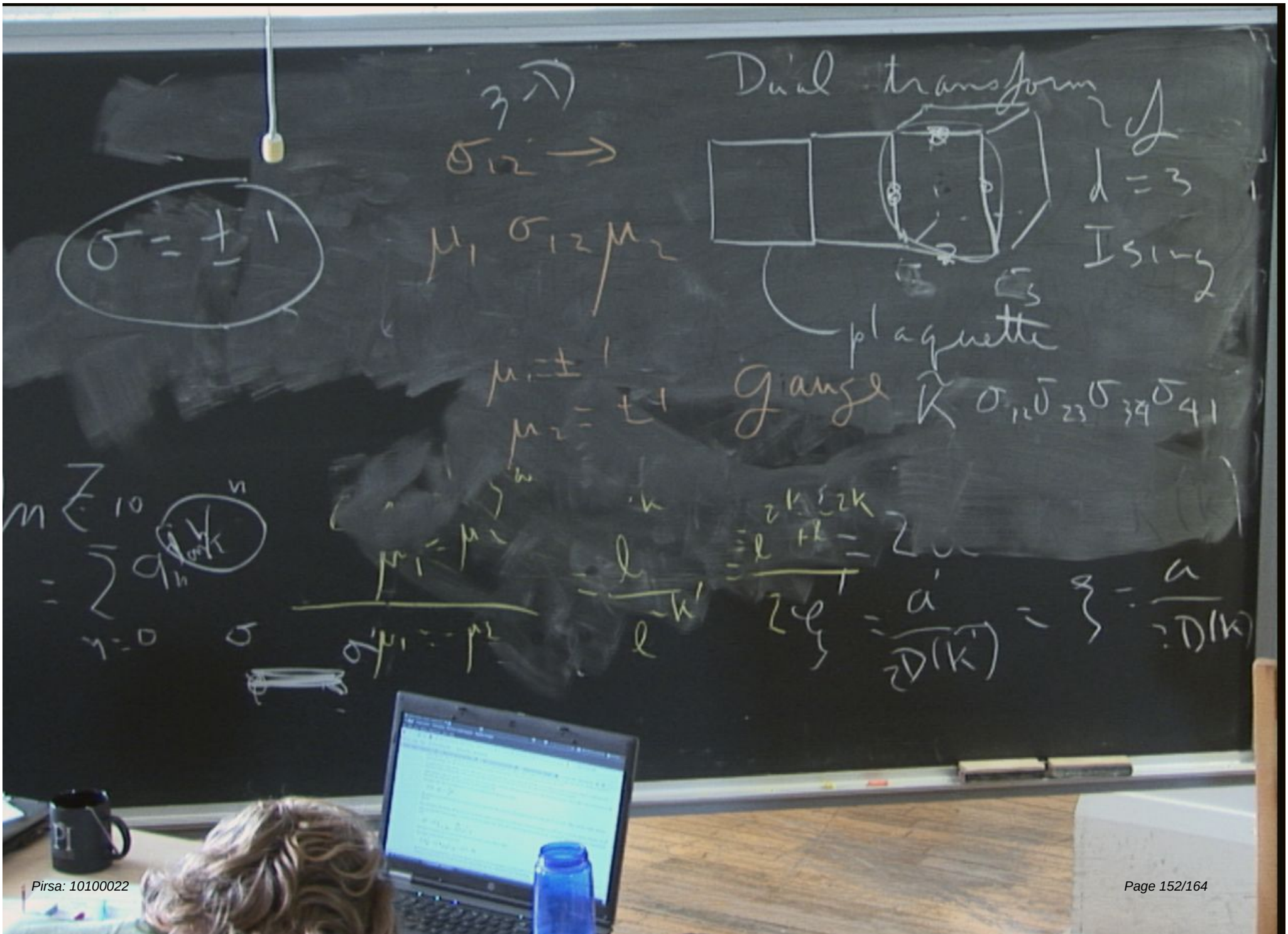
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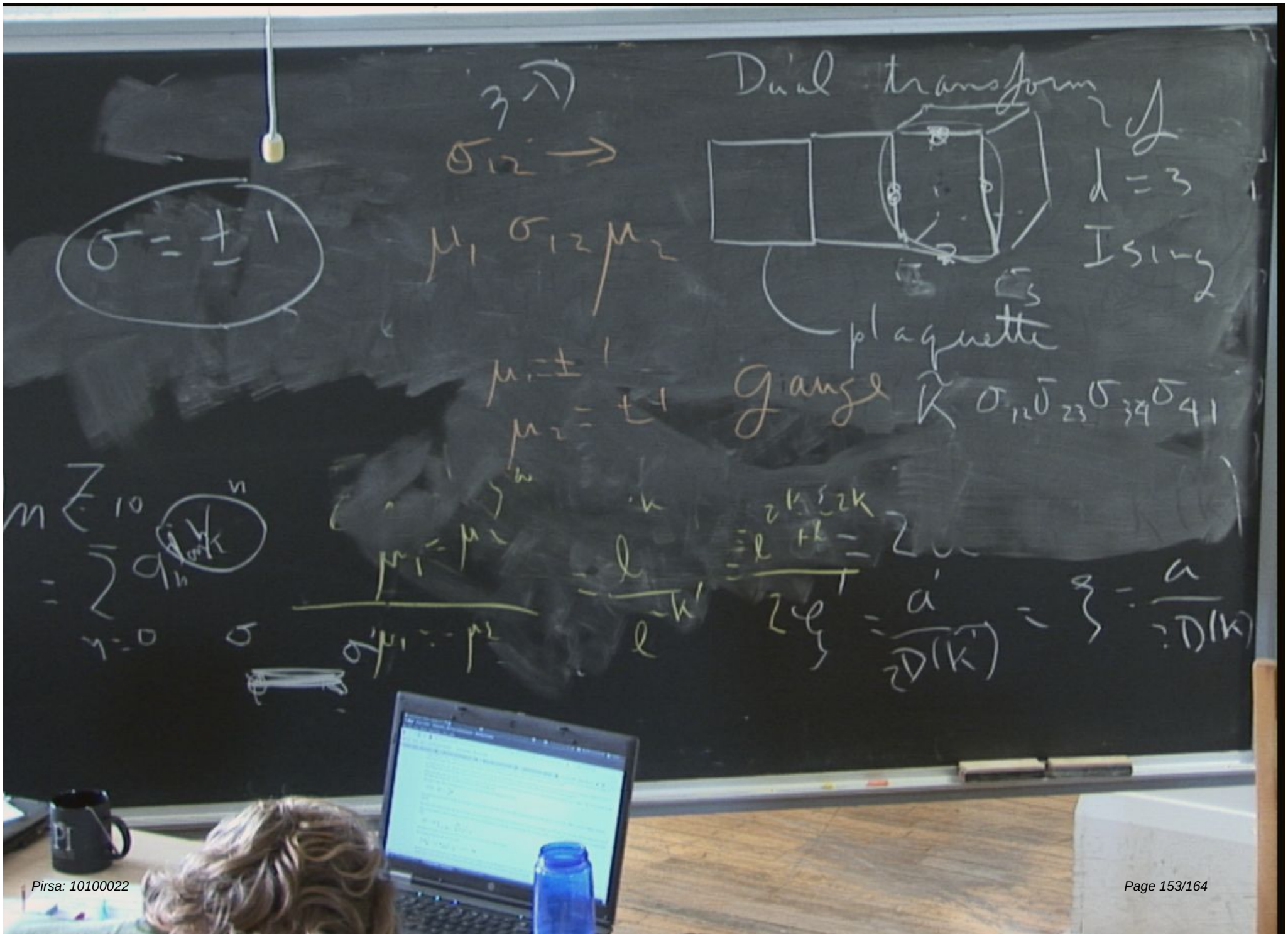
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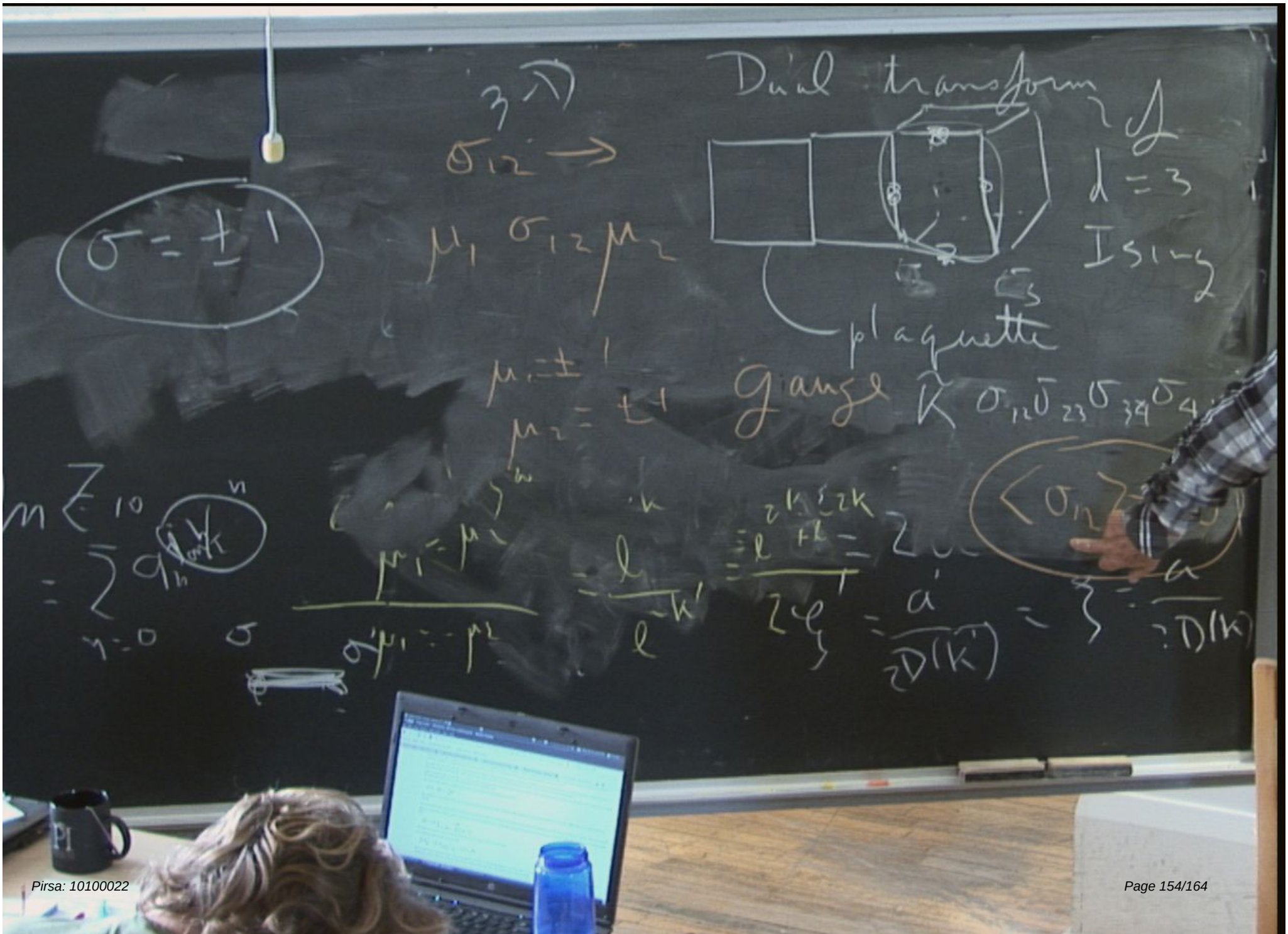
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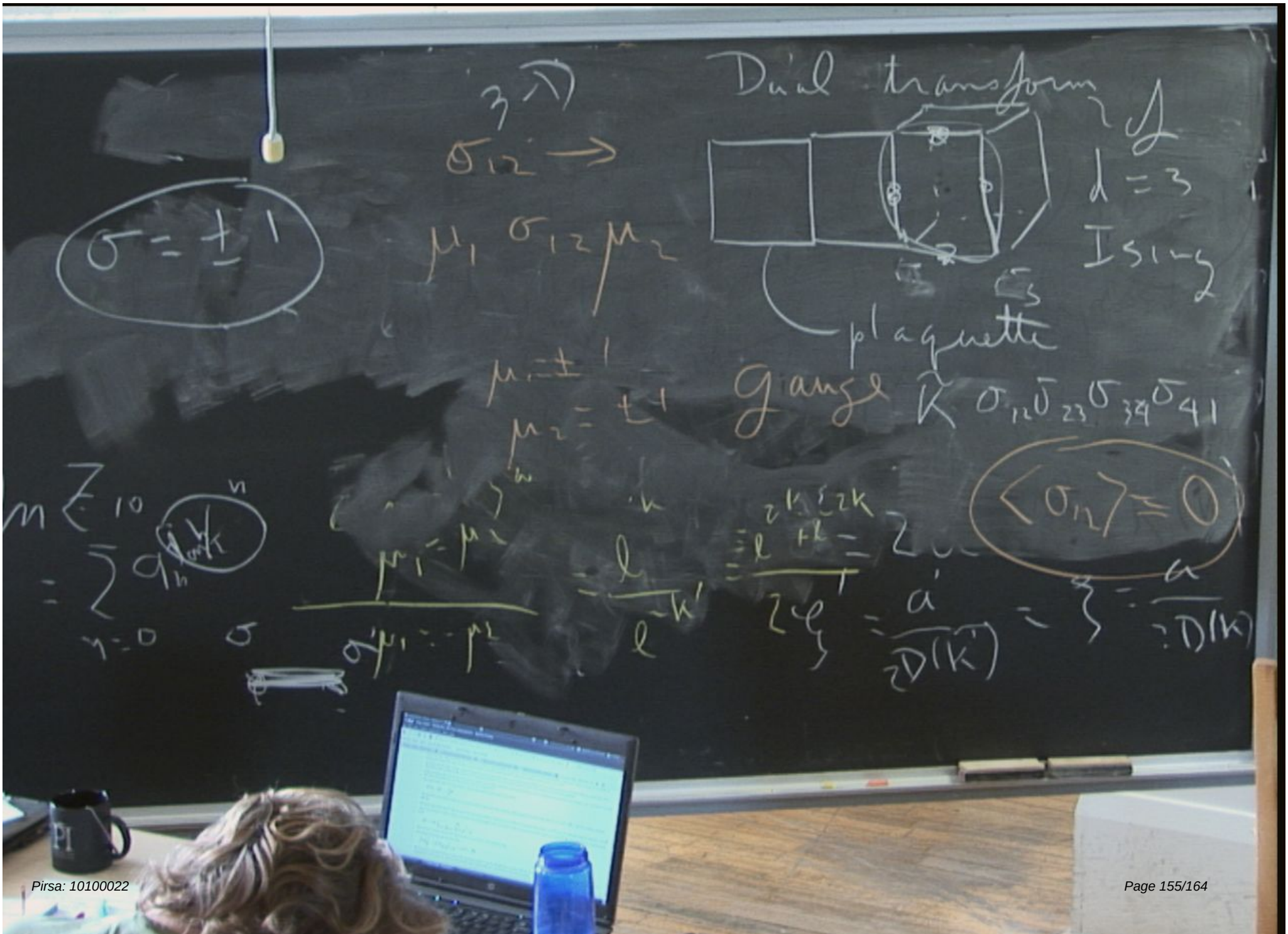
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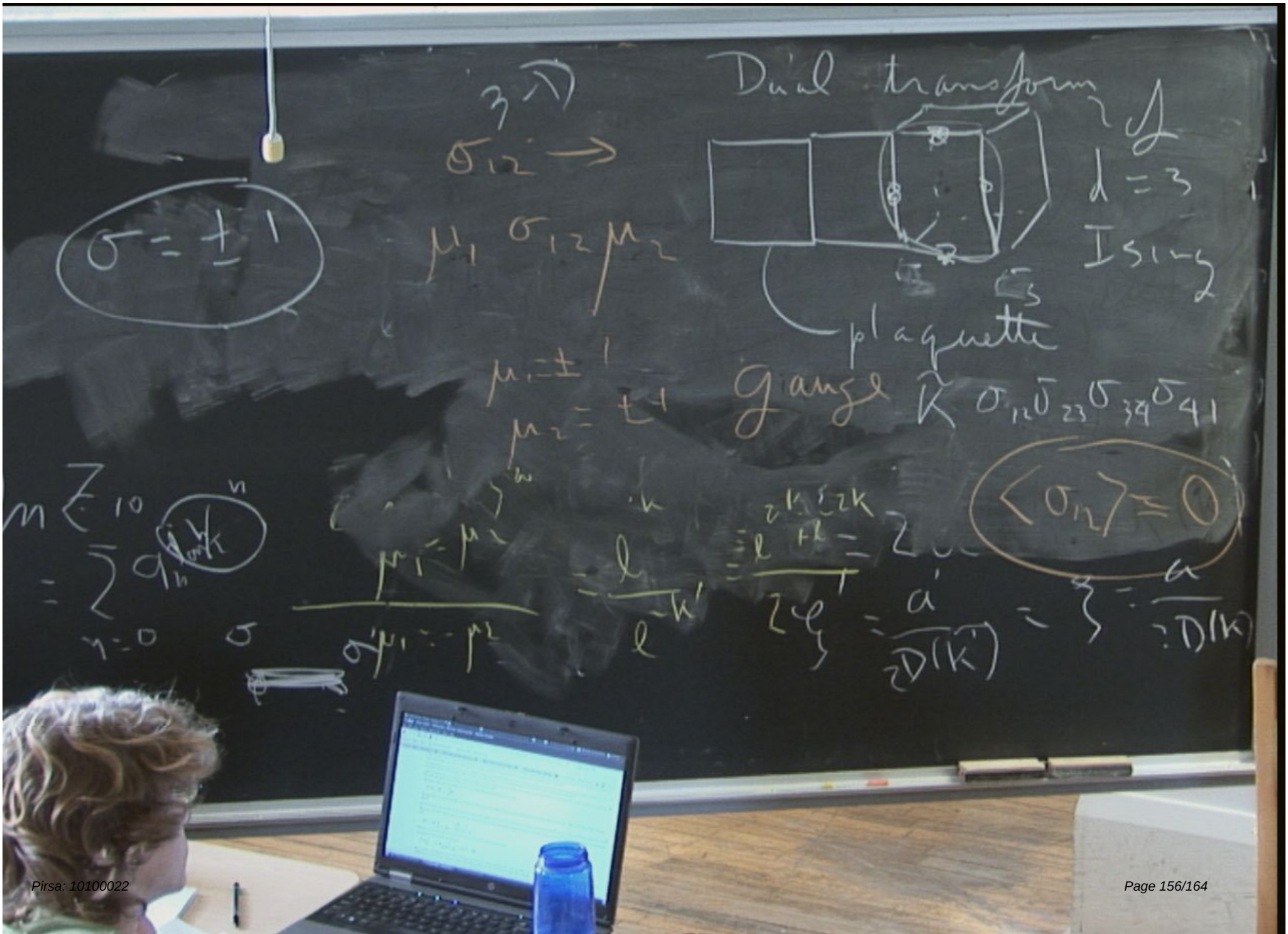
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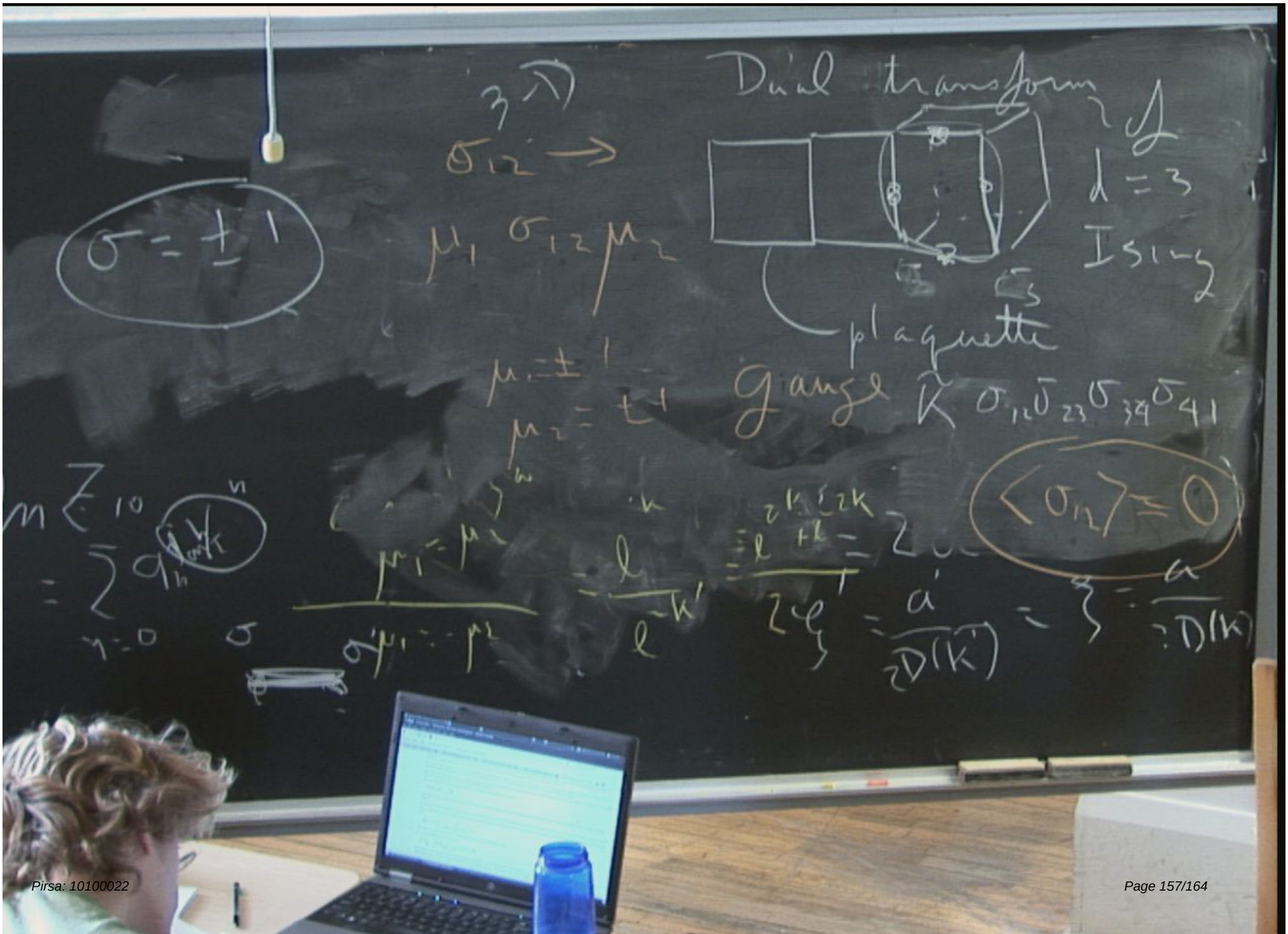
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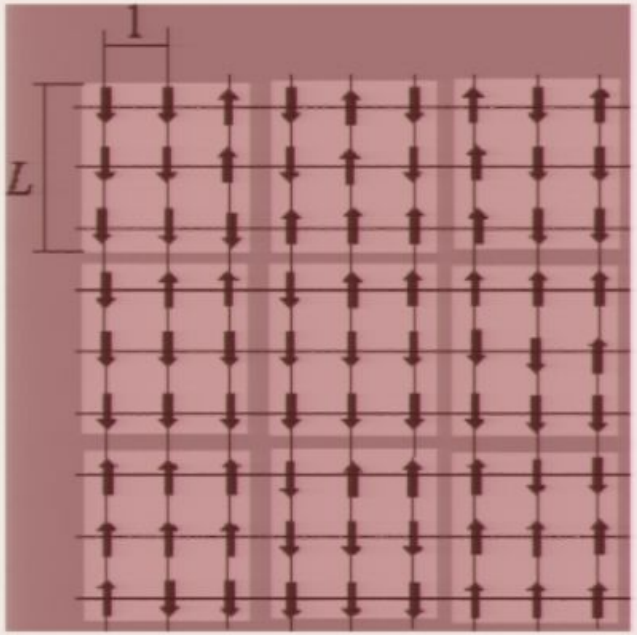

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Renormalization for d-2 Ising model

Ben Widom, myself, Kenneth Wilson.

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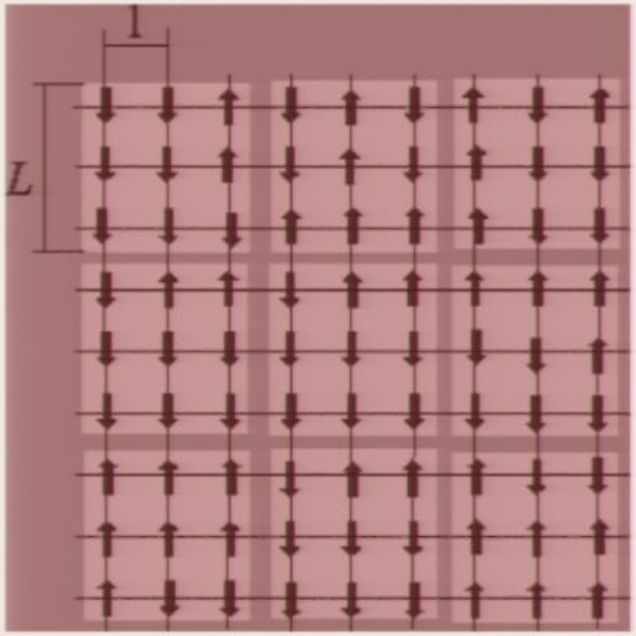

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Renormalization: $a \dashrightarrow 3a = a'$ $\quad W_K\{\sigma\} \dashrightarrow W_{K'}\{\mu\}$ $\quad Z' = Z$ $\quad K' = R(K)$

Scale Invariance at the critical point: $\dashrightarrow$ $K_c = R(K_c)$

Temperature Deviation: $K = K_c + t$ $\quad K' = K_c + t'$

if $t=0$ then $t'=0$

ordered region ($t>0$) goes into ordered region ($t'>0$)

disordered region goes into disordered region

if t is small, $t'=bt$. $\quad b=(a'/a)^y$ defines y. b can be found through a numerical calculation.

coherence length: $\xi = \xi_0\, a\, t^{-\nu}$ 2d Ising has $\nu=1$; 3d has $\nu \approx 0.64$....

$\xi = \xi'$ $\qquad \xi_0\, a\, t^{-\nu} = \xi_0\, a'\, (t')^{-\nu}$

so $\nu = 1/y$

number of lattice sites: $N = \Omega/a^d$ $\quad N' = \Omega/a'^d$

$N'/N = a^d / a'^d = (a'/a)^{-d}$

Free energy: F = non-singular terms $+Nf_c(t) = F'$ = non-singular terms $+N'f_c(t')$

$f_c(t) = f_c^0\, t^{dy}$

Specific heat: $C = d^2F/dt^2 \sim t^{dy-2}$ form of singularity determined by y

One can do many more roughly analogous calculations and compare with experiment and numerical simulation. Everything works!

However notice that this is not a complete theory. It is a *phenomenological* theory. We have no way to find y from theory.

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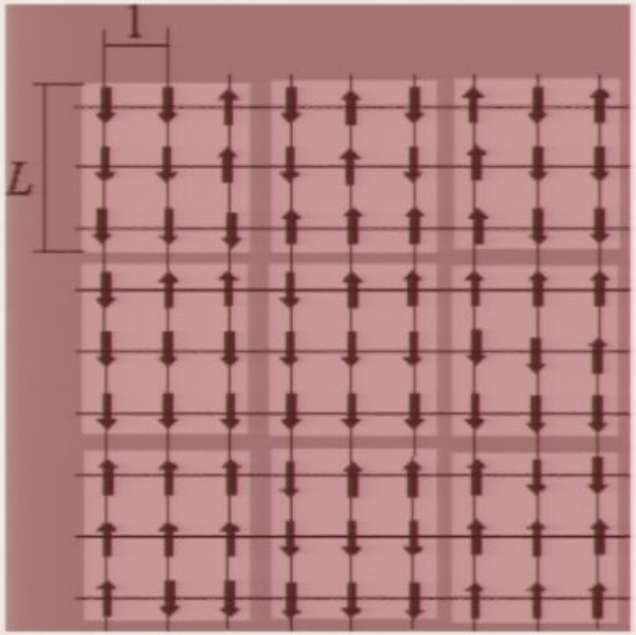

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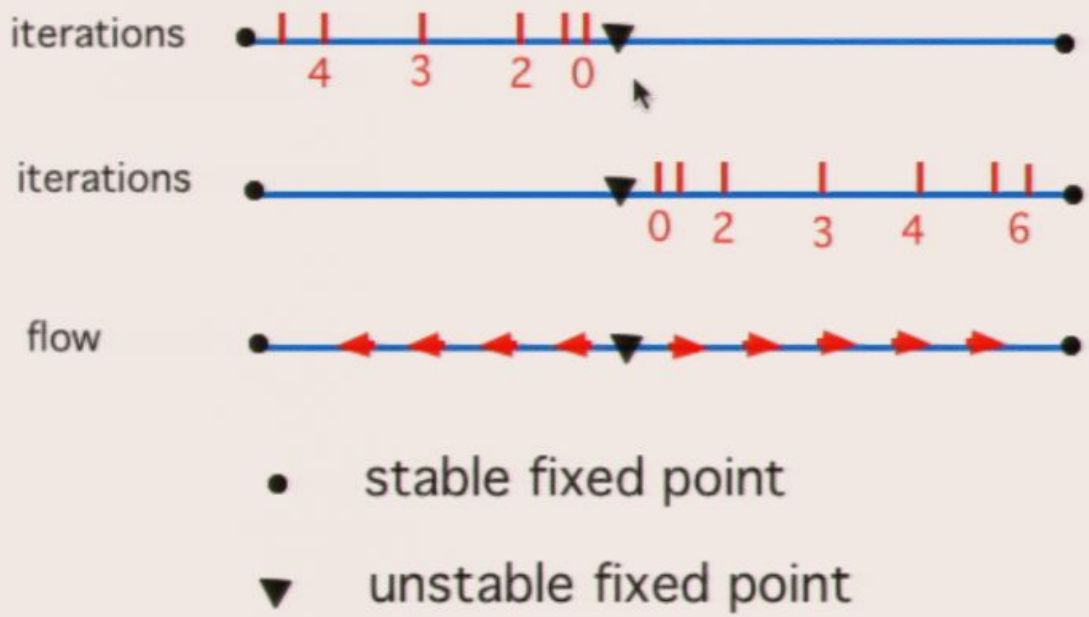
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